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AN INTRODUCTION TO PROJECTIVE GEOMETRY

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By

C. W. O'HARA, S.J.

and

D. R. WARD, S.J.

OXFORD

AT THE CLARENDON PRESS

1937

OXFORD UNIVERSITY PRESS
AMEN HOUSE, E.C. 4
London Edinburgh Glasgow New York
Toronto Melbourne Capetown Bombay
Calcutta Madras
HUMPHREY MILFORD
PUBLISHER TO THE UNIVERSITY

PRINTED IN GREAT BRITAIN

PREFACE

The aim of this book is, as its title implies, to give the reader an introductory course in Projective Geometry. It aims, therefore, at removing the Euclidean prejudices which the study of elementary Geometry begets, and at substituting for them what may be called the Projective mentality; at the same time it seeks to familiarize the reader with most of the important methods used in the subject. Since these include not only what are called pure or Synthetic methods, but also the Algebraic method, the latter is included in the book. And in order that the subject-matter may be kept as simple as possible until facility in the use of these methods is attained, the work is confined to two-dimensional Projective Geometry.

In the first six chapters of the book, after a short historical introduction, the synthetic method is developed as far as the investigation of the more complex properties of the conic. In the next two chapters coordinate systems are introduced projectively, and the Algebraic method is developed. This introduction of coordinates makes possible the definition of metrical concepts, and these are discussed in the ninth and tenth chapters, their true place in the scheme of Geometry being shown. After a short treatment of the theory of transformations, the work is brought to a close by a chapter which indicates the possible developments of the subject from the point reached.

It cannot with truth be said that the book has been written in order to 'supply a long-felt want'. There seems, unfortunately, to be very little demand for the teaching of Projective Geometry in this country. In default of this excuse, therefore, the authors must fall back on another, namely the hope that their work may do something to stimulate a demand for more widespread familiarity with the subject. It is surely time that scholarship candidates in Mathematics and first-year University students should be allowed to know that the classical Geometry which they assimilate occupies but a subsidiary place in the scheme of Geometry. An acquaintance with Projective

Geometry shows them what things are fundamental and what are subsidiary in that scheme; it prepares them too for Geometries even more general than Projective Geometry, and for some at least of the subtleties of modern mathematical Physics.

C. W. O'H.

14 *September* 1936

D. R. W.

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CHAPTER I

HISTORICAL AND CRITICAL

1.1. Historical Outline

It is not a waste of time to begin the subject of Projective Geometry with a short study of its history, and there are two reasons for doing this. The first is that Projective Geometry has behind it a very long history which is well worth knowing for its own sake. The second is that the reader who takes up the subject already knowing something of its history is better equipped to understand it, since he has at least seen in what direction the logical development of the subject is going to take him. In such a book as this it is, naturally, impossible to give a detailed account of the history, but enough can be said to indicate the main courses along which it has developed, and to show the significance of the more important turning-points in those courses.

Looking back from the standpoint of our present-day knowledge, we can distinguish three important lines along which Geometry has developed. In the earliest period, which extends from the time of the Greek geometers or earlier up to the middle of the seventeenth century, only two of these appear. The work of Euclid may be taken as typical of the first of these two. His object was to elaborate a science of the measurement of physical space, and to this end, starting from intuitional ideas of such terms as *point*, *line*, and *distance* or *length*, and their properties, he deduced a number of geometrical theorems which he classified. It should be noticed that to Euclid and those who followed him the notion of distance was fundamental and all-pervasive; it was taken to be so obvious an idea that it did not need to be defined, and it underlay everything in his geometrical science. This will be better realized when it is remembered that the Geometry which is still taught to-day in schools differs very little from that first elaborated by Euclid. Because of this fundamental importance in his Geometry of the notion of distance, the line of development of which Euclid is typical is called the line of *metrical* development.

An examination of the work of Pappus (*c.* 200 B.C.) shows that he was interested in a type of theorem that is not concerned with distance, but with such things as concurrence of lines and collinearity of points. This type of theorem may be called for the present the Projective type, and for this reason the second line of development is called the line of *projective* development. It must not, however, be thought that Pappus excluded from his Geometry the notion of distance; he and his followers did not hesitate to use metrical ideas in proving these non-metrical theorems. At the time, and indeed for many centuries, the distinction between the two types of theorem was not seen.

If the subsequent history of the metrical line of development be followed, it will be found that very early in the history of Geometry geometers directed their efforts towards one particular object in addition to the general aim of proving new geometrical theorems. This was to remove what had come to be considered a blemish on Euclid's system. Euclid himself, and all who came after him, had found it absolutely necessary to assume some sort of 'parallel postulate'† in order to deduce any theorems about parallel lines. The necessity of assuming such a postulate was looked upon as a fault in the system, and geometers were convinced that somehow or other it could be proved from the other axioms. Outstanding amongst those who attempted to do this was Saccheri (1733). His method was the well-known method of *reductio ad absurdum*, and in following it he unwittingly investigated what is now sometimes known as non-Euclidean Geometry. With him must be mentioned Lobatchewskij (1829) and Bolyai (1833) who, convinced by the unsuccessful attempts of geometers to prove the parallel postulate that it was definitely not deducible from the other axioms of elementary Geometry, built up a system of metrical Geometry based on the logical alternative to the parallel postulate.

So far, two of the main lines of development of Geometry have been noticed, and with Descartes (1637) the third begins. Descartes introduced into the study of metrical Geometry a

† In modern text-books of Geometry which are in use in schools the postulate known as Playfair's Axiom is usually assumed. This runs: *Through any point not on a given line, one and only one line can be drawn parallel to the given line.*

method which is now known as *Coordinate Geometry* or *Analytical Geometry*. The essence of this method is that the concepts of elementary Geometry are expressed in algebraic language, and the various theorems are proved therefrom by the application of the laws of Algebra. It is quite clear that Descartes thought the traditional geometrical methods cumbrous and unphilosophical, since they all seemed to him to depend on the happy intuition of a construction which would lead to the proof of the desired theorem. For this apparently haphazard method he wished to substitute the universally applicable, certain, and abstract method of proof by Algebra. Descartes's new method was still a metrical method, although it marks the beginning of a new line of geometrical development; to distinguish it from what has just been considered, it will be called the *Metrical-Analytical* line, and Euclid's the *Metrical-Synthetic* line.

A question naturally arises here: is it licit to call by the name of Geometry such a method as the analytical method? As an historical fact, mathematicians were very sharply divided in their opinions, and it was because of this that there arose the terms *Pure* (or *Synthetic*) *Geometry*, and *Analytical Geometry.* Those on the extreme right would not admit that any proof of a geometrical theorem by algebraic methods was a valid proof, though they admitted that these methods might be used to suggest problems for the pure geometer. Those on the extreme left did not indeed condemn the methods of Pure Geometry as invalid, but they certainly despised them as elephantine.†

The next critical phase in the development of metrical Geometry occurs with the publication of Poncelet's work (1822). Although he was convinced of the autonomy of Pure Geometry, so that his contribution to Geometry is really an advance along the Metrical-Synthetic line, he did not disdain to learn something from Analytical Geometry. Analytical Geometry, we have seen, took over from Pure Geometry the fundamental ideas, and translated them into algebraic language; Poncelet,

† An interesting example of the division of opinion is shown in the editorial policies of two of the leading mathematical journals of a century or more ago. *Crelle's Journal* would never admit to its pages any algebraic proof of a geometrical theorem; *Liouville's Journal*, on the other hand, refused to print anything but algebraic proofs of geometrical theorems.

however, took over from Algebra and Analytical Geometry, and translated into pure-geometrical terms, certain ideas and principles which it seems hard to believe the pure geometer would have discovered for himself until very much later. This point, because of its importance, needs to be illustrated by a concrete example.

In Algebra, until the theory of complex numbers was founded on a firm basis, all that could be said of a quadratic equation was that it had two distinct, or two coincident, or no roots. But once complex numbers had been admitted with full rights into Mathematics, it could be said that every quadratic equation without exception had two roots. Now it will easily be seen that the problem of finding the points common to a straight line and a conic by methods of Analytical Geometry leads always to the solution of a quadratic equation. Hence, once complex numbers were placed on the same level as other numbers, the analytical geometer became convinced that any conic and any straight line always had two points, distinct or coincident, in common. This conviction was not overthrown by its apparent contradiction with the inference to be drawn from the figure on paper; in fact, it led the analytical geometer to the further conviction that the points which he could draw physically on paper were not all the points of which Geometry should treat. In passing, it should be realized that this in itself was an important step forward, for it meant that mathematicians were beginning to get a more abstract idea of the terms *point* and *line*.

Poncelet's advance on his predecessors was to take over this analytical discovery into Pure Geometry, and to state definitely as a principle that any conic and any line had a pair of common points, and that from any point two tangents could be drawn to a conic. These new *complex points* were taken over into Pure Geometry and their interrelations discussed by pure-geometrical methods in just the same way as the other points of the geometrical field, even though they could not be represented by marks on paper. This did not, as a matter of fact, exhaust Poncelet's contribution to Geometry, but the remainder of his work belongs rather to the projective line of development, to which we now return.

It has been said about Pappus, whose name is the first to occur in the second, or Projective, line of development, that while the theorems he collected were non-metrical, his proofs of them were based on metrical theorems. It can also be said that all investigation of theorems of this type was for a long time marred by the fact that they were proved metrically. Development along the projective line was thus a mixed or hybrid development, and it is perhaps because of this that progress was slow. Even the contributions made by Desargues (1593) and Pascal (1623), though they were purely projective and non-metrical theorems, were proved metrically.

It was not until the publication of *Geometrie der Lage* (1847) and *Beiträge zur Geometrie der Lage* (1856) by von Staudt that Projective Geometry began to emerge as a geometrical science entirely independent of the notion of distance. It is scarcely an exaggeration to say that von Staudt's work began to open mathematicians' eyes to the real nature of Geometry, and made them begin to suspect that length, hitherto looked upon as a fundamental geometrical notion, was not so in fact. An English mathematician, Cayley (1859), brought von Staudt's work to completion by showing that distance or length could be defined in simpler terms, and moreover that what had hitherto been accepted as the idea of distance was in fact only a particular case of the much more general projective definition. This led to the further and not less important conclusions that not only was the Metrical Geometry which had been studied since the time of Euclid merely a part of the more general science of Projective Geometry, but that the non-Euclidean Geometries elaborated by Lobatchewskij, Bolyai, Riemann, and others were also only sub-sciences of Projective Geometry. In fact, Cayley was led to declare that 'Projective Geometry is all Geometry'.

Cayley's work was done by algebraic methods, and it was left to Klein (1872) to translate it into the language of Pure Geometry. Thenceforward, Projective Geometry, whether pure or analytical, gradually came to be recognized as the fundamental geometrical science of the simplest type, and Metrical Geometry as the expression in other terms of some of its theorems. It is interesting to note that in Cayley's general projective definition of distance was contained the solution of a number of problems in the very modern Theory of Special Relativity; so that, as often happens, the pure mathematician had anticipated the needs of modern physical science.

With the completion of the work of Cayley and Klein it can safely be said that all the most important results and principles of Projective Geometry were finally made explicit. But development was not on that account arrested. For there still remained the task of ordering the science of Projective Geometry hierarchically. That is to say, its propositions had to be classified and arranged in order of dependence, and the fundamental, unproved initial propositions stated and reduced to their simplest terms. This process has gone on almost to the present day. But even so there remained one more important development. For it remained to show that the lines and points of which Projective Geometry speaks were not necessarily the particular ones which had helped the historical development, but rather any objects of which the initial propositions of Projective Geometry are true. This particular result has only come about through the awakened interest in the philosophy of Mathematics which the past half-century has shown. Mathematicians have now realized that Geometry deals not with the points and lines of physical space necessarily, but with something much more abstract; it is no longer their aim to measure physical space; that task is now left to the physicist, who may or may not find the mathematician's theorems of value in performing it.

1.2. The Characteristics of a Mathematical Science

At the end of the foregoing paragraphs it was said that Geometry no longer concerned itself necessarily with the points and lines of physical space. It is necessary to explain this statement carefully, for plainly it is essential that the reader should know from the outset what sort of objects Projective Geometry is dealing with. In order to do this we consider for a moment what a science is, and how a branch of Mathematics differs from other sciences.

In Physics we start from certain *observed facts*, usually the results of experiments in the laboratory. From these results, and from the admitted principle of uniformity in nature, we *argue* to the causes of the observed facts, and so formulate physical theories to account for them. We may also argue from them that certain other experiments, not yet performed, will have certain results.

In History, similarly, we start with *recorded facts*; from these we can sometimes argue that certain unrecorded facts must have occurred, and we may also trace the effects of these recorded facts, even though they are not recorded as effects.

In both these examples, and indeed in any example of a science that may be chosen, two parts are clearly distinguishable; the first may be conveniently called the *initial propositions*; the second is the process of *inference* from these initial propositions.

In any branch of Mathematics the same two parts can be distinguished, for there too are initial propositions and inferences drawn from them. But, in a branch of Mathematics, the initial propositions are not statements of observed or recorded facts; indeed, they are not statements of physical fact at all. It is true that they may have been *suggested* by observed facts, but, nevertheless, the initial propositions of Mathematics are always, in essence, propositions about *ideas* or *concepts* whose full connexion with physically existing reality is a matter of secondary importance to the mathematician.

To the Greek geometers, and indeed to the geometers of many subsequent centuries, the points and lines of Geometry were the points and lines of physical space, and the science of Geometry was a set of deductions from certain observed facts about them. Nobody ever pretended that it was anything else. It was only as a result of Poncelet's work that mathematicians began to see that when they used the words *point* and *line* they were really talking of something more general and more abstract than their predecessors had done. For nobody believed that the newly introduced complex or 'imaginary' points were points of physical space; nevertheless they were points in the sense in which Geometry made use of the term. And so, after Poncelet's time, the terms *point* and *line* stood for things of which the points and lines of physical space were but particular cases.

But the real opportunity of framing a completely abstract and mathematical definition of the terms had occurred earlier, and been missed, when Descartes introduced Analytical Geometry into Mathematics. Indeed, the full significance of Analytical Geometry was not realized by mathematicians for two and a half centuries. They are not to be blamed for this, for the point is somewhat subtle; nevertheless it is worth trying to grasp.

To Descartes the coordinates (x, y) of a point were but a label distinguishing that point from the rest. Similarly, the equation

$$lx+my+n = 0$$

was an equation which was associated with a certain line, and which distinguished it from all the rest. Had he made a slight addition to his terminology he might have reached a conclusion which would have prevented all the subsequent acrimony between analysts and synthetists, and which would also have anticipated modern work.

Let us suppose that in addition to speaking of 'the point (x, y)' he had also spoken of 'the line $[l, m, n]$', meaning thereby the line whose equation is

$$lx+my+n = 0.$$

$[l, m, n]$ would then have been a label attached to a line in just the same way that (x, y) was a label attached to a point. Analytical Geometry would then have dealt with *number-pairs*—labels attached to points—and *number-triples*—labels attached to lines. But another point of view would have been possible; for instead of considering spatial Geometry as the principal science and Analytical Geometry as but its algebraical translation, Analytical Geometry could have been looked on as the principal science—the science of these number-pairs and number-triples—and spatial Geometry as merely a spatial representation of it. It is quite clear that either of the two can be considered the principal science, and the other as the ancillary science, and that neither has any real claim to priority. It is quite clear too that any theorem about spatial points and lines can be translated into a theorem about number-pairs and number-triples, and vice versa. The real, abstract, science of Geometry is the same whichever of these two kinds of things is supposed to be its object, for ultimately the reasoning processes involved are exactly the same in form. And if somebody else had thought of some other way of representing spatial points and lines, the very things used to represent them could have been looked on as the objects of a new science, and the reasoning processes about them as a science in its own right.

It is clear then that Analytical Geometry, considered as a science concerned with number-pairs and number-triples, and Synthetic Geometry have equal claims to be considered as sciences in their own right; and, on account of their similarity, have equal claims to be called Geometry. The first principles of Geometry—the *initial propositions* of which we spoke above—can be formulated in terms of either of the two, and the subsequent reasoning can with equal validity be in terms of either of the two. Now because of all this, it follows that the whole complexus of points and lines of Synthetic Geometry with their interrelations and the whole complexus of number-pairs and number-triples of Analytical Geometry with their interrelations must have some property or properties in common. It is precisely on account of this which is in common between them that there is a science common to the two sets of things. And if there is a third set of things into terms of which Geometry can be translated, this third set has something in common with the other two; and so on. Clearly, Geometry is concerned with what is common to the various possible sets of things and not with the particular properties of each set which differentiate it from the others.

Now though it would be possible to enunciate Geometry in terms of either the points and lines which Euclid thought of, or the number-pairs and number-triples which Descartes might have thought of, or in terms of any other adequate set of things, to do so would be cumbrous. It would be rather like stating Geometry in two or more different languages at once. We therefore agree to use terms which *by convention* mean any of these different things which we know have something in common, just as we have agreed in ordinary everyday language to use the term *animal* indifferently of cats, dogs, and elephants.

Instead of having to speak of points and number-pairs, we agree to use the term *point* indifferently not only of both of these but of any other objects which can play the role these play in Geometry; similarly the term *line* is used to signify anything which can play the role which number-triples play. In other words, the meaning attached to the words *point* and *line* has at one and the same time been made wider and more abstract.

This modification of the meaning of these terms must be carefully noted, and two mistakes avoided. On the one hand, it is not true that in modern Geometry the terms mean something entirely different from what they meant, say, to Euclid; on the other hand, there has been a very definite and important change in their meanings. They have, in fact, been *generalized.*

Having now seen what meaning is attached to the terms *point* and *line* in the science of Geometry, we return to the question of the initial propositions. Plainly, these will be propositions about points and lines in the geometrical sense of those terms, and not in any other sense. But since the terms stand for a much wider class of things than physical points and lines, the initial propositions cannot merely be statements of observed physical facts. Indeed, the mathematician as such is not very much concerned whether or not his initial propositions have any physical application. In this, Geometry resembles every other mathematical science, and it is precisely here that Mathematics differs from other sciences. In Physics, for instance, the initial propositions are very definitely statements of physical fact; in Mathematics they may or may not be, and the mathematician is not concerned even to know whether they are or not.

It may occur to the reader to suggest that since the initial propositions of a mathematical science are as abstract as the foregoing remarks imply, it is open to the mathematician to lay down any arbitrary set of propositions as the initial propositions of a new branch of Mathematics. Even apart from the question whether such an arbitrarily founded science would be fruitful, there is an important condition to be fulfilled by the initial propositions, and this prevents an entirely arbitrary set being chosen. For it is essential that they should be *self-consistent*; that is to say, they must not lead to contradictions.

This is equivalent to saying that a set of initial propositions must not be such that the contradictory of any one of them can be logically deduced from the rest. The necessity of this condition need not be enlarged upon. In Physics and all other natural sciences it is automatically fulfilled, but because Mathematics is not one of the natural sciences, the mathematician must see to it before he starts his work that his initial propositions are consistent.

To do this, he must prove what is called an *existence-theorem.* This is done by finding a set of things of which his initial propositions are true, for if they are simultaneously true in even one such case, they cannot be inherently self-contradictory. In other words, *he must know of at least one particular instance of the general concepts with which his science deals.*

In the second half of this chapter an attempt has been made to sketch the logical and conceptual basis on which a mathematical science rests, and in the following chapters the science of Projective Geometry is worked out in accordance with the principles here laid down. The full significance of what has been said will appear more clearly as the subject is developed, and the reader will be well advised to return to this section in the course of his reading.

CHAPTER II

THE PROPOSITIONS OF INCIDENCE

PROJECTIVE GEOMETRY does not start where elementary Geometry leaves off; that is to say, it does not presuppose any of the results of elementary Geometry. It stands by itself, and is developed logically from its own initial propositions. The reader will find, however, that the two subjects are not entirely unconnected, for it will appear that elementary Geometry is a particular case of Projective Geometry. As a consequence of the fact that it is not dependent on elementary Geometry he must not expect to find that the initial propositions are familiar to him from what he already knows. Indeed, it is only at the end of the development that he will see elementary Geometry emerging. But it must not be supposed that the only aim of Projective Geometry is to establish the results of elementary Geometry; it does this incidentally, but at the same time it shows them in their true perspective, for it shows clearly what places in the hierarchy of Geometry this and other Geometries occupy.

In this chapter are laid down the first few of the initial propositions of Projective Geometry, and the first elementary deductions from them are made.

2.1. Undefined Elements and Initial Propositions

Projective Geometry deals with two kinds of things to which are given the names of *point* and *line*.† No definition of these terms is given save that which is implied by the initial propositions. These state certain relations between points and lines, and since there is contained in them the only definitions of the terms, any things between which the relations stated by them can exist are amongst the possible sets of objects studied by the science.

The first three of the initial propositions are termed the *initial propositions of incidence*:

2.11. The Initial Propositions of Incidence‡

† But see 2.7.

‡ In order to distinguish initial propositions clearly from propositions which are deduced from them, they are printed always in heavy type.

2.111. There is at least one line on which are both of two distinct points.

2.112. There is not more than one line on which are both of two distinct points.

2.113. There is at least one point which is on both of two distinct lines.

2.12. Remarks on the Propositions of Incidence

(*a*) The propositions of incidence speak of but one kind of relation between a point and a line, and to this relation is given the name of being 'on'. While they do not state how many points have this relation to any particular line, the first two state that two different points always have it to one and only one line. The third states that of all the points which are severally on two different lines, there is at least one which is on both.

(*b*) As in elementary Geometry, it is convenient to refer to points by means of Roman capital letters: A, B, C, etc. In addition to this usage, lines will be referred to by means of small letters: a, b, c, etc.

(*c*) Because a point P is on a line q, there is a converse relation between q and P, and to this relation a name must be given. For reasons which will appear very soon, it is most convenient to say that if a point P is on a line q, then the line q is *on* the point P. This may be put formally as a definition.

DEFINITION. *A line q is said to be on a point P if and only if the point P is on the line q.*†

(*d*) It may occur to some readers to object at this point that the statement made earlier on that elementary Geometry is a particular case of Projective Geometry cannot be true. For it may be argued that since in elementary Geometry pairs of lines can be found which are not on a common point, namely parallel lines, the initial proposition 2.113 is not true in elementary Geometry. Hence since one of the initial propositions is not verified, elementary Geometry cannot be a particular case of Projective Geometry.

† Many text-books do not adopt this terminology; instead they speak of a line 'passing through' a point.

The real answer to this difficulty lies in the fact that since elementary Geometry is based on the fundamental notion of *distance* it can only consider points which are at a finite distance from the points under consideration. Hence in elementary Geometry all that can be asserted with certainty is that parallel lines are lines which are not on a common point *at a finite distance*; whether or not they are on a point which is not at a finite distance, elementary Geometry is incapable of saying. And so it cannot assert categorically that parallel lines are not on a common point. While what has been said cannot pretend to be a full answer to the difficulty raised, it at any rate contains the substance of the answer, and indeed all that can usefully be said at this point. Later on, the reader will see for himself the full answer, and he will see too that what has been said is not merely a verbal quibble.

2.2. Existence Theorems

In order to show that the propositions of incidence are not mutually contradictory, it is necessary to prove an *existence theorem*. As has been explained in the preceding chapter, to do this two sets of objects to which can be given respectively the names 'points' and 'lines' must be shown to exist, having relations between them to which can be given the name of 'on'. When this has been done, it is necessary to prove that with these interpretations of the terms *point*, *line*, and *on* the initial propositions of incidence are verified. Such a set of objects will then be called a *representation* or a *verification*.

A number of representations are given, not so much with the idea of convincing the reader by repeated argument that the initial propositions of incidence are compatible, as of showing him the variety of different kinds of things to which the concepts of Projective Geometry are applicable.

2.21. The Algebraic Representation

(*a*) A *number-triple* (x, y, z), where x, y, and z are any numbers whatever, with the sole proviso that not all of them are zero, will be called a 'point'. Since, however, not the numbers themselves but only their ratios are considered, the *number-triple*

(kx, ky, kz), where k is any number different from zero, is not considered to be different from (x, y, z).

(*b*) A *number-triple* $[l, m, n]$, where l, m, and n are any numbers not all equal to zero will be called a 'line'. As before, the triple $[kl, km, kn]$, where k is any number different from zero will not be considered different from $[l, m, n]$. To distinguish 'lines' from 'points' the number-triples representing the former will be enclosed in square brackets.

(*c*) A 'point' (x, y, z) will be said to be 'on' a 'line' $[l, m, n]$ if and only if

$$lx+my+nz = 0.$$

(*d*) It remains to show that with this representation of the terms the propositions 2.111–2.113 are verified.

First, let (x, y, z) and (x', y', z') be two different points, so that not all of the equations

$$x/x' = y/y' = z/z'$$

are true; that is to say, not all of the expressions

$$(yz'-zy'), \quad (zx'-xz'), \quad (xy'-yx')$$

are equal to zero.

Suppose now that

$$l = (yz'-zy'),$$
$$m = (zx'-xz'),$$
$$n = (xy'-yx'),$$

so that not all of the numbers l, m, n are zero. Then the line $[l, m, n]$ is on both of the points (x, y, z) and (x', y', z'), for it is easily verifiable by direct substitution that

$$lx+my+nz = 0,$$

and

$$lx'+my'+nz' = 0.$$

Hence, *on two distinct points there is at least one line*, and so proposition 2.111 is verified.

Secondly, suppose that $[l', m', n']$ is a line distinct from $[l, m, n]$, and that this too is on both of the points (x, y, z) and (x', y', z') which are by supposition distinct. Then

$$l'x+m'y+n'z = 0,$$

and

$$l'x'+m'y'+n'z' = 0.$$

When these two simultaneous equations are solved for l', m', and n', it is found that

$$l' = k(yz'-zy'),$$
$$m' = k(zx'-xz'),$$
$$n' = k(xy'-yx'),$$

where k is an arbitrary constant. That is to say, $l' = kl$, $m' = km$, and $n' = kn$, so that $[l, m, n]$ and $[l', m', n']$ are not distinct lines. Hence, *on two distinct points there is not more than one line*, and so proposition 2.112 is verified.

Finally, let $[l, m, n]$ and $[l', m', n']$ be two distinct lines, so that not all of the numbers

$$(mn'-nm'), \qquad (nl'-ln'), \qquad (lm'-ml')$$

are zero; then since under these conditions the simultaneous equations

$$lx+my+nz = 0,$$
$$l'x+m'y+n'z = 0$$

have a solution, namely

$$x = k(mn'-nm'),$$
$$y = k(nl'-ln'),$$
$$z = k(lm'-ml'),$$

and not all of these numbers are zero, there is a point (x, y, z) on both of the lines. Hence, *there is at least one point on both of two distinct lines*; that is to say, proposition 2.113 is verified.

When these three results are combined, an existence theorem for the three initial propositions of incidence emerges, and so those propositions cannot be mutually contradictory.

The reader who is familiar with what is known as general homogeneous coordinate Geometry will recognize in the algebraic representation something similar to this part of Analytical Geometry, but a very fundamental distinction between the two must be noticed. In the algebraic representation the 'points' and 'lines' *are* number-triples, and nothing more; that is, they are essentially only sets of numbers. Because of the laws of algebra it is possible to find a relation between 'points' and 'lines' to which is given the name 'on'. In other words, certain algebraic properties of number-triples are renamed with geometrical names. But in general homogeneous coordinate Geometry the process is reversed. Here points and lines are the fundamental things, and they are labelled by means of number-triples; the geometrical relations between the points and lines are interpreted as algebraic relationships between the number-triples by means of which they are labelled. In other words, certain geometrical properties of lines and points are renamed with algebraic names. The algebraic representation is the application of Geometry to Algebra; general homogeneous coordinate Geometry, on the other hand, is the application of Algebra to Geometry.

2.22. A Representation from Elementary Solid Geometry

In this representation the diameters of a sphere are 'points', and the great-circles of the same sphere are 'lines'. A 'point' will be said to be 'on' a 'line' if the diameter of the sphere representing the point is also a diameter of the great-circle representing the line. Elementary geometrical considerations establish the truth of the following results:

(*a*) there is at least one great-circle which has two different diameters of the sphere as diameters of itself,

(*b*) there is not more than one great-circle which has two different diameters of the sphere as diameters of itself,

(*c*) there is at least one diameter of the sphere which is also a diameter of each of two different great-circles.

In these three results the propositions 2.111–2.113 are verified.

2.23. A Physical Representation

At the inaugural meeting of a Lunch Club the members decided to formulate the following rules:

(1) Periodical lunches were to be given by the club, and they were to be attended only by members of the club.

(2) Every member of the club was to meet every other member at least once, but not more than once, at one of the club's lunches.

(3) The lists of members selected by the Secretary to attend any two lunches were never to be entirely different, at least one member was to be present at both.

(4) The President, the Treasurer, and the Secretary were to be the only members present at the first lunch, and at all subsequent lunches there were to be at least three members present.

How many members were there in the club?

How many lunches were given?

How many members were present at each of the remaining lunches?

How many lunches did each member attend?

That the above problem is a representation of the propositions of incidence may seem at first sight to be somewhat far-fetched, nevertheless it is true. In order to show that it is a representation it is first of all necessary to show that the rules agreed upon could, in fact, be kept, and that they did not impose an impossible set of conditions. It is left to the reader to show that they were not impossible, and that the four questions have definite answers; this is an interesting problem in logic. Once it is solved it is easy to show that the whole thing is a representation of the propositions of incidence.

If a member of the club be a 'point' and a club-lunch be a 'line', and if by definition a 'point' is on a 'line' when the member in question is present at the lunch in question, then the following propositions are verified:

(*a*) *two distinct points are on at least one line*, by Rule (2);

(*b*) *two distinct points are on not more than one line*, by Rule (2) also;

(*c*) *there is at least one point which is on both of two distinct lines*, by Rule (3).

2.24. The Drawn Figure as a Representation

It is customary in elementary Geometry to illustrate theorems, constructions, and the like by figures drawn on paper, and it is natural to ask whether Projective Geometry can be illustrated in the same way. This question is really equivalent to the following. Are dots made on paper and marks made with a pencil drawn across the paper in contact with a straight-edge valid representations of the points and lines of Projective Geometry, when the obvious convention is made about the relation 'on'? In order to have names for these two kinds of marks let us call them, for the moment, *drawn points* and *drawn lines* respectively.

Now in elementary Geometry the drawn figure fulfils two functions. In the first place it is a help to the imagination, in that it enables the mind to concentrate on the work in hand;

indeed it may suggest methods of proof and so on. In the second place it is an *approximate* representation, and this statement is best illustrated by an example. It is proved in elementary Geometry that the sum of the angles of a triangle is equal to two right angles; if a triangle be drawn on paper, and the angles between the drawn lines be measured and added together, it is found that their sum is approximately two right angles. Owing, however, to the deficiencies of the instruments used to measure the angles, they can only be measured to a known degree of accuracy, so that the sum is only known to lie within certain limits; usually it is found that these limits enclose two right angles. Nearly every theorem in elementary Geometry has the same sort of *approximate verification* in the drawn figure. But besides these two, the drawn figure is sometimes made to fulfil another and an illegitimate function in elementary Geometry. It is, for instance, nearly always taken for granted that the diagonals of a parallelogram necessarily intersect and are not parallel, merely because the drawn diagonals intersect. That is to say, inferences are made from the figure, when they should be made from the data of a problem.

In Projective Geometry it is plain that if in some sense drawn lines and drawn points are a representation of points and lines, then the drawn figure can fulfil the first of these functions exactly as in elementary Geometry. But when we come to inquire whether it can fulfil the second function, that of being at least an approximate representation, it becomes obvious that the question is without meaning. For measurement of distance and angle does not figure in the initial propositions, and there can be approximation only where there is measurement. The question 'Are drawn points and drawn lines valid representations of the points and lines of Projective Geometry?' is impossible to answer either affirmatively or negatively, and the answer that they are an approximate representation is nonsense. Hence it appears that no use can be made of the drawn figure, since we cannot be certain that it is a valid representation. Nevertheless, a glance at the remaining pages of this book will suffice to show that drawn figures are used, and this use is justified, not by *a priori* considerations of the nature of drawn points and drawn lines, but rather by *a posteriori* considerations. For it is found, in practice, that a careful and reasonable use of drawn figures does not lead to absurd results, and that it does fulfil the first and main function of a drawn figure, in that it helps the mind to concentrate on the problem as a whole, and suggests new results for proof. The third and illegitimate function of the figure is as illegitimate in Projective Geometry as in elementary Geometry.

2.3. First Deductions

THEOREM. *There is only one point which is on both of two distinct lines.*

Let l and m be two distinct lines, then by 2.113 there is a point P which is on both of them.

Suppose now that there is a second point Q, distinct from P, which is on both l and m. That is to say, there are two distinct lines l and m which are both on the two distinct points P and Q.

But this conclusion contradicts 2.112, which states that on two distinct points there is not more than one line.

Hence the supposition that a second point Q, distinct from P, is also on l and m must be false, and this proves the theorem.

2.31. The Principle of Duality

So far, there are the three initial propositions of incidence, and one theorem which has been deduced from them. These four propositions are now set out together:

(2.111) *On two distinct points there is at least one line.*

(2.112) *On two distinct points there is not more than one line.*

(2.113) *On two distinct lines there is at least one point.*

(2.3) *On two distinct lines there is not more than one point.*

It will be seen that by interchanging the terms *point* and *line* the first two of these propositions are changed into the last two, and vice versa. This fact has important consequences.

Let us suppose that there is a proposition about *points, lines,* and the relation *on* which is deducible from the propositions 2.111–2.113; then by interchanging the terms *point* and *line* in the enunciation, a proposition is obtained which is plainly deducible from the propositions 2.111, 2.113, and 2.3. For the necessary proof is obtained by making the same interchange in the original proof. But since 2.3 is deducible from the other three, the new proposition must also be deducible from them.

Hence if any proposition about *lines*, *points*, and the relation *on* is deducible from the propositions of incidence, the proposition obtained by interchanging the terms *line* and *point* is also deducible from the propositions of incidence.

As Projective Geometry proceeds it becomes necessary to define new terms which are complications of the elementary concepts contained in the propositions of incidence; at the same time corresponding terms will also be defined, the definitions being obtained by the above-mentioned interchange. In this way a vocabulary of corresponding terms will be elaborated, and such pairs of terms will be said to be *dual* terms. In the terms *point* and *line* we have a first pair of dual terms. The process of changing a proposition by substituting for every term its dual term is called *dualizing* a proposition. Hence if any proposition is deducible from the propositions of incidence, its dual is also deducible from them.

This important result is known as the *Principle of Duality*.

Projective Geometry, however, is not based on only the propositions of incidence, for other initial propositions are added as the work proceeds. It will be necessary then, if the Principle of Duality is to be preserved, either to add also the dual of every other initial proposition, or to prove it. It will be found, as a matter of fact, that with the initial propositions which are to be added, it is an easy matter to prove the required dual propositions.

2.4. Extension

In elementary Geometry it is taken for granted that the number of points on any line is not finite, and it is natural to ask whether anything of the same sort is true in Projective Geometry. The initial propositions of incidence, however, do not enable us to give any definite answer to the question 'How many points are there on the line?' For in the representation given in 2.23 there are only three points on any line, while in the algebraic representation there is not a finite number of points on any line; yet both of these are compatible with the propositions of incidence. Indeed, it is not difficult to construct a representation in which there are any desired finite number of points on a line.† Hence it is possible to say, roughly, that the propositions of incidence are compatible with there being any number of points, finite or infinite, on a line. Incidentally, this may help to show the reader that in Projective Geometry the terms *point* and *line* have a much wider significance than they have in elementary Geometry.

Now it does not require much thought to see that the propositions of incidence are compatible with even simpler systems than that given in 2.23. They are true, for instance, of the following systems:

(i) a system consisting of three points which are not all on the same line, and the three lines which are on pairs of these points;

(ii) a system consisting of n points all on one line, one other point not on this line, and the n lines each of which is on this special point and one of the other n points.

The discussion of these extremely simple systems is not very fruitful, and so we shall lay down, provisionally, two initial propositions which will exclude them. These propositions will be the following: *Not all points are on the same line*, and *There are at least three points on every line.*

If we call the totality of points and lines in any system the *field*, it will be seen that the above propositions state something about the *extent* of the field, and so they are called *propositions of extension.* The effect of the two propositions of extension here stated is to put, as it were, a lower bound to the simplicity of the field. This is merely a matter of convenience, for it obviates the necessity of constantly stating annoying and trivial exceptions to general theorems which are true in the less simple fields. But it must be noticed that these two propositions of extension are not definite, that is to say, they do not state that the field has any definite extension; they merely say

† But there is one proviso. If there are, say, n points and no more on one line, there are n points and no more on every line. See Ex. 3 of the examples which follow.

that it must not be too simple. For the present they will suffice, but later more definite propositions will be substituted.

2.41. Provisional Initial Propositions of Extension

2.411. Not all points are on the same line.

2.412. There are at least three points on every line.

In accordance with what has been said about the Principle of Duality, it is necessary to prove the duals of these two propositions and to show that they are compatible with the propositions of incidence. These theorems are not difficult, and they are left to the reader as examples.

Examples

1. Using the propositions of incidence and 2.411, show that not all lines are on the same point.

2. Using the propositions of incidence, 2.411 and 2.412, show that there are at least three lines on every point.

3. Show that if there are precisely n points on one line, then

(i) there are precisely n points on every line,
(ii) there are precisely n lines on every point,
(iii) there are precisely n^2-n+1 points in all,
(iv) there are precisely n^2-n+1 lines in all.

4. Show that the geometrical reasoning used to solve the last example provides a solution of the puzzle quoted in 2.23.

5. Show that the provisional propositions of extension are compatible with the propositions of incidence by showing that they are verified in (i) the algebraic representation, and (ii) the representation of 2.23.

2.42. A Note on 2.412

The second of the propositions of extension states that there are at least three points on every line, but in the work which follows it is assumed in every theorem that there are sufficient points on every line to make the theorem significant. Thus if there is a theorem about six points on a line, it would be a meaningless theorem in a field where there are only three points on every line. Strictly speaking, the enunciation of such a theorem should be qualified by the phrase 'Provided there are at least six points on every line'. These qualifications will, however, be systematically omitted, for it is plain that they are implied.

2.5. Notation

In 2.12 (*b*) it was remarked that capital letters would be used to designate points and small letters to designate lines, and these beginnings of the notation used in Projective Geometry are now developed.

The first two propositions of incidence ensure that there is a unique line on two distinct points. Hence there is no ambiguity in referring to the line on the two points A and B as the line AB. This usage is familiar from elementary Geometry.

Dually, by 2.113 and 2.3 there is a unique point on two distinct lines a and b, and this point will be referred to as the point ab.

Often, however, it will be necessary to speak of the common point of two lines which are known only as, say, XY and AB; and though it might be possible, with care, to speak of the point $XYAB$, it is not very desirable to do so. Instead, we use the notation: 'the point $\begin{pmatrix} XY \\ AB \end{pmatrix}$', and, dually, 'the line $\begin{pmatrix} xy \\ ab \end{pmatrix}$'.

The reader may find it useful in his written work to shorten the phrases 'The point P is on the line q' and 'The line x is on the point Y' to the symbolic statements

$$P \mid q \quad \text{and} \quad x \mid Y,$$

respectively. The negative statements 'The point P is not on the line q' and 'The line x is not on the point Y' may be shortened to

$$P \nmid q \quad \text{and} \quad x \nmid Y,$$

respectively. This notation is not adopted in this book.

What has been said is completed by giving definitions of the two dual terms *collinear* and *concurrent.*

2.51. Definition

Three or more points which are all on the same line are said to be collinear.

2.52. Definition

Three or more lines which are all on the same point are said to be concurrent.†

† The etymologically dual term is *copunctual* or *compunctual*; it seems to savour of pedantry to prefer one of these to the well-known term *concurrent,* even though this has in it an erroneous suggestion of motion.

2.6. Figures, Theorems, Constructions

2.61. Figures

In elementary Geometry, the term *figure* is used to signify indifferently either the drawn figure or an assemblage of lines and points, and there is no reason to make any careful distinction between the two. But in Projective Geometry, whose connexion with the drawn figure is rather tenuous, the term is restricted to the second of these two meanings. For convenience, figures are classified into point-figures, line-figures, and mixed figures; the following definition gives the principle of this classification.

2.611. DEFINITION. *Any set of points and lines is termed a figure; if it consists of points only, it is termed a point-figure; if it consists of lines only, it is termed a line-figure; otherwise it is termed a mixed figure.*

2.612. DEFINITION. *A point-figure all of whose points are collinear is termed a collinear point-figure.*

2.613. DEFINITION. *A line-figure all of whose lines are concurrent is termed a concurrent line-figure.*

2.614. Simple Figures. The following is a list of some of the simpler figures which occur in Projective Geometry:

1*a*. *The Point.*

1*b*. *The Line.*

2. *The Point-on-Line.* A self-dual figure consisting of a single point and a single line, the point being on the line.

3. *The Point-and-Line.* Self-dual. As 2, except that the point is not on the line.

4*a*. *The Point-pair.* A point-figure consisting of two distinct points.

4*b*. *The Line-pair.* The dual of 4*a*.

5*a*. *The Point-pair on a line.* This is 4*a*, together with the line on the point-pair.

5*b*. *The Line-pair on a point.* The dual of 5*a*.

6*a*. *The Three-point.* A point-figure consisting of three non-collinear points.

6*b*. *The Three-line.* The dual of 6*a*.

7. *The Triangle.* A mixed, self-dual figure consisting of three non-collinear points, together with the three lines which are on pairs of these.

The triangle is an extremely important figure in the sequel; in order to be able to refer to its constituent parts, the three points are called the *points* or *vertices* of the triangle, the three lines are called the *lines* or *sides* of the triangle. It should be noted that there are only three points in a triangle; any other points on any of the sides are not points of the figure.

The side BC of a triangle ABC will be said to be *opposite* to the point A, and vice versa. Similarly, B is opposite to CA, C to AB.

2.62. Theorems

The word theorem has been used before this, and it is taken for granted that its meaning is known; if, however, a formal definition of the term be needed, the following will suffice: *A theorem is any true statement about the points and lines of the field.*

The word *true* merits notice; a proposition is said to be true in Projective Geometry if it is a logical consequence of the initial propositions. A proposition may be true of a certain representation without being true in Projective Geometry; but a proposition cannot be true in Projective Geometry without being true of all the representations. Thus the proposition 'There are only three points on a line' is true of the representation of 2.23, but it is not a theorem of Projective Geometry; whereas the proposition 'Two lines have only one point in common', being a theorem of Projective Geometry, is true in all the representations.

Some cautions must be given about the proving of theorems. The proof must proceed by strictly logical deduction from the initial propositions or from theorems already proved. Hence

(i) To say 'It is obvious from the figure that . . .' is not a sound reason for the statement which this phrase precedes. At best, this only shows that the conclusion is true in a representation, not that it is true in Projective Geometry; and even this assumes what has not been proved, namely that the drawn figure is a representation.

(ii) Reasoning which is based on considerations of *length* or *angle*, in any form whatever, is not logical deduction, for these terms are not yet defined. When they are defined they may be used like any other term.

(iii) Propositions which are true only under certain conditions should not be stated as theorems without stating the conditions definitely.

2.63. Constructions

In elementary Geometry there are, besides theorems to be proved, exercises known as *constructions*. In these some figure is given, and it is necessary (i) to give practical rules for determining by means of ruler and compass certain points and lines or some other figure which has some specified property, and (ii) to prove that the figure so determined has the desired property.

From what has already been said, it will be plain that the first of these two cannot be a part of any constructions which appear in Projective Geometry. Its place is, however, taken by something else. In a construction in Projective Geometry a figure is given, and it is necessary (i) to specify exactly some other figure (by stating relations which exist between it and the given figure) which has some desired property, and (ii) to prove that this figure has the desired property.

2.7. Projective Geometry of Many Dimensions

At the beginning of this chapter it was stated that Projective Geometry deals with two kinds of things, namely *points* and *lines*, and that it discusses the relations which exist between them as a consequence of the relations of incidence. While this statement is true, it is not the whole truth, for the Projective Geometry of points and lines is not the whole of Projective Geometry; it is only that part known as two-dimensional Projective Geometry.

It would have been possible to start with three different kinds of things, *points*, *lines*, and *planes*, and after stating propositions of incidence about these, to have discussed their interrelations. This would have been three-dimensional Projective Geometry. And in complete generality it would have been possible to start with n kinds of things, which might have been called T_0's (points), T_1's (lines), T_2's (planes), T_3's, T_4's,..., T_{n-1}'s; this would have been n-dimensional Projective Geometry. It will be plain that these Projective Geometries are classified according to the number of different kinds of things they deal with; and that a Projective Geometry dealing with k different kinds of things is called k-dimensional Projective Geometry.

Suppose now, for simplicity's sake, that we proposed to study three-dimensional Projective Geometry. It would obviously be desirable to spend some time in studying the interrelations of only those points and lines which are on a single plane. Of this restricted field of points and lines the initial propositions of incidence, 2.111–2.113, would be true, and the Projective Geometry of the plane would be two-dimensional Projective Geometry. Similarly, in n-dimensional Projective Geometry the study of the interrelations of the T_0's, T_1's, T_2's,..., T_{k-1}'s which are on a T_k is the same as the study of k-dimensional Projective Geometry.

All this has been mentioned, not for the purpose of mystifying the reader, but rather in order to show him two important things. First, that what is given in this book, namely two-dimensional Projective Geometry, is only a part of the whole subject. Secondly, that nevertheless it is an extremely important part. For in two-dimensional Projective Geometry most of the basic ideas are developed which pervade the whole subject. It is better to become acquainted with these ideas in the simplified field of points and lines only, rather than in the more complicated fields with which the higher dimensions are concerned. Once these ideas are familiar, it is not a difficult matter to pass on to higher dimensions. To do so is indeed the natural generalization to which the elementary two-dimensional work points the way.

EXAMPLES

1. Show that if the point A is on the line BC, then the point B is on the line CA, and the point C is on the line AB.

2. Show that if the three points A, B, C are not collinear, then the lines BC, CA, AB are not concurrent.

3. Show that the points ax, bx, cx, dx,... are collinear.

4. State the duals of the last three examples, and prove them without appeal to the principle of duality.

5. Show that the initial proposition 2.113 is a consequence of 2.111, 2.112, and 2.3.

6. If A, B, C, D are four points, no three of which are collinear, show that there are three and only three points, E, F, G, each of which is collinear with two pairs of the four given points.

7. In the last example show that if there are only three points on every line, then E, F, G are collinear. (The reader is advised to try to draw a figure illustrating this.)

8. In the Algebraic Representation show that the necessary and sufficient condition that the three distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) should be collinear is that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

9. Show that, in the Algebraic Representation, the three points (x, y, z), (ξ, η, ζ), and $(lx+\lambda\xi, ly+\lambda\eta, lz+\lambda\zeta)$ are collinear.

CHAPTER III

PERSPECTIVITY AND PROJECTIVITY

3.1. Perspective Figures

3.11. Definition

Two point-figures $ABCD....$, $A'B'C'D'....$ *are said to be centrally perspective on the point* O, *or in central perspective from the point* O, *if the lines* AA', BB', CC' *are all on* O.

The point O is called the *centre of perspective.*

3.12. Definition

Two line-figures $abcd....$, $a'b'c'd'....$ *are said to be axially perspective on the line* o, *or in axial perspective from the line* o, *if the points* aa', bb', cc',... *are all on* o.

The line o is called the *axis of perspective.*

The terms *central(ly) perspective* and *axial(ly) perspective* are, plainly, dual terms; so also are *centre of perspective* and *axis of perspective.*

3.13. Triangles in Perspective

The consideration of figures in perspective is divided into two parts:

(i) perspective point-figures whose points are not all collinear, and the dual of this;

(ii) perspective point-figures whose points are all collinear, and the dual of this.

The second part is of greater importance, and the first is only considered in so far as it helps in the consideration of the second.

The simplest figure whose points are not all collinear is the triangle, and perspective triangles are therefore considered here. In fact it is not necessary to consider any other types of figure in perspective in this first part.

Since the triangle is a mixed figure, consisting of points and lines, it would appear at first sight that two triangles could be centrally perspective, or axially perspective, or both, or neither. In 1639 Desargues published a theorem in elementary Geometry which stated that any two triangles which were centrally perspective were also axially perspective, and vice versa. This theorem can be easily deduced from the initial propositions of incidence and extension in the Projective Geometry of three or more dimensions, but it is not a consequence of the corresponding propositions in the Projective Geometry of two dimensions. In fact there are systems of points and lines in which all the initial propositions so far stated are verified, but in which Desargues's theorem is not verified. Similarly there are systems in which not only the initial propositions of incidence and extension but also Desargues's proposition are all verified.

Since one of the objects of this book is to provide an introduction to Projective Geometry of three or more dimensions, no useful purpose will be served by considering systems in which Desargues's proposition is not true, for these cannot occur as sub-systems in a Geometry of more than two dimensions. We therefore confine ourselves to those systems in which it is true, and this is tantamount to laying down Desargues's proposition as an initial proposition. When this has been done, it will be necessary to show that it is compatible with the other initial propositions; this is done by showing that it is verified in the algebraic representation; it will also be necessary to prove the dual of Desargues's proposition.

3.14. Desargues's Proposition

If two triangles are centrally perspective, they are also axially perspective.

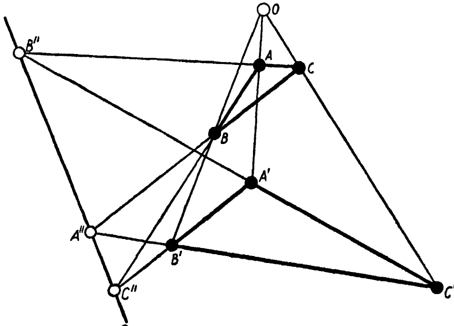


FIG. 1.

In the figure the two triangles ABC, $A'B'C'$ are centrally perspective on the point O and axially perspective on the line o.

3.141. Verification in the Algebraic Representation. Let A, B, C be (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) respectively, and A', B', C' be (x_1', y_1', z_1'), (x_2', y_2', z_2'), and (x_3', y_3', z_3') respectively. Let O, the centre of perspective, be (ξ, η, ζ).

Since O, A, A' are collinear, O, B, B' are collinear, and O, C, C' are collinear, numbers $\lambda_1, \lambda_1', \lambda_2, \lambda_2', \lambda_3, \lambda_3'$ exist such that

$$\xi = \lambda_1 x_1 + \lambda_1' x_1' = \lambda_2 x_2 + \lambda_2' x_2' = \lambda_3 x_3 + \lambda_3' x_3',$$
$$\eta = \lambda_1 y_1 + \lambda_1' y_1' = \lambda_2 y_2 + \lambda_2' y_2' = \lambda_3 y_3 + \lambda_3' y_3',$$
$$\zeta = \lambda_1 z_1 + \lambda_1' z_1' = \lambda_2 z_2 + \lambda_2' z_2' = \lambda_3 z_3 + \lambda_3' z_3'.$$

Hence

$$\left.\begin{aligned} \lambda_2 x_2 - \lambda_3 x_3 &= -\lambda_2' x_2' + \lambda_3' x_3', \\ \lambda_2 y_2 - \lambda_3 y_3 &= -\lambda_2' y_2' + \lambda_3' y_3', \\ \lambda_2 z_2 - \lambda_3 z_3 &= -\lambda_2' z_2' + \lambda_3' z_3', \end{aligned}\right\} \quad (1)$$

and

and there are two similar sets of three equations.

Consider now the point

$$(\lambda_2 x_2 - \lambda_3 x_3, \lambda_2 y_2 - \lambda_3 y_3, \lambda_2 z_2 - \lambda_3 z_3).$$

Not all of these numbers are zero, for then either B and C would coincide, or λ_2 and λ_3 would both be zero and so B' and C' would coincide.

This point plainly lies on BC; but from the equations (1) it can also be specified as

$$(-\lambda_2' x_2' + \lambda_3' x_3', -\lambda_2' y_2' + \lambda_3' y_3', -\lambda_2' z_2' + \lambda_3' z_3'),$$

so that it also lies on $B'C'$. Hence it is the point A''.

Similarly, B'' is

$$(\lambda_3 x_3 - \lambda_1 x_1, \lambda_3 y_3 - \lambda_1 y_1, \lambda_3 z_3 - \lambda_1 z_1),$$

and C'' is

$$(\lambda_1 x_1 - \lambda_2 x_2, \lambda_1 y_1 - \lambda_2 y_2, \lambda_1 z_1 - \lambda_2 z_2).$$

It remains to show that A'', B'', and C'' are collinear. This will be so if the determinant

$$\begin{vmatrix} \lambda_2 x_2 - \lambda_3 x_3 & \lambda_2 y_2 - \lambda_3 y_3 & \lambda_2 z_2 - \lambda_3 z_3 \\ \lambda_3 x_3 - \lambda_1 x_1 & \lambda_3 y_3 - \lambda_1 y_1 & \lambda_3 z_3 - \lambda_1 z_1 \\ \lambda_1 x_1 - \lambda_2 x_2 & \lambda_1 y_1 - \lambda_2 y_2 & \lambda_1 z_1 - \lambda_2 z_2 \end{vmatrix}$$

vanishes. On adding the second and third rows of this determinant to the first, it is easily seen that it does, in fact, vanish; hence A'', B'', and C'' are collinear.

Desargues's proposition is therefore verified in the algebraic representation.

3.142. Dual of Desargues's Proposition

THEOREM. *If two triangles are axially perspective, they are also centrally perspective.*

Suppose ABC, $A'B'C'$ be two axially perspective triangles;

that is, suppose there exists a line l and on it three points A'', B'', C'', such that A'' is the point $\begin{pmatrix} BC \\ B'C' \end{pmatrix}$, B'' is the point $\begin{pmatrix} CA \\ C'A' \end{pmatrix}$, C'' is the point $\begin{pmatrix} AB \\ A'B' \end{pmatrix}$.

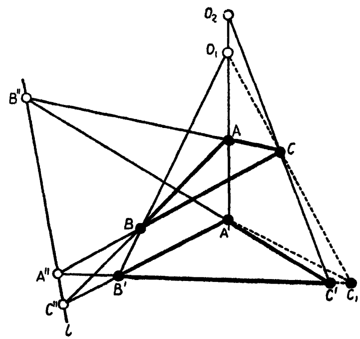


FIG. 2.

Suppose now that ABC and $A'B'C'$ are not centrally perspective. Let O_1 be the point $\begin{pmatrix} AA' \\ BB' \end{pmatrix}$, and O_2 be the point $\begin{pmatrix} CC' \\ AA' \end{pmatrix}$, so that by supposition O_1 and O_2 are not coincident.

Let C_1 be the point $\begin{pmatrix} O_1C \\ B'C' \end{pmatrix}$, distinct from C', by supposition.

Then the triangles ABC, $A'B'C_1$ are centrally perspective on O_1, and so by Desargues's proposition they are axially perspective also.

But $\begin{pmatrix} A'B' \\ AB \end{pmatrix}$ is on l, and $\begin{pmatrix} B'C_1 \\ BC \end{pmatrix}$ is on l, hence l is the axis of perspective of ABC, $A'B'C_1$. Now AC and l are on B'', and therefore $A'C_1$ and l are on B''.

Since, however, $A'C'$ is also on B'', the two lines $A'C_1$ and $A'C'$ have the two points A' and B'' in common; by 2.112 they are therefore not distinct lines, and so C' and C_1 must coincide.

Hence O_1 and O_2 must also coincide, so that ABC and $A'B'C'$ are centrally perspective.

Examples

1. Using the dual of Desargues's proposition as an initial proposition, prove Desargues's proposition.

2. If ABC, $A'B'C'$ are two perspective triangles show that the three points $\begin{pmatrix} BC \\ B'C' \end{pmatrix}$, $\begin{pmatrix} C'A \\ CA' \end{pmatrix}$, and $\begin{pmatrix} A'B \\ AB' \end{pmatrix}$ are collinear.

3. If ABC, $A'B'C'$ are two triangles which are centrally perspective on O, and if A'', B'', C'' are the collinear points $\begin{pmatrix} BC \\ B'C' \end{pmatrix}$, $\begin{pmatrix} CA \\ C'A' \end{pmatrix}$, and $\begin{pmatrix} AB \\ A'B' \end{pmatrix}$ respectively, show that the following pairs of triangles are also perspective:

(i) $CC'B''$ and $BB'C''$,
(ii) $AA'C''$ and $CC'A''$,
(iii) $BB'A''$ and $AA'B''$,
(iv) OBC and $A'B''C''$,
(v) OCA and $B'C''A''$,
(vi) OAB and $C'A''B''$,
(vii) $OB'C'$ and $AB''C''$,
(viii) $OC'A'$ and $BC''A''$,
(ix) $OA'B'$ and $CA''B''$.

State the centre and axis of perspective of each pair.

4. The triangles ABC, $A'B'C'$ are perspective on O and o as centre and axis of perspective. If the triangles BCD and $B'C'D'$ are perspective on the same centre and axis, show that both of the pairs of triangles ABD, $A'B'D'$, and ACD, $A'C'D'$ are perspective on O and o.

3.2. Projectivity

3.21. Collinear Point Figures in Central Perspective

Let $ABCD...$ be a set of points on a line l, and $A'B'C'D'...$ another set on a line m, distinct from l. If these two figures are centrally perspective on the point O, then AA', BB', CC', DD',... are all on O. The two figures are clearly related, and the relation between them is called a *central perspectivity*. The fact that there is a central perspectivity between $ABCD...$ and $A'B'C'D'...$ is expressed by writing

$$l(ABCD...) \overset{O}{\barwedge} m(A'B'C'D'...).$$

Clearly the definition of a central perspectivity implies that also $m(A'B'C'D'...) \overset{O}{\barwedge} l(ABCD...)$.

The relation of central perspectivity is not only a relation between the two figures $ABCD...$, $A'B'C'D'...$, each taken as a whole, it implies a relation between each pair of corresponding

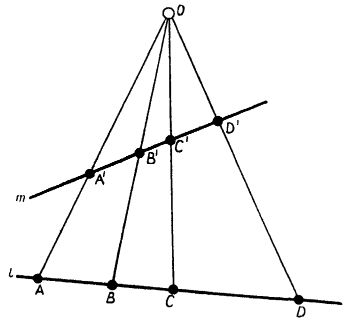


FIG. 3.

points in the two figures, so that A and A' are related, B and B' are related, and so on, *and all these pairs are related in the same way.* The relation between each pair is that of being collinear with the point O. The statement

$$l(ABCD...) \overset{O}{\barwedge} m(A'B'C'D'...)$$

implies these separate relations by the order assigned to the letters on each side of the relation.

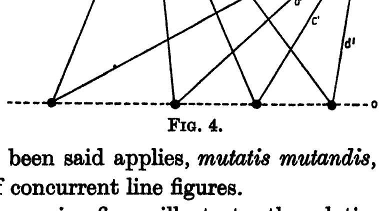


FIG. 4.

What has been said applies, *mutatis mutandis*, to axial perspectivity of concurrent line figures.

The accompanying figure illustrates the relation

$$L(abcd...) \overset{o}{\barwedge} M(a'b'c'd'...).$$

3.22. Successive Perspectivities

Suppose that

$$a(A_1 A_2 A_3 \ldots) \overset{C}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

and

$$b(B_1 B_2 B_3 \ldots) \overset{A}{\barwedge} c(C_1 C_2 C_3 \ldots);$$

it is natural to ask whether there exists a point B such that

$$c(C_1 C_2 C_3 \ldots) \overset{B}{\barwedge} a(A_1 A_2 A_3 \ldots),$$

in other words, whether two successive perspectivities are equivalent to one. To this question only a partial answer can be given at present. Later, it will be shown that unless the three lines a, b, and c are concurrent, and the three centres of perspectivity A, B, and C are collinear, the two perspectivities are not equivalent to a single perspectivity. For the present, it is shown that if the three lines a, b, and c are concurrent, then there exists a third centre of perspectivity on which $c(C_1 C_2 C_3 \ldots)$ and $a(A_1 A_2 A_3 \ldots)$ are perspective.

3.221. THEOREM. *If a, b, and c are three concurrent lines, and if*

$$a(A_1 A_2 A_3 \ldots) \overset{C}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

and

$$b(B_1 B_2 B_3 \ldots) \overset{A}{\barwedge} c(C_1 C_2 C_3 \ldots),$$

then there exists a point B collinear with C and A such that

$$c(C_1 C_2 C_3 \ldots) \overset{B}{\barwedge} a(A_1 A_2 A_3 \ldots).$$

Since the lines a, b, and c are concurrent at the point V, the triangles $A_1 B_1 C_1$ and $A_2 B_2 C_2$ are centrally perspective on V; hence by Desargues's proposition they are axially perspective.

But by supposition $\begin{pmatrix} A_1 B_1 \\ A_2 B_2 \end{pmatrix}$ is the point C,

and $\begin{pmatrix} B_1 C_1 \\ B_2 C_2 \end{pmatrix}$ is the point A;

hence $\begin{pmatrix} C_1 A_1 \\ C_2 A_2 \end{pmatrix}$ is on the line AC.

Similarly, since the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are centrally perspective on V they are axially perspective on AC, and so $\begin{pmatrix} C_2A_2 \\ C_3A_3 \end{pmatrix}$ is on the line AC.

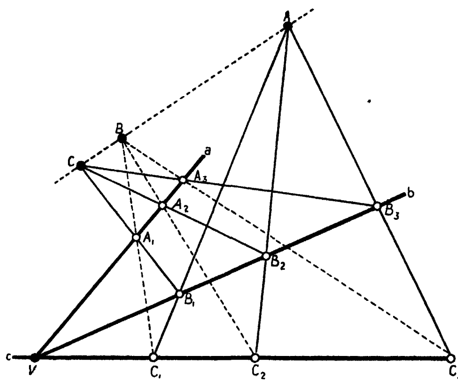


FIG. 5.

Hence the lines A_1C_1, A_2C_2, A_3C_3 are concurrent at a point B on AC; that is to say

$$c(C_1C_2C_3...) \overset{B}{\barwedge} a(A_1A_2A_3...).$$

3.222. THEOREM. *If A, B, and C are three collinear points, and if*

$$A(a_1a_2a_3...) \overset{c}{\barwedge} B(b_1b_2b_3...)$$

and

$$B(b_1b_2b_3...) \overset{a}{\barwedge} C(c_1c_2c_3...),$$

then there exists a line b concurrent with c and a such that

$$C(c_1c_2c_3...) \overset{b}{\barwedge} A(a_1a_2a_3...).$$

This theorem is the dual of the preceding theorem; the same figure illustrates it.

Ex. Show that if

$$a(A_1A_2A_3...) \overset{C}{\barwedge} b(B_1B_2B_3...),$$

$$b(B_1B_2B_3...) \overset{A}{\barwedge} c(C_1C_2C_3...),$$

and $$c(C_1C_2C_3\ldots) \overset{B}{\barwedge} a(A_1A_2A_3\ldots),$$

and if the three points A, B, and C are collinear, then the three lines a, b, and c are concurrent. Dualize this theorem.

3.23. Three or More Successive Perspectivities

It has been said that two successive perspectivities do not necessarily reduce to one, and it might be expected from this that three successive perspectivities do not necessarily reduce to two, and so on. This, however, is not so, and the next stage in the inquiry into the theory of perspectivity is to show that three or more successive perspectivities are always equivalent to at most two. To prove this important theorem it is first necessary to prove a subsidiary one. The subsidiary theorem and the main theorem are proved in the next two sections.

3.231. THEOREM. *If*

$$a(A_1A_2A_3\ldots) \overset{V_1}{\barwedge} b(B_1B_2B_3\ldots)$$

and $$b(B_1B_2B_3\ldots) \overset{V_2}{\barwedge} c(C_1C_2C_3\ldots),$$

and if b' *be any line concurrent with* a *and* b, *then there exists a point* V_1' *collinear with* V_1 *and* V_2 *and a collinear point-figure* $B_1'B_2'B_3'\ldots$ *on* b', *such that*

$$a(A_1A_2A_3\ldots) \overset{V_1'}{\barwedge} b'(B_1'B_2'B_3'\ldots)$$

and $$b'(B_1'B_2'B_3'\ldots) \overset{V_2}{\barwedge} c(C_1C_2C_3\ldots).$$

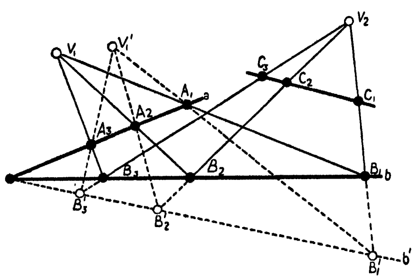


FIG. 6.

Let $B_1' B_2' B_3' \ldots$ be the point-figure on b' perspective with $C_1 C_2 C_3 \ldots$ from the centre V_2. Then plainly

$$b(B_1 B_2 B_3 \ldots) \overset{V_2}{\barwedge} b'(B_1' B_2' B_3' \ldots).$$

But also,

$$a(A_1 A_2 A_3 \ldots) \overset{V_1}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

and so, since the lines a, b, and b' are concurrent, it follows from Theorem 3.221 that there is a point V_1' collinear with V_1 and V_2 such that

$$a(A_1 A_2 A_3 \ldots) \overset{V_1'}{\barwedge} b'(B_1' B_2' B_3' \ldots).$$

This proves the theorem.

3.232. THEOREM. *If*

$$a(A_1 A_2 A_3 \ldots) \overset{V_1}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

$$b(B_1 B_2 B_3 \ldots) \overset{V_2}{\barwedge} c(C_1 C_2 C_3 \ldots),$$

$$c(C_1 C_2 C_3 \ldots) \overset{V_3}{\barwedge} d(D_1 D_2 D_3 \ldots),$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$m(M_1 M_2 M_3 \ldots) \overset{V_r}{\barwedge} n(N_1 N_2 N_3 \ldots),$$

then there exist a line x and points U and V such that

$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} x(X_1 X_2 X_3 \ldots)$$

and

$$x(X_1 X_2 X_3 \ldots) \overset{V}{\barwedge} n(N_1 N_2 N_3 \ldots).$$

The theorem states that any number of perspectivities can always be reduced to two, but it is plainly only necessary to prove that three successive perspectivities can always be reduced to two, for if this is proved, it is possible to reduce any number successively until there are only two.

Accordingly, for the above theorem may be substituted the following:

If

$$a(A_1 A_2 A_3 \ldots) \overset{V_1}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

$$b(B_1 B_2 B_3 \ldots) \overset{V_2}{\barwedge} c(C_1 C_2 C_3 \ldots),$$

and $$c(C_1 C_2 C_3 \ldots) \overset{V_3}{\barwedge} d(D_1 D_2 D_3 \ldots),$$

then there exist a line x and points U and V such that

$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} x(X_1 X_2 X_3 \ldots)$$

and $$x(X_1 X_2 X_3 \ldots) \overset{V}{\barwedge} d(D_1 D_2 D_3 \ldots).$$

Firstly, if the lines a, b, c are concurrent, the first two perspectivities may be reduced to one by 3.221. Similarly, if the lines b, c, and d are concurrent, the last two perspectivities may be reduced to one; hence these cases may be dismissed.

Three cases now remain to be considered:

(i) no three of the lines a, b, c, and d concurrent,

(ii) a, b, and d concurrent,

(iii) a, c, and d concurrent.

Suppose first that no three of the lines a, b, c, and d are concurrent.

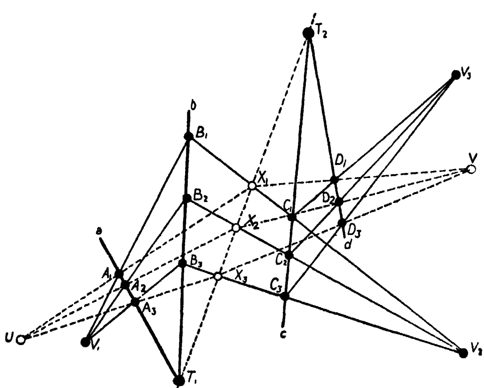


FIG. 7.

Let T_1 be the common point of a and b, and T_2 that of c and d. Let x be the line $T_1 T_2$.

Let $X_1 X_2 X_3 \ldots$ be points on x such that

$$c(C_1 C_2 C_3 \ldots) \overset{V_2}{\barwedge} x(X_1 X_2 X_3 \ldots),$$

so that also
$$x(X_1 X_2 X_3 \ldots) \overset{V_2}{\barwedge} b(B_1 B_2 B_3 \ldots).$$
But since
$$b(B_1 B_2 B_3 \ldots) \overset{V_1}{\barwedge} a(A_1 A_2 A_3 \ldots),$$
it follows by 3.221 that there is a point U such that
$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} x(X_1 X_2 X_3 \ldots).$$
By exactly similar reasoning there is a point V such that
$$x(X_1 X_2 X_3 \ldots) \overset{V}{\barwedge} d(D_1 D_2 D_3 \ldots).$$
Hence there is a line x and two points U and V such that
$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} x(X_1 X_2 X_3 \ldots)$$
and
$$x(X_1 X_2 X_3 \ldots) \overset{V}{\barwedge} d(D_1 D_2 D_3 \ldots).$$

Secondly, suppose that a, b, and d are concurrent. Let b' be on bc but not on V_1. Then by 3.231 there exist a point V_2' and points B_1', B_2', B_3',... on b' such that
$$a(A_1 A_2 A_3 \ldots) \overset{V_1}{\barwedge} b'(B_1' B_2' B_3' \ldots)$$
and
$$b'(B_1' B_2' B_3' \ldots) \overset{V_2'}{\barwedge} c(C_1 C_2 C_3 \ldots).$$

But no three of the lines a, b', c, d are concurrent, and so the previous part of the theorem is applicable; hence there exist a line x and points U and V satisfying the required conditions.

The third case, when a, c, and d are concurrent, is treated in exactly the same way. In place of c a line c' is substituted and the first part of the theorem again used.

3.233. Theorem. *If*
$$A(a_1 a_2 a_3 \ldots) \overset{v_1}{\barwedge} B(b_1 b_2 b_3 \ldots)$$
and
$$B(b_1 b_2 b_3 \ldots) \overset{v_2}{\barwedge} C(c_1 c_2 c_3 \ldots),$$

and if B' be any point on AB but not on v_2, then there exist a line v_1' on $v_1 v_2$ and a concurrent line-figure $b_1' b_2' b_3' \ldots$ such that

$$A(a_1 a_2 a_3 \ldots) \overset{v_1'}{\barwedge} B'(b_1' b_2' b_3' \ldots)$$

and

$$B'(b_1' b_2' b_3' \ldots) \overset{v_2}{\barwedge} C(c_1 c_2 c_3 \ldots).$$

3.234. THEOREM. *If*

$$A(a_1 a_2 a_3 \ldots) \overset{v_1}{\barwedge} B(b_1 b_2 b_3 \ldots),$$

$$B(b_1 b_2 b_3 \ldots) \overset{v_2}{\barwedge} C(c_1 c_2 c_3 \ldots),$$

$$C(c_1 c_2 c_3 \ldots) \overset{v_3}{\barwedge} D(d_1 d_2 d_3 \ldots),$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot \qquad \cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$M(m_1 m_2 m_3 \ldots) \overset{v_r}{\barwedge} N(n_1 n_2 n_3 \ldots),$$

then there exist a point X and lines u and v such that

$$A(a_1 a_2 a_3 \ldots) \overset{u}{\barwedge} X(x_1 x_2 x_3 \ldots),$$

and

$$X(x_1 x_2 x_3 \ldots) \overset{v}{\barwedge} N(n_1 n_2 n_3 \ldots).$$

These last two theorems are the duals of the two preceding ones. It will be useful practice for the reader if he proves them on their own merits, and not merely by appeal to the principle of duality.

3.235. Collinear Point-figures on the same Line

In Theorem 3.232 it was assumed that the line n, on which was the final collinear point-figure of the sequence considered, was distinct from the line a, the first of the sequence. If, however, a and n are not distinct, a slight change has to be made in the theorem, which then reads:

If $a(A_1 A_2 A_3 \ldots)$ and $a(A_1' A_2' A_3' \ldots)$ are two collinear point-figures on the same line which are connected by a sequence of perspectivities, then the sequence may be reduced to one of three perspectivities at most.

The formal proof of this theorem is left to the reader.

3.24. Definition of Projectivity

In 3.21 it was pointed out that a relation exists between two collinear point-figures which are centrally perspective, and that this relation between the two figures implies relations between corresponding points of the two figures. If now

$$a(A_1 A_2 A_3 ...) \overset{U}{\barwedge} b(B_1 B_2 B_3 ...),$$

and

$$b(B_1 B_2 B_3 ...) \overset{V}{\barwedge} c(C_1 C_2 C_3 ...),$$

it is clear that the two figures $A_1 A_2 A_3 ...$ and $C_1 C_2 C_3 ...$ are related, although the relation between them is not, in general, a perspectivity. The relation between two collinear point-figures which are connected by a finite number of perspectivities is called a *projectivity*, and by 3.221 a perspectivity is a particular case of a projectivity.

A projectivity between two collinear point-figures $A_1 A_2 A_3 ...$ and $B_1 B_2 B_3 ...$ is symbolized by writing

$$a(A_1 A_2 A_3 ...) \sim b(B_1 B_2 B_3 ...).$$

Just as a perspectivity is not only a relation between the two figures in perspective, each taken as a whole, so a projectivity is not only a relation between the two figures each taken as a whole. It implies that to each point of one figure there corresponds a single, determinate point of the other, and vice versa; in the case contemplated above, A_1 and B_1 are related, A_2 and B_2 are related, and so on, *and all these pairs are related in the same way*. The relation between each pair is that determined by the perspectivities which specify the projectivity.

It is plain from what has been said that a projectivity can exist between two point-figures which are on the same line.

This also applies, *mutatis mutandis*, to projectivities between concurrent line-figures. No distinguishing adjective is used to differentiate between projectivities between collinear point-figures and projectivities between concurrent line-figures.

Now that the term *projectivity* has been defined, it may be used to restate Theorems 3.232 and 3.235 and their duals.

3.241. THEOREM. *A projectivity between two collinear point-figures on different lines can always be specified as a sequence of at most two perspectivities.*

3.242. THEOREM. *A projectivity between two concurrent line-figures on different points can always be specified as a sequence of at most two perspectivities.*

3.243. THEOREM. *A projectivity between two collinear point-figures on the same line can always be specified as a sequence of at most three perspectivities.*

3.244. THEOREM. *A projectivity between two concurrent line-figures on the same point can always be specified as a sequence of at most three perspectivities.*

3.245. Projectivities between Point-figures and Line-figures.

So far, only the notions of a projectivity between two collinear point-figures and a projectivity between two concurrent line-figures have been considered. It is convenient, however, to supplement these by the notion of a projectivity between a collinear point-figure and a concurrent line-figure.

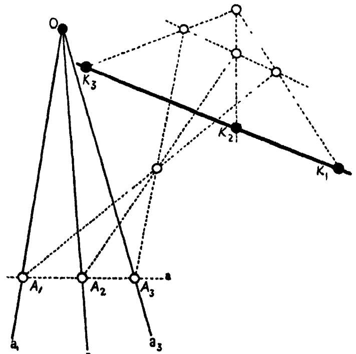


FIG. 8.

Let $A_1 A_2 A_3$... be a collinear point-figure on the line a and O any point not on a; let OA_1, OA_2, OA_3,... be $a_1 a_2 a_3$..., a concurrent line-figure. Then if $K_1 K_2 K_3 \ldots$ be a collinear point-figure and

$$a(A_1 A_2 A_3 \ldots) \sim k(K_1 K_2 K_3 \ldots)$$

in the sense already defined, the concurrent line-figure $a_1 a_2 a_3 \ldots$ is said to be projective with the collinear point-figure $K_1 K_2 K_3 \ldots$. The relation is expressed by writing

$$O(a_1 a_2 a_3 \ldots) \sim k(K_1 K_2 K_3 \ldots).$$

The principle of duality leads us to formulate the definition of a projectivity between a collinear point-figure and a concurrent line-figure, and having formulated this dual definition, we can then *prove* (for it must be proved) that

$$O(a_1 a_2 a_3 \ldots) \sim k(K_1 K_2 K_3 \ldots)$$

implies

$$k(K_1 K_2 K_3 \ldots) \sim O(a_1 a_2 a_3 \ldots).$$

Examples

1. a, b, and c are three concurrent lines; U, V, and W are three collinear points. Show that if

$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

$$b(B_1 B_2 B_3 \ldots) \overset{V}{\barwedge} c(C_1 C_2 C_3 \ldots),$$

and

$$a(A_1 A_2 A_3 \ldots) \overset{W}{\barwedge} b(B_1' B_2' B_3' \ldots),$$

then there is a point X collinear with U, V, and W and such that

$$b(B_1' B_2' B_3' \ldots) \overset{X}{\barwedge} c(C_1 C_2 C_3 \ldots).$$

2. Show that if

$$a(A_1 A_2 A_3 \ldots) \overset{U}{\barwedge} b(B_1 B_2 B_3 \ldots),$$

$$b(B_1 B_2 B_3 \ldots) \overset{V}{\barwedge} c(C_1 C_2 C_3 \ldots),$$

and if b' be any line other than a, c, and UV, then there exist points U', V' on UV such that

$$a(A_1 A_2 A_3 \ldots) \overset{U'}{\barwedge} b'(B_1' B_2' B_3' \ldots),$$

and

$$b'(B_1' B_2' B_3' \ldots) \overset{V'}{\barwedge} c(C_1 C_2 C_3 \ldots).$$

3. Show that if $A(a_1 a_2 a_3 \ldots) \sim B(b_1 b_2 b_3 \ldots)$

and $B(b_1 b_2 b_3 \ldots) \sim C(c_1 c_2 c_3 \ldots)$,

then $A(a_1 a_2 a_3 \ldots) \sim C(c_1 c_2 c_3 \ldots)$. Dualize.

4. Show that if $A(a_1 a_2 a_3 ...) \sim b(B_1 B_2 B_3 ...)$
and $b(B_1 B_2 B_3 ...) \sim c(C_1 C_2 C_3 ...)$,
then $A(a_1 a_2 a_3 ...) \sim c(C_1 C_2 C_3 ...)$. Dualize.

5. Show that if $A(a_1 a_2 a_3 ...) \sim B(b_1 b_2 b_3 ...)$
and $B(b_1 b_2 b_3 ...) \sim c(C_1 C_2 C_3 ...)$,
then $A(a_1 a_2 a_3 ...) \sim c(C_1 C_2 C_3 ...)$. Dualize.

6. Show that if $A(a_1 a_2 a_3 ...) \sim b(B_1 B_2 B_3 ...)$
and $b(B_1 B_2 B_3 ...) \sim C(c_1 c_2 c_3 ...)$,
then $A(a_1 a_2 a_3 ...) \sim C(c_1 c_2 c_3 ...)$. Dualize.

7. Is it true that $A(a_1 a_2 a_3 ...) \sim A(a_1 a_2 a_3 ...)$? Justify the answer given.

3.25. Perspectivity and Projectivity in the Algebraic Representation

If (x_1, y_1, z_1) and (x_1', y_1', z_1') be any two points, then any point on their join is

$$(\lambda_1 x_1 + \lambda_1' x_1', \lambda_1 y_1 + \lambda_1' y_1', \lambda_1 z_1 + \lambda_1' z_1').$$

The numbers (λ_1, λ_1') will be termed the *coordinates of the point*

$$(\lambda_1 x_1 + \lambda_1' x_1', \lambda_1 y_1 + \lambda_1' y_1', \lambda_1 z_1 + \lambda_1' z_1')$$

relative to the base points (x_1, y_1, z_1) and (x_1', y_1', z_1'). It can easily be verified that the point whose coordinates are $(k\lambda_1, k\lambda_1')$, where k is any constant other than zero, is the same as the point whose coordinates are (λ_1, λ_1'), provided the same base points are used. To avoid cumbrous phraseology, we shall refer in future simply to *the point* (λ_1, λ_1').

3.251. Perspectivity

Let (λ_1, λ_1') be any point on a line l_1 on which the base points are (x_1, y_1, z_1) and (x_1', y_1', z_1').

Let (λ_2, λ_2') be any point on a line l_2 on which the base points are (x_2, y_2, z_2) and (x_2', y_2', z_2').

We now ask what is the algebraic relation connecting the four numbers $\lambda_1, \lambda_1', \lambda_2, \lambda_2'$ which is the counterpart of the geometrical relation known as central perspectivity on the point (ξ, η, ζ).

Now if the three points (λ_1, λ_1') on l_1, (λ_2, λ_2') on l_2, and (ξ, η, ζ) are collinear,

$$\begin{vmatrix} \xi & \eta & \zeta \\ \lambda_1 x_1 + \lambda_1' x_1' & \lambda_1 y_1 + \lambda_1' y_1' & \lambda_1 z_1 + \lambda_1' z_1' \\ \lambda_2 x_2 + \lambda_2' x_2' & \lambda_2 y_2 + \lambda_2' y_2' & \lambda_2 z_2 + \lambda_2' z_2' \end{vmatrix} = 0;$$

that is, $A\lambda_1 \lambda_2 + B\lambda_1 \lambda_2' + C\lambda_1' \lambda_2 + D\lambda_1' \lambda_2' = 0$, where

$$A = \begin{vmatrix} \xi & \eta & \zeta \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \equiv \xi(y_1 z_2 - y_2 z_1) + \eta(z_1 x_2 - z_2 x_1) + \zeta(x_1 y_2 - x_2 y_1), \quad (1)$$

$$B = \begin{vmatrix} \xi & \eta & \zeta \\ x_1 & y_1 & z_1 \\ x_2' & y_2' & z_2' \end{vmatrix} \equiv \xi(y_1 z_2' - y_2' z_1) + \eta(z_1 x_2' - z_2' x_1) + \zeta(x_1 y_2' - x_2' y_1), \quad (2)$$

$$C = \begin{vmatrix} \xi & \eta & \zeta \\ x_1' & y_1' & z_1' \\ x_2 & y_2 & z_2 \end{vmatrix} \equiv \xi(y_1'z_2-y_2z_1')+\eta(z_1'x_2-z_2x_1')+\zeta(x_1'y_2-x_2y_1'), \quad (3)$$

$$D = \begin{vmatrix} \xi & \eta & \zeta \\ x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \end{vmatrix} \equiv \xi(y_1'z_2'-y_2'z_1')+\eta(z_1'x_2'-z_2'x_1')+\zeta(z_1'y_2'-x_2'y_1'). \quad (4)$$

When ξ, η, and ζ are eliminated from these equations, the equation

$$\begin{vmatrix} A & y_1z_2-y_2z_1 & z_1x_2-z_2x_1 & x_1y_2-x_2y_1 \\ B & y_1z_2'-y_2'z_1 & z_1x_2'-z_2'x_1 & x_1y_2'-x_2'y_1 \\ C & y_1'z_2-y_2z_1' & z_1'x_2-z_2x_1' & x_1'y_2-x_2y_1' \\ D & y_1'z_2'-y_2'z_1' & z_1'x_2'-z_2'x_1' & x_1'y_2'-x_2'y_1' \end{vmatrix} = 0 \quad (5)$$

remains, and this may be written

$$A\rho_1+B\rho_2+C\rho_3+D\rho_4 = 0,$$

ρ_1, ρ_2, ρ_3, and ρ_4 being the minors of A, B, C, and D in the above determinant.

Hence, if there is a central perspectivity between the points (λ_1, λ_1') on l_1 and points (λ_2, λ_2') on l_2, then there are four numbers A, B, C, and D such that

$$A\lambda_1\lambda_2+B\lambda_1\lambda_2'+C\lambda_1'\lambda_2+D\lambda_1'\lambda_2' = 0, \quad (6)$$

and

$$A\rho_1+B\rho_2+C\rho_3+D\rho_4 = 0. \quad (7)$$

It should also be observed that $AD-BC \neq 0$, for were this so, we could write $A/C = B/D = k$, and the equation (6) would reduce to

$$(C\lambda_2+D\lambda_2')(k\lambda_1+\lambda_1') = 0,$$

and plainly this equation cannot be satisfied by all pairs of points in the perspectivity.

Conversely, if the two equations (6) and (7) are satisfied, and if $AD-BC \neq 0$, there is a perspectivity between points (λ_1, λ_1') on l_1, and points (λ_2, λ_2') on l_2. No difficulty should be experienced in proving this.

The following theorem may therefore be enunciated:

The necessary and sufficient condition that there should be a central perspectivity between points (λ_1, λ_1') on l_1 and points (λ_2, λ_2') on l_2 is that there should be four numbers A, B, C, and D, such that $AD-BC \neq 0$, and

$$A\lambda_1\lambda_2+B\lambda_1\lambda_2'+C\lambda_1'\lambda_2+D\lambda_1'\lambda_2' = 0,$$

$$A\rho_1+B\rho_2+C\rho_3+D\rho_4 = 0.$$

3.252. Projectivity

A projectivity between points on a line l_1 and points on a line l_3 is defined as a sequence of perspectivities, and this can, in fact, be reduced to a sequence of two perspectivities if the two lines are distinct. Suppose then that a perspectivity is specified between points on l_1 and points on l_2, and that a second perspectivity is specified between the points on l_2 and those on l_3.

Let the base points on l_1 and l_2 be as before, and let those on l_3 be (x_3, y_3, z_3) and (x_3', y_3', z_3').

Since there is a perspectivity between points on l_1 and l_2, numbers A, B, C, and D exist such that $AD-BC \neq 0$, and

$$A\lambda_1\lambda_2+B\lambda_1\lambda_2'+C\lambda_1'\lambda_2+D\lambda_1'\lambda_2' = 0 \tag{1}$$

and

$$A\rho_1+B\rho_2+C\rho_3+D\rho_4 = 0. \tag{2}$$

Similarly, there are numbers E, F, G, and H, such that $EH-FG \neq 0$, and

$$E\lambda_2\lambda_3+F\lambda_2\lambda_3'+G\lambda_2'\lambda_3+H\lambda_2'\lambda_3' = 0 \tag{3}$$

and

$$E\sigma_1+F\sigma_2+G\sigma_3+H\sigma_4 = 0. \tag{4}$$

When λ_2 and λ_2' are eliminated from equations (1) and (3), the resulting equation is

$$(BE-AG)\lambda_1\lambda_3+(BF-AH)\lambda_1\lambda_3'+(DE-CG)\lambda_1'\lambda_3+(DF-CH)\lambda_1'\lambda_3' = 0,$$

and this may be written

$$a\lambda_1\lambda_3+b\lambda_1\lambda_3'+c\lambda_1'\lambda_3+d\lambda_1'\lambda_3' = 0.$$

Further, the expression $ad-bc$ is equal to $(AD-BC)(EH-FG)$, and therefore it cannot vanish.

Hence, if there is a projectivity between points (λ_1, λ_1') on l_1, and points (λ_3, λ_3') on l_3, then there are four numbers a, b, c, and d, such that $ad-bc \neq 0$ and

$$a\lambda_1\lambda_3+b\lambda_1\lambda_3'+c\lambda_1'\lambda_3+d\lambda_1'\lambda_3' = 0. \tag{5}$$

The converse theorem is also true, for if the equation (5) be satisfied, and if $ad-bc \neq 0$, then there are eight numbers A, B, C, D, E, F, G, and H, such that neither $(AD-BC)$ nor $(EH-FG)$ vanishes and, in addition,

$$\begin{aligned} a &= BE-AG, \\ b &= BF-AH, \\ c &= DE-CG, \\ d &= DF-CH, \\ A\rho_1+B\rho_2+C\rho_3+D\rho_4 &= 0, \\ E\sigma_1+F\sigma_2+G\sigma_3+H\sigma_4 &= 0. \end{aligned}$$

In fact, it will be found that there are an infinity of solutions of these equations. Hence the relation implied by equation (5) can be specified as a sequence of perspectivities; hence it is a projectivity.

The following theorem may therefore be enunciated:

The necessary and sufficient condition that a projectivity should exist between points (λ_1, λ_1') on a line l_1 and points (λ_3, λ_3') on a line l_3 is that there should be four numbers a, b, c, and d, such that $ad-bc \neq 0$, and

$$a\lambda_1\lambda_3+b\lambda_1\lambda_3'+c\lambda_1'\lambda_3+d\lambda_1'\lambda_3' = 0.$$

3.3. Projectivity of Ranges and Pencils

In considering perspectivities and projectivities between collinear point-figures, we have so far taken these figures to be merely selections of points from the whole set of points on the line. In order to simplify the work, we now pay special attention to one particular collinear point-figure, namely that consisting of *all* the points on a line. Similarly, we pay special attention to the concurrent line-figure consisting of *all* the lines on a point.

DEFINITION. *The point-figure consisting of all the points on a line is called a range of points on a line, or, simply, a range; the line on which a range is, is called the base of the range.*

DEFINITION. *The line-figure consisting of all the lines on a point is called a pencil of lines on a point, or, simply, a pencil; the point on which a pencil is, is called the base of a pencil.*

Since the range is a collinear point-figure, all the theorems about perspectivity and projectivity of collinear point-figures are true of ranges.

3.31. Projectivities between Ranges

A projectivity between two ranges on different bases sets up relations between the points of one and the points of the other in such a way that to one point of one there is made to correspond a unique point of the other, and vice versa. A question here presents itself: Can a projectivity between a range on the base a and a range on the base b exist in which certain arbitrary points $A_1 A_2 A_3 A_4 \ldots$ chosen on a correspond to arbitrary points $B_1 B_2 B_3 B_4 \ldots$ chosen on b?

A partial answer to this question is given in the following theorems.

3.311. THEOREM. *There is a projectivity between a range on a base a and another range on a base b in which three arbitrarily chosen distinct points on the first range correspond to three arbitrarily chosen distinct points on the second.*

Let $A_1 A_2 A_3$ be three arbitrarily chosen points of the range on a, and $B_1 B_2 B_3$ be three arbitrarily chosen points of the range on b.

Let U be any point on $A_1 B_1$, and c any line on B_1, other than b. Then points C_2, C_3 exist such that

$$a(A_1 A_2 A_3) \overset{U}{\barwedge} c(B_1 C_2 C_3).$$

Let V be the point $\begin{pmatrix} B_2 C_2 \\ B_3 C_3 \end{pmatrix}$. Then, clearly,

$$c(B_1 C_2 C_3) \overset{V}{\barwedge} b(B_1 B_2 B_3),$$

so that $$a(A_1 A_2 A_3) \sim b(B_1 B_2 B_3).$$

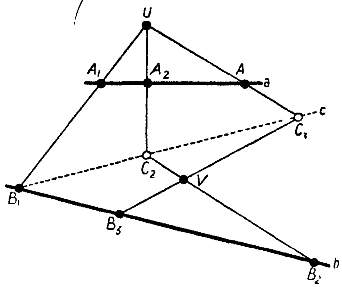


FIG. 9.

Hence the projectivity which is specified by the perspectivities on U and V has the required property, and the theorem is proved.

It is clear that the two perspectivities specifying the projectivity are by no means unique.

3.312. THEOREM. *There is a projectivity between a pencil on a base A and another on a base B in which three arbitrarily chosen distinct lines of the first pencil correspond to three arbitrarily chosen lines on the second.*

If certain ideas which are elaborated in the next chapter were now at our disposal, it could be proved that Theorems 3.311 and 3.312 are not in general true when the word *three* is changed into *four*. In fact, it could be proved that if A, B, and C are three arbitrary points of one range and A', B', C' are three arbitrary points of a second range, and if a projectivity is set up which makes A, B, C correspond to A', B', C', then *by whatever perspectivities the projectivity is specified* certain points of the first range will always correspond to certain other points of the second. That is to say, there are points D, E, F,... which will always correspond to certain points D', E', F',.... In the Algebraic Representation it is found that if in a projectivity three points of one range are chosen to correspond to three points of another, then the point corresponding to *any* other point is fixed—however the projectivity be specified.

These facts lead us to lay down an initial proposition which states that if there are two differently specified projectivities in each of which three points A, B, C of one range correspond to A', B', C' of the second, then the two projectivities are entirely equivalent; that is to say, if in one of the projectivities D, E, F,... correspond to D', E', F',..., then in the other D, E, F,... correspond to D', E', F',... .

3.313. The Projective Proposition

If A, B, C, D are four points of a range, and A', B', C' are three points of another range, then there is a unique point D' of the second such that any projectivity in which $(ABC) \sim (A'B'C')$ is also a projectivity in which $(ABCD) \sim (A'B'C'D')$.

3.314. Verification in the Algebraic Representation

Let (λ_1, λ_1') be a typical point on a line l_1, and (λ_2, λ_2') a typical point on a line l_2. By 3.252, if a projectivity exists between the two ranges, the coordinates of corresponding points are connected by an algebraic relation of the type

$$a\lambda_1\lambda_2 + b\lambda_1\lambda_2' + c\lambda_1'\lambda_2 + d\lambda_1'\lambda_2' = 0 \quad (ad - bc \neq 0).$$

Suppose now that three points (α_1, α_1'), (β_1, β_1'), and (γ_1, γ_1') on l_1 correspond respectively to the three points (α_2, α_2'), (β_2, β_2'), and (γ_2, γ_2') on l_2.

Then $$a\alpha_1\alpha_2 + b\alpha_1\alpha_2' + c\alpha_1'\alpha_2 + d\alpha_1'\alpha_2' = 0,$$

$$a\beta_1\beta_2 + b\beta_1\beta_2' + c\beta_1'\beta_2 + d\beta_1'\beta_2' = 0,$$

and $$a\gamma_1\gamma_2 + b\gamma_1\gamma_2' + c\gamma_1'\gamma_2 + d\gamma_1'\gamma_2' = 0;$$

from these three equations the ratios of the four numbers a, b, c, d can be determined uniquely.

Let (δ_1, δ_1') be any fourth point on l; then since a, b, c, d are known—apart from a constant factor—it follows that (δ_2, δ_2'), the corresponding point on l_2, is uniquely determined from the equation

$$a\delta_1\delta_2 + b\delta_1\delta_2' + c\delta_1'\delta_2 + d\delta_1'\delta_2' = 0,$$

however the projectivity be specified in terms of perspectivities.

The Projective Proposition is therefore verified in the Algebraic Representation, and so it is compatible with the other initial propositions.

3.315. Dual of the Projective Proposition

THEOREM. *If a, b, c, d, are four lines of a pencil and a', b', c' are three lines of another pencil, then there is a unique line d' of the second such that any projectivity in which $(abc) \sim (a'b'c')$ is also a projectivity in which $(abcd) \sim (a'b'c'd')$.*

Let U and V be the bases of the two pencils, and suppose that there are two lines d' and d'' on V and that there are two projectivities, in one of which

$$U(abcd) \sim V(a'b'c'd'),$$

and in the other

$$U(abcd) \sim V(a'b'c'd'').$$

Let x and y be any two lines; let the points xa, xb, xc, xd be A, B, C, D respectively; and the points ya', yb', yc', yd', yd'' be A', B', C', D', D''.

Then clearly there are two projectivities, in one of which

$$x(ABCD) \sim y(A'B'C'D'),$$

and in the other

$$x(ABCD) \sim y(A'B'C'D'').$$

This contradicts the Projective Proposition, hence the supposition that d' and d'' are distinct is false. This proves the theorem.

3.32. Elementary Deductions from the Projective Proposition

A number of important theorems can be deduced at once from the Projective Proposition, and great use will be made of them in the sequel.

3.321. THEOREM. *If X is the common point of two lines a and b, and if a projectivity between the range on a and the range on b is such that the point X, considered as a point of a, corresponds to the point X, considered as a point of b, then the projectivity is a perspectivity: that is, the lines on pairs of corresponding points are all concurrent.*

Let A_1 and A_2 be any two points on a, and let B_1 and B_2 be the points corresponding to them in the projectivity.

Let V be the point $\begin{pmatrix} A_1 B_2 \\ A_2 B_2 \end{pmatrix}$.

Then the central perspectivity on V between the two ranges involves the relation

$$a(A_1 A_2 X) \overset{V}{\barwedge} b(B_1 B_2 X).$$

But the projectivity also involves, by supposition,

$$a(A_1 A_2 X) \sim b(B_1 B_2 X).$$

Hence, by the Projective Proposition, the projectivity is equivalent to the perspectivity.

3.322. THEOREM. *If x is the line on two points A and B, and if a projectivity between the pencil on A and that on B is such that the line x, considered as a line of the pencil on A, corresponds to the line x, considered as a line of the pencil on B, then the projectivity reduces to a perspectivity; that is, the common points of corresponding lines of the two pencils are collinear.*

The somewhat long enunciation of these two theorems may tend to make the reader lose his grasp of their significance, and so they are re-enunciated in less careful words.

A projectivity between two ranges, in which the common point is self-corresponding, is always a perspectivity.

Or, with even greater economy:

A projectivity with a common self-corresponding point is a perspectivity.

This theorem is constantly in use when it is desired to prove that three or more lines are concurrent; for the lines which are on pairs of corresponding points in a perspectivity are concurrent at the centre of perspective. The dual theorem is used for the dual purpose.

3.323. Pappus's Theorem

If A, B, C are three points on a line x, and A', B', C' are three points on a line y, and if A'', B'', C'' are the points $\begin{pmatrix} BC' \\ B'C \end{pmatrix}$, $\begin{pmatrix} CA' \\ C'A \end{pmatrix}$, $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$ *respectively, then A'', B'', and C'' are collinear.*

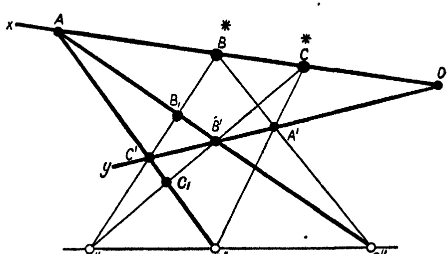


FIG. 10.

Let O be the point xy, and B_1, C_1 the points $\begin{pmatrix} BC' \\ AB' \end{pmatrix}$, $\begin{pmatrix} C'A \\ B'C \end{pmatrix}$ respectively.

Then $$y(A'B'C'O) \overset{B}{\barwedge} (C''B'B_1A)$$

and $$y(A'B'C'O) \overset{C}{\barwedge} (B''C_1C'A),$$

hence $$(C''B'B_1A) \sim (B''C_1C'A).$$

But this is a projectivity between ranges on different bases in which there is a common self-corresponding point A. Hence it is a perspectivity, by the last theorem. Hence $B''C''$, $B'C_1$, B_1C' are concurrent, but the common point of the last two lines is A''. Hence A'', B'', C'' are collinear.

The reader should notice how this theorem is proved, and should study carefully the figure accompanying it. Pappus's theorem is not very difficult, but time is often wasted in futile attempts to reproduce the proof. Of the many available ways of proving the theorem, that given here has been chosen as being the most 'automatic'. It will appear later on as the proof of another important theorem.

It is left to the reader to state the dual of Pappus's theorem; it is useful practice to prove it without using the principle of duality.

The line $A''B''C''$ in Pappus's theorem is called the *Pappus line* of the two sets of three collinear points A, B, C, and A', B', C'.

3.324. Utility of Pappus's Theorem

At first sight, Pappus's theorem appears to have little bearing on the work that has preceded it, nor does it answer any question which that work has raised. Nevertheless, it is of great value in that it provides a simple method of determining corresponding points in a projectivity when three pairs of corresponding points are known. One way of doing this is already known, namely by setting up two perspectivities; but this is cumbrous. The following theorem is the foundation of a simpler method.

THEOREM. *If*

(i) *l be the Pappus line of two sets of three collinear points A, B, C and A', B', C' on x and y respectively,*

(ii) *D be any other point on x, and D″ the point common to A′D and l,*

(iii) *D′ be the point common to y and AD″,*

then $$(ABCD) \sim (A'B'C'D').$$

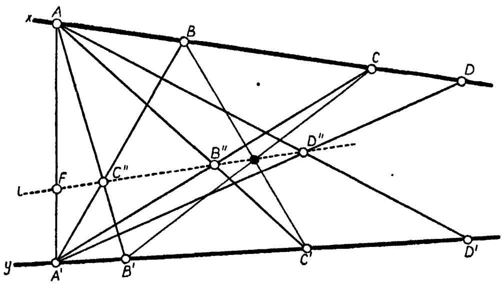


FIG. 11.

Let F be the point common to AA' and l.

Then $$x(ABCD) \overset{A'}{\barwedge} l(FC''B''D''),$$

but $$l(FC''B''D'') \overset{A}{\barwedge} y(A'B'C'D'),$$

hence $$x(ABCD) \sim y(A'B'C'D').$$

The reader will have no difficulty in devising a construction for finding corresponding points in a projectivity in which three pairs of corresponding points are given.

3.325. The Permutation Theorem

THEOREM. *If* $$x(ABCD) \sim y(A'B'C'D'),$$

then also $$x(ABCD) \sim y(B'A'D'C').$$

Let V be any point not on x, and z any line other than x, and

suppose $$x(ABCD) \overset{V}{\barwedge} z(A''B''C''D'').$$

Let U be the point $\begin{pmatrix} CA'' \\ BD'' \end{pmatrix}$, then UV is the Pappus line of the two sets ABC and $B''A''D''$.

For $\begin{pmatrix} AA'' \\ BB'' \end{pmatrix}$ is V, and $\begin{pmatrix} CA'' \\ BD'' \end{pmatrix}$ is U.

Hence in the projectivity in which $(ABC) \sim (B''A''D'')$ the point corresponding to D is C'', for CC'' is on V and DD'' is on V.

Hence $$(ABCD) \sim (B''A''D''C'').$$

Now $$(B''A''D''C'') \overset{V}{\barwedge} (BADC)$$

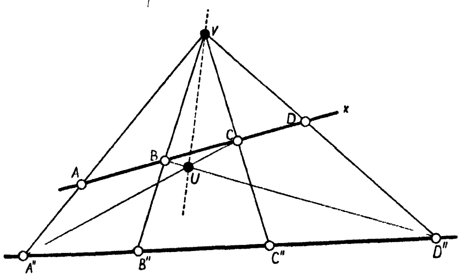


FIG. 12.

and $$(BADC) \sim (B'A'D'C'),$$
so that $$(ABCD) \sim (B'A'D'C').$$

The permutation theorem is constantly in use in subsequent work.

EXAMPLES

1. If $$a(A_1 A_2 A_3 \ldots) \overset{C}{\barwedge} b(B_1 B_2 B_3 \ldots),$$
and $$b(B_1 B_2 B_3 \ldots) \overset{A}{\barwedge} c(C_1 C_2 C_3 \ldots),$$
and $$c(C_1 C_2 C_3 \ldots) \overset{B}{\barwedge} a(A_1 A_2 A_3 \ldots),$$
show that a, b, and c are concurrent, and that A, B, and C are collinear, Dualize.

2. If the lines joining three pairs of corresponding points of two ranges in a projectivity are concurrent, show that the projectivity is a perspectivity.

3. $ABCD$ and $A'B'C'D'$ are two sets of four points on x and y respectively. If l is the Pappus line of ABC and $A'B'C'$ and also of ABD and $A'B'D'$, show that it is also the Pappus line of BCD and $B'C'D'$ and of ABD and $A'B'D'$.

4. ABC and $A'B'C'$ are two sets of three collinear points and A'', B'', C'' are the points $\binom{BC'}{CB'}$, $\binom{CA'}{AC'}$, and $\binom{AB'}{BA'}$ respectively. Show that the Pappus line of $A'B'C'$ and $A''B''C''$ is ABC.

5. In a certain projectivity between two ranges, ABC correspond respectively to $A'B'C'$, and l is the Pappus line of these two sets. If DEF correspond to $D'E'F'$ respectively, show that l is also the Pappus line of these two sets.

6. If in the last example X is the point common to the line l and the base of the first range, what point in the second range corresponds to X?

7. If ABC and $A'B'C'$ are two sets of three collinear points, and if AA', BB', CC' are concurrent, show that the Pappus line of ABC and $A'B'C'$ and these two lines are concurrent.

8. Prove the converse of the last example.

9. State the dual of Theorem 3.324, and prove it without appeal to the principle of duality.

10. State the dual of Theorem 3.325, and prove it without appeal to the principle of duality.

3.4. Cobasal Ranges and Pencils

Strictly speaking, there is but one range of points on a line and one pencil of lines on a point, and therefore to speak of more than one range or pencil on a given base is a contradiction in terms. Nevertheless, these expressions are used, and they are used with a very definite meaning and for a definite purpose.

It will be quite clear that a projectivity can exist between a range and itself, and if A, B, C, A', B', C' are points of the *same range*, it is possible to set up a projectivity in which the points A, B, C correspond to the points A', B', C' in just the same sort of way as any other projectivity is set up. A little thought will show that if we are going to speak of a projectivity between a range and itself, we shall have to be constantly qualifying terms and using cumbersome language; moreover the terminology used when speaking of ranges not on the same base in projectivity will not be applicable without qualification to a projectivity between a range and itself. For these and other reasons we are led to speak not of a projectivity between a range and itself, but between one range and another range on the same base. The two ranges are not really distinct, but it is a help to thought and language to think of them as distinct. Just as in Algebra it is usual to say that the equation

$$x^2-2x+1 = 0$$

has *two* roots, which we say are equal, so here we say that on the one base there are two (or more) coincident ranges and these ranges are called *cobasal ranges*.

The term *cobasal pencils* has a similar meaning, and is used for similar reasons.

3.41. Projectivities between Cobasal Ranges and Pencils

THEOREM. *If A, B, C are three points of a range, and A', B', C' are three points of a cobasal range, then a projectivity in which A and A', B and B', C and C' correspond is unique.*

Let A'', B'', C'' be three points on another line perspective with A', B', C'.

By considering the projectivity in which $(ABC) \sim (A''B''C'')$ the proof of the theorem becomes plain. The details are left to the reader.

3.42. The reader may find it helpful in thinking of projectivities between cobasal ranges to ascribe colours to the different ranges, even though in formal written work the practice might be deprecated. Thus there will be (say) a red range and a blue range on a certain base, and because of a projectivity, to each red point there corresponds a definite blue point, and vice versa.

The theorem of the last section shows that when three distinct red points are specified as corresponding to three distinct blue points, the projectivity is completely determined, that is to say, the blue point corresponding to any other red point is determined.†

A question naturally arises here. Are there any red points, say A, B, C,... which correspond to *the same* blue points A, B, C,...? In formal language, if a projectivity exists between two cobasal ranges are there any points which are self-corresponding? This question is answered in the next section.

3.43. Self-corresponding Points of Cobasal Ranges in Projectivity

3.431. THEOREM. *If in a projectivity between two cobasal ranges there are three self-corresponding points, then every point is a self-corresponding point.*

† Another helpful way of considering a projectivity between cobasal ranges is to think of one range as a range of electric lights and the other as a range of switches. When a projectivity exists, the switch at the point A switches on the light at A'. But the switch at A' does *not* switch on the light at A—except under certain special circumstances which will be investigated in the next chapter.

Let A, B, C, be the self-corresponding points.

Now there is a projectivity between two cobasal ranges, the *identical projectivity*, in which

$$(ABCDEF...) \sim (ABCDEF...),$$

and plainly, in this projectivity, every point is self-corresponding.

But a projectivity is determined uniquely when three pairs of corresponding points are specified, hence if there is a projectivity in which $(ABC) \sim (ABC)$ it cannot be other than the identical projectivity. Hence every point is a self-corresponding point.

3.432. THEOREM. *There are projectivities between cobasal ranges in which there are only two self-corresponding points.*

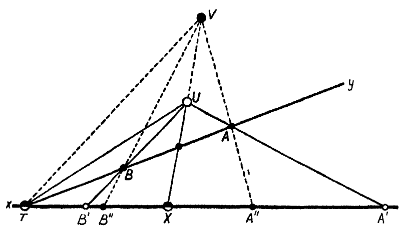


FIG. 13.

Let x be the base of the cobasal ranges.

Let y be any other line, and let T be the common point of x and y.

Let U and V be any two points neither of which is on x or y, and which are not collinear with T.

Let X be the point common to UV and x.

Let $ABCD...$ be a range on y.

Consider the two ranges $x(A'B'C'D'...)$ and $x(A''B''C''D''...)$, defined by the perspectivities

$$y(ABCD...) \overset{U}{\overline{\wedge}} x(A'B'C'D'...),$$

$$y(ABCD...) \overset{V}{\overline{\wedge}} x(A''B''C''D''...),$$

so that $$x(A'B'C'D'...) \sim x(A''B''C''D''...).$$

Then clearly X and T are self-corresponding points.

Moreover there are points on x which are not self-corresponding. Hence by the previous theorem there cannot be more than two self-corresponding points.

3.433. THEOREM. *There are projectivities between cobasal ranges in which there is only one self-corresponding point.*

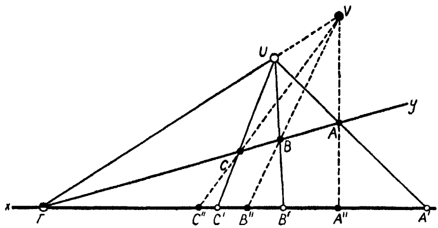


FIG. 14.

This theorem is very similar to the last, and the only difference in the preliminary construction is that the points U and V must be collinear with T.

3.434. The Number of Self-corresponding Points

The last three theorems have shown that if a projectivity exists between two cobasal ranges (i) there cannot be more than two self-corresponding points unless all points are self-corresponding, (ii) there can be two self-corresponding points without all points being self-corresponding, (iii) there can be only one self-corresponding point. It is impossible to say definitely at this stage whether there can or cannot be projectivities between cobasal ranges in which there are no self-corresponding points. To answer this question definitely it would be necessary to lay down an initial proposition on extension in place of 2.412, and more definite than 2.412. This will be done eventually, but for the present there is no need to enter into a further discussion of the question of extension.

In working examples the reader may always assume that in the projectivities between cobasal ranges with which he deals there are two self-corresponding points. These two may be *coincident* (as in Theorem 3.433), and this possibility should not be overlooked.

Examples

1. Show how two establish a projectivity between two cobasal ranges, given (i) three pairs of corresponding points, (ii) two pairs of corresponding points and one self-corresponding point, (iii) two distinct self-corresponding points and one other pair of corresponding points.

2. Given that in a certain projectivity between two cobasal ranges the two self-corresponding points are coincident, show that the projectivity is uniquely determined if the coincident self-corresponding points are known, and one other pair of corresponding points is known. (Use **3.433.**)

3. If X and X' are the distinct self-corresponding points of a projectivity between two cobasal ranges on the line x, and if A, A' and B, B' are two pairs of corresponding points, show that there is a projectivity between cobasal ranges in which $x(XX'AA') \sim x(XX'BB')$.

4. Show that if $k(XYAB) \sim k(XYBA)$, then $k(XYAB) \sim k(YXAB)$.

5. If there are two distinct self-corresponding points in a projectivity between two cobasal ranges, show that the projectivity may be specified by two perspectivities.

6. ABC is a triangle, and P, Q, R are three points not on any of its sides. Show that if the self-corresponding points of a certain projectivity are known, then a triangle $A'B'C'$ can be constructed such that

(i) A' is on BC, B' is on CA, C' is on AB; and

(ii) $B'C'$ is on P, $C'A'$ is on Q, and $A'B'$ is on R.

How many triangles are there which fulfil these conditions?

CHAPTER IV

THE FOUR-POINT AND THE FOUR-LINE

4.1. Definitions and Elementary Properties

4.11. The Simple Four-Point

DEFINITION. *Any set of four points, no three of which are collinear, is termed a simple four-point.*

4.12. The Simple Four-Line

DEFINITION. *Any set of four lines, no three of which are concurrent, is termed a simple four-line.*

The four-point and the four-line are, plainly, dual figures.

4.13. The Complete Four-Point

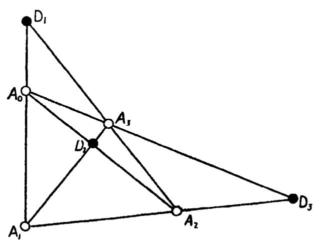


FIG. 15.

In the figure A_0, A_1, A_2, A_3 are the points of a simple four-point.

(*a*) Since any four things can be associated in pairs in six different ways, there are six lines associated with any simple four-point, each of the lines being on two of the points. In the figure A_0A_1, A_2A_3, A_0A_2, A_1A_3, A_0A_3, A_1A_2 are the six lines in question. They are called *sides* of the four-point.

(*b*) These six sides can be arranged in three pairs; two sides will belong to a pair if and only if their common point is *not* a point of the four-point. Two sides which belong to the same pair are called *opposite sides.*

(*c*) The common point of a pair of opposite sides will be called a *diagonal point*; there are thus three diagonal points in all.

In the figure, A_0A_1 and A_2A_3 are opposite sides and their common point D_1 is a diagonal point. The other pairs of opposite sides and the corresponding diagonal points are:

$$A_0A_2 \text{ and } A_1A_3;\ D_2$$

$$A_0A_3 \text{ and } A_1A_2;\ D_3.$$

(*d*) The method of assigning the letters to the various points should be noticed. The four points of the four-point are $A_0A_1A_2A_3$; in naming any pair of opposite sides, all the suffixes 0, 1, 2, 3 are mentioned; the suffix of the diagonal point associated with any pair of opposite sides is the suffix of the letter associated with A_0 in that line of the pair which is on A_0.

DEFINITION. *The four points of a simple four-point, together with the six sides and the three diagonal points is termed a complete four-point.*

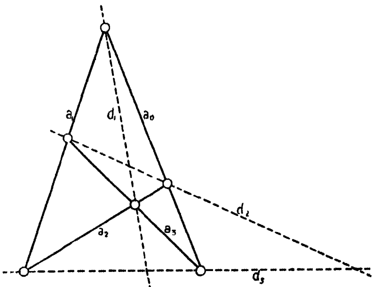


FIG. 16.

What has been said in this section can all be dualized; associated with a simple four-line are six points which can be classified into three pairs, and three diagonal lines. This leads to the definition of the complete four-line as follows:

DEFINITION. *The four lines of a simple four-line, together with the six points and the three diagonal lines, is termed a complete four-line.*

4.131. Elementary Properties

The following elementary properties of the complete four-point, and their duals, are simple consequences of the definitions, and are scarcely worth enunciating as theorems. The reader should satisfy himself that they are true, and that not by merely verifying them in the drawn figure.

(*a*) No two of the six sides of a complete four-point can coincide.

(*b*) No four of the six sides can be concurrent.

(*c*) A diagonal point cannot coincide with any of the four points of the four-point.

(*d*) nor can it be on three of the sides.

(*e*) None of the four points of a complete four-point can be collinear with a pair of diagonal points.

4.14. The Four-Point in the Algebraic Representation

Let (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the four points A_0, A_1, A_2, and A_3 respectively.

Then the six sides of the four-point are:

$$\begin{aligned}
A_0A_1;\ & [y_0z_1-y_1z_0,\ z_0x_1-z_1x_0,\ x_0y_1-x_1y_0],\\
A_2A_3;\ & [y_2z_3-y_3z_2,\ z_2x_3-z_3x_2,\ x_2y_3-x_3y_2],\\
A_0A_2;\ & [y_0z_2-y_2z_0,\ z_0x_2-z_2x_0,\ x_0y_2-x_2y_0],\\
A_1A_3;\ & [y_1z_3-y_3z_1,\ z_1x_3-z_3x_1,\ x_1y_3-x_3y_1],\\
A_0A_3;\ & [y_0z_3-y_3z_0,\ z_0x_3-z_3x_0,\ x_0y_3-x_3y_0],\\
A_1A_2;\ & [y_1z_2-y_2z_1,\ z_1x_2-z_2x_1,\ x_1y_2-x_2y_1].
\end{aligned}$$

In order to find the diagonal points, the common points of these pairs of lines may be found in the usual way, or more simply thus:

Numbers λ_0, λ_1, λ_2, λ_3 exist such that the three equations

$$\left.\begin{aligned}
\lambda_0x_0+\lambda_1x_1+\lambda_2x_2+\lambda_3x_3 &= 0,\\
\lambda_0y_0+\lambda_1y_1+\lambda_2y_2+\lambda_3y_3 &= 0,\\
\lambda_0z_0+\lambda_1z_1+\lambda_2z_2+\lambda_3z_3 &= 0
\end{aligned}\right\}\qquad(1)$$

are satisfied. The ratios of λ_0, λ_1, λ_2, λ_3 may be determined from these equations by the elementary theory of determinants; none of them can be zero, since no three points of the four-point can be collinear.

From the equations (1) it follows that

$$\left.\begin{aligned}
\lambda_0x_0+\lambda_1x_1 &= -\lambda_2x_2-\lambda_3x_3,\\
\lambda_0y_0+\lambda_1y_1 &= -\lambda_2y_2-\lambda_3y_3,\\
\lambda_0z_0+\lambda_1z_1 &= -\lambda_2z_2-\lambda_3z_3.
\end{aligned}\right\}\qquad(2)$$

Consider now the point $(\lambda_0x_0+\lambda_1x_1, \lambda_0y_0+\lambda_1y_1, \lambda_0z_0+\lambda_1z_1)$. Since not all these numbers are zero (see 3.141), it is plain that this point is collinear with (x_0, y_0, z_0) and (x_1, y_1, z_1). But, from the equations (2), this point may be specified as $(\lambda_2x_2+\lambda_3x_3, \lambda_2y_2+\lambda_3y_3, \lambda_2z_2+\lambda_3z_3)$, and this specification shows that it is also collinear with (x_2, y_2, z_2) and (x_3, y_3, z_3). Hence it is the diagonal point D_1.

Similarly the other diagonal points may be found. Hence

D_1 is the point $(\lambda_0 x_0+\lambda_1 x_1,\ \lambda_0 y_0+\lambda_1 y_1,\ \lambda_0 z_0+\lambda_1 z_1)$,
or $(\lambda_2 x_2+\lambda_3 x_3,\ \lambda_2 y_2+\lambda_3 y_3,\ \lambda_2 z_2+\lambda_3 z_3)$;

D_2 is the point $(\lambda_0 x_0+\lambda_2 x_2,\ \lambda_0 y_0+\lambda_2 y_2,\ \lambda_0 z_0+\lambda_2 z_2)$,
or $(\lambda_1 x_1+\lambda_3 x_3,\ \lambda_1 y_1+\lambda_3 y_3,\ \lambda_1 z_1+\lambda_3 z_3)$;

D_3 is the point $(\lambda_0 x_0+\lambda_3 x_3,\ \lambda_0 y_0+\lambda_3 y_3,\ \lambda_0 z_0+\lambda_3 z_3)$,
or $(\lambda_1 x_1+\lambda_2 x_2,\ \lambda_1 y_1+\lambda_2 y_2,\ \lambda_1 z_1+\lambda_2 z_2)$.

4.15. Diagonal Points

The initial propositions so far laid down do not make it possible to deduce that the diagonal points of a four-point are collinear or that they are not collinear. They are in fact compatible with both of these possibilities. For instance, in the finite Geometry in which there are only three points on every line† the whole field consists of one complete four-point in which the three diagonal points are collinear. On the other hand, in the Algebraic Representation there is no complete four-point whose diagonal points are collinear.

It is possible to prove, with the material at our disposal, the following proposition and its dual: *If the diagonal points of any one complete four-point are collinear, the diagonal points of every complete four-point are collinear.* From this, the negative proposition follows at once: *If the diagonal points of one complete four-point are not collinear, then the diagonal points of every complete four-point are not collinear.*

The study of systems in which the diagonal points of every four-point are collinear is less simple than the study of those in which they are not; moreover it is best left until the simpler systems have been investigated. Hence, in this book we confine ourselves to systems in which the diagonal points of a complete four-point are not collinear. To do this, we could lay down the initial proposition: *There is one complete four-point whose diagonal points are not collinear.* We should then go on to prove the second of the propositions stated above. To save time, however, we lay down instead an initial proposition which is equivalent to these two.

† See 2.23.

4.151. The Harmonic Proposition.† There is no complete four-point whose diagonal points are collinear.

4.152. Verification in the Algebraic Representation

Let (x_0, y_0, z_0), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the points A_0, A_1, A_2, A_3, as in 4.14. Then if λ_0, λ_1, λ_2, λ_3 have the meanings there assigned, the diagonal points are:

$$\begin{array}{ll} D_1 & (\lambda_0 x_0+\lambda_1 x_1,\ \lambda_0 y_0+\lambda_1 y_1,\ \lambda_0 z_0+\lambda_1 z_1), \\ \text{or} & (\lambda_2 x_2+\lambda_3 x_3,\ \lambda_2 y_2+\lambda_3 y_3,\ \lambda_2 z_2+\lambda_3 z_3); \\ D_2 & (\lambda_0 x_0+\lambda_2 x_2,\ \lambda_0 y_0+\lambda_2 y_2,\ \lambda_0 z_0+\lambda_2 z_2), \\ \text{or} & (\lambda_1 x_1+\lambda_3 x_3,\ \lambda_1 y_1+\lambda_3 y_3,\ \lambda_1 z_1+\lambda_3 z_3); \\ D_3 & (\lambda_0 x_0+\lambda_3 x_3,\ \lambda_0 y_0+\lambda_3 y_3,\ \lambda_0 z_0+\lambda_3 z_3), \\ \text{or} & (\lambda_1 x_1+\lambda_2 x_2,\ \lambda_1 y_1+\lambda_2 y_2,\ \lambda_1 z_1+\lambda_2 z_2). \end{array}$$

Suppose now that the three diagonal points are all on the line $[l, m, n]$.

Let
$$\begin{aligned} k_0 &= lx_0+my_0+nz_0, \\ k_1 &= lx_1+my_1+nz_1, \\ k_2 &= lx_2+my_2+nz_2, \\ k_3 &= lx_3+my_3+nz_3. \end{aligned}$$

Now since all the diagonal points are on $[l, m, n]$, it follows that

$$\begin{array}{ll} \lambda_0 k_0+\lambda_1 k_1 = 0; & \lambda_2 k_2+\lambda_3 k_3 = 0; \\ \lambda_0 k_0+\lambda_2 k_2 = 0; & \lambda_1 k_1+\lambda_3 k_3 = 0; \\ \lambda_0 k_0+\lambda_3 k_3 = 0; & \lambda_1 k_1+\lambda_2 k_2 = 0; \end{array}$$

and from these six equations that

$$\lambda_0 k_0 = \lambda_1 k_1 = \lambda_2 k_2 = \lambda_3 k_3 = 0.$$

Now since none of the numbers λ_0, λ_1, λ_2, λ_3 can be zero,

$$k_0 = k_1 = k_2 = k_3 = 0;$$

that is to say,

$$\begin{aligned} lx_0+my_0+nz_0 &= 0, \\ lx_1+my_1+nz_1 &= 0, \\ lx_2+my_2+nz_2 &= 0, \\ lx_3+my_3+nz_3 &= 0. \end{aligned}$$

But this implies that all the points A_0, A_1 A_2, A_3 are collinear, and this is contrary to the definition of a four-point. Hence the supposition that the three diagonal points are collinear is false.

The Harmonic Proposition is therefore verified in the Algebraic Representation.

† The word *harmonic* probably appears to be used here, and elsewhere in this chapter, for no reason at all. It was applied to certain properties of the four-point when these were being investigated metrically, and its aptness can only be explained in metrical or analytical terms. The word is retained simply because of its historical associations.

4.153. Dual of the Harmonic Proposition

THEOREM. *There is no complete four-line whose diagonal lines are concurrent.*

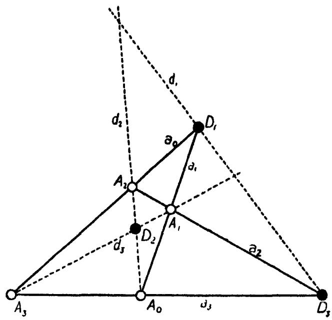


FIG. 17.

Let a_0, a_1, a_2, a_3 be any four-line, and d_1, d_2, d_3 its three diagonal lines.

Then its six points are, in pairs, a_0a_1, a_2a_3; a_0a_2, a_1a_3; a_0a_3, a_1a_2.

Consider the four-point whose points are specified as follows:

A_0 is the point a_1a_3;
A_1 is the point a_1a_2;
A_2 is the point a_0a_2;
A_3 is the point a_0a_3.

No three of these points are collinear.

The six sides of this four-point and the three diagonal points may be tabulated thus:

$$\left.\begin{array}{l} A_0A_1 \text{ is the line } a_1 \\ A_2A_3 \text{ is the line } a_0 \end{array}\right\} D_1 \text{ is the point } a_0a_1, \text{ on } d_1;$$

$$\left.\begin{array}{l} A_0A_2 \text{ is the line } d_2 \\ A_1A_3 \text{ is the line } d_3 \end{array}\right\} D_2 \text{ is the point } d_2d_3;$$

$$\left.\begin{array}{l} A_0A_3 \text{ is the line } a_3 \\ A_1A_2 \text{ is the line } a_2 \end{array}\right\} D_3 \text{ is the point } a_3a_2, \text{ on } d_1.$$

(The reader should verify that these statements are true, by finding the reasons for them, and not merely by reference to the figure.)

Now since D_1, D_2, D_3 are not collinear, and since D_1 and D_3 are on d_1, it follows that D_2 is not on d_1. But D_2 is the common point of d_2 and d_3; and so d_2, d_3, and d_1 are not concurrent.

Hence the diagonal lines of a four-line are never concurrent, and so the theorem is proved.

4.16. The Diagonal Triangles

Since the diagonal points of a complete four-point are not collinear, they are the vertices of a triangle; dually, the diagonal lines of a complete four-line are the sides of a triangle. To these triangles a special name is given.

DEFINITION. *The triangle whose vertices are the diagonal points of a complete four-point is termed the diagonal triangle of the four-point.*

The triangle whose sides are the three diagonal lines of a complete four-line is termed the diagonal triangle of the four-line.

4.17. The Harmonic Points

The sides A_0A_1 and A_2A_3 of a four-point are both on the diagonal point D_1. Hence, since D_1 is not on D_2D_3, the common

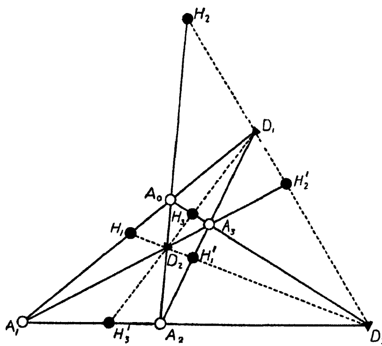


FIG. 18.

point of A_0A_1 and D_2D_3 is distinct from D_1; and, similarly, the common point of A_2A_3 and D_2D_3 is distinct from D_1 and from $\begin{pmatrix} A_0A_1 \\ D_2D_3 \end{pmatrix}$. On each of the six sides of the four-point there is therefore such a point, and to them are assigned the letters H_1, H_1', H_2, H_2', H_3, H_3', in the following way:

H_1 is on A_0A_1 and is the point $\begin{pmatrix} A_0A_1 \\ D_2D_3 \end{pmatrix}$,

H_1' is on A_2A_3 and is the point $\begin{pmatrix} A_2A_3 \\ D_2D_3 \end{pmatrix}$,

H_2 is on A_0A_2 and is the point $\begin{pmatrix} A_0A_2 \\ D_1D_3 \end{pmatrix}$,

H_2' is on A_1A_3 and is the point $\begin{pmatrix} A_1A_3 \\ D_1D_3 \end{pmatrix}$,

H_3 is on A_0A_3 and is the point $\begin{pmatrix} A_0A_3 \\ D_1D_2 \end{pmatrix}$,

H_3' is on A_1A_2 and is the point $\begin{pmatrix} A_1A_2 \\ D_1D_2 \end{pmatrix}$.

To these six points is given the name of *harmonic points* of the four-point; dually, there are *harmonic lines* of a four-line.

DEFINITION. *The point common to a side of a four-point and the line on the two diagonal points which are not on that side is termed a harmonic point of the four-point.*

The line on a point of a four-line and on the common point of the two diagonal lines which are not on that point is termed a harmonic line of the four-line.

There is an important theorem about the harmonic points of a four-point; this is given at once.

4.171. THEOREM. *The six harmonic points of a four-point are the points of a four-line.*

Since the lines A_1H_2', A_2H_1', and A_0D_3 are all on the point A_3, the triangles $A_0A_1A_2$ and $D_3H_2'H_1'$ are centrally perspective on the point A_3. Hence they are axially perspective.

Now $\begin{pmatrix} A_0A_1 \\ H_2'D_3 \end{pmatrix}$ is the point D_1, and $\begin{pmatrix} A_0A_2 \\ H_1'D_3 \end{pmatrix}$ is the point D_2. Hence the common point of A_1A_2 and $H_1'H_2'$ is on D_1D_2. That is to say, H_1', H_2', H_3' are collinear.

It may be proved in a similar way that (i) H_1, H_2, and H_3', (ii) H_1, H_2', and H_3, and (iii) H_1', H_2, H_3 are collinear.

This shows that the six harmonic points are in threes on four distinct lines, no three of which are collinear. The theorem is therefore proved.

4.172. THEOREM. *The six harmonic lines of a four-line are the sides of a four-point.*

4.2. Harmonic Tetrads

DEFINITION. *The two pairs of points consisting of* (1) *two diagonal points of a four-point and* (2) *the two harmonic points collinear with them are termed a harmonic point-tetrad.*†

DEFINITION. *The two pairs of lines consisting of* (1) *two diagonal lines of a four-line and* (2) *the two harmonic lines concurrent with them are termed a harmonic line-tetrad.*

Thus with the ordinary lettering which has been used for the four-point, the following are harmonic point-tetrads:

$$(D_2 D_3, H_1 H_1'),\ (D_3 D_1, H_2 H_2'),\ \text{and}\ (D_1 D_2, H_3 H_3').$$

It is clear from the definition that if (XY, LM) is a harmonic tetrad, then (XY, ML), (YX, LM), and (YX, ML) are also harmonic point-tetrads.

Where there is no danger of ambiguity, the qualifying words *point* and *line* in the terms *harmonic point-tetrad* and *harmonic line-tetrad* will be omitted.

4.21. Permutation Property of Harmonic Tetrads

THEOREM. *If* (XY, LM) *is a harmonic tetrad, then* (LM, XY) *is also a harmonic tetrad.*

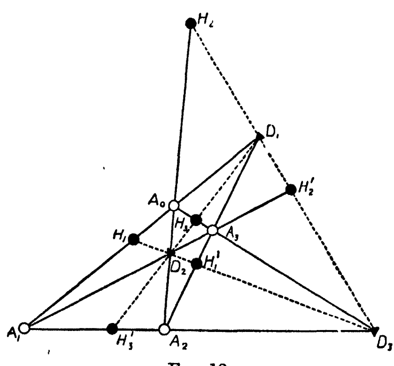


FIG. 18.

† A harmonic point-tetrad is sometimes called a *harmonic range*. The dual term is a *harmonic pencil*. These names are not used here.

With the lettering usually applied to the complete four-point, (D_2D_3, H_1H_1') is a harmonic tetrad.

Consider the four-point whose points are $H_2H_2'H_3H_3'$. Then the diagonal points of this four-point are H_1, H_1', and D_1.

The harmonic points on the line H_1H_1' are D_2 and D_3.

Hence (H_1H_1', D_2D_3) is a harmonic tetrad.

This proves the theorem.

4.22. The Unicity Theorem

A harmonic point-tetrad has been defined as a set of four collinear points which fulfil certain conditions, but it has not yet been proved that these conditions are not fulfilled by any arbitrary set of four collinear points. In the following theorem it is shown that the harmonic tetrad is, in fact, something special and that not every set of four collinear points is a harmonic tetrad.

THEOREM. *If X, Y, and L are three distinct collinear points, then there is a unique point M, collinear with them, such that (XY, LM) is a harmonic tetrad.*

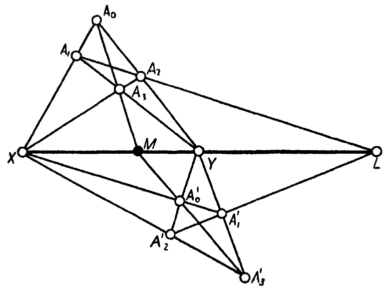


FIG. 19.

Let X, Y, and L be any three collinear points.

Let A_0 and A_0' be any two distinct points not collinear with them. Let A_1 and A_2 be points on A_0X, A_0Y respectively, such that A_1, A_2, and L are collinear. Let A_3 be the point $\begin{pmatrix} A_2X \\ A_1Y \end{pmatrix}$.

Let the points A_1', A_2', A_3' be defined similarly, relative to the point A_0'.

Now the triangles $A_0A_1A_2$ and $A'_0A'_1A'_2$ are axially perspective, since $\begin{pmatrix}A_0A_1\\A'_0A'_1\end{pmatrix}$, $\begin{pmatrix}A_0A_2\\A'_0A'_2\end{pmatrix}$, $\begin{pmatrix}A_1A_2\\A'_1A'_2\end{pmatrix}$ are the collinear points X, Y, and L respectively. Hence the lines $A_0A'_0$, $A_1A'_1$, and $A_2A'_2$ are concurrent.

Similarly, the triangles $A_1A_2A_3$ and $A'_1A'_2A'_3$ are axially perspective, so that the lines $A_1A'_1$, $A_2A'_2$, $A_3A'_3$ are concurrent.

Hence the triangles $A_0A_2A_3$ and $A'_0A'_2A'_3$ are centrally perspective, and they are therefore axially perspective also.

Hence the point $\begin{pmatrix}A_0A_3\\A'_0A'_3\end{pmatrix}$ is collinear with X and Y: let this point be M.

But X and Y are diagonal points and L and M are harmonic points of both of the four-points $A_0A_1A_2A_3$ and $A'_0A'_1A'_2A'_3$. Hence the fourth point M of a harmonic tetrad, of which one pair is X, Y, and the third point is L, is the same, however the four-point is constructed, and this proves the theorem.

The dual theorem is:

THEOREM. *If x, y, and l are three distinct concurrent lines, then there is a unique fourth line on their common point such that (xy, lm) is a harmonic tetrad.*

4.221. Harmonic Conjugates

The theorem just proved shows that if X and Y are one of the pairs of a harmonic tetrad, then corresponding to every point L there is a unique point M such that (XY, LM) is a harmonic tetrad. It is useful to have a name for such pairs of points.

DEFINITION. *If X, Y, L, M be four collinear points, and (XY, LM) be a harmonic tetrad, then L and M are said to be harmonic conjugates relative to the pair X, Y.*

If x, y, l, m be four concurrent lines, and (xy, lm) be a harmonic tetrad, then l and m are said to be harmonic conjugates relative to the pair x, y.

It is plain that if L and M are harmonic conjugates relative to X and Y, then X and Y are harmonic conjugates relative to L and M.

The phrase '*relative to*' which is used in the above definitions, is deserving of some notice, for it will frequently occur. To say that L and M are harmonic conjugates is a meaningless form of words unless another pair of points is mentioned with which this pair forms a harmonic tetrad. L and M can only be harmonically conjugate when there is another pair of points to which they are *related* in such a way that the two pairs form a harmonic tetrad. *Harmonically conjugate* is an example of a *relative term*, that is, a term which does not acquire precise meaning unless taken in conjunction with another term, the choice of which is, within certain limits, arbitrary.

4.222. Singular Harmonic Tetrads

Given three collinear points X, Y, L there is a unique fourth point M, collinear with the other three, which is the harmonic conjugate of L relative to X and Y. It is not a completely trivial question to ask what the harmonic conjugate of Y is, relative to X and Y; in fact, later on, the answer to this question is important.

It is an easy matter to verify, by carrying out the construction of 4.22, that *the harmonic conjugate of Y relative to X and Y is Y.*

Similarly, the harmonic conjugate of X relative to X and Y is X.

A tetrad such as that just considered, in which three of the points are coincident, is aptly called a *singular harmonic tetrad.* Dually, there are singular harmonic line-tetrads.

In the paragraphs which follow immediately it is consistently supposed that the harmonic tetrads dealt with are not singular.

4.23. Projective Properties of Harmonic Tetrads

It has been shown that not every set of four collinear points is a harmonic point-tetrad, and, dually, that not every set of four concurrent lines is a harmonic line-tetrad. It follows that a harmonic tetrad has special properties which distinguish it from other tetrads, and it is therefore natural to ask whether a tetrad (point- or line-) which is projective with a harmonic tetrad (point- or line-) is itself harmonic. This general question is answered in the theorems which follow, and in them the projective properties of the harmonic tetrad are investigated; it is these projective properties which make the harmonic tetrad important in Projective Geometry.

4.231. THEOREM. *If (XY, LM) is a harmonic point-tetrad, then*

$$(XYLM) \sim (XYML) \sim (YXLM) \sim (YXML).$$

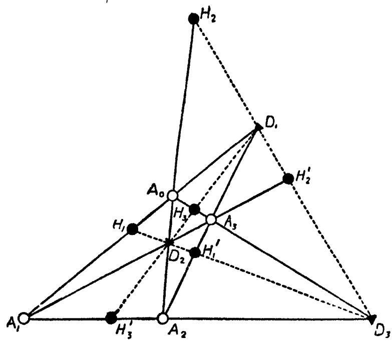


FIG. 18.

With the usual lettering of the complete four-point, $(D_2 D_3, H_1 H_1')$ is a harmonic tetrad.

Now $$(D_2 D_3 H_1 H_1') \overset{H_3'}{\barwedge} (D_1 D_3 H_2 H_2'),$$

but $$(D_1 D_3 H_2 H_2') \overset{H_3}{\barwedge} (D_2 D_3 H_1' H_1).$$

Hence, if (XY, LM) is a harmonic tetrad,

$$(XYLM) \sim (XYML).$$

The other results follow by applying the permutation theorem (3.325).

4.232. THEOREM. *If X, Y, L, and M are four distinct collinear points, and if $(XYLM) \sim (XYML)$, then (XY, LM) is a harmonic tetrad.*

Suppose that (XY, LM) is not a harmonic tetrad. Let Y' be the point such that (XY', LM) is a harmonic tetrad.

Then by the last theorem $(XY'LM) \sim (XY'ML)$; and by supposition $(XYLM) \sim (XYML)$.

Now these two projectivities have three pairs of corresponding points in common, hence they are the same projectivity.

It follows that $(XYY'LM) \sim (XYY'ML)$. But this is a projectivity in which there are three distinct self-corresponding points, and it is not a projectivity in which every point is self-corresponding. This is a contradiction in terms. Hence Y' must coincide with Y. That is to say, (XY, LM) is a harmonic tetrad.

The last two theorems may be re-enunciated as one:

4.233. Necessary and Sufficient Condition for a Harmonic Tetrad

THEOREM. *The necessary and sufficient condition that the four distinct collinear points X, Y, L, and M should form a harmonic tetrad (XY, LM) is that*

$$(XYLM) \sim (XYML).$$

THEOREM. *The necessary and sufficient condition that the four distinct concurrent lines x, y, l, and m should form a harmonic tetrad (xy, lm) is that $(xylm) \sim (xyml)$.*

It is now possible to prove the full projective properties of harmonic tetrads. This is done in the following theorem.

4.234. THEOREM. *If a tetrad is projective with a harmonic tetrad, it is itself a harmonic tetrad.*

Conversely, any two harmonic tetrads are projective with each other.

The enunciation does not specify whether the tetrads in question are point-tetrads or line-tetrads. In the proof which follows that case only is considered in which one is a point-tetrad and the other a line-tetrad; the proofs of the other cases are entirely similar.

Suppose first that (XY, LM) is a harmonic tetrad, and that $(XYLM) \sim (xylm)$, where x, y, l, and m are four concurrent lines.

Then by the last theorem $(XYLM) \sim (XYML)$, and from this it follows that $(xylm) \sim (xyml)$.

Hence by the last theorem (xy, lm) is a harmonic tetrad.

Conversely, suppose that both (XY, LM) and (xy, lm) are harmonic tetrads. If it is not true that $(XYLM) \sim (xylm)$, let m^* be the line on the same point as the other four, and such that $(XYLM) \sim (xylm^*)$.

Then by the first part of the theorem (xy, lm^*) is a harmonic tetrad. That is to say, m and m^* coincide. This contradicts the supposition that m^* is not the same line as m. Hence

$$(XYLM) \sim (xylm).$$

The theorem is thus proved.

4.24. Harmonic Tetrads in the Algebraic Representation

The object of this paragraph is to answer the question: Given four collinear points X, Y, L, M, whose coordinates relative to some base points are, respectively, (μ_1, μ_1'), (μ_2, μ_2'), (μ_3, μ_3'), and (μ_4, μ_4'), what is the necessary and sufficient algebraic condition that (XY, LM) shall be a harmonic tetrad?

The usual notation for the four-point in the Algebraic Representation being supposed, it is not difficult to verify that the harmonic points H_1 and H_1' are, respectively, $(\lambda_0 x_0 - \lambda_1 x_1, \lambda_0 y_0 - \lambda_1 y_1, \lambda_0 z_0 - \lambda_1 z_1)$ and $(\lambda_2 x_2 - \lambda_3 x_3, \lambda_2 y_2 - \lambda_3 y_3, \lambda_2 z_2 - \lambda_3 z_3)$.

Hence the harmonic tetrad $(D_2, D_3, H_1 H_1')$ consists of the following points: $(\lambda_0 x_0 + \lambda_2 x_2, \lambda_0 y_0 + \lambda_2 y_2, \lambda_0 z_0 + \lambda_2 z_2)$, $(\lambda_0 x_0 + \lambda_3 x_3, \lambda_0 y_0 + \lambda_3 y_3, \lambda_0 z_0 + \lambda_3 z_3)$, $(\lambda_0 x_0 - \lambda_1 x_1, \lambda_0 y_0 - \lambda_1 y_1, \lambda_0 z_0 - \lambda_1 z_1)$, and $(\lambda_2 x_2 - \lambda_3 x_3, \lambda_2 y_2 - \lambda_3 y_3, \lambda_2 z_2 - \lambda_3 z_3)$.

If now the points D_2 and D_3 be taken as base points, these four points have the coordinates $(1, 0)$, $(0, 1)$, $(1, 1)$, and $(1, -1)$ respectively.

Suppose now that the points X, Y, L, and M have coordinates (μ_1, μ_1'), (μ_2, μ_2'), (μ_3, μ_3'), and (μ_4, μ_4') relative to some base points. Then since the necessary and sufficient condition that (XY, LM) is a harmonic tetrad is that $(XYLM) \sim (D_2 D_3 H_1 H_1')$, a necessary and sufficient condition is that there should be numbers a, b, c, and d such that $ad - bc \neq 0$, and

$$\begin{aligned} a\mu_1 + b\mu_1' \qquad\qquad &= 0, \\ c\mu_2 + d\mu_2' &= 0, \\ a\mu_3 + b\mu_3' + c\mu_3 + d\mu_3' &= 0, \\ a\mu_4 + b\mu_4' - c\mu_4 - d\mu_4' &= 0, \end{aligned}$$

by 3.252; that is to say, that

$$\begin{vmatrix} \mu_1 & \mu_1' & 0 & 0 \\ 0 & 0 & \mu_2 & \mu_2' \\ \mu_3 & \mu_3' & \mu_3 & \mu_3' \\ \mu_4 & \mu_4' & -\mu_4 & -\mu_4' \end{vmatrix} = 0.$$

This determinant, on being simplified, is found to be

$$(\mu_1 \mu_3' - \mu_1' \mu_3)(\mu_2 \mu_4' - \mu_2' \mu_4) + (\mu_1 \mu_4' - \mu_1' \mu_4)(\mu_2 \mu_3' - \mu_2' \mu_3),$$

and so the equation may be written in the form

$$\frac{(\mu_1 \mu_3' - \mu_1' \mu_3)(\mu_2 \mu_4' - \mu_2' \mu_4)}{(\mu_1 \mu_4' - \mu_1' \mu_4)(\mu_2 \mu_3' - \mu_2' \mu_3)} = -1.$$

Hence the following theorem may be enunciated:

The necessary and sufficient condition that four distinct collinear points X, Y, L, M, whose coordinates relative to some base points collinear with them are (μ_1, μ_1'), (μ_2, μ_2'), (μ_3, μ_3'), and (μ_4, μ_4') respectively, should be a harmonic tetrad (XY, LM) is that

$$\frac{(\mu_1\mu_3' - \mu_1'\mu_3)(\mu_2\mu_4' - \mu_2'\mu_4)}{(\mu_1\mu_4' - \mu_1'\mu_4)(\mu_2\mu_3' - \mu_2'\mu_3)} = -1.$$

EXAMPLES

1. With the usual notation applied to the four-point, show that (A_0A_1, D_1H_1) and (A_2A_3, D_1H_1') are harmonic tetrads. Hence enumerate nine harmonic tetrads associated with the four-point.

2. x and y are two lines and V is a point not on either, and l, m, n are three lines on V. $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are the points lx, mx, nx, ly, my, ny respectively. If Z_1, Z_2, Z_3 are points on l, m, n respectively, such that (VX_1, Y_1Z_1), (VX_2, Y_2Z_2), and (VX_3, Y_3Z_3) are all harmonic tetrads, show that Z_1, Z_2, Z_3, and xy are collinear.

3. A, B, C, D, X, and Y are six collinear points, and A', B', C', D' are the harmonic conjugates relative to XY of the points A, B, C, D respectively. Show that $(ABCD) \sim (A'B'C'D')$, and that X and Y are the self-corresponding points of the projectivity.

4. Show that the four-line whose six points are the harmonic points of a four-point has the same diagonal triangle as the four-point.

5. ABC is any triangle; A' and A'' are two points on BC such that $(BC, A'A'')$ is a harmonic tetrad. The points B' and B'' on CA, C' and C'' on AB are similarly defined. Show that corresponding sides of the three triangles ABC, $A'B'C'$, $A''B''C''$ are concurrent.

6. ABC is any triangle, and O any other point. A', B', C' are the points $\binom{BC}{AO}$, $\binom{CA}{BO}$, $\binom{AB}{CO}$ respectively, and A'', B'', C'' are points such that $(BC, A'A'')$, $(CA, B'B'')$, $(AB, C'C'')$ are all harmonic tetrads.

Show that (i) A'', B', C' are collinear, and that there are two other, similar, sets; (ii) AA', BB'', CC'' are concurrent, and that there are two other, similar, sets; and (iii) A'', B'', C'' are collinear.

7. Show that the diagonal triangle of a four-point is perspective with any triangle whose vertices are any three of the four points of the four-point.

8. If $(ABCD) \sim (BCDA)$, show that (AC, BD) is a harmonic tetrad. Show that the converse is also true.

9. If two harmonic point-tetrads on two different lines have a point in common, show that they are perspective in two different ways.

10. Given four points X, Y, Z, A, no three of which are collinear, construct a four-point $ABCD$ whose diagonal points are X, Y, and Z.

How many four-points satisfy the conditions?

4.3. Involutory Hexads

The harmonic tetrad is a set of four collinear points which is defined in terms of concepts which arise from the consideration of the four-point; the involutory hexad is a set of six collinear points which arises in a similar way. In a certain sense the harmonic tetrad is a particular case of the involutory hexad, but this is not the real reason why the latter is important in Projective Geometry. Its real importance lies in the fact that it leads on at once to the notion of the *involution*, without which Projective Geometry would be extremely handicapped.

4.31. Definitions

A set of six collinear points such that each is on one of the six sides of a complete four-point is termed an involutory point-hexad.

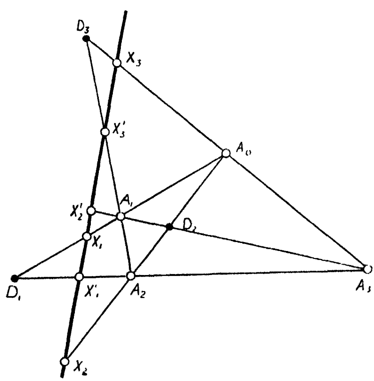


Fig. 20.

In the figure X_1, X_2, X_3, X_1', X_2', X_3' are the six points. The method of assigning suffixes will be easily understood.

A set of six concurrent lines such that each is on one of the six points of a complete four-line is termed an involutory line-hexad.

The words *point* and *line* in the terms *point-hexad* and *line-hexad* will be omitted when no ambiguity arises.

Since the six sides of a four-point can be classified into three pairs, the six points of an involutory hexad can also be classified into three pairs. This fact will have been realized by the way in which suffixes and dashes have been assigned to the points. Thus X_1 and X_1', X_2 and X_2', X_3 and X_3' are the three pairs. The

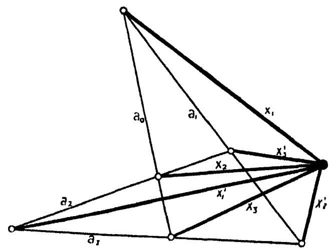


FIG. 21.

fact that three pairs of points constitute an involutory hexad is expressed by writing† $(X_1X_2X_3, X_1'X_2'X_3')$.

It is clear that if $(X_1X_2X_3, X_1'X_2'X_3')$ is an involutory hexad, then

$$(X_1'X_2X_3, X_1X_2'X_3'), \qquad (X_1X_2'X_3, X_1'X_2X_3'),$$
$$(X_1X_2X_3', X_1'X_2'X_3), \qquad (X_1X_2'X_3', X_1'X_2X_3),$$
$$(X_1'X_2X_3', X_1X_2'X_3), \qquad (X_1'X_2'X_3, X_1X_2X_3'),$$

and

$$(X_1'X_2'X_3', X_1X_2X_3)$$

are also involutory hexads.

The same questions arise about involutory hexads as arose about harmonic tetrads. These are: (1) Are any six collinear points an involutory hexad? (2) If there are six collinear points, and there is a projectivity between them and an involutory hexad, are they themselves an involutory hexad? (3) What is the necessary and sufficient condition that six points should be an involutory hexad? These questions are answered in the theorems which follow.

† The reader will probably find it more helpful in written work to symbolize an involutory hexad by the scheme $\begin{bmatrix} X_1X_2X_3 \\ X_1'X_2'X_3' \end{bmatrix}$. He may also find it helpful to name the three pairs of points: *Father, Mother; Brother, Sister; Uncle, Aunt.* The initial letters of those words will be the letters assigned to the points, so that the hexad will be $\begin{bmatrix} FBU \\ MSA \end{bmatrix}$. This device is not so childish as it may appear at first sight.

4.32. The Unicity Theorem

THEOREM. *If $X_1, X_2, X_3, X'_1, X'_2$ are five distinct collinear points, then there is a unique point X'_3 collinear with them, such that*

$$(X_1 X_2 X_3, X'_1 X'_2 X'_3)$$

is an involutory hexad.

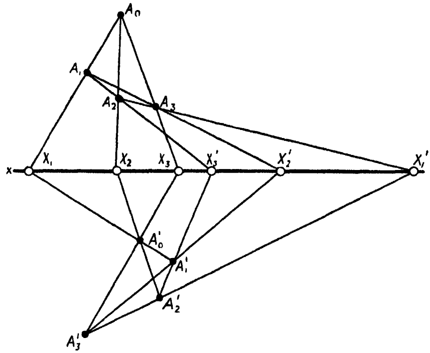


FIG. 22.

Let x be the line on the five given points.

Let A_0 be any point not on x, and A_1 any other point on A_0X_1.

Let A_3 be the point $\begin{pmatrix} A_1 X'_2 \\ A_0 X_3 \end{pmatrix}$, and A_2 the point $\begin{pmatrix} A_3 X'_1 \\ A_0 X_2 \end{pmatrix}$.

Let A'_0 be another point not on x, and distinct from A_0. Let A'_1, A'_2, A'_3 be three points defined in a similar way, relative to A'_0, as A_1, A_2, A_3 were defined relative to A_1.

The proof of the theorem now proceeds very similarly to that of 4.22. The outline only is given here; the details may be filled in by the reader.

The triangles $A_0A_1A_3$ and $A'_0A'_1A'_3$ are axially perspective, and so they are also centrally perspective. Similarly, the triangles $A_0A_2A_3$ and $A'_0A'_2A'_3$ are axially perspective, and so they too are centrally perspective.

Hence the triangles $A_1A_2A_3$ and $A'_1A'_2A'_3$ are centrally perspective, and so they are axially perspective, and x is their axis of perspective. That is, the common point of A_1A_2 and

$A_1'A_2'$ is on x. This point is the unique sixth point of the involutory hexad.

4.33. Necessary and Sufficient Condition for an Involutory Hexad

THEOREM. *The necessary and sufficient condition that six collinear points X_1, X_2, X_3, X_1', X_2', X_3' should form an involutory hexad in which X_1 and X_1', X_2 and X_2', X_3 and X_3' are pairs is that*

$$(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3' X_1),$$

or

$$(X_1 X_2 X_3 X_2') \sim (X_1' X_2' X_3' X_2),$$

or

$$(X_1 X_2 X_3 X_3') \sim (X_1' X_2' X_3' X_1).$$

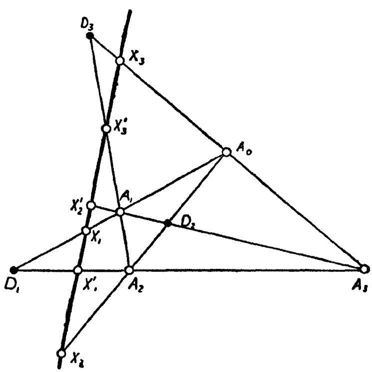


FIG. 20.

First suppose that $(X_1 X_2 X_3, X_1' X_2' X_3')$ is an involutory hexad. Then

$$(X_1 X_2 X_3 X_1') \overset{A_0}{\barwedge} (D_1 A_2 A_3 X_1')$$

$$\overset{A_1}{\barwedge} (X_1 X_3' X_2' X_1').$$

But by the permutation theorem (3.325)

$$(X_1 X_3' X_2' X_1') \sim (X_1' X_2' X_3' X_1),$$

hence

$$(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3' X_1).$$

The other two results are proved similarly. Hence the conditions are necessary.

Next suppose that

$$(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3' X_1),$$

and that the six points are not an involutory hexad. Suppose then that $(X_1 X_2 X_3, X_1' X_2' X_3'')$ is an involutory hexad. Then by the first part of the theorem

$$(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3'' X_1).$$

Hence $$(X_1' X_2' X_3' X_1) \sim (X_1' X_2' X_3'' X_1);$$

but this last projectivity has three self-corresponding points, so that X_3' and X_3'' must coincide. Hence the six points form an involutory hexad.

This proves that the first of the conditions mentioned is sufficient, and it may also be proved in a similar manner that either of the others is sufficient.

The theorem which has just been proved is of very great importance in the work which follows; it occurs again, with a slightly different enunciation in the theory of involutions, and it will constantly be encountered in the theory of conics.

4.34. Projective Properties

It is easy to see that if

$$(X_1 X_2 X_3, X_1' X_2' X_3') \quad \text{and} \quad (Y_1 Y_2 Y_3, Y_1' Y_2' Y_3')$$

are two involutory hexads, there is not necessarily a projectivity such that

$$(X_1 X_2 X_3 X_1' X_2' X_3') \sim (Y_1 Y_2 Y_3 Y_1' Y_2' Y_3').$$

For in the projectivity in which $(X_1 X_2 X_3) \sim (Y_1 Y_2 Y_3)$, X_1' and Y_1' need not necessarily be corresponding points, since Y_1' may be chosen arbitrarily; and even if they are, X_2' and Y_2' need not necessarily be corresponding points, for a similar reason. But if these five pairs of points be, in fact, pairs of corresponding points in a projectivity, then it can easily be proved that the sixth pair, X_3' and Y_3', are also a pair of corresponding points in the projectivity.

The converse of the theorem, namely, that if

$$(X_1 X_2 X_3, X_1' X_2' X_3')$$

be an involutory hexad, and if there be a projectivity in which

$$(X_1 X_2 X_3 X_1' X_2' X_3') \sim (Y_1 Y_2 Y_3 Y_1' Y_2' Y_3'),$$

then $(Y_1Y_2Y_3, Y_1'Y_2'Y_3')$ is also an involutory hexad, is an easy corollary of 4.33. Both of these are left to the reader.

4.35. Singular Involutory Hexads

In all that has been said about involutory point-hexads it has been tacitly supposed that the line on which the six points are is not on any of the diagonal points, and something must be added in order to cover the cases when it is.

In the first place, suppose that the line is on one, but only one, diagonal point, and, for definiteness, suppose this is the point D_1. It is at once obvious that there are now not six, but five, distinct points; nevertheless, it is also obvious that the point D_1 can be looked on as two coincident points, one of which is on A_0A_1, the other on A_2A_3. There is therefore still an involutory hexad, but one of the constituent pairs is a pair of coincident points; D_1 is the coincident pair X_1 and X_1'.

Next, suppose that the line is on two diagonal points D_2 and D_3. As before, each of these may be looked on as a pair of coincident points, and the involutory hexad consists of one pair of distinct points and two pairs of coincident points.

Involutory hexads in which one or more pairs are pairs of coincident points may be called *singular involutory hexads*. The reader will easily prove that there cannot be more than two pairs of coincident points. All the theorems about involutory hexads are true of the singular cases, and the reader should verify this.

The four points which constitute the singular hexad in which there are two pairs of coincident points together form a harmonic tetrad, and so the remark that the harmonic tetrad is, in a certain sense, a particular case of the involutory hexad is justified. But the full significance of this will be more obvious when the notion of the involution has become familiar.

4.4. Involutions

If X_1, X_2, X_3 are three collinear points, and X_1', X_2', X_3' are any other three points collinear with the first three, there is a projectivity in which $(X_1X_2X_3) \sim (X_1'X_2'X_3')$, and this projectivity is completely determined. If now the two sets of three points together form an involutory hexad, it is natural to suppose that the projectivity has special properties which distinguish it from other projectivities. This is, in fact, true, and the following theorems bring out the properties of such a projectivity. First of all, however, it is useful to have a name for a projectivity between cobasal ranges or pencils in which three pairs of corresponding points are the three pairs of an involutory hexad.

4.41. Definition

Any projectivity between cobasal ranges or cobasal pencils in which there are three pairs of corresponding points which are the three pairs of an involutory hexad is termed an involution.†

The name *involution* is primarily a name for the projectivity between two cobasal ranges or pencils, and so we speak of two ranges *in involution*, meaning thereby that there is an involution between them. In a secondary sense the word *involution* is sometimes applied to the two ranges between which there is an involution; it will be found that there is no ambiguity in this usage.

4.42. The Fundamental Theorem on Involutions

THEOREM. *If X_1 and X_1', X_2 and X_2', X_3 and X_3' be three pairs of corresponding points in a projectivity, the necessary and sufficient condition that this projectivity be an involution is that*

$$(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3' X_1).$$

First suppose that the projectivity is an involution so that there are three pairs of corresponding points which form an involutory hexad. Let this hexad be $(Y_1 Y_2 Y_3, Y_1' Y_2' Y_3')$. Then

$$(Y_1 Y_1 Y_2 Y_1' Y_2' Y_3') \sim (Y_1' Y_2' Y_3' Y_1 Y_2 Y_3).$$

Consider now the five points Y_1, Y_2, X_3, Y_1', Y_2'; let X_3'' be a sixth point such that $(Y_1 Y_2 X_3, Y_1' Y_2' X_3'')$ is an involutory hexad. Then

$$(Y_1 Y_2 X_3 Y_1') \sim (Y_1' Y_2' X_3'' Y_1)$$

But by supposition

$$(Y_1 Y_2 X_3 Y_1') \sim (Y_1' Y_2' X_3' Y_1),$$

so that $(Y_1 Y_2 X_3, Y_1' Y_2' X_3')$ is an involutory hexad.

† The name was that given by Desargues (1639) when he studied involutions metrically. The reason for the name is a little obscure, and it is certainly not a very apt term. It is retained because it is now universally used. As the sequel will show, the term *reciprocal projectivity* would perhaps be more apt.

Similarly, it may be proved that $(Y_1 X_2 X_3, Y_1' X_2' X_3')$ and finally $(X_1 X_2 X_3, X_1' X_2' X_3')$ are also involutory hexads.

Hence any three pairs of corresponding points in the projectivity form an involutory hexad.

Hence, by 4.33, for any three pairs of corresponding points

$$(X_1 X_2 X_3 X_1') \sim (X_1 X_2 X_3 X_1').$$

This proves the necessity of the condition.

The sufficiency of the condition follows at once from 4.33, for if $(X_1 X_2 X_3 X_1') \sim (X_1' X_2' X_3' X_1)$, then $(X_1 X_2 X_3, X_1' X_2' X_3')$ is an involutory hexad, and the projectivity is an involution.

4.421. Definition

A pair of corresponding points in an involution is termed a pair of mates of the involution.

4.422. Remarks on Theorem 4.42

The theorem just proved shows that pairs of mates of an involution have a remarkable property, which is probably most simply explained by using the ideas suggested in 3.42. There it was suggested that cobasal ranges should be thought of as possessing distinguishing colours. A projectivity sets up a correspondence between red points and blue points. In a general projectivity a red point A corresponds to the blue point B, say; but the red point B does not necessarily correspond to the blue point A. When the projectivity is an involution, however, Theorem 4.42 shows that if the red point A corresponds to the blue point B, then the red point B corresponds to the blue point A. There is thus a *reciprocal* correspondence in an involution.

4.43. Another Sufficient Condition

THEOREM. *If in a projectivity between cobasal ranges a pair of distinct points correspond reciprocally, that is to say, if*

$$(AB...) \sim (BA...),$$

then the projectivity is an involution.

Let C be any other point of the first range and D the corresponding point of the second. Let X be that point of the second range which corresponds to D, considered as a point of the first range.†

† If the reader will substitute *red* and *blue* for *first* and *second* respectively, the thought underlying this formal language will become clearer.

Then $(ABCD) \sim (BADX)$.

But, by the permutation theorem,

$$(ABCD) \sim (BADC),$$

so that X and C are the same point.

Similarly, if E be any other point of the first range and F the corresponding point in the second,

$$(ABEF) \sim (BAFE).$$

Hence $(ACEB) \sim (BDFA)$,

and so, by the previous theorem, the projectivity is an involution.

4.431. Note on the Sequence of Theorems

It is clear that an involution could have been defined as a projectivity in which there is a pair of distinct points which correspond reciprocally. It could then have been shown that in an involution every pair of corresponding points correspond reciprocally, exactly as in 4.43. From this result it is a simple conclusion that three pairs of mates in an involution together form an involutory hexad.

This line of approach to the subject is in many ways more satisfying, since it starts with something much simpler than the involutory hexad, and leads up to the connexion between involutions and involutory hexads. It has not been adopted here in order that the theory of involutions might appear as an immediate extension of the matter preceding it.

4.44. Self-corresponding Points of Involutions

Since an involution is a special case of a projectivity, it is natural to inquire about the self-corresponding points, if there are any, of an involution. In the following theorems this inquiry is undertaken.

4.441. THEOREM. *If X and Y be any two distinct points, then the pairs of points which are harmonic conjugates relative to X and Y are pairs of mates in an involution of which X and Y are the self-corresponding points.*

Let A and A' be any pair of conjugate points relative to X and Y.

Consider the projectivity in which $(XYA) \sim (XYA')$, and suppose that in this projectivity $(XYAA') \sim (XYA'A'')$.

Now (XY, AA') is a harmonic tetrad, therefore by 4.23 $(XY, A'A'')$ is a harmonic tetrad. Hence A'' is the same point as A.

Hence in the projectivity A and A' are reciprocally corresponding points, and the projectivity is therefore an involution, by 4.43.

Let B and B' be another pair of corresponding points, so that they are, by 4.43, reciprocally corresponding points.

Then $(XYBB') \sim (XYB'B)$, that is to say, by 4.24, B and B' are harmonic conjugates relative to X and Y.

Similarly, any pair of mates are harmonic conjugates relative to X and Y.

Hence the pairs of points harmonically conjugate relative to X and Y are pairs of mates in an involution whose self-corresponding points are X and Y.

4.442. THEOREM. *If an involution have a pair of distinct self-corresponding points, then every pair of mates of the involution is a pair of harmonic conjugates relative to the self-corresponding points.*

Let X and Y be the self-corresponding points of the involution, and let A and A' be a pair of mates.

Then $(XYAA') \sim (XYA'A)$, and so, by 4.24, A and A' are a pair of harmonic conjugates relative to X and Y.

4.443. THEOREM. *If an involution have one self-corresponding point, then it has a second, distinct from the first.*

Let X be the self-corresponding point, and let A and A' be any pair of mates of the involution.

Then the involution is a projectivity in which

$$(XAA') \sim (XA'A).$$

Suppose that Y is the harmonic conjugate of X relative to AA', and suppose that Y' is the mate of Y in the involution.

Then $(XYAA') \sim (XY'A'A)$.

But (XY, AA') is a harmonic tetrad, and so, by 4.23, $(XY', A'A)$ is also a harmonic tetrad; hence Y' and Y are the same point. Therefore Y, which must be distinct from X, is also a self-corresponding point of the involution.

4.444. Summary

It has been proved that there are involutions which have two self-corresponding points, and that no involution can have a single self-corresponding point. It is therefore natural to ask whether there can be involutions which have no self-corresponding points. This question cannot be answered definitely, for there are some systems in which every involution has two self-corresponding points, and there are others in which some involutions have no self-corresponding points. Both types of system are compatible with the initial propositions so far adopted. Later an initial proposition about extension will be added, and this will exclude all systems of the second type.

4.45. Conditions Determining an Involution

A projectivity is completely determined when three pairs of corresponding points are known, and these three pairs may be chosen arbitrarily. But if the projectivity is to be an involution, then obviously the three pairs cannot be chosen arbitrarily, for they must form an involutory hexad. The following theorem shows that when two pairs of mates are known, the involution is completely determined.

THEOREM. *An involution is completely specified when two pairs of mates are known; either or both of the pairs may be coincident.*

Let A and A', B and B' be the two pairs of mates.

Suppose there are two involutions in which these two pairs are pairs of mates.

Let C and C' be any other pair of mates in the first involution, and let C'' be the mate of C in the second involution.

Then $(ABC, A'B'C')$ and $(ABC, A'B'C'')$ are both involutory hexads; hence, by 4.32, C' and C'' are the same point.

This shows that the two involutions are identical.

4.46. Common Mates of Two Involutions

A problem that is continually occurring in subsequent work is the following: If there are two different involutions on a line, what pairs of mates are the same in the two involutions?

The following theorem shows that if the two involutions are really different, then there is one and only one pair of mates common to the two. By the time this theorem is needed, an initial proposition of extension will have been added which will ensure that every projectivity has two self-corresponding points, distinct or coincident. But since this initial proposition has not yet been laid down, the following theorem is enunciated conditionally.

THEOREM. *If every involution has two self-corresponding points, then there is one and only one pair of points, distinct or coincident, which is a pair of mates in both of two involutions on the same line.*

It is clear from the last theorem that two different involutions cannot have more than one pair of mates in common.

Suppose now that X and X' are the self-corresponding points of one involution and that Y and Y' are those of the other.

If one of the first pair coincide with either of the second, this is the pair of (coincident) mates common to the two involutions. Suppose, however, that the four points are all distinct.

Consider the involution in which X and X' are a pair of mates and Y and Y' are a pair of mates. Let Z and Z' be the self-corresponding points of this involution.

Then, by 4.442, (XX', ZZ') is a harmonic tetrad, hence Z and Z' are a pair of mates in the first involution. By entirely similar reasoning Z and Z' are also a pair of mates in the second involution.

This proves the theorem.

4.47. Involutions in the Algebraic Representation

If (λ_1, λ_1') and (λ_2, λ_2') are the coordinates of two points on a line relative to some chosen base points, and if these two points are a pair of corresponding points in a projectivity, then there exists an algebraic relation

$$a\lambda_1\lambda_2+b\lambda_1\lambda_2'+c\lambda_1'\lambda_2+d\lambda_1'\lambda_2' = 0 \quad (ad-bc \neq 0)$$

connecting λ_1, λ_1', λ_2, λ_2'.

If now the projectivity be an involution, then every pair of corresponding points is a pair of reciprocally corresponding points, and so

$$a\lambda_2\lambda_1+b\lambda_2\lambda_1'+c\lambda_2'\lambda_1+d\lambda_2'\lambda_1' = 0.$$

Subtracting these two,

$$(b-c)\lambda_1\lambda_2'+(c-b)\lambda_1'\lambda_2 = 0,$$

or

$$(b-c)(\lambda_1\lambda_2'-\lambda_1'\lambda_2) = 0.$$

But $(\lambda_1\lambda_2'-\lambda_1'\lambda_2)$ does not vanish, and so $b = c$.

Hence, *if a projectivity specified by the equation*

$$a\lambda_1\lambda_2+b\lambda_1\lambda_2'+c\lambda_1'\lambda_2+d\lambda_1'\lambda_2' = 0 \quad (ad-bc \neq 0)$$

be an involution, then $b = c$.

Conversely, in any projectivity in which $b = c$ every pair of points is a pair of reciprocally corresponding points, so that the projectivity is an involution.

The two results may be stated as one, thus:

The necessary and sufficient condition that a projectivity specified by the equation

$$a\lambda_1\lambda_2+b\lambda_1\lambda_2'+c\lambda_1'\lambda_2+d\lambda_1'\lambda_2' = 0 \quad (ad-bc \neq 0)$$

should be an involution is that $b = c$.

4.5. Concurrence and Collinearity in Triangles

In elementary Geometry it is proved that the three medians of a triangle are concurrent; this theorem is only one of a number of such theorems, which assert that three lines, one on each of the points of a triangle, are concurrent provided certain conditions are satisfied. Similarly, there are theorems which assert that three points, one on each of the sides of a triangle, are collinear. There are two theorems in metrical Geometry, known as Ceva's and Menelaus's theorems, which state general conditions for the concurrence of such lines and the collinearity of such points, but these theorems state metrical conditions, and therefore they cannot be given here, for the term *length* has not been defined. But it is to be expected, since concurrence of lines and collinearity of points are notions which do not involve the notion of length, that non-metrical conditions can be stated. In the following two theorems, which are dual theorems, necessary and sufficient conditions are stated for the concurrence of lines on the points of a triangle, and the collinearity of points on the lines of a triangle. The theorems are a simple corollary of the work that has been done on involutions, and for that reason they are included here. It will be found that all problems of concurrence and collinearity in triangles can be solved by their use.

4.51. THEOREM. *If*

(i) *ABC is a triangle whose sides BC, CA, AB are the lines a, b, c respectively,*

(ii) *l, m, n are three lines on A, B, C respectively,*

(iii) *x is any other line distinct from these six,*

(iv) *A', B', C', A'', B'', C'' are the six collinear points ax, bx, cx, lx, mx, nx respectively,*

then the three lines l, m, n are concurrent if and only if A', A''; B', B''; C', C'' are pairs of mates in an involution.

The details of the proof of this theorem are left to the reader, and only the outline is given here.

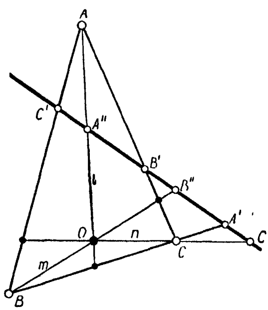


FIG. 23.

First suppose that l, m, n are concurrent, and that O is their common point. Then the necessity of the condition at once follows from the fact that the six points are an involutory hexad associated with the four-point $ABCO$.

Next suppose that the condition is fulfilled, but that l, m, and n are not concurrent. Let O be the point mn, and l' the line AO. Further, let K be the point xl'. Then by the first part of the theorem $(A'B'C', KB''C'')$ is an involutory hexad, and so K and A'' are the same point. That is to say l is on O, and the three lines are concurrent.

4.52. THEOREM. *If*

(i) *ABC is a triangle whose sides BC, CA, AB are the lines a, b, c respectively,*

(ii) *L, M, N are three points on a, b, c respectively,*

(iii) *X is any other point distinct from these six,*

(iv) *a', b', c', a'', b'', c'' are the concurrent lines AX, BX, CX, LX, MX, NX respectively,*

then the three points L, M, N are collinear if and only if a', a''; b', b''; c', c'' are three pairs of mates in an involution.

Examples

1. Prove Theorem 4.32 when A_0 and A_0' coincide but A_1 and A_1' do not.

2. If $(X_1 X_2 X_3, X_1' X_2' X_3')$ is an involutory hexad, and if V is any point not collinear with these six, show that

$$(VX_1, VX_2, VX_3;\ VX_1', VX_2', VX_3')$$

is also an involutory hexad.

3. X and X', Y and Y' are two pairs of mates in an involution in which M and N are the self-corresponding points. Show that X and Y', X' and Y, M and N are three pairs of mates in another involution.

4. If $$x(A_1 A_2 A_3 A_4 \ldots) \overset{U}{\barwedge} y(B_1 B_2 B_3 B_4 \ldots)$$

and $$x(A_1 A_2 A_3 A_4 \ldots) \overset{V}{\barwedge} y(C_1 C_2 C_3 C_4 \ldots),$$

determine a necessary and sufficient condition that the projectivity

$$y(B_1 B_2 B_3 B_4 \ldots) \sim y(C_1 C_2 C_3 C_4 \ldots)$$

shall be an involution.

5. If l is the Pappus line of the projectivity

$$x(X_1 X_2 X_3 X_4 \ldots) \sim y(Y_1 Y_2 Y_3 Y_4 \ldots),$$

and if $y(Y_1 Y_2 Y_3 Y_4 \ldots) \overset{U}{\barwedge} x(Z_1 Z_2 Z_3 Z_4 \ldots)$, show that the projectivity $x(X_1 X_2 X_3 X_4 \ldots) \sim x(Z_1 Z_2 Z_3 Z_4 \ldots)$ is an involution if and only if U is on l.

6. Prove the dual of Theorem 4.22 without appealing to the principle of duality.

7. A and A' are a pair of mates in an involution. Assuming that every involution has two self-corresponding points, devise a construction for finding a second pair of mates, B and B', such that (AA', BB') is a harmonic tetrad.

8. If $(XYABCD) \sim (XYBCDA)$, show that A, C and B, D are two pairs of mates in an involution whose self-corresponding points are X and Y.

CHAPTER V
THE CONIC

5.1. Introductory

5.11. Notation

It will be found, in dealing with the conic, that it is often necessary to speak of projectivities between a set of concurrent lines, say XA, XB, XC,... and some other set of points or lines. In symbolizing such projectivities it would be legitimate, though cumbrous, to write $X(XA, XB, XC,...) \sim ....$ Similarly it would be cumbrous to say: 'Let a be the line XA, b be the line XB, etc.', and then to write $X(abc...) \sim ....$

To obviate these difficulties, an addition is made to the notation already in use. By the set of symbols $X(ABC...)$ is denoted the set of concurrent lines XA, XB, XC,.... No confusion will arise between this and the set of symbols $x(ABC...)$. It should be noted that the points A, B, C, in $X(ABC...)$ need not be collinear.

With this a modification of the existing notation is made. Hitherto the symbol $\barwedge$ has been used to denote a perspectivity, and the symbol $\sim$ to denote a projectivity that is not a perspectivity. In future the symbol $\sim$ will be used for both, and the fact that a projectivity is a perspectivity will be denoted if necessary by placing a letter (large or small) over the sign $\sim$.

Thus $\overset{O}{\sim}$ will denote a central perspectivity on the point O, and $\overset{o}{\sim}$ will denote an axial perspectivity on the line o.

5.12. A Provisional Initial Proposition of Extension

We have so far been content with the indefinite initial propositions of extension: *Not all points are on the same line,* and *There are at least three points on every line.* To enter into a discussion of the question of extension at this point would not be very fruitful, and it would take us a long way from the line of development that is being followed. At the same time, to attempt to study the conic having only the above indefinite initial propositions of extension would be very laborious, for it would entail the constant enumeration of exceptions to general theorems. Moreover, it would be seen, when the time came for the complete discussion of extension, that most of these exceptions were in reality trivial. Hence it is convenient and useful at this point to lay down a provisional proposition of extension as follows:

5.121. Every projectivity between cobasal ranges has two self-corresponding points which are either distinct or coincident.

EXAMPLES

1. Show that the dual of 5.121 follows from 5.121.
2. Show that 5.121 is verified in the Algebraic Representation.

The reader may be surprised to find that 5.121 is called a proposition of extension, and not a second projective proposition. The full reason for this cannot be given here, but it can be explained at least roughly. Let us suppose first of all that we are dealing with a field in which 5.121 is verified, so that every projectivity between cobasal ranges has two distinct or coincident self-corresponding points. Now suppose that instead of considering the whole field, we consider only a part of it, yet a part in which every one of the initial propositions so far laid down is verified, with the exception of 5.121. It is quite plausible to suppose that in thus cutting off from our consideration a part of the original field, we get rid of the self-corresponding points of some of the projectivities between cobasal ranges in the part that is left. And so it may be seen, in a rough sort of way, that the effect of 5.121 is to ensure that the field is extensive enough to include the double points of every projectivity between cobasal ranges.

5.13. Loci and Envelopes

DEFINITION. *A locus is a point-figure to which a point does or does not belong, according as it does or does not satisfy some given condition.*

DEFINITION. *An envelope is a line-figure to which a line does or does not belong, according as it does or does not satisfy some given condition.*

It should be noticed that in Projective Geometry a locus is not 'the path traced out by a point moving according to some given law'; points do not move.

The repeated alternatives 'does or does not . . . does or does not . . .' in these definitions are important. In proving that a certain point-figure is the locus corresponding to some given condition it is necessary to prove two things: (i) that every point of the figure satisfies the condition, and (ii) that no other point satisfies the condition. This is what is implied by the repeated alternatives. *It is therefore insufficient to prove that every point satisfying the condition is a point of the figure which is asserted to be the locus.* Dual remarks apply to envelopes.

5.2. Definition and Basic Properties of the Conic†

DEFINITION. *A point-conic is the locus of the points which are common to pairs of corresponding lines of two pencils between which there is a projectivity.*

The bases of the two pencils are called the *generating bases.*

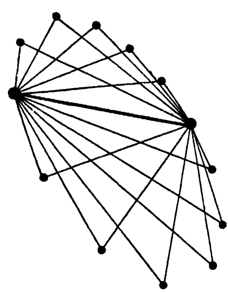


FIG. 24.

DEFINITION. *A line-conic is the envelope of the lines which are on pairs of corresponding points of two ranges between which there is a projectivity.*

The bases of the two ranges are called the *generating bases.*

† The name *conic* is given to these loci and envelopes for historical reasons. The properties of plane sections of the right circular cone were investigated very early in the history of Mathematics, and to these curves was given the name *conic section.* Later, these curves were defined by the familiar focus-and-directrix property ($SP = ePM$) instead of being defined as sections of a cone, and the name was shortened to *conic.* The name is taken over into Projective Geometry because eventually, when certain metrical ideas are introduced, it can be proved that these loci and envelopes can be identified with loci and envelopes having the focus-directrix property.

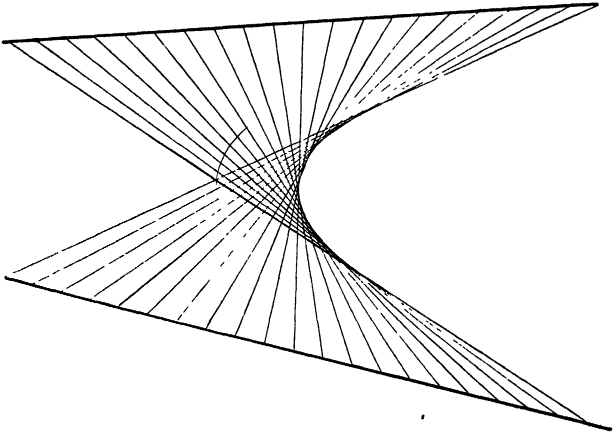


FIG. 25.

5.21. Non-singular and Singular Point- and Line-conics

The definition of the point-conic makes no stipulation about the two pencils by which it is specified; while there may be no particular relation between them beyond the projectivity spoken of, they may, on the other hand, be specially related, as for instance when they are cobasal. It is therefore necessary to classify point-conics, and this is done here.

(i) Point-conics specified by a projectivity between pencils which are not cobasal and whose common line is not self-corresponding. These point-conics are called *non-singular point-conics*. All other point-conics are *singular point-conics*.

(ii) Point-conics specified by a projectivity between pencils which are not cobasal but whose common line is self-corresponding. According to the definition of the point-conic, every point on this common self-corresponding line is a point of the locus. The points common to other pairs of corresponding lines are all collinear by 3.322. *Hence the locus consists of two ranges of points on different bases.*

(iii) Point-conics specified by a projectivity between two cobasal pencils which have two distinct self-corresponding lines. It is clear that every point on each of the two self-corresponding lines is a point of the locus. The only point common to other pairs of corresponding lines is the common base of the two pencils. *Hence the locus consists of two ranges of points on different lines.*

(iv) Point-conics specified by a projectivity between two cobasal pencils which have two coincident self-corresponding lines. *Hence the locus consists of two cobasal ranges, or two coincident ranges of points.*

There is a dual classification of line-conics.

What has been said is not sufficient to show that the non-singular point-conic is not, as a matter of fact, a pair of ranges; that it is not is a simple consequence of a theorem which is about to be proved.

5.22. Elementary Deductions

Before making any deductions from the definitions of the point-conic and the line-conic, an addition is made to the terminology in use.

DEFINITION. *If A be any point of a point-conic, then the point-conic is said to be on A, and A is said to be on the point-conic.*

DEFINITION. *If a be any line of a line-conic, then the line-conic is said to be on a, and a is said to be on the line-conic.*

The following theorems are elementary deductions from the definitions of point-conic and line-conic.

5.221. THEOREM. *Every point-conic is on its two generating bases.*

Let U and V be the two generating bases of a point-conic.

Consider the pencil on U; one of its lines is the line UV.

The line corresponding to this in the pencil on V is some line VX, say. Now V is on both UV and VX, hence V is on the point-conic. Similarly, U is on the point-conic.

5.222. THEOREM. *If U, V, A, B, C are five points, no four of which are collinear, then there is one and only one point-conic which is on A, B, C, and whose generating bases are U and V.*

Since no four of the five points are collinear, of the three pairs of lines UA, VA; UB, VB; UC, VC, at most one pair can coincide, and the rest are distinct.

Consider the projectivity specified by

$$U(ABC...) \sim V(ABC...).$$

Three pairs of corresponding lines being here specified, the projectivity is determined. Hence there is a point-conic satisfying the conditions, and it is unique.

The reader should investigate why it is that the theorem breaks down when four of the five points are collinear; there are two cases to consider.

5.223. THEOREM. *Two distinct points on a non-singular point-conic cannot be collinear with a generating base.*

Let U and V be the generating bases of a non-singular point-conic.

(i) Let A be any other point on the conic. Then it is asserted that A, U, and V are not collinear.

For if they were, the lines UA and VA being corresponding lines in the two pencils, these pencils would have a common self-corresponding line, and so the point-conic would be singular; this contradicts the supposition.

(ii) Let A and B be two points on the point-conic distinct from both U and V. Then it is asserted that U, A, and B are not collinear.

For if they were, the lines VA and VB would both correspond to the same line in the pencil on U, and this is impossible.

The theorem is thus proved.

Incidentally this theorem proves also that a non-singular point-conic does not consist of two ranges of points, and so a non-singular point-conic differs from a singular point-conic.

5.23. The First Basic Theorem

At first sight it would appear that the two generating bases, which, by 5.221, are on the point-conic, are points on the point-conic which have some special properties distinguishing them from the rest. The next theorem shows that this is not so.

THEOREM. *Any two distinct points on a point-conic can be the generating bases.*

Let U and V be the generating bases of a non-singular point-conic, and let A, B, C, D,... be any other distinct points on it. It will be shown that A and B can be taken as the generating bases.

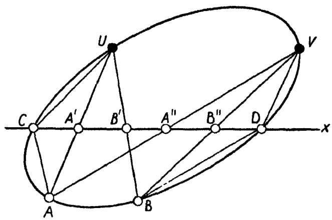


FIG. 26.

Let x be the line CD, so that by 5.223 neither U nor V is on x.

Let A', B', A'', B'' be the points $\begin{pmatrix} CD \\ AU \end{pmatrix}$, $\begin{pmatrix} CD \\ BU \end{pmatrix}$, $\begin{pmatrix} CD \\ AV \end{pmatrix}$, $\begin{pmatrix} CD \\ BV \end{pmatrix}$ respectively, so that by 5.223 A', B', C, D, A'', B'' are all distinct.

Then $\qquad U(ABCD) \sim x(A'B'CD)$,

and $\qquad V(ABCD) \sim x(A''B''CD)$.

But by the permutation theorem

$$x(A''B''CD) \sim x(B''A''DC).$$

Hence, since by the definition of a point-conic

$$U(ABCD) \sim V(ABCD),$$

it follows that $\qquad x(A'B'CD) \sim x(B''A''DC)$.

Now this is a projectivity between two cobasal ranges in which there is a pair of reciprocally corresponding points, and so by 4.43 it is an involution in which A' and B'', B' and A'', C and D are pairs of mates.

Hence $\qquad x(A'A''CD) \sim x(B''B'DC)$.

But $\qquad x(A'A''CD) \sim A(UVCD)$,

and $\qquad x(B''B'DC) \sim x(B'B''CD)$

$\qquad\qquad \sim B(UVCD)$.

Hence $\qquad A(UVCD) \sim B(UVCD)$.

Similarly, $\qquad A(UVCE) \sim B(UVCE)$,

and so $\qquad A(UVCDE...) \sim B(UVCDE...)$.

Hence A and B can be taken as the generating bases of the point-conic.

In this proof it has been assumed that the point-conic is non-singular; the theorem remains true when it is singular, but the proof is then extremely easy, and it is left to the reader to prove it for himself.

5.24. The Second Basic Theorem

The second basic theorem about the point-conic is an immediate consequence of the first, and the proof can be omitted. It is enunciated because it is constantly being used, while the first basic theorem is, by comparison, seldom used.

THEOREM. *The necessary and sufficient condition that six points A, B, C, D, E, F should all be on a point-conic is that*

$$A(CDEF) \sim B(CDEF)$$

The sufficiency of the condition is proved by the method of *reductio ad absurdum.*

5.25. Other Deductions from the First Basic Theorem

5.251. THEOREM. *On five points, no four of which are collinear, there is one and only one point-conic.*

This is an immediate deduction from 5.222 and the first basic theorem.

5.252. THEOREM. *No three points of a non-singular point-conic are collinear.*

This is an immediate deduction from 5.223 and the first basic theorem.

EXAMPLES

1. A, B, C, D is a four-point, and X, Y, Z, W are any four points on a line x. Find the locus of a point P such that $P(ABCD) \sim x(XYZW)$.

2. Prove the dual of the first basic theorem without appeal to the Principle of Duality, and draw an appropriate figure.

3. Show that if every point-conic which is on four points A, B, C, D is singular, then at least three of these points are collinear.

4. *abcd* is a simple four-line. How many singular line-conics are there on these four lines? Dualize.

5. Ranges $x(X_1X_2X_3...)$ and $y(Y_1Y_2Y_3...)$ are centrally perspective on a point Z. X' is any point on x, Y' any point on y; P_n is the point $\begin{pmatrix} X'Y_n \\ Y'X_n \end{pmatrix}$. Show that the locus of P_n is a point-conic. Under what circum-

stances is this point-conic singular? (This is known as *Maclaurin's construction* for the point-conic.)

Give a dual construction for a line-conic.

6. Show that if there are precisely n points on every line of the field, then

(i) there are precisely n points on every non-singular point-conic,
(ii) there are precisely n lines on every non-singular line-conic,
(iii) there are precisely n point-conics (singular and non-singular) on every simple four-point,
(iv) there are precisely n line-conics (singular and non-singular) on every simple four-line.

7. A certain figure is asserted to be the locus corresponding to a given condition. Show that it is necessary and sufficient to prove that (i) every point of the figure is a point of the locus, and (ii) every point of the locus is a point of the figure.

5.26. The Point- and Line-conic in the Algebraic Representation

Let $[l_1, m_1, n_1]$, $[l'_1, m'_1, n'_1]$ be two lines on a point U, so that any other line on U is $[\lambda_1 l_1 + \lambda'_1 l'_1,\ \lambda_1 m_1 + \lambda'_1 m'_1,\ \lambda_1 n_1 + \lambda'_1 n'_1]$ and its coordinates are (λ_1, λ'_1) relative to these base lines.

Similarly, let (λ_2, λ'_2) be the coordinates of a line on V, relative to the base lines $[l_2, m_2, n_2]$ and $[l'_2, m'_2, n'_2]$.

Let (x, y, z) be the common point of the two lines (λ_1, λ'_1) and (λ_2, λ'_2), so that

$$\lambda_1(l_1 x + m_1 y + n_1 z) + \lambda'_1(l'_1 x + m'_1 y + n'_1 z) = 0 \tag{1}$$

and

$$\lambda_2(l_2 x + m_2 y + n_2 z) + \lambda'_2(l'_2 x + m'_2 y + n'_2 z) = 0. \tag{2}$$

If now the lines (λ_1, λ'_1) on U and (λ_2, λ'_2) on V are a pair of corresponding lines in a projectivity between the two pencils, there are numbers a, b, c, and d such that $ad - bc \neq 0$ and

$$a\lambda_1\lambda_2 + b\lambda_1\lambda'_2 + c\lambda'_1\lambda_2 + d\lambda'_1\lambda'_2 = 0. \tag{3}$$

When λ_1, λ'_1, λ_2, and λ'_2 are eliminated from the equations (1), (2), and (3), the equation

$$\begin{aligned} a(l'_1 x + m'_1 y + n'_1 z)(l'_2 x + m'_2 y + n'_2 z) &+ \\ + b(l'_1 x + m'_1 y + n'_1 z)(l_2 x + m_2 y + n_2 z) &+ \\ + c(l_1 x + m_1 y + n_1 z)(l'_2 x + m'_2 y + n'_2 z) &+ \\ + d(l_1 x + m_1 y + n_1 z)(l_2 x + m_2 y + n_2 z) &= 0 \end{aligned} \tag{4}$$

is left.

This equation may be written in the form

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0. \tag{5}$$

Now from the definition of a point-conic (x, y, z) is a point on a point-conic. Hence every point (x, y, z) on a point-conic is such that

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$$

where the coefficients A, B, etc., are determined by the projectivity which specifies the point-conic.

Suppose now that (x, y, z) is any point satisfying (5). Then clearly (x, y, z) satisfies (4). That is to say, there are numbers λ_1, λ'_1, λ_2, and λ'_2

which satisfy (1), (2), and (3). Hence (x, y, z) is the common point of two corresponding lines in the projectivity between the pencils on U and V.

We may therefore say that in the Algebraic Representation all the points on a point-conic satisfy an equation similar to (5), and that all the points which satisfy this equation are points of the point-conic.

The complementary theorem, that *every* equation of the form given is satisfied by the points of some point-conic is more complicated, and it is not proved here.

Dually, there is the theorem that in the Algebraic Representation all the lines of a line-conic satisfy an equation of the form

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0.$$

5.3. The Incidence of Lines and Point-conics, and Dual

5.31. Fundamental Theorem of Incidence

In 5.242 it was proved that no three points of a non-singular point-conic were collinear; this theorem is only a part of a more general theorem which is now proved.

THEOREM. *Every line of the field has two and only two points in common with a non-singular point-conic.*

Let x be any line of the field, and let U and V be any two distinct points of the point-conic not on x.

Let $U(u_1 u_2 u_3 \ldots) \sim V(v_1 v_2 v_3 \ldots)$ be the projectivity between the two pencils which specifies the point-conic, so that $u_1 v_1$, $u_2 v_2$, $u_3 v_3, \ldots$ are points on the point-conic.

Let X_1, X_2, $X_3, \ldots$ be the points xu_1, xu_2, $xu_3, \ldots$, and let X_1', X_2', $X_3', \ldots$ be the points xv_1, xv_2, $xv_3, \ldots$. Then

$$x(X_1 X_2 X_3 \ldots) \sim x(X_1' X_2' X_3' \ldots),$$

and since this is a projectivity between cobasal ranges, it has, by 5.121, two self-corresponding points. Let these be X_n and X_m.

Then plainly the point $u_n v_n$ is X_n, and $u_m v_m$ is X_m.

Hence there are two points (which may, however, coincide) on x which are also on the point-conic.

Clearly there cannot be more than two, by 5.242.

The theorem just proved deals only with non-singular point-conics; the corresponding theorem for the singular cases runs:

THEOREM. *Every line of the field, with certainly one exception, and possibly two, has two and only two points in common with a singular point-conic. The exceptional lines have all their points in common with the singular point-conic.*

5.32. Tangents to a Point-conic

In the proof of the last theorem it was stated in passing that the two points which a line has in common with a non-singular point-conic might be coincident. Before discussing the implications of this coincidence, it is important to prove that there are such lines.

5.321. THEOREM. *On every point of a non-singular point-conic there is one and only one line which is on two coincident points of the point-conic.*

Let A and B be two points of a non-singular point-conic; then there is a projectivity between the two pencils on A and B, the common points of corresponding lines being the points of the point-conic.

Consider the line AB of the pencil on A.

To this corresponds some line on B, BX say.

Suppose now that D is a point on BX distinct from B, and on the point-conic. Then, by the definition of a point-conic, AD is the line on A corresponding to BD, i.e. to BX.

But, by supposition, AB corresponds to BX; hence there is no other point than B on BX which is a point of the point-conic. This shows that there is a line on B which satisfies the conditions of the theorem.

It remains to show that BX is the only line on B which satisfies the conditions.

Suppose then that there is another, BY say. Then both of the lines BX, BY correspond, in the projectivity, to the line AB of the pencil on A. This being impossible, the second part of the theorem is also proved.

DEFINITION. *Any line which is on two coincident points of a non-singular point-conic is termed a tangent to the point-conic; the point common to a tangent and a point-conic is termed the point of contact of the tangent.*

An immediate consequence of the last theorem is the following.

5.322. THEOREM. *If A, B, C, D, and P be five distinct points on a non-singular point-conic, and if AA', BB', CC', DD' be the* *tangents to this point-conic on A, B, C, and D respectively, then*

$$P(ABCD) \sim A(A'BCD) \sim B(AB'CD) \sim C(ABC'D) \sim D(ABCD').$$

In order to enunciate the last theorem it was necessary to name another point on each tangent; but, to save this trouble, in future we shall write

$$P(ABCD) \sim A(ABCD) \sim B(ABCD) \sim C(ABCD) \sim D(ABCD),$$

provided there is no danger of ambiguity. The lines AA, BB, CC, DD denote the tangents to the point-conic at A, B, C, and D respectively.

5.33. Duals of the Preceding Theorems

The results of the last paragraphs are important enough to merit the explicit statement of the dual results.

5.331. THEOREM. *Every point of the field is on two and only two lines of a non-singular line-conic.*

5.332. THEOREM. *Every point of the field, with certainly one exception, and possibly two, is on two and only two lines of a singular line-conic. All the lines on the exceptional points are lines of the line-conic.*

5.333. THEOREM. *On every line of a non-singular line-conic there is one and only one point which is on two coincident lines of the line-conic.*

DEFINITION. *Any point which is on two coincident lines of a non-singular line-conic is termed a tangent-point to the line-conic; the line of the line-conic which is on the tangent-point is termed the line of contact of the tangent-point.*

5.334. THEOREM. *If a, b, c, d, and p be five distinct lines on a non-singular line-conic, then*

$$p(abcd) \sim a(abcd) \sim b(abcd) \sim c(abcd) \sim d(abcd).$$

In the enunciation of this theorem, aa, bb, etc., denote the tangent-points to the line-conic on a, b, etc.

5.4. Desargues's (Conic) Theorem, and Pascal's Theorem

In this section of the chapter are proved two extremely important theorems about the conic. They are named after their discoverers, who, however, employed metrical ideas in their proofs.

5.41. Desargues's (Conic) Theorem

THEOREM. *If $A_0A_1A_2A_3$ be a four-point, and x any line distinct from the six lines of this four-point, then every point-conic on $A_0A_1A_2A_3$ is also on a pair of mates of an involution on x.*

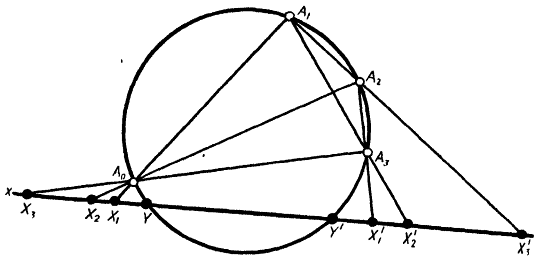


FIG. 27.

With the usual convention, let X_1, X_2, X_3, X_1', X_2', X_3' be the points common to the six sides of the four-point and the line x.

Let Y and Y' be the two points common to x and any non-singular point-conic on the four points of the four-point.

Then $\quad A_0(A_1A_3YY') \sim x(X_1X_3YY')$;

also $\quad A_2(A_1A_3YY') \sim x(X_3'X_1'YY')$

$$\sim x(X_1'X_3'Y'Y),$$

by the permutation theorem.

Hence, since $A_0(A_1A_3YY') \sim A_2(A_1A_3YY')$,

$$x(X_1X_3YY') \sim x(X_1'X_3'Y'Y).$$

By 4.43 this projectivity is an involution in which X_1 and X_1', X_3 and X_3', Y and Y' are three pairs of mates.

Hence Y and Y' are mates in an involution of which X_1 and X_1', X_3 and X_3' are two pairs of mates. Similarly for any other point-conic on the four-point.

It may be noticed that X_1 and X_1' and the other two pairs of the involutory hexad are themselves pairs of points common to x and a point-conic on the four-point. The point-conics in question are the three singular point-conics on the four-point $A_0A_1A_2A_3$.

It will be recognized that this theorem was virtually proved in 5.23.

5.42. Singular Cases of Desargues's Theorem

There are two other theorems, similar to Desargues's (conic) theorem. These can be proved by considering, not a simple four-point on a point-conic, but singular four-points, that is, four-points in which two or more of the points coincide. They are, in a sense, singular cases of Desargues's theorem.

5.421. Theorem. *If $A_1A_2A_3$ be a three-point, t any line on A_1, and x any other line not on any of the points of the three-point, then any point-conic on the three-point, such that t is a tangent to it, is also on a pair of mates of an involution on x.*

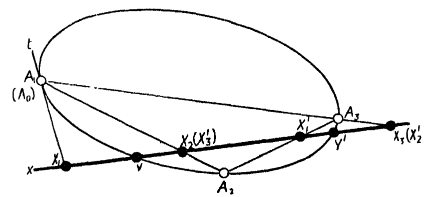


Fig. 28.

This is proved in exactly the same way as Desargues's theorem, by considering the singular four-point $(A_0)A_1A_2A_3$, in which A_0 and A_1 coincide, but the line A_0A_1 is fixed as t.

The figure shows the assignment of the various letters.

5.422. Theorem. *If A_1A_2 be a point-pair, and t_1 and t_2 be two lines on A_1 and A_2 respectively, each being distinct from*

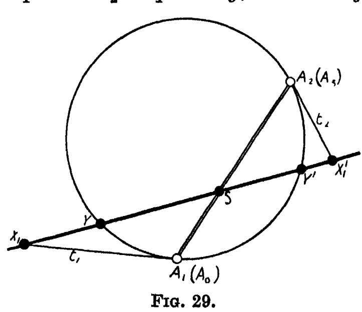


Fig. 29.

A_1A_2, and if x be any other line, then any point-conic on A_1 and A_2, such that t_1 and t_2 are tangents to it, is also on a pair of mates of an involution on x, and one of the self-corresponding points of this involution is the common point of x and A_1A_2.

The method of assigning the letters in the figure will make the method of proof clear. S is a self-corresponding point of the involution.

Examples

1. x is a line on which is an involution, and P, Q, R are three distinct points not on x, and not collinear. Show that all point-conics on P, Q, R and a pair of mates of the involution are also on a certain other point S.

2. P, Q, R, S are the points of a simple four-point, and t is a line which is not on any of them. Show that there are two and only two point-conics on these four points to which t is a tangent. What significance has this theorem when t is on one or more of the diagonal points of the four-point $PQRS$?

3. ABC is a triangle, x a line distinct from its sides, and O a point not on any of these four lines. A', B', C' are the common points of x and BC, CA, AB respectively. A'' is the second of the two points common to x and the point-conic on A, B, C, O, A'. B'', C'' are similarly defined. Show that AA'', BB'', CC'' are concurrent.

4. A, B, C, D, E are five points no three of which are collinear, and x is a line on A. Use Desargues's theorem to find the other point on x which is on the point-conic on the five points.

5. $ABCD$ and $A'B'C'D'$ are two simple four-points, and x is any line not on any of these eight points. Show that there is, in general, one and only one point-pair X, X' on x such that the six points $ABCDXX'$ are all points of a point-conic, and the points $A'B'C'D'XX'$ are all points of another point-conic.

6. a, b, c are three non-concurrent lines on the points A, B, C respectively. a and b are tangents to a point-conic on A, B, C. Find a second point on c which is also on the point-conic.

7. D and E are two diagonal points of a complete four-point $WXYZ$. Show that any point-conic on this four-point is also on a pair of points on DE which are harmonic conjugates relative to D and E.

8. The usual notation for the points associated with a complete four-point being supposed, show that the six points $A_0A_1H_2H_3H_2'H_3'$ are all on a non-singular point-conic, and that $A_2A_3H_2H_3H_2'H_3'$ are all on a non-singular point-conic.

5.43. Pascal's Theorem

THEOREM. *If A, B, C, A', B', C' be six points on a non-singular point-conic, and if A'', B'', C'' be the points*

$\begin{pmatrix} BC' \\ B'C \end{pmatrix}$, $\begin{pmatrix} CA' \\ C'A \end{pmatrix}$, $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$ *respectively, then* A'', B'' *and* C'' *are collinear.*

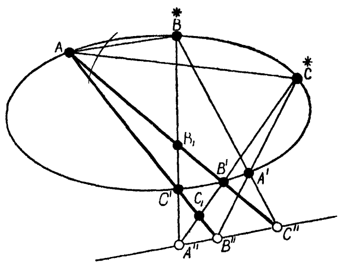


FIG. 30.

Let B_1 be the point $\begin{pmatrix} BC' \\ AB' \end{pmatrix}$, and C_1 the point $\begin{pmatrix} C'A \\ B'C \end{pmatrix}$.

Then $\qquad B(A'B'C'A) \sim (C''B'B_1A),$

and $\qquad C(A'B'C'A) \sim (B''C_1C'A).$

But $\qquad B(A'B'C'A) \sim C(A'B'C'A),$

hence $\qquad (C''B'B_1A) \sim (B''C_1C'A).$

This being a projectivity between ranges on different bases, and there being a common self-corresponding point, it must be a perspectivity.

Hence $B''C''$, $B'C_1$, B_1C' must be concurrent. But the common point of the last two of these three lines is A'', hence A'', B'', and C'' are collinear.

If this theorem and its proof be compared with Pappus's theorem (3.323), the similarity can hardly escape notice. Indeed, Pappus's theorem is only a particular case of Pascal's theorem.

Pascal's theorem is sometimes enunciated thus: *If a hexagon be inscribed in a (point-)conic, the intersections of pairs of opposite sides are collinear.* If the obvious meanings be ascribed to the terms *hexagon, inscribe, intersect,* and if the hexagon considered be $AB'CA'BC'$, it will be seen that this enunciation is equivalent to that given. For the opposite sides of this hexagon are AB' and $A'B$, $B'C$ and BC', CA' and $C'A$.

5.431. Converse of Pascal's Theorem

THEOREM. *If A, B, C, A', B', C' be six points, no three of which are collinear, and if A'', B'', C'' be defined as in Pascal's theorem, and be collinear, then A, B, C, A', B', C' are all on a non-singular point-conic.*

This theorem is proved by the method of *reductio ad absurdum*. With this indication, the reader should find no difficulty in proving it.

5.432. Dual of Pascal's Theorem and its Converse

THEOREM. *If a, b, c, a', b', c' be six lines on a non-singular line-conic, and if a'', b'', c'' be the lines $\binom{bc'}{b'c}$, $\binom{ca'}{c'a}$, $\binom{ab'}{a'b}$ respectively, then a'', b'', c'' are concurrent.*

It will be found useful to draw a figure appropriate to this theorem.

THEOREM. *If a, b, c, a', b', c' be six lines, no three of which are concurrent, and a'', b'', c'' be defined as in the dual of Pascal's theorem, and be concurrent, then a, b, c, a', b', c' are all on a line-conic.*

5.433. Utility of Pascal's Theorem and its Converse

It will be found that the converse of Pascal's theorem is often the simplest method of proving that six points are all on a point-conic, although there are always two other methods of doing this. The dual theorem is useful for the dual purpose.

The converse of Pascal's theorem is also a very convenient method of determining other points on a point-conic, five of whose points are known. This construction is important enough to merit a formal enunciation and proof.

CONSTRUCTION. *A, B, C, D, E are five points, no three of which are collinear; EF is any line on E, but not on any of the other four points. Determine the other point on EF which is on the point-conic on $ABCDE$.*

Let L be the point $\binom{AB}{DE}$, and M the point $\binom{BC}{EF}$.

Let N be the point $\binom{CD}{LM}$, and X the point $\binom{AN}{EF}$.

Then X is the required point.

For L, M, N are, respectively, the points $\begin{pmatrix} AB \\ DE \end{pmatrix}$, $\begin{pmatrix} BC \\ EX \end{pmatrix}$, $\begin{pmatrix} CD \\ XA \end{pmatrix}$,

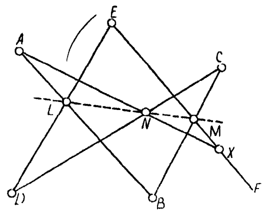


FIG. 31.

and these points are collinear by construction, hence $ABCDEX$ are all on a point-conic.

5.434. Singular Cases of Pascal's Theorem. Just as with Desargues's theorem so with Pascal's theorem there are singular cases. These arise when there are not six points but five or less on the point-conic, and these are counted as six by considering one, or more, as pairs of coincident points. The line joining coincident points will then be the tangent to the point-conic at the point in question.

In distinguishing the various possible cases that can arise it is useful to consider the six points in the cyclic ordering $AB'CA'BC'A$. Two or more points will be *consecutive* if they are consecutive in this ordering. It can easily be verified that the following assertions are true.

(i) If two non-consecutive points are coincident, Pascal's theorem is true, but trivial.

(ii) If more than two consecutive points are coincident, the theorem is again true, but trivial.

(iii) It is true and *not* trivial when one or more pairs of consecutive points are coincident, so long as these pairs have not a common point. (If they had, more than two consecutive points would be coincident, and so, by (ii), the theorem would become trivial.)

This leaves four non-trivial theorems:

(*a*) When e.g. A and B' are coincident.

(*b*) When e.g. A and B' are coincident and C and A' are coincident.

(*c*) When e.g. A and B' are coincident and A' and B are coincident.

(*d*) When e.g. A and B' are coincident, C and A' are coincident, B and C' are coincident.

All these non-trivial theorems are proved just as Pascal's theorem itself is proved, and it is a useful exercise to work out a complete proof. The last of the four types of theorem can be stated in other terms thus:

5.435. THEOREM. *If the sides BC, CA, AB of a triangle are tangents to a non-singular point-conic, and their points of contact are A', B', C' respectively, then the two triangles ABC, $A'B'C'$ are perspective.*

EXAMPLES

1. X, Y, Z are three collinear points, and P and Q are two other points not collinear with them. Show that the other six intersections of the six lines PX, PY, PZ, QX, QY, QZ are six points of a point-conic.

2. $A_0A_1A_2A_3$ is a four-point which is on a point-conic. Show that the common point of tangents, whose points of contact are two of the points $A_0A_1A_2A_3$, is collinear with two of the diagonal points of the four-point.

3. $A_0A_1A_2A_3$ is a four-point which is on a point-conic, and $a_0a_1a_2a_3$ is the four-line composed of the four tangents to the point-conic at A_0, A_1, A_2, and A_3 respectively. Show that the four points D_2, D_3, a_0a_1, and a_2a_3 are collinear. Dualize.

4. In the last example show that the six points A_0, A_1, A_2, A_3, a_0a_1, a_2a_3 are all on a point-conic. Hence show that a_0a_1 and a_2a_3 are harmonic conjugates relative to D_2 and D_3.

5. Use Pascal's theorem to show that three tangents to a non-singular point-conic cannot be concurrent. Dualize.

6. With the usual notation, $X_1X_2X_3X_1'X_2'X_3'$ is an involutory hexad associated with a four-point, and $Y_1Y_2Y_3Y_1'Y_2'Y_3'$ are, respectively, the harmonic conjugates of these points relative to the two points of the four-point with which each is collinear. Show that $Y_1Y_2Y_3Y_1'Y_2'Y_3'$ are all on a point-conic. Under what circumstances is this point-conic singular?

7. Use Pascal's theorem to prove Ex. 8 of the last set of examples.

8. ABC and $A'B'C'$ are two perspective triangles. Show that the six points $\binom{BC}{C'A'}$, $\binom{BC}{A'B'}$, $\binom{CA}{A'B'}$, $\binom{CA}{B'C'}$, $\binom{AB}{B'C'}$, $\binom{AB}{C'A'}$ are all on a non-singular point-conic. Dualize.

9. Given two tangents to a point-conic with their points of contact and one other point on the point-conic, find the second point of the point-conic which is on a line on the given point.

10. Given four points A, B, C, D of a point-conic, and d the tangent to the point-conic at D, determine (i) the second point of the point-conic which is on any line on A, (ii) the second point of the point-conic which is on any line on D.

11. A is a point, and x a line not on it. $P_1 P_2 P_3 P_4 P_5 P_6$ are six distinct points on a point-conic which is non-singular. The common points of x and the six lines AP_1, AP_2,..., AP_6 are $Q_1 Q_2 Q_3 \dots Q_6$. $R_1 R_2 \dots R_6$ are six points, one on each of these lines, such that

$$(AP_1 Q_1 R_1) \sim (AP_2 Q_2 R_2) \sim (AP_3 Q_3 R_3) \sim \dots \sim (AP_6 Q_6 R_6).$$

Show that the six points $R_1 R_2 \dots R_6$ are all on a non-singular point-conic.

5.5. Pole and Polar

5.51. THEOREM. *The locus of the harmonic conjugates of a point, relative to pairs of points of a non-singular point-conic which are collinear with it, is a range of points.*

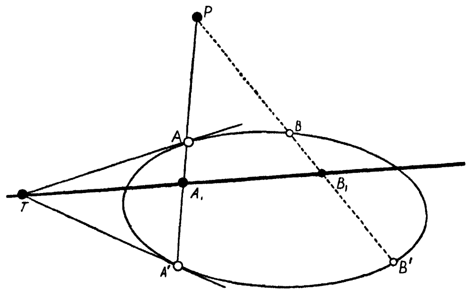


FIG. 32.

Let A and A' be a pair of points on a non-singular point-conic and collinear with P; and let A_1 be the harmonic conjugate of P relative to AA'.

Let T be the common point of the tangents at A and A'. Similarly, let B and B' be another pair of points on the point-conic, collinear with P; and let B_1 be the harmonic conjugate of P relative to BB'.

Let X and Y be the points $\begin{pmatrix} AB \\ A'B' \end{pmatrix}$, $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$ respectively, so that X, Y, and P are the diagonal points of the four-point $AA'BB'$.

Then, as a consequence of a singular case of Pascal's theorem (type (c)), the three points $\binom{A'A'}{AA}$, $\binom{BA}{B'A'}$, $\binom{AB'}{A'B}$ are collinear. That is, T, X, Y are collinear.

But X, Y, A_1, B_1 are collinear, hence B_1 is collinear with A and A_1.

Similarly, if CC' be any other pair, then the appropriate point C_1 is collinear with T and A.

Hence all points of the locus are on TA_1.

Suppose now that L_1 is any point of TA_1, and that LL' is the pair of points on the point-conic collinear with P and L_1.

Then by what has just been proved (PL_1, LL') is a harmonic tetrad. Hence every point of the range on TA is a point of the locus, and vice versa.

The proof just given has tacitly supposed that P is not on the point-conic. If P is on the point-conic, it is easily verified that the locus is the tangent at P. (See 4.222.)

5.511. Definition. *If p be the base of the range which is the locus in the preceding theorem, then p is said to be the polar of P, and P the pole of p, relative to the point-conic in question.*

Notice that pole and polar are *not* dual terms.

5.512. Theorem. *The envelope of the harmonic conjugates of a line relative to pairs of lines of a non-singular line-conic which are concurrent with it is a pencil of lines.*

5.513. Definition. *If P be the base of the pencil which is the envelope in the preceding theorem, then P is said to be the pole of p, and p the polar of P relative to the line-conic in question.*

Here again polar and pole are not dual terms; but 'pole and polar relative to a point-conic' is the dual of 'polar and pole relative to a line-conic'. The apparent ambiguity will shortly be removed.

5.52. Elementary Properties of Pole and Polar

5.521. Theorem. *If P and p be pole and polar relative to a non-singular point-conic, and if A and B be the two points common to p and the point-conic, then the tangents at A and B are both on P.*

Suppose that AP is not a tangent to the point-conic. Let A' be the other point of the point-conic on AP, and let A'' be such that $(AA', A''P)$ is a harmonic tetrad.

Then A'' is on p. That is p and AP are the same line.

This is absurd except when P is on the point-conic. Hence when P is not on the point-conic the theorem is true.

And when P is on the point-conic the theorem is plainly true.

As direct corollaries of this theorem, the following may be enunciated:

5.522. Theorem. *If P and P' are distinct points, their polars relative to any non-singular point-conic are distinct.*

5.523. Theorem. *Two and only two tangents to a point-conic are on any point which is not on the point-conic.*

5.53. Projective Properties

In this section certain deeper properties of pole and polar are investigated. These investigations give an answer to the general question: 'Given a set of points which has certain projective properties, what projective properties has the set of their polars relative to any non-singular point-conic?' It is to be expected that the polars will have dual properties; this is, as a matter of fact, the answer. Before coming to the investigation proper two preliminary theorems are needed.

5.531. Theorem. *If the polar of P relative to any non-singular point-conic is on Q, then the polar of Q relative to the same point-conic is on P.*

Let R and S be the two points of the point-conic which are collinear with P and Q.

Then because Q is on the polar of P, (RS, PQ) is a harmonic tetrad. But this shows that P is on the polar of Q.

5.532. Theorem. *If P, Q, R, S,... be collinear points, then their polars relative to any non-singular point-conic are concurrent lines.*

Let p, q, r, s,... be the polars, and let X be the point pq.

Then by the last theorem the polar of X is on both P and Q.

Hence it is also on R, S,....

By the last theorem the polars of R, S,... are therefore on X. That is to say, p, q, r, s,... are all concurrent.

5.533. THEOREM. *If P, Q, R, S are four collinear points on the line x, and p, q, r, s are the four concurrent lines on the point X, which are their polars respectively relative to a non-singular point-conic, then*

$$x(PQRS) \sim X(pqrs).$$

Let A and B be the points of the point-conic which are on x.

Let P', Q', R', S' be the harmonic conjugates of P, Q, R, S respectively, relative to AB.

Then XP', XQ', XR', XS' are the polars of P, Q, R, S respectively, i.e. they are the lines p, q, r, s.

Hence $$X(pqrs) \sim x(P'Q'R'S').$$

But P and P', Q and Q', R and R', S and S' are mates in an involution, of which the double points are A and B; so that

$$x(PQRS) \sim x(P'Q'R'S').$$

Hence $$x(PQRS) \sim X(pqrs).$$

The reader should verify that this proof remains valid when one, or two, of the points $PQRS$ are on the point-conic.

The following important theorems are direct consequences of the preceding theorems. In order to avoid prolix enunciations the following convention is used: by A, B, C, D,... p, q, r, s,... are denoted the poles and polars of a, b, c, d,... P, Q, R, S,... relative to a definite non-singular point-conic. The truth of these theorems should be verified.

5.534. THEOREM. *If*

$$x(X_1X_2X_3...) \,\mathcal{Q}\, y(Y_1Y_2Y_3...),$$

then $$X(x_1x_2x_3...) \,\mathcal{Q}\, Y(y_1y_2y_3...).$$

5.535. THEOREM. *If P_1, P_2, P_3,... be collinear points on the respective lines of the pencil $X(x_1x_2x_3...)$, then p_1, p_2, p_3,... are concurrent lines on the respective points of the range $x(X_1X_2X_3...)$.*

5.536. THEOREM. *If*

$$x(X_1X_2X_3...) \sim y(Y_1Y_2Y_3...) \sim Z(z_1z_2z_3...) \sim W(w_1w_2w_3...),$$

then

$$X(x_1x_2x_3...) \sim Y(y_1y_2y_3...) \sim z(Z_1Z_2Z_3...) \sim w(W_1W_2W_3...).$$

5.537. THEOREM. *If P, Q, R, S, T, U are six points on a point-conic, then p, q, r, s, t, u are six lines on a line-conic. The line-conic is or is not singular according as the point-conic is or is not singular.*

If x is the tangent to the point-conic on any one of these points, then X is the tangent-point to the line-conic on the corresponding line.

This theorem can be proved in two ways at least. The first way is to make use of the definition of the point-conic, in virtue of which

$$P(RSTU) \sim Q(RSTU),$$

so that

$$p(rstu) \sim q(rstu).$$

The second way is to use Pascal's theorem and the converse of its dual.

5.54. Equivalence of the Point-conic and Line-conic

5.541. THEOREM. *The set of all tangents to a non-singular point-conic is a non-singular line-conic, and the set of all tangent-points to a non-singular line-conic is a non-singular point-conic.*

The two parts of the theorem are plainly dual, and only the first is proved here.

Let P, Q, R, S, T, U be any six points on a non-singular point-conic.

By 5.537 their polars relative to any non-singular point-conic are all lines on a line-conic.

In particular, therefore, their polars *relative to the point-conic which they are on* are lines on a line-conic.

But these polars are the tangents to the point-conic in question, hence any six tangents to a point-conic are lines on a line-conic, and this line-conic cannot be singular.

Hence all the tangents to a non-singular point-conic are lines on a non-singular line-conic, and there can be no lines on this line-conic which are not tangents to the point-conic.

This important theorem, whose existence the reader has probably suspected, leads to the definition of the term *conic*, as distinct from point-conic and line-conic.

5.542. DEFINITION. *The mixed self-dual figure consisting of a non-singular point-conic and the non-singular line-conic which is the set of tangents to it is known as a conic (non-singular).*

5.543. Classification of Conics. It is now no longer necessary to distinguish carefully between point-conics and line-conics, and even the singular conics will not be distinguished in this way in future. It is worth while at this stage to make a list of the various types of conics.

Type I. *The non-singular conic.* A mixed, self-dual figure.

Type II. *Singular conics of the first class.*

(*a*) Two ranges of points on different bases. Point-conic. Not self-dual.

(*b*) Two pencils of lines on different bases. Line-conic. Not self-dual.

Type III. *Singular conics of the second class.*

(*a*) Two ranges of points on the same base. Point-conic. Not self-dual.

(*b*) Two pencils of points on the same base. Line-conic. Not self-dual.

This classification is important, for it is the only division which is inherent in the nature of conics. Later it will be possible to classify conics of Type I into sub-classes, but these classifications are all relative to some arbitrarily chosen standard external to the conic itself.†

5.544. Duality of Pole and Polar. There remains one theorem to be proved. If P is any point, and its polar relative to the *points* of a conic is p, is its polar relative to the *lines* of the conic also p? It is not at once obvious that it is, nor has it yet been proved.

Theorem. *If P be any point, and p be its polar relative to the points of a conic, then p is also its polar relative to the lines of the conic.*

The theorem is obviously true if P be on the conic. Suppose then that it is not on the conic. Let x and y be the two tangents to the point-conic of the conic which, by 5.523, are on the point P; let X and Y be their points of contact. Then the polar of P relative to the point-conic is XY.

† Thus we could, even now, classify conics of Type I into (*a*) those which are on an arbitrarily chosen line x, and (*b*) those which are not. But this is a classification *relative to* x.

Consider now the line-conic alone. x and y are two lines of this, and X and Y are the tangent-points on them. The polar of X relative to the line-conic is x, and y is the polar of Y.

Hence the pole of XY relative to the line-conic is xy, i.e. P; and this is the same as saying that XY is the polar of P relative to the line-conic.

This proves the theorem, and removes the slight ambiguity which so far has been involved in the use of the terms pole and polar.

5.545. Conjugate Points and Lines. DEFINITIONS. *If the polar of P relative to a conic is on Q, then P and Q are termed conjugate points relative to the conic.*

If the pole of p relative to a conic is on q, then p and q are termed conjugate lines relative to the conic.

This new term is frequently useful in dealing with conics.

EXAMPLES

1. $ABCDE$ are five points on a non-singular conic. Show that the necessary and sufficient condition that AB and CD should be conjugate lines relative to the conic is that $E(AB, CD)$ should be a harmonic line-tetrad.

2. P is a point not on a certain conic. Show that pairs of lines on P which are conjugate relative to the conic are mates in an involution. What are the self-corresponding lines of this involution?

3. P is a point not on a certain conic. The two tangents to the conic which are on P have X and Y as their points of contact. Show that XY is the polar of P. Why was it impossible to adopt this as a definition of the term *polar* instead of 5.511?

4. P is a point not on a certain conic. A_1A_1', A_2A_2', A_3A_3',... are pairs of points on the conic collinear with P. T_1 is the common point of the tangents at A_1 and A_1'; T_2, T_3,... are defined similarly. Show that the locus of the points T is the polar of P.

What objection was there to adopting this as a definition of the term *polar*, instead of 5.511?

5. X, Y, C are three distinct points on a conic, and Z is the pole of XY relative to this conic. ZT is any line on Z. Show that $\begin{pmatrix}CX\\ZT\end{pmatrix}$, $\begin{pmatrix}CY\\ZT\end{pmatrix}$ are conjugate points relative to the conic.

6. X, Y, Z, W are four distinct points on a conic; A and B are the poles of XY and ZW relative to this conic. Show that

$$A(XYZW) \sim B(XYZW).$$

Hence show that $ABXYZW$ are six points on a second conic.

(This theorem has already been proved in another way. See 5.435, Ex. 4.)

7. A_0, A_1, A_2, A_3 are four distinct points on a conic. If D_1, D_2, D_3 be the diagonal points of the four-point $A_0A_1A_2A_3$, show that D_1 is the pole of D_2D_3.

8. ABC, $A'B'C'$ are two triangles centrally perspective on O and axially perspective on o.

Show that if $ABCA'B'C'$ are six points on a conic, the following pairs are pole and polar relative to this conic: (i) O and o; (ii) $\binom{BC}{B'C'}$ and AA'; (iii) $\binom{CA}{C'A'}$ and BB'; (iv) $\binom{AB}{A'B'}$ and CC'.

5.6. Ranges and Pencils on a Conic

DEFINITIONS. *The set of points on a non-singular conic is termed a range on a conic.*

The set of lines on a non-singular conic is termed a pencil on a conic.

The notion of ranges on a line led to the notion of projectivities between ranges and thence to important results, and in just the same way the notion of ranges and pencils on a conic leads to further results about the conic. The first task, obviously, is to define projectivity between ranges on a conic and other ranges.

Before doing this it is necessary to introduce an addition to our notation.

A range on a line x is denoted by $x(ABC...)$; it will obviously be convenient to have a similar notation for ranges (and pencils) on a conic. We therefore use Greek capital letters to denote conics, and to avoid ambiguity only those Greek capitals which are different from Roman capitals, namely, Γ, Δ, Θ, Ξ, Π, Σ, Φ, Ψ, Ω. It will be found that these are quite sufficient.

By $\Phi(ABC...)$ is meant the range $ABC...$ on the conic Φ. By $\Phi(abc...)$ the pencil $abc...$ on Φ.

5.61. Projectivity

5.611. DEFINITIONS. *A range $\Phi(ABC...)$ on a conic will be said to be perspective with a range $x(A'B'C'...)$ on a line if and only if there is a point O* **on the conic, but not on x,** *such that AA', BB', CC'... are all on O.*

A pencil $\Phi(abc...)$ on a conic will be said to be perspective with a pencil $X(a'b'c'...)$ on a point if and only if there is a line o **on the conic, but not on X,** *such that aa', bb', cc'... are all on o.*

5.612. Definitions

(*a*) *A range* $\Phi(ABC...)$ *on a conic will be said to be projective with a range* $x(A'B'C'...)$ *on a line if and only if there exists a range* $y(A''B''C''...)$ *such that*

$$y(A''B''C''...) \sim x(A'B'C'...),$$

and $\Phi(ABC...)$ *is perspective with* $y(A''B''C''...)$.

(*b*) *There is also said to be a projectivity between the two ranges.*

(*c*) *Two ranges* $\Phi(ABC...)$, $\Phi'(A'B'C'...)$ *on different conics are said to be projective if both are projective with the same range on a line.*

(*d*) *Two ranges* $\Phi(ABC...)$, $\Phi(A'B'C'...)$ *on the same conic are said to be projective if both are projective with the same range on a line.*

Examples

1. Give a reason for the insistence on the fact that in a perspectivity between a range on a conic and a range on a line, the centre of perspective must be on the conic.

2. Frame suitable definitions for projectivities between
(*a*) a range on a conic and a pencil on a point,
(*b*) a range on a conic and a pencil on the same or another conic,
(*c*) a pencil on a conic and a range on a line.

3. Show that if $\Phi(ABC...) \overset{O}{\wedge} x(A'B'C'...)$, where O is a point on the conic but not on x, then the two points common to Φ and x are common self-corresponding points.

4. Show that a projectivity between a range on a conic and a range on a line is completely determined when three pairs of corresponding points are given.

5. Show that if the common points of x and Φ are U and V, and $\Phi(UVABC...) \sim x(UVA'B'C'...)$, then there is a point O on Φ such that AA', BB', CC' are all on O; that is, the projectivity is a perspectivity.

6. Show that in a projectivity between two ranges on the same conic there cannot be more than two self-corresponding points, and that there are always two, which may not, however, be distinct.

7. If $a, b, c,...$ are the tangents to a conic Φ at the points $A, B, C,...$, show that
$$\Phi(ABC...) \sim \Phi(abc...).$$

8. If $\Phi(ABCD...) \sim \Phi(A'B'C'D'...)$ and X be any point on Φ, show that
$$X(ABCD) \sim X(A'B'C'D').$$

9. Show that
$$\Phi(ABCD) \sim \Phi(BADC).$$

5.62. Self-corresponding Points and Lines

The first important consequence of the idea of projectivities between ranges on a conic and other ranges is that it makes possible a construction for determining the self-corresponding points of a projectivity between cobasal ranges.†

5.621. THEOREM. *If*

$$\Phi(ABC...) \sim \Phi(A'B'C'...)$$

then the Pascal line of the two triads ABC, $A'B'C'$ is identical with the Pascal line of any other two corresponding triads DEF, $D'E'F'$.

Further, if R and S are the points common to this Pascal line and Φ, then R and S are self-corresponding points of the projectivity, and they are the only ones.

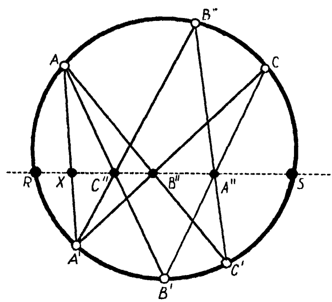


FIG. 33.

The second part of the theorem is proved first, and from it the first part is deduced.

Let l be the Pascal line of the two triads ABC, $A'B'C'$, so that l is the line RS.

Let A'', B'', C'' be the points $\begin{pmatrix} BC' \\ B'C \end{pmatrix}$, $\begin{pmatrix} CA' \\ C'A \end{pmatrix}$, $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$ respectively, so that these three points are on l.

Let X be the point $\begin{pmatrix} RS \\ AA' \end{pmatrix}$.

Then $$\Phi(RABCS) \overset{A'}{\sim} l(RXC''B''S),$$

and $$\Phi(RA'B'C'S) \overset{A}{\sim} l(RXC''B''S),$$

and so $$\Phi(RABCS) \sim \Phi(RA'B'C'S).$$

† The meaning of the word *base* can now be extended so as to include not only points (the bases of pencils) and lines (the bases of ranges), but also conics (the bases of ranges and pencils on conics).

Hence in the projectivity determined by the three pairs of corresponding points A and A', B and B', C and C', R and S are self-corresponding points. It is clear that they are the only ones.

Similarly, if D and D', E and E', F and F' are three other pairs of corresponding points in the projectivity, their Pascal line is on two self-corresponding points of the projectivity. But there cannot be more than two self-corresponding points in a projectivity, hence RS is the Pascal line of DEF and $D'E'F'$.

This proves the first part of the theorem.

The dual theorem should be proved independently, and an appropriate figure should be drawn.

The theorem just proved provides an easy method of determining the self-corresponding points of two ranges on the same conic. The next theorem may be used to find the self-corresponding points of two ranges on the same line.

5.622. THEOREM. *If*

(i) $x(ABC...) \sim x(A'B'C'...)$,

(ii) Φ *be any non-singular conic and* O *any point on it but not on* x,

(iii) $$\Phi(A_1 B_1 C_1...) \overset{O}{\sim} x(ABC...)$$

and $$\Phi(A_1' B_1' C_1'...) \overset{O}{\sim} x(A'B'C'...),$$

(iv) R *and* S *be the self-corresponding points of these two ranges on* Φ,

(v) T *and* U *be points on* x *collinear with* R *and* O, S *and* O *respectively,*

then T *and* U *are the self-corresponding points in the projectivity* $x(ABC...) \sim x(A'B'C'...)$.

The proof of this theorem is left to the reader.

The line joining the two self-corresponding points in a projectivity between two ranges on a conic is often called the *axis of the projectivity*. The dual term is *centre of projectivity*, but this latter must be carefully distinguished from a centre of perspectivity.

5.623. A Note on Constructions. The problem, or construction:

Given three pairs of corresponding points in a projectivity between two cobasal ranges, find the self-corresponding points, can evidently be solved by means of the last theorem. It should be noticed, however, that in solving it a conic Φ is an integral part of the construction.

So far, no construction has involved more than points and lines, and this construction is therefore very different. There is no logical objection to using conics in problems and constructions, but it is better to avoid using them when possible. This advice can be justified by remembering that the conic is a very complex construct; points and lines are the simple elements. A construction is better when it uses simpler materials.

5.63. Involutions on Conics

DEFINITION. *A projectivity between two ranges on a conic, in which there is a pair of reciprocally corresponding points, is termed an involution.*

That is, if $\Phi(AB...) \sim \Phi(BA...)$ the projectivity is an involution.

Involutions between pencils on conics are defined dually.

The properties of involutions on a conic are proved in the following theorems.

5.631. THEOREM. *Every pair of corresponding points in an involution on a conic is a pair of reciprocally corresponding points.*

Distinguish the two ranges by calling one the *first* range, the other the *second*.

Let A and B be the pair of reciprocally corresponding points.

Let C in the first range correspond to D in the second.

Let D in the first range correspond to X in the second.

Then $$\Phi(ABCD...) \sim \Phi(BADX);$$

but by the permutation theorem

$$\Phi(ABCD) \sim \Phi(BADC).$$

Hence X is C, and so every pair is a pair of reciprocally corresponding points.

5.632. THEOREM. *The lines which are on pairs of mates of an involution on a conic are concurrent.*

Let A and A' be a pair of mates of the involution.

Let l be the axis of the involution, and let A_1 be the point on l which is collinear with A and A'.

Let P be the harmonic conjugate of A_1, relative to A and A'. Let B and B' be another pair of mates.

Then $\begin{pmatrix} AB' \\ A'B \end{pmatrix}$ is on l; and since the projectivity is an involution, $\begin{pmatrix} AB \\ A'B' \end{pmatrix}$ is also on l.

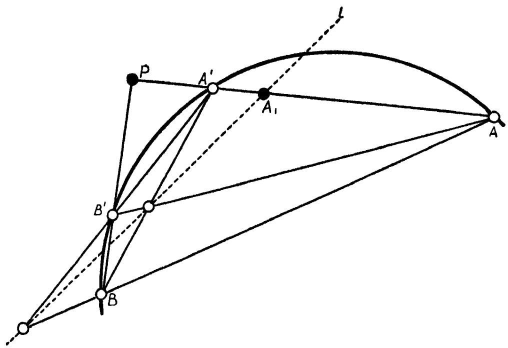


FIG. 34.

These are two of the diagonal points of the four-point $AA'BB'$. The third is plainly P.

Hence BB' is on P.

Similarly, the line on any other pair of mates is on P.

This proves the theorem.

5.633. THEOREM. *If*

$$\Phi(ABC...) \sim \Phi(A'B'C'...)$$

and if AA', BB', CC' are concurrent, the projectivity is an involution.

Let O be the point of concurrence of AA', BB', CC'.

Consider the involution in which A and A', B and B' are two pairs of mates.

Then in this involution, by the last theorem, C and C' must be a pair of mates.

Hence the projectivity

$$\Phi(ABC...) \sim \Phi(A'B'C'...)$$

is an involution.

EXAMPLES

1. A and A' are mates of an involution on a conic, and R and S are the self-corresponding points. If E is any other point on the conic, show that $E(AA', RS)$ is a harmonic tetrad.

2. X and Y are two points on a conic, and Z is the polar of XY relative to this conic. A and A' are two points on the conic and collinear with Z. Show that A and A' are mates in an involution of which X and Y are the self-corresponding points.

3. l, m, n are three non-concurrent lines, P and Q are two points on n, but not on l or m. Show how to construct two lines r and s each on the point lm such that (lm, rs) and (PQ, RS) are both harmonic tetrads, where R and S are the points nr, ns respectively.

4. P, Q, R, S are four points on a conic Φ. Find a fifth point T on Φ such that

$$\Phi(PQRT) \sim \Phi(PQTS).$$

How many points satisfy the condition?

5. If $\Phi(PQABC...) \sim \Phi(PQA'B'C'...)$ and if Z is the pole of PQ relative to Φ, show that A', B' and Z are collinear if and only if A, B, and Z are collinear.

6. If $\Phi(ABC...) \sim \Phi(A'B'C'...)$ and the projectivity is not an involution, show that the envelope of lines on pairs of corresponding points is a conic.

7. P, Q, R are three points not on a conic Φ. Show how to construct a triangle ABC whose sides BC, CA, AB shall be on P, Q, R respectively and whose points shall be on Φ. How many such triangles are there?

8. Give a suitable definition of the harmonic conjugate of the point A on a conic Φ, relative to two other points on the conic.

Show that mates in an involution on a conic Φ are harmonic conjugates relative to the self-corresponding points.

9. Show that every involution on a conic has distinct self-corresponding points.

CHAPTER VI

FURTHER THEOREMS ON CONICS

In the preceding chapter the five principal methods of dealing with the conic in synthetic Projective Geometry were elaborated. These five are enumerated here for convenience:

(i) *Projectivities between pencils or ranges whose bases are points or lines on the conic.*

(ii) *Desargues's theorem and its converse.*

(iii) *Pascal's theorem and its converse.*

(iv) *Theory of poles and polars.*

(v) *Ranges and pencils on the conic as base.*

In this chapter these methods are applied in various ways in order to prove certain well known theorems about the conic.

6.1. Pencils and Ranges of Conics

6.11. Conics on Four Distinct Points or Lines

If A_0, A_1, A_2, A_3 are four points, no three of which are collinear, and if X is any fifth point, then there is one and only one conic on these five points.

The set of all conics on the points A_0, A_1, A_2, A_3 is termed a *pencil of conics on four points*, or a *pencil of conics of Type I.*

Given any fifth point X, other points of the conic on A_0, A_1, A_2, A_3, X can be found by any of the first three of the methods enumerated above.

Given any line l, not on any of the four points, then every conic of the pencil is, by Desargues's theorem, on a pair mates of an involution on l. In particular, l is on two and only two conics of the pencil.

Dually, if a_0, a_1, a_2, a_3 are four lines no three of which are concurrent, the set of conics which are on these four lines is called a *range of conics on four lines*, or a *range of conics of Type I.*

There is one and only one conic of the range on any fifth line l; two and only two on any point X not on any of the four lines.

Various singular cases of pencils and ranges of conics arise, and two of these are considered in the following paragraphs.

6.12. Conics on Three Distinct Points and a Line on one of them

If A_1, A_2, A_3 are three distinct non-collinear points, and t is a line on one of them, and if X be any fourth point, then there is one and only one conic on these four points and on t.

The set of conics on A_1, A_2, A_3 and on t is termed *a pencil of conics on three points and having single contact at one of them*, or, more simply, *a pencil of conics of Type II*. It may be looked upon as a singular case of Type I, wherein A_0 and A_1 are coincident, but the line A_0A_1 is determined as t.

Given any fourth point X, other points of that conic of the pencil which is on X may be found by any of the first three methods enumerated.

Given any line l not on any of the three points, there are two and only two conics of the pencil on l.

The dual of the pencil of Type II is the range of Type II.

Any two conics of the pencil are said to have *single contact* at A_0.

6.13. Conics having Double Contact

If we now suppose that not only A_0 and A_1 coincide, but also A_2 and A_3, and that the lines A_0A_1 and A_2A_3 are determined as t_1 and t_2 respectively, we have a second singular case of a pencil of conics.

If A_1 and A_2 are points on t_1 and t_2 respectively, then the set of all conics on A_1, A_2, t_1, and t_2 is termed a *pencil of conics having double contact*, and sometimes a *pencil of conics of Type V*.

Singular cases of Pascal's or Desargues's theorem establish the result that on any point X, not on A_1 or A_2, there is one and only one conic of the pencil.

A singular case of Desargues's theorem establishes the result that on any line l, not on A_1, A_2, or t_1t_2, there is one and only one *non-singular* conic of the pencil.

It may also be verified that the pencil of conics having double contact is a self-dual figure.

Other singular cases of pencils of conics exist, but their discussion is omitted here.

Examples

1. Enumerate the singular conics of the three types of pencils.
2. Explain the apparent exception to Desargues's theorem contained in the statement: *There is one and only one non-singular conic of a pencil of conics having double contact on any line not on* A_1, A_2, or $t_1 t_2$. Why are there not two?

6.2. Further Theory of Poles and Polars

The following group of theorems about poles and polars is selected as being useful in application, and important enough to merit explicit proof.

6.21. Theorem

If $D_1 D_2 D_3$ *be the diagonal triangle of the four-point* $A_0 A_1 A_2 A_3$, *then* $D_2 D_3$ *is the polar of* D_1 *relative to any conic of the pencil on* $A_0 A_1 A_2 A_3$.

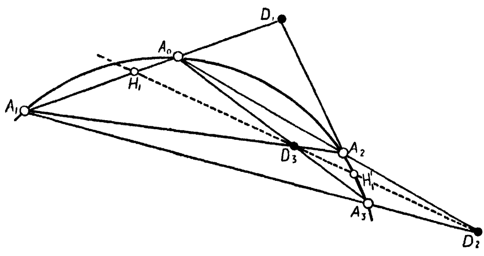


Fig. 35.

Let Φ be any non-singular conic on $A_0 A_1 A_2 A_3$.

Then, with the usual lettering used for the four-point, H_1 is the harmonic conjugate of D_1 relative to $A_0 A_1$, and H_1' is the harmonic conjugate of D_1 relative to $A_2 A_3$.

Hence, by the definition of polar, $H_1 H_1'$ is the polar of D_1 relative to Φ.

But $H_1 H_1'$ is $D_2 D_3$, and Φ is any non-singular conic of the pencil. Hence the theorem is proved.

It should be noticed that, in addition, D_2 is the pole of $D_3 D_1$ relative to Φ, and D_3 the pole of $D_1 D_2$ relative to Φ.

The triangle $D_1 D_2 D_3$ therefore has the remarkable property that each side is the polar, relative to Φ, of the opposite point.

Such a triangle is called a *self-polar* triangle relative to Φ. Other properties of self-polar triangles will be investigated later.

6.22. Theorem

If two four-points $A_0 A_1 A_2 A_3$, $B_0 B_1 B_2 B_3$ have the same diagonal points $D_1 D_2 D_3$, then there is a conic which is on the eight points $A_0 A_1 A_2 A_3 B_0 B_1 B_2 B_3$.

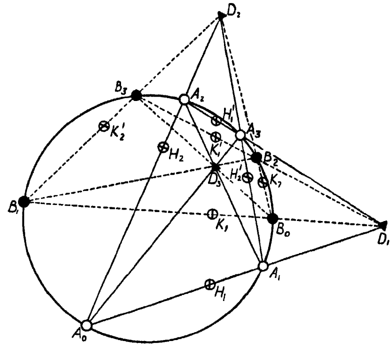


Fig. 36.

Let $H_1 H_2 H_3 H_1' H_2' H_3'$ be the harmonic points of the four-point $A_0 A_1 A_2 A_3$, and $K_1 K_2 K_3 K_1' K_2' K_3'$ those of $B_0 B_1 B_2 B_3$.

Suppose first that B_0 is not on any of the sides of $A_0 A_1 A_2 A_3$.

Consider the conic on $A_0 A_1 A_2 A_3 B_0$.

Since D_1 is the pole of $D_2 D_3$ relative to this conic, and since K_1 is the point on $D_2 D_3$ which is collinear with B_0 and D_1, and since, further, B_1 is the harmonic conjugate of B_0 relative to $B_1 K_1$, it follows that B_1 is on the conic.

Similarly, B_2 and B_3 are on the conic.

This proof breaks down when B_0 is on one of the sides of $A_0 A_1 A_2 A_3$, but when this is so, it is a simple matter to show that B_1, B_2, and B_3 are also on sides of $A_0 A_1 A_2 A_3$. Not only this, but that the two four-points have a pair of opposite sides in common. When B_0 is on one of the sides of $A_0 A_1 A_2 A_3$ there is therefore a singular point-conic on the eight points.

6.23. Theorem

If $A_0 A_1 A_2 A_3$ be a four-point, and X be any point not on one of its sides, then the polars of X relative to conics of the pencil on $A_0 A_1 A_2 A_3$ are all concurrent.

Consider the non-singular conic Φ on $A_0A_1A_2A_3X$.

Let XT be the tangent to Φ at X.

By Desargues's theorem, conics of the pencil are on pairs of mates in an involution on XT, and X is one of the self-corresponding points of this involution.

Let X' be the other, and let Ψ be any non-singular conic of the pencil other than Φ.

Let R and R' be the pair of mates of the involution on XT which is on Ψ.

Then (XX', RR') is a harmonic tetrad.

Hence the polar of X relative to Ψ is on X'.

But the polar of X relative to Φ is on X'.

Hence all the polars of X relative to conics of the pencil are on X'.

The point-pair XX' has the property of being a conjugate point-pair relative to every conic of the pencil.

6.24. Theorem

If ABC be any triangle and A', B', C' be the poles of BC, CA, AB relative to any non-singular conic Φ, then the triangles ABC, $A'B'C'$ are perspective.

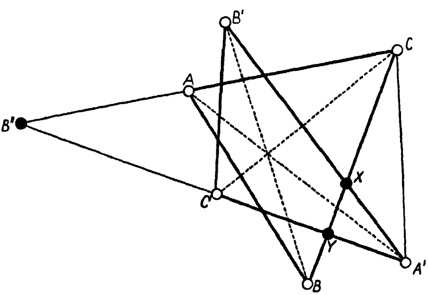


Fig. 37.

Let a, b, c, a', b', c' be the lines BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ respectively; these are the polars of A', B', C', A, B, C respectively.

Let B'', X, Y be the points bb', ac', ab' respectively.

Let x be the line $A'C$, so that x is the polar of X.

Then $B(A'B'AX) \sim b'(aba'x)$,

and $b'(aba'x)$ is $b'(YB''C'A')$.

Now
$$b'(YB''C'A') \sim C(YB''C'A') \sim C(A'C'B''Y).$$

This last pencil is identical with $C(A'C'AY)$.

Hence $B(A'B'AX) \sim C(A'C'AY)$,

or $B(A'B'AC) \sim C(A'C'AB)$.

These two pencils have the common corresponding line BC, hence they are perspective. Hence $\begin{pmatrix}BB'\\CC'\end{pmatrix}$ is on AA'; and so the triangles are perspective.

6.25. Hesse's Theorem

THEOREM. *If $A_0A_1A_2A_3$ be any four-point, and if two pairs of opposite sides are pairs of conjugate lines relative to a conic Φ, then the third pair of sides is also a pair of conjugate lines relative to Φ.*

The theorem is a direct consequence of the previous theorem and is in fact merely a restatement of it. It is obtained by considering the four-point A, B, C, $\begin{pmatrix}BB'\\CC'\end{pmatrix}$.

6.26. THEOREM

If the sides BC, CA, AB of a triangle are tangents to a conic Φ at A', B', C' respectively, then the triangles ABC, $A'B'C'$ are perspective.

This is a particular case of 6.24. It has already been noticed in the discussion of the singular cases of Pascal's theorem (5.434).

EXAMPLES

1. Φ is a conic, and P, Q, A are three distinct non-collinear points not on it. X and Y are two points on Φ collinear with A. Show that there is a second pair of points X' and Y' on Φ and collinear with A such that $XYX'Y'PQ$ are all on a conic.

Discuss the case when A and P, A and Q are both pairs of conjugate points relative to Φ.

2. P, Q, A are three distinct non-collinear points, and a is a line not on any of them. Show that conics on P and Q, relative to which A and a are pole and polar, form a pencil on four points.

3. If A and a are pole and polar relative to every non-singular conic of a pencil on four points, show that A is a diagonal point of the four-point.

4. $IJKL$ are four points on a conic Φ. P is the polar of IJ relative to Φ, Q that of KL. Show that the poles of IJ and KL relative to the conic on $IJKLPQ$ are on PQ. (See 5.545, Ex. 6.)

5. $A_0A_1A_2A_3$ is a four-point, Φ any conic on A_0 and A_1. Show that any conic of the pencil on $A_0A_1A_2A_3$ is on a second pair of points B and B' of Φ which are collinear with a certain point on A_2A_3.

6. $ABCDE$ are five points, no three of which are collinear. x is a line not on any of them. Give a construction for determining the points on x which are also on the conic on $ABCDE$.

7. AB and l are two conjugate lines relative to a conic Φ; A' and B' are points on l which are conjugate to A and B respectively relative to Φ. Show that the point $\begin{pmatrix}AA'\\BB'\end{pmatrix}$ is on Φ.

8. ABC is a three-point on a conic Φ, and RS is a line conjugate to BC relative to Φ. Show that the points $\begin{pmatrix}RS\\CA\end{pmatrix}$, $\begin{pmatrix}RS\\AB\end{pmatrix}$ are conjugate relative to Φ.

9. $XYZPQ$ are five points, no three of which are collinear. Show that the set of conics which are on X, Y, and Z, and relative to which P and Q are conjugate, is a pencil on four points.

6.3. Conics and Triangles

Under this heading are grouped a number of theorems which, though they might be spread about under different headings, are more convenient when found together.

6.31. Theorem

If two triangles ABC, $A'B'C'$ are both self-polar relative to a conic Φ, then the six points A, B, C, A', B', C' are all on a second conic Ψ.

Let BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ be a, b, c, a', b', c' respectively, so that these lines are the polars, relative to Φ, of A, B, C, A', B', C' respectively.

Let U, V, X, Y be the points ab', ac', $\begin{pmatrix}BC\\AC'\end{pmatrix}$, $\begin{pmatrix}BC\\AB'\end{pmatrix}$ respectively. The polars of these collinear points are AY, AX, AV, AU respectively.

Now
$$A(BCB'C') \sim a(BCYX)$$
$$\sim A(CBVU).$$

But C, B, V, U are collinear, so that

$$A(CBVU) \sim A'(CBVU)$$
$$\sim A'(BCUV).$$

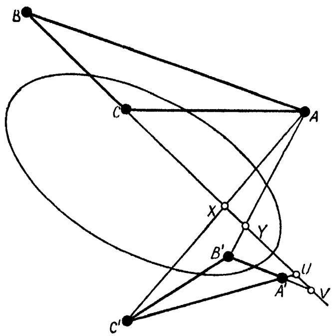


FIG. 38.

This last pencil is $A'(BCB'C')$.

Hence $A'(BCB'C') \sim A(BCB'C')$, and so the six points are on a second conic.

The converse of this theorem will be proved later (6.422). As a corollary the following theorem can be proved.

6.311. THEOREM. *If $D_1D_2D_3$ be the diagonal triangle of the four-point $A_0A_1A_2A_3$, and if x be any line not on any of these seven points, then the poles of x relative to conics of the pencil on $A_0A_1A_2A_3$ are all on a conic.*

Two non-singular conics of the pencil are on x; let I and J, on x, be the points of contact of these two conics.

Let Φ be any other non-singular conic of the pencil. It is on two points L and L', say, on x, such that (IJ, LL') is a harmonic tetrad.

Let the tangents at L and L' to Φ have the common point T.

Then IJ is the polar of T relative to Φ,

TI ,, ,, J ,, Φ,

and TJ ,, ,, I ,, Φ,

so that TIJ is a self-polar triangle relative to Φ.

But, by 6.21, $D_1 D_2 D_3$ is also a self-polar triangle relative to Φ. Hence T is on the conic on $IJD_1 D_2 D_3$.

6.32. The Eleven-point Conic

The conic on the five points D_1, D_2, D_3, I, J spoken of in the last paragraph has certain interesting properties, and six other points on it can be specified at once. That is to say, given a four-point, and a line not on any of these points nor on the diagonal points, there is a conic determined by this four-point and this line, and eleven points of this conic can be specified.

The conic may be aptly spoken of as the *eleven-point conic relative to the four-point and the line.*

The reader may justly ask why this is included in the section devoted to conics and triangles. The reason is as follows. The four-point $A_0 A_1 A_2 A_3$ may be looked upon as a triangle $A_1 A_2 A_3$, and a fourth point A_0. $D_1 D_2 D_3$ will then be three points on the sides $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ of the triangle, and the lines $A_1 D_1$, $A_2 D_2$, $A_3 D_3$ are concurrent at A_0. The eleven-point conic is usually spoken of as the eleven-point conic relative to a triangle ($A_1 A_2 A_3$), a line (x), and a point (A_0). Once it has been realized that all questions about concurrence and collinearity in triangles are really questions about four-points and four-lines respectively, much has been gained.

THEOREM. *If*

(i) $A_0 A_1 A_2 A_3$ *be a four-point whose diagonal points are* $D_1 D_2 D_3$,

(ii) x *be a line not on any of these seven points,*

(iii) $X_1 X_2 X_3 X_1' X_2' X_3'$ *be the involutory hexad on* x *determined by the four-point, and* I *and* J *be the self-corresponding points of the associated involution,*

(iv) Y_1, Y_2, Y_3, Y_1', Y_2', Y_3' *be the harmonic conjugates of* X_1, X_2, X_3, X_1', X_2', X_3' *relative to the pairs of points of the four-point collinear with each,*

then I, J, D_1, D_2, D_3, Y_1, Y_2, Y_3, Y_1', Y_2', Y_3' *are all on a conic.*

It is necessary to prove that the six points $Y_1 Y_2 Y_3 Y_1' Y_2' Y_3'$ are on the conic on $IJD_1 D_2 D_3$.

Let P be the point on x collinear with A_2 and Y_1.

Let Q be the harmonic conjugate of A_2, relative to the pair PY_1. Consider that conic Φ of the pencil on $A_0 A_1 A_2 A_3$ which is on Q. Then since P is the harmonic conjugate of Y_1 relative

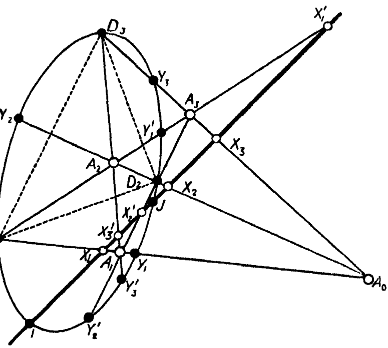


FIG. 39.

to $A_2 Q$, and since X_1 is the harmonic conjugate of Y_1 relative to $A_1 A_0$, PX_1, i.e. x, is the polar of Y_1 relative to Φ. Hence, by 6.31, Y_1 is on the conic on $D_1 D_2 D_3 I J$.

Similarly, all the other five points are on the same conic.

6.33. THEOREM. *If*

(i) Φ *be any non-singular conic,*

(ii) ABC *be a triangle whose points are on* Φ,

(iii) X, Y, Z *be three collinear points on* BC, CA, AB *respectively,*

(iv) R *and* S *be the two points on* Φ *collinear with* X, Y, Z,

then $\Phi(ABCR) \sim x(XYZS)$, *where* x *is the line* XY.

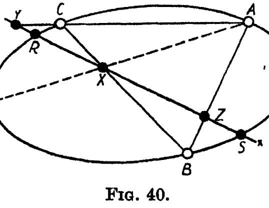


FIG. 40.

Let T be the point on Φ collinear with A and X. Then

$$\begin{aligned} x(XYZS) &\sim A(XYZS) \\ &\sim A(TCBS) \\ &\sim \Phi(TCBS). \end{aligned}$$

Now since there is an involution on Φ such that A and T, R and S, B and C, are three pairs of mates,

$$\Phi(TCBS) \sim \Phi(ABCR).$$

Hence $$x(XYZS) \sim \Phi(ABCR).$$

6.34. Theorem

If A, B, C, A', B', C' be six points on a non-singular conic, then the six sides BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ of the two triangles ABC, $A'B'C'$ are on a second conic.

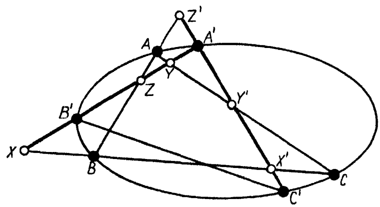


Fig. 41.

Let a, b, c, a', b', c' be the lines BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ respectively.

Let X, Y, Z be the points $c'a$, $c'b$, $c'c$ respectively, and X', Y', Z' the points $b'a$, $b'b$, $b'c$ respectively.

It follows from the last theorem that

$$c'(XYZB') \sim \Phi(ABCA')$$

and $$b'(X'Y'Z'C') \sim \Phi(ABCA'),$$

so that $$c'(XYZB') \sim b'(X'Y'Z'C').$$

But these last two ranges are, respectively,

$$c'(abca') \quad \text{and} \quad b'(abca'),$$

and so $$c'(abca') \sim b'(abca').$$

From the definition of a line-conic it follows that a, b, c, a', b', c' are all on a conic.

The dual of this theorem is its converse.

6.35. Theorem. *If*

(i) *ABC be any triangle and Φ be any non-singular conic not on any of its points or sides,*

(ii) *X and X' be points on BC such that AX and AX' are on Φ,*

(iii) *Y and Y' on CA, Z and Z' on AB be points similarly defined,*

then X, Y, Z, X', Y', Z' are six points on a conic.

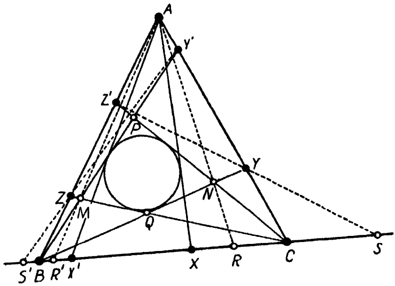


FIG. 42.

Let P, Q, M, N be the points $\begin{pmatrix}CZ'\\BY'\end{pmatrix}$, $\begin{pmatrix}CZ\\BY\end{pmatrix}$, $\begin{pmatrix}CZ\\BY'\end{pmatrix}$, $\begin{pmatrix}CZ'\\BY\end{pmatrix}$ respectively.

Let R, S, R', S' be the points $\begin{pmatrix}AN\\BC\end{pmatrix}$, $\begin{pmatrix}YZ'\\BC\end{pmatrix}$, $\begin{pmatrix}AM\\BC\end{pmatrix}$, $\begin{pmatrix}Y'Z\\BC\end{pmatrix}$ respectively.

Consider the range of conics on the four-line whose points are $BCMNPQ$.

The pairs of tangents on the point A to conics of this range are mates in an involution on A.

Now the singular conics of this range are the pencils on the three point-pairs B, C; M, N; P, Q.

Hence AB and AC, AR and AR', AX and AX' are three pairs of mates of the involution.

It follows that B and C, R and R', X and X' are pairs of mates in an involution on BC.

Consider now the four-point $ANZ'Y$; two of its diagonal points are B and C, while R and S are the two harmonic points collinear with them.

Hence (BC, RS) is a harmonic tetrad.

Similarly, $(BC, R'S')$ is a harmonic tetrad.

Hence $$(BCRS) \sim (BCS'R')$$ $$\sim (CBR'S').$$

It follows that S and S' are mates in the involution also. Altogether, there is an involution on BC, in which X and X', R and R', S and S', B and C are pairs of mates.

Now any conic on $YY'ZZ'$ is on a pair of mates of an involution on BC, and in this involution B and C, S and S' are pairs of mates. It is therefore identical with the previous involution, and so there is a conic on $XX'YY'ZZ'$.

The dual of this theorem is its converse.

6.4. Conic Constructions

Under this heading are collected certain problems in which it is required to construct a conic, but in which the data require something more than the direct application of Desargues's or Pascal's theorem. In general, the method is to reduce the problem to a simpler one in which either of the above theorems may be applied. The problems are enunciated as theorems, and the proof is often left to the reader once the construction has been given.

6.41. Data involving Poles and Polars

In general, it may be taken that to be given a line and a point which are polar and pole is equivalent to being given two points on the conic. Thus sufficient data are the following:

(i) a pole and polar, and three points on the conic;

(ii) two poles and polars, and one point on the conic.

If three poles and polars are given, there is in general no solution to the problem, unless the triangle formed by the poles and that formed by the polars are perspective.

6.411. THEOREM. *If A, P, Q, R be four points, no three of which are collinear, and a be any line, not on any of these four points, then there is one and only one conic on P, Q, and R relative to which A and a are pole and polar.*

Let X be the point on a collinear with A and P and Y be the point on a collinear with A and Q.

Let P' and Q' be points such that (AX, PP'), (AY, QQ') are harmonic tetrads.

Then the conic on $PQRP'Q'$ fulfils the conditions, and it is the only one which does so.

The following cases should be examined:

(i) a on A.

(ii) a on P or Q or R, but not on A.

(iii) a on P and Q, or Q and R, or R and P, but not on A.

(iv) A, P, Q collinear.

These are not covered by the theorem.

6.412. THEOREM. *If A, B, P are three non-collinear points, a and b two lines not on any of these three points, then there is one and only one conic on P and relative to which A and a, B and b are two poles and polars.*

The method used in 6.411 may be applied here. The following is an alternative method.

Let X be the point ab.

Let A' and B' be points on a and b respectively which are collinear with AB.

Let R and S be the double points of the involution on AB in which A and A', B and B' are two pairs of mates.

Then any conic relative to which A and a, B and b are poles and polars must be on R and S, and moreover XR and XS must be tangents to it.

Hence all these form, together, a pencil of Type V, i.e. they have double contact.

The theorem now reduces to the proof that there is one and only one conic of the pencil on P.

The case when A, B, and P are collinear is not covered by the theorem and should be examined. The case when a and b are on B and A respectively involves the fact that XAB is self-polar relative to the required conic; cases involving self-polar triangles are treated later.

6.413. THEOREM. *If ABC, $A'B'C'$ are two perspective triangles and no three of these points are collinear, then there is one and only one conic relative to which A and $B'C'$, B and $C'A'$, C and $A'B'$ are poles and polars.*

From the work done in the preceding theorem it is only necessary to prove that there is one and only one conic of a pencil of Type V relative to which a given line and point are polar and pole.

6.42. Cases involving Self-polar Triangles

6.421. THEOREM. *If ABC be a triangle and P and Q be two other points, one at least of which is not on any of the sides of the* *triangle ABC, then there is one and only one conic on P and Q, and relative to which ABC is a self-polar triangle.*

Let P be the point not on any of the sides of ABC.

Let RST be the three other points of that four-point whose diagonal triangle is ABC, and one of whose vertices is P.

Then the conic on $PQRST$ is the required conic, by 6.21.

That this conic is unique is left to the reader to prove.

The case when both P and Q are on sides of ABC should be examined.

6.422. THEOREM. *If A, B, C, A', B', C' be six points on a non-singular conic, then there is one and only one conic relative to which ABC and $A'B'C'$ are both self-polar triangles.*

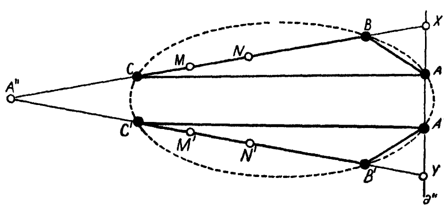


FIG. 43.

Let BC, CA, AB, $B'C'$, $C'A'$, $A'B'$ be a, b, c, a', b', c' respectively.

Let A'' be the point aa', a'' the line AA'.

Let X and Y be the points $a''a$, $a''a'$ respectively.

Let M and N on a be the self-corresponding points of an involution in which B and C, A'' and X are two pairs of mates.

Let M' and N' be similarly specified points on a'.

Let Φ be the conic on $ABCA'B'C'$.

Consider that conic on $MNM'N'$ which is also on AM; let it be Ψ.

Since $(A''X, MN)$ and $(A''Y, M'N')$ are harmonic tetrads, XY, i.e. a'', is the polar of A'' relative to Ψ, and the point-pair B, C is a conjugate pair relative to it.

Since also (BC, MN) is a harmonic tetrad, and AM is on Ψ, AN is also on it. Hence MN, i.e. a, is the polar of A relative to Ψ; that is, ABC is a self-polar triangle relative to Ψ.

Now for similar reasons B' and C' are conjugate points relative to Ψ, and the pole of $B'C'$ must be on a''. That is, there is a point K on a'' such that $KB'C'$ is self-polar relative to Ψ.

But by 6.31 $ABCKB'C'$ must be on a conic, and this conic cannot be other than Φ. Hence K is A'.

Hence Ψ fulfils the requirements.

Plainly, also, it is the only conic which does so.

6.43. Data involving Lines and Points

A different class of problem is typified by the following: *Given two points and three lines, not specially related, construct a conic which shall be on all five.*

There are four cases of this problem; the data in these are:

(i) four points and one line,
(ii) three points and two lines,
(iii) two points and three lines,
(iv) one point and four lines.

It will be realized that (iii) and (iv) are the duals of (ii) and (i) respectively, and so consideration of this type of problem is limited to the consideration of (i) and (ii).

6.431. THEOREM. *If $A_0A_1A_2A_3$ is a four-point and x any line, not on any of these points, there are two and only two conics on* $A_0A_1A_2A_3$ and x.

Let $X_1X_2X_3$, $X_1'X_2'X_3'$ be the involutory hexad on x determined by the four-point.

Let U and V be the self-corresponding points of the involution in which these are mates.

Then the conics on $A_0A_1A_2A_3U$, $A_0A_1A_2A_3V$ are the required conics, and they are the only two. The proof is left to the reader.

The following cases, covered by the theorem, deserve attention:

(i) when x is on one and only one diagonal point of the four-point;
(ii) when x is on two diagonal points.

The following cases, not covered by the theorem, should be investigated:

(i) x on one and only one of the four points;
(ii) x on two of the four points.

6.432. THEOREM. *If A, B, C are three points and x, y two lines not on any of them, and if no three of the points A, B, C, xy are collinear, then there are four and only four conics on A, B, C, x, and y.*

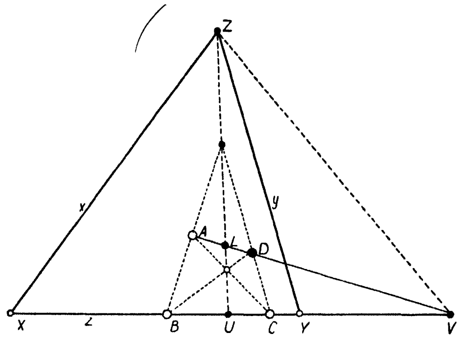


FIG. 44.

Let z be the line BC, X and Y the points zx, zy.

Let Z be the point xy.

Let U and V on z be the self-corresponding points of the involution in which B and C, X and Y are mates.

Let L be the point $\begin{pmatrix} AV \\ ZU \end{pmatrix}$, and let D be the harmonic conjugate of A relative to LV.

Then V is a diagonal point of the four-point $ABCD$, and the other two diagonal points are on ZU.

Consider now the pencil of conics on $ABCD$.

Relative to any conic of the pencil, V and ZU are pole and polar.

Now x and y are harmonic conjugates relative to ZU and ZV, hence any conic of the pencil which is on x is on y also.

Two conics of this pencil therefore fulfil the conditions.

By interchanging the roles of U and V in the construction, a second pencil of conics is obtained; of this pencil two more conics fulfil the required conditions.

There are therefore four conics which fulfil the conditions. It is plain that there are only four.

The following cases, not covered by the theorem, should be examined:

(i) A, B, C, and xy collinear;

(ii) A, B, and C collinear, but not collinear with xy;

(iii) B, C, and xy collinear, but not collinear with A;

(iv) A on x, B and C not on y;

(v) A on x, B on y;

(vi) A, B, and C on x;

(vii) B and C on y, A not on y.

It will be found that in the last case no conic satisfies the conditions unless B and C coincide.

Examples

1. ABC is a triangle and x a line not on any of its points; X, a point on x, is not on any of the sides of the triangle. Give a construction for finding a conic Φ on x and X and relative to which ABC is self-polar.

2. If in the last example X is not on x, and if Y and Z are points on x which are also on a conic on A, B, C, X, show that x and X are polar and pole relative to the conic which has ABC and XYZ as self-polar triangles. Show that this conic is the only one which has ABC as a self-polar triangle, and relative to which X and x are pole and polar.

3. Give a construction for finding points of a conic which shall be on two given points and a given line, and relative to which a given point and line shall be pole and polar. How many conics satisfy the conditions?

4. Give a construction for finding points of a conic which shall be on a point X and a line x, and relative to which ABC shall be self-polar. How many conics fulfil the conditions?

5. Use the converse of the theorem that the points of two triangles which are self-polar relative to a conic Φ are on a conic Ψ, to prove that if the points of two triangles are on one conic the sides are on a second conic.

MISCELLANEOUS EXAMPLES

1. $A_2B_2C_2$ and $A_3B_3C_3$ are two triangles perspective on the point X_1; $A_3B_3C_3$ and $A_1B_1C_1$ are perspective on X_2; $A_1B_1C_1$ and $A_2B_2C_2$ are perspective on X_3. If X_1, X_2, X_3 are collinear, show that the triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ are perspective in pairs on three collinear centres.

Hence show that if three triangles are perspective from the same centre, the three axes of perspective are concurrent.

2. (PQ, AB) is a harmonic tetrad on a line x, and Φ is any non-singular conic. The four-line formed by the two pairs of tangents to Φ on A and B has $WXYZ$ as its other four points. Show that $PQWXYZ$ are six points on a conic.

A' and B' is another pair of harmonic conjugates relative to P and Q, and $W'X'Y'Z'$ are defined in a similar way. Show that the ten points $PQWXYZW'X'Y'Z'$ are all on the same conic.

3. If two different pencils of conics on four points have a conic in common, show that the two four-points have the same diagonal points.

4. ABC, $A'B'C'$ are two triangles perspective on X and x. A'', B'', C'' are the three points on x collinear with A and A', B and B', C and C' respectively. If (XA'', AA'), (XB'', BB'), (XC'', CC') are all harmonic tetrads, show that $ABCA'B'C'$ are all on a conic.

5. $a(OA_1A_2A_3) \sim b(OB_1B_2B_3) \sim c(OC_1C_2C_3) \sim d(OD_1D_2D_3)$ and $A_1B_1C_1D_1$ are all on a line XY. Show that if $A_2B_2C_2D_2XY$ are six points on a conic, $A_3B_3C_3D_3XY$ are six points on another.

6. Three conics Φ, Ψ, Ω are on two points I and J. P_1 and Q_1 are the other two points on both Ψ and Ω, P_2 and Q_2 those on Ω and Φ, P_3 and Q_3 those on Φ and Ψ. Show that P_1Q_1, P_2Q_2, P_3Q_3 are concurrent.

7. X, Y, Z are the three points on a line m which are also on the sides BC, CA, AB of a triangle ABC. $X'Y'Z'$ are similar points on a line m'. $O_1O_2O_3O_4$ are the points of contact with m of the four conics on A, B, C, m, and m'; $O_1'O_2'O_3'O_4'$ are similar points on m'. Show that

$$m(XYZO_1O_2O_3O_4) \sim m'(X'Y'Z'O_1'O_2'O_3'O_4').$$

8. $A_1A_2A_3$ is any triangle and x a line not on any of its vertices. $X_1X_2X_3$ on x are collinear with A_2A_3, A_3A_1, A_1A_2 respectively. I and J are any other distinct points on x.

Y_1 and Y_1' are the self-corresponding points in an involution on x in which X_2 and X_3, I and J are two pairs of mates. Y_2 and Y_2', Y_3 and Y_3' are similarly defined.

Show that (i) $Y_1Y_2Y_3Y_1'Y_2'Y_3'$ is an involutory hexad, (ii) the six lines A_1Y_1, A_1Y_1', A_2Y_2, A_2Y_2', A_3Y_3, A_3Y_3' are six lines of a four-point $B_0B_1B_2B_3$.

9. $A_1A_2A_3$, x, X_1, X_2, X_3, I, and J are as specified in the last example. X_1', X_2', X_3' are the harmonic conjugates of X_1, X_2, X_3 respectively rela-

tive to I and J. T is any point on the conic on $A_1A_2A_3IJ$. Show that the points $\begin{pmatrix}TX'_1\\A_2A_3\end{pmatrix}$, $\begin{pmatrix}TX'_2\\A_3A_1\end{pmatrix}$, $\begin{pmatrix}TX'_3\\A_1A_2\end{pmatrix}$ are collinear.

10. $A_1A_2A_3$, x, X_1, X_2, X_3, I, and J are as specified in Example 8; X'_1, X'_2, X'_3 are as specified in Example 9. M_1, M_2, M_3 are the harmonic conjugates of X_1, X_2, X_3 relative to A_2A_3, A_3A_1, A_1A_2 respectively. Show that $M_1X'_1$, $M_2X'_2$, $M_3X'_3$ are concurrent.

11. If P is the common point of the three lines $M_1X'_1$, $M_2X'_2$, $M_3X'_3$ in the last example, show that PI, PJ are tangents to the conic on $A_1A_2A_3IJ$.

12. A conic Φ is on the sides of a triangle ABC which is self-polar relative to a second conic Ψ. x is any other line on Φ and X is its pole relative to Ψ. y and z are the two tangents to Φ which are on X. Show that the triangle xyz is self-polar relative to Ψ.

13. Φ_1, Φ_2, and Φ_3 are three conics. Φ_2 and Φ_3 have double contact and X_1, X'_1 are their common points; Φ_3 and Φ_1 have double contact and X_2, X'_2 are their common points; Φ_1 and Φ_2 have double contact and X_3, X'_3 are their common points. Show that $\begin{pmatrix}X_2X'_2\\X_1X'_1\end{pmatrix}$ and $\begin{pmatrix}X_3X'_3\\X_1X'_1\end{pmatrix}$ are harmonic conjugates relative to X_1 and X'_1.

14. P and Q are two points not on a line x. R and R' are a typical pair of mates of an involution on x. U and V are the other two points of the four-line PR, PR', QR, QR'. Show that the locus of U and V is a conic.

When and how is this conic singular?

15. If $x(ABA_1A_2A_3...) \sim x(ABB_1B_2B_3...)$, find a third range $x(C_1C_2C_3...)$, such that both the projectivities

$$x(A_1A_2A_3...) \sim x(C_1C_2C_3...)$$

and

$$x(B_1B_2B_3...) \sim x(C_1C_2C_3...)$$

are involutions.

Is this third range unique?

16. x and y are two lines conjugate relative to a conic. P and Q are two points on x; P' and Q' are two points on y. If P and P', Q and Q' are both conjugate relative to Φ, show that $\begin{pmatrix}PP'\\QQ'\end{pmatrix}$ is on Φ.

17. Two conics Φ and Ψ are each on the sides of a triangle ABC. Show that the six points of contact are all on a third conic Ω. Dualize this result, and also examine singular cases.

18. A, B, C are any three points on a conic Φ. T is the polar of BC relative to Φ. Show that if TU be any line on T, the points $\begin{pmatrix}TU\\CA\end{pmatrix}$, $\begin{pmatrix}TU\\AB\end{pmatrix}$ are conjugate relative to Φ. Dualize.

19. A, B, C, P, Q are five points no three of which are collinear. Show that the set of conics on A, B, and C and relative to which P and Q are conjugate points is, in general, a pencil on four points.

Hence devise a construction for a conic on four points and relative to which two other points are conjugate points.

20. x and y are two lines, P and Q are two points not on either. Find the locus of poles of PQ relative to conics on x, y, P, and Q.

Examine the case when PQ, x, and y are concurrent.

21. $x(X_1 X_2 X_3 \ldots) \sim y(Y_1 Y_2 Y_3 \ldots)$, and $Z_1 Z_2 Z_3 \ldots$ is a range on a third line z such that $X_1 Y_1 Z_1$ are collinear, $X_2 Y_2 Z_2$ are collinear, and so on. Show that, in general, it is not true that

$$z(Z_1 Z_2 Z_3 \ldots) \sim y(Y_1 Y_2 Y_3 \ldots).$$

Under what conditions is it true?

22. A, B, C, D, P, Q are six points, no three of which are collinear. There is a unique conic on $ABCD$ relative to which P and Q are conjugate points. Give a construction for the polar of P relative to this conic.

23. $ABCD$ are the points common to two conics Φ and Φ'. $abcda'b'c'd'$ are the eight tangents to these conics at these points. Show that these eight lines are all on a third conic.

24. p and q are conjugate lines relative to a conic Φ. P and P' are the common points of p and Φ, Q and Q' those of q and Φ. Show that $\Phi(PP', QQ')$ is a harmonic tetrad.

25. O is a point, Φ a conic not on it, X, Y, Z, W four points on a line x. R and R' are the common points of Φ and a typical line p of the pencil on O. S is a fourth point on this line such that

$$p(ORR'S) \sim x(XYZW).$$

Show that the locus of S is a conic which has double contact with Φ, except when (YZ, XW) is a harmonic tetrad.

26. $A_0 A_1 A_2 A_3$ is a four-point and Φ is any conic on its diagonal points. With the usual convention for assigning letters, $Y_1 Y_2 Y_3 Y_1' Y_2' Y_3'$ are the six other points on Φ which are on the sides of the four-point. $X_1 X_2 X_3 X_1' X_2' X_3'$ are the harmonic conjugates of these relative to the pairs of points of the four-point with which they are collinear. Show that these six points are collinear, and that the self-corresponding points of the involution in which they are mates are on Φ.

27. $X_1 X_2 X_3 Z_1 Z_2 Z_3 Z_1' Z_2' Z_3'$ are nine collinear points such that $(X_2 X_3, Z_1 Z_1')$, $(X_3 X_1, Z_2 Z_2')$, $(X_1 X_2, Z_3 Z_3')$ are harmonic tetrads. Show that $(Z_1 Z_2 Z_3, Z_1' Z_2' Z_3')$ is an involutory hexad.

28. A non-singular conic is on the points of the triangle ABC and on the sides of $A'B'C'$. Show that there is another conic on the points of $A'B'C'$ and on the sides of ABC.

CHAPTER VII

THE NON-HOMOGENEOUS MESH GAUGE

IN the preceding chapters of this book the subject of Projective Geometry has been studied by what is usually known as the Synthetic method. This method consists in the direct deduction, by the ordinary processes of deductive logic, of the consequences of the initial propositions laid down, no special technique being evolved to simplify the process. Now although it is possible to continue the study of Projective Geometry by the Synthetic method far beyond the point we have reached, it is convenient to introduce here a new method which can be used along with it. This new method is, in fact, the application of algebraic language and symbolism to the concepts of Geometry in a way similar to that in which Descartes and those who followed him applied Algebra to elementary Geometry. There is, however, a very important difference between Algebraic Projective Geometry and what is usually known as Analytical or Coordinate Geometry; this difference will be noted and commented on in due course.

This chapter and the succeeding one are taken up with the foundations and elaboration of the new method; incidentally, the question of extension is reviewed, and a final and definitive initial proposition of extension laid down. The reader may find that the introduction to the algebraic method is long and at times a little uninteresting, but it has been thought better to study the question thoroughly, rather than to sacrifice rigour to interest.

7.1. Number-Systems

It is necessary to consider first some basic and rather abstract ideas which belong properly to the subject of Algebra, and have, at first sight, no bearing on what has gone before. In all that is to be said it is taken for granted that the reader is familiar with the concept of *complex numbers*, and with the elementary properties of complex numbers. That is to say, it is assumed that he knows not only what a complex number is, but also what is meant by the *sum*, *difference*, *product*, and *quotient* of two complex numbers.

7.11. Simple Number-Systems

We now ask the question: Are there any sets of numbers which can be selected from the whole domain of complex numbers such that, if a and b are any two numbers of the set, then

(1) $a+b$ is also a number of the set,

(2) $a-b$ is also a number of the set,

(3) $a\times b$ is also a number of the set,

and (4) $a\div b$ is also a number of the set, except, obviously, when b is zero?

Not every arbitrarily chosen set of numbers has all these properties; it is easy to prove that, for instance, the set consisting of all the positive integers does not possess them.

On the other hand, it is not difficult to prove that such sets can, in fact, be selected; the simplest of them all is the set consisting of the number zero only. In addition to this trivial example, the following are given; it is not difficult to show that they have the properties enumerated:

(1) the set consisting of all the rational numbers (positive and negative, zero included);

(2) the set consisting of all the numbers of the form $a+b\sqrt{2}$, where a and b are rational numbers;

(3) the set consisting of all the numbers of the form

$$a+b\sqrt{3}+c\sqrt{5}+d\sqrt{15},$$

where a, b, c, and d are rational numbers;

(4) the set consisting of all the real numbers;

(5) the set consisting of all the complex numbers.

The examples given do not exhaust all the possibilities nor even all the types of possible sets. In the sequel we shall require a name for these sets of numbers, and so we call them *systems of numbers*, or, more simply, *number-systems*.

Now of the numbers belonging to a number-system the following propositions are plainly true:

I. *If a and b are numbers of the system, then there is a unique number of the system which is their sum.*

II. *If* a, b, *and* c *are numbers of the system and* $a+b = c$, *then* $b+a = c$.

III. *There is a unique number*, 0, *of the system, such that if* a *is a number of the system, then* $a+0 = a$, *for every* a.

IV. *If* a, b, *and* c *are any three numbers of the system, then* $a+(b+c) = (a+b)+c$.

V. *If* a *and* b *are any two numbers of the system, then there is a unique number*, c, *of the system, such that* $a+c = b$.

VI. *If* a *and* b *are any numbers of the system, then there is a unique number of the system which is their product.*

VII. *If* a, b, *and* c *are numbers of the system, and if* $a\times b = c$, *then* $b\times a = c$.

VIII. *There is a unique number*, 1, *of the system, such that if* a *is a number of the system*, $a\times 1 = a$, *for every* a.

IX. *If* a, b, *and* c *are any three numbers of the system, then* $a\times(b\times c) = (a\times b)\times c$.

X. *If* a, b, *and* c *are any three numbers of the system, then* $a\times(b+c) = (a\times b)+(a\times c)$.

XI. *If* a *is any number of the system other than zero, and* b *is any number whatever of the system, then there is a unique number* c *of the system, such that* $a\times c = b$.

These propositions will be recognized as the fundamental propositions on which the whole of Algebra is based, although they do not ensure that all algebraic processes can be carried out in any particular number-system. For instance, if the number-system is that consisting of all rational numbers, the process of extracting the square root of a number will not always be possible.

7.12. Generalized Number-Systems

Having noticed the basic properties of number-systems, we now make use of them to introduce deeper ideas. The process which is undertaken is that of *generalizing* the concepts of *number* and *number-system*.

It will be observed that a number-system is a set of entities—numbers—whose members are interrelated by two types of relation, which may be called the *sum-relation* and the *product-relation*. For if $a+b = c$, the number c bears to the two numbers a and b the relation of being their sum; and if $a \times b = d$, the number d bears to the two numbers a and b the relation of being their product. These two relations are, respectively, the sum-relation and the product-relation. Of them the eleven propositions of the preceding section are true.

We now ask the question: Is it possible to find sets of entities, other than the number-systems we have been considering, interrelated by two types of relation, similar to the sum-relation and product-relation, and of which propositions similar to those of the preceding section are true? If it is possible to do so, then we are in a position to lay down the initial propositions of a completely abstract science, *of which the number-systems of* 7.11 *are representations*. If it is impossible to do so, then there is nothing to be gained by attempting to conceive of something more abstract than these number-systems; but it is our aim to show that a certain set of points on a line is (when suitable analogues of the sum-relation and product-relation have been specified) a representation of the more abstract concepts. Without waiting to find out if this is possible or not, we lay down the definition of an abstract number-system as follows.

DEFINITION. *A set of entities, identifiable by the symbols a, b, c, d,..., is said to be a representation of an abstract number-system if and only if the following propositions are true:*

I. *If a and b are any members of the set, then there is a unique member, c say, of the set, related to them by what is termed the S-relation. This S-relation is precisely specified for every pair a and b, and it is symbolized by writing $a+b = c$.*

II. *If a and b are any members of the set, and if $a+b = c$, then $b+a = c$.*

III. *There is a unique member, Z, of the set, such that if a is a member of the set, than $a+Z = a$, for every a.*

IV. *If a, b, and c are any members of the set, then*

$$a+(b+c) = (a+b)+c.$$

V. *If a and b are any two members of the set, then there is a unique member, c, such that $a+c = b$.*

VI. *If a and b are any members of the set, then there is a unique member, d say, related to them by what is termed the P-relation.*

This P-relation is precisely specified for every pair a and b, and it is symbolized by writing $a \times b = d$.

VII. *If a and b are any members of the set, and if* $a \times b = d$, *then* $b \times a = d$.

VIII. *There is a unique member, U, of the set, such that if a is a member of the set,* $a \times U = a$, *for every a.*

IX. *If a, b, and c are any three members of the set, then*

$$a \times (b \times c) = (a \times b) \times c.$$

X. *If a, b, and c are any three members of the set, then*

$$a \times (b + c) = (a \times b) + (a \times c).$$

XI. *If a is any member of the set other than Z, and b is any member of the set whatever, then there is a unique member, c, of the set such that* $a \times c = b$.

It is plain that these eleven propositions, stated as they are in abstract form, are a set of initial propositions, and since we already have an existence theorem for them in the simple number-systems of 7.11, they can, in fact, form the basis of a possible science. There, *entity* was interpreted as *number*, *S-relation* as *sum*, and *P-relation* as *product*. The way is now open to inquire whether a set of entities, apparently quite unconnected with the numbers of 7.11, satisfy these initial propositions when appropriate meanings are given to the terms *S*-relation and *P*-relation.

One further point may be noticed here. If we find such a set of entities, by what name are they to be called? There seems to be no reason for not calling them *number-systems*. It is, of course, open to the reader to reserve the term *number* for the entities of 7.11, and to invent some new term, such as 'abstract number', for other representations of 7.12. But no useful purpose is served by such a verbal distinction as this; it is rather like reserving the term animal for the familiar dog, cat, horse, etc., and using some other term to denote unfamiliar animals like the *Teratoscincus Scincus*. For the truth is that the *numbers* of 7.11 have the properties which Mathematics takes cognisance of, solely because they are, in fact, representations of the initial propositions of 7.12.

In the same way, no useful purpose is served by distinguishing between *S-relation* and *sum*, and between *P-relation* and *product*, and henceforward the simpler terms will be used.

7.2. A Geometrical Number-System

The object of this section is to show that there is a geometrical representation of the concepts of the last section. In order to do this it will be necessary (1) to choose a suitable set of entities, (2) to specify what is meant by the *sum* and *product* of a pair of these entities, and (3) to show that the eleven initial propositions are verified.

7.21. The Gauge-points and the Open Set on a Line

Let l be any line of the field, and let A_0, A_1, and A_ω be any three arbitrary but distinct points on it; in subsequent work it will be necessary to refer to these points, and so they are called the *gauge-points*.

The set of all points on the line l, *with the exception of the point* A_ω, will be called the *open set* on the line.

The open set on the line l is chosen as the set of entities for the representation of the initial propositions of 7.12.

7.22. The Sum and Product of a Pair of Points

DEFINITION. *The* **sum** *of any two points, A_x and A_y, of the open set is defined to be a mate of A_0 in the involution in which*

(1) *A_ω is a self-corresponding point, and*

(2) *A_x and A_y are a pair of mates.*†

DEFINITION. *The* **product** *of any two points of the open set, A_x and A_y, both of which are distinct from A_0, is defined to be the mate of A_1 in the involution in which*

(1) *A_0 and A_ω are a pair of mates, and*

(2) *A_x and A_y are a pair of mates.*

If either or both of the points A_x, A_y coincide with A_0 their **product** *is defined to be the point A_0.*

It should be noted that both of the terms *sum* and *product* of a pair of points are relative terms (see 4.221); they are meaningless unless gauge-points have been specified.

† The suffixes attached to letters labelling points of the open set are not necessarily numbers; any distinguishing suffix is sufficient for the purpose.

If A_z is the sum of A_x and A_y relative to the gauge-points A_0, A_1, and A_ω, the involution specifying this relation may be symbolized by the scheme $\begin{pmatrix} A_0 A_x A_\omega \\ A_z A_y A_\omega \end{pmatrix}$. The accompanying figure illustrates the definition; from it the reader will be able to elaborate a formal construction for determining the sum of two points relative to any gauge-points.

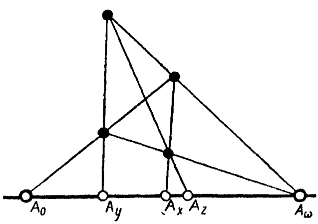


FIG. 45.

The involution specifying the product of two points may be symbolized by the scheme $\begin{pmatrix} A_1 A_x A_0 \\ A_z A_y A_\omega \end{pmatrix}$, where A_z is the product of A_x and A_y. The accompanying figure illustrates this.

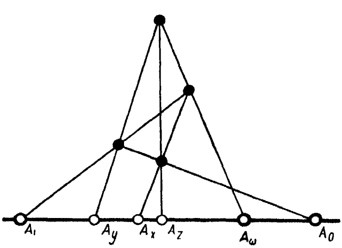


FIG. 46.

The symbolic propositions $A_x + A_y = A_z$ and $A_x \times A_y = A_z$ will henceforward bear their obvious meanings in terms of the above definitions, and the symbols $(A_x + A_y)$ and $(A_x \times A_y)$ will denote the *points* which are, respectively, the sum and product of A_x and A_y.

Before verifying the eleven initial propositions, it is necessary to prove two theorems about the sum-relation and product-relation; these theorems are given below, and the reader will find that the formal proof of them is an almost immediate consequence of the definitions.

7.221. THEOREM. *If*

$$A_x+A_a = A_p, \quad A_x+A_b = A_q, \quad A_x+A_c = A_r, \ldots,$$

or if $$A_a+A_x = A_p, \quad A_b+A_x = A_q, \quad A_c+A_x = A_r, \ldots,$$

then $$(A_\omega A_0 A_a A_b A_c \ldots) \sim (A_\omega A_x A_p A_q A_r \ldots).$$

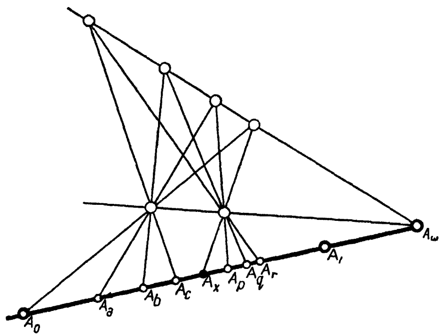


FIG. 47.

7.222. THEOREM. *If*

$$A_x \times A_a = A_p, \quad A_x \times A_b = A_q, \quad A_x \times A_c = A_q, \ldots,$$

or if $$A_a \times A_x = A_p, \quad A_b \times A_x = A_q, \quad A_c \times A_x = A_r, \ldots,$$

then $$(A_\omega A_0 A_1 A_a A_b A_c \ldots) \sim (A_\omega A_0 A_x A_p A_q A_r \ldots).$$

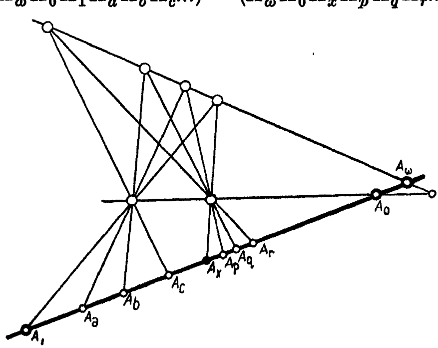


FIG. 48.

7.23. Verification of the Initial Propositions

The following eleven theorems show that the eleven initial propositions of 7.12 are verified. Most of them are extremely simple.

7.2301. THEOREM. *If A_x and A_y are any two points of the open set, then there is a unique point of the open set which is their sum.*

7.2302. THEOREM. *If A_x and A_y are any two points of the open set, and if $A_x+A_y = A_z$, then $A_y+A_x = A_z$.*

7.2303. THEOREM. *There is a unique point of the open set, namely A_0, such that if A_x is any point of the open set,*

$$A_x+A_0 = A_x.$$

These three theorems are simple consequences of the definition of the term *sum*.

7.2304. THEOREM. *If A_x, A_y, and A_z are any three points of the open set, then $A_x+(A_y+A_z) = (A_x+A_y)+A_z$.*

Suppose that

$$A_x+A_y = A_u, \quad A_y+A_z = A_v, \quad \text{and } A_u+A_z = A_s.$$

Then since $A_y+A_z = A_v$ and $A_u+A_z = A_s$, by 7.221,

$$(A_\omega A_0 A_u A_y) \sim (A_\omega A_z A_s A_v). \tag{1}$$

Since also $A_x+A_y = A_u$, by the definition of the term sum,

$$(A_\omega A_0 A_u A_y A_x) \sim (A_\omega A_u A_0 A_x A_y);$$

and so, from (1), it follows that

$$(A_\omega A_u A_0 A_x) \sim (A_\omega A_z A_s A_v). \tag{2}$$

But since $A_u+A_z = A_s$,

$$(A_\omega A_0 A_s A_u A_z) \sim (A_\omega A_s A_0 A_z A_u). \tag{3}$$

This last involution is plainly identical with the projectivity (2); that is to say, A_v and A_x are mates in the involution (3).

Hence $A_x+A_v = A_s$, or

$$A_x+(A_y+A_z) = (A_x+A_y)+A_z,$$

and this proves the theorem.

7.2305. THEOREM. *If A_x and A_y are any two points of the open set, then there is a unique point A_z, also of the open set, such that $A_x+A_z = A_y$.*

This theorem is a simple consequence of the definition of the term *sum.*

7.2306. THEOREM. *If A_x and A_y are any two points of the open set, then there is a unique point of the open set which is their product.*

7.2307. THEOREM. *If A_x and A_y are any two points of the open set, and if $A_x \times A_y = A_z$, then $A_y \times A_x = A_z$.*

7.2308. THEOREM. *There is a unique point of the open set, namely A_1, such that if A_x is any point of the open set, then $A_x \times A_1 = A_x$.*

The last three theorems are simple consequences of the definition of the term *product*; but in verifying them the reader should not omit to notice the second half of the definition.

7.2309. THEOREM. *If A_x, A_y, and A_z are any three points of the open set, then $A_x \times (A_y \times A_z) = (A_x \times A_y) \times A_z$.*

When one or more of the points mentioned coincides with A_0, the theorem is trivial. When this is not so, the proof proceeds in a similar way to that of 7.2304, save that 7.222 is used in place of 7.221. The details are left to the reader.

7.2310. THEOREM. *If A_x, A_y, and A_z are any points of the open set, then $A_x \times (A_y+A_z) = (A_x \times A_y)+(A_x \times A_z)$.*

The theorem is trivial if A_x coincides with A_0, and so this case is at once excluded.

Suppose then that $A_y+A_z = A_r$, $A_x \times A_y = A_s$, $A_x \times A_z = A_t$, and $A_x \times A_r = A_u$; then by 7.222

$$(A_\omega A_0 A_1 A_y A_z A_r) \sim (A_\omega A_0 A_x A_s A_t A_u). \qquad (1)$$

But since $A_y+A_z = A_r$,

$$(A_\omega A_0 A_r A_y A_z) \sim (A_\omega A_r A_0 A_z A_y),$$

and hence, from (1),

$$(A_\omega A_0 A_u A_s A_t) \sim (A_\omega A_u A_0 A_t A_s).$$

That is to say, $A_s + A_t = A_u$, or

$$A_x \times (A_y + A_z) = (A_x \times A_y) + (A_x \times A_z),$$

and this proves the theorem.

7.2311. THEOREM. *If A_x is any point of the open set other than the point A_0, and A_y is any point whatever of the open set, then there is a unique point A_z, also of the open set, such that $A_x \times A_z = A_y$.*

This theorem is a simple consequence of the definition of the term *product*.

The eleven propositions of 7.12 are thus verified of the open set of points on a line, when the sum-relation and product-relation are interpreted according to the definitions which have been given. This extremely important result is worth stating as a formal theorem.

7.24. THEOREM

The open set of points on a line is a representation of the abstract number-system of 7.12, *when the sum-relation and product-relation are interpreted according to the definitions in* 7.22.

EXAMPLES

1. Φ is any non-singular conic, and P_0, P_1, and P_ω are any three distinct points on it. If these are the *gauge-points* on Φ, and if the terms *open set, sum of two points, product of two points* are defined analogously to the corresponding terms for the line, prove the analogues of Theorems 7.221 and 7.222.

2. Hence show that the open set of points on a non-singular conic is a representation of the abstract number-system of 7.12.

3. Prove analogues of Theorems 7.2304, 7.2308, and 7.2310, without appealing to the theorems proved in Example 1.

7.3. Extension

The preceding sections have shown that the open set of points on any line of the field is a representation of the abstract number-system, and this result enables us to lay down a final and definitive initial proposition of extension. It is essential to do this before introducing the algebraic method; it is moreover convenient to dispose finally of the question of extension at this point. In order to do this it is necessary to introduce the notion of *isomorphism*.

7.31. Isomorphism

DEFINITION. *Two representations of the abstract number-system are said to be isomorphic if and only if to every entity in one there corresponds one and only one entity in the other, and vice versa, in such a way that if A and B are two entities in one which correspond respectively to A' and B' in the other, then*

(i) $A+B$ *corresponds to* $A'+B'$, *and*

(ii) $A\times B$ *corresponds to* $A'\times B'$.

It will be seen from this definition that if two number-systems are isomorphic, there is complete parallelism between them, in the sense that if any operations consisting of successive sums and products are performed on the entities of one, and if the corresponding operations are performed on the corresponding entities of the other, the results will be corresponding entities.

7.311. THEOREM. *If l and m are any two distinct lines of the field, and on them gauge-points L_0, L_1, L_ω, and M_0, M_1, M_ω, respectively, are chosen, then the open sets on l and m are isomorphic.*

Consider the projectivity in which

$$l(L_0\,L_1\,L_\omega) \sim m(M_0\,M_1\,M_\omega);$$

by means of this projectivity, every point of the open set on l is made to correspond to a unique point of the open set on m, and vice versa. The first condition of isomorphism is therefore fulfilled.

Let L_x and L_y be any two points on l, and suppose that $L_x+L_y = L_p$ and $L_x\times L_y = L_q$. Further, suppose

$$l(L_0\,L_1\,L_\omega\,L_x\,L_y\,L_p\,L_q) \sim m(M_0\,M_1\,M_\omega\,M_u\,M_v\,M_r\,M_s).$$

Then since

$$l(L_\omega\,L_0\,L_p\,L_x\,L_y) \sim l(L_\omega\,L_p\,L_0\,L_y\,L_x)$$

and $$l(L_\omega\,L_0\,L_1\,L_q\,L_x\,L_y) \sim l(L_0\,L_\omega\,L_q\,L_1\,L_y\,L_x),$$

it follows that

$$m(M_\omega\,M_0\,M_r\,M_u\,M_v) \sim m(M_\omega\,M_r\,M_0\,M_v\,M_u)$$

and $$m(M_\omega\,M_0\,M_1\,M_s\,M_u\,M_v) \sim m(M_0\,M_\omega\,M_s\,M_1\,M_v\,M_u).$$

That is to say, $M_u+M_v = M_r$ and $M_u+M_v = M_s$. Hence the second and third conditions for isomorphism are fulfilled, and the theorem is proved.

7.32. Isomorphism with Simple Number-Systems

The open set of points on any line of the field being a representation of the abstract number-system, the question at once arises: Is this set of points isomorphic with any of the simple number-systems of 7.11? A precise answer to this question would settle the question of extension at once; but it is impossible to give a precise answer to it, for the initial propositions of Projective Geometry are compatible with its being isomorphic with any one of a variety of simple number-systems. It is therefore necessary to lay down as an *initial proposition* the isomorphism of the open set with some chosen simple number-system. This will be an initial proposition of extension.

For our purposes it is simplest to choose as the initial proposition of extension the isomorphism of the open set with the set of all complex numbers. Some reasons why this choice is made are given below.

If, in studying Algebra, we confine ourselves to the *real numbers*, that is to say, if we study the Algebra of the Real Number-System, it is necessary to say that many problems have therein no solution. For instance, the quadratic equation $x^2+x+1 = 0$ has no roots in the system. It is therefore necessary to be constantly enumerating exceptions, and stating conditions. But if we study the Algebra of the Complex Number-System, it is easily proved that therein *every* algebraic equation has a root, and so, roughly, that every significant elementary problem has a solution. Moreover, the history of Mathematics shows that the introduction of complex number into Algebra enabled mathematicians to increase considerably their knowledge of the Real Number-System.

In just the same sort of way, if we agreed to lay down as an initial proposition of extension the isomorphism of the open set with e.g. the real number-system, we should find that some problems were without a solution. Not every line, for instance, would have points in common with an arbitrary non-singular conic. In consequence we should have to hedge our theorems about with conditions, and progress would be retarded. But if we agree to lay down that the open set shall be isomorphic with the set of all complex numbers, we may confidently expect that all elementary problems in Geometry will have a solution, and that there will be no necessity to enumerate irritating exceptions. Moreover, as in Algebra, the study of the consequences of this initial proposition of extension will give us knowledge of other systems which we could not obtain otherwise.

7.33. Initial Proposition of Extension

The open set of points on a line is isomorphic with the set of all complex numbers.

As before, it will be necessary to prove that this particular initial proposition of extension is compatible with the initial propositions laid down up to this. This is done by proving an existence theorem; we show that there is a representation of the other initial propositions of which this initial proposition is also true. As usual, we choose the Algebraic Representation for this purpose.

7.331. Verification in the Algebraic Representation

THEOREM. *In the Algebraic Representation the open set of points on a line is isomorphic with the complex number-system.*

Let (x_0, y_0, z_0), (x_1, y_1, z_1), and $(x_\omega, y_\omega, z_\omega)$ be three distinct collinear points in the Algebraic Representation, so that numbers p and q exist such that $x_1 = px_0+qx_\omega$, $y_1 = py_0+qy_\omega$, and $z_1 = pz_0+qz_\omega$. Plainly there is no loss in generality if it be supposed that p and q are unity.

These three points are taken as the gauge-points A_0, A_1, A_ω, and any other point on the line is $(\lambda x_0+\mu x_\omega, \lambda y_0+\mu y_\omega, \lambda z_0+\mu z_\omega)$.

If now we take the points A_0 and A_ω as *reference* points, the coordinates of any point on the line relative to these reference points are (λ, μ). In particular, the coordinates of A_0, A_1, and A_ω are, respectively, $(1,0)$, $(1,1)$, and $(0,1)$, and any point of the open set has coordinates $(1,k)$, where k is any complex number.

The open set may therefore be put into correspondence with the complex number-system by making the *point* $(1,k)$ correspond to the *number* k, and vice versa. The first condition for isomorphism is therefore fulfilled.

Further, if the two points (λ,μ) and (ρ,σ) are mates in an involution, then numbers a, b, and c exist such that

$$a\lambda\rho+b(\lambda\sigma+\mu\rho)+c\mu\sigma = 0,$$

where $ac-b^2 \neq 0$.

Firstly, suppose that (i) A_ω is a self-corresponding point, and (ii) the points $(1,x)$ and $(1,y)$ are mates in this involution. It is easily verified that these conditions entail (i) $c = 0$, and therefore $b \neq 0$, and (ii) $a = -b(x+y)$. The involution is therefore specified by the equation

$$-(x+y)\lambda\rho+(\lambda\sigma+\mu\rho) = 0,$$

and the mate of $(1,0)$ in this is $(1,x+y)$. Hence the sum of the two points $(1,x)$ and $(1,y)$ is $(1,x+y)$; the second condition for isomorphism is therefore fulfilled.

Secondly, suppose that (i) A_0 and A_ω are mates, and (ii) the points $(1,x)$ and $(1,y)$ are mates in the involution. It may be verified that these conditions entail (i) $b = 0$, and therefore $ac \neq 0$, and (ii) $a = -cxy$.

The involution is therefore specified by the equation $-xy\lambda\rho+\mu\sigma = 0$, and the mate of $(1,1)$ in this is $(1,xy)$. Hence the product of the two points $(1,x)$ and $(1,y)$ is $(1,xy)$, and therefore the third condition for isomorphism is fulfilled. The theorem is therefore proved.

7.332. Duality. In order to maintain the Principle of Duality it is necessary, as usual, to prove the dual of the initial proposition of extension. The reader should find no difficulty in doing this, once he has, by dualizing, elaborated definitions of *gauge-lines on a point*, *sum of two lines*, *product of two lines*. It is then only necessary to prove that the open set of lines on a point is isomorphic with the open set of points on a line.

7.4. The Algebraic Method

We are now in a position to introduce the algebraic method, and the first step in this is to attach algebraic labels to the points of the field. In the first six chapters of this book it was found necessary, as in every treatise on Geometry, to attach labels to the various points and lines which were considered in any theorem. Their function was simply to identify the points and lines in question, and so alphabetical labels were sufficient for the purpose. Moreover, the practice of labelling points never pretended to be exhaustive; only those were labelled which were relevant to the theorem considered.

It is now our purpose, however, to undertake a much more comprehensive system of labelling. In this every point of the field will receive a label, and this label will be not merely an empty symbol useful for purposes of identification only but one which gives certain information about the relations of the point to other points of the field.

7.41. Labelling the Points on a Line

We begin by assigning a simple form of label to the points on a line. Let l be any line, and on this, let A_0, A_1, and A_ω be the gauge-points. Now since the open set on this line is isomorphic with the complex number-system, and since in this isomorphism the point A_0 corresponds† to the number 0, and A_1 to the

† This correspondence of A_0 and A_1 to 0 and 1 respectively is not a consequence of Theorem 7.331, but of Theorems 7.2303 and 7.2308.

number 1, it is only natural to assign the label A_x to the point which corresponds to the number x. In this way, to every point of the open set is attached a label of the form A_x. Moreover, because of the isomorphism,

$$A_x + A_y = A_{x+y}$$

and

$$A_x \times A_y = A_{xy}.$$

Two important features of this system of labels must be noticed. First, the labels have nothing to do with the notion of *distance*, for the simple reason that distance has not yet been defined; it is, however, worth remarking that number has been introduced into Geometry without appeal to the notion of distance. Secondly, the label attached to any particular point is entirely dependent on the choice of the gauge-points, and it will, in general, vary when the gauge-points are varied. In other words, the label attached to any particular point is its label *relative to the gauge-points*.

7.411. The Determination of Particular Labels. The reader may feel that at this point two questions call for an answer. They are: (1) Given any number, z say, how is it possible to determine the point whose label is A_z? (2) Given any arbitrary point, how is it possible to determine what label is attached to it?

(1) If z is a rational number, the point A_z may be determined by a finite number of repetitions of the constructions for sum and product. For by successive applications of the construction for the sum of two points, the points A_2, A_3, A_4, A_5, A_6,... can be constructed, and if $z = m/n$, where m and n are integers, A_z is easily found from these.

If z is not a rational number, it may be possible to give a construction for determining A_z if constructions using conics are employed. But conics are not the only type of locus, and though we have not done so, it is possible to define more complicated loci and to study their properties, and with their aid to determine all the other points. The answer to the question is therefore: With the knowledge at present at our disposal it may not be possible to determine the point A_z.

(2) To the questioner who asks 'What is the label of this

particular point?' the answer must be given: 'If you will tell me exactly which point you mean, i.e. give me a construction by means of which I can be certain which point you are talking about, I will tell you its label.' Any construction which will identify it will give its relation to the gauge-points, and when this is known its label is determinable.

7.42. Dropping the Pilot

In the process of labelling the points of the open set we have used labels consisting of a letter of the alphabet, to which a numerical suffix was attached. It is plain, however, that the important and significant part of this combination is the suffix, and that the letter plays no useful role whatever. This being so, the useless letter may without loss be omitted: instead of speaking of the points A_0, A_1, $A_{\frac{1}{2}}$, A_x, etc., we may therefore for the future speak of the points 0, 1, $\frac{1}{2}$, x, etc. The effect of this technical simplification is that every point of the open set has now a number-label instead of a suffixed letter-label.

At the same time we may change the label of the gauge-point A_ω to ω. More will be said about this point later, but the reader has probably realized that in some sense it corresponds to the improper number ∞. For the present we shall call it the *unlabelled point*, not because it has no label at all, but because its label is altogether outside the numerical labelling system which has been elaborated.

7.43. Complex Points

It may occur to the reader to ask what significance is to be attached to points whose labels are 'strictly complex numbers', i.e. numbers of the form $x+iy$, where x and y are real numbers, $i^2 = -1$, and y is not zero. The question may be put in another way: Is there any essential difference between points whose labels are *real* numbers, and those whose labels are not real numbers? The answer is that there is no essential difference whatever; in fact, if the gauge-points are changed in a suitable way, points which before had real-number-labels can be made to have labels which are not real numbers.

The preoccupation of mathematicians in past years with the Geometry which is apparently applicable to the *physical* space in which we live led them to make a sharp, but unnecessary and misleading distinction between so-called *real* points and so-called *imaginary* points in the *conceptual* space studied in Analytical Geometry. The method by which number has been introduced into Projective Geometry, and the nature of Projective Geometry itself, should convince the reader that points with so-called imaginary labels are on precisely the same footing as points with real labels. The only difference between them is in their labels; this difference has as much significance as the difference between points whose labels are odd integers and those whose labels are even integers. The precise label which a point has is entirely dependent on the choice of the gauge-points.

7.44. Labelling the Field

Having labelled the points of a single line we now amplify the process in order to attach labels to other points of the field.

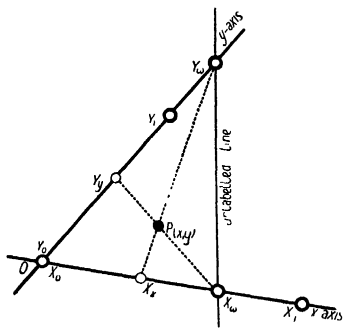


FIG. 49.

Let l and m be any two distinct lines of the field, and let O be their common point.

On l choose three gauge-points X_0, X_1, and X_ω in such a way that X_0 coincides with O. Similarly, on m choose three gauge-points Y_0, Y_1, and Y_ω in such a way that Y_0 coincides with O.

Suppose the open sets on these lines labelled in the way described above.

Let P be any point of the field not on the line $X_\omega Y_\omega$; let x be the label of the point on l which is also on PY_ω; let y be the label of the point on m which is also on PX_ω.

Then to the point P is given the *double number-label* (x, y). Points on the line l will have labels of the form $(0, y)$, and points on the line m will have labels of the form $(x, 0)$; these replace the temporary labels first attached.

It is clear that two distinct points of the field have different labels, and that points with different labels are distinct.

The points of the line $X_\omega Y_\omega$ do not receive any label in this system, and for this reason the line is termed the *unlabelled line*.

This system of giving double number-labels to all the points of the field save those of the line $X_\omega Y_\omega$ is called the *non-homogeneous mesh system*, or the *non-homogeneous mesh gauge*. The significance of the qualifying word *non-homogeneous* will become clearer in due course.

There are certain terms connected with the non-homogeneous mesh gauge which are in normal use; these are defined below.

DEFINITIONS. *The double number-label attached to any point is called the coordinates of the point; the first of the two numbers is called the x-coordinate; the second, the y-coordinate.*

The lines OX_ω, OY_ω are termed the axes of coordinates; the first is the x-axis; the second, the y-axis.

The point O is called the origin of coordinates.

7.441. The Unlabelled Line. The non-homogeneous mesh gauge leaves the line $X_\omega Y_\omega$ unlabelled, nor can this line be included by any extension of the labelling system as it stands. This line is sometimes spoken of as the *ideal line*, or the *vanishing line*, and there are other terms in use. None of them is adopted here, since they imply that this particular line is something rather special in itself; the truth is, however, that it is in no way special. It is just as much a line of the field as any other line; it is the labelling system which breaks down on this line, and with another choice of axes and gauge-points, another line would be thus apparently singular. It is only singular relative to the labelling system; it is not singular in itself.

The fact that the labelling system is thus defective has important consequences later, but it makes it desirable that a system free from this defect should be devised. This will be done in the next chapter. It is convenient, however, to make some use of this labelling system, even though it is defective; and this is done in the following sections.

7.5. The Algebraic Specification of the Projectivity

Since the work to be done on the projectivity is concerned only with the points on a single line, it is easier to use the single number-labels of 7.41, relative to gauge-points on the line, rather than the double number-labels of 7.44.

The problem being attacked may be stated in general terms thus: If there is a projectivity between a range on a line and a range on another or the same line, and if, in this projectivity, points whose labels are x, y, z, t,..., correspond to points whose labels are x', y', z', t',..., what *algebraic* relation connects x with x', y with y', and so on? In the language of analysis, what function of x is x'?

Four simple theorems, three of which are restatements of known results, are prefixed to the theorem which answers this question.

7.51. THEOREM

If l and m are two distinct lines of the field, and if gauge-points A_0, A_1, A_ω are chosen on l, and B_0, B_1, B_ω on m, then

$$l(A_0 A_1 A_\omega A_x) \sim m(B_0 B_1 B_\omega B_x)$$

for every x.

This is an immediate consequence of the definition of isomorphism.

7.52. THEOREM

If l and m are the same or distinct lines of the field, there is a projectivity in which

$$l(A_\omega A_0 A_x A_y A_z \ldots) \sim m(B_\omega B_t B_{x+t} B_{y+t} B_{z+t} \ldots).$$

Case 1. If l and m coincide, and the gauge-points also coincide, this theorem is a restatement of 7.221.

Case 2. If l and m do not coincide, it is a consequence of Case 1 and the preceding theorem.

Case 3. If l and m coincide, but the gauge-points do not, it is a consequence of Case 2 and the preceding theorem.

7.53. THEOREM

If l and m are the same or distinct lines of the field, there is a projectivity in which

$$l(A_\omega A_1 A_x A_y A_z \ldots) \sim m(B_\omega B_t B_{tx} B_{ty} B_{tz} \ldots),$$

provided $t \neq 0$.

Case 1. If l and m coincide, and the gauge-points coincide, this theorem is a restatement of 7.222.

Cases 2 *and* 3. As in the preceding theorem.

7.54. THEOREM

If l and m are the same or distinct lines of the field, then there is a projectivity in which

$$l(A_\omega A_0 A_1 A_t A_x A_y A_z \ldots) \sim m(B_0 B_\omega B_t B_1 B_{t/x} B_{t/y} B_{t/z} \ldots),$$

provided $t \neq 0$.

Case 1. If l and m coincide, and the gauge-points coincide, this is the definition of the product of two points. The projectivity is, in fact, the involution specifying the product.

Cases 2 *and* 3. As in the previous theorems.

We are now in a position to prove the theorem to which this preliminary work has been leading.

7.55. THEOREM

If A_x, A_y, A_z, and A_t are four distinct collinear points, all of which are distinct from A_ω, and if $B_{x'}$, $B_{y'}$, $B_{z'}$, and $B_{t'}$ are four other distinct collinear points, all of which are distinct from B_ω, then the necessary and sufficient condition that

$$l(A_x A_y A_z A_t) \sim m(B_{x'} B_{y'} B_{z'} B_{t'})$$

is that four numbers a, b, c, and d should exist such that

$$ad-bc \neq 0,$$

and

$$\left.\begin{aligned} x' &= \frac{ax+b}{cx+d}, & y' &= \frac{ay+b}{cy+d}, \\ z' &= \frac{az+b}{cz+d}, & t' &= \frac{at+b}{ct+b}. \end{aligned}\right\} \qquad (1)$$

The sufficiency of the condition is proved first. Suppose then that $ad-bc \neq 0$, and that the four equations (1) are satisfied.

By 7.52, $l(A_x A_y A_z A_t) \sim l(A_\alpha A_\beta A_\gamma A_\delta)$, where

$$\alpha = x+\frac{d}{c}, \quad \beta = y+\frac{d}{c}, \quad \gamma = z+\frac{d}{c}, \quad \delta = t+\frac{d}{c}.$$

By 7.53, $l(A_\alpha A_\beta A_\gamma A_\delta) \sim l(A_\epsilon A_\zeta A_\eta A_\theta)$, where

$$\epsilon = \frac{\alpha c^2}{(bc-ad)}, \quad \zeta = \frac{\beta c^2}{(bc-ad)}, \quad \text{etc.}$$

By 7.54, $l(A_\epsilon A_\zeta A_\eta A_\theta) \sim l(A_{1/\epsilon} A_{1/\zeta} A_{1/\eta} A_{1/\theta})$.

By 7.52, $l(A_{1/\epsilon} A_{1/\zeta} A_{1/\eta} A_{1/\theta}) \sim l(A_\lambda A_\mu A_\nu A_\pi)$, where

$$\lambda = \frac{1}{\epsilon}+\frac{a}{c}, \quad \mu = \frac{1}{\zeta}+\frac{a}{c}, \quad \text{etc.}$$

Hence $\qquad l(A_x A_y A_z A_t) \sim l(A_\lambda A_\mu A_\nu A_\pi).$

But $\qquad \lambda = \frac{\epsilon a+c}{\epsilon c} = \frac{\alpha ac+bc-ad}{\alpha c^2} = \frac{ax+b}{cx+d} = x',$

and, similarly, $\mu = y'$, $\nu = z'$, $\pi = t'$.

Hence $l(A_x A_y A_z A_t) \sim l(A_{x'} A_{y'} A_{z'} A_{t'})$, and so the condition is sufficient.

It will be noticed that this proof is only valid when $c \neq 0$; when $c = 0$, the proof is as follows.

$$l(A_x A_y A_z A_t) \sim l(A_\alpha A_\beta A_\gamma A_\delta), \quad \text{where} \quad \alpha = x+\frac{b}{a}, \text{ etc.,}$$

$$l(A_\alpha A_\beta A_\gamma A_\delta) \sim l(A_\epsilon A_\zeta A_\eta A_\theta), \quad \text{where} \quad \epsilon = \frac{a\alpha}{d}, \text{ etc.,}$$

that is to say, $l(A_x A_y A_z A_t) \sim l(A_{x'} A_{y'} A_{z'} A_{t'})$.

We have assumed throughout the proof that the same base and gauge-points are retained throughout; the transference to other gauge-points on the same or a different base is effected in the obvious manner.

The necessity of the condition follows at once from this by *reductio ad absurdum*. For suppose that

$$l(A_x A_y A_z A_t) \sim l(A_{x'} A_{y'} A_{z'} A_{t'}),$$

and that a, b, c, and d are four numbers satisfying the equations

$$x' = \frac{ax+b}{cx+d}, \quad y' = \frac{ay+b}{cy+d}, \quad z' = \frac{az+b}{cz+d}.$$

Suppose also that $\qquad t'' = \frac{at+b}{ct+d}.$

Then, by what has just been proved,

$$l(A_x A_y A_z A_t) \sim l(A_{x'} A_{y'} A_{z'} A_{t''}),$$

that is to say, $A_{t'}$ and $A_{t''}$ coincide. Hence

$$t' = \frac{at+b}{ct+d},$$

and it is clear that $ad-bc \neq 0$.

This theorem may be expressed more fully as follows.

7.56. Theorem

If a projectivity between two ranges on the same or different ranges is symbolized by

$$l(A_\omega A_p A_x A_y A_z A_t \ldots) \sim m(B_{p'} B_\omega B_{x'} B_{y'} B_{z'} B_{t'} \ldots),$$

then numbers a, b, c, and d exist such that $ad-bc \neq 0$, *and*

$$x' = \frac{ax+b}{cx+d}, \quad y' = \frac{ay+b}{cy+d}, \quad z' = \frac{az+b}{cz+d},$$

$$t' = \frac{at+b}{ct+d}, \quad p' = \frac{a}{c}, \quad p = \frac{-d}{c}.$$

Conversely, if any three of the last six equations are satisfied, then

$$l(A_\omega A_p A_x A_y A_z A_t ...) \sim m(B_{p'} B_\omega B_{x'} B_{y'} B_{z'} B_{t'} ...),$$

and the other three equations are also satisfied.

The details of the proof of this theorem are left to the reader. The value of p' is found by tracing the points corresponding to A_ω in the first range through the various projectivities of the last theorem. p is found in a similar way.

7.57. The Equation of a Projectivity

The theorems just proved show that, in general, if x is the label of any point of a range, and x' is the label of the point corresponding to it in a projectivity, then

$$x' = \frac{ax+b}{cx+d}, \tag{1}$$

or

$$cxx'-ax+dx'-b = 0. \tag{2}$$

Corresponding to every such equation there is a projectivity, and corresponding to every projectivity there is an equation of this type. We therefore speak of it as the *equation of the projectivity*. It is plain that it may also be written in the form

$$x = \frac{-dx'+b}{cx'-a}. \tag{3}$$

The equations (1) and (3) show that x is a one-valued function of x', and that x' is a one-valued function of x. It is interesting to notice that the equation of the projectivity is the most general equation between two variables, such that each is a one-valued analytic function of the other.

7.58. Self-corresponding Points

If the two ranges between which there is a projectivity are cobasal, and if the same gauge-points are chosen for both, the self-corresponding points of the projectivity are those points whose labels satisfy the quadratic equation

$$cx^2+(d-a)x-b = 0.$$

This is obtained by putting $x = x'$ in (2). Exceptional cases arise when the unlabelled point is one (or both) of the self-corresponding points of the projectivity. The conditions for this are left as examples for the reader.

EXAMPLES

1. Determine the necessary and sufficient algebraic condition that a projectivity shall be an involution.

2. Determine the necessary and sufficient condition that A_ω shall be a self-corresponding point of a projectivity. (The following 'proof' is insufficient: The necessary and sufficient condition that one root of the quadratic equation $cx^2+(d-a)x-b = 0$ should be 'infinity' is that $c = 0$; hence this is the required condition that A_ω be a self-corresponding point.)

3. Determine the necessary and sufficient condition that the two self-corresponding points of a projectivity shall coincide. Hence prove algebraically that the self-corresponding points of an involution are always distinct.

4. What is the equation of the projectivity $l(A_\alpha A_\beta A_\gamma) \sim l(A_\beta A_\gamma A_\alpha)$? Determine the self-corresponding points.

5. What is the equation of the involution

$$l(A_\omega A_0 A_\alpha A_\beta) \sim l(A_\omega A_0 A_\beta A_\alpha)?$$

Show that $\beta+\alpha = 0$.

6. Find the harmonic conjugate of A_s relative to A_x and A_y.

7. Prove algebraically that if A_s and A_t are harmonic conjugates relative to A_x and A_y, then A_x and A_y are harmonic conjugates relative to A_s and A_t.

7.6. Loci

The general problem of which particular instances are studied in this section may be stated thus: Given a locus in the field, what relation exists between the non-homogeneous coordinates (relative to some axes and gauge-points) x and y of any labelled point of the locus? Conversely, given a relation between two variables x and y, what is the locus of points whose coordinates satisfy this relation?

The relation between the coordinates x and y is usually in the form of a single equation, $f(x,y) = 0$, but this is by no means necessary.

DEFINITION. *If there is an equation $f(x,y) = 0$, such that*

(i) *the coordinates (x,y) of every labelled point of a certain locus satisfy this equation, and*

(ii) *every point whose coordinates (x,y) satisfy this equation is a point of the locus,*

then the equation $f(x,y) = 0$, or any equivalent of it, is termed the equation of the locus, and the point (x,y) is said to satisfy the equation.

DEFINITION. *If there are two equations, $x = x(t)$ and $y = y(t)$, such that*

(i) *the coordinates (x,y) of every labelled point of a certain locus satisfy these equations, and*

(ii) *every point whose coordinates (x,y) satisfy these equations is a point of the locus,*

then the equations are termed the parametric equations of the locus, and the point (x,y) of the locus is said to correspond to the value t of the parameter.

In what follows, the only loci considered are the line and the conic. A general theorem is prefixed.

7.61. THEOREM

If $f(x,y) = 0$ and $g(x,y) = 0$ are the equations of two loci, then those **labelled** *points which are common to the two loci are points whose coordinates are solutions of the simultaneous equations $f(x,y) = 0$, $g(x,y) = 0$, and vice versa.*

This theorem should not require formal proof.

7.62. The Equation of the Line

THEOREM. *The equation of any line of the field, other than the unlabelled line, is of the form*

$$lx+my+n = 0,$$

where not both of l and m are zero.

Conversely, any locus whose equation is of this form is a line.

(1) Consider any line on Y_ω, other than the unlabelled line, and suppose it is also on the point $(k,0)$. From the definition of the non-homogeneous mesh gauge it is plain that the x-coordinate of every point on this line is k, and that every

point whose x-coordinate is k is on this line. Hence the equation of the line is $x-k = 0$, and this is of the form specified.

(2) Similarly, any line on X_ω other than the unlabelled line has an equation of the form specified.

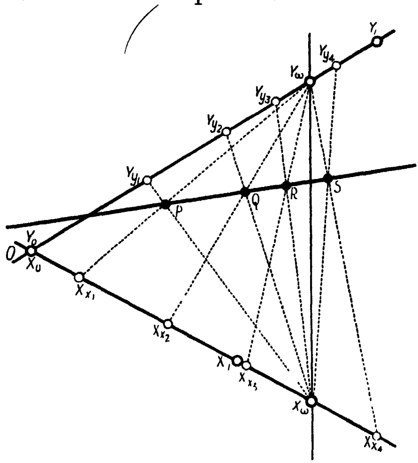


FIG. 50.

(3) Let p be any line not on X_ω or Y_ω, and consider the *perspectivities*

$$p(PQRS...) \overset{Y_\omega}{\sim} l(X_{x_1} X_{x_2} X_{x_3} X_{x_4} ...),$$

$$p(PQRS...) \overset{X_\omega}{\sim} m(Y_{y_1} Y_{y_2} Y_{y_3} Y_{y_4} ...).$$

It follows at once that (i) the coordinates of P, Q, R, S,... are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)..., and (ii)

$$l(X_{x_1} X_{x_2} X_{x_3} X_{x_4} ...) \sim m(Y_{y_1} Y_{y_2} Y_{y_3} Y_{y_4} ...).$$

Moreover, in this projectivity, the point X_ω corresponds to the point Y_ω, and so numbers a and b exist such that

$$x_1 = ay_1+b, \qquad x_2 = ay_2+b, \quad \text{etc.},$$

and, in general, if (x, y) are the coordinates of any point on p,

$$x = ay+b.$$

Further, suppose that (ξ, η) is any point such that

$$\xi = a\eta+b;$$

then clearly X_ξ and Y_η are corresponding points in the above projectivity, and so the point is on p. Hence the equation of the line is

$$x-ay-b = 0,$$

and this is of the form specified.

This proves the first part of the theorem, namely, that every line other than the unlabelled line has an equation of the form $lx+my+n = 0$; it does not show that every locus whose equation is of this form is a line.

Consider then any locus whose equation is $lx+my+n = 0$; let (x_1, y_1) and (x_2, y_2) be any two distinct points on this locus. Let p be the line on these two points, so that the equation of p is (say) $l'x+m'y+n' = 0$. It follows that

$$lx_1+my_1+n = 0,$$

$$lx_2+my_2+n = 0,$$

$$l'x_1+m'y_1+n' = 0,$$

$$l'x_2+m'y_2+n' = 0.$$

From these equations it follows at once that

$$\frac{l'}{l} = \frac{m'}{m} = \frac{n'}{n},$$

and this implies that $lx+my+n = 0$ and $l'x+m'y+n' = 0$ are equations of the same locus. Hence every equation of the form $lx+my+n = 0$ is the equation of a line.

7.621. Particular Cases. The general equation of the line found in the last section may be put into a more convenient form in certain particular cases. These are given in the following theorems.

THEOREM. *The equation of the line on the points* $(a, 0)$ *and* $(0, b)$, *where* $ab \neq 0$, *is*

$$\frac{x}{a}+\frac{y}{b} = 1.$$

THEOREM. *The equation of the line on the origin and the point* (a, b), *where not both of* a *and* b *are zero; is*

$$bx-ay = 0.$$

THEOREM. *If the points* (x_1, y_1) *and* (x_2, y_2) *are distinct, the equation of the line on the two is*

$$(y_2-y_1)(x-x_1)-(x_2-x_1)(y-y_1) = 0.$$

All three theorems are proved by assuming that

$$lx+my+n = 0$$

is the equation of the line in question and writing down the condition that the points named shall be on it. Alternatively, the third of the three may be proved in this way; the other two are particular cases of the third.

7.622. Parametric Form. THEOREM. *The equations* $x = lr+a$, $y = mr+b$ *are parametric equations of a line in the field (in terms of the parameter* r*). Conversely, any line of the field has parametric equations of this type.*

The proof of this theorem should present no difficulty after what has already been proved.

EXAMPLES

1. Why is it impossible to give an equation for the unlabelled line?

2. Show that the necessary and sufficient condition that the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) should be collinear is that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

3. Show that the equation of the line on the points (x_1, y_1) and (x_2, y_2) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

4. Show that for all values of λ the point $(\lambda x_1+\{1-\lambda\}x_2, \lambda y_1+\{1-\lambda\}y_2)$ is collinear with the points (x_1, y_1) and (x_2, y_2). Conversely, show that any point not on the unlabelled line, and collinear with (x_1, y_1) and (x_2, y_2) has coordinates $(\lambda x_1+\{1-\lambda\}x_2, \lambda y_1+\{1-\lambda\}y_2)$.

5. Find the coordinates of the point common to the two lines whose equations are $lx+my+n = 0$ and $l'x+m'y+n' = 0$. Show that if $lm'-ml' = 0$, there is no point (x, y) common to these two lines unless $ln'-nl' = 0$. Why is it that this fact does not contradict the initial proposition that two distinct lines have a common point?

6. Show that any line on the common point of the two lines whose equations are $lx+my+n = 0$ and $l'x+m'y+n' = 0$ is

$$\lambda(lx+my+n)+\mu(l'x+m'y+n') = 0.$$

7. Show that the necessary and sufficient condition for the concurrence of the three lines whose equations are

$$lx+my+n = 0, \quad l'x+m'y+n' = 0, \quad \text{and} \quad l''x+m''y+n'' = 0$$

is that

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 0.$$

8. Use Example 4 to determine the coordinates of the diagonal points of the simple four-point

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3).$$

Determine also the coordinates of the harmonic points.

7.63. The Equation of the Conic

THEOREM. *The equation of any point-conic, singular or non-singular, other than the singular point-conic consisting of two coincident ranges on the unlabelled line, is of the form*

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

Conversely, any locus whose equation is of this form is a point-conic.

(i) Suppose that the conic is singular, and that it consists of the ranges on the two lines whose equations are $lx+my+n = 0$ and $l'x+m'y+n' = 0$. (These two lines may be identical.) Then the equation of the conic is

$$(lx+my+n)(l'x+m'y+n') = 0,$$

and this is of the form specified.

(ii) Suppose that the conic is singular, and that it consists of one range on the unlabelled line, and one on the line whose equation is $lx+my+n = 0$. Then its equation is $lx+my+n = 0$, and this is of the form specified.

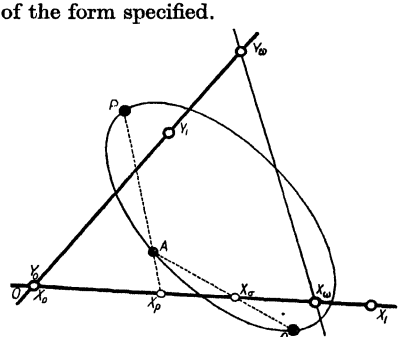


FIG. 51.

(iii) Finally, suppose that Φ is a non-singular point-conic. Let P and Q be two distinct points on it which are not on the unlabelled line or the x-axis. Let (ξ, η) and (ξ', η') respectively be the coordinates of these points.

Consider the perspectivities

$$\left.\begin{aligned} \Phi(A\ldots) &\overset{P}{\sim} l(X_\rho\ldots), \\ \Phi(A\ldots) &\overset{Q}{\sim} l(X_\sigma\ldots). \end{aligned}\right\} \tag{1}$$

From these it follows at once that

$$l(X_\rho\ldots) \sim l(X_\sigma\ldots),$$

and hence that

$$\rho = \frac{\alpha\sigma+\beta}{\gamma\sigma+\delta}, \quad \text{where} \quad \alpha\delta-\beta\gamma \neq 0. \tag{2}$$

Now if (x,y) are the coordinates of A, since A, P, and X are collinear,

$$\begin{vmatrix} x & y & 1 \\ \xi & \eta & 1 \\ \rho & 0 & 1 \end{vmatrix} = 0; \quad \text{that is,} \quad \rho = \frac{\eta x-\xi y}{y-\eta}.$$

Similarly,
$$\sigma = \frac{\eta' x-\xi' y}{y-\eta'}.$$

It follows from (2) that

$$\frac{\eta x-\xi y}{y-\eta} = \frac{\alpha(\eta' x-\xi' y)+\beta(y-\eta')}{\gamma(\eta' x-\xi' y)+\delta(y-\eta')},$$

or
$$\gamma(\eta x-\xi y)(\eta' x-\xi' y)+\delta(\eta x-\xi y)(y-\eta')-$$
$$-\alpha(\eta' x-\xi' y)(y-\eta)-\beta(y-\eta)(y-\eta') = 0,$$

or
$$\gamma\eta\eta' x^2+(\delta\eta-\alpha\eta'-\gamma\xi\eta'-\gamma\xi'\eta)xy+(\gamma\xi\xi'+\delta\xi-\alpha\xi'-\beta)y^2+$$
$$+\eta\eta'(\alpha-\delta)x+(\delta\xi\eta'-\alpha\xi'\eta+\beta\eta+\beta\eta')y-\beta\eta\eta' = 0, \tag{3}$$

so that the coordinates of every point on the point-conic satisfy an equation of the type specified.

On the other hand, if (x,y) be any point satisfying (3), it follows at once, by reasoning in the reverse direction, that X_ρ and X_σ are corresponding points in the projectivity (1), and hence that the point (x,y) is on the point-conic.

Hence the equation of every point-conic with the single exception mentioned is of the form specified.

The converse part of the theorem is more complicated than the corresponding part of 7.62.

Consider the equation $ax^2+2hxy+by^2+2gx+2fy+c = 0$. (i) If the left-hand side of this equation factorizes, and its factors are $lx+my+n$ and $l'x+m'y+n'$, then the equation is plainly that of the locus consisting of the two lines $lx+my+n = 0$ and $l'x+m'y+n' = 0$, and this is a point-conic.

(ii) If the left-hand side does not factorize, let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5) be any five distinct points on the locus whose equation this is. Then

$$ax_\nu^2+2hx_\nu y_\nu+by_\nu^2+2gx_\nu+2fy_\nu+c = 0 \quad (\nu = 1, 2, 3, 4, 5),$$

and these five equations determine uniquely the ratios of the coefficients a, b, c, f, g, and h in terms of x_1, y_1, etc. (If the left-hand side factorizes, there is not necessarily a unique solution.)

Now on these five points there is a point-conic; let its equation be $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c' = 0$. Hence

$$a'x_\nu^2+2h'x_\nu y_\nu+b'y_\nu^2+2g'x_\nu+2f'y_\nu+c = 0 \quad (\nu = 1, 2, 3, 4, 5),$$

and these five equations determine uniquely the ratios of the coefficients a', b', c', f', g', and h' in terms of x_1, y_1, etc.

It is plain that as a consequence the two equations are equivalent, and hence that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

is the equation of a point-conic.

7.631. Singular Point-conics. In this section the criterion whereby the equation of a singular point-conic may be distinguished from that of a non-singular point-conic is given.

THEOREM. *The necessary and sufficient condition that*

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

should be the equation of a singular point-conic is that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

The necessity of the condition is proved first. Suppose then that the conic is singular, and that it consists of the two lines whose equations are $lx+my+n = 0$ and $l'x+m'y+n' = 0$, Then the equation is equivalent to

$$(lx+my+n)(l'x+m'y+n') = 0,$$

that is to say, $a = kll'$, $b = kmm'$, $c = knn'$, $f = \frac{1}{2}k(mn'+m'n)$, $g = \frac{1}{2}k(nl'+n'l)$, and $h = \frac{1}{2}k(lm'+l'm)$, and so the determinant in question is

$$k^3\begin{vmatrix} ll', & \frac{1}{2}(lm'+l'm), & \frac{1}{2}(nl'+n'l) \\ \frac{1}{2}(lm'+l'm), & mm', & \frac{1}{2}(mn'+m'n) \\ \frac{1}{2}(nl'+n'l), & \frac{1}{2}(mn'+m'n), & nn' \end{vmatrix}.$$

It is easily verified that this determinant in fact vanishes. Hence the condition is necessary.

If, on the other hand, the determinant vanishes, then *either* (i) there are unique numbers ξ and η such that

$$a\xi+h\eta+g = h\xi+b\eta+f = g\xi+f\eta+c = 0,$$

or (ii) $a/h = h/b = g/f$.

In the first case, since

$$a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c$$
$$\equiv \xi(a\xi+h\eta+g)+\eta(h\xi+b\eta+f)+g\xi+f\eta+c,$$

the point (ξ,η) is on the conic. And if (ξ',η') is any other point on the conic, it is easily verified that every point collinear with these two is also on the conic; hence the conic is singular.

In the second case, $ab = h^2$ and $af = gh$, and so, if $a \neq 0$,

$$a(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$\equiv a^2x^2+2ahxy+h^2y^2+2agx+2ghy+ac$$
$$\equiv (ax+hy)^2+2g(ax+hy)+ac.$$

This expression can plainly be factorized into two linear factors; hence the equation $ax^2+2hxy+by^2+2gx+2fy+c = 0$ is that of a locus consisting of two ranges on distinct or coincident lines; the conic is therefore singular.

But if $a = 0$, then $h = g = 0$, and the equation of the conic reduces to $by^2+2fy+c = 0$; the left-hand side of this equation being factorizable, it is the equation of a singular conic.

7.632. Tangents, Pole and Polar. THEOREM. *If*

$$ax^2+2hxy+by^2+2gx+2fy+c = 0$$

be the equation of a non-singular conic, then

(i) *if* (x_1, y_1) *be the coordinates of any point on the conic, the equation of the tangent on* (x_1, y_1) *is*

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c = 0,$$

(ii) *if* (x_1, y_1) *be the coordinates of any point not on the conic, the equation of the pair of tangents to the conic which are on* (x_1, y_1) *is*

$$(ax^2+2hxy+by^2+2gx+2fy+c)\times$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)-$$
$$-(axx_1+h[xy_1+yx_1]+byy_1+g[x+x_1]+f[y+y_1]+c)^2 = 0,$$

(iii) *the equation of the polar of any point* (x_1, y_1) *relative to the conic is*

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c = 0.$$

The three theorems are taken together because the first two make use of the same principle, and the third is an immediate deduction from them.

Consider any line p which is on the point (x_1, y_1). (It is not at this stage supposed that this point is on or not on the conic.) Let (x, y) be the coordinates of any point on p; then the coordinates of any point collinear with these two are

$$\left(\frac{\lambda x+\mu x_1}{\lambda+\mu}, \frac{\lambda y+\mu y_1}{\lambda+\mu}\right);$$

and if this point is on the conic,

$$a(\lambda x+\mu x_1)^2+2h(\lambda x+\mu x_1)(\lambda y+\mu y_1)+b(\lambda y+\mu y_1)^2+$$
$$+2g(\lambda x+\mu x_1)(\lambda+\mu)+2f(\lambda y+\mu y_1)(\lambda+\mu)+c(\lambda+\mu)^2 = 0,$$

or $\lambda^2[ax^2+2hxy+by^2+2gx+2fy+c]+$

$$+2\lambda\mu[axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c]+$$
$$+\mu^2[ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c] = 0. \quad (1)$$

This is a quadratic equation for determining the ratio of λ to μ, and so for determining those points on p which are on the conic. We make use of it in two different ways in what follows.

(i) Suppose first that the point (x_1, y_1) is on the conic, so that one of the values of λ/μ is zero. If, in addition, the line p is a tangent, the other value of λ/μ is also zero; for if it were not, there would be a point on p, distinct from (x_1, y_1), on the conic.

The condition that both the values of λ/μ shall be zero is that

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0,$$

and $axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c = 0.$

The first of these equations merely shows that (x_1, y_1) is on the conic, but the second is a relation between the coordinates (x, y) of any point on p, the tangent. It is the equation of a line, and so it is the equation of the tangent on (x_1, y_1).

This proves the first part of the theorem.

(ii) Suppose next that (x_1, y_1) is not on the conic, then if p is either of the tangents to the conic which are on (x_1, y_1), both values of the ratio λ/μ are equal, and so

$$(ax^2+2hxy+by^2+2gx+2fy+c)\times$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)-$$
$$-(axx_1+h[xy_1+yx_1]+byy_1+g[x+x_1]+f[y+y_1]+c)^2 = 0.$$

This last is a relation between the coordinates (x, y) of any point on either of the tangents; it is the equation of a point-conic, and so it is the equation of the pair of tangents.

(iii) If $F(x, y) = 0$ and $G(x, y) = 0$ be the equations of any two loci, it is plain that the locus whose equation is

$$\lambda F(x, y)+\mu G(x, y) = 0$$

is on all the points common to the two loci. We use this fact to determine the equation of the polar of any point (x, y) relative to the conic. For if P be any point, and Q and R be the points of contact of the two tangents to the conic on P, then QR is the polar of P.

The equation

$$\lambda(ax^2+2hxy+by^2+2gx+2fy+c)+$$
$$+(ax^2+2hxy+by^2+2gx+2fy+c)\times$$
$$\times(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)-$$
$$-(axx_1+h[xy_1+yx_1]+byy_1+g[x+x_1]+f[y+y_1]+c)^2 = 0$$

is that of a locus on the points common to the conic and to the two tangents to the conic which are on (x_1, y_1). It is plainly the equation of a conic; moreover, if

$$-\lambda = ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c,$$

it reduces to

$$(axx_1+h[xy_1+yx_1]+byy_1+g[x+x_1]+f[y+y_1]+c)^2 = 0.$$

This is the equation of a pair of coincident lines, each of which is therefore the polar of (x_1, y_1). Hence

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c = 0$$

is the equation of the polar of (x_1, y_1) when that point is not on the conic. The first part of the theorem shows that it is also the equation of the polar of (x_1, y_1) when that point is on the conic. Hence the theorem is proved.

7.7. The Non-homogeneous Mesh Gauge and Elementary Geometry

By means of the non-homogeneous mesh gauge, a label has been given to every point of the field save the points of the unlabelled line. This system of labels has enabled us to apply Algebra to Geometry, but even in the limited amount which has been done in this chapter, the unlabelled line has shown itself to be a source of trouble. Whenever it or any point on it was mentioned special treatment was required. For this reason a labelling system without this defect will be elaborated in the next chapter.

But the work done with the non-homogeneous mesh gauge serves to give us a first hint about elementary Geometry. The reader cannot have failed to notice the extreme similarity between the Algebraic Projective Geometry, which uses the non-homogeneous mesh gauge, and the ordinary algebraic treatment of elementary Geometry known as Analytical Geometry. There is, so to speak, isomorphism between the two. And since the non-homogeneous mesh gauge leaves one line and all the points on it out of consideration, this suggests that elementary Geometry may be all along doing the same thing. It is true that this line is sometimes 'added' to the field of elementary Geometry under the name of the 'line at infinity', but it is nevertheless not amenable to treatment in the same way as other lines, since none of its points are at a finite distance from the rest of the field. The mesh gauge of elementary Analytical Geometry fails to label the points on the 'line at infinity', simply because that mesh gauge is defined in terms of length. Hence the 'line at infinity' always remains exceptional in elementary Geometry. Hence too there are such anomalies as parallel lines, circular points at infinity, and other puzzling things.

It is not possible to do more here than adumbrate the explanation of the apparently exceptional things in elementary Geometry; in a later chapter the relations between it and Projective Geometry are more fully discussed.

7.8 The Non-homogeneous Mesh Gauge and Cartesian Coordinates

The similarity between the non-homogeneous mesh gauge and what are called in Analytical Geometry *Cartesian coordinates* makes it necessary to emphasize the radical differences between the two.

In Cartesian coordinates the points on the x-axis and the points on the y-axis are given number-labels by a method which involves the notion of distance; the two axes are chosen at right angles to each other, and this choice involves the notion of angle. Further, if it be allowed that in elementary Analytical Geometry there is an unlabelled line, this line is fixed definitely and cannot be chosen at will.

In the non-homogeneous mesh gauge the points on the axes are given number-labels by a process which makes no appeal to the concepts of distance or angle, and the unlabelled line may be chosen to be any line of the field whatever.

The similarity between the two in the algebraic processes involved, and even to some extent in the terminology in use, is a similarity only in form; the thought underlying this external form is different.

Examples

1. Show that with the recently added initial proposition of extension two non-singular conics always have four distinct or coincident points in common.

2. Write down the equation of the pair of tangents on the origin to the conic whose equation is $ax^2+2hxy+by^2+2gx+2fy+c = 0$.

3. If $ax^2+2hxy+by^2+2gx+2fy+c = 0$ is the equation of a pair of distinct lines, determine the coordinates of their common point.

4. Determine a necessary and sufficient condition that the line $lx+my+n = 0$ shall be a tangent to the conic whose equation is $x^2+y^2 = k^2$.

5. Determine a necessary and sufficient condition that the two points (x_1, y_1) and (x_2, y_2) shall be conjugate points relative to the conic whose equation is $ax^2+2hxy+by^2+2gx+2fy+c = 0$.

6. If $ax^2+2hxy+by^2+2gx+2fy+c = 0$ is the equation of a non-singular conic, determine a necessary and sufficient condition that the x-axis shall be a tangent to it, and that X_ω shall be its point of contact.

7. Show that the unlabelled line is the polar of the origin relative to the conic whose equation is $x^2+y^2 = k^2$.

CHAPTER VIII

THE HOMOGENEOUS MESH GAUGE

8.1. Homogeneous Coordinates on a Line

In developing the non-homogeneous mesh gauge in the last chapter we first gave number-labels to the points of a line, and later extended the labelling system to other points of the field. The homogeneous mesh gauge is developed in the same way, labels being first attached to the points of a line only.

The homogeneous coordinates of a point on a line are defined in the following way:

If A_0, A_1, and A_ω are gauge-points on the line, then (i) to any point A_x of the open set (x being its number-label in the sense of the last chapter) is attached a *double* label (x_1, x_2), where $x_1/x_2 = x$; (ii) to the point A_ω is attached the label $(r, 0)$, where r is any number whatever.

It is clear that the double number-label attached to any point is not unique, and that if (a_1, a_2) is a suitable label for a given point, then (ka_1, ka_2) is equally suitable, provided $k \neq 0$. This ambiguity causes no ambiguity in the work done with the homogeneous coordinates.

It may be observed that (i) every point of the line, A_ω included, has been labelled, and (ii) to every possible number-pair (a_1, a_2), with the single exception of the pair $(0, 0)$, there corresponds a point on the line.

Now though by this means A_ω has been included in the labelling system, it is not *therefore* obvious that it is not still an apparently exceptional point. The first of the following theorems, which deal with the projectivity in terms of homogeneous coordinates, shows that in this labelling system A_ω is on exactly the same footing as all the other points.

8.2. The Projectivity on a Line

8.21. The Equation of a Projectivity

THEOREM. *The general equation of a projectivity between two ranges is*

$$ax_1x_1' + bx_1x_2' + cx_2x_1' + dx_2x_2' = 0,$$

where (x_1, x_2) *are the homogeneous coordinates of any point of one range, and* (x_1', x_2') *are the homogeneous coordinates of the corresponding point of the other, and* $bc-ad \neq 0$.

The enunciation includes, in reality, two theorems: first, that if there is a projectivity between two ranges, the homogeneous coordinates of every pair of corresponding points are related by the equation $ax_1x_1'+bx_1x_2'+cx_2x_1'+dx_2x_2' = 0$, and, secondly, that if two ranges of points are thus connected in pairs, the correspondence is a projectivity.

First then, suppose that there is a projectivity between the two ranges. Then, by 7.56 and the definition of homogeneous coordinates, if (x_1, x_2) and (x_1', x_2') are a pair of corresponding points, both of which belong to the open set, numbers a, b, c, and d exist, such that

$$a\frac{x_1}{x_2}\frac{x_1'}{x_2'}+b\frac{x_1}{x_2}+c\frac{x_1'}{x_2'}+d = 0,$$

and $bc-ad \neq 0$.

Since neither of x_2, x_2' is equal to zero, it follows that

$$ax_1x_1'+bx_1x_2'+cx_2x_1'+dx_2x_2' = 0.$$

The equation is therefore true for all such pairs.

It remains to show that the equation is still satisfied when A_ω is one or both of a pair of corresponding points.

By 7.56, if A_p is the point of the first range corresponding to A_ω of the second, $p = -c/a$, and so the homogeneous coordinates of A_p are $(-c, a)$. Clearly the equation is satisfied when $x_1 = -c$, $x_2 = a$, and $x_2' = 0$.

Similarly A_ω in the first range corresponds to $A_{p'}$ in the second, where the homogeneous coordinates of $A_{p'}$ are $(-b, a)$, and the equation is satisfied in this case also.

Finally, if A_ω in the first range corresponds to A_ω in the second, $a = 0$, and so the equation of the projectivity is $bx_1x_2'+cx_2x_1'+dx_2x_2' = 0$, and this equation is plainly satisfied when $x_2 = x_2' = 0$.

The first part of the theorem is therefore proved. The second part is proved by *reductio ad absurdum*; the details are left to the reader. The theorem shows that there is no need to pay special attention to the point A_ω when dealing with a projectivity in homogeneous coordinates.

8.22. Second Form of the Equation of a Projectivity

THEOREM. *The algebraic relation between the homogeneous coordinates of a pair of corresponding points in a projectivity may be expressed by equations of the form*

$$kx_1' = \alpha x_1 + \beta x_2,$$
$$kx_2' = \gamma x_1 + \delta x_2,$$

where $k \neq 0$ *and* $\beta\gamma - \alpha\delta \neq 0$.

This theorem is a very simple consequence of the last.

8.23. Self-corresponding Points

THEOREM. *If* $ax_1x_1' + bx_1x_2' + cx_2x_1' + dx_2x_2' = 0$ *be the equation of a projectivity between two cobasal ranges, each referred to the same gauge-points, then the self-corresponding points of the projectivity are*

(i) $(-b-c+\sqrt{(b+c)^2-4ad},\ 2a)$ *and*

$(-b-c-\sqrt{(b+c)^2-4ad},\ 2a)$

if neither a nor d is zero,

(ii) $(-d, b+c)$ *and* $(1, 0)$ *if* $a = 0$ *and* $d \neq 0$,

(iii) $(b+c, -a)$ *and* $(0, 1)$ *if* $a \neq 0$ *and* $d = 0$,

(iv) $(0, 1)$ *and* $(1, 0)$ *if* $a = d = 0$.

If (x_1, x_2) is a self-corresponding point of the projectivity,

$$ax_1x_1 + bx_1x_2 + cx_2x_1 + dx_2x_2 = 0,$$

or

$$ax_1^2 + (b+c)x_1x_2 + dx_2^2 = 0.$$

This last is a quadratic equation for determining the ratios of x_1 to x_2, and the results follow at once from the theory of the quadratic equation.

8.24. Condition for an Involution

THEOREM. *The necessary and sufficient condition that the projectivity between two cobasal ranges, referred to the same gauge-points whose equation is* $ax_1x_1' + bx_1x_2' + cx_2x_1' + dx_2x_2' = 0$, *shall be an involution is that* $b = c$.

First suppose that the projectivity is an involution. Then its equation must remain unaltered by the interchange of (x_1, x_2) and (x_1', x_2'). Hence $b = c$.

Next suppose that $b = c$, then since the equation remains unaltered by the interchange of (x_1, x_2) and (x_1', x_2') the projectivity must be an involution.

Examples

1. Show that $(a, 1)$ and $(1, a)$ are mates in the involution in which $(1, 1)$ and $(-1, 1)$ are self-corresponding points.

2. Determine a necessary and sufficient condition that the projectivity whose equation is $ax_1 x_1' + bx_1 x_2' + cx_2 x_1' + dx_2 x_2' = 0$ shall have coincident self-corresponding points.

8.3. The Cross-ratio

Suppose that P, Q, R, and S are four distinct collinear points whose homogeneous coordinates relative to some gauge-points are respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Similarly, suppose that P', Q', R', and S' are four other distinct collinear points with coordinates (x_1', y_1'), (x_2', y_2'), (x_3', y_3'), and (x_4', y_4').

The necessary and sufficient condition that

$$(PQRS) \sim (P'Q'R'S')$$

is that four numbers a, b, c, and d should exist, such that $ax_\nu x_\nu' + bx_\nu y_\nu' + cy_\nu x_\nu' + dy_\nu y_\nu' = 0$ $(\nu = 1, 2, 3, 4)$ and $bc - ad \neq 0$.

An equivalent statement of this necessary and sufficient condition is that

$$\begin{vmatrix} x_1 x_1' & x_1 y_1' & y_1 x_1' & y_1 y_1' \\ x_2 x_2' & x_2 y_2' & y_2 x_2' & y_2 y_2' \\ x_3 x_3' & x_3 y_3' & y_3 x_3' & y_3 y_3' \\ x_4 x_4' & x_4 y_4' & y_4 x_4' & y_4 y_4' \end{vmatrix} = 0,$$

this being a simple deduction from the first condition.

Neither of these two statements of the necessary and sufficient condition for projectivity is, in practice, very convenient to use, but from the second it is possible to deduce, by mere Algebra, a very simple and convenient condition. As, however, this deduction is not a very interesting piece of work, it is deduced from geometrical considerations in the following theorem.

Theorem. *The necessary and sufficient condition that the four distinct points whose homogeneous coordinates are respectively*

(x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) should be projective with four other points whose coordinates are respectively (x_1', y_1'), (x_2', y_2'), (x_3', y_3'), and (x_4', y_4') is that

$$\frac{(x_1y_3-x_3y_1)(x_2y_4-x_4y_2)}{(x_1y_4-x_4y_1)(x_2y_3-x_3y_2)} = \frac{(x_1'y_3'-x_3'y_1')(x_2'y_4'-x_4'y_2')}{(x_1'y_4'-x_4'y_1')(x_2'y_3'-x_3'y_2')}.$$

Let the first four points be P, Q, R, and S respectively, and let T be a point such that

$$(PQRS) \sim (A_0 A_\omega A_1 T).$$

Then if (x, y) be the coordinates of T, it follows that

$$\begin{vmatrix} 0 & x_1 & 0 & y_1 \\ x_2 & 0 & y_2 & 0 \\ x_3 & x_3 & y_3 & y_3 \\ xx_4 & yx_4 & xy_4 & yy_4 \end{vmatrix} = 0,$$

or

$$xx_4\begin{vmatrix} x_1 & 0 & y_1 \\ 0 & y_2 & 0 \\ x_3 & y_3 & y_3 \end{vmatrix} + xy_4\begin{vmatrix} 0 & x_1 & y_1 \\ x_2 & 0 & 0 \\ x_3 & x_3 & y_3 \end{vmatrix} - yx_4\begin{vmatrix} 0 & 0 & y_1 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & y_3 \end{vmatrix} - yy_4\begin{vmatrix} 0 & x_1 & 0 \\ x_2 & 0 & y_2 \\ x_3 & x_3 & y_3 \end{vmatrix} = 0,$$

or

$$x(x_4y_2-x_2y_4)(x_1y_3-x_3y_1)-y(x_4y_1-x_1y_4)(x_2y_3-x_3y_2) = 0.$$

Hence $$\frac{y}{x} = \frac{(x_1y_3-x_3y_1)(x_2y_4-x_4y_2)}{(x_1y_4-x_4y_1)(x_2y_3-x_3y_2)}.$$

If now the second four points be P', Q', R', and S' respectively, and if $(PQRS) \sim (P'Q'R'S')$, it follows that

$$(P'Q'R'S') \sim (A_0 A_\omega A_1 T),$$

and hence, by precisely similar reasoning, that

$$\frac{y}{x} = \frac{(x_1'y_3'-x_3'y_1')(x_2'y_4'-x_4'y_2')}{(x_1'y_3'-x_4'y_1')(x_2'y_3'-x_3'y_2')}.$$

Hence

$$\frac{(x_1y_3-x_3y_1)(x_2y_4-x_4y_2)}{(x_1y_4-x_4y_1)(x_2y_3-x_3y_2)} = \frac{(x_1'y_3'-x_3'y_1')(x_2'y_4'-x_4'y_2')}{(x_1'y_4'-x_4'y_1')(x_2'y_3'-x_3'y_2')},$$

and so the condition is necessary.

Suppose, on the other hand, that the condition is fulfilled. Then if $(PQRS) \sim (A_0 A_\omega A_1 T)$ the coordinates of the point T are

$$(\{x_1y_4-x_4y_1\}\{x_2y_3-x_3y_2\}, \{x_1y_3-x_3y_1\}\{x_2y_4-x_4y_2\}).$$

And if $(P'Q'R'S') \sim (A_0 A_\omega A_1 U)$, the coordinates of U are

$$(\{x_1' y_4' - x_4' y_1'\}\{x_2' y_3' - x_3' y_2'\}, \{x_1' y_3' - x_3' y_1'\}\{x_2' y_4' - x_4' y_2'\}).$$

But the initial supposition shows that T and U are the same point, and so $(PQRS) \sim (P'Q'R'S')$.

Hence the condition is sufficient, and the theorem is proved.

The expression

$$\frac{(x_1 y_3 - x_3 y_1)(x_2 y_4 - x_4 y_2)}{(x_1 y_4 - x_4 y_1)(x_2 y_3 - x_3 y_2)}$$

occurring in this theorem is of the utmost importance in what follows, and it is therefore essential to have a name for it; it is called the *cross-ratio* of the four points in question. The following is a formal definition.

DEFINITION. *If P, Q, R, and S are four distinct collinear points, and if their homogeneous coordinates relative to some gauge-points are respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), then the function*

$$\frac{(x_1 y_3 - x_3 y_1)(x_2 y_4 - x_4 y_2)}{(x_1 y_4 - x_4 y_1)(x_2 y_3 - x_3 y_2)}$$

is termed the cross-ratio of the four points, in that order.

The symbols $R\!\!\!/(PQRS)$ or $R\!\!\!/\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ are used to denote the cross-ratio of the four points. If the *non*-homogeneous coordinates of the four points are z_1, z_2, z_3, and z_4 respectively, the cross-ratio is symbolized by

$$R\!\!\!/(z_1, z_2, z_3, z_4).$$

It is easily verified, by direct substitution of the corresponding homogeneous coordinates, that

$$R\!\!\!/(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

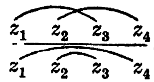
(The fact that all four points have non-homogeneous coordinates implies that none of them coincides with A_ω.)

8.31. Some Practical Notes.

(1) *Structure of the Cross-ratio.*

$$R\!\!\!/(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

It is important to be able to write this fraction rapidly and accurately, and so its structure should be carefully noticed. The numerator is the product of the differences of alternates; the denominator is the product of (i) the difference of extremes, and (ii) the difference of means. A practical way of memorizing this structure is by means of the following diagram, whose significance is plain:



Neither of these is of very much help when homogeneous coordinates are being used, but in practice it is wiser (because *safer*) to use *non*-homogeneous coordinates whenever possible. The only difficulty that can arise with these is when A_ω is one of the four points whose cross-ratio is sought. A method of surmounting this difficulty is given in the next paragraph.

(2) *Limiting Forms.* The reader should verify the following four propositions:

$$(a)\qquad \text{℞}(A_\omega QRS) \equiv \lim_{z_1\to\infty}\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{z_2-z_4}{z_2-z_3};$$

$$(b)\qquad \text{℞}(PA_\omega RS) \equiv \lim_{z_2\to\infty}\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{z_1-z_3}{z_1-z_4};$$

$$(c)\qquad \text{℞}(PQA_\omega S) \equiv \lim_{z_3\to\infty}\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{z_2-z_4}{z_1-z_4};$$

$$(d)\qquad \text{℞}(PQRA_\omega) \equiv \lim_{z_4\to\infty}\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \frac{z_1-z_3}{z_2-z_3}.$$

The verification is accomplished by showing that in each case the homogeneous form of the cross-ratio is equal to the limit given. This gives a method of using non-homogeneous coordinates for the cross-ratio, even when one of the points is A_ω.

For this reason it is permissible to write, for instance, $\text{℞}(z_1, z_2, \infty, z_4)$, meaning thereby the fraction $\dfrac{z_2-z_4}{z_1-z_4}$. It is even permissible to write $\dfrac{(z_1-\infty)(z_2-z_4)}{(z_1-z_4)(z_3-\infty)}$ for this cross-ratio, provided we do not attempt to manipulate the symbol ∞ as if it were a number; it is, however, better to avoid this last usage in formal work.

(3) *Consistency.* Some writers adopt a different definition of the cross-ratio. For example, $R\!\!\!/(z_1, z_2, z_3, z_4)$ is sometimes defined as $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$.

The definition adopted in this book is in agreement with that of standard modern writers on the subject, and it fits in best with the developments made in the next chapter. But if the reader consults other works on Projective Geometry, he should make certain which definition the writers use.

Examples

1. Evaluate $R\!\!\!/(0, \infty, t, 1)$, and $R\!\!\!/(0, \infty, t, -t)$.
2. If 1, ρ, and ρ^2 are the roots of the equation $x^3-1 = 0$, show that $R\!\!\!/(1, \rho, -\rho^2, t) = \frac{\rho(t-\rho)}{(t-1)}$.
3. Show that if $l(PQBCDE...) \sim l(PQB'C'D'E'...)$, then $R\!\!\!/(PQBB') = R\!\!\!/(PQCC') = R\!\!\!/(PQDD') = R\!\!\!/(PQEE')$, etc.
4. Prove that the converse theorem is also true.
5. If the equation of the projectivity in Ex. 3 is
$$axx'+bxy'+cyx'+dyy' = 0,$$
show that $R\!\!\!/(PQAA')+R\!\!\!/(QPAA') = -\frac{b^2+c^2-2ad}{ad-bc}$.

8.32. Permutations

There are twenty-four ways in which the four letters $PQRS$ can be permuted amongst themselves, but though the value of the cross-ratio of four points is not independent of their order, there are not, in fact, twenty-four different values of the cross-ratio. By the permutation theorem of 3.325
$$(PQRS) \sim (QPSR) \sim (SRQP) \sim (RSPQ),$$
and so the cross-ratios corresponding to these four permutations are equal. In this way the twenty-four arrangements of the letters can be grouped into six sets of four, and there are in fact six different cross-ratios. Given one of these six, it is possible to deduce from the following theorems the other five.

8.321. Theorem.
$$R\!\!\!/(PQRS) = R\!\!\!/(QPSR) = R\!\!\!/(SRQP) = R\!\!\!/(RSPQ).$$

This is a consequence of the permutation theorem (3.325) and 8.3.

The theorem may be stated in words thus: *If any pair of* *letters be interchanged, and the remaining pair be also interchanged, the value of the cross-ratio is unaltered.*

8.322. THEOREM. *If* ℞$(PQRS) = \lambda$, *then*

$$℞(QPRS) = ℞(PQRS) = ℞(RSQP) = ℞(SRPQ) = \frac{1}{\lambda}.$$

There is no loss in generality if the points P, Q, R, S are taken as 0, 1, ∞, and t respectively. Then ℞$(PQRS) = (t-1)/t$, and ℞$(QPRS) = t/(t-1)$. This proves the first result; the rest follow by the last theorem.

The theorem may be partially stated in words thus: *If either the first pair or the second pair be interchanged, the new cross-ratio is the reciprocal of the old.*

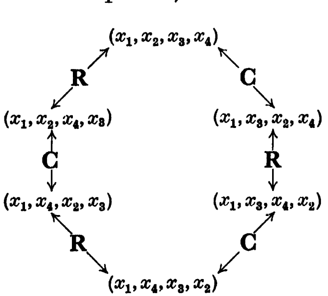
8.323. THEOREM. *If* ℞$(PQRS) = \lambda$, *then*

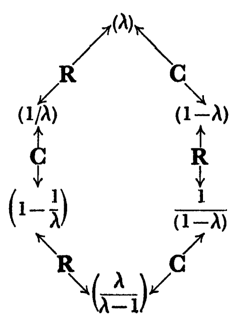
$$℞(SQRP) = ℞(RPSQ) = ℞(QSPR) = ℞(PRQS) = 1-\lambda.$$

The theorem may be proved in the same way as the last. It may be partially stated in words thus: *If either the outer pair or the inner pair be interchanged, the sum of the old cross-ratio and the new cross-ratio is unity.*

The six possible values of the cross-ratio may now be all deduced by successive application of the last two theorems. They are: λ, $1/\lambda$, $1-1/\lambda$, $\lambda/(\lambda-1)$, $1/(1-\lambda)$, and $1-\lambda$.

The proof that this is so is best shown by a schematic diagram, as under. In this —**C**→ denotes a permutation which changes a cross-ratio λ into $(1-\lambda)$; —**R**→ denotes a permutation which changes a cross-ratio into its reciprocal. (**C** for complement, **R** for reciprocal.)





8.33. Cross-ratios of Singular Tetrads

The cross-ratio of four collinear points has been defined only when the four points are all distinct. It is natural to make an attempt to generalize this definition so as to cover the cases when the four points are not all distinct, i.e. when they form a singular tetrad.

The cross-ratio $\text{R}(x_1, x_2, x_3, x_4)$ is, by definition, the fraction $\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$; the obvious generalization is to say that the cross-ratio of *any* four points, distinct or not, is the fraction just written down. This is, however, impossible, since the fraction is not always significant when two or more of the four points coincide. Subject to a proviso to be made almost immediately, the definition is given: *If the tetrad* (x_1, x_2, x_3, x_4) *is singular, its cross-ratio is defined to be the fraction*

$$\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)},$$

provided this is significant.

The following cases should be noticed:

(i) If three or more of the points coincide, the fraction assumes the indeterminate form 0/0.

(ii) If the first pair or the second pair coincide, and, *a fortiori*, if the first pair coincide and the second pair coincide, but separately, the value of the fraction is unity.

(iii) If the first and the third coincide, or the second and the fourth coincide, and, *a fortiori*, if each pair coincide separately, the value of the cross-ratio is zero.

(iv) In all other cases of coincidence the fraction assumes the meaningless form 1/0.

The proviso mentioned above is noticed here; Theorem 8.3 is not true of singular tetrads, and must never be applied to them. Thus, if P, Q, R, and S are four distinct collinear points, $\text{R}(PPRS) = \text{R}(PQRR) = 1$, but it is not true that $(PPRS) \sim (PQRR)$; similarly, $\text{R}(PQPS) = \text{R}(PQRQ) = 0$, but it is not true that $(PQPS) \sim (PQRQ)$. Caution must therefore be exercised when the cross-ratios of singular tetrads are in question. In what follows it will always be supposed that a cross-ratio is that of a non-singular tetrad, unless the contrary is explicitly stated.

The following theorem is sometimes of value.

THEOREM. *If* ℞$(PQRS)$ *is equal to zero or unity, the tetrad is singular.*

8.34. The Cross-ratio of a Harmonic Tetrad

THEOREM. *The necessary and sufficient condition that the four distinct collinear points* P, Q, R, *and* S *should form a harmonic tetrad* (PQ, RS) *is that* ℞$(PQRS) = -1$.

First suppose that ℞$(PQRS) = -1$. Then, by 8.322,

$$℞(QPRS) = -1.$$

Hence by 8.3, $(PQRS) \sim (QPRS)$, so that (PQ, RS) is a harmonic tetrad. The condition is therefore sufficient.

Next suppose that (PQ, RS) is a harmonic tetrad, so that $(PQRS) \sim (QPRS)$. It follows that if ℞$(PQRS) = \lambda$, $\lambda = 1/\lambda$, so that $\lambda = \pm 1$. But since the tetrad is not singular, $\lambda \neq 1$; that is to say, $\lambda = -1$, and so the condition is necessary.

8.35. The Multiplication Theorem

THEOREM. *If the five collinear points* O, U, P, Q, R *are all distinct, then* ℞$(OUPQ)$. ℞$(OUQR) =$ ℞$(OUPR)$.

Choose gauge-points on the line so that none of the points is the unlabelled point; there is no loss in generality if these are so chosen that O and U have the labels 0 and 1 respectively. Let p, q, and r be the labels of the other points.

Then

$$℞(OUPQ).℞(OUQR) = \frac{(-p)(1-q)}{(-q)(1-p)}\frac{(-q)(1-r)}{(-r)(1-q)}$$
$$= \frac{(-p)(1-r)}{(-r)(1-q)}$$
$$= ℞(OUPR).$$

This proves the theorem.

8.36. Theorems involving the Mesh Gauge

The theorems on cross-ratios have so far been confined to the points on a single line; the two theorems which follow deal with the cross-ratio of four collinear points in terms of their non-homogeneous coordinates relative to some non-homogeneous mesh gauge imposed on the field.

8.361. THEOREM. *If the non-homogeneous coordinates of four distinct collinear points P, Q, R, and S are, respectively,* (x_1, y_1), (x_2, y_2), (x_3, y_3), *and* (x_4, y_4), *then*

$$\text{R}(PQRS) = \text{R}(x_1, x_2, x_3, x_4) = \text{R}(y_1, y_2, y_3, y_4),$$

provided these cross-ratios are significant.

The two cross-ratios cannot both be indeterminate, and one of them will be indeterminate only if the four points are all on a line whose equation is $x = k$ or $y = k$.

The reader should have no difficulty in proving this theorem if he bears in mind the definition of the non-homogeneous coordinates of a point of the field.

8.362. THEOREM. *If P, Q, R, and S are four distinct points on the line whose parametric specification is* $x = lr+a$, $y = mr+b$, *and if the values of the parameter r corresponding to these four points are respectively* r_1, r_2, r_3, *and* r_4, *then*

$$\text{R}(PQRS) = \text{R}(r_1, r_2, r_3, r_4).$$

l and m cannot both be zero; suppose then that $l \neq 0$. Then by the previous theorem

$$\begin{aligned} \text{R}(PQRS) &= \text{R}(lr_1+a, lr_2+a, lr_3+a, lr_4+a), \\ &= \text{R}(lr_1, lr_2, lr_3, lr_4) \quad \text{by 7.221}, \\ &= \text{R}(r_1, r_2, r_3, r_4) \quad \text{by 7.222}. \end{aligned}$$

This proves the theorem if $l \neq 0$; if $l = 0$, the proof is entirely similar, save that it starts from

$$\text{R}(mr_1+b, mr_2+b, mr_3+b, mr_4+b).$$

EXAMPLES

1. Show that if the collinear points O, U, P, Q, R are distinct, and if $\text{R}(OUPQ).\text{R}(OUQR).\text{R}(OURT) = 1$, then T coincides with P.

2. Examine the conditions under which Theorem 8.35 is true when one or more of the points coincide.

3. If $\text{R}(ABCP) = \lambda$, and $\text{R}(ABCQ) = \mu$, show that

$$\text{R}(BCPQ) = \text{R}(0, 1, \lambda, \mu).$$

4. If $\text{R}(x_1, x_2, x_3, \infty) = \text{R}(y_1, y_2, y_3, \infty)$, show that the three points whose non-homogeneous coordinates are (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear.

8.4. The Homogeneous Mesh Gauge

In 8.1 the homogeneous coordinates of a point on a line, relative to gauge-points chosen on the line, were defined; in this section the labelling system is extended to all the points of the field.

8.41. The Triangle of Reference

Choose any four-point $XYZI$ in the field, and let X_1, Y_1, and Z_1 be its diagonal points, X_1 being the point $\begin{pmatrix} XI \\ YZ \end{pmatrix}$, Y_1 the point $\begin{pmatrix} YI \\ ZX \end{pmatrix}$, and Z_1 the point $\begin{pmatrix} ZI \\ XY \end{pmatrix}$.

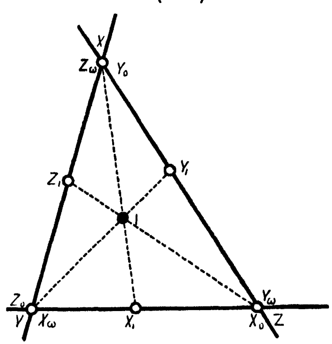


FIG. 52.

Gauge-points are now chosen on the three lines YZ, ZX, and XY as follows:

(i) on YZ, gauge-points X_0, X_1, and X_ω, coinciding with Z, X_1, and Y respectively;

(ii) on ZX, gauge-points Y_0, Y_1, and Y_ω, coinciding with X, Y_1, and Z respectively;

(iii) on XY, gauge-points Z_0, Z_1, and Z_ω, coinciding with Y, Z_1, and X respectively.

The *triangle* XYZ so gauged will be called the *triangle of reference*; the point I will be called the *gauging point.*

Before making use of the triangle of reference to label the points of the field it is necessary to prove a preliminary theorem about it.

8.411. Theorem. *If X', Y', and Z' be three points on the sides of YZ, ZX, XY of a triangle of reference, and distinct from its points, the necessary and sufficient condition that the three lines XX', YY', ZZ' should be concurrent is that*

$$\text{℞}(X_0 X_\omega X' X_1) \,.\, \text{℞}(Y_0 Y_\omega Y' Y_1) \,.\, \text{℞}(Z_0 Z_\omega Z' Z_1) = 1.$$

The necessity of the condition is proved first. Suppose then that the lines XX', YY', ZZ' are all on the point P.

(i) Suppose first that P coincides with I. Then each of the tetrads mentioned in the enunciation is singular, and its cross-ratio is unity; the theorem is therefore true.

(ii) Suppose next that P and I do not coincide, but that P is on one of the lines XX_1, YY_1, ZZ_1. For the sake of definiteness take it to be on XX_1; then X' and X_1 coincide.

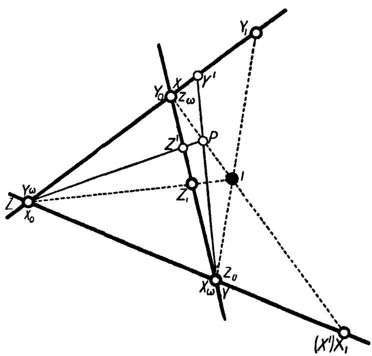


Fig. 53.

Hence $\quad (Y_0 Y_\omega Y' Y_1) \overset{Y}{\barwedge} (XX_1 PI) \overset{Z}{\barwedge} (Z_\omega Z_0 Z' Z_1)$,

and so $\quad \text{℞}(Y_0 Y_\omega Y' Y_1) = \text{℞}(Z_\omega Z_0 Z' Z_1)$,

or $\quad \text{℞}(Y_0 Y_\omega Y' Y_1) \,.\, \text{℞}(Z_0 Z_\omega Z' Z_1) = 1.$

Hence, since $\text{℞}(X_0 X_\omega X' X_1) = 1$,

$$\text{℞}(X_0 X_\omega X' X_1) \,.\, \text{℞}(Y_0 Y_\omega Y' Y_1) \,.\, \text{℞}(Z_0 Z_\omega Z' Z_1) = 1.$$

(iii) Lastly, suppose that P is not on any of the lines XX_1, YY_1, ZZ_1, so that none of the pairs X_1 and X', Y_1 and Y', Z_1 and Z' coincide.

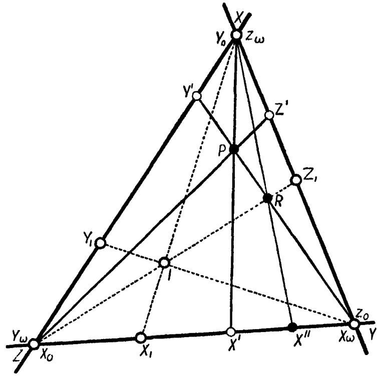


FIG. 54.

Let R be the point $\begin{pmatrix} YY' \\ ZZ_1 \end{pmatrix}$, and X'' the point $\begin{pmatrix} YZ \\ XR \end{pmatrix}$. Then

$$(Y_\omega Y_0 Y'Y_1) \overset{Y}{\barwedge} (ZZ_1 RI) \overset{X}{\barwedge} (X_0 X_\omega X''X_1),$$

so that $$℞(Y_\omega Y_0 Y'Y_1) = ℞(X_0 X_\omega X''X_1). \qquad (1)$$

Also $$(Z_0 Z_\omega Z'Z_1) \overset{Z}{\barwedge} (YY'PR) \overset{X}{\barwedge} (X_\omega X_0 X'X''),$$

and so

$$\begin{aligned} ℞(Z_0 Z_\omega Z'Z_1) &= ℞(X_\omega X_0 X'X''), \\ &= ℞(X_\omega X_0 X'X_1) \,.\, ℞(X_\omega X_0 X_1 X''), &&\text{by 8.35,} \\ &= ℞(X_\omega X_0 X'X_1) \,.\, ℞(X_0 X_\omega X''X_1), &&\text{by 8.321,} \\ &= ℞(X_\omega X_0 X'X_1) \,.\, ℞(Y_\omega Y_0 Y'Y_1), &&\text{by (1).} \end{aligned}$$

Hence

$$℞(X_0 X_\omega X'X_1) \,.\, ℞(Y_0 Y_\omega Y'Y_1) \,.\, ℞(Z_0 Z_\omega Z'Z_1) = 1.$$

This proves that the condition is necessary. That it is also sufficient follows at once by the *reductio ad absurdum* argument.

It is perhaps worth noticing that the above theorem is the equivalent in Projective Geometry of the metrical theorem usually known as Ceva's theorem. The companion theorem, Menelaus's theorem, has also a projective equivalent, and, though it is not needed, it is enunciated here. The proof is left as an exercise to the reader.

8.412. THEOREM. *If X', Y', and Z' are, respectively, on the sides YZ, ZX, and XY of a gauged triangle of reference XYZ,* *and are distinct from X, Y, and Z, then the necessary and sufficient condition that they should be collinear is that*

$$\text{R}(X_0 X_\omega X' X_1) . \text{R}(Y_0 Y_\omega Y' Y_1) . \text{R}(Z_0 Z_\omega Z' Z_1) = -1.$$

8.42. Labelling the Points of the Field

If (y, z) are the homogeneous coordinates of the point X' in 8.411, relative to the gauge-points X_0, X_1, and X_ω, then neither y nor z is zero, and $\text{R}(X_0 X_\omega X' X_1) = y/z$. Similarly, if (z, x) and (x, t) are the homogeneous coordinates of Y' and Z' respectively, relative to the gauge-points collinear with them, then $\text{R}(Y_0 Y_\omega Y' Y_1) = z/x$, and $\text{R}(Z_0 Z_\omega Z' Z_1) = x/t$. The theorem just proved states that the necessary and sufficient condition that XX', YY', and ZZ' should be concurrent is that

$$\frac{y . z . x}{z . x . t} = 1,$$

that is to say, $t = y$.

In other words, given three numbers x, y, and z, none of which are zero, there is a unique point P, common to the three concurrent lines XX', YY', and ZZ', where X' is the point (y, z) on YZ, Y' the point (z, x) on ZX, Z' the point (x, y) on XY. Conversely, given any point P, not on YZ, ZX, or XY, there are three numbers x, y, and z, such that X' is (y, z), Y' is (z, x), and Z' is (x, y). These facts are used in the definition of the homogeneous mesh gauge, which is now given.

DEFINITION. *If XYZ is a gauged triangle of reference, the homogeneous coordinates of any point of the field relative to this triangle of reference are defined as follows:*

(i) *if P be any point on YZ, and if its homogeneous coordinates relative to X_0, X_1, and X_ω be (y, z), its homogeneous coordinates relative to the triangle of reference are $(0, y, z)$;*

(ii) *if P be the point (z, x) on ZX, its homogeneous coordinates relative to the triangle of reference are $(x, 0, z)$;*

(iii) *if P be the point (x, y) on XY, its homogeneous coordinates relative to the triangle of reference are $(x, y, 0)$;*

(iv) *if P be any point not on YZ, ZX, or XY, and if (y, z), (z, x), (x, y) be the coordinates of the points X', Y',* and *Z' respectively, then the homogeneous coordinates of P relative to the triangle of reference are (x, y, z).*

8.43. Observations on the Definition

(i) In the above definition the homogeneous coordinates of the three points X, Y, and Z are defined twice over. For example, the coordinates of the point X are defined in (ii) and (iii). It is important to notice that both definitions give the same coordinates for these points. X has coordinates $(1, 0, 0)$; Y, $(0, 1, 0)$; Z, $(0, 0, 1)$.

(ii) In the homogeneous mesh gauge, a *triple* number-label is given to every point of the field. It is clear that this label is not unique, and that if (x, y, z) is a suitable label for a certain point, the label (kx, ky, kz) is equally suitable, provided k is different from zero.

(iii) If P and Q are two different points, their homogeneous coordinates relative to any triangle of reference are plainly different.

(iv) If (x, y, z) is any triple number-label other than the label $(0, 0, 0)$, there is a point of the field whose homogeneous coordinates are (x, y, z). There is no point whose homogeneous coordinates are $(0, 0, 0)$.

(v) If (x, y, z) and (x', y', z') are two different labels, and if each is different from $(0, 0, 0)$, they are the labels of different points of the field, unless $x/x' = y/y' = z/z'$.

(vi) It has already been noticed that the homogeneous coordinates of the points X, Y, and Z are, respectively, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. It is easily verified that the coordinates of the points X_1, Y_1, Z_1, and I are, respectively, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and $(1, 1, 1)$.

8.44. Homogeneous Equations

The utility of the non-homogeneous mesh gauge lay in the fact that the condition to be fulfilled by the points of a locus could be expressed as an equation between the coordinates of those points, and the same is true of the homogeneous mesh gauge. But the reader who is familiar with homogeneous coordinates in Analytical Geometry will be aware that not every equation connecting three variables x, y, and z is usefully significant there; in fact, only those equations of the type known as *homogeneous equations* are of any value. The same is true of the homogeneous mesh gauge in Projective Geometry. For the sake of the reader who is not familiar with the use of homogeneous coordinates, this point is explained in detail.

In the *non*-homogeneous mesh gauge any equation of the form $f(x, y) = 0$, where $f(x, y)$ is an algebraic function, is the equation of some locus, and not every point of the field satisfies it. But the corresponding proposition for the homogeneous mesh gauge is not true. For consider any algebraic equation $f(x, y, z) = 0$, and suppose that the point P, (x_0, y_0, z_0), satisfies it; it is not difficult to see that usually the point (kx_0, ky_0, kz_0) will not satisfy it; that is to say, the point P both satisfies and does not satisfy it. The equation $x^2+y+z-3 = 0$, for instance, is satisfied by the point $(1, 1, 1)$ but not by the point $(2, 2, 2)$, though these are the coordinates of the same point. Moreover, it may be shown that if P is any point of the field other than $(0, 1, -1)$ there is a specification of P which satisfies the equation, and there is also one which does not. Hence every point of the field save one satisfies the equation and yet does not satisfy it; the remaining point does not satisfy it. In other words, the equation is valueless as the expression of a relation between the coordinates of points on a locus. The only kind of equation which can possibly be of any value is an equation such that if (x_0, y_0, z_0) satisfies it, then (kx_0, ky_0, kz_0) also satisfies it, for all values of k other than zero. Such equations exist, and are called *homogeneous equations.*

DEFINITION. *An algebraic function $f(x, y, z)$ is said to be homogeneous if and only if for all values of x, y, and z, other than simultaneous zeroes, the value of the fraction $\frac{f(kx, ky, kz)}{f(x, y, z)}$ is a constant depending only on the value of k.*

An algebraic equation $f(x, y, z) = 0$ is said to be homogeneous if and only if $f(x, y, z)$ is a homogeneous function.

The following theorem is an immediate consequence of this definition.

THEOREM. *If $f(x, y, z) = 0$ is a homogeneous equation, and if (x_0, y_0, z_0) satisfies it, then (kx_0, ky_0, kz_0) also satisfies it, for every value of k other than zero.*

In practice, the only homogeneous functions which need to be considered are polynomials, and a polynomial in the variables x, y, and z is homogeneous if and only if every term of it is of the same degree in x, y, and z. In the sequel we shall be concerned only with polynomials of the first and second degrees. These will occur, respectively, in (i) the *linear equation* $lx+my+nz = 0$, and (ii) the *quadratic equation*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$

8.5. Loci in the Homogeneous Mesh Gauge

DEFINITION. *If there is an equation $f(x, y, z) = 0$, such that*

(i) *the homogeneous coordinates (kx, ky, kz) of every point of a certain locus satisfy this equation, for all values of k other than the value zero, and*

(ii) *every point whose homogeneous coordinates (kx, ky, kz) satisfy this equation, for all values of k other than the value zero, is a point of the locus,*

then the equation $f(x, y, z) = 0$, or any equivalent of it, is termed the equation of the locus, and the point (x, y, z) is said to satisfy the equation.

This definition should be compared with the corresponding definition for the non-homogeneous mesh gauge (7.6). As has been pointed out above, the equation of a locus is in reality an algebraic statement of the condition of a locus, and a definition of it might have been framed along these lines.

The equations of the line and the point-conic, which are now given, can be deduced from the corresponding equations in the non-homogeneous mesh gauge, but the process, though not intrinsically difficult, is a little complicated; it seems easier and more natural to deduce them anew from first principles.

8.6. The Equation of the Line†

8.61. THEOREM

The equation in homogeneous coordinates of any line of the field is of the form

$$lx+my+nz = 0,$$

where not all of l, m, and n are zero.

† It will probably have been already realized that it is not strictly accurate to speak of the 'equation of a line'; the locus in question is the range of points on the line, and so the equation is the equation of the range of points. The distinction is pointed out because it is needed later.

(i) XYZ being the triangle of reference, let P be any point other than these three; let (x_0, y_0, z_0) be the coordinates of P. Then from the definition of homogeneous coordinates, any point on the line XP, distinct from X must have coordinates (k, y_0, z_0).

Hence the coordinates (x, y, z) of any point on XP, other than X, must satisfy the equation $z_0 y - y_0 z = 0$; moreover, this equation is satisfied by the coordinates of X, $(1, 0, 0)$. Hence the coordinates of every point on the line XP satisfy this equation.

Further, any point which satisfies this equation must have coordinates (k, y_0, z_0), and from the definition of homogeneous coordinates, this is on XP.

Hence the equation of XP is $z_0 y - y_0 z = 0$, and this is of the form specified.

Similarly, the equation of any line on Y or Z is of the form specified.

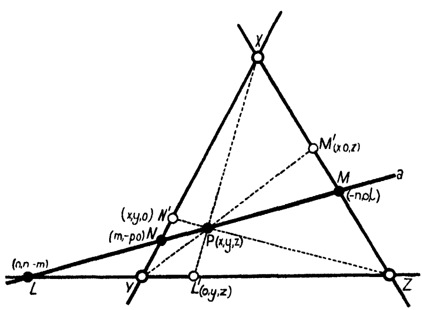


FIG. 55.

(ii) Consider now any line a, not on X, Y, or Z. Let L, M, and N be the three points common to a and YZ, ZX, XY respectively; let their coordinates be $(0, n, -m)$, $(-n, 0, l)$, $(m, -p, 0)$ respectively.†

Let P, (x, y, z), be any point on a other than L, M, and N, and let L', M', N' be, respectively, the points $(0, y, z)$, $(x, 0, z)$, and

† It is a consequence of the projective equivalent of Menelaus's theorem (8.412) that $p = l$; this fact is not used in the proof, but appears as a subsidiary result.

$(x, y, 0)$, so that X, P, and L' are collinear, Y, P, and M' are collinear, Z, P, and N' are collinear.

Then $$(LMNP) \overset{X}{\sim} (LZYL'),$$

hence

$$\begin{aligned} ℞(LMNP) &= ℞(LZYL') = ℞(LX_0 X_\omega L') \\ &= ℞((n, -m), (0,1), (1,0), (y,z)) \\ &= \frac{my}{(my+nz)}. \end{aligned}$$

Similarly, since $$(LMNP) \overset{Y}{\sim} (ZMXM'),$$

$$℞(LMNP) = \frac{(lx+nz)}{lx};$$

and also, since $$(LMNP) \overset{Z}{\sim} (YXNN'),$$

$$℞(LMNP) = \frac{-my}{px}.$$

It follows that

$$\frac{my}{(my+nz)} = \frac{(lx+nz)}{lx} = \frac{-my}{px}.$$

From these two equations it can be at once deduced that $p = l$, and that $lx+my+nz = 0$, and since the points L, M, and N also satisfy this last equation, every point on a satisfies the homogeneous linear equation

$$lx+my+nz = 0.$$

It remains to show that every point which satisfies this equation is a point on a; this is done by the method of *reductio ad absurdum.*

Suppose then that (x_0, y_0, z_0) satisfies the equation, but is not on a. Let Q be the point on a which is collinear with this point and X; then the coordinates of Q must be (t, y_0, z_0), where t is some number different from x_0.

Now since Q is on a,

$$lt+my_0+nz_0 = 0,$$

but, by hypothesis, $$lx_0+my_0+nz_0 = 0,$$

hence $$l(x_0-t) = 0,$$

and since $l \neq 0$, it follows that $t = x_0$. As this is contradictory to the supposition that $t \neq x_0$, that supposition must be absurd. Hence (x_0, y_0, z_0) is on a.

It follows that the equation of any line of the field is of the form $lx+my+nz = 0$.

8.62. Theorem

Any locus whose equation is a homogeneous linear equation is a line of the field.

This theorem is the converse of the last.

Consider the equation $lx+my+nz = 0$. Let (x_0, y_0, z_0) and (x_1, y_1, z_1) be two distinct points on it, so that

$$lx_0+my_0+nz_0 = 0,$$

and

$$lx_1+my_1+nz_1 = 0;$$

it follows that

$$\frac{l}{(y_0 z_1 - y_1 z_0)} = \frac{m}{(z_0 x_1 - z_1 x_0)} = \frac{n}{(x_0 y_1 - x_1 y_0)}.$$

If now $l'x+m'y+n'z = 0$ is the equation of the line on (x_0, y_0, z_0) and (x_1, y_1, z_1), it may be proved in precisely the same way that

$$\frac{l'}{(y_0 z_1 - y_1 z_0)} = \frac{m'}{(z_0 x_1 - z_1 x_0)} = \frac{n'}{(x_0 y_1 - x_1 y_0)};$$

from these two equations it follows that $l/l' = m/m' = n/n'$, so that the two equations $lx+my+nz = 0$ and $l'x+m'y+n'z = 0$ are equivalent. Hence the former is the equation of a line of the field.

The following theorems are important enough to merit formal enunciation, but since they are simple consequences of previous work they are left to the reader as examples.

8.63. Theorem

The equation of the line on the two distinct points (x_0, y_0, z_0) and (x_1, y_1, z_1) is

$$(y_0 z_1 - y_1 z_0)x + (z_0 x_1 - z_1 x_0)y + (x_0 y_1 - x_1 y_0)z = 0,$$

or

$$\begin{vmatrix} x & y & z \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{vmatrix} = 0.$$

8.64. Theorem

The necessary and sufficient condition that the three distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *should be collinear is that*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

8.65. Theorem

If (x_0, y_0, z_0) *and* (x_1, y_1, z_1) *be two distinct points, then*

(i) $(\lambda x_0 + \mu x_1, \lambda y_0 + \mu y_1, \lambda z_0 + \mu z_1)$ *are the coordinates of a point collinear with them, for all values of* λ *and* μ, *save only the single pair* $\lambda = \mu = 0$, *and*

(ii) *the coordinates of any point collinear with these two may be expressed in the form* $(\lambda x_0 + \mu x_1, \lambda y_0 + \mu y_1, \lambda z_0 + \mu z_1)$.

8.66. Theorem

If P_1, P_2, P_3, *and* P_4 *be four distinct collinear points, having respectively the coordinates* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), *and* (x_4, y_4, z_4), *then the cross-ratio* $℞(P_1 P_2 P_3 P_4)$ *is equal to whichever of the following cross-ratios are significant:*

$$℞\{(y_1, z_1), (y_2, z_2), (y_3, z_3), (y_4, z_4)\};$$
$$℞\{(z_1, x_1), (z_2, x_2), (z_3, x_3), (z_4, x_4)\};$$
$$℞\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}.$$

It should be noticed that at least one of the above cross-ratios must be significant.

8.67. Theorem

If (x', y', z') *and* (x'', y'', z'') *be two distinct points, and if* P_1, P_2, P_3, *and* P_4 *be four distinct points collinear with these,* P_n *having coordinates*

$$(\lambda_n x' + \mu_n x'', \lambda_n y' + \mu_n y'', \lambda_n z' + \mu_n z'') \quad (n = 1, 2, 3, 4),$$

then $℞(P_1 P_2 P_3 P_4) = ℞\{(\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3), (\lambda_4, \mu_4)\}$.

Examples

1. Determine the equation of the line on the points $(1, 0, -1)$, and $(-1, 1, 0)$ in the form $lx + my + nz = 0$, and show that the point $(0, 1, -1)$ is also on it.

2. Find the coordinates of the point common to the distinct lines whose equations are $l_1 x + m_1 y + n_1 z = 0$ and $l_2 x + m_2 y + n_2 z = 0$.

3. Show that the necessary and sufficient condition that the three distinct lines $l_\nu x+m_\nu y+n_\nu z = 0$ $(\nu = 1,2,3)$ should be concurrent is that

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

4. For what values of t are the three points $(t,n,-m)$, $(-n,t,l)$, $(m,-l,t)$ collinear? Show that if these three points are collinear and $t \neq 0$, the point (l,m,n) is collinear with them.

5. Determine the coordinates of the three diagonal points of the four-point (x_0,y_0,z_0), (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3).

6. Verify algebraically the harmonic proposition (4.151) that there is no non-singular four-point whose diagonal points are collinear.

8.7. The Dual Mesh Gauge

In the last chapter no attempt was made to define a mesh gauge which was the dual of the non-homogeneous mesh gauge there studied. This was not because it was impossible, but because it would have fulfilled no useful purpose to do so. It is, however, extremely useful to elaborate a system of coordinates which is the dual of the homogeneous mesh gauge, for if this is done, the algebraic method becomes a much more flexible instrument for the study of Projective Geometry. For this reason the dual system is now developed; it follows exactly the same lines as the development of the homogeneous mesh gauge in the earlier stages of this chapter.

8.71. Labelling the Lines on a Point

Gauge-lines l_0, l_1, and l_ω are chosen on any point P, and the open set of lines on P is labelled, each with a *single* number-label; this is the dual of the process of labelling the points of the open set on a line, as in the last chapter.

Double number-labels are now given to all the lines on P in exactly the same way as double number-labels were given to the points of a line in 8.1. That is to say, (i) to a line whose single number-label is x, the double number-label (x_1,x_2) is given, where $x_1/x_2 = x$, and (ii) to the line l_ω the double number-label $(r,0)$ is given, where r is any number different from zero.

It is plain that with this labelling of the pencil of lines on a point, the duals of Theorems 8.21, 8.22, 8.23, and 8.24 are all true, and, in addition, the following self-dual theorem.

THEOREM. *The general equation of a projectivity between a range and a pencil is* $ax_1l_1+bx_1l_2+cx_2l_1+dx_2l_2 = 0$, *where* (x_1, x_2) *is the double number-label of any point of the range, and* (l_1, l_2) *is the double number-label of the corresponding line of the pencil, and* $bc-ad \neq 0$.

The cross-ratio of any four concurrent lines is now defined, and the duals of Theorems 8.3, 8.321, 8.322, 8.323, 8.34, and 8.35 are true. The following self-dual theorem is also a consequence of the definition.

THEOREM. *If* P_1, P_2, P_3, *and* P_4 *be four distinct collinear points, and* l_1, l_2, l_3, *and* l_4 *four concurrent lines, the necessary and sufficient condition that* $(P_1P_2P_3P_4) \sim (l_1l_2l_3l_4)$ *is that*

$$℞(P_1P_2P_3P_4) = ℞(l_1l_2l_3l_4).$$

8.72. Labelling the Lines of the Field

In order to label all the lines of the field a triangle of reference is first defined.

Choose any non-singular four-line *lmni* in the field, and let l_1, m_1, and n_1 be its diagonal lines, l_1 being the line $\begin{pmatrix} li \\ mn \end{pmatrix}$, m_1 the line $\begin{pmatrix} mi \\ nl \end{pmatrix}$, and n_1 the line $\begin{pmatrix} ni \\ lm \end{pmatrix}$.

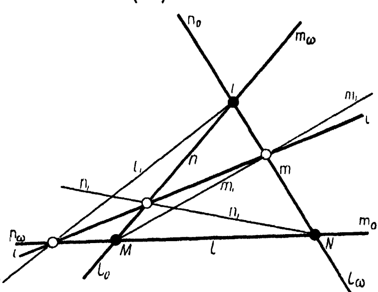


FIG. 56.

Calling the three points *mn*, *nl*, and *lm*, *L*, *M*, and *N* respectively, gauge-lines are now chosen on these three points as follows:

(i) on *L*, gauge-lines l_0, l_1, and l_ω, coinciding with *n*, l_1, and *m* respectively;

(ii) on M, gauge-lines m_0, m_1, and m_ω, coinciding with l, m_1, and n respectively;

(iii) on N, gauge-lines n_0, n_1, and n_ω, coinciding with m, n_1, and l respectively.

The triangle LMN (lmn) so gauged will be called the *triangle of reference* for the dual gauge, and the line i will be called the *gauging line*.

The dual of Theorem 8.411 can now be proved, and from it will follow, exactly as from 8.411, that any line of the field p, not on L, M, or N, determines three numbers λ, μ, and ν, whose ratios are unique and which are such that (i) the line $\begin{pmatrix} lp \\ mn \end{pmatrix}$ has the label (μ, ν) in the labelled pencil on L, (ii) the line $\begin{pmatrix} mp \\ nl \end{pmatrix}$ has the label (ν, λ) in the labelled pencil on M, and (iii) the line $\begin{pmatrix} np \\ lm \end{pmatrix}$ has the label (λ, μ) in the labelled pencil on N. This is illustrated in the accompanying figure.

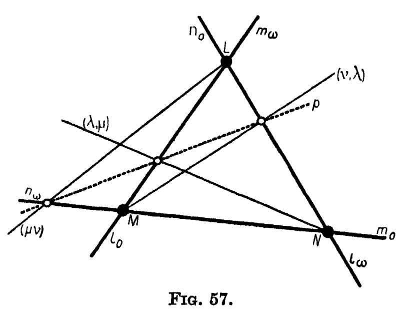


FIG. 57.

Homogeneous coordinates of any line of the field are now defined, the definition being the dual of that given in 8.42.

EXAMPLES

1. Prove the dual of Theorem 8.411 without appeal to the principle of duality.

2. Prove the dual of Theorem 8.412 without appeal to the principle of duality.

8.73. Envelopes in the Dual Homogeneous Mesh Gauge

DEFINITION. *If there is an equation $F(\lambda, \mu, \nu) = 0$, such that*

(i) *the homogeneous coordinates $(k\lambda, k\mu, k\nu)$ of every line of a certain envelope satisfy this equation, for all values of k other than the value zero, and*

(ii) *every line whose homogeneous coordinates $(k\lambda, k\mu, k\nu)$ satisfy this equation, for all values of k other than the value zero, is a line of the envelope,*

then the equation $F(\lambda, \mu, \nu) = 0$ or any equivalent of it is termed the equation of the envelope, and the line (λ, μ, ν) is said to satisfy the equation.

This definition is the dual of the definition of 8.5, and from the principle of duality the following theorems are true:

8.731. THEOREM. *The equation in dual homogeneous coordinates of the pencil of lines on any point of the field is a linear homogeneous equation.*

8.732. THEOREM. *Any envelope, whose equation is a linear homogeneous equation, is a pencil of lines on some point of the field.*

8.733. THEOREM. *The equation of the pencil on the common point of two distinct lines $(\lambda_0, \mu_0, \nu_0)$ and $(\lambda_1, \mu_1, \nu_1)$ is*

$$(\mu_0\nu_1-\mu_1\nu_0)\lambda+(\nu_0\lambda_1-\nu_1\lambda_0)\mu+(\lambda_0\mu_1-\lambda_1\mu_0)\nu = 0,$$

or

$$\begin{vmatrix} \lambda & \mu & \nu \\ \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \end{vmatrix} = 0.$$

8.734. THEOREM. *The necessary and sufficient condition that the three distinct lines $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$ should be concurrent is that*

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0.$$

8.735. THEOREM. *If $(\lambda_0, \mu_0, \nu_0)$ and $(\lambda_1, \mu_1, \nu_1)$ be two distinct lines, then*

(i) *$(p\lambda_0+q\lambda_1, p\mu_0+q\mu_1, p\nu_0+q\nu_1)$ are the coordinates of a line* *which is on their common point, for all values of p and q, save only the single pair $p = q = 0$, and*

(ii) *the coordinates of any line which is on the common point of these two may be expressed in the form*

$$(p\lambda_0+q\lambda_1, p\mu_0+q\mu_1, p\nu_0+q\nu_1).$$

8.736. THEOREM. *If l_1, l_2, l_3, and l_4 be four distinct concurrent lines, having respectively the coordinates (λ_1,μ_1,ν_1), (λ_2,μ_2,ν_2), (λ_3,μ_3,ν_3), and (λ_4,μ_4,ν_4), then the cross-ratio $R(l_1\,l_2\,l_3\,l_4)$ is equal to whichever of the following cross-ratios are significant:*

$$R\{(\mu_1,\nu_1), (\mu_2,\nu_2), (\mu_3,\nu_3), (\mu_4,\nu_4)\};$$

$$R\{(\nu_1,\lambda_1), (\nu_2,\lambda_2), (\nu_3,\lambda_3), (\nu_4,\lambda_4)\};$$

$$R\{(\lambda_1,\mu_1), (\lambda_2,\mu_2), (\lambda_3,\mu_3), (\lambda_4,\mu_4)\}.$$

8.737. THEOREM. *If (λ, μ, ν) and (λ', μ', ν') be two distinct lines, and if l_1, l_2, l_3, and l_4 be four distinct lines on the common point of these two, l having coordinates*

$$(p_n\lambda+q_n\lambda', p_n\mu+q_n\mu', p_n\nu+q_n\nu') \quad (n = 1,2,3,4),$$

then

$$R(l_1\,l_2\,l_3\,l_4) = R\{(p_1,q_1), (p_2,q_2), (p_3,q_3), (p_4,q_4)\}.$$

The preceding theorems, 8.731–8.737, are the duals of Theorems 8.61–8.67 respectively; the reader should examine them carefully, and satisfy himself that they are true; it is a useful exercise to prove the first two of them without appealing to the principle of duality.

8.74. Simultaneous Dual Mesh Gauges

The simplicity and beauty of the algebraic method in Projective Geometry is the result, in large measure, of the simultaneous use of the two dual mesh gauges which have been defined in this chapter. To achieve this, then, it is plainly necessary to impose two mesh gauges simultaneously on the field; in one of these, every point of the field is labelled, in the other, every line. When this has been done, the relations between the two must be found; that is to say, given the *coordinates* of a point in the first mesh gauge, the *equation* of the pencil on it in the second must be found, and vice versa, and given the *equation* of a line in the first mesh gauge, its *coordinates* in the second must be found, and vice versa.

To avoid confusion in dealing with two mesh gauges simultaneously, the following additions to terminology and notation are made.

(i) The mesh gauge wherein every *point* of the field is labelled will be called the *point mesh gauge*, or the system of *point coordinates*; the dual mesh gauge wherein every *line* of the field is labelled will be called the *line mesh gauge*, or the system of *line coordinates*; the term *complete mesh gauge*, or simply, *mesh gauge*, will denote both of these two simultaneously.

(ii) A triple number-label which is the coordinates of a *point* will, as heretofore, be enclosed in round brackets; a triple number-label which is the coordinates of a *line* will in future be enclosed in square brackets, thus: $[\lambda, \mu, \nu]$.

In imposing the two mesh gauges on the field it is obviously possible to choose two arbitrary triangles (that is, triangles not specially related) as the triangles of reference for the two mesh gauges. If this is done, there is no gain in generality, and there is a distinct loss in simplicity; we therefore choose the same triangle of reference for both.

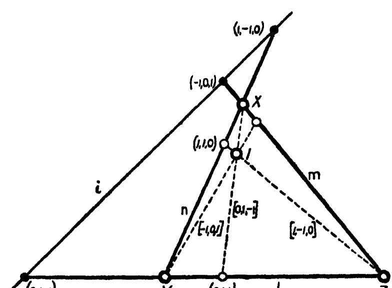


FIG. 58.

Let $XYZI$ be any non-singular four-point; the triangle XYZ is taken as the triangle of reference for the point mesh gauge, the point I as the gauging point. For the line mesh gauge the non-singular four-line $lmni$ is chosen, where l, m, and n are, respectively, the lines YZ, ZX, and XY, and i is the line whose equation is (in the point mesh gauge already set up) $x+y+z = 0$. The triangle XYZ (lmn) is the triangle of reference for this mesh gauge, the line i the gauging line.

It will be noticed that the line i is on three of the harmonic points, $(0,1,-1)$, $(-1,0,1)$, and $(1,-1,0)$, of the four-point $XYZI$, and that the point I is on three of the harmonic lines, $[0,1,-1]$, $[-1,0,1]$, and $[1,-1,0]$, of the four-line $lmni$.

The following theorem gives the relations between the two mesh gauges.

8.741. Theorem. *In the complete mesh gauge set up,*

(i) *the base of the range whose equation in the point mesh gauge is* $\lambda x+\mu y+\nu z = 0$ *has coordinates* $[\lambda, \mu, \nu]$ *in the line mesh gauge, and vice versa, and, dually,*

(ii) *the base of the pencil whose equation in the line mesh gauge is* $\lambda x+\mu y+\nu z = 0$ *has coordinates* (x, y, z) *in the point mesh gauge, and vice versa.*

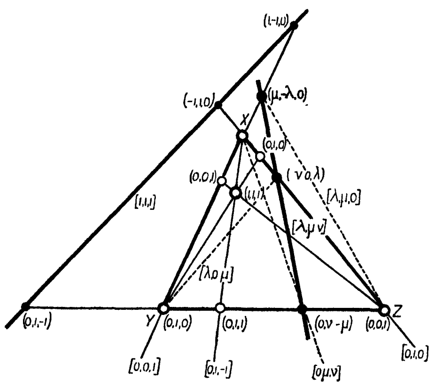


Fig. 59.

Consider the labelled pencil of lines on X, and the labelled range of points on YZ. There is a perspectivity between these two, and in this perspectivity,

(1) the line $[0,1,0]$ corresponds to the point $(0,0,1)$,
(2) the line $[0,0,1]$ corresponds to the point $(0,1,0)$, and
(3) the line $[0,1,1]$ corresponds to the point $(0,1,-1)$.

Hence if, in this perspectivity, the line $[0, \mu, \nu]$ corresponds to the point $(0, y, z)$, and the equation of the perspectivity is

$$a\mu y + b\mu z + c\nu y + d\nu z = 0,$$

the conditions (1), (2), and (3) above entail that $b = c = 0$, and $a = d$, so that the equation is $\mu y + \nu z = 0$.

Hence the line $[0, \mu, \nu]$ corresponds to the point $(0, \nu, -\mu)$.

Similarly, in the perspectivity between the pencil on Y and the pencil on ZX, the line $[\lambda, 0, \nu]$ corresponds to the point $(-\nu, 0, \lambda)$; and in the perspectivity between the pencil on Z and the range on XY, the line $[\lambda, \mu, 0]$ corresponds to the point $(\mu, -\lambda, 0)$.

Take now any line whose equation in the point mesh gauge is $\lambda x + \mu y + \nu z = 0$. This line is on the points $(0, \nu, -\mu)$, $(-\nu, 0, \lambda)$, and $(\mu, -\lambda, 0)$; hence, by what has just been próved, the lines on X, Y, and Z respectively which determine its coordinates in the line mesh gauge are $[0, \mu, \nu]$, $[\lambda, 0, \nu]$, and $[\lambda, \mu, 0]$. That is to say, its coordinates in the line mesh gauge are $[\lambda, \mu, \nu]$.

This proves the first part of the theorem; the second part is the dual of this.

This theorem may be stated in another way thus:

8.742. THEOREM. *In the complete mesh gauge the necessary and sufficient condition that the point (x, y, z) should be on the line $[\lambda, \mu, \nu]$ is that $\lambda x + \mu y + \nu z = 0$.*

8.743. The Complete Mesh Gauge and the Algebraic Representation. It should not be necessary at this stage to point out that the complete mesh gauge is something different from the Algebraic Representation, though there are formal likenesses between the two. (See the end of 2.31.) The fact that it is now possible to refer to points and lines by triple number-labels does not mean, however, that we have now only one possible representation of Projective Geometry, namely the Algebraic Representation; but it does mean that now, because of the initial proposition of extension added in the last chapter, we are confined to those representations which are isomorphic with the Algebraic Representation.

EXAMPLES

1. Prove part (ii) of Theorem 8.741 directly, and without appealing to the principle of duality.

2. Determine the coordinates of the line on both of the points

(x_1, y_1, z_1) and (x_2, y_2, z_2). Determine also the coordinates of the point on both of the lines $[\lambda_1, \mu_1, \nu_1]$ and $[\lambda_2, \mu_2, \nu_2]$.

3. P is the point (a, b, c); show that three of the harmonic points of the four-point $XYZP$ are all on the line $[a^{-1}, b^{-1}, c^{-1}]$. Show also that three of the harmonic lines of the four-line, $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, and $[a, b, c]$, are all on the point (a^{-1}, b^{-1}, c^{-1}). Does the second proposition follow from the first by the principle of duality?

4. $A_0 A_1 A_2 A_3$ is a non-singular four-point whose diagonal points are D_1, D_2, and D_3 and whose harmonic points are H_1, H_1', H_2, H_2', H_3, H_3'. A complete mesh gauge is set up as follows: (i) the triangle of reference XYZ for the point mesh gauge is the triangle $A_1 A_2 A_3$ (in that order), and the gauging point is A_0; (ii) the triangle of reference LMN for the line mesh gauge is the triangle $D_1 D_2 D_3$ (in that order), and the gauging line is the line on H_1, H_2, and H_3. Show that in this mesh gauge the necessary and sufficient condition that a point (x, y, z) shall be on a line $[\lambda, \mu, \nu]$ is that

$$x(\lambda-\mu-\nu)+y(\mu-\nu-\lambda)+z(\nu-\lambda-\mu) = 0.$$

Determine also, in this mesh gauge, the coordinates of the line on both of the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

8.8. The Equations of the Conic

The conic is a self-dual figure consisting of a point-conic and the set of all tangents to it, this last forming a line-conic. The same conic is therefore associated with two equations; one of these is the equation of the point-conic, in the point mesh gauge, the other is the equation of the line-conic, in the line mesh gauge. The first will be called the *point equation*, the second the *line equation*. The two are not alternative forms of the same equation; one is the equation of a locus, i.e. a point-figure, the other is the equation of an envelope, i.e. a line-figure.

8.81. The Point Equation

THEOREM. *The point equation of any conic is of the form*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$

Let Φ be any point-conic, P and Q any two distinct points on it. Then there is a projectivity between the pencils on these two points such that the common points of corresponding lines are all on Φ.

Let $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ be any two distinct lines on P; then any line on P has coordinates

$$[\lambda l_1+\mu l_2, \lambda m_1+\mu m_2, \lambda n_1+\mu n_2].$$

Take now two distinct lines $[l_3, m_3, n_3]$ and $[l_4, m_4, n_4]$ on Q, and choose their coordinates in such a way that in the projectivity between the two pencils the line

$$[\lambda l_3+\mu l_4, \lambda m_3+\mu m_4, \lambda n_3+\mu n_4] \text{ on } Q$$

corresponds to the line

$$[\lambda l_1+\mu l_2, \lambda m_1+\mu m_2, \lambda n_1+\mu n_2] \text{ on } P.$$

It is plainly possible to make this choice.

Now the common point of these two is (x, y, z), where

$$kx = \lambda^2(m_1 n_3-m_3 n_1)+\lambda\mu(m_2 n_3-m_3 n_2+m_1 n_4-m_4 n_1)+ +\mu^2(m_2 n_4-m_4 n_2), \quad (1)$$

$$ky = \lambda^2(n_1 l_3-n_3 l_1)+\lambda\mu(n_2 l_3-n_3 l_2+n_1 l_4-n_4 l_1)+ +\mu^2(n_2 l_4-n_4 l_2), \quad (2)$$

$$kz = \lambda^2(l_1 m_3-l_3 m_1)+\lambda\mu(l_2 m_3-l_3 m_2+l_1 m_4-l_4 m_1)+ +\mu^2(l_2 m_4-l_4 m_2), \quad (3)$$

and $k \neq 0$.

This point (x, y, z) is a point on Φ, whatever be the values of λ and μ. Hence, if λ and μ be eliminated from the three equations above, an equation will be obtained which is satisfied by all the points on Φ.

Rewriting the three equations in the form

$$kx = a_{11}\lambda^2+a_{12}\lambda\mu+a_{13}\mu^2,$$
$$ky = a_{21}\lambda^2+a_{22}\lambda\mu+a_{23}\mu^2,$$
$$kz = a_{31}\lambda^2+a_{32}\lambda\mu+a_{33}\mu^2,$$

and denoting by A_{rs} the minor of a_{rs} in the determinant† $|a_{rs}|$, we have

$$(A_{11}x+A_{21}y+A_{31}z)(A_{13}x+A_{23}y+A_{33}z) = (A_{12}x+A_{22}y+A_{32}z)^2,$$

and this is a homogeneous quadratic equation, that is, it is of the form specified.

It remains to prove that any point which satisfies the equation is a point of Φ. Suppose then that (x_0, y_0, z_0) satisfies the equation.

† This determinant is not, in general, zero.

Write

$$A_{11}x_0+A_{21}y_0+A_{31}z_0 = \lambda^2 k',$$
$$A_{12}x_0+A_{22}y_0+A_{32}z_0 = \lambda\mu k',$$
$$A_{13}x_0+A_{23}y_0+A_{33}z_0 = \mu^2 k';$$

this is permissible since (x_0, y_0, z_0) satisfies the equation.

Then clearly the equations (1), (2), and (3) are satisfied for some value of k by x_0, y_0, and z_0. That is to say, the point (x_0, y_0, z_0) is the common point of two corresponding lines of the pencils on P and Q; hence it is on Φ.

Hence the equation of any point-conic is a homogeneous quadratic equation.

8.82. The Line Equation

THEOREM. *The line equation of any conic is of the form*

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0.$$

This is the dual of the last theorem.

8.83. The Converse Theorems

THEOREM. *Any locus whose (point) equation is a homogeneous quadratic equation is a point-conic.*

THEOREM. *Any envelope whose (line) equation is a homogeneous quadratic equation is a line-conic.*

The first of these two theorems is proved in precisely the same way as the second part of Theorem 7.63; the second is the dual of the first.

8.84. Singular Conics

THEOREM. *The necessary and sufficient condition that the point-conic whose equation is*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$$

should be singular is that the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

should vanish.

THEOREM. *The necessary and sufficient condition that the line-conic whose equation is*

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0$$

should be singular is that the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

should vanish.

The proof of the first of these theorems is very similar to that of Theorem 7.631; the second is the dual of the first.

8.85. Tangents and Points of Contact

THEOREM. *If* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ *be the point equation of a non-singular conic, then*

(i) $axx_0+byy_0+czz_0+f(yz_0+zy_0)+g(zx_0+xz_0)+$
$$+h(xy_0+yx_0) = 0$$

is the equation of the tangent at (x_0, y_0, z_0) *on the conic, and*

(ii) $(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)\times$
$$\times(ax_0^2+by_0^2+cz_0^2+2fy_0z_0+2gz_0x_0+2hx_0y_0)$$
$$= \{axx_0+byy_0+czz_0+f(yz_0+zy_0)+g(zx_0+xz_0)+$$
$$+h(xy_0+yx_0)\}^2$$

is the equation of the pair of tangents to the conic which are on the point (x_0, y_0, z_0).

THEOREM. *If* $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0$ *be the line equation of any non-singular conic, then*

(i) $All_0+Bmm_0+Cnn_0+F(mn_0+nm_0)+G(nl_0+ln_0)+$
$$+H(lm_0+ml_0) = 0$$

is the equation of the point of contact of any line $[l_0, m_0, n_0]$ *on the conic, and*

(ii) $(Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm)\times$
$$\times(Al_0^2+Bm_0^2+Cn_0^2+2Fm_0n_0+2Gn_0l_0+2Hl_0m_0)$$
$$= \{All_0+Bmm_0+Cnn_0+F(mn_0+nm_0)+$$
$$+G(nl_0+ln_0)+H(lm_0+ml_0)\}^2$$

is the equation of the pair of points common to the line $[l_0, m_0, n_0]$ *and the conic.*

The proof of the first of these two theorems is very similar to the proof of the first two parts of Theorem 7.632; the second is the dual of the first.

It should be noticed that the expression

$$axx_0+byy_0+czz_0+f(yz_0+zy_0)+g(zx_0+xz_0)+h(xy_0+yx_0)$$

may be written in either of the forms:

$$x(ax_0+hy_0+gz_0)+y(hx_0+by_0+fz_0)+z(gx_0+fy_0+cz_0),$$
$$x_0(ax+hy+gz)+y_0(hx+by+fz)+z_0(gx+fy+cz).$$

The dual expression can, plainly, be written in similar ways.

8.86. Point Equation and Line Equation of the same Conic

The preceding theorem enables us to solve the following problem: Given the point equation of a non-singular conic, what is its line equation, and vice versa? The answer is contained in the following theorems.

THEOREM. *If* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ *be the point equation of a non-singular conic, its line equation is*

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0,$$

where A, B, C, F, G, *and* H *are the minors of* a, b, c, f, g, *and* h *respectively in the non-vanishing determinant*

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

THEOREM. *If* $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0$ *be the line equation of a non-singular conic, its point equation is* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$, *where* a, b, c, f, g, *and* h, *are the minors of* A, B, C, F, G, *and* H *respectively in the non-vanishing determinant*

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

The two theorems are dual; only the first is proved, the second then follows by the principle of duality.

Let (x_0, y_0, z_0) be any point on the non-singular conic whose point equation is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$. Then by the preceding theorem the tangent at (x_0, y_0, z_0) to this conic has the equation

$$(ax_0+hy_0+gz_0)x+(hx_0+by_0+fz_0)y+(gx_0+fy_0+cz_0)z = 0;$$

so that the coordinates of this tangent are

$$[(ax_0+hy_0+gz_0), (hx_0+by_0+fz_0), (gx_0+fy_0+cz_0)].$$

Hence if $[l, m, n]$ are the coordinates of any line on the conic, a point (x_0, y_0, z_0) must exist such that

$$kl+ax_0+hy_0+gz_0 = 0, \qquad (1)$$

$$km+hx_0+by_0+fz_0 = 0, \qquad (2)$$

$$kn+gx_0+fy_0+cz_0 = 0. \qquad (3)$$

Moreover, since the point (x_0, y_0, z_0) is on the line $[l, m, n]$,

$$lx_0+my_0+nz_0 = 0. \qquad (4)$$

If x_0, y_0, z_0, and k be eliminated from the equations (1)–(4), the equation

$$\begin{vmatrix} l & a & h & g \\ m & h & b & f \\ n & g & f & c \\ 0 & l & m & n \end{vmatrix} = 0$$

is left.

On expansion, this becomes

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0.$$

This being the line equation of a conic, and being satisfied by every line on the conic in question, must be the line equation of that conic. This proves the theorem.

8.87. Pole and Polar

THEOREM. *The polar of the point* (x_0, y_0, z_0), *relative to the non-singular conic whose point equation is*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$

is the line whose equation is

$$(ax_0+hy_0+gz_0)x+(hx_0+by_0+fz_0)y+(gx_0+fy_0+cz_0)z = 0.$$

THEOREM. *The pole of the line* $[l_0, m_0, n_0]$, *relative to the non-singular conic whose line equation is*

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm = 0,$$

is the point whose equation is

$$(Al_0+Hm_0+Gn_0)l+(Hl_0+Bm_0+Fn_0)m+ \\ +(Gl_0+Fm_0+Cn_0)n = 0.$$

The second of these theorems follows from the first by the principle of duality. The first has already been proved in

Theorem 8.85 when the point (x_0, y_0, z_0) is on the conic; when it is not on the conic, the proof given in part (iii) of Theorem 7.632 can be adapted to the homogeneous mesh gauge, or the following independent proof may be used.

Suppose that P is the point (x_0, y_0, z_0), and that Q, (x, y, z), is any point on the polar of P. Then if U and V are the two points on the conic collinear with P and Q, (PQ, UV) is a harmonic tetrad, that is to say, $\text{R}(PQUV) = -1$.

Now any point collinear with (x_0, y_0, z_0) and (x, y, z) has coordinates $(\lambda x + \mu x_0, \lambda y + \mu y_0, \lambda z + \mu z_0)$ and if this point is on the conic,

$$a(\lambda x+\mu x_0)^2+b(\lambda y+\mu y_0)^2+c(\lambda z+\mu z_0)^2+2f(\lambda y+\mu y_0)(\lambda z+\mu z_0)+$$
$$+2g(\lambda z+\mu z_0)(\lambda x+\mu x_0)+2h(\lambda x+\mu x_0)(\lambda y+\mu y_0) = 0,$$

or

$$\lambda^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)+$$
$$+2\lambda\mu\{x(ax_0+hy_0+gz_0)+y(hx_0+by_0+fz_0)+z(gx_0+fy_0+cz_0)\}+$$
$$+\mu^2(ax_0^2+by_0^2+cz_0^2+2fy_0z_0+2gz_0x_0+2hx_0y_0) = 0.$$

This is a quadratic equation which gives the two possible values of the ratio of λ to μ; if these be denoted by λ_1, μ_1, and λ_2, μ_2, the necessary and sufficient condition that (x, y, z) should be on the polar of (x_0, y_0, z_0) is that

$$\text{R}(\infty, 0, \lambda_1/\mu_1, \lambda_2/\mu_2) \equiv \frac{\lambda_1\mu_2}{\lambda_2\mu_1} = -1,$$

that is,
$$\frac{\lambda_1}{\mu_1}+\frac{\lambda_2}{\mu_2} = 0.$$

From the theory of the quadratic equation it follows that the necessary and sufficient condition that (x, y, z) be on the polar of (x_0, y_0, z_0) is that

$$(ax_0+hy_0+gz_0)x+(hx_0+by_0+fz_0)y+(gx_0+fy_0+cz_0)z = 0;$$

this is therefore the equation of the polar of (x_0, y_0, z_0).

8.88. Notation

It is convenient in working examples and in subsequent work to have shortened forms for the various algebraical expressions which occur in the theory of the conic, and for this reason the following are introduced.

(i) The expression

$$(ax_0+hy_0+gz_0)x+(hx_0+by_0+fz_0)y+(gx_0+fy_0+cz_0)z$$

may be shortened to $f(x,x_0;\, y,y_0;\, z,z_0)$, and this may be still further shortened without ambiguity to $f(x,x_0)$. This latter form will be used henceforward, together with its variants. Thus the equation $f(x,x) = 0$ is the short form of

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0;$$

moreover, it is plain that $f(x_0,x_1) \equiv f(x_1,x_0)$.

(ii) If $f(x,x) = 0$ is the point equation of a conic, the line equation of the same conic will be written $F(l,l) = 0$, and the same variants of $F(l,l)$ will be used.

(iii) By X will be denoted the expression $ax+hy+gz$, by Y the expression $hx+by+fz$, by Z the expression $gx+fy+cz$.

X_0, Y_0, Z_0, etc., will have their obvious meanings. But if there is danger of these *numbers* being confused with the suffixed *letters* used for the gauge-points, their use should be avoided.

8.881. A Note on Examples. Many quite general theorems, independent of any mesh gauge, can be proved algebraically, that is, by imposing a mesh gauge on the field. For example, it is possible to prove Pascal's theorem algebraically. In doing so, it is allowable to select any conic of the field, and in practical work this would reduce to choosing the conic whose point equation is $f(x,x) = 0$. But the actual algebra involved will be very much simpler if a conic with a simpler equation is chosen. This is tantamount to choosing a conic of the field, and then imposing the mesh gauge in such a way that its equation relative to this mesh gauge is simple. There is plainly no loss in generality in this method. In the examples which follow immediately a number of conics with simple point or line equations will be found. At the end of the set some general examples, in which the method here explained can be used, are given.

Examples

1. Show that the point equation of any conic which is on the three points of the triangle of reference is of the form $fyz+gzx+hxy = 0$, and deduce the line equation.

2. By the principle of duality, write down the line equation of any conic which is on the three sides of the triangle of reference.

3. Determine the equations of the conic on the five points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, (x_1,y_1,z_1), and (x_2,y_2,z_2).

4. Prove that the necessary and sufficient condition that the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) shall be conjugate points relative to the conic whose point equation is $f(x, x) = 0$ is that $f(x_1, x_2) = 0$.

5. Show that the triangle of reference is self-polar relative to any conic whose equation is of the form $ax^2+by^2+cz^2 = 0$.

6. What are the coordinates of the pole of the line on (x_0, y_0, z_0) and (x_1, y_1, z_1) relative to the conic whose point equation is $f(x, x) = 0$?

7. Show that two conics have four points in common.

8. If $f_1(x, x)$ and $f_2(x, x)$ denote respectively the expressions

$$a_1 x^2+b_1 y^2+c_1 z^2+2f_1 yz+2g_1 zx+2h_1 xy$$

and

$$a_2 x^2+b_2 y^2+c_2 z^2+2f_2 yz+2g_2 zx+2h_2 xy,$$

show that any conic which is on the four common points of the two conics, whose point equations are $f_1(x, x) = 0$ and $f_2(x, x) = 0$, has a point equation of the form $\lambda f_1(x, x)+\mu f_2(x, x) = 0$.

9. Determine the coordinates of the common points of the two conics whose point equations are

$$f_1 yz+g_1 zx+h_1 xy = 0 \quad \text{and} \quad f_2 yz+g_2 zx+h_2 xy = 0.$$

10. Dualize Ex. 8.

11. Show that the necessary and sufficient condition that the conic whose point equation is $f(x, x) = 0$ should consist of two coincident ranges is that

$$bc-f^2 = ca-g^2 = ab-h^2 = gh-af = hf-bg = fg-ch = 0.$$

12. Determine the point equation and the line equation of a conic on the point (a, b, c) and such that the lines YZ and ZX are tangents to it, Y and Z being their respective points of contact.

13. Show that for all values of t and u, save simultaneous zeros, the point $[i(t^2+u^2), (t^2-u^2), 2tuR]$ is on the non-singular conic whose point equation is $R^2x^2+R^2y^2+z^2 = 0$. Show also that any point on this conic has coordinates which may be expressed in this form.

14. P is the pole of p relative to a conic Φ. Show that if the locus of P is a range of points on a line, the envelope of p is a pencil on a point. Show also that if the locus of P is a point-conic, the envelope of p is a line-conic.

15. Show that if two triangles are self-polar relative to a certain conic, their six vertices are on a second conic. (Use Ex. 5.)

16. Find the locus of the poles of a line relative to the conics of a pencil on four points, no three of which are collinear. (Take the four points as $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, and $(1, 1, 1)$.)

17. The sides BC, CA, AB of a triangle are tangents to a conic, and D, E, F are, respectively, their points of contact. Show that the three lines AD, BE, CF are concurrent.

18. Show that if two different non-singular conics have four distinct points in common, there is one and only one triangle which is self-polar relative to both.

CHAPTER IX

THE METRIC GAUGE

9.1. Distance and Angle in Elementary Geometry

THE terms *length* (or *distance*) and *angle* are familiar in elementary Geometry, where they are taken as intuitive notions; we are now in a position to introduce the same terms into Projective Geometry, not as intuitive notions, but as terms with a perfectly definite meaning, and to show the relationship between these well-defined terms of Projective Geometry and the intuitive notions of elementary Geometry.

In elementary Geometry *distance* is something measurable associated with two points, *angle* something measurable associated with two lines. We speak of the distance between two points being so many inches or centimetres, the angle between two lines being so many degrees or radians. But though it is difficult to speak more precisely than this (simply because the notions are intuitive and not defined), there are certain fundamental properties of distance and angle which are recognizable as being characteristic. If we denote the distance between two points P and Q by the symbol (PQ), these fundamental properties may be expressed thus:

(i) $(PP) = 0$,

(ii) $(PQ) = -(QP)$,

(iii) if P, Q, and R be three collinear points, then

$$(PQ)+(QR)+(RP) = 0.$$

The corresponding properties of angle are slightly different. If by $[pq]$ is denoted the angle between two lines p and q, and if this angle be measured in radians, then

(i) $[pp] = 2n\pi$, where n is an integer, positive, negative, or zero,†

(ii) $[pq] = -[qp]+2n\pi$, where n is an integer,

(iii) if p, q, and r be three concurrent lines, then

$$[pq]+[qr]+[rp] = 2n\pi,$$

where n is an integer.

† The phrase 'where n is an integer, positive, negative, or zero' will always henceforward be shortened to 'where n is an integer'.

Now, plainly, it is desirable that when the terms *distance* and *angle* are defined in Projective Geometry their meanings shall bear some resemblance to the meanings these same terms bear in elementary Geometry. This is effected by so choosing their definitions that propositions analogous to the above six are true of them. It will be found that the meanings given to the terms are in fact generalizations of the meanings they bear in elementary Geometry.

9.11. Distance on a Line: Angle on a Point

A theorem was proved in the last chapter which can be restated in a form which resembles the third of the three propositions stated above about angle in elementary Geometry. The dual of Theorem 8.35 states that if o, u, p, q, and r be five concurrent lines, the first two being distinct and coinciding with none of the last three, then

$$℞(oupq) \, . \, ℞(ouqr) \, . \, ℞(ourp) = 1,$$

and this is equivalent to

$$\log ℞(oupq) + \log ℞(ouqr) + \log ℞(ourp) = 2n\pi i,$$

where n is an integer,† and $i^2 = -1$.

This form of the theorem provides the basis of a definition of the term *angle* in Projective Geometry.

Let L be any point, o and u a pair of distinct lines on it; these lines will be called the *metric gauge-lines* on L. The angle between two lines on L is now defined.

Definition. *If p and q be two lines (distinct or not) on a point L, both of which are distinct from the metric gauge-lines o and u on L, the angle $[pq]$ is defined to be any one of the values of $k \log ℞(oupq)$, where k is a constant different from zero.*

The ambiguity of the logarithmic function leaves the exact measure of an angle ambiguous by an integral multiple of $2k\pi i$, just as in elementary Geometry the exact measure of an angle (in radians) is ambiguous by an integral multiple of 2π. In actual work this ambiguity is avoided by using periodic functions of the angle, or by a suitable convention; such a convention will be introduced later.

† Owing to the ambiguity of the logarithmic function, it is impossible to say that n is necessarily zero.

The principle of duality suggests that the dual concept be defined, and though it is further removed (apparently) from the notion of *distance* in elementary Geometry than the projectively defined angle is from angle in elementary Geometry, it is given the name *distance.*

Let l be any line, O and U a pair of distinct points on it; these points will be called the metric gauge-points on l.

DEFINITION. *If P and Q be two points (distinct or not) on a line l, both of which are distinct from the metric gauge-points O and U on l, the distance (PQ) is defined to be any one of the values of $k \log \mathrm{R}(OUPQ)$, where k is a constant different from zero.*

9.12. Notes on the Definitions of Distance and Angle

(i) In Projective Geometry the term *distance* is a relative term; it is meaningless unless metric gauge-points have been specified. Hence the distance between two points in Projective Geometry is not, as in elementary Geometry, an apparently inherent property of the two points; it is a property of these two points and the metric gauge-points.

(ii) In Projective Geometry the distance between two points may be any number whatever, real or complex; in elementary Geometry it is usually taken for granted that the distance between two points is always a real number.

(iii) It should be observed that the distance between two points is independent of the mesh gauge imposed on the field, since the cross-ratio of four collinear points is independent of the mesh gauge.

(iv) The constant k appearing in the definition is termed the *scale constant*; its function is not very important.

Similar remarks may be made about the projective definition of *angle.*

EXAMPLES

1. If P is a point on l distinct from the metric gauge-points thereon, show that $(PP) = 2kn\pi i$, where n is any integer. Dualize.

2. If P and Q are distinct points on l, both of which are distinct from the metric gauge-points thereon, show that $(PQ) = -(QP)+2kn\pi i$, where n is any integer. Dualize.

3. If P, Q, and R be three points on a line l, all distinct from the

metric gauge-points thereon, show that $(PQ)+(QR)+(RP) = 2kn\pi i$, where n is any integer. Dualize.

4. If P, Q, and R be three points on a line l, all distinct from the metric gauge-points thereon, and if $(PQ) \neq (PR)+2kn\pi i$, show that Q and R do not coincide. Dualize.

5. If A, B, C, and D are four points on a line l, all distinct from the metric gauge-points thereon, show that

$$(AC)+(BD) = (AD)+(BC)+2kn\pi i.$$

9.2. The Metric Gauge-conic

If the notions of *distance* and *angle* are to be of general use in Projective Geometry, it is clear that metric gauge-points must be assigned on every line of the field and metric gauge-lines on every point of the field. Plainly, it is possible to choose these metric gauge-points and gauge-lines in any way we please, but for the sake of simplicity it is advisable to choose them in a simple and orderly way. Now, however they be chosen, the mixed figure consisting of all the metric gauge-points and gauge-lines is such that at least two of its points are on every line of the field, and at least two of its lines are on every point of the field. The simplest mixed figure which has both of these properties is the non-singular conic, and so we choose the metric gauge-points and gauge-lines in such a way that the mixed figure which they form is a non-singular conic. This non-singular conic is called the *metric gauge-conic*, or, more simply, the *metric gauge*.†

It will be realized that if a non-singular conic be taken as the method of specifying the metric gauge-points and gauge-lines there will be certain lines of the field on which the metric gauge-points coincide, and certain points of the field on which the metric gauge-lines coincide. These points and lines will be noticed in detail in due course.

The general definitions of distance and angle for the whole field are now given formally in the following terms.

DEFINITION. *The non-singular conic Φ being the metric gauge,*

(i) *the distance (PQ) between any two points P and Q is defined to be any one of the values of $k \log Ɍ(M_1 M_2 PQ)$ (pro-*

† Cayley and others call this conic the ***absolute conic*** or the ***absolute***. The term is not adopted here because of the false implication in the word ***absolute***; there is nothing absolute or fixed about the metric gauge.

vided this expression is significant), where M_1 *and* M_2 *are the points on* Φ *collinear with* P *and* Q, *and* k *is a fixed scale-constant different from zero;*

(ii) *the angle* $[pq]$ *between any two lines* p *and* q *is defined to be any one of the values of* $k' \log ℞(m_1 m_2 pq)$ *(provided this expression is significant), where* m_1 *and* m_2 *are the lines on* Φ *concurrent with* p *and* q, *and* k' *is a fixed scale-constant, different from zero.*

9.21. Notes on the Definition.

(i) *Break-down of the Definition.* The expression for (PQ) ceases to be significant if either or both of P and Q are on the metric gauge. The dual proposition is also true.

(ii) *Isotropic Points and Lines.* If l be a line on the metric gauge, and P and Q be two points on l, the distance (PQ) is $2kn\pi i$, however P and Q be chosen, since on l the metric gauge-points coincide, and so $℞(M_1 M_2 PQ) = 1$. Lines which have this peculiarity are called *isotropic lines*; the dual term is *isotropic point.* Clearly all isotropic lines are on the metric gauge, and vice versa.

Points and lines which are not isotropic will be called *non-isotropic* or *ordinary* points and lines. Even at the risk of labouring the obvious, it may be remarked that there is no essential difference between ordinary and isotropic points; they have different properties relative to the metric gauge.

(iii) *Order of the Metric Gauge-points.* If P and Q are a pair of distinct ordinary points, and M_1, M_2 are the metric gauge-points collinear with them, the definition of distance does not make it clear whether (PQ) is equal to $k \log ℞(M_1 M_2 PQ)$ or $k \log ℞(M_2 M_1 PQ)$; that is to say, it does not specify in what order the two metric gauge-points are to be taken in the cross-ratio. The question is left open, but the convention is adopted here that when more than one distance is measured on the same line the same order of the metric gauge-points is kept for all of them. Thus if $P, Q, R, S, \ldots$ are collinear points, and M_1, M_2 are the metric gauge-points collinear with them, and if (PQ) is taken as $k \log ℞(M_2 M_1 PQ)$, then (RS) will be taken as $k \log ℞(M_2 M_1 RS)$, etc.

The choice of a particular order for the metric gauge-points on any line corresponds to the choice in elementary Geometry of a positive direction on a line; for plainly,

$$k \log \mathrm{R}(M_1 M_2 PQ) = -k \log \mathrm{R}(M_2 M_1 PQ) + 2kn\pi i,$$

where n is any integer. The dual convention is also made.

9.22. Laguerre's Theorem in Elementary Geometry

The difference between the metrical notions of Projective Geometry and those of elementary Geometry is not so great as it may appear to be at first sight. There is a theorem in elementary Geometry, known as Laguerre's theorem, which shows the similarity between the two.

Laguerre's theorem states that if P, Q, and R be any three points, and if I and J be the circular points at infinity, the angle PQR is equal to $-\frac{1}{2}i \log \mathrm{R}(PI, PJ, PQ, PR)$.

This theorem is tantamount to saying that angle in elementary Geometry may be measured in exactly the same way as in Projective Geometry, the metric gauge-lines on any point P being the lines PI and PJ. The line-figure formed by all these metric gauge-lines consists of the pencils of lines on I and J; that is to say, it is a singular conic. In a certain sense, then, it can be said that in elementary Geometry the gauge-conic is a singular conic. Though we have in this chapter confined ourselves to non-singular conics as metric gauges, we shall see in the next chapter how the definitions of distance and angle can be extended in such a way that singular conics can be used as metric gauges.

9.3. Deductions from the Definitions

9.31. Pairs of Orthogonal Points and Lines

THEOREM. *The necessary and sufficient condition that two ordinary points, P and Q, should be conjugate points relative to the metric gauge is that $(PQ) = (2n+1)k\pi i$, where n is any integer.*

The necessary and sufficient condition that two ordinary lines, p and q, should be conjugate lines relative to the metric gauge is that $[pq] = (2n+1)k\pi i$, where n is any integer.

The two theorems being dual, only the first is proved.

Let M_1 and M_2 be the points on the metric gauge collinear with P and Q. Then the necessary and sufficient condition that P and Q should be conjugate points relative to the metric gauge is that $(M_1 M_2, PQ)$ should be a harmonic tetrad; that is, that $\mathrm{R}(M_1 M_2 PQ) = -1$; that is, that $(PQ) = (2n+1)k\pi i$.

It is convenient to have a name for pairs of points or lines which are conjugate relative to the metric gauge; two points thus related will be said to be *orthogonal* to each other, or to be a pair of *orthogonal* points. Dually, two lines so related will be said to be *orthogonal* to each other, or to be a pair of *orthogonal* lines. They have properties analogous to those of perpendicular lines in elementary Geometry.

9.32. Distance and Angle in the Homogeneous Mesh Gauge

THEOREM. *If a homogeneous mesh gauge be imposed on the field, and if in this the equations of the metric gauge be* $f(x,x) = 0$ *and* $F(l,l) = 0$, *then*

(i) *the distance* (PQ) *between two ordinary points* P, (x_1, y_1, z_1), *and* Q, (x_2, y_2, z_2), *satisfies the equation*

$$\cosh^2\frac{(PQ)}{2k} = \frac{f^2(x_1,x_2)}{f(x_1,x_1)f(x_2,x_2)},$$

(ii) *the angle* $[pq]$ *between the two ordinary lines* p, $[l_1, m_1, n_1]$, *and* q, $[l_2, m_2, n_2]$, *satisfies the equation*

$$\cosh^2\frac{[pq]}{2k'} = \frac{F^2(l_1,l_2)}{F(l_1,l_1)F(l_2,l_2)}.$$

Suppose that M_1 and M_2 are the two points on the metric gauge collinear with P and Q. Let their coordinates be

$$(\lambda_1 x_1+\mu_1 x_2, \lambda_1 y_1+\mu_1 y_2, \lambda_1 z_1+\mu_1 z_2)$$

and $$(\lambda_2 x_1+\mu_2 x_2, \lambda_2 y_1+\mu_2 y_2, \lambda_2 z_1+\mu_2 z_2).$$

Since these two points are on the metric gauge, the two ratios λ_1 to μ_1 and λ_2 to μ_2 satisfy the quadratic equation

$$\lambda^2 f(x_1,x_1)+2\lambda\mu f(x_1,x_2)+\mu^2 f(x_2,x_2) = 0,$$

and we may therefore write

$$\lambda_1\lambda_2 = f(x_2,x_2),$$

$$\lambda_1\mu_2+\lambda_2\mu_1 = 2f(x_1,x_2),$$

and $$\mu_1\mu_2 = f(x_1,x_1),$$

Now $$\text{R}\!\!\!/(M_1 M_2 PQ) = \text{R}\!\!\!/\left(\frac{\lambda_1}{\mu_1},\frac{\lambda_2}{\mu_2},0,\infty\right) = \frac{\lambda_1\mu_2}{\mu_1\lambda_2};$$

and, similarly, $$\text{R}\!\!\!/(M_1 M_2 QP) = \frac{\lambda_2\mu_1}{\mu_2\lambda_1};$$

it follows that

$$e^{\frac{(PQ)}{k}}+e^{-\frac{(PQ)}{k}}=\frac{\lambda_1\mu_2}{\mu_1\lambda_2}+\frac{\lambda_2\mu_1}{\mu_2\lambda_1}=\frac{(\lambda_1\mu_2)^2+(\lambda_2\mu_1)^2}{\lambda_1\lambda_2\mu_1\mu_2}.$$

Hence

$$\cosh^2\frac{(PQ)}{2k}\equiv\tfrac{1}{4}\left(e^{\frac{(PQ)}{k}}+2+e^{-\frac{(PQ)}{k}}\right)$$

$$=\frac{(\lambda_1\mu_2+\lambda_2\mu_1)^2}{4\lambda_1\lambda_2\mu_1\mu_2}$$

$$=\frac{f^2(x_1,x_2)}{f(x_1,x_1)f(x_2,x_2)}.$$

This proves the first part of the theorem; the second part is the dual of the first.

9.33. The Equidistance Locus and the Equiangular Envelope

THEOREM. *The locus of points which are all at a distance* $2kn\pi i+d$ *(where n is any integer) from a given ordinary point P is the figure consisting of all the ordinary points on*

(i) *the polar of P relative to the metric gauge if* $d=(2m+1)k\pi i$, *where m is any integer,*

(ii) *a non-singular point-conic having double contact with the metric gauge if* $d\neq(2m+1)k\pi i$, *where m is any integer.*

The first part of the theorem is an immediate consequence of 9.31.

The second part of the theorem is a consequence of 9.32; it may also be proved in other ways. The details are left to the reader. The dual theorem is worth enunciating formally:

THEOREM. *The envelope of lines, all of which make an angle* $2k'n\pi i+\theta$ *(where n is any integer) with a given ordinary line p, is the figure consisting of all the ordinary lines on*

(i) *the pole of p relative to the metric gauge if* $\theta=(2m+1)k'\pi i$, *where m is any integer,*

(ii) *a non-singular line-conic having double contact with the metric gauge if* $\theta\neq(2m+1)k'\pi i$, *where m is any integer.*

The locus and envelope here given should be compared with the corresponding locus and envelope in elementary Geometry.

Examples

1. If P, Q_1, Q_2, Q_3 are four collinear ordinary points, and if
$$(PQ_1) = (PQ_2) = (PQ_3),$$
show that at least two of the three points Q_1, Q_2, Q_3 coincide, unless all the points are on an isotropic line.

2. l is an ordinary line, and P an ordinary point on it. Show that, in general, there are two and only two points Q and Q' on l such that $(PQ) = (PQ') = d$. Show also that Q and Q' coincide in two cases.

3. P and P' are a pair of orthogonal points; Q and Q' are a pair of distinct points on the line PP', such that $(PQ) = (PQ')$; show that $(P'Q) = (P'Q')$.

4. P, P', Q, Q' are four distinct collinear ordinary points on an ordinary line; show that if $(PQ) = (P'Q)$ and $(P'Q) = (P'Q')$, then either P and P' are orthogonal, or Q and Q' are orthogonal, or both pairs are orthogonal. Show that in the last case $(PQ) = (P'Q')$.

5. If XYZ be the triangle of reference, and $f(x,x) = 0$ the point equation of the metric gauge, determine the value of $\cosh^2 \frac{(YZ)}{2k}$.

6. The distance between a point P and a line l is said to be d if and only if $(PQ) = d$, where Q is the point common to l and the line on P orthogonal to l. Show that the distance of a point from its polar is $(2n+1)k\pi i$, where n is any integer.

7. If $f(x,x) = 0$, $F(l,l) = 0$ are the equations of the metric gauge, show that the distance d between the ordinary point (x,y,z) and the ordinary line $[l,m,n]$ satisfies the equation
$$\cosh^2 \frac{d}{2k} = 1 - \frac{(lx+my+nz)^2}{f(x,x)F(l,l)} \Delta,$$
where
$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

8. Dualize the definition in Ex. 6 to obtain the definition of the term angle *between a point and a line*. Show that the angle between a point and a line is equal to the distance between them. (Use Ex. 7.)

9. Show that if two ordinary points are equidistant from an ordinary line they are equidistant from the pole of the line relative to the metric gauge. Hence or otherwise determine the locus of points all of which are equidistant from a given ordinary line. (This locus should be compared with the corresponding locus in elementary Geometry.)

10. Show that, if $k = k'$, the distance between two ordinary points is equal to the angle between their polars relative to the metric gauge.

11. Φ is a non-singular conic distinct from the metric gauge, and A, B, C, and D are four distinct ordinary points on it. Show that, if $k = \frac{1}{2}$, then for all points P on Φ, the expression
$$\frac{\sinh^2 APC \sinh^2 BPD}{\sinh^2 APD \sinh^2 BPC}$$
is constant. (By APC is meant the angle between PA and PC.)

12. M_1 and M_2 are the metric gauge-points on a line l. Show that if A and A', B and B' are two pairs of corresponding points in a projectivity on l whose self-corresponding points are M_1 and M_2, then $(AA') = (BB')+2kn\pi i$, where n is any integer.

9.4. Preamble to Particular Metrical Geometries

We have so far investigated the notions of length and angle in the most general case possible, and though this general investigation can be taken farther, it is more interesting and useful to consider certain special cases, and to pay special attention to certain points and lines of the field. To do this we require some preliminary notions.

9.41. Real and Complex Points and Lines

DEFINITION. *A homogeneous mesh gauge being imposed on the field, a point* (x,y,z) *is said to be a real point if and only if there is a number* c, *not equal to zero, such that all the numbers* cx, cy, cz *are real numbers. All other points are said to be complex points.*

In other words, a real point is a point whose coordinates in the mesh gauge can be specified by three real numbers.

The terms *real line, complex line,* are defined dually.

The warning is repeated here against imagining that there is any inherent difference between real and complex points or lines; the distinction arises solely because of the imposition of the mesh gauge. Strictly speaking, real points are real *relative to the mesh gauge.*

It should be clear that the choice of the triangle of reference and the gauging point determines which points of the field are real, and which complex. But it should be noticed (though the fact is not proved here) that different triangles of reference and gauging points can determine the same set of points to be the real points of the field. In fact, if one triangle of reference and gauging point makes a certain set of points the real points of the field, then there are an infinity of other triangles of reference and gauging points which make the same set of points the real points of the field.

EXAMPLES

1. The line on two distinct real points is real. Dualize.

2. Is it true to say that the line on two distinct complex points is complex?

3. Show that there are complex points on every line. Dualize.

4. Show that on a complex line there is one and only one real point. Dualize.

5. Show that the line on the two distinct points

$$(a+ia', b+ib', c+ic') \quad \text{and} \quad (a-ia', b-ib', c-ic'),$$

where a, a', b, b', c, c' are real and not all zero is a real line. Dualize.

6. Show that if a real line is on the complex point

$$(a+ia', b+ib', c+ic'),$$

it is also on the complex point $(a-ia', b-ib', c-ic')$. Dualize.

9.42. Classification of Non-singular Conics

DEFINITION. *A homogeneous mesh gauge being imposed on the field, a conic whose point equation is $f(x,x) = 0$ is said to be a real conic if and only if there is a number d, different from zero, such that all the numbers da, db, dc, df, dg, dh are real numbers. All other conics are said to be complex conics.*

There is no loss in duality in speaking only of the point equation in the definition; if the point equation of a conic has real coefficients, the line equation also has real coefficients.

We are concerned in the sequel with real conics only, and these can be divided again into two classes. The distinction which is about to be made can be easily illustrated. Consider the two conics whose point equations are

$$x^2+y^2+z^2 = 0$$

and

$$x^2+y^2-z^2 = 0.$$

It will be seen that both of these conics are real conics, but while on the second there certainly are some real points, e.g. $(\sqrt{2}, \sqrt{2}, 2)$, on the first there are no real points. For if x, y, and z are any real numbers, not all of which are zero,

$$x^2+y^2+z^2 > 0.$$

We may therefore divide real conics into two classes: (1) those on which there are no real points, and (2) those on which there are some real points. The algebraic criterion whereby a given real conic may be classified in practice is irrelevant here, so also are the technical names for the two classes. It will be sufficient to give them temporary, working, names, and we call the first class *real-complex conics*, the second, *real-real conics*. The connotation of these terms is plain.

9.43. Metrical Geometries

By the term *Metrical Geometry* is meant, strictly, the interpretation, in terms of the defined concepts of distance and angle, of the theorems of Projective Geometry when a metric gauge-conic has been chosen and fixed. When this process is undertaken without an auxiliary mesh gauge it is known as Synthetic Metrical Geometry; when it is undertaken with a mesh gauge it is known as Algebraic Metrical Geometry.

The method by which we have approached Metrical Geometry makes it impossible to distinguish between the real and complex points of the field without impressing a mesh gauge, and since this distinction is very important in the sequel, it is necessary for us to proceed by the algebraic method. For the choice of a real-real conic as metric gauge gives rise to one set of metrical theorems about the real points of the field, while the choice of a real-complex metric gauge gives rise to a different set.

The first of these sets of theorems is identical with the metrical Geometry discovered by Lobatchewskij and Bolyai, and sometimes called Hyperbolic Geometry; the second set is identical with the metrical Geometry discovered by Riemann, and sometimes called Elliptic Geometry.

These metrical Geometries are sometimes called non-Euclidean Geometries, but the term is misleading in so far as it conveys the idea that there is no connexion between Hyperbolic and Elliptic metrical Geometries on the one hand, and on the other the Euclidean metrical Geometry which we are familiar with as elementary Geometry. The next chapter will show in detail the extent of the connexion that there is between them, and it is sufficient here to say that metric gauges exist for Euclidean metrical Geometry, but that they are singular conics, while the metric gauges of this chapter are non-singular conics.

Even at this stage it is useful to mention a point which is vital to the true understanding of Geometry. It should be clear that there are certain theorems of Projective Geometry which can be proved either synthetically or algebraically, and which presuppose no metric gauge; such theorems are aptly called projective theorems. But there are other theorems which are only significant when a metric gauge has been impressed; that is to say, they presuppose a metric gauge of some kind. Such theorems are called metrical theorems. If this is borne in mind, it should be clear that the distinction between Elliptic, Hyperbolic, and Euclidean Metrical Geometries is not a distinction between the fields of which they are true; it is a distinction which arises solely from the choice of a particular metric gauge. On one and the same field, an Elliptic, a Hyperbolic, or a Euclidean metric gauge may be imposed at choice. The projective theorems are the same whatever metric gauge be chosen, but the metrical theorems differ.

9.5. Elliptic Metrical Geometry

Elliptic Metrical Geometry results from the choice of a real-complex conic as metric gauge. The choice may be made in a variety of ways, but for simplicity we choose that conic whose point equation is

$$cx^2+cy^2+z^2 = 0,$$

where c is a positive number. There is no loss in generality if we take $c = 1$, and this we do. The symbol $f(x,x)$ will stand for $x^2+y^2+z^2$, and for the sake of simplicity we shall shorten such expressions as $f(x_1,x_2)$, $f(x_3,x_3)$, etc., to f_{12}, f_{33}, etc.

The line equation of the conic is plainly $l^2+m^2+n^2 = 0$, and the expressions F_{11}, F_{23}, etc., will have their obvious meanings.

We agree, since we are dealing only with real points, to specify the coordinates of all real points by real numbers only; e.g. we shall exclude such specifications of the point $(\frac{1}{2}, 1, 5)$ as $(\frac{1}{2}i, i, 5i)$.

An algebraic theorem, of great importance in what follows, is proved first.

9.51. Lagrange's Identity

THEOREM. *If (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two real points, then*

$$(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)-(x_1x_2+y_1y_2+z_1z_2)^2$$
$$\equiv (y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2+(x_1y_2-x_2y_1)^2 \geqslant 0,$$

the two expressions being equal to zero if and only if the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *coincide.*

The first expression on simplification and rearrangement becomes

$$(y_1^2 z_2^2 + y_2^2 z_1^2 - 2y_1 y_2 z_1 z_2) + (z_1^2 x_2^2 + z_2^2 x_1^2 - 2z_1 z_2 x_1 x_2) + \\ + (x_1^2 y_2^2 + x_2^2 y_1^2 - 2x_1 x_2 y_1 y_2) \\ \equiv (y_1 z_2 - y_2 z_1)^2 + (z_1 x_2 - z_2 x_1)^2 + (x_1 y_2 - x_2 y_1)^2.$$

This expression, being a sum of squares of real numbers, cannot be negative, and it is zero if and only if the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) coincide.

9.52. Distance on a Line

If P and Q be two real points whose coordinates are respectively (x_1, y_1, z_1), (x_2, y_2, z_2), it follows from the last theorem that $f_{11} f_{22} - (f_{12})^2 \geqslant 0$, that is to say,

$$0 < \frac{(f_{12})^2}{f_{11} f_{22}} \leqslant 1,$$

the first of these two inequalities being a consequence of the fact that none of the numbers involved in $(f_{12})^2/(f_{11} f_{22})$ is negative.

From this it follows that there is a real number θ such that $0 \leqslant \theta < \pi$, and $\cos\theta = (f_{12})/(f_{11} f_{22})^{\frac{1}{2}}$, the *positive* square root being taken in the denominator of this fraction.

Now if M_1 and M_2 are the metric gauge-points on the line PQ, their coordinates being, respectively,

$$(\lambda_1 x_1 + \mu_1 x_2,\ \lambda_1 y_1 + \mu_1 y_2,\ \lambda_1 z_1 + \mu_1 z_2),$$

and

$$(\lambda_2 x_1 + \mu_2 x_2,\ \lambda_2 y_1 + \mu_2 y_2,\ \lambda_2 z_1 + \mu_2 z_2),$$

the ratios λ_1/μ_1 and λ_2/μ_2 satisfy

$$\lambda^2 f_{11} + 2\lambda\mu f_{12} + \mu^2 f_{22} = 0. \quad \text{(See 9.32.)}$$

We now make the convention that

$$\frac{\lambda_1}{\mu_1} = \frac{-f_{12} - \sqrt{\{(f_{12})^2 - f_{11} f_{22}\}}}{f_{11}}$$

and

$$\frac{\lambda_2}{\mu_2} = \frac{-f_{12} + \sqrt{\{(f_{12})^2 - f_{11} f_{22}\}}}{f_{12}}.$$

From the analysis in 9.32 it follows that

$$(PQ) = k\log\left\{\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\right\} = k\log\left\{\frac{-f_{12}-\sqrt{\{(f_{12})^2-f_{11}f_{22}\}}}{-f_{12}+\sqrt{\{(f_{12})^2-f_{11}f_{22}\}}}\right\}$$

$$= k\log\left\{\frac{-\cos\theta-\sqrt{(\cos^2\theta-1)}}{-\cos\theta+\sqrt{(\cos^2\theta-1)}}\right\} = k\log\left\{\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right\}$$

$$= k\log e^{2i\theta} = k(2i\theta+2n\pi i),$$

where n is any integer.

In order that distances between real points shall be real, we take k to be a multiple of i, the most convenient being $-\frac{1}{2}i$. We therefore fix, once for all, that $k = -\frac{1}{2}i$, so that

$$(PQ) = \theta+n\pi,$$

where n is any integer.

In the limited amount of work on Elliptic Metrical Geometry which is done here, there will be no loss in generality and no ambiguity if n is taken as zero, and hence, for all real points P and Q,

(i) $0 \leqslant (PQ) < \pi$,

(ii) $\cos(PQ) = f_{12}/(f_{11}f_{22})^{\frac{1}{2}}$, *the positive square root being taken always in this fraction,*

(iii) $0 \leqslant \sin(PQ) \leqslant 1$.

9.521. Direction on a Line. It will probably have been noticed that if we had taken $(-x_2, -y_2, -z_2)$ as the coordinates of Q throughout the preceding work, the sign of f_{12} would have been changed, and this would have given, as the value of (PQ), the supplement of that obtained. At first sight this appears to be an ambiguity, but it is not so in fact. For the effect of changing the sign of the coordinates of Q is to interchange the order of the metric gauge-points M_1 and M_2; this interchange occurs because of the method we adopted of specifying their coordinates. Hence the apparent ambiguity is explained. In actual work all pitfalls will be avoided if, once the coordinates of a point have been specified by a particular triple number-label, this specification is rigidly adhered to throughout.

Another apparent ambiguity may occur to the reader. If we had undertaken to evaluate the distance (QP) in the preceding section, the result we should have obtained, following the same process, would have been exactly the same as that obtained for

(PQ). The reason for this is that again we should have interchanged the gauge-points. Hence the rule must be followed: take and fix two points on the line, express all other points on the line in terms of the coordinates of these two; determine and fix the metric gauge-points and the order in which they are to be taken; determine all distances on the line by applying the definition for distance.

It may be observed that there are, so to speak, two finite paths from P to Q on the line PQ. By fixing one order of metric gauge-points we measure the length of one of these paths; by fixing the other order we measure the other. Still speaking roughly, the whole set of real points on the line PQ form a closed path. The fixing of the order of the metric gauge-points determines which direction shall be followed in going from P to Q. More than this need not be said here, since it is not our purpose to enter deeply into Metrical Geometry.

9.53. Angle on a Point

Since the expression $F(l, l)$ has the same algebraic form as $f(x, x)$, the whole of 9.52 can be dualized. It follows that, if p and q be two real lines whose coordinates are, respectively, $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$,

(i) $0 \leqslant [pq] < \pi$,

(ii) $\cos[pq] = F_{12}/(F_{11}F_{22})^{\frac{1}{2}}$, *the positive square root being taken in this fraction,*

(iii) $0 \leqslant \sin[pq] \leqslant 1$.

9.531. Convention about the Coordinates of a Line on Two Points. The remarks of 9.521 about the need for consistency in specifying the coordinates of a point may be dualized, and this makes it essential to adopt a uniform convention for specifying the coordinates of a line on two points whose coordinates are known.

If P and Q are the points (x_1, y_1, z_1), (x_2, y_2, z_2) respectively, we agree to specify the coordinates of the line PQ as

$$(y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1)$$

and the coordinates of the line QP as

$$(y_2 z_1 - y_1 z_2,\ z_2 x_1 - z_1 x_2,\ x_2 y_1 - x_1 y_2).$$

9.54. The Triangle in Elliptic Metrical Geometry

As an illustration of Elliptic Metrical Geometry, we consider the metrical properties of a triangle therein.

Take any triangle ABC whose vertices are, respectively, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). The sides BC, CA, AB will have coordinates

$$[y_2 z_3 - y_3 z_2,\ z_2 x_3 - z_3 x_2,\ x_2 y_3 - x_3 y_2],$$
$$[y_3 z_1 - y_1 z_3,\ z_3 x_1 - z_1 x_3,\ x_3 y_1 - x_1 y_3],$$
$$[y_1 z_2 - y_2 z_1,\ z_1 x_2 - z_2 x_1,\ x_1 y_2 - x_2 y_1]$$

respectively. These will also be referred to as $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$, $[l_3, m_3, n_3]$ respectively.

We use the symbols a, b, and c for the lengths of BC, CA, AB respectively.

By the angle BAC we mean the angle between the lines AC and AB; that is, between $[-l_2, -m_2, -n_2]$ and $[l_3, m_3, n_3]$; this angle is referred to as the angle A. Hence

$$\cos A = -F_{23}/(F_{22} F_{33})^{\frac{1}{2}},$$

and not $F_{23}/(F_{22} F_{33})^{\frac{1}{2}}$. Similarly for the angles B and C.

9.541. Preliminary Algebraic Identities. Lagrange's identity states that $F_{11} = f_{22} f_{33} - (f_{23})^2$, and there is a similar identity, which the reader should verify, namely,

$$F_{23} = f_{12} f_{31} - f_{11} f_{23}.$$

It follows from these that $F_{11}, F_{22}, F_{33}, F_{23}, F_{31}, F_{12}$ are the minors of $f_{11}, f_{22}, f_{33}, f_{23}, f_{31}, f_{12}$ respectively in the symmetrical determinant

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}.$$

This determinant will be symbolized by Δ; if it is written out in full, it will be recognized at once as

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2;$$

it is therefore positive.

From the expressions for $\cos a$ and $\cos A$ it is now possible to deduce the values of all the trigonometrical functions of the sides and angles. In fact

$$\cos a = \frac{f_{23}}{(f_{22}f_{33})^{\frac{1}{2}}}, \quad \sin a = \frac{(F_{11})^{\frac{1}{2}}}{(f_{22}f_{33})^{\frac{1}{2}}}, \quad \tan a = \frac{(F_{11})^{\frac{1}{2}}}{f_{23}};$$

$$\cos A = \frac{-F_{23}}{(F_{22}F_{33})^{\frac{1}{2}}}, \quad \sin A = \frac{(\Delta f_{11})^{\frac{1}{2}}}{(F_{22}F_{33})^{\frac{1}{2}}}, \quad \tan A = \frac{(\Delta f_{11})^{\frac{1}{2}}}{-F_{23}}.$$

Similar results for the other sides and angles are obtained by cyclic permutation of letters and suffixes.

9.542. The Cosine Formula. THEOREM. *In a triangle ABC in Elliptic Metrical Geometry*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

and

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

$$\sin b \sin c \cos A = \frac{(F_{22})^{\frac{1}{2}}}{(f_{33}f_{11})^{\frac{1}{2}}} \frac{(F_{33})^{\frac{1}{2}}}{(f_{11}f_{22})^{\frac{1}{2}}} \frac{(-F_{23})}{(F_{22}F_{33})^{\frac{1}{2}}} = \frac{-F_{23}}{f_{11}(f_{22}f_{33})^{\frac{1}{2}}}$$

$$= \frac{f_{11}f_{23} - f_{12}f_{31}}{f_{11}(f_{22}f_{33})^{\frac{1}{2}}} = \cos a - \cos b \cos c.$$

This proves the first part of the theorem; the second part is proved in a similar way.

9.543. The Sine Formula. THEOREM. *In a triangle ABC in Elliptic Metrical Geometry*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \left\{\frac{\Delta f_{11} f_{22} f_{33}}{F_{11}F_{22}F_{33}}\right\}^{\frac{1}{2}}.$$

This theorem is an immediate consequence of the results of 9.541.

9.544. The Sum of the Angles of a Triangle. THEOREM. *In a triangle ABC in Elliptic Metrical Geometry*

$$A+B+C > \pi.$$

By 9.542,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

and since

$$\cos(B+C) = \cos B \cos C - \sin B \sin C,$$

$$\cos A + \cos(B+C) = -\sin B \sin C(1-\cos a),$$

or

$$\cos \tfrac{1}{2}(A+B+C) \cos \tfrac{1}{2}(B+C-A) = -\sin B \sin C \sin^2 \tfrac{1}{2}a.$$

The right-hand side of this equation being negative, the factors of the left-hand side must have opposite signs.

Now we may suppose that $A \geqslant B \geqslant C$, so that

$$-\pi < B+C-A < \pi;$$

hence $\cos\frac{1}{2}(B+C-A) > 0$.

It follows that $\cos\frac{1}{2}(A+B+C) < 0$, that is to say,

$$A+B+C > \pi.$$

9.55. Elliptic Metrical Geometry and Spherical Geometry

The reader who is acquainted with Spherical Trigonometry will recognize that these formulae which have been deduced for the triangle in Elliptic Metrical Geometry are identical with the formulae for a triangle in Spherical Trigonometry. It is not, however, legitimate to deduce from this that the Geometry on the surface of a sphere is identical with Elliptic Metrical Geometry. It may be found useful to deduce other formulae for the triangle in Elliptic Metrical Geometry; these other formulae will be found in any book of Spherical Trigonometry, and to these the reader is referred.

9.56. A Representation of Elliptic Metrical Geometry

The fact of the identity of form between Elliptic Metrical Geometry and Spherical Trigonometry makes it possible to give a simple representation of the former. This is given to help the reader to *visualize* the metrical conditions when a real-complex metric gauge is imposed on the field.

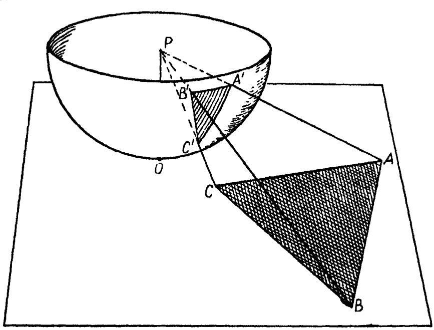


FIG. 60.

The representation is illustrated in the figure. A hemisphere whose centre is P rests on the plane, touching it at the point O. ABC is any triangle in the plane, and A', B', C' are points on the surface of the hemisphere such that PAA' are collinear, PBB' are collinear, PCC' are collinear. The sides and angles of the triangle ABC in the plane when an elliptic metric gauge is imposed may be measured by determining (in the usual way) the sides and angles of the spherical triangle $A'B'C'$.

Examples

Apply the results of 9.541 to prove the following formulae for the triangle in Elliptic Metrical Geometry:

(i) $\cos C \cos b + \cot A \sin C = \sin b \cot a$.

(ii) $\cos b \cos c \cos A + \sin b \sin c = \sin B \sin C - \cos B \cos C \cos a$.

9.6. Hyperbolic Metrical Geometry

Hyperbolic Metrical Geometry results from the choice of a real-real conic as metric gauge. For simplicity, we take the conic whose point equation is $cx^2+cy^2+z^2 = 0$, where c is a negative number; we take c to be -1, since this involves no loss in generality.

The symbol $f(x,x)$ will stand for $-x^2-y^2+z^2$, and f_{11}, f_{23}, etc., will have their obvious meanings.

The line equation of the metric gauge is $l^2+m^2-n^2 = 0$, and so $F(l,l)$ stands for the expression on the left of this equation.

As in Elliptic Metrical Geometry, we agree to specify the coordinates of all real points by real number-labels only.

9.61. Interior and Exterior Points and Lines

Definition. *A real point* (x_1, y_1, z_1) *will be termed an interior point if and only if* $f_{11} > 0$*; it will be termed an exterior point if and only if* $f_{11} < 0$.

Definition. *A real line* $[l_1, m_1, n_1]$ *will be termed an interior line if and only if* $F_{11} > 0$*; it will be termed an exterior line if and only if* $F_{11} < 0$.

Hyperbolic Metrical Geometry is concerned almost entirely with interior points and lines, and some preliminary theorems concerning them are necessary.

9.611. Lagrange's Identity. Theorem.

$$(-x_1^2-y_1^2+z_1^2)(-x_2^2-y_2^2+z_2^2)-(-x_1x_2-y_1y_2+z_1z_2)^2 \equiv -(y_1z_2-y_2z_1)^2-(z_1x_2-z_2x_1)^2+(x_1y_2-x_2y_1)^2.$$

The proof of this is left to the reader,

9.612. THEOREM. *If* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are both interior points, then* $f_{11}f_{22}-(f_{12})^2 \leqslant 0$, *equality only holding if the two points coincide.*

Since the two points are both interior points, neither z_1 nor z_2 can be zero; we may therefore without loss of generality take $z_1 = z_2 = 1$.

Write $x_1 = r_1 \cos A$, $y_1 = r_1 \sin A$, where $r_1 > 0$; then since (x_1, y_1, z_1) is an interior point, $r_1 < 1$.

Similarly, write $x_2 = r_2 \cos B$, $y_2 = r_2 \sin B$, where $r_2 > 0$; as before, $r_2 < 1$. Then

$$\begin{aligned} f_{11}f_{22}-(f_{12})^2 &= (-r_1^2+1)(-r_2^2+1)-\{-r_1 r_2 \cos(A-B)+1\}^2 \\ &= (1-r_1^2)(1-r_2^2)-\{1-r_1 r_2+2r_1 r_2 \sin^2 \tfrac{1}{2}(A-B)\}^2 \\ &= -(r_1-r_2)^2-4r_1 r_2(1-r_1 r_2)\sin^2 \tfrac{1}{2}(A-B)- \\ &\qquad -4r_1^2 r_2^2 \sin^4 \tfrac{1}{2}(A-B). \end{aligned}$$

All the terms of this are negative or zero, and they are all zero only when the two points coincide; hence the theorem is proved.

9.613. THEOREM. *If* $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ *are two distinct real lines which are both on the interior point* (x, y, z), *then*

$$F_{11}F_{22}-(F_{12})^2 > 0.$$

By Lagrange's identity

$$\begin{aligned} &F_{11}F_{22}-(F_{12})^2 \\ &\quad = -(m_1 n_2-m_2 n_1)^2-(n_1 l_2-n_2 l_1)^2+(l_1 m_2-l_2 m_1)^2 \\ &\quad = d^2(-x^2-y^2+z^2), \end{aligned}$$

where d is a real number.

Since (x, y, z) is an interior point, this last expression is positive; hence $F_{11}F_{22}-(F_{12})^2 > 0$.

9.614. THEOREM. *The line on two distinct interior points is an interior line.*

The proof depends on Lagrange's identity, and is very similar to that of the last theorem. Compare this theorem with 9.617.

9.615. THEOREM. *If* (x_1, y_1, z_1), (x_2, y_2, z_2) *be two interior points, and if* $z_1 > 0$ *and* $z_2 > 0$, *then* $-x_1 x_2-y_1 y_2+z_1 z_2 > 0$.

Without loss of generality, we may take $z_1 = z_2 = 1$. Write, as in 9.612, $x_1 = r_1 \cos A$, $y_1 = r_1 \sin A$, $x_2 = r_2 \cos B$, $y_2 = r_2 \sin B$, where $0 < r_1 < 1$, $0 < r_2 < 1$.

Then $-x_1x_2 - y_1y_2 + z_1z_2 = r_1r_2\cos(A-B)+1$, and this is plainly positive.

The following theorems are not needed in the sequel, but they are set down to enable the reader to obtain a fuller idea of the relations between interior and exterior points and lines. The details of proof are omitted.

9.616. THEOREM. *On every interior line there is an infinity of interior points.*

If $[l, m, n]$ is an interior line, all the points whose coordinates are $(-nl+\lambda m, -nm-\lambda l, l^2+m^2)$ are on it, and they are interior points if $-(l^2+m^2-n^2)^{\frac{1}{2}} < \lambda < +(l^2+m^2-n^2)^{\frac{1}{2}}$.

THEOREM. *On every exterior point there is an infinity of exterior lines.*

9.617. THEOREM. *Any real line on an interior point is an interior line.*

THEOREM. *Any real point on an exterior line is an exterior point.*

9.618. THEOREM. *The isotropic points on an interior line are real; those on an exterior line are complex.*

THEOREM. *The isotropic lines on an interior point are complex; those on an exterior point are real.*

9.619. THEOREM. *The polar of an interior point relative to the metric gauge is an exterior line; that of an exterior point is an interior line.*

THEOREM. *The pole of an interior line relative to the metric gauge is an exterior point; that of an exterior line is an interior point.*

9.62. Distance and Angle in Hyperbolic Metrical Geometry

9.621. Distance on an Interior Line. Let P and Q be two interior points whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

From now onwards we make the convention that in specifying the coordinates of all interior points, the third coordinate (z) shall always be a positive number. This is possible, since all the points on the line $z = 0$ are exterior points.

By 9.612, $(f_{12})^2 - f_{11}f_{22} \geqslant 0$, so that

$$\frac{(f_{12})^2}{f_{11}f_{22}} \geqslant 1,$$

and by 9.615, $f_{12} > 0$.

There is therefore a positive number θ, such that

$$\cosh\theta = f_{12}/(f_{11}f_{22})^{\frac{1}{2}},$$

the positive square root being taken in the denominator; and it follows that

$$\sinh\theta = \left\{\frac{(f_{12})^2 - f_{11}f_{12}}{f_{11}f_{22}}\right\}^{\frac{1}{2}} > 0.$$

If now M_1 and M_2 are the metric gauge points on the line PQ, their coordinates being, respectively,

$$(\lambda_1 x_1 + \mu_1 x_2,\ \lambda_1 y_1 + \mu_1 y_2,\ \lambda_1 z_1 + \mu_1 z_2)$$

and $$(\lambda_2 x_1 + \mu_2 x_2,\ \lambda_2 y_1 + \mu_2 y_2,\ \lambda_2 z_1 + \mu_2 z_2),$$

the ratios λ_1/μ_1 and λ_2/μ_2 satisfy the equation

$$\lambda^2 f_{11} + 2\lambda\mu f_{12} + \mu^2 f_{22} = 0.$$

We make the convention that

$$\frac{\lambda_1}{\mu_1} = \frac{-f_{12} - \sqrt{\{(f_{12})^2 - f_{11}f_{22}\}}}{f_{11}}, \text{ and } \frac{\lambda_2}{\mu_2} = \frac{-f_{12} + \sqrt{\{(f_{12})^2 - f_{11}f_{22}\}}}{f_{11}}.$$

Now

$$\begin{aligned}(PQ) &= k\log\left\{\frac{\lambda_1\mu_2}{\mu_1\lambda_2}\right\} = k\log\left\{\frac{-f_{12} - \sqrt{\{(f_{12})^2 - f_{11}f_{22}\}}}{-f_{12} + \sqrt{\{(f_{12})^2 - f_{11}f_{22}\}}}\right\} \\ &= k\log\left\{\frac{-\cosh\theta - \sinh\theta}{-\cosh\theta + \sinh\theta}\right\} = k\log(e^{2\theta}) \\ &= k(2\theta + 2n\pi i).\end{aligned}$$

We take $k = \frac{1}{2}$, and $n = 0$, so that $(PQ) = \theta$, where $\theta > 0$, and $\cosh\theta = f_{12}/(f_{11}f_{22})^{\frac{1}{2}}$, the positive square root being taken in this fraction.

9.622. Angle on an Interior Point. If $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ are the coordinates of two interior lines p and q which are on a common interior point, the method of determining the angle between them is identical with the method adopted in Elliptic Metrical Geometry. By Theorem 9.613, $(F_{12})^2 - F_{11}F_{22}$

is not positive, and so the conventions of 9.52 are adopted, the scale constant being $-\frac{1}{2}i$. The results may be stated thus:

(i) $0 \leqslant [pq] < \pi$,

(ii) $\cos[pq] = F_{12}/(F_{11}F_{22})^{\frac{1}{2}}$, *the positive square root being taken in the denominator.*

9.63. The Triangle in Hyperbolic Metrical Geometry

Take any triangle ABC whose vertices are, respectively, the interior points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), where z_1, z_2, and z_3 are all positive.

The same symbols and conventions will be used as in the corresponding part of Elliptic Metrical Geometry (9.54).

9.631. Preliminary Algebraic Identities. The analysis of 9.541 may be repeated in almost identical terms. It should be noticed, however, that the minors of f_{11}, f_{22}, etc., in the determinant Δ are $-F_{11}$, $-F_{22}$, etc.; also that Δ is equal to

$$\begin{vmatrix} ix_1 & iy_1 & z_1 \\ ix_2 & iy_2 & z_2 \\ ix_3 & iy_3 & z_3 \end{vmatrix}^2,$$

and is therefore positive.

The corresponding results are:

$$\cosh a = \frac{f_{23}}{(f_{22}f_{33})^{\frac{1}{2}}}, \qquad \sinh a = \frac{(F_{11})^{\frac{1}{2}}}{(f_{22}f_{33})^{\frac{1}{2}}}, \qquad \tanh a = \frac{(F_{11})^{\frac{1}{2}}}{f_{23}};$$

$$\cos A = \frac{-F_{23}}{(F_{22}F_{33})^{\frac{1}{2}}}, \qquad \sin A = \frac{(\Delta f_{11})^{\frac{1}{2}}}{(F_{22}F_{33})^{\frac{1}{2}}}, \qquad \tan A = \frac{(\Delta f_{11})^{\frac{1}{2}}}{-F_{23}}.$$

Corresponding results for the other sides and angles are obtained by cyclic permutation of the letters and suffixes.

9.632. The Cosine Formula. THEOREM. *In a triangle ABC in Hyperbolic Metrical Geometry*

$$\cosh a = \cosh b \cosh c + \sinh b \sinh c \cos A,$$

and $$\cos A = -\cos B \cos C + \sin B \sin C \cosh a.$$

9.633. The Sine Formula. THEOREM. *In a triangle ABC in Hyperbolic Metrical Geometry*

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} = \left[\frac{\Delta f_{11}f_{22}f_{33}}{F_{11}F_{22}F_{33}}\right]^{\frac{1}{2}}.$$

9.634. The Sum of the Angles of a Triangle. THEOREM. *In a triangle ABC in Hyperbolic Metrical Geometry*

$$A+B+C < \pi.$$

The proofs of the three foregoing theorems are analogous to the proofs of the corresponding theorems in Elliptic Metrical Geometry.

9.64. The Metrical Geometry of Exterior Points and Lines

In the foregoing we have confined our attention to interior points and lines, and it is natural to ask whether there is any corresponding metrical Geometry of the exterior points and lines. The answer to this is that the metrical Geometry of the exterior points and lines is the dual of that of the interior points and lines. It may be verified, for instance, that if ABC is a triangle of exterior points, such that all three of the lines BC, CA, and AB are exterior lines, and if suitable conventions are adopted,

$$\cos a = -\cos b \cos c + \sin b \sin c \cosh A,$$

and $$\cosh A = \cosh B \cosh C + \sinh B \sinh C \cos a.$$

For this reason the study of the metrical Geometry of the exterior points and lines gives us nothing that is really new.

It may be noticed that if we wish to stay within the realm of real numbers there is a complete barrier between interior and exterior points, and between interior and exterior lines. For by no convention about the scale constant can it be ensured that the distances between all pairs of real points one of which is interior and one exterior are all real; a similar statement can be made about the angles between pairs of lines one of which is interior, the other exterior.

CHAPTER X

SINGULAR METRIC GAUGES

In the preceding chapter it has been shown how distance and angle can be defined in terms of the concepts of Projective Geometry. In a certain sense it is true to say that, thus defined, these notions are more general than the corresponding notions in elementary Geometry, but it is not to be inferred from this that Euclidean Metrical Geometry can be elaborated from them in the same way that Elliptic and Hyperbolic Metrical Geometries were elaborated. In other words, Euclidean Metrical Geometry is not a particular case of the general Metrical Geometry to which Projective Geometry gives rise. It is, in fact, a limiting (and singular) case of the general Metrical Geometry, and in this chapter it is shown how the limiting process is undertaken, and what its results are. Incidentally, it will appear that it occupies, so to speak, the borderline position between Elliptic and Hyperbolic Metrical Geometries.

From another point of view, Euclidean Metrical Geometry may be said to be a metrical Geometry which arises from the use of singular conics as metric gauges. This statement is not, however, the whole truth, and it needs very precise qualification. What exactly that qualification is will appear from the sequel.

10.1. The Limiting Process

We start by taking as metric gauge the non-singular real conic whose point and line equations are, respectively,

$$cx^2+cy^2+z^2 = 0,$$

and

$$l^2+m^2+cn^2 = 0,$$

where c is a real number. Plainly, if c is positive, this will give rise to Elliptic Metrical Geometry, and if c is negative, to Hyperbolic Metrical Geometry.

It is convenient to write $c = R^2$, where R^2 is a real number, positive or negative.

The limiting process is to make R^2 tend to zero, and in carrying it out, three separate processes must be examined: (i) the effect on the metric gauge, (ii) the effect on the definition of angle, and (iii) the effect on the definition of distance.

10.11. The Limiting Process on the Metric Gauge

As R^2 tends to zero, the point equation

$$R^2x^2+R^2y^2+z^2 = 0$$

tends to $$z^2 = 0.$$

This is the equation of two coincident ranges of points on the line [0, 0, 1], and is therefore the equation of a singular point-conic.

A little thought will show that such a conic cannot be used for the definition of distance in the ordinary way, since if it were, the metric gauge points on every line of the field would coincide.

We call the line [0, 0, 1] the *special line*,† since it is metrically special. The points on it may be called the *isotropic points*, since they have the properties of isotropic points in other Metrical Geometries, but the name is not usual.

As R^2 tends to zero, the line equation of the metric gauge tends to $l^2+m^2 = 0$.

This is the equation of the two pencils of lines on the points $(i, 1, 0)$ and $(-i, 1, 0)$; it is therefore the equation of a singular line-conic.

This singular conic can be used in the ordinary way for the definition of angle, since two of its lines are on every ordinary point of the field. In practice, for the sake of uniformity, we derive the expression for the angle between two lines by the limiting process, and not by the direct application of the definition of angle. The results are the same, whichever method be used.

The two points $(i, 1, 0)$ and $(-i, 1, 0)$ will be given the permanent labels I and J, and will be called the *special points*.‡ It will be noticed that both are on the special line.

The isotropic lines of the field are the two pencils on I and J;

† It is sometimes called the *line at infinity*, or the *vanishing line*, or the *absolute*; none of these terms is without false implications.

‡ They are also called the *circular points at infinity*; this name is misleading, and it has another disadvantage in that it is based on a very subsidiary property of the points.

hence the isotropic lines on any ordinary point P are the lines PI and PJ.

10.12. The Limiting Process on the Definition of Angle

If p and q be the two real lines whose coordinates are $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ respectively, and if the line equation of the metric gauge be

$$l^2+m^2+R^2n^2 = 0,$$

then, by working through the usual analysis and taking $-\tfrac{1}{2}i$ as the scale-constant, we find that

$$[pq] = -\tfrac{1}{2}i\log\left\{\frac{-F_{12}\pm\sqrt{\{(F_{12})^2-F_{11}F_{22}\}}}{-F_{12}\mp\sqrt{\{(F_{12})^2-F_{11}F_{22}\}}}\right\}.$$

If we make R tend to zero in the expression on the right, its limit is

$$-\tfrac{1}{2}i\log\left\{\frac{(l_1l_2+m_1m_2)\pm i(l_1m_2-l_2m_1)}{(l_1l_2+m_1m_2)\mp i(l_1m_2-l_2m_1)}\right\}. \tag{1}$$

Now provided neither p nor q coincides with the special line, it follows from Lagrange's identity and the inequality attached to it (9.51) that

$$0 \leqslant \frac{(l_1l_2+m_1m_2)^2}{(l_1^2+m_1^2)(l_2^2+m_2^2)} \leqslant 1,$$

and

$$0 \leqslant \frac{(l_1m_2-l_2m_1)^2}{(l_1^2+m_1^2)(l_2^2+m_2^2)} \leqslant 1.$$

We make no convention as yet about the sign to be taken in the numerator and denominator of (1).

Now write

$$\cos\theta = \frac{(l_1l_2+m_1m_2)}{(l_1^2+m_1^2)^{\frac{1}{2}}(l_2^2+m_2^2)^{\frac{1}{2}}}, \tag{2}$$

and

$$\sin\theta = \frac{(l_1m_2-l_2m_1)}{(l_1^2+m_1^2)^{\frac{1}{2}}(l_2^2+m_2^2)^{\frac{1}{2}}}, \tag{3}$$

positive square roots being taken in the denominators of both fractions, and θ satisfying $0 \leqslant \theta < 2\pi$.

On substituting these values in (1), we find that $[pq] = n\pi+\theta$ if the *upper sign* be taken in (1), and $[pq] = n\pi-\theta$ if the *lower sign* be taken in (1), n being any integer.

These results are only compatible with (2) and (3) if n is an even integer or zero. Hence

$[pq] = 2n\pi+\theta$ if the upper sign be taken,

and $[pq] = 2n\pi-\theta$ if the lower sign be taken.

It will be recognized that the choice of sign corresponds to the choice of a definite sense of rotation as the positive sense in elementary Geometry.

There is no loss in generality in elementary work if n is so chosen that $0 \leqslant [pq] < 2\pi$.

These results may be summed up as follows:

$$\text{(i)}\ 0 \leqslant [pq] < 2\pi,$$

$$\text{(ii)}\ \cos[pq] = \frac{(l_1 l_2 + m_1 m_2)}{(l_1^2+m_1^2)^{\frac{1}{2}}(l_2^2+m_2^2)^{\frac{1}{2}}},$$

$$\text{(iii)}\ \sin[pq] = \frac{\pm(l_1 m_2 - l_2 m_1)}{(l_1^2+m_1^2)^{\frac{1}{2}}(l_2^2+m_2^2)^{\frac{1}{2}}},$$

$$\text{(iv)}\ \tan[pq] = \frac{\pm(l_1 m_2 - l_2 m_1)}{(l_1 l_2 + m_1 m_2)}.$$

A further useful result may be given here; if we write

$$\cos\theta_1 = \frac{l_1}{(l_1^2+m_1^2)^{\frac{1}{2}}} \quad \text{and} \quad \sin\theta_1 = \frac{m_1}{(l_1^2+m_1^2)},$$

and similar expressions for $\cos\theta_2$ and $\sin\theta_2$, it is easily verified that $[pq] = \pm(\theta_1 - \theta_2)$.

It will be noticed that the results given above are similar to those of ordinary Analytical Geometry.

10.13. The Limiting Process on the Definition of Distance

The fact that as R tends to zero the point equation of the metric gauge breaks down more completely than the line equation leads us to expect that the limiting process on the definition of distance is a more involved and delicate process than the preceding. This expectation is verified.

We start with the point equation of the metric gauge

$$R^2x^2 + R^2y^2 + z^2 = 0,$$

where R is not zero.

Let P and Q be two points not on the special line, whose coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let T be the greater of the two positive numbers

$$\frac{x_1^2+y_1^2}{z_1^2} \quad \text{and} \quad \frac{x_2^2+y_2^2}{z_2^2};$$

we restrict $|R^2|$ to be less than T if T is positive; if T is zero,

$|R^2|$ is unrestricted. The object of this restriction will become apparent later.

Now if $(PQ) = d$, it follows from 9.32 that

$$\sinh^2\frac{d}{2k} = \frac{(f_{12})^2 - f_{11}f_{22}}{f_{11}f_{22}},$$

where k is the scale-constant.

We now choose k equal to $1/(2iR)$, so that

$$-\sin^2(Rd) = \frac{(f_{12})^2 - f_{11}f_{22}}{f_{11}f_{22}}$$

$$= \frac{-R^2[(z_1x_2 - z_2x_1)^2 + (z_1y_2 - z_2y_1)^2] + R^4L}{z_1^2z_2^2 + R^2M + R^4N},$$

where L, M, and N are polynomials in the coordinates.

As R tends to zero, the right-hand side of this equation tends to zero (and the denominator does not vanish in the process owing to the restriction on R^2).

Hence† $$\lim_{R\to 0}\sin^2(Rd) = 0,$$

and so $$\lim_{R\to 0}\frac{\sin^2(Rd)}{R^2d^2} = 1.$$

Now

$$d^2\frac{\sin^2(Rd)}{R^2d^2} = \frac{(z_1x_2 - z_2x_1)^2 + (z_1y_2 - z_2y_1)^2 - R^2L}{z_1^2z_2^2 + R^2M + R^4N},$$

and from this it is obvious that

$$\lim_{R\to 0}(d^2) = \frac{(z_1x_2 - z_2x_1)^2 + (z_1y_2 - z_2y_1)^2}{z_1^2z_2^2}.$$

We therefore *define* the distance between the two points P and Q by the equation

$$(PQ)^2 = \frac{(z_1x_2 - z_2x_1)^2 + (z_1y_2 - z_2y_1)^2}{z_1^2z_2^2}.$$

Since neither P nor Q is on the special line $[0, 0, 1]$, we may, without loss of generality, take $z_1 = z_2 = 1$; the expression for distance then takes the familiar form

$$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

In practice it is not always useful to make this simplification.

† Since d is itself a function of R, we cannot infer that $\lim_{R\to 0}\sin^2(Rd) = 0$, except by the method used, or some equivalent of it.

From the above equations it follows that

$$(PQ) = \pm\sqrt{\left(\frac{(z_1x_2-z_2x_1)^2+(z_1y_2-z_2y_1)^2}{z_1^2z_2^2}\right)},$$

and when $z_1 = z_2 = 1$

$$(PQ) = \pm\sqrt{\{(x_2-x_1)^2+(y_2-y_1)^2\}}.$$

In the work undertaken here we shall take the positive square root always.

10.2. Euclidean Metrical Geometry

The limiting processes for the metric gauge, the angle formula, and the distance formula having been effected, it is now possible to give a formal definition of Euclidean Metrical Geometry.

DEFINITION. *A Euclidean metric is said to be imposed on the projective field, when*

(i) *two distinct points I and J are chosen as the special points, and the line on them as the special line, and*

(ii) *a homogeneous mesh gauge having been imposed on the field in such a way that the points I and J have the coordinates* $(i, 1, 0)$ *and* $(-i, 1, 0)$ *respectively, the angle between two ordinary lines is defined by the expression given in* 10.12, *and the distance between two ordinary points is defined by the expression given in* 10.13.

10.3. The Triangle in Euclidean Metrical Geometry

Let ABC be a triangle whose vertices are the ordinary points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) respectively. For simplicity, we take $z_1 = z_2 = z_3 = 1$.

The coordinates of the sides of the triangle will then be

$$a:\quad [y_2-y_3,\ x_3-x_2,\ x_2y_3-x_3y_2] \text{ or } [l_1, m_1, n_1];$$

$$b:\quad [y_3-y_1,\ x_1-x_3,\ x_3y_1-x_1y_3] \text{ or } [l_2, m_2, n_2];$$

$$c:\quad [y_1-y_2,\ x_2-x_1,\ x_1y_2-x_2y_1] \text{ or } [l_3, m_3, n_3].$$

The usual convention that the angle A will be the angle between the lines AC and AB will be observed; the choice of a sense of rotation will be made later.

The lengths of all sides will be taken as positive.

10.31. Preliminary Algebraic Identities

From 10.13 it follows that

$$a = [(x_2-x_3)^2+(y_2-y_3)^2]^{\frac{1}{2}},$$

and there are similar expressions for b and c.

From 10.12

$$\cos A = \frac{-[(y_3-y_1)(y_1-y_2)+(x_1-x_3)(x_2-x_1)]}{[(y_3-y_1)^2+(x_1-x_3)^2]^{\frac{1}{2}}[(y_1-y_2)^2+(x_2-x_1)^2]^{\frac{1}{2}}},$$

and $$\sin A = \frac{\pm[(y_3-y_1)(x_2-x_1)-(x_1-x_3)(y_1-y_2)]}{[(y_3-y_1)^2+(x_1-x_3)^2]^{\frac{1}{2}}[(y_1-y_2)^2+(x_2-x_1)^2]^{\frac{1}{2}}},$$

the positive or negative sign being taken in the numerator according as one or the other sense of rotation is chosen as the positive sense.

If the numerator of this last fraction be expanded and simplified, it is found to be $\pm\, \Delta$, where

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix};$$

we agree to take the positive sign if Δ is positive, the negative sign if Δ is negative. It follows that $\sin A$, $\sin B$, $\sin C$ are all positive, so that all the angles of the triangle are less than π.

10.32. The Cosine Formula

THEOREM. *In a triangle ABC in Euclidean Metrical Geometry*

$$a^2 = b^2+c^2-2bc\cos A,$$

and $$\cos A = -\cos B\cos C+\sin B\sin C.$$

These two formulae are the counterparts in Euclidean Metrical Geometry of the cosine formulae in Elliptic and Hyperbolic Metrical Geometries.

To prove the first formula we observe that

$$\begin{aligned} -2bc\cos A &= 2(y_3-y_1)(y_1-y_2)+2(x_1-x_3)(x_2-x_1) \\ &= [(y_3-y_1)-(y_2-y_1)]^2+[(x_3-x_1)-(x_2-x_1)]^2- \\ &\qquad -(y_3-y_1)^2-(y_2-y_1)^2-(x_3-x_1)^2-(x_2-x_1)^2 \\ &= a^2-b^2-c^2. \end{aligned}$$

The second formula may be proved by substitution of the appropriate algebraic expressions for $\cos B$, $\cos C$, etc.

10.33. The Sine Formula

THEOREM. *In a triangle ABC in Euclidean Metrical Geometry*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\pm\Delta}{abc},$$

that sign being taken in the last member which makes the whole expression positive.

This theorem follows at once from the fact that

$$\sin A = \pm\Delta/(bc).$$

10.34. The Sum of the Angles of a Triangle

THEOREM. *In a triangle ABC in Euclidean Metrical Geometry*

$$A+B+C = \pi.$$

This theorem follows from the second part of 10.32; the proof is similar to that of the corresponding theorem in Elliptic Metrical Geometry (9.542).

EXAMPLES

1. From the results of 10.31 show that in a triangle ABC in Euclidean Metrical Geometry $a = b\cos C+c\cos B$.

2. Show that in Euclidean Metrical Geometry two ordinary lines make equal angles with a third ordinary line if and only if the common point of the first two is on the special line but distinct from I and J.

3. A and B are a pair of mates in an involution on the special line of which I and J are the self-corresponding points; P is any ordinary point. Show that the lines PA and PB are orthogonal.

4. Determine necessary and sufficient algebraic conditions that a real conic whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$$

shall be

(i) on a pair of real points of the special line,
(ii) on a pair of complex points of the special line,
(iii) on a pair of coincident points of the special line,
(iv) on I and J,
(v) on the points A and B, where (IJ, AB) is a harmonic tetrad.

5. The point R is said to be the mid-point of PQ if and only if $(PR) = (RQ)$. Show that if S be the isotropic point collinear with P and Q, the necessary and sufficient condition that R should be the mid-point of PQ is that (PQ, RS) should be a harmonic tetrad.

6. Determine the coordinates of the mid-points of the sides of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

7. Determine the equation of the conic on the five points I, J, (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

8. Φ is a conic on the two points I and J. Show that the ordinary points on Φ are all equidistant from the pole of the special line relative to Φ.

9. A and B are the self-corresponding points of an involution on the special line, in which I and J are a pair of mates. Show that if C and D are another pair of mates in this involution, and if P is any ordinary point, $\angle CPA = \angle APD$ and $\angle CPB = \angle BPD$; that is, that the lines PA and PB are the bisectors of the two angles CPD and DPC.

10. In Euclidean Metrical Geometry two lines are said to be parallel if their common point is on the special line (cf. Ex. 2), and a four-line is said to be a parallelogram if one of its diagonal lines is the special line. With these definitions prove the theorems of elementary Geometry which deal with the parallelogram. (The most important are (i) opposite sides and angles equal, and (ii) diagonals bisect each other.)

12. Taking the definition of a *median* of a triangle as it is given in elementary Geometry, prove that the medians of a triangle are concurrent.

13. In Euclidean Metrical Geometry the centre of a non-singular conic is defined to be the pole of the special line relative to the conic, and any line on the centre of a conic is said to be a diameter of the conic. Show that

(i) If A and B are the two points common to a conic and one of its diameters, then the centre is the mid-point of AB.

(ii) The locus of the mid-points of chords of a conic which are all parallel to a given diameter is a diameter which is conjugate to the first relative to the conic.

14. In Euclidean Metrical Geometry a *central conic* is defined to be a non-singular conic whose centre is an ordinary point; a *parabola* is defined to be a conic whose centre is on the special line. Show that

(i) all diameters of a parabola are parallel;

(ii) on four distinct ordinary points, no three of which are collinear, there are in general two and only two parabolas.

15. In Euclidean Metrical Geometry the *asymptotes* of a central conic are defined to be the tangents to the conic which are on the centre of the conic. Show that the common points of a conic and its asymptotes are on the special line. Show also that pairs of diameters which are conjugate relative to the conic are pairs of mates in an involution in which the asymptotes are the self-corresponding lines.

16. Determine the coordinates of the centre, and the equation of the asymptotes of the conic whose equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0.$$

17. Show that the necessary and sufficient condition that the asymptotes of a conic should be orthogonal is that I and J should be conjugate points relative to the conic.

10.4. The Use of the Non-homogeneous Mesh Gauge

Because in Euclidean Metrical Geometry one line is the special line, it is sometimes convenient to use a non-homogeneous mesh gauge wherein the special line is the unlabelled line. The change from the homogeneous mesh gauge to the non-homogeneous mesh gauge is effected, as usual, by choosing the coordinates of all ordinary points so that the z-coordinate is 1. The vertex Z of the triangle of reference will then be the origin, the lines ZY and ZX will be the axes of x and y respectively.

But it should be noted that though this is often a simplification of the algebraic work involved in a problem, it may not be a useful simplification of the geometrical problem. For the non-homogeneous mesh gauge so chosen cannot lead to any information about the special line; and in one sense the special line is the most important line of the field in Euclidean Metrical Geometry. To leave it out of account may therefore be a serious blemish.

10.5. Euclidean Metrical Geometry and Elementary Geometry

The reader cannot have failed to notice the similarity between the results of this chapter and the various results with which he is familiar from elementary Geometry, Trigonometry, and Analytical Geometry. This similarity raises the question: What are the relations between Euclidean Metrical Geometry (i.e. Projective Geometry with a Euclidean metric imposed) and elementary Geometry?

It should be recognized at this stage that if we impose on the projective field a homogeneous mesh gauge, and then consider the set of points which are (i) real and (ii) not on a certain definite line, we have a set of points of which the points of elementary Geometry are a representation. The propositions of incidence which can be stated of this restricted set of points in the projective field are identical with the propositions of incidence which can be stated in elementary Geometry. Owing to the fact that we have deliberately left out of consideration a whole range of points, the initial propositions of Projective

Geometry cannot be predicated of the restricted field without qualification, and this is the reason why the initial propositions of Projective Geometry are not true, as they stand, of elementary Geometry.

If now we impose a Euclidean metric on the original Projective field, making the line we considered exceptional to be the special line, and two appropriate complex points on it the special points, then the metrical relations between the points and lines of the restricted field will be identical with the metrical relations between the points and lines of elementary Geometry.

Hence we may say that elementary Geometry is *one* representation of a restricted portion of the Projective field on which a Euclidean metric has been imposed.

Another important distinction between Euclidean Metrical Geometry and elementary Geometry should be noticed. If we wish to impose a Euclidean metric on the Projective field, we may choose any line we please as the special line, and any pair of distinct points on it as the special points. The metrical relations of the field will then be Euclidean. There are thus an infinity of ways in which a Euclidean metric may be imposed on the field; with one of them a certain triangle may be right-angled, with another the same triangle may be equilateral, and so on. But in elementary Geometry the metric is definitely and unchangeably fixed by the presuppositions of the Geometry.

10.51. The Circular Points at Infinity

The analytical treatment of elementary Geometry led mathematicians to the conclusion that the field of elementary Geometry lacked an important line, which they called the 'line at infinity', and the device of homogeneous coordinates in Analytical Geometry enabled them to add this line to the field and discuss it in much the same way as other lines. In doing so they realized they were stepping outside the boundaries of elementary Geometry, and so the line was spoken of as an 'ideal line'—something not really there, but convenient to imagine as being there. When a further widening of the field was made, and complex points were added, it was recognized that there were two very important complex points on the line at infinity; and because every circle passed through these points, they were called the 'circular points at infinity'.

These points we have seen as the special points of the Euclidean metric; the approach to them from Projective Geometry shows that they are not different from any other point of the field, though from the nature of things they do not share certain metrical properties with other points. But, since the reason for these exceptional *metrical* properties is clearly shown, the special points are not the anomalous and contradictory things they usually appear to be in Analytical Geometry. The projective approach shows that, *because of the definitions*, the special line and all points on it are metrically exceptional; Analytical Geometry cannot show why this should be so, and sometimes does not point out that it is so.

By way of illustrating the use of definitions, we may here attempt to answer the question: 'Does every circle pass through the circular points at infinity?' The true answer is: 'It depends how the term *circle* is defined.' If it is defined as being the locus of points which are equidistant from a given point, then I and J are not on this locus. If it is defined as the conic whose ordinary points are all equidistant from a given point, then I and J are on this conic. (See 10.34, Ex. 8.) It is best to define a circle in Euclidean Metrical Geometry either as a conic which is on I and J, or as a conic whose ordinary points are all equidistant from a given ordinary point. (The two definitions are not quite equivalent; the second definition excludes the singular conics which satisfy the first.)

10.52. The Role of the Special Points

The role which the special points play in Euclidean Metrical Geometry has already been illustrated in the examples given after 10.34, and it may be further emphasized. A number of examples and theorems in Chapters V and VI are projective theorems which can easily be translated into metrical theorems in Euclidean Metrical Geometry, when a certain pair of points are taken as the special points. The theorems and examples which admit of easy translation in this manner have been so worded that the letters I and J appear in them; when these points are taken as the special points, and the projective relations of the other points and lines with them are stated in metrical terms, a metrical theorem (usually well known) appears.

Thus if in Theorem 6.31 I and J are taken as the special points of Euclidean Metrical Geometry, then

(i) Y_1', Y_2', Y_3' are the mid-points of A_2A_3, A_3A_1, A_1A_2 respectively;

(ii) A_0 is the orthocentre of the triangle $A_1A_2A_3$;

(iii) D_1, D_2, D_3 are the feet of the perpendiculars from A_1, A_2, A_3 on to A_2A_3, A_3A_1, A_1A_2 respectively;

(iv) Y_1, Y_2, Y_3 are the mid-points of A_0A_1, A_0A_2, A_0A_3 respectively.

In metrical terms the theorem states that there is a circle on the nine points D_1, D_2, D_3, Y_1, Y_2, Y_3, Y_1', Y_2', Y_3'. This circle is known as the nine-points circle in elementary Geometry.

The converse process, that of producing a projective theorem from a metrical theorem in Euclidean Metrical Geometry is also instructive, and the reader should attempt to translate some of the easier metrical theorems of elementary Geometry into projective terms, and then to prove them projectively.

It is not to be inferred from this, however, that the whole object of Projective Geometry is to produce metrical theorems. The foregoing remarks are intended to show that the metrical interpretation of theorems gives only one aspect of those theorems, and that not the fundamental aspect. The metrical properties of the field are accidental and subsidiary; they are not the fundamental properties that elementary Geometry makes them. The whole truth is expressed when it is said that metrical properties are not absolute but relative properties; they are relative to a metric gauge (singular or non-singular), the choice of which is entirely at our disposal.

10.6. Simultaneous Metrics

This relativity of metrical properties may be illustrated by imposing simultaneously on the field three distinct sets of metrical properties.

First we impose on the field a homogeneous mesh gauge, and choose three real points of the field which are interior points relative to the real-real conic whose equation is

$$-(x^2+y^2-z^2) = 0.$$

Calling these points A, B, and C, we impose a hyperbolic metric by taking this point-conic and the corresponding line-conic as metric gauge.

Let the measures of the sides and angles of this triangle with this metric be a_1, b_1, c_1, A_1, B_1, C_1.

Next we impose an elliptic metric, by taking the usual conic as metric gauge; let the measures of the sides and angles with this metric be a_2, b_2, c_2, A_2, B_2, C_2.

Finally, we impose a Euclidean metric, by taking the line [0, 0, 1] as the special line and the points I and J as the special points. Let the measures of the sides and angles in this metric be a_3, b_3, c_3, A_3, B_3, C_3.

Then at one and the same time it is true that

$$\cosh a_1 = \cosh b_1 \cosh c_1 + \sinh b_1 \sinh c_1 \cos A_1,$$

$$\cos a_2 = \cos b_2 \cos c_2 + \sin b_2 \sin c_2 \cos A_2,$$

and

$$a_3^2 = b_3^2 + c_3^2 - 2b_3 c_3 \cos A_3;$$

nor can it be said that any one of the foregoing represents the Metrical Geometry of the triangle more faithfully than any other.

10.7. Parallelism

It is sometimes said that the fundamental difference between Elliptic, Hyperbolic, and Euclidean Metrical Geometries lies in what is called the *parallel postulate* which each makes. It has been shown that the fundamental difference between the three lies *not* in the choice of this or that parallel postulate, but in the choice of this or that metric gauge. Nevertheless, the question of parallelism in the three Metrical Geometries deserves some consideration.

In elementary Geometry a common form of parallel postulate is what is called Playfair's Axiom: Through any point not on a given line, there is one and only one line parallel to the given line. And the corresponding postulates for the other Metrical Geometries are (1) for Elliptic Metrical Geometry: Through any point not on a given line there are no lines parallel to the given line, and (2) for Hyperbolic Metrical Geometry: Through any point not on a given line there are two lines parallel to the given line.

Whether or not these postulates are verified in the three Metrical Geometries depends on the definition given of parallel lines. We therefore test various definitions.

DEF. I. *Parallel lines are lines that do not meet (at a finite point).*

The postulates are verified for Euclidean and Elliptic Metrical Geometries. That for Hyperbolic Metrical Geometry is not verified, since, given an interior line, there are an infinity of interior lines which do not meet it.

DEF. II. *Parallel lines are lines which meet on the absolute.*

If 'absolute' be translated into the term 'point-conic of the metric gauge' the three postulates are verified in their respective Metrical Geometries.

DEF. III. *Parallel lines are lines which make equal angles with any transversal.*

The postulate of Euclidean Metrical Geometry is verified. Lines having the requisite property are non-existent in the other two Metrical Geometries.

DEF. IV. *A line is said to be parallel to a given line if it is everywhere equidistant from it.*

The postulate of Euclidean Metrical Geometry is verified. In the other two Metrical Geometries the equidistance locus is a conic, not a line.

The four definitions of parallel lines are equivalent in Euclidean Metrical Geometry; if they have any meaning in the other Metrical Geometries, they are not necessarily equivalent. Hence the meaning of the term *parallel* must be carefully defined before it can be said whether or not a given 'parallel postulate' is verified in Elliptic or Hyperbolic Metrical Geometry.

10.8. Complex and Real Euclidean Metrical Geometries

The Euclidean Metrical Geometry which has been investigated so far in this chapter is the result of a limiting process on the non-singular conic whose equations are

$$R^2x^2+R^2y^2+z^2 = 0$$

and

$$l^2+m^2+R^2n^2 = 0.$$

This limiting process produces the real-complex singular line-conic whose equation is

$$l^2+m^2 = 0,$$

as the metric gauge for the measurement of angle.

Now it is quite clear that if we had started with the non-singular conic whose equations are

$$-R^2x^2+R^2y^2+z^2 = 0$$

and $$l^2-m^2-R^2n^2 = 0,$$

and carried out the same limiting process, the resulting singular line-conic for the measurement of angle would have been that whose equation is

$$l^2-m^2 = 0.$$

This is, plainly, a real-real singular line-conic. It is in fact the line-conic consisting of the pencils of lines on the two points $(1,1,0)$ and $(1,-1,0)$.

Hence this singular Metrical Geometry is very similar to the Euclidean Metrical Geometry which has already been developed. There is a special line, the line $[0,0,1]$, and on it two special points $(1,1,0)$ and $(1,-1,0)$, and from these are derived the metrical properties of the field.

These two singular Metrical Geometries bear to each other a relation very similar to that between Elliptic and Hyperbolic Metrical Geometries. They may be distinguished by calling them, respectively, Complex Euclidean Metrical Geometry and Real Euclidean Metrical Geometry; the reason for the names is obvious.

The reader is very strongly advised to work out the details of Real Euclidean Metrical Geometry in exactly the same way as those of Complex Euclidean Metrical Geometry. The following is a synopsis of the results.

(i) *Distance.*

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\pm\sqrt{\left(\frac{(z_1x_2-z_2x_1)^2-(z_1y_2-z_2y_1)^2}{z_1^2z_2^2}\right)},$$

or, if $z_1 = z_2 = 1$, $\pm\{(x_2-x_1)^2-(y_2-y_1)^2\}^{\frac{1}{2}}$.

(ii) *Angle.*

It being agreed that the coordinates of all lines shall be specified only by three real numbers, it will be found that the ordinary lines of the field are divided into two distinct classes. If $[l, m, n]$ are the coordinates of an ordinary line, it will belong to one class if $l^2-m^2 > 0$, to the other if $l^2-m^2 < 0$. This corresponds to the distinction in Hyperbolic Metrical Geometry between interior and exterior lines.

The scale-constant is taken as $\frac{1}{2}$.

If $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ are two lines such that $l_1^2 > m_1^2$ and $l_2^2 > m_2^2$, and if θ be the angle between them, then if suitable conventions be adopted,

$$\cosh\theta = \frac{(l_1 l_2 - m_1 m_2)}{(l_1^2-m_1^2)^{\frac{1}{2}}(l_2^2-m_2^2)^{\frac{1}{2}}}$$

and

$$\sinh\theta = \frac{(l_1 m_2 - l_2 m_1)}{(l_1^2-m_1^2)^{\frac{1}{2}}(l_2^2-m_2^2)^{\frac{1}{2}}}.$$

If, on the other hand, $l_1^2 < m_1^2$ and $l_2^2 < m_2^2$, the denominators in these fractions are changed to

$$(m_1^2-l_1^2)^{\frac{1}{2}}(m_2^2-l_2^2)^{\frac{1}{2}}.$$

From these results the Real Euclidean Metrical Geometry of the triangle can be deduced.

10.9. Dual Euclidean Metrical Geometries

The two Euclidean Metrical Geometries were evolved from the Metrical Geometries of Chapter IX by limiting processes in which the metric gauge became a singular conic. The line-conic became a pair of pencils of lines on distinct points, the point-conic a pair of coincident ranges on a line. It is quite clear that we could have made the limiting processes work the other way, so that the point-conic of the metric gauge became a pair of ranges on distinct lines, and the line-conic a pair of coincident pencils on a point, the common point of the bases of the ranges.

The resulting Metrical Geometries would have been the exact duals of those elaborated in this chapter. Save for the fact that they illustrate a fifth and a sixth simple Metrical Geometry, they have no particular interest.

CHAPTER XI

TRANSFORMATIONS OF THE MESH GAUGE AND THE FIELD

THE problem with which we are first concerned in this chapter may be stated thus: Given that the field is labelled by two homogeneous mesh gauges simultaneously, one having the triangle XYZ as triangle of reference and I as gauging-point, the other having the triangle $X'Y'Z'$ as triangle of reference and I' as gauging-point, what will be the relation between the coordinates of any point relative to the first mesh gauge, and its coordinates relative to the second mesh gauge? Put another way the problem is: If the triangle of reference and gauging-point be changed from XYZ and I to $X'Y'Z'$ and I', how will the coordinates of any point of the field be changed?

11.1. Transformations of the Mesh Gauge

It is necessary to have a uniform notation so that confusion may be avoided. The coordinates of *all* points relative to the mesh gauge fixed by the triangle of reference XYZ and the gauging-point I will be written in the normal way, (x,y,z), $(2,7,-10)$, etc. The coordinates of *all* points relative to the mesh gauge fixed by the triangle of reference $X'Y'Z'$ and the gauging-point I' will be written with primes affixed to the coordinates, thus: (x',y',z'), $(2',3',14')$, etc.

In order to have a short distinguishing word for each mesh gauge, we shall speak of the *plain* mesh gauge and the *prime* mesh gauge, rather than of the mesh gauge fixed by the triangle of reference XYZ and the gauging-point I and the mesh gauge fixed by the triangle of reference $X'Y'Z'$ and the gauging-point I'.

11.11. The Equations of Transformation

THEOREM. *If the coordinates of any point in the plain mesh gauge be (x,y,z), and the coordinates of the same point in the prime mesh gauge be (x',y',z'), the two are connected by equations of the form*

$$\begin{aligned} k'x' &= a_{11}x+a_{12}y+a_{13}z, & kx &= A_{11}x'+A_{21}y'+A_{31}z', \\ k'y' &= a_{21}x+a_{22}y+a_{23}z, & ky &= A_{12}x'+A_{22}y'+A_{32}z', \\ k'z' &= a_{31}x+a_{32}y+a_{33}z, & kz &= A_{13}x'+A_{23}y'+A_{33}z', \end{aligned}$$

where (i) *the determinant*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is not equal to zero,

(ii) A_{11}, A_{12}, A_{13} *are the minors of* a_{11}, a_{12}, a_{13}, *etc., respectively, in this determinant,*

(iii) k *and* k' *are constants, different from zero.*

Suppose that XYZ is the triangle of reference for the plain mesh gauge, and that I is the gauging-point; let (x_1', y_1', z_1'), (x_2', y_2', z_2'), (x_3', y_3', z_3'), and (x_0', y_0', z_0'), respectively, be the prime coordinates of these points. Then since no three of them are collinear, there is a set of four numbers, λ_0, λ_1, λ_2, λ_3, none of which are zero, which satisfies the three equations

$$\lambda_0 x_0' + \lambda_1 x_1' + \lambda_2 x_2' + \lambda_3 x_3' = 0,$$
$$\lambda_0 y_0' + \lambda_1 y_1' + \lambda_2 y_2' + \lambda_3 y_3' = 0,$$
and
$$\lambda_0 z_0' + \lambda_1 z_1' + \lambda_2 z_2' + \lambda_3 z_3' = 0.$$

Moreover, it is plain that if the actual numbers specifying the coordinates of these four points are suitably chosen, one solution of these equations will be $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 1$, and it will henceforward be supposed that this is so.

From this it follows that the point whose prime coordinates are $(x_2' + x_3', y_2' + y_3', z_2' + z_3')$ may also be specified as

$$(x_0' + x_1', y_0' + y_1', z_0' + z_1');$$

that is to say it is collinear with Y and Z, and it is also collinear with X and I.

If then I_x, I_y, I_z, are the points $\begin{pmatrix} YZ \\ XI \end{pmatrix}$, $\begin{pmatrix} ZX \\ YI \end{pmatrix}$, $\begin{pmatrix} XY \\ ZI \end{pmatrix}$, respectively, their prime coordinates are $(x_2' + x_3', y_2' + y_3', z_2' + z_3')$, $(x_3' + x_1', y_3' + y_1', z_3' + z_1')$, $(x_1' + x_2', y_1' + y_2', z_1' + z_2')$, respectively.

Suppose now that P is a point of the field, and that its plain and prime coordinates are, respectively, (x, y, z) and (x', y', z'). Then, as before, there is a set of four numbers, μ, μ_1, μ_2, μ_3, which satisfies the three equations

$$\mu x' + \mu_1 x_1' + \mu_2 x_2' + \mu_3 x_3' = 0,$$
$$\mu y' + \mu_1 y_1' + \mu_2 y_2' + \mu_2 z_2' = 0,$$
and
$$\mu z' + \mu_1 z_1' + \mu_2 z_2' + \mu_3 z_3' = 0.$$

Moreover, if it be supposed that no three of the five points $XYZIP$ are collinear, none of the numbers μ, μ_1, μ_2, μ_3 are zero, and their ratios are unique.

If now P_x, P_y, P_z, are the points $\binom{YZ}{XP}$, $\binom{ZX}{YP}$, $\binom{XY}{ZP}$, respectively, their prime coordinates are

$$(\mu_2 x_2'+\mu_3 x_3', \mu_2 y_2'+\mu_3 y_3', \mu_2 z_2'+\mu_3 z_3'),$$
$$(\mu_3 x_3'+\mu_1 x_1', \mu_3 y_3'+\mu_1 y_1', \mu_3 z_3'+\mu_1 z_1'),$$
$$(\mu_1 x_1'+\mu_2 x_2', \mu_1 y_1'+\mu_2 y_2', \mu_1 z_1'+\mu_2 z_2'),$$

respectively.

Consider now the cross-ratio of the four distinct collinear points, $R\!\!\!/(YZI_x P_x)$. If their plain coordinates are used to evaluate this cross-ratio, it is found to be

$$R\!\!\!/\{(1,0), (0,1), (1,1), (y,z)\};$$

but if their prime coordinates are used to evaluate it, it is found to be $R\!\!\!/\{(1,0), (0,1), (1,1), (\mu_2, \mu_3)\}$. From this, and from a similar consideration of the two cross-ratios $R\!\!\!/(ZXI_y P_y)$, $R\!\!\!/(XYI_z P_z)$, it follows that $cx = \mu_1$, $cy = \mu_2$, $cz = \mu_3$, where c is a number which is not zero.

When these values of μ_1, μ_2, and μ_3 are substituted in the equations given above, the result is

$$\mu x'+cxx_1'+cyx_2'+czx_3' = 0,$$

and two similar equations, and if a_{11}, a_{12}, a_{13}, etc. be written for $-cx_1'$, $-cx_2'$, $-cx_3'$, and k' for μ, these take the form given in the enunciation; namely

$$k'x' = a_{11}x+a_{12}y+a_{13}z,$$
$$k'y' = a_{21}x+a_{22}y+a_{23}z,$$
$$k'z' = a_{31}x+a_{32}y+a_{33}z.$$

It is plain that the determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -cx_1' & -cx_2' & -cx_3' \\ -cy_1' & -cy_2' & -cy_3' \\ -cz_1' & -cz_2' & -cz_3' \end{vmatrix}$$

are identical, and therefore that the former does not vanish; from this it follows that the second set of equations given in the enunciation are also true.

This proves the theorem when P is a point which is not collinear with any pair of the four points $XYZI$. That the equations remain true even when this is so is a simple consequence which is left to the reader to verify.

11.12. The Equations of Transformation of the Line

THEOREM. *The equations of transformation being those given in the last theorem, if the coordinates of any line in the plain mesh gauge be $[l, m, n]$ and the coordinates of the same line in the prime mesh gauge be $[l', m', n']$, then the two are connected by the equations*

$$k'l' = A_{11}l+A_{12}m+A_{13}n, \qquad kl = a_{11}l'+a_{21}m'+a_{31}n',$$
$$k'm' = A_{21}l+A_{22}m+A_{23}n, \qquad km = a_{12}l'+a_{22}m'+a_{32}n',$$
$$k'n' = A_{31}l+A_{32}m+A_{33}n, \qquad kn = a_{13}l'+a_{23}m'+a_{33}n'.$$

Consider any line $[l, m, n]$; then the plain coordinates of the points on it satisfy the equation $lx+my+nz = 0$.

Then their prime coordinates satisfy the equation

$$l(A_{11}x'+A_{21}y'+A_{31}z')+m(A_{12}x'+A_{22}y'+A_{32}z')+$$
$$+n(A_{13}x'+A_{23}y'+A_{33}z') = 0.$$

Hence, if the line is $[l', m', n']$ specified in prime coordinates,

$$k'l' = A_{11}l+A_{12}m+A_{13}n,$$

and there are two similar equations for m' and n', as in the enunciation of the theorem. From these three equations the other three plainly follow.

11.2. Real Transformations

The theorems so far proved have done no more than show how the two sets of coordinates of a point in two mesh gauges are connected. We now consider in more detail a certain special type of transformation to which is given the name of *real transformation.*

DEFINITION. *A transformation from one mesh gauge to another is said to be a real transformation if in both mesh gauges the same set of points are real points.*

Two questions arise: (1) Are there such transformations? and (2) What are the necessary and sufficient conditions that a transformation should be a real transformation?

11.21. Existence of Real Transformations

THEOREM. *A transformation is a real transformation if all the coefficients in the equations of transformation* (11.11) *are real numbers.*

It may be observed first, that if all the nine coefficients a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} are real numbers, then the corresponding minors, A_{11}, etc., are also real numbers, and vice versa.

The theorem is an obvious consequence of 11.11.

11.22. Necessary and Sufficient Conditions

THEOREM. *A necessary and sufficient condition that a transformation shall be a real transformation is that X', Y', Z', and I' shall be real points in the plain mesh gauge.*

That the condition is necessary is an obvious consequence of the definition of a real transformation. That it is sufficient follows from 11.11 and 11.21. For by supposition the plain coordinates of X', Y', Z', and I' may all be expressed by real numbers; from 11.11 it follows that A_{11}, etc., may be expressed as real numbers, and so, from 11.21, the transformation is a real transformation.

This theorem is of interest, not because it is a particularly useful theorem but because it shows that if four points of the field, no three of which are collinear, are chosen as real points, then all the real points of the field are fixed by this choice. It therefore shows the number of degrees of freedom there are in the choice of which set of points shall be the real points of the field.

11.23. Another Necessary and Sufficient Condition

THEOREM. *The necessary and sufficient condition that a transformation shall be a real transformation is that all the coefficients in the equations of transformation shall be expressible as real numbers.*

The word *expressible* needs some explanation. The equations of transformation $k'x' = a_{11}x+a_{12}y+a_{13}z$, etc., may be written $ck'x' = ca_{11}x+ca_{12}y+ca_{13}z$, etc.; hence the equations of a transformation are not unique. The theorem states that if a transformation is a real transformation, there is a constant c such that ca_{11}, ca_{12}, etc., are all real numbers, and conversely.

That the condition is sufficient is obvious from 11.11; this part of the theorem merely restates Theorem 11.21.

That the condition is necessary is proved as follows. If the transformation is a real transformation, then by 11.22 the plain coordinates of X', Y', Z', and I' may all be specified by real numbers; hence, by 11.11, the coefficients in the equations of transformation may all be expressed as real numbers.

11.3. Application to Metrical Geometry

In the chapter on the metric gauge it was first supposed that a mesh gauge was imposed on the field, and then a non-singular conic with a conveniently simple equation was chosen as the metric gauge. This was, however, an unnecessarily narrow restriction, which can now be removed.

In the first place, it is plain that if in the plain mesh gauge a non-singular conic is a real-real or a real-complex conic, then it is a real-real or a real-complex conic respectively in the prime mesh gauge if the transformation between the two is a real transformation.

Let us suppose now that a certain real conic (and for definiteness we may suppose it to be a real-real conic) is chosen as the metric gauge, and that its equation is not $-x^2-y^2+z^2 = 0$. We choose any real self-polar triangle of this conic as the prime triangle of reference, and any real point not on the sides of this triangle as the prime gauging-point. In the prime mesh gauge the conic will have an equation of the form $ax'^2+by'^2+cz'^2 = 0$, where a, b, and c are real and have not all the same sign. For definiteness suppose that a and c are negative, and that b is positive. Then the real transformation given by the equations

$$kx'' = +\sqrt{(-a)}x',$$
$$ky'' = +\sqrt{(-c)}z',$$
$$kz'' = +\sqrt{b}\,y'$$

transforms the mesh gauge into one wherein the conic has the equation $-x''^2-y''^2+z''^2 = 0$.

In this final mesh gauge the metrical properties of the real points may be investigated by the methods of Chapter IX. For instance, suppose that the distance between two points whose plain coordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) is required. Their coordinates in the double-prime mesh gauge are first found, the distance is then found by the methods of Chapter IX. This is the distance between them relative to the chosen metric gauge.

The problems for Elliptic and Euclidean Metrical Geometries are treated in a similar fashion.

11.4. Transformations of the Field: Homographies

The fundamental concept of Projective Geometry, the projectivity, was defined in Chapter III as being a relationship between two ranges or two pencils, or a range and a pencil, which could be specified by a sequence of perspectivities. A projectivity between two ranges was seen to be a multiple relationship between them, whereby to any point of one range was related one and only one point of the other. It was seen also that projectivities could exist between two ranges on the same base.

We now introduce a corresponding, but wider, multiple relationship, called a *homography*, not between range and range, but between all the points of the field and themselves.

In order to grasp the notion clearly it is convenient to think of every point of the field as being two coincident points, one red, one blue. We may then speak of the red field and the blue field. A homography is said to exist between these two fields when (i) to every point of the red field there corresponds one and only one point of the blue field, and vice versa, (ii) if three points of the red field are collinear, the corresponding points of the blue field are collinear, and (iii) if $PQRS$ are four collinear points of the red field, and $P'Q'R'S'$ are the corresponding points of the blue field then $(PQRS) \sim (P'Q'R'S')$. This may suffice as a definition, for though it is not in formal language, it is rigorous.

It is possible to discuss homographies by the methods of Synthetic Projective Geometry; here, however, the algebraic method is used, and so a homogeneous mesh gauge is imposed on the field. Clearly, in any such discussion, it is necessary to prove first that the multiple relationship called a homography is possible.

We shall regularly make use of an informal terminology, since this makes for brevity and clarity. We shall therefore speak of *plain* points, the *plain* field, and *prime* points, the *prime* field. These adjectives are preferred to the adjectives *red* and *blue*, since we agree to specify plain points by plain coordinates, (x, y, z), and prime points by coordinates to which a prime is affixed, (x', y', z').

The algebraic work in the discussion of the homography is almost identical with that used in discussing the transformation of mesh gauges, though the thought behind that work is different. It is therefore necessary to distinguish clearly between the two, since they are easily confused. In treating of the transformation of mesh gauges we are dealing with two mesh gauges and one set of points; in treating of homographies we are dealing with one mesh gauge and two sets of points.

11.41. The Equations of a Homography

THEOREM. *The necessary and sufficient condition that a multiple relationship between the points of the field shall be a homography is that the coordinates of corresponding points shall be connected by equations of the form*

$$\begin{aligned} k'x' &= a_{11}x+a_{12}y+a_{13}z, & kx &= A_{11}x'+A_{21}y'+A_{31}z',\\ k'y' &= a_{21}x+a_{22}y+a_{23}z, & ky &= A_{12}x'+A_{22}y'+A_{32}z',\\ k'z' &= a_{31}x+a_{32}y+a_{33}z, & kz &= A_{13}x'+A_{23}y'+A_{33}z', \end{aligned}$$

where (i) *the determinant*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is not equal to zero,

(ii) A_{11}, A_{12}, A_{13}, *etc., are the minors of* a_{11}, a_{12}, a_{13}, *etc., respectively, in this determinant, and*

(iii) k *and* k' *are constants, different from zero.*

The sufficiency of the condition is proved first.

Suppose then that the two sets of points are connected by the equations given, and the determinant does not vanish.

Then clearly to every plain point there corresponds one and only one prime point. Moreover, if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are three collinear plain points, then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

But, from the equations, this determinant is equal to

$$\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} \begin{vmatrix} x_1' & y_1' & z_1' \\ x_2' & y_2' & z_2' \\ x_3' & y_3' & z_3' \end{vmatrix},$$

and since by supposition the first of these two determinants does not vanish, the second must vanish. Hence the three corresponding prime points are collinear.

Finally, let $PQRS$ be any four distinct collinear points of the plain field, and $P'Q'R'S'$ the corresponding points of the prime field. If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of P and Q, those of R and S may be written

$$(\lambda x_1+\mu x_2, \lambda y_1+\mu y_2, \lambda z_1+\mu z_2) \text{ and } (\rho x_1+\sigma x_2, \rho y_1+\sigma y_2, \rho z_1+\sigma z_2),$$

respectively, and so $\text{R}(PQRS) = \text{R}\{(1,0), (0,1), (\lambda,\mu), (\rho,\sigma)\}$.

Moreover, if (x_1', y_1', z_1') and (x_2', y_2', z_2') are the coordinates of P' and Q', it is plain from the equations connecting the coordinates of corresponding points of the plain and prime fields that those of R' and S' may be written

$$(\lambda x_1'+\mu x_2', \lambda y_1'+\mu y_2', \lambda z_1'+\mu z_2') \text{ and } (\rho x_1'+\sigma x_2', \rho y_1'+\sigma y_2', \rho z_1'+\sigma z_2'),$$

respectively.

It follows at once from this that

$$\text{R}(P'Q'R'S') = \text{R}\{(1,0), (0,1), (\lambda,\mu), (\rho,\sigma)\},$$

and so $(PQRS) \sim (P'Q'R'S')$. This proves the sufficiency of the condition.

The necessity of the same condition is proved as follows.

Suppose that there is a homography between the plain and prime fields. Let XYZ be the triangle of reference of the mesh gauge, and I the gauging-point. If these points are thought of as belonging to the plain field, let $X'Y'Z'I'$ be the points of the prime field which correspond to them in the homography.

Let P be any point of the plain field, and P' the corresponding point of the prime field.

If now I_x, I_y, I_z, P_x, P_y, P_z be the six points $\begin{pmatrix}YZ\\XI\end{pmatrix}$, $\begin{pmatrix}ZX\\YI\end{pmatrix}$, $\begin{pmatrix}XY\\ZI\end{pmatrix}$, $\begin{pmatrix}YZ\\XP\end{pmatrix}$, $\begin{pmatrix}ZX\\YP\end{pmatrix}$, $\begin{pmatrix}XY\\ZP\end{pmatrix}$, respectively, and if these be considered as points of the plain field, it is not difficult to see that the corresponding points of the prime field are I'_x, I'_y, I'_z, P'_x, P'_y, and P'_z respectively, that is, $\begin{pmatrix}Y'Z'\\X'I'\end{pmatrix}$, $\begin{pmatrix}Z'X'\\Y'I'\end{pmatrix}$, $\begin{pmatrix}X'Y'\\Z'I'\end{pmatrix}$, $\begin{pmatrix}Y'Z'\\X'P'\end{pmatrix}$, $\begin{pmatrix}Z'X'\\Y'P'\end{pmatrix}$, and $\begin{pmatrix}X'Y'\\Z'P'\end{pmatrix}$, respectively.

From this it follows that (i) $(YZI_xP_x) \sim (Y'Z'I'_xP'_x)$ if the four points are distinct, and (ii) P'_x coincides with Y', Z', or I'_x, according as P coincides with Y, Z, or I_x, if the four are not distinct. There are similar conclusions about the other two tetrads $Z'X'I'_yP'_y$ and $X'Y'I'_zP'_z$.

Consider now a second mesh gauge imposed on *both* fields; for purposes of reference this will be called the *star mesh gauge.* Let its triangle of reference be $X'Y'Z'$, and its gauging-point I'. From what has been already deduced, it follows that if in the plain mesh gauge the coordinates of P are (x,y,z), the coordinates of P' in the star mesh gauge are (x^*,y^*,z^*), where $x^* = kx$, $y^* = ky$, $z^* = kz$, k being a number different from zero.

If therefore *in the plain mesh gauge* the coordinates of P' are (x',y',z'), the two sets of coordinates (x^*,y^*,z^*) and (x',y',z') are connected by equations of the type given in 11.11. It follows that the *plain coordinates* of P and the *plain coordinates* of P' are connected by equations of the same type, and this proves the necessity of the condition.

The equations determined here may be called the *equations of the homography*; they should be compared with the equations of the projectivity found in 8.22.

Since the equations of a homography are exactly the same as the equations of transformation of the mesh gauge (11.11), a homography is sometimes called a transformation (of the field). The prime point corresponding to a plain point is called its *transform*. This latter term will be used here.

11.42. The Homography of Lines

The homography has been defined as a relationship between points, and we therefore inquire about the dual concept, the homography of lines. Actually, however, this is not different from the homography defined, and it is easily proved that the homography of points is at the same time a homography of lines.

For consider the plain line $[l, m, n]$; the points on it satisfy the equation $lx+my+nz = 0$, and hence the prime points corresponding satisfy the equation

$$l(A_{11}x'+A_{21}y'+A_{31}z')+m(A_{12}x'+A_{22}y'+A_{32}z')+ \\ +n(A_{13}x'+A_{23}y'+A_{33}z') = 0.$$

Hence the coordinates $[l', m', n']$ of the prime line corresponding to $[l, m, n]$ satisfy the equations given in 11.12. The homography of points is therefore a homography of lines also.

11.43. The Determination of a Homography

THEOREM. *A homography is completely determined when four prime points, no three of which are collinear, are specified as the transforms of four plain points, no three of which are collinear.*

The proof of this theorem is a simple example of the solution of simultaneous linear equations by means of determinants; it should present no difficulty.

11.44. Self-corresponding Points and Lines

In a projectivity between two ranges on the same base there are two self-corresponding points, which may, however, coincide. Similarly, in a projectivity between two pencils on the same base there are, in general, two self-corresponding lines which may coincide. It is natural to ask the question: How many self-corresponding points and lines are there in a homography? The following theorems supply a partial answer to this question.

11.441. THEOREM. *In a homography there cannot be four self-corresponding points, no three of which are collinear, unless every point is self-corresponding.*

By the last theorem a unique homography is determined when four points, no three of which are collinear, are specified as the transforms of four points, no three of which are collinear.

If then the four points X, Y, Z, and W, no three of which are collinear, are specified as being self-corresponding points, the homography having this property is uniquely determined. But the 'identical' homography, that is, the homography in which every point is self-corresponding, has this property. Hence, if a homography has four self-corresponding points, no three of which are collinear, it is the identical homography. This proves the theorem.

It should be noticed, however, that if the words 'no three of which are collinear' are omitted from the enunciation of this theorem, it ceases to be true. For consider the homography whose equations are

$$kx = x', \quad ky = y', \quad kz = bz',$$

where b is not equal to zero or unity. In this homography the transform of the plain point (r, s, t) is the prime point $(r, s, t/b)$; hence *all* points on the line $z = 0$ are self-corresponding.

The dual of Theorem 11.441 is plainly true.

11.442. Determination of Self-corresponding Points. THEOREM. *In a homography whose equations are those given in* 11.41 *the coordinates of the self-corresponding points satisfy the equations*

$$(a_{11}-k)x+a_{12}y+a_{13}z = 0,$$
$$a_{21}x+(a_{22}-k)y+a_{23}z = 0,$$
$$a_{31}x+a_{32}y+(a_{33}-k)z = 0,$$

where k is a root of the cubic equation

$$\begin{vmatrix} a_{11}-k & a_{12} & a_{13} \\ a_{21} & a_{22}-k & a_{23} \\ a_{31} & a_{32} & a_{33}-k \end{vmatrix} = 0.$$

If the point (x, y, z) is a self-corresponding point of the homography, x, y, and z satisfy the equations

$$kx = a_{11}x + a_{12}y + a_{13}z, \text{ etc.}$$

That is to say, they satisfy the equations

$$(a_{11}-k)x + a_{12}y + a_{13}z = 0,$$

and the other two given in the enunciation. These equations are not, however, compatible unless the determinant

$$\begin{vmatrix} a_{11}-k & a_{12} & a_{13} \\ a_{21} & a_{22}-k & a_{23} \\ a_{31} & a_{32} & a_{33}-k \end{vmatrix} = 0;$$

hence k must be a root of this last equation.

A full discussion of self-corresponding points of a homography is beyond the scope of this book, but it will easily be recognized from the last theorem and its dual that there are in general three and only three self-corresponding points and three and only three self-corresponding lines in a homography. In certain cases, however, the value of k found from the determinantal equation does not give a unique solution for x, y, and z. In these cases a whole range of points is self-corresponding. This occurs when two roots of the equation for k coincide.

11.443. Invariant Figures. DEFINITION. *A figure is said to be an invariant figure of a homography, or to be invariant in the homography, if and only if it is identical with the figure formed by its transforms in the homography.*

If l is a self-corresponding line of a homography, then the range of points on l is an invariant figure. Similarly, if P is a self-corresponding point of the homography, the pencil of lines on P is an invariant figure.

This does not imply that all the points on l and all the lines on P are self-corresponding; it merely implies that the homography permutes amongst themselves, so to speak, the points on l and the lines on P. Care must be taken to distinguish between invariant figures and self-corresponding figures; the two are not the same.

It is possible to have other invariant figures than ranges of points and pencils of lines, and we shall meet non-singular conics which are invariant figures in a homography.

11.45. Further Theorems on Homographies

The following theorems on homographies are simple consequences of what has preceded; the proofs should present no difficulty to the reader.

11.451. THEOREM. *The transform of a non-singular conic in a homography is a non-singular conic.*

11.452. THEOREM. *If P, Q, R, and S are four distinct points on a non-singular conic Φ, and if P', Q', R', S', and Φ' are, respectively, the transforms of these, then*

$$\Phi(PQRS) \sim \Phi'(P'Q'R'S').$$

11.453. THEOREM. *The transforms of pole and polar relative to a non-singular conic are pole and polar relative to the transform of the conic; the transform of a tangent to a conic is a tangent to the transform of the conic.*

11.5. Real Homographies

DEFINITION. *A homography is said to be a real homography if and only if the transforms of all real points in it are real points.*

We shall be concerned in the sequel only with real homographies, and the two following theorems are proved about them. The first of these proves that there are real homographies.

11.51. Necessary and Sufficient Condition for a Real Homography

THEOREM. *The necessary and sufficient condition that a homography shall be a real homography is that all the coefficients in the equations of the homography shall be expressible as real numbers.*

That the condition is sufficient is plain. Its necessity may be proved in much the same way as Theorems 11.21 and 11.22, or as follows.

Suppose that the homography is a real homography, and that (x', y', z') is the transform of (x, y, z), so that

$$k'x' = a_{11}x + a_{12}y + a_{13}z, \text{ etc.},$$

x, y, z, x', y', z' being all real numbers.

Suppose now that

$$k' = k'' + ik^*, \quad a_{11} = b_{11} + ic_{11}, \quad a_{12} = b_{12} + ic_{12}, \quad \text{etc.},$$

where k'', k^*, b_{11}, b_{12}, etc., c_{11}, c_{12}, etc., are all real numbers. Then

$$(k''+ik^*)x' = (b_{11}+ic_{11})x+(b_{12}+ic_{12})y+(b_{13}+ic_{13})z.$$

If this equation be multiplied throughout by $(k''-ik^*)$, it follows that

$$(k''^2+k^{*2})x'$$
$$= (b_{11}k''+c_{11}k^*)x+(b_{12}k''+c_{12}k^*)y+(b_{13}k''+c_{13}k^*)z-$$
$$-i(b_{11}k^*-c_{11}k'')x-i(b_{12}k^*-c_{12}k'')y-i(b_{13}k^*-c_{13}k'')z.$$

From this it follows that

$$(b_{11}k^*-c_{11}k'')x+(b_{12}k^*-c_{12}k'')y+(b_{13}k^*-c_{13}k'')z = 0,$$

for every x, y, and z; hence the coefficients in this last equation are all zero. It follows that

$$(k''^2-k^{*2})x'$$
$$= (b_{11}k''+c_{11}k^*)x+(b_{12}k''+c_{12}k^*)y+(b_{13}k''+c_{13}k^*)z,$$

and there are two other similar equations. These must be equivalent to the original equations of the homography, and so the equations of the homography are expressible with real coefficients.

11.52. THEOREM.

The transform of a real-real conic in a real homography is a real-real conic; the transform of a real-complex conic in a real homography is a real-complex conic.

This theorem should scarcely require proof.

11.6. Invariant Non-singular Conics

The term *invariant figure* has been defined already, in 11.443, and in this section we consider briefly one type of invariant figure, the conic.

Consider, in the first place, a homography in which the self-corresponding points are the three non-collinear points X, Y, and Z, so that the three self-corresponding lines are YZ, ZX, and XY. It is at once possible to name six *singular* point-conics which are invariant figures in this homography. They are

(i) the two ranges on ZX and XY,
(ii) the two ranges on XY and YZ,
(iii) the two ranges on YZ and ZX,

(iv) the two coincident ranges on YZ,
(v) the two coincident ranges on ZX,
(vi) the two coincident ranges on XY.

Similarly, six *singular* line-conics can be named, all of which are invariant figures. In general it is true to say that these singular conics are the only invariant conics in a homography, but under certain special conditions it sometimes happens that non-singular conics are invariant.

Consider, for example, the homography determined by the equations $$k'x' = y,\quad k'y' = z,\quad k'z' = x.$$

The transform of the conics whose equations are
$$x^2+y^2+z^2+\lambda(yz+zx+xy) = 0,$$
where λ has any arbitrary value, are plainly
$$x'^2+y'^2+z'^2+\lambda(y'z'+z'x'+x'y') = 0,$$
so that every one of these conics is an invariant figure in this particular homography, and, save when $\lambda = 2$ and $\lambda = -1$, they are not singular conics.

Two questions naturally arise from this fact: (1) Given a homography, what non-singular conics, if any, are invariant figures in it? (2) Given a non-singular conic, in what homographies is it an invariant figure?

It is impossible in this book to give a complete answer to either question, and, in fact, no attempt is made to answer the first. A partial answer to the second is made, by taking a certain set of non-singular conics and determining the homographies in which they are invariant figures. The non-singular conics selected are those whose point equations are
$$R^2x^2+R^2y^2+z^2 = 0.$$

This particular set of conics is admittedly selected because of the applicability of the results to Metrical Geometry. Before proving the main theorem about these conics, two subsidiary theorems are necessary.

11.61. THEOREM.

For all values of t and u, save simultaneous zeros, the point whose coordinates are $(i(t^2+u^2), (t^2-u^2), 2tuR)$ is on the non-singular conic whose point equation is
$$R^2x^2+R^2y^2+z^2 = 0\quad (R \neq 0).$$

Conversely, any point on this conic has coordinates which may be expressed in this form.

The proof of this theorem is left to the reader; it should already have been done in 8.881, Ex. 13.

The theorem shows that we may legitimately speak of the point (t, u) on the conic, meaning thereby the point whose coordinates in the mesh gauge are those specified in the enunciation of the theorem.

11.62. Theorem

If there is a projectivity between two ranges on the non-singular conic whose point equation is $R^2x^2+R^2y^2+z^2 = 0$, *where* $R \neq 0$, *and if in this projectivity the points* (t_1, u_1), (t_2, u_2), $(t_3, u_3),\ldots,$ (t, u) *of one range correspond to the points* (t_1', u_1'), (t_2', u_2'), $(t_3', u_3'),\ldots,$ (t', u') *of the other, then constants* A, B, C, D *exist such that for every pair of corresponding points,*

$$t' = At+Bu, \quad u' = Ct+Du, \quad \text{and} \quad AD-BC \neq 0.$$

Let P_1, P_2, $P_3,\ldots$ be the points (t_1, u_1), (t_2, u_2), $(t_3, u_3),\ldots,$ and P_1', P_2', $P_3',\ldots$ be the points (t_1', u_1'), (t_2', u_2'), $(t_3', u_3'),\ldots$; let B be the point $(1, i, 0)$. Plainly B is on the conic.

The coordinates of the line BP are $[2iRtu, 2Rtu, 2t^2]$, or $[iRu, Ru, t]$. The common point of this line and the line $[1, 0, 0]$ is $(0, Ru, t)$. Let this point be Q.

Similarly, the coordinates of the line BP' are iRu', Ru', t', and the common point of this line and $[1, 0, 0]$ is $(0, Ru', t')$.

Now since $\quad (P_1 P_2 P_3 P_4\ldots) \sim (P_1' P_2' P_3' P_4'\ldots),$

and $\quad (P_1 P_2 P_3 P_4\ldots) \overset{B}{\sim} (Q_1 Q_2 Q_3 Q_4\ldots),$

and $\quad (P_1' P_2' P_3' P_4'\ldots) \overset{B}{\sim} (Q_1' Q_2' Q_3' Q_4'\ldots),$

it follows that

$$(Q_1 Q_2 Q_3 Q_4\ldots) \sim (Q_1' Q_2' Q_3' Q_4'\ldots),$$

and so, from 8.22, that constants A, B, C, and D exist such that $AD-BC \neq 0$, and for all pairs of corresponding points $t' = At+Bu$ and $u' = Ct+Du$.

This proves the theorem.

We are now in a position to prove the main theorem about homographies which leave the conic whose point equation is $R^2x^2+R^2y^2+z^2 = 0$ invariant.

11.63. THEOREM

Any homography whose equations may be put into the form

$$k'x' = R(\rho^2+\lambda^2-\mu^2-\nu^2)x+2R(\lambda\mu-\rho\nu)y+2(\nu\lambda+\rho\mu)z, \quad (1)$$

$$k'y' = 2R(\lambda\mu+\rho\nu)x+R(\rho^2+\mu^2-\nu^2-\lambda^2)y+2(\mu\nu-\rho\lambda)z, \quad (2)$$

$$k'z' = 2R^2(\nu\lambda-\rho\mu)x+2R^2(\mu\nu+\rho\lambda)y+R(\rho^2+\nu^2-\lambda^2-\mu^2)z, \quad (3)$$

where $\rho^2+\lambda^2+\mu^2+\nu^2 \neq 0$ *has the conic whose point equation is* $R^2x^2+R^2y^2+z^2 = 0$ *as an invariant figure.*

Conversely, the equations of any homography which leaves this conic invariant may be put into this form.

That any homography whose equations are of the form stated leaves the conic whose point equation is $R^2x^2+R^2y^2+z^2 = 0$ invariant, may be verified by mere algebra. The converse theorem may be proved by *reductio ad absurdum*, or, directly, as follows.

Consider any homography which leaves this conic invariant. By 11.453 there is a projectivity between the range of plain points of the conic, and the range of prime points on it. If (t, u) be a typical plain point on it, and (t', u') be a typical prime point on it, by 11.62 there are constants A, B, C, and D, such that $AD-BC \neq 0$, and

$$t' = At+Bu, \qquad u' = Ct+Du.$$

These equations may be written

$$t' = (\rho+i\nu)t-(\lambda+i\mu)u,$$
$$u' = (\lambda-i\mu)t+(\rho-i\nu)u,$$

where $\rho^2+\lambda^2+\mu^2+\nu^2 \neq 0$.

Now the coordinates in the mesh gauge of the points (t, u), (t', u') are, respectively,

$$(i(t^2+u^2), (t^2-u^2), 2Rtu), \quad \text{and} \quad (i(t'^2+u'^2), (t'^2-u'^2), 2Rt'u'),$$

and these two are corresponding points in the homography. We write their coordinates (x, y, z) and (x', y', z') respectively.

But

$$\begin{aligned} x' &= i(t'^2+u'^2) \\ &= i(\rho+i\nu)^2t^2-2i(\rho+i\nu)(\lambda+i\mu)tu+i(\lambda+i\mu)^2u^2+ \\ &\qquad +i(\lambda-i\mu)^2t^2+2i(\lambda-i\mu)(\rho-i\nu)tu+i(\rho-i\nu)^2u^2 \\ &= t^2(i\rho^2+i\lambda^2-i\mu^2-i\nu^2-2\rho\nu+2\lambda\mu)+4ut(\nu\lambda+\mu\rho)+ \\ &\qquad +u^2(i\rho^2+i\lambda^2-i\mu^2-i\nu^2+2\rho\nu-2\lambda\mu) \\ &= i(t^2+u^2)(\rho^2+\lambda^2-\mu^2-\nu^2)+2(t^2-u^2)(\lambda\mu-\rho\nu)+ \\ &\qquad +4tu(\nu\lambda+\mu\rho). \end{aligned}$$

Hence

$$x' = (\rho^2+\lambda^2-\mu^2-\nu^2)x+2(\lambda\mu-\rho\nu)y+2(\nu\lambda+\rho\mu)R^{-1}z. \tag{1}$$

Similarly, it may be proved that

$$y' = 2(\lambda\mu+\rho\nu)x+(\rho^2+\mu^2-\nu^2-\lambda^2)y+2(\mu\nu-\rho\lambda)R^{-1}z, \tag{2}$$

and $$z' = 2R(\nu\lambda-\rho\mu)x+2R(\mu\nu+\rho\lambda)y+(\rho^2+\nu^2-\lambda^2-\mu^2)z. \tag{3}$$

Now these equations give the relations between the coordinates of points on the conic and their transforms; but since no three of these points are collinear, by 11.43 they must be the equations of the homography.

If they are multiplied by R, and if a constant be substituted for R on the left-hand side, they take the form specified in the enunciation.

This proves the theorem.

The equations (1), (2), and (3) deduced in this theorem are a slightly more general form of what are known as the *Euler-Rodrigues Equations*. They reduce to the Euler-Rodrigues equations when $R = 1$. In this form they are well known in the theory of transformation of axes in three-dimensional Analytical Geometry.

The group of homographies determined by these equations for various values of the parameters λ, μ, ν, and ρ may be called the *Congruence Group of Homographies* for the conic whose point equation is $R^2x^2+R^2y^2+z^2 = 0$; the reason for the name will appear very shortly. If it is necessary to refer to any particular homography of the group by name, it may be called the homography $(R;\rho,\lambda,\mu,\nu)$; clearly, the homography $(R;\rho,\lambda,\mu,\nu)$

is identical with the homography $(R; k\rho, k\lambda, k\mu, k\nu)$, where k is any constant different from zero.

11.64. Real Congruence Homographies

When $R = 1$, so that the invariant conic is that whose point equation is $x^2+y^2+z^2 = 0$, those and only those homographies of the congruence group for this conic are real for which all the ratios of the parameters ρ, λ, μ, and ν are real numbers. In practice, the real homographies of this group are obtained by making all the parameters real numbers.

When $R = i$, so that the invariant conic is that whose point equation is $-x^2-y^2+z^2 = 0$, those and only those homographies of the congruence group for this conic are real for which all the ratios of $i\rho$, λ, μ, and $i\nu$ are real numbers. If in the equations of Theorem 11.63 R is made equal to i, and $i\rho$, $i\nu$, substituted for ρ and ν respectively, they take the form (when multiplied throughout by $-i$)

$$k'x' = (-\rho^2+\lambda^2-\mu^2+\nu^2)x+2(\lambda\mu+\rho\nu)y+2(\nu\lambda+\rho\mu)z,$$
$$k'y' = 2(\lambda\mu-\rho\nu)x+(-\rho^2+\mu^2+\nu^2-\lambda^2)y+2(\mu\nu-\rho\lambda)z,$$
$$k'z' = -2(\nu\lambda-\rho\mu)x-2(\mu\nu+\rho\lambda)y+(-\rho^2-\nu^2-\lambda^2-\mu^2)z.$$

In this form these equations are the equations of the real homographies of the congruence group for the conic whose point equation is $-x^2-y^2+z^2 = 0$, when all the parameters ρ, λ, μ, and ν are real numbers, and $\rho^2+\nu^2-\lambda^2-\mu^2 \neq 0$.

11.7. Congruence in Metrical Geometry

The concept of congruence is familiar in elementary Geometry, where two triangles are congruent if and only if the corresponding sides and angles are equal. The concept may be taken over without any modification into the Metrical Geometry developed from Projective Geometry.

It is possible to elaborate a set of theorems which would give necessary and sufficient conditions for the congruence of two triangles, and which would be very similar to the corresponding theorems in elementary Geometry. There is, however, a simple method which is made possible by the foregoing work on homographies. This is given in the following theorem.

11.71. The Necessary and Sufficient Condition for Congruence

THEOREM. *In either Elliptic Metrical Geometry or Hyperbolic Metrical Geometry the necessary and sufficient condition that two real triangles ABC, $A'B'C'$ should be congruent is that there should be a real homography of the congruence group for the metric gauge, in which $A'B'C'$ is the transform of ABC.*

We content ourselves with proving the theorem for Elliptic Metrical Geometry; the proof for Hyperbolic Metrical Geometry is very similar, but simpler.

The sufficiency of the condition is proved first. Suppose then that $A'B'C'$ is the transform of ABC in a real homography of the congruence group for the conic whose point equation is $x^2+y^2+z^2 = 0$.

Let M_1 and M_2 be the metric gauge-points on the line BC. Let M_1' and M_2' be the transforms of M_1 and M_2 respectively in the homography. Then M_1' and M_2' are on the metric gauge, and $\text{R}(M_1 M_2 BC) = \text{R}(M_1' M_2' B'C')$. Hence the two segments $B'C'$ are equal to the two segments BC; similarly, the two segments $C'A'$ are equal to the two segments CA, and the two segments $A'B'$ are equal to the two segments AB.

Hence each of the eight triangles ABC has its sides equal to the sides of one of the triangles $A'B'C'$. If a corresponding pair be selected, the fact that their angles are equal follows at once from the second part of 9.542. Hence the two triangles are congruent.

That the condition is necessary is proved as follows. Suppose that the triangles are congruent.

Let L_1, L_2 be the metric gauge-points on BC, the order being so chosen that $(BC) = -\frac{1}{2}i\log\text{R}(L_1 L_2 BC)$.

Similarly, let M_1 and M_2, N_1 and N_2, L_1' and L_2', M_1' and M_2', N_1' and N_2' be the metric gauge-points on CA, AB, $B'C'$, $C'A'$, and $A'B'$ respectively, the order being chosen in a similar way each time.

Since no three of the points M_1, M_2, N_1, N_2 are collinear, and no three of the points M_1', M_2', N_1', N_2' are collinear, there is a unique homography in which these latter four points are the transforms of the first four, respectively. And since A is the point $\begin{pmatrix} M_1 M_2 \\ N_1 N_2 \end{pmatrix}$, and A' is the point $\begin{pmatrix} M_1' M_2' \\ N_1' N_2' \end{pmatrix}$, A' is the transform of A.

Since $(AB) = (A'B')$, so that

$$\text{R}(N_1 N_2 AB) = \text{R}(N_1' N_2' A'B'),$$

B' is the transform of B. Similarly, C' is the transform of C.

It remains to prove that this homography is one of the congruence group, that is, that L_1' and L_2' are the transforms of L_1 and L_2.

Suppose then, that L_1^* and L_2^* are the transforms of L_1 and L_2. Then on the four points M_1', M_2', N_1', N_2' are three conics, and these are respectively on (i) L_1' and L_2', (ii) L_1^* and L_2^*, and (iii) B' and C'. Hence, by Desargues's (conic) Theorem, there is an involution

$$(L_1' L_1^* B' L_2' L_2^* C') \sim (L_2' L_2^* C' L_1' L_1^* B'). \qquad (1)$$

But $\quad (L_1' L_2' B'C') \sim (L_1 L_2 BC) \sim (L_1^* L_2^* B'C'),$

so that $\quad (L_1' L_1^* B'C') \sim (L_2' L_2^* B'C'),$

and so, by (1) $\quad (L_1' L_1^* B'C') \sim (L_2' L_2^* C'B').$

Hence *either* (i) $(L_1 L_1^* B'C')$ and $(L_2 L_2^* B'C')$ are both harmonic tetrads, *or* (ii) B' and C' coincide, *or* (iii) L_1' and L_1^* coincide, and L_2' and L_2^* coincide.

The first of these is impossible, for it entails that the two distinct involutions (1) and $(B'C'L_1' L_2' L_1^* L_2^*) \sim (B'C'L_2' L_1' L_2^* L_1^*)$ have two pairs of mates in common. The second is absurd, and therefore the third is true. Hence the homography is one of the congruence group. That it is a real homography is left to the reader to prove.

11.8. Congruence in Euclidean Metrical Geometry

Corresponding to the congruence group of homographies for Elliptic and Hyperbolic Metrical Geometries there is a congruence group of homographies for Euclidean Geometry. The equations for these cannot be satisfactorily deduced by a limiting process (making R tend to zero) from the equations found in 11.63.

For a homography to be one of the congruence group for

Euclidean Metrical Geometry it is clearly necessary that the line [0, 0, 1] shall be a self-corresponding line, and, in addition, *either* that the points I and J shall be self-corresponding points, *or* that they shall be transforms of each other. These conditions are necessary, but they are not sufficient. In addition it is necessary that the distance between one pair of points shall be equal to the distance between their transforms.

It is left to the reader to deduce the equations and to prove the congruence theorem corresponding to 11.71.

It will be found that the equations take the form

$$k'x' = x\cos\theta - y\sin\theta + az,$$
$$k'y' = x\sin\theta + y\cos\theta + bz,$$
$$k'z' = z,$$

when I and J are both self-corresponding points, and

$$k'x' = x\cos\theta + y\sin\theta + az,$$
$$k'y' = x\sin\theta - y\cos\theta + bz,$$
$$k'z' = z,$$

when I and J are transforms of each other.

EXAMPLES

1. The plain coordinates of the points X', Y', Z', I' are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and (a, b, c) respectively, where $abc \neq 0$. Deduce the equations of transformation from one mesh gauge to the other.

2. If they are $(1, 1, 1)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 0, 0)$ respectively, what are the equations of transformation?

3. Show that in the last example the plain coordinates of all points on the line YZ are identical with the prime coordinates.

4. Determine the coordinates of the self-corresponding points and lines in the homography whose equations are $k'x' = y-z$, $k'y' = z-x$, $k'z' = x-y$.

5. Show that if the conic whose point equation is $-x^2-y^2+z^2 = 0$ is invariant in a real homography, the transforms of interior points† are interior points, and the transforms of exterior points are exterior points.

6. A non-singular conic Φ is invariant in a certain homography, and two distinct points A and B on it are self-corresponding. Show that (i) the common point of the tangents at A and B is also a self-corresponding point, and (ii) every conic having double contact with Φ at A and B is also invariant in the homography.

† An interior point is here to be defined as a point such that the two tangents to the conic which are on it are complex lines.

CHAPTER XII

FURTHER DEVELOPMENTS

THE investigation of the elementary theory of homographies carried out in the last chapter brings to a close that part of the subject of Projective Geometry with which this book deals. The aim has been to give the reader a wider viewpoint of Geometry, and to acquaint him with the methods used in the subject. This strictly limited aim makes it a useful thing to add a closing chapter in which it is pointed out how the subject can be extended once the preliminary work is done. These possible developments are very many, and so this account cannot pretend to give more than an outline of some of the more important ones.

12.1. Projective Geometry of Many Dimensions

In this book we have confined ourselves to studying the Projective Geometry of the two-dimensional field, that is to say, a field of points and lines. By far the most important development of the subject is its extension to a field of many dimensions. The starting-point of many-dimensional Projective Geometry was outlined in 2.7, and it need not be repeated here. It will be sufficient to say that instead of confining attention to but two types of fundamental element, the point and the line, many-dimensional Projective Geometry deals with many types of fundamental element. These are inter-related in the first place by initial propositions of incidence which are generalizations of those adopted here, and are, naturally, more complicated than those of the two-dimensional field.

In addition to these initial propositions the Projective Proposition (3.313), the Harmonic Proposition (4.151), and, clearly, some suitable initial proposition about extension are needed as initial propositions. It is unnecessary to take Desargues's proposition as an initial proposition, since this is a consequence of the initial propositions of incidence in the Projective Geometry of many dimensions.

There is no particular reason for confining oneself to any particular number of dimensions, and possibly the easiest method of attacking this extension is to proceed from two-dimensional to n-dimensional Projective Geometry at once, n being an unspecified positive integer.

Many-dimensional Projective Geometry throws considerable light on a number of questions which are, strictly, two-dimensional by nature. Thus, the deeper theory which underlies Pascal's theorem (5.43) can only be fully grasped when it is approached from the starting-point of many-dimensional Projective Geometry. The theory of homographies is also greatly simplified by this method of approach; its synthetic treatment in two dimensions is laborious, but in many-dimensional Projective Geometry the synthetic treatment is the natural and obvious one.

12.2. Finite Geometries

In Chapter VII the question of extension was closed, once for all, by taking as an initial proposition the isomorphism of the open set of points on a line with the complex number-system. It will now be seen that though other initial propositions of extension might have been adopted, some of them would have made the work unnecessarily laborious. Such a one would have been the proposition that the open set of points on a line was isomorphous with the system of real numbers, or with the system of rational numbers. It was far easier to take the proposition we did take, and then when necessary confine our attention to the real points or the rational points. The reason of this is that the system of real numbers is itself isomorphous with a part of the system of complex numbers; and, similarly, the system of rational numbers is isomorphous with a part of the system of complex numbers.

But there are number-systems which are not isomorphous with a part of the complex number-system; and there are Projective Geometries which correspond. One such system was encountered in the representation given in 2.23; there, there were not more than three points on any line. Systems such as this were definitely excluded by our initial proposition of extension, but they are, for all that, a part of Projective Geometry. Their analytical treatment involves the theory of numbers, and, in particular, the theory of numerical congruences; it may be assumed that the synthetic treatment of them is correspondingly complicated.

12.3. General Loci and Envelopes

In the preceding chapters the only loci considered were the range of points on a line and the point-conic; dually, the only envelopes considered were the duals of these. It is quite clear, however, that these are not the only types of locus and envelope which Projective Geometry is capable of handling. The fact that there is a mesh gauge at our disposal may suggest that a fruitful development of the preceding work would be the investigation of more complicated types of locus and envelope by the algebraic method. For instance, it would be possible to investigate algebraically the properties of loci and envelopes with cubic, quartic, quintic,... equations. This is certainly a possible method of studying these more complicated loci and envelopes, but it is not the best method. Many-dimensional Projective Geometry is the proper and most natural starting-point for the study of these more complicated loci and envelopes; the algebraic method may be used in conjunction with this, but alone it is not very fruitful. This is another example of the fact that many-dimensional Projective Geometry can throw light on a strictly two-dimensional question.

12.4. Generalized Metrical Geometries

In Chapter IX the notions of distance and angle were defined projectively, and from them Metrical Geometry was built up. By taking certain simple conics as metric gauges, and by confining our attention to the real points of the field, a number of simple Metrical Geometries were developed. Contrary to expectation, there is not much to be gained from the study of the general Metrical Geometry, in which a general (complex) conic is taken as metric gauge, and the metrical relations of the whole field are considered. But there is a most important generalization of Metrical Geometry which is worth outlining here.

In the Metrical Geometries we considered one conic was taken as the metric gauge for all the points and lines of the field; in the generalization, not one, but a whole system of conics is taken. In fact, with every point of the field is associated a metric gauge-conic which is used for the measurement of *small* distances from the point in question.

By way of illustrating how this is done, and of giving a precise interpretation of the vague word *small*, the following example is taken.

With every point of the field (x_1, y_1, z_1) is associated (in this example) the metric gauge-conic whose point equation is $x_1 x^2 + y_1 y^2 + z_1 z^2 = 0$. Clearly, a point will only be on its own metric gauge if $x_1^3 + y_1^3 + z_1^3 = 0$, so that points whose coordinates satisfy this equation correspond to the isotropic points of Projective Metrical Geometry. With these points we do not deal, and so, in order to fix definitely the specification of the coordinates of the points with which we do deal, we stipulate that the coordinates of these points be so chosen that $x_1^3 + y_1^3 + z_1^3 = 1$.

Consider now two points whose coordinates are (x_1, y_1, z_1) and $(x_1 + \delta x, y_1 + \delta y, z_1 + \delta z)$; if the distance between these two points be written δs, then, by 9.32, if the scale-constant is $-\frac{1}{2}i$,

$$\cos^2(\delta s) = \frac{[x_1^2(x_1+\delta x) + y_1^2(y_1+\delta y) + z_1^2(z_1+\delta z)]^2}{x_1(x_1+\delta x)^2 + y_1(y_1+\delta y)^2 + z_1(z_1+\delta z)^2}.$$

If δx, δy, δz, and therefore δs be small, this equation after simplification may be written

$$(\delta s)^2 = (3x_1^4 + x_1)(\delta x)^2 + (3y_1^4 + y_1)(\delta y)^2 + (3z_1^4 + z_1)(\delta z)^2 + \\ + 2(y_1 z_1)^2 \delta y \delta z + 2(z_1 x_1)^2 \delta z \delta x + 2(x_1 y_1)^2 \delta x \delta y,$$

to the second order of small numbers. In the language of differentials,

$$(ds)^2 = (3x_1^4 + x_1)(dx)^2 + (3y_1^4 + y_1)(dy)^2 + (3z_1^4 + z_1)(dz)^2 + \\ + 2y_1^2 z_1^2\, dydz + 2z_1^2 x_1^2\, dzdx + 2x_1^2 y_1^2\, dxdy,$$

where $$x_1^2\, dx + y_1^2\, dy + z_1^2\, dz = 0.$$

In this generalized Metrical Geometry, therefore, we obtain in the first instance, not an expression for the distance between two points, but an expression for the differential of distance at any point. The *length* between two points of a given locus will then be $\int ds$, where the integral is taken along the locus; the *distance* between two points will be the minimum length between these two points for all the possible loci. These minimum-length loci are called *geodesics*; in the Metrical Geometries considered in Chapters IX and X the lines of the field are the geodesics, but this is not necessarily so in these more general Metrical Geometries.

This generalization of metrical ideas is most important, since it is the true foundation of what is called *Differential Geometry*; in fact, Differential Geometry is only the study of the metrical relations of a perfectly normal projective field, or a part of it, upon which has been imposed a rather complicated metric.

Differential Geometry must of its nature be treated analytically, and the analytical weapon most suited for the purpose is the Tensor Calculus.

12.5. Applications to Physics

The science of Geometry was, in the first instance, the science of the measurement of the physical space in which we live; it therefore was, strictly speaking, a physical science. To-day it is no longer a physical science; nevertheless, it has evolved to its present state of development from its original state by successive generalization and abstraction. There is therefore at least a genealogical connexion between Geometry and the science of the measurement of physical space. It is only natural to ask whether there is any closer connexion; whether, in fact, Projective Geometry is a 'pure' mathematical science, corresponding to the 'applied' science of Physical Geometry.

Physical Geometry is of its nature a metrical Geometry, since it is concerned with measurement of space. The first ordering of the results of these measurements led men to formulate the propositions of Euclidean Metrical Geometry, and this Euclidean Metrical Geometry continued to be applied to the measurements of space, since it seemed to fit the facts of observation and its predictions were uniformly verified.

But as the observed facts multiplied, and measurement became more precise, it became clear that sometimes, at any rate, the predictions of Euclidean Metrical Geometry were not verified. The source of this discrepancy was at first attributed to faulty observation, and later to some hitherto unknown physical law, but with the advance in the development of Geometry it became at least a tenable hypothesis that the cause of the discrepancy did not lie in either of these, but in the Metrical Geometry that was being used to describe the universe. Men had become so accustomed to the classical Euclidean Metrical Geometry that they could not imagine any other being verified in nature.

The preceding chapters have shown that there are various Metrical Geometries, each with precise definitions of distance and angle, and none of them with any intrinsic claim to be considered more important than the rest; it is therefore at least possible that one of them may fit the facts of nature better than Euclidean Metrical Geometry. Until Geometry had developed to the point of realizing the possibility of different Metrical Geometries, any variant on the classical Euclidean scheme was unthinkable; when the possibility had been realized, the question arose: Which of the possible Metrical Geometries best describes the universe?

12.6. The Special Theory of Relativity

How this question is answered can be indicated by a simple concrete example.

The physicist is concerned not only with the measurement of space but with that of time. He found it convenient to represent his simultaneous space- and time-measurements by a four-dimensional field of points. There is no need to attempt the imagination of a four-dimensional field; it is sufficient to confine ourselves to a field of two dimensions. This two-dimensional field the physicist used to 'map' the events of the universe; he took one axis to represent one dimension of space, and one to represent time. He thus set up on his map a non-homogeneous mesh gauge. On this map a line parallel to the time-axis represents a stationary point, and a line at an angle to the time-axis represents a moving point—moving, that is, relative to the 'observer', the man who makes the map.

Consider now a second observer, moving relatively to the first; in his map of the universe his axes of space and time will be at an angle to those of the first observer. The first observer will describe an event occurring at a certain place and time by coordinates (x, t) relative to his mesh gauge; the second will describe the same event by coordinates (x', t') relative to his mesh gauge. How are the two descriptions connected?

It was always assumed that the Metrical Geometry of this map was the classical Euclidean (i.e. Complex Euclidean), and therefore that the equations of transformation from one mesh gauge to the other were of the form

$$x' = x\cos\theta - t\sin\theta + a,$$
$$t' = x\sin\theta + t\cos\theta + b,$$

where $\tan\theta = v$, the velocity of the second observer relative to the first.

The classical Euclidean Metrical Geometry therefore predicts that if the first observer sees a body moving with a velocity u, the second will see it moving with a velocity $u-v$. This prediction seemed to be verified in fact, within the limits of observational error, so long as small velocities were being observed. But when velocities of the order of magnitude of that of light were observed, the equations were found to be inaccurate. It was then found that the equations which fitted the facts best were not those given above, but

$$x' = x\cosh\theta - t\sinh\theta + a,$$
$$t' = -x\sinh\theta + t\cosh\theta + b,$$

where $\tanh\theta = v$, the velocity of the second observer relative to the first.

This discrepancy between prediction and observed fact was at first interpreted by physicists as being due to the contraction of 'rigid' bodies when in motion, and was called the 'Fitzgerald-Lorentz Contraction'. But there is a far simpler interpretation than this.

The first set of equations of transformation are those which leave invariant the special points of Complex Euclidean Metrical Geometry, the second are those that leave invariant the special points of Real Euclidean Metrical Geometry. The inference is that the Metrical Geometry of the map is not the classical (Complex) Euclidean Geometry, but the Real Euclidean Metrical Geometry discussed in Chapter X, and that this is the Metrical Geometry best fitted to describe the physical universe of space and time.

The fact which upset the traditional theories was the invariance of the velocity of light. The Michelson-Morley and other experiments showed that however an observer was moving his estimate of the velocity of light was always the same. On the classical theory, if one observer found that the velocity of a ray of light was c, an observer moving with a velocity v relative to the first should observe the velocity of the same ray of light as $c-v$. But in fact he observes it to be c. If this physical fact be translated into terms of the map made above, it reads that a certain set of lines (those parallel to a line representing the velocity of light) make the same angle with all other lines of the field. This should have warned mathematicians that these exceptional lines were the isotropic lines of the field; and since they are real lines, that the Metrical Geometry of the map was not Complex Euclidean Metrical Geometry as had been supposed, but Real Euclidean Metrical Geometry.

The theory which has been outlined here was called, when it was discovered, the Special Theory of Relativity; in its turn it had to give place to a more general theory still, as the facts of observation were multiplied. Physicists have now realized that the Metrical Geometry which best describes the universe is not even Real Euclidean Metrical Geometry, but one of the more general Metrical Geometries outlined earlier in this chapter.

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PRINTED IN
GREAT BRITAIN
AT THE
UNIVERSITY PRESS
OXFORD
BY
JOHN JOHNSON
PRINTER
TO THE
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