A TREATISE

ON THE

ANALYTIC GEOMETRY

OF

THREE DIMENSIONS.

BY

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PREFACE.

In writing a preface, what I am most tempted to do is to enumerate and account for the omissions of this treatise; if it were not that the size to which the volume has swelled, renders it needless for me to apologize for not having made it larger. It may be right however to mention that the chapters of this work were written and sent to press at intervals as I found leisure, and that the earlier part of the book has been in type more than a year. This will explain why no use has been made of some recent works and memoirs. In particular, I must express my regret that Hesse's "Lectures on the Analytic Geometry of Space" came too late to be of service to me.

In treating of the less modern parts of the Science, I have usually had Leroy's and Gregory's Treatises before me. The parts of this work which correspond to the contents of theirs are, the Theory of Surfaces of the Second Order, pp. 1—88; of the Curvature of Surfaces, pp. 197—223; of what I have called the Non-Projective Properties of Curves of Double Curvature, pp. 259—277 and of Families of Surfaces, pp. 312—338. Junior readers will probably find all the information they require, if to the course here marked out they add part of the Theory of Confocal Surfaces, pp. 129—138, and the General Theory of Surfaces, Chap. x.

I have to acknowledge with thanks the kind readiness with which assistance was afforded me by any of my friends whose help I claimed. Those to whom I am most indebted are Dr. Hart and the Messrs. Roberts; but I have received occasional assistance from Messrs. Townsend, Williamson, and Gray, to the latter of whom I owe the list of Errata which follows the Table of Contents.

I have to thank the Board of Trinity College, for their liberality in contributing to the expense of publication.

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ERRATA.

Page	Line	
3,	5,	*for* C, D, E, *read* A, B, C.
47,	4,	*for* drawn, *read* be drawn.
80,	5,	*for* plane of, *read* "plane, if.
91,	4,	*for* eighth fixed point, *read* eight fixed points.
112,		last line, third group, *for* c^2b^2, *read* a^2b^2.
143,	15,	the theorem here ascribed to Jacobi, had been previously published by Chasles, *Liouville*, XI. 121.
189,	11,	*for* $\lambda^{n-1}\mu^2$, *read* $\lambda^{n-2}\mu^2$.
192,	10,	*for* 57, *read* 51.
196,	9,	*for* $3n(n-1)(n-2)$, *read* $n(n-1)(n-2)$.
199,	15,	*for* $\frac{1}{2}R$, *read* $\frac{1}{2R}$.
201,	8,	*after* Dxz, *insert* $\cos\phi$, *after* Fz^2, *insert* $\cos^2\phi$.
202,		The determinant at foot of page ought to be bordered horizontally and vertically with L, M, N.
206,	22,	*for* $a - l$, *read* $c - l$.
210,	4,	*for* $\frac{1}{2}A$, $\frac{1}{2}C$, *read* $\frac{1}{2A}$, $\frac{1}{2C}$.
210,	5 of note,	after $2Cy$, *insert* 1.
215,	13,	*for* 260, *read* 270.
222,	9,	*for* 278, *read* 280.
231,	9,	*for* if, *read* when.
236,	10,	and 243, line 7 from bottom, *for* double points, *read* double edges.
250,	6,	In the values for $2y$ and $2x$, a factor $\mu\nu$ is omitted. $2y = \mu\nu\{\mu\nu\,(3\mu + \&c.,\quad 2x = \mu\nu\{\mu\nu\,(\mu + \&c.$
253,		last line, *for* "principal," *read* "common conjugate."
254,	2,	U and V mean tangent planes to U and V at the point.
254,	17,	*for* "meets the quadrics," *read* "meets the quartic."
255,		last line but two, *for* "distant," *read* "distinct."
259,	14,	*for* $4m$, *read* $2m$.
263,	15,	transpose $y'x$ and $x'y$.
267,		Note, *for* CD, *read* BD.

ANALYTIC GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

THE POINT.

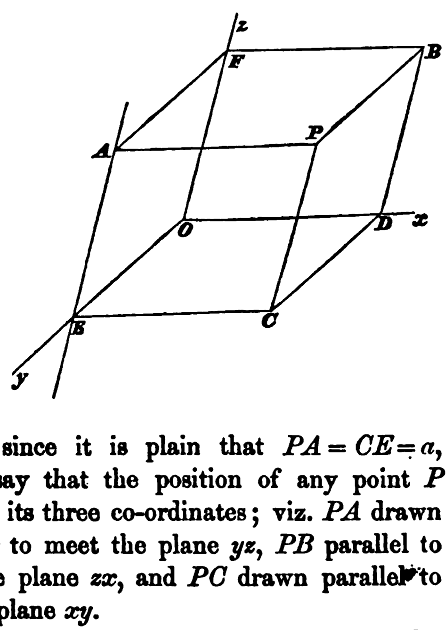
1. We have seen already how the position of a point C in a plane is determined, by referring it to two co-ordinate axes OX, OY drawn in the plane. To determine the position of any point P in space, we have only to add to our apparatus a third axis OZ not in the plane (see figure next page). Then if we knew the distance of the point P from the plane XOY, measured parallel to the line OZ, and also knew the x and y co-ordinates of the point C, where PC parallel to OZ meets the plane, it is obvious that the position of P would be completely determined.

Thus, if we were given the three equations $x = a$, $y = b$, $z = c$, the first two equations would determine the point C, and then drawing through that point a parallel to OZ, and taking on it a length $PC = c$, we should have the point P.

We have seen already how a change in the sign of a or b affects the position of the point C. The sign of c will determine on which side of the plane XOY the line PC is to be measured. If it be settled that lines on one side of the plane are to be considered as positive, then those in the other direction must be considered as negative. Thus, if we conceive the plane XOY to be horizontal, it is customary to consider as positive the z of every point *above* that plane, in which case the z of every point *below* it must be counted as negative. It is obvious that every point *on* the plane has its $z = 0$.

The angles between the axes may be any whatever; but the axes are said to be rectangular when the lines OX, OY are at right angles to each other, and the line OZ perpendicular to the plane XOY.

2. We have stated the method of representing a point in space, in the manner which seemed most simple for readers already acquainted with Plane Analytic Geometry. We proceed now to state the same more symmetrically. Our apparatus evidently consists of *three* co-ordinate axes OX, OY, OZ meeting in a point O, which, as in Plane Geometry, is called the origin. The three axes are called the axes of x, y, z respectively. These three *axes* determine also three co-ordinate *planes*, namely, the planes XOY, YOZ, ZOX, which we shall call the planes xy, yz, zx respectively. Now since it is plain that $PA = CE = a$, $PB = CD = b$, we may say that the position of any point P is known if we are given its three co-ordinates; viz. PA drawn parallel to the axis of x to meet the plane yz, PB parallel to the axis of y to meet the plane zx, and PC drawn parallel to the axis of z to meet the plane xy.



Again, since $OD = a$, $OE = b$, $OF = c$, the point given by the equations $x = a$, $y = b$, $z = c$ may be found by the following symmetrical construction: measure on the axis of x, the length $OD = a$, and through D draw the plane $PBCD$ parallel to the plane yz: measure on the axis of y, $OE = b$, and through

E draw the plane $PACE$ parallel to zx: measure on the axis of z, $OF = c$, and through F draw the plane $PABF$ parallel to xy: the intersection of the three planes so drawn is the point P, whose construction is required.

3. The points A, B, C, are called the *projections* of the point P on the three co-ordinate planes; and when the axes are rectangular they are its *orthogonal* projections. In what follows we shall be almost exclusively concerned with orthogonal projections, and therefore when we speak simply of projections, are to be understood to mean orthogonal projections, unless the contrary is stated. There are some properties of orthogonal projections which we shall often have occasion to employ, and which we therefore collect here, though we have given the proof of some of them already. (See *Conics*, p. 315.)

The length of the orthogonal projection of a finite right line on any plane is equal to the line multiplied by the cosine of the angle which it makes with the plane.*

* The angle a line makes with a plane is measured by the angle which the line makes with its orthogonal projection on that plane.

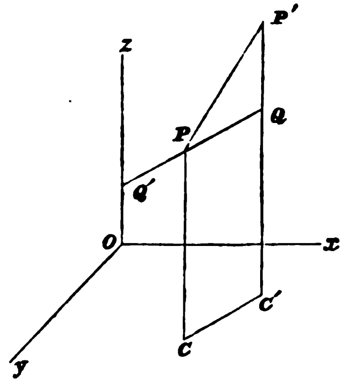
The angle between two planes is measured by the angle between the perpendiculars drawn in each plane to their line of intersection at any point of it. It may also be measured by the angle between the perpendiculars let fall on the planes from any point.

The angle between two lines which do not intersect, is measured by the angle between parallels to both drawn through any point.

When we speak of the angle between two lines, it is desirable to express without ambiguity whether we mean the acute or the obtuse angle which they make with each other. When therefore we speak of the angle between two lines (for instance PP', CC' in the figure, next page), we shall understand that these lines are measured in the *direction* from P to P' and from C to C', and that the parallel PQ is measured in the same direction. The angle then between these lines is acute. But if we spoke of the angle between PP' and $C'C$, we should draw the parallel PQ' in the opposite direction, and should wish to express the obtuse angle made by the lines with each other.

When we speak of the angle made by any line OP with the axes, we shall always mean the angle between OP and the *positive* direction of the axes, viz. OX, OY, OZ.

Let PC, $P'C'$ be drawn perpendicular to the plane XOY; and CC' is the orthogonal projection of the line PP' on that plane. Complete the rectangle by drawing PQ parallel to CC', and PQ will also be equal to CC'. But $PQ = PP' \cos P'PQ$.



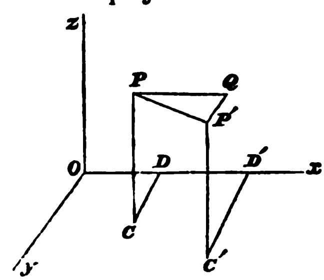
4. *The projection on any plane of any area in another plane is equal to the original area multiplied by the cosine of the angle between the planes.* (See *Conics*, p. 315.)

For if ordinates of both figures be drawn perpendicular to the intersection of the two planes, then, by the last article, every ordinate of the projection is equal to the corresponding ordinate of the original figure multiplied by the cosine of the angle between the planes. But it was proved (*Conics*, p. 293,) that when two figures are such that the ordinates corresponding to equal abscissæ have to each other a constant ratio, then the areas of the figures have to each other the same ratio.

5. The projection of a point on any *line*, is the point where the line is met by a plane drawn through the point perpendicular to the line. Thus, in figure, p. 2, if the axes be rectangular, D, E, F are the projections of the point P on the three axes.

The projection of a finite right line upon another right line is equal to the first line multiplied by the cosine of the angle between the lines.

Let PP' be the given line, and DD' its projection on OX. Through P draw PQ parallel to OX to meet the plane $P'C'D'$; and since it is perpendicular to this plane, the angle PQP' is right, and $PQ = PP' \cos P'PQ$. But PQ and DD' are equal, since they are the intercepts made by two parallel planes on two parallel right lines.



6. *If there be any three points P, P', P'', the projection of PP'' on any line will be equal to the sum of the projections on that line of PP' and $P'P''$.*

Let the projections of the three points be D, D', D'', then if D' lie between D and D'', DD'' is evidently the sum of DD' and $D'D''$. If D'' lie between D and D', DD'' is the *difference* of DD' and $D'D''$; but since the direction from D' to D'' is the opposite of that from D to D', DD'' is still the algebraic sum of DD' and $D'D''$. It may be otherwise seen that the projection of $P'P''$ is in the latter case to be taken with a negative sign from the consideration that in this case the length of the projection is found by multiplying $P'P''$ by the cosine of an *obtuse* angle (see note, p. 3). In general, if there be any number of points P, P', P'', P''', &c., the projection of PP''' on any line is equal to the sum of the projections of PP', $P'P''$, $P''P'''$.

7. We shall have constant occasion to make use of the following particular case of the preceding.

If the co-ordinates of any point P be projected on any line, the sum of the three projections is equal to the projection of the radius vector on that line.

For consider the points O, D, C, P (see figure, p. 2) and the projection of OP must be equal to the sum of the projections of $OD\ (=x)$, $DC\ (=y)$, and $CP\ (=z)$.

8. Having established those principles concerning projections which we shall constantly have occasion to employ, we return now to the more immediate subject of this chapter.

The co-ordinates of the point dividing in the ratio $m : n$ the distance between two points $x'y'z'$, $x''y''z''$, are

$$x = \frac{mx'' + nx'}{m+n},\quad y = \frac{my'' + ny'}{m+n},\quad z = \frac{mz'' + nz'}{m+n}.$$

The proof is precisely the same as that given at *Conics*, p. 5, for the corresponding theorem in Plane Analytic Geometry. The lines PM, QN in the figure there given, now represent the ordinates drawn from the two points to any one of the co-ordinate planes.

If we consider the ratio $m : n$ as indeterminate, we have the co-ordinates of *any* point in the line joining the two given points.

9. *Any side of a triangle is cut in the ratio* $m : n$, *and the line joining this point to the opposite vertex is cut in the ratio* $m + n : l$, *to find the co-ordinates of the point of section.*

Ans.

$$x = \frac{lx' + mx'' + nx'''}{l + m + n}, \quad y = \frac{ly' + my'' + ny'''}{l + m + n}, \quad z = \frac{lz' + mz'' + nz'''}{l + m + n}.$$

This is proved as in Plane Analytic Geometry (see *Conics*, p. 6). If we consider l, m, n as indeterminate, we have the co-ordinates of *any* point in the plane determined by the three points.

Ex. The lines joining middle points of opposite edges of a tetrahedron meet in a point. The x's of two such middle points are $\frac{x' + x''}{2}$, $\frac{x''' + x''''}{2}$, and the x of the middle point of the line joining them is $\frac{x' + x'' + x''' + x''''}{4}$. The other co-ordinates are found in like manner, and their symmetry shows that this is also a point on the line joining the other middle points. Through this same point will pass the line joining each vertex to the centre of gravity of the opposite triangle. For the x of one of these centres of gravity is $\frac{x' + x'' + x'''}{3}$, and if the line joining this to the opposite vertex be cut in the ratio of $3 : 1$, we get the same value as before.

10. *To find the distance between two points* P, P', *whose rectangular co-ordinates are* $x'y'z'$, $x''y''z''$.

Evidently (see figure, p. 4) $PP'^2 = P'Q^2 + PQ^2$. But $P'Q = z' - z''$, and $PQ^2 = CC'^2$ is by Plane Analytic Geometry $= (x' - x'')^2 + (y' - y'')^2$. Hence

$$PP'^2 = (x' - x'')^2 + (y' - y'')^2 + (z' - z'')^2.$$

Cor. The distance of any point $x'y'z'$ from the origin is given by the equation

$$OP^2 = x'^2 + y'^2 + z'^2.$$

11. The position of a point is sometimes expressed by its radius vector and the angles it makes with three rectangular axes. Let these angles be α, β, γ. Then since the co-ordinates x, y, z are the projections of the radius vector on the three axes, we have

$$x = \rho \cos\alpha, \quad y = \rho \cos\beta, \quad z = \rho \cos\gamma.$$

And, since $x^2 + y^2 + z^2 = \rho^2$, the three cosines (which are sometimes called the direction-cosines of the radius vector) are connected by the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.*$$

The position of a point is also sometimes expressed by the following polar co-ordinates—the radius vector, the angle γ which the radius vector makes with a fixed axis OZ, and the angle COD $(=\phi)$ which OC the projection of the radius vector on a plane perpendicular to OZ (see figure, p. 4) makes with a fixed line OX in that plane. Since then $OC = \rho \sin\gamma$, the formulæ for transforming from rectangular to these polar co-ordinates are

$$x = \rho \sin\gamma \cos\phi, \quad y = \rho \sin\gamma \sin\phi, \quad z = \rho \cos\gamma.$$

12. *The square of the area of any plane figure is equal to the sum of the squares of its projections on three rectangular planes.*

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* I have followed the usual practice in denoting the position of a line by these angles, but in one point of view there would be an advantage in using instead the complementary angles, namely, the angles which the line makes with the co-ordinate planes. This appears from the corresponding formulæ for oblique axes which I have not thought it worth while to give in the text, as we shall not have occasion to use them afterwards. Let α, β, γ be the angles which a line makes with the planes yz, zx, xy, and let A, B, C be the angles which the axis of x makes with the plane of yz, of y with the plane of zx, and of z with the plane of xy, then the formulæ which correspond to those in the text, are

$$x \sin A = \rho \sin\alpha, \quad y \sin B = \rho \sin\beta, \quad z \sin C = \rho \sin\gamma.$$

These formulæ are proved by the principle of Art. 7. If we project on a line perpendicular to the plane of yz, since the projections of y and of z on this line vanish, the projection of x must be equal to that of the radius vector, and the angles made by x and ρ with this line are the complements of A and α.

Let the area be A, and let a perpendicular to its plane make angles α, β, γ with the three axes; then (Art. 4) the projections of this area on the planes yz, zx, xy respectively, are $A\cos\alpha$, $A\cos\beta$, $A\cos\gamma$. And the sum of the squares of these three $= A^2$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

13. *To express the cosine of the angle θ between two lines OP, OP' in terms of the direction-cosines of these lines.*

We have proved (Art 10),

$$PP'^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

But also
$$PP'^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos\theta.$$

And since
$$\rho^2 = x^2 + y^2 + z^2,\quad \rho'^2 = x'^2 + y'^2 + z'^2,$$

we have
$$\rho\rho'\cos\theta = xx' + yy' + zz',$$

or
$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

Cor. The condition that two lines should be at right angles to each other is

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$$

14. The following formula is also sometimes useful:

$$\sin^2\theta = (\cos\beta\cos\gamma' - \cos\gamma\cos\beta')^2 + (\cos\gamma\cos\alpha' - \cos\alpha\cos\gamma')^2 + (\cos\alpha\cos\beta' - \cos\beta\cos\alpha')^2.$$

This may be derived from the following elementary theorem for the sum of the squares of three determinants (*Lessons on Higher Algebra*, Art. 21), but which can also be verified at once by actual expansion,

$$(bc'-cb')^2 + (ca'-ac')^2 + (ab'-ba')^2 = (a^2+b^2+c^2)(a'^2+b'^2+c'^2) - (aa'+bb'+cc')^2.$$

For when a, b, c; a', b', c' are the direction-cosines of two lines, the right-hand side becomes $1 - \cos^2\theta$.

Ex. To find the perpendicular distance from a point $x'y'z'$ to a line through the origin whose direction-angles are α, β, γ.

Let P be the point $x'y'z'$, OQ the given line, PQ the perpendicular, then it is plain that $PQ = OP\sin POQ$; and using the value just obtained for $\sin POQ$, and remembering that $x' = OP\cos\alpha'$, &c., we have

$$PQ^2 = (y'\cos\gamma - z'\cos\beta)^2 + (z'\cos\alpha - x'\cos\gamma)^2 + (x'\cos\beta - y'\cos\alpha)^2.$$

15. *To find the direction-cosines of a line perpendicular to two given lines, and therefore perpendicular to their plane.*

Let $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ be the direction-angles of the given line, and $\alpha\beta\gamma$ of the required line, then we have to find $\alpha\beta\gamma$ from the three equations

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0,$$
$$\cos\alpha\cos\alpha'' + \cos\beta\cos\beta'' + \cos\gamma\cos\gamma'' = 0,$$
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

From the first two equations we can easily derive, by eliminating in turn $\cos\alpha$, $\cos\beta$, $\cos\gamma$,

$$\lambda\cos\alpha = \cos\beta'\cos\gamma'' - \cos\beta''\cos\gamma',$$
$$\lambda\cos\beta = \cos\gamma'\cos\alpha'' - \cos\gamma''\cos\alpha',$$
$$\lambda\cos\gamma = \cos\alpha'\cos\beta'' - \cos\alpha''\cos\beta',$$

where λ is indeterminate; and substituting in the third equation, we get (see Art. 14)

$$\lambda^2 = \sin^2\theta.$$

This result may be also obtained as follows: take any two points P, Q, or $x'y'z'$, $x''y''z''$, one on each of the two given lines. Now double the area of the projection on the plane of xy of the triangle POQ, is (see *Conics*, p. 25) $x'y'' - y'x''$, or $\rho'\rho''(\cos\alpha'\cos\beta'' - \cos\alpha''\cos\beta')$. But double the area of the triangle is $\rho'\rho''\sin\theta$, and therefore the projection on the plane of xy is $\rho'\rho''\sin\theta\cos\gamma$. Hence, as before,

$$\sin\theta\cos\gamma = \cos\alpha'\cos\beta'' - \cos\alpha''\cos\beta',$$

and in like manner

$$\sin\theta\cos\alpha = \cos\beta'\cos\gamma'' - \cos\beta''\cos\gamma';$$
$$\sin\theta\cos\beta = \cos\gamma'\cos\alpha'' - \cos\gamma''\cos\alpha'.$$

TRANSFORMATION OF CO-ORDINATES.

16. *To transform to parallel axes through a new origin, whose co-ordinates referred to the old axes are x', y', z'.*

The formulæ of transformation are (as in Plane Geometry)

$$x = X + x', \quad y = Y + y', \quad z = Z + z'.$$

For let a line drawn through the point P parallel to one of the axes (for instance z) meet the old plane of xy in a point C, and the new in a point C'; then $PC = PC' + C'C$.

But PC is the old z, PC' is the new z; and since parallel planes make equal intercepts on parallel right lines, CC' must be equal to the line drawn through the new origin O' parallel to the axis of z, to meet the old plane of xy.

17. *To pass from a rectangular system of axes to another system of axes having the same origin.*

Let the angles made by the new axes of x, y, z with the old axes be α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$ respectively. Then if we project the new co-ordinates on one of the old axes, the sum of the three projections will (Art. 7) be equal to the projection of the radius vector, which is the corresponding old co-ordinate. Thus we get the three equations

$$\left.\begin{aligned} x &= X\cos\alpha + Y\cos\alpha' + Z\cos\alpha'' \\ y &= X\cos\beta + Y\cos\beta' + Z\cos\beta'' \\ z &= X\cos\gamma + Y\cos\gamma' + Z\cos\gamma'' \end{aligned}\right\} \ldots\ldots\ldots (A).$$

We have, of course, (Art. 11)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1, \quad \cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' = 1,$$
$$\cos^2\alpha'' + \cos^2\beta'' + \cos^2\gamma'' = 1 \ldots\ldots\ldots\ldots\ldots (B).$$

If the new axes be also rectangular, we have also (Art. 13)

$$\left.\begin{aligned} \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' &= 0 \\ \cos\alpha'\cos\alpha'' + \cos\beta'\cos\beta'' + \cos\gamma'\cos\gamma'' &= 0 \\ \cos\alpha''\cos\alpha + \cos\beta''\cos\beta + \cos\gamma''\cos\gamma &= 0 \end{aligned}\right\} \ldots (C).$$

By the help of these relations we can verify that when we pass from one system of rectangular axes to another, we have, as is geometrically evident, $x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2$.

When the new axes are rectangular, since α, α', α'' are the angles made by the old axis of x with the new axes, we must have

$$\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1, \quad \cos^2\beta + \cos^2\beta' + \cos^2\beta'' = 1,$$
$$\cos^2\gamma + \cos^2\gamma' + \cos^2\gamma'' = 1 \ldots\ldots\ldots\ldots\ldots (D),$$

$$\left.\begin{aligned} \cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha &= 0 \\ \cos\alpha'\cos\beta' + \cos\beta'\cos\gamma' + \cos\gamma'\cos\alpha' &= 0 \\ \cos\alpha''\cos\beta'' + \cos\beta''\cos\gamma'' + \cos\gamma''\cos\alpha'' &= 0 \end{aligned}\right\} \ldots (E),$$

and the new co-ordinates expressed in terms of the old are

$$\left.\begin{aligned} X &= x\cos\alpha + y\cos\beta + z\cos\gamma \\ Y &= x\cos\alpha' + y\cos\beta' + z\cos\gamma' \\ Z &= x\cos\alpha'' + y\cos\beta'' + z\cos\gamma'' \end{aligned}\right\} \ldots\ldots\ldots (F).$$

It would not be difficult to derive analytically equations D, E, F, from equations A, B, C, but we shall not spend time on what is geometrically evident.

18. When we transform rectangular axes to a system not rectangular, let λ, μ, ν be the angles between the new axes of y and z, of z and x, of x and y respectively, then (Art. 13)

$$\cos\lambda = \cos\alpha' \cos\alpha'' + \cos\beta' \cos\beta'' + \cos\gamma' \cos\gamma'',$$
$$\cos\mu = \cos\alpha'' \cos\alpha + \cos\beta'' \cos\beta + \cos\gamma'' \cos\gamma,$$
$$\cos\nu = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'.$$

Hence

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2 + 2YZ\cos\lambda + 2ZX\cos\mu + 2XY\cos\nu.$$

Thus we obtain the radius vector from the origin to any point expressed in terms of the oblique co-ordinates of that point. It is proved in like manner that the square of the distance between two points, the axes being oblique, is

$$(x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2 + 2(y'-y'')(z'-z'')\cos\lambda$$
$$+ 2(z'-z'')(x'-x'')\cos\mu + 2(x'-x'')(y'-y'')\cos\nu.*$$

19. *The degree of any equation between the co-ordinates is not altered by transformation of co-ordinates.*

This is proved, as at *Conics*, p. 8, from the consideration that the expressions just given for x, y, z, only involve the new co-ordinates *in the first degree.*

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* As we shall never require in practice the formulæ for transforming from one set of oblique axes to another, we only give them in a note.

Let A, B, C have the same meaning as at note, p. 7, and let a, β, γ; a', β', γ'; a'', β'', γ'' be the angles made by the new axes with the old co-ordinate *planes*; then by projecting on lines perpendicular to the old co-ordinate planes, as in the note referred to, we find

$$x\sin A = X\sin a + Y\sin a' + Z\sin a'',$$
$$y\sin B = X\sin\beta + Y\sin\beta' + Z\sin\beta'',$$
$$z\sin C = X\sin\gamma + Y\sin\gamma' + Z\sin\gamma''.$$

CHAPTER II.

INTERPRETATION OF EQUATIONS.

20. It appears from the construction of Art. 1 that if we were given merely the two equations $x = a$, $y = b$, and if the z were left indeterminate, the two given equations would determine the point C, and we should know that the point P lay *somewhere* on the line PC. These two equations then are considered as representing that right line, it being the locus of all points whose $x = a$, and whose $y = b$. We learn then that any two equations of the form $x = a$, $y = b$ represent a right line parallel to the axis of z. In particular, the equations $x = 0$, $y = 0$ represent the axis of z itself. Similarly for the other axes.

Again, if we were given the single equation $x = a$, we could determine nothing but the point D. Proceeding, as at the end of Art. 2, we should learn that the point P lay *some where* in the plane $PBCD$, but its position in that plane would be indeterminate. This plane then being the locus of all points whose $x = a$ is represented analytically by that equation. We learn then that any equation of the form $x = a$ represents a plane parallel to the plane yz. In particular, the equation $x = 0$ denotes the plane yz itself. Similarly, for the other two co-ordinate planes.

21. In general, *any single equation between the co-ordinates represents a surface of some kind; any two simultaneous equations between them represent a line of some kind, either straight or curved; and any three equations denote one or more points.*

I. If we are given a *single* equation, we may take for x and y any arbitrary values; and then the given equation solved for z will determine one or more corresponding values of z. In other words, if we take arbitrarily any point C in the plane of xy, we can always find on the line PC one or more points whose co-ordinates will satisfy the given equation. The assemblage then of points so found on the lines PC will form a surface which will be the geometrical representation of the given equation (see *Conics*, p. 13).

II. When we are given *two* equations, we can, by eliminating y and z alternately between them, throw them into the form $y = \phi(x)$, $z = \psi(x)$. If then we take for x any arbitrary value, the given equations will determine corresponding values for y and z. In other words, we can no longer take the point C *anywhere* on the plane of xy, but this point is limited to a certain locus represented by the equation $y = \phi(x)$. Taking the point C anywhere on this locus, we determine as before on the line PC a number of points P, the assemblage of which is the locus represented by the two equations. And since the points C which are the projections of these latter points, lie on a certain line, straight or curved, it is plain that the points P must also lie on a line of some kind, though of course they do not necessarily lie all in any one plane.

Otherwise thus: when two equations are given, we have seen in the first part of this article that the locus of points whose co-ordinates satisfy either equation separately, is a surface. Consequently, the locus of points whose co-ordinates satisfy *both* equations is the assemblage of points common to the two surfaces which are represented by the two equations considered separately: that is to say, the locus is the line of intersection of these surfaces.

III. When *three* equations are given, it is plain that they are sufficient to determine absolutely the values of the three unknown quantities x, y, z, and therefore that the given equations represent one or more *points*. Since each equation taken separately represents a surface, it follows hence that any three surfaces have one or more common points of intersection, real or imaginary.

22. Surfaces, like plane curves, are classed according to the degrees of the equations which represent them. Since every point in the plane of xy has its $z = 0$, if in any equation

we make $z = 0$, we get the relation between the x and y co-ordinates of the points in which the plane xy meets the surface represented by the equation: that is to say, we get the equation of the plane curve of section, and it is obvious that the equation of this curve will be in general of the same degree as the equation of the surface. It is evident, in fact, that the degree of the equation of the section cannot be *greater* than that of the surface, but it appears at first as if it might be *less*. For instance, the equation

$$zx^2 + ay^2 + b^2x = c^3$$

is of the third degree, but when we make $z = 0$, we get an equation of the second degree. But since the original equation would have been unmeaning if it were not homogeneous, every term must be of the third dimension in some linear unit (see *Conics*, p. 61), and therefore when we make $z = 0$, the remaining terms must still be regarded as of three dimensions. They will form an equation of the second degree multiplied by a constant, and denote (see *Conics*, p. 61) a conic and a line at infinity. If then we take into account lines at infinity, we may say that the section of a surface of the n^{th} degree by the plane of xy will be *always* of the n^{th} degree, and since any plane may be made the plane of xy, and since transformation of co-ordinates does not alter the degree of an equation, we learn that *every plane section of a surface of the n^{th} degree is a curve of the n^{th} degree.*

In like manner it is proved that *every right line meets a surface of the n^{th} degree in n points.* The right line may be made the axis of z, and the points where it meets the surface are found by making $x = 0$, $y = 0$ in the equation of the surface, when in general we get an equation of the n^{th} degree to determine z. If the degree of the equation happened to be less than n, it would only indicate that some of the n points where the line meets the surface are at infinity.

23. *Curves in space* are classified according to the number of points in which they are met by any plane. *Two equations of the m^{th} and n^{th} degrees respectively represent a curve of the mn^{th} degree.* For the surfaces represented by the equations

are cut by any plane in curves of the m^{th} and n^{th} degrees respectively, and these curves intersect in mn points.

Three equations of the m^{th}, n^{th}, and p^{th} degrees respectively, denote mnp points.

This follows from the theory of elimination, since if we eliminate y and z between the equations, we obtain an equation of the mnp^{th} degree to determine x (see *Lessons on Higher Algebra*, p. 26). This proves also that *three surfaces of the m^{th}, n^{th}, p^{th} degrees respectively, intersect in mnp points.*

24. If an equation only contain two of the variables $\phi(x, y) = 0$, the learner might at first suppose that it represents a curve in the plane of xy, and so that it forms an exception to the rule that it requires *two* equations to represent a curve. But it must be remembered that the equation $\phi(x, y) = 0$ will be satisfied not only for any point of this curve in the plane of xy, but also for any other point having the same x and y though a different z: that is to say, for any point of the surface generated by a right line moving along this curve, but remaining parallel to the axis of z.* The curve in the plane of xy can only be represented by *two* equations, namely, $z = 0$, $\phi(x, y) = 0$.

If an equation contain only *one* of the variables x, we know by the theory of equations, that it may be resolved into n factors of the form $x - a = 0$, and therefore (Art. 20) that it represents n planes parallel to one of the co-ordinate planes.

* A surface generated by a right line moving parallel to itself is called a *cylindrical* surface.

CHAPTER III.

THE PLANE.

25. In the discussion of equations we commence of course with equations of the first degree, and the first step is to prove that *every equation of the first degree represents a plane*, and conversely, that *the equation of a plane is always of the first degree.* We commence with the latter proposition, which may be established in two or three different ways.

In the first place we have seen (Art. 20) that the plane of xy is represented by an equation of the first degree, viz. $z=0$; and transformation to any other axes cannot alter the *degree* of this equation (Art. 19).

We might arrive at the same result by forming the equation of the plane determined by three given points, which we can do by eliminating l, m, n from the three equations given Art. 9, when we should arrive at an equation of the first degree. The following method however of expressing the equation of a plane leads to one of the forms most useful in practice.

26. *To find the equation of a plane, the perpendicular on which from the origin* $=p$, *and makes angles* α, β, γ *with the axes.*

The length of the projection on the perpendicular of the radius vector to any point of the plane is of course $=p$, and (Art. 7) this is equal to the sum of the projections on that line of the three co-ordinates. Hence we obtain for the equation of the plane

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p.\text{*}$$

* In what follows we suppose the axes rectangular, but this equation is true whatever be the axes.

27. Now, conversely, any equation of the first degree

$$Ax + By + Cz + D = 0,$$

can be reduced to the form just given, by dividing it by a factor R. We are to have $A = R\cos\alpha$, $B = R\cos\beta$, $C = R\cos\gamma$, whence, by Art. 11, R is determined to be $= \sqrt{(A^2 + B^2 + C^2)}$. Hence any equation $Ax + By + Cz + D = 0$ may be identified with the equation of a plane, the perpendicular on which from the origin $= \dfrac{-D}{\sqrt{(A^2 + B^2 + C^2)}}$, and makes angles with the axes whose cosines are A, B, C, respectively divided by the same square root. We are to give to the square root the sign which will make the perpendicular positive, and then the signs of the cosines will determine whether the angles which the perpendicular makes with the positive directions of the axes are acute or obtuse.

28. *To find the angle between two planes*

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0.$$

The angle between the planes is the same as the angle between the perpendiculars on them from the origin. By the last article we have the angles these perpendiculars make with the axes, and thence, Arts. 13, 14, we have

$$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{\{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)\}}},$$

$$\sin^2\theta = \frac{(AB' - A'B)^2 + (BC' - B'C)^2 + (CA' - C'A)^2}{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}.$$

Hence the condition that the planes should cut at right angles is $AA' + BB' + CC' = 0$.

They will be parallel if we have the conditions

$$AB' = A'B, \quad BC' = B'C, \quad CA' = C'A;$$

in other words, if the coefficients A, B, C be proportional to A', B', C', in which case it is manifest from the last article that the direction of the perpendicular on both will be the same.

29. *To express the equation of a plane in terms of the intercepts a, b, c, which it makes on the axes.*

The intercept made on the axis of x by the plane

$$Ax + By + Cz + D = 0$$

is found by making y and z both $= 0$, when we have $Aa + D = 0$. And similarly, $Bb + D = 0$, $Cc + D = 0$. Substituting in the general equation the values just found for A, B, C, it becomes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If in the general equation any term be wanting, for instance, if $A = 0$, the point where the plane meets the axis of x is at infinity, or the plane is parallel to the axis of x. If we have both $A = 0$, $B = 0$, then two axes meet at infinity the given plane which is therefore parallel to the plane of xy (see also Art. 20). If we have $A = 0$, $B = 0$, $C = 0$, all three axes meet the plane at infinity, and we see, as at *Conics*, p. 61, that an equation $D = 0$ must be taken to represent a plane at infinity.

30. *To find the equation of the plane determined by three points.*

Let the equation be $Ax + By + Cz + D = 0$; and since this is to be satisfied by the co-ordinates of each of the given points, A, B, C, D must satisfy the equations

$$Ax' + By' + Cz' + D = 0,\ Ax'' + By'' + Cz'' + D = 0,$$
$$Ax''' + By''' + Cz''' + D = 0.$$

Eliminating A, B, C, D between the four equations, the result is the determinant

$$\begin{vmatrix} x, & y, & z, & 1 \\ x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & z''', & 1 \end{vmatrix} = 0.$$

Expanding this by the common rule, the equation is

$$\begin{aligned} & x\{y'(z'' - z''') + y''(z''' - z') + y'''(z' - z'')\} \\ & + y\{z'(x'' - x''') + z''(x''' - x') + z'''(x' - x'')\} \\ & + z\{x'(y'' - y''') + x''(y''' - y') + x'''(y' - y'')\} \\ & = x'(y''z''' - y'''z'') + x''(y'''z' - y'z''') + x'''(y'z'' - y''z'). \end{aligned}$$

If we consider x, y, z as the co-ordinates of any fourth point, we have the condition that four points should lie in one plane.

31. The coefficients of x, y, z in the preceding equation are evidently double the areas of the projections on the co-ordinate planes of the triangle formed by the three points.

If now we take the equation (Art. 26)

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p,$$

and multiply it by twice A, (A being the area of the triangle formed by the three points) the equation will become identical with that of the last article, since $A\cos\alpha$, $A\cos\beta$, $A\cos\gamma$ are the projections of the triangle on the co-ordinate planes (Art. 4). The absolute term then must be the same in both cases. Hence the quantity

$$x'(y''z''' - y'''z'') + x''(y'''z' - y'z''') + x'''(y'z'' - y''z')$$

represents double the area of the triangle formed by the three points multiplied by the perpendicular on its plane from the origin: or, in other words, *six times the volume of the triangular pyramid, whose base is that triangle, and whose vertex is the origin.**

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* If in the preceding values we substitute for x', y', z'; $\rho'\cos\alpha'$, $\rho'\cos\beta'$, $\rho'\cos\gamma'$, &c., we find that six times the volume of this pyramid $= \rho'\rho''\rho'''$ multiplied by the determinant

$$\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma' \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'' \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''' \end{vmatrix}.$$

Now let us suppose the three radii vectores cut by a sphere whose radius is unity, having the origin for its centre, and meeting it in a spherical triangle $R'R''R'''$. Then if a denote the side $R'R''$, and p the perpendicular on it from R''', six times the volume of the pyramid will be $\rho'\rho''\rho'''\sin a\sin p$; for $\rho'\rho''\sin a$ is double the area of one face of the pyramid, and $\rho'''\sin p$ is the perpendicular on it from the opposite vertex. It follows then that the determinant above written is equal to double the function

$$\sqrt{\{(\sin s\sin(s-a)\sin(s-b)\sin(s-c)\}}$$

of the sides of the above-mentioned spherical triangle. The same thing

We can at once express A itself in terms of the co-ordinates of the three points by Art. 12, and must have $4A^2$ equal to the sum of the squares of the coefficients of x, y, and z, in the equation of the last article.

32. *To find the length of the perpendicular from a given point $x'y'z'$ on a given plane.*

If we draw through $x'y'z'$ a plane parallel to the given plane and let fall on the two planes a common perpendicular from the origin, then the intercept on this line will be equal to the length of the perpendicular required, since parallel planes make equal intercepts on parallel lines. But the length of the perpendicular on the plane through $x'y'z'$ is, by definition, (Art. 5) the projection on that perpendicular of the radius vector to $x'y'z'$, and therefore (Art. 26) is equal to

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma.$$

The length required is therefore

$$x'\cos\alpha + y'\cos\beta + z'\cos\gamma - p.$$

N.B. This supposes the perpendicular on the plane through $x'y'z'$ to be greater than p, or, in other words, that $x'y'z'$ and

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may be proved by forming the square of the same determinant according to the ordinary rule; when if we write

$$\cos\alpha''\cos\alpha''' + \cos\beta''\cos\beta''' + \cos\gamma''\cos\gamma''' = \cos a, \text{ \&c.}$$

we get

$$\begin{vmatrix} 1 & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1 \end{vmatrix},$$

which expanded is $1 + 2\cos a\cos b\cos c - \cos^2 a - \cos^2 b - \cos^2 c$, which is known to have the value in question.

It is useful to remark that when the three lines are at right angles to each other the determinant

$$\begin{vmatrix} \cos\alpha', & \cos\beta', & \cos\gamma' \\ \cos\alpha'', & \cos\beta'', & \cos\gamma'' \\ \cos\alpha''', & \cos\beta''', & \cos\gamma''' \end{vmatrix}$$

has unity for its value. In fact we see, as above, that its square is

$$\begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix}.$$

the origin are on opposite sides of the plane. If they were on the same side, the length of the perpendicular would be $p-(x'\cos\alpha+y'\cos\beta+z'\cos\gamma)$. If the equation of the plane were given in the form $Ax+By+Cz+D$, it is reduced to the other form, as in Art. 27, and the length of the perpendicular is

$$\frac{Ax'+By'+Cz'+D}{\sqrt{(A^2+B^2+C^2)}}.$$

It is plain that all points for which $Ax'+By'+Cz'+D$ has the same sign as D, will be on the same side of the plane as the origin, and *vice versâ* when the sign is different.

33. *To find the co-ordinates of the intersection of three planes.*

This is only to solve three equations of the first degree for three unknown quantities (see *Lessons on Higher Algebra*, Art. 24). The value of the co-ordinates will become infinite if the determinant $(AB'C'')$ vanishes, or

$$A(B'C''-B''C')+A'(B''C-BC'')+A''(BC'-B'C)=0.$$

This then is the condition that the three planes should be parallel to the same line. For in such a case the line of intersection of any two would be also parallel to this line, and could not meet the third plane at any finite distance.

34. *To find the condition that four planes should meet in a point.*

This is evidently obtained, by eliminating x, y, z between the equations of the four planes, and is therefore the determinant $(AB'C''D''')$, or

$$\begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix}=0.$$

35. *To find the volume of the tetrahedron whose vertices are any four given points.*

If we multiply the area of the triangle formed by three points, by the perpendicular on their plane from the fourth, we obtain three times the volume. The length of the perpendicular on the plane whose equation is given, (Art. 30) is formed by substituting in that equation the co-ordinates of the fourth point, and dividing by the square root of the sum of the squares of the coefficients of x, y, z. But (Art. 31) that square root is double the area of the triangle formed by the three points. Hence *six times the volume of the tetrahedron in question is equal to the determinant*

$$\begin{vmatrix} x', & y', & z', & 1 \\ x'', & y'', & z'', & 1 \\ x''', & y''', & z''', & 1 \\ x'''', & y'''', & z'''', & 1 \end{vmatrix}.*$$

36. It is evident, as in Plane Geometry, (see *Conics*, Art. 36) that if S, S', S'' represent any three surfaces, then $aS+bS'$ where a and b are any constants, represents a surface passing through the line of intersection of S and S'; and that $aS+bS'+cS''$ represents a surface passing through the points of intersection of S, S', and S''. Thus then if L, M, N denote any three planes, $aL+bM$ denotes a plane passing through the line of intersection of the first two, and $aL+bM+cN$ denotes a plane passing through the point common to all three. As a particular case of the preceding $aL+b$ denotes a plane parallel to L, and $aL+bM+c$ denotes a plane parallel to the intersection of L and M (see Art. 29).

So again, four planes L, M, N, P will pass through the same point if their equations are connected by an identical relation

$$aL+bM+cN+dP=0,$$

* The volume of the tetrahedron formed by four planes, whose equations are given, can be found by forming the co-ordinates of its angular points, and then substituting in the formula given above. The result is, (see *Lessons on Higher Algebra*, Art. 25) that six times the volume is equal to

$$\frac{R^3}{(AB'C'')(A'B''C''')(A''B'''C)(A'''BC')}$$

where R is the determinant $(AB'C''D''')$ Art. 34, and the factors in the denominator express the conditions (Art. 33) that any three of the planes should be parallel to the same line.

for then any co-ordinates which satisfy the first three must satisfy the fourth. Conversely, given any four planes intersecting in a common point, it is easy to obtain such an identical relation. For multiply the first equation by the determinant $(A'B''C''')$, the second by $-(A''B'''C)$, the third by $(A'''BC')$, and the fourth by $-(AB'C'')$, and add: then (*Lessons on Higher Algebra*, Art. 7) the coefficients of x, y, z vanish identically; and the remaining term is the determinant which vanishes (Art. 34), because the planes meet in a point. Their equations are therefore connected by the identical relation

$$L(A'B''C''') - M(A''B'''C) + N(A'''BC') - P(AB'C'') = 0.$$

37. Given any four planes L, M, N, P not meeting in a point, it is easy to see (as at *Conics*, Arts. 58, 59) that the equation of any other plane can be thrown into the form

$$aL + bM + cN + dP = 0.$$

And in general the equation of any surface of the n^{th} degree can be expressed by a homogeneous equation of the n^{th} degree between L, M, N, P (see *Conics*, Art. 270). For the number of terms in the *complete* equation of the n^{th} order between *three* variables is the same as the number of terms in the *homogeneous* equation of the n^{th} order between *four* variables.

Accordingly, in what follows, we shall use these *quadriplanar* co-ordinates whenever by so doing our equations can be materially simplified.

Ex. 1. To find the equation of the plane passing through $x'y'z'$, and through the intersection of the planes

$Ax + By + Cz + D$, $A'x + B'y + C'z + D'$ (see *Conics*, Ex. 3, p. 29).

Ans. $(A'x' + B'y' + C'z' + D')(Ax + By + Cz + D)$

$$= (Ax' + By' + Cz' + D)(A'x + B'y + C'z + D').$$

Ex. 2. Find the equation of the plane passing through the points ABC, figure, p. 2.

The equations of the line BC are evidently $\frac{x}{a} = 1$, $\frac{y}{b} + \frac{z}{c} = 1$. Hence obviously the equation of the required plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2$, since this passes through each of the three lines joining the three given points.

Ex. 3. Find the equation of the plane *PEF* in the same figure.

The equations of the line *EF* are $x = 0, \frac{y}{b} + \frac{z}{c} = 1$, and forming as above the equation of the plane joining this line to the point *abc*, we get $\frac{y}{b} + \frac{z}{c} - \frac{x}{a} = 1$.

38. *If four planes which intersect in a right line be met by any plane, the anharmonic ratio of the pencil so formed will be constant.* For we could by transformation of co-ordinates make the transverse plane the plane of xy, and then by making $z = 0$ in the equations would have the equations of the intersections of the four planes with this plane. These will be of the form $aL + M$, $bL + M$, $cL + M$, $dL + M$, whose anharmonic ratio (see *Conics*, Art. 56) depends solely on the constants a, b, c, d; and does not alter when by transformation of co-ordinates L and M come to represent different lines.

THE RIGHT LINE.

39. The equations of any two planes taken together will represent their line of intersection which will include all the points whose co-ordinates satisfy *both* the equations. By eliminating x and y alternately between the equations we reduce them to a form commonly used, viz.

$$x = mz + a, \quad y = nz + b.$$

The first represents the projection of the line on the plane of xz and the second that on the plane of yz. The reader will observe that *the equations of a right line include four independent constants.*

We might form independently the equations of the line joining two points; for taking the values given (Art. 8) of the co-ordinates of any point on that line, solving for the ratio $m : n$ from each of the three equations there given, and equating results, we get

$$\frac{x - x'}{x' - x''} = \frac{y - y'}{y' - y''} = \frac{z - z'}{z' - z''}$$

for the required equations of the line. It thus appears that the equations of the projections of the line are the same as the equations of the lines joining the projections of two points on the line, as is otherwise evident.

40. Two right lines in space will in general not intersect. If the first line be represented by any two equations $L=0$, $M=0$, and the second by any other two $N=0$, $P=0$, then if the two lines meet in a point, each of these four planes must pass through that point, and the condition that the lines should intersect is the same as that already given (Art. 34).

Two intersecting lines determine a plane whose equation can easily be found. For we have seen (Art. 36) that when the four planes intersect, their equations satisfy an identical relation

$$aL+bM+cN+dP=0.$$

The equations therefore $aL+bM=0$, and $cN+dP=0$ must be identical and must represent the same plane. But the form of the first equation shows that this plane passes through the line L, M, and that of the second equation shows that it passes through the line N, P.

Ex. When the given lines are represented by equations of the form

$$x=mz+a,\quad y=nz+b;\quad x=m'z+a',\quad y=n'z+b',$$

the condition that they should intersect is easily found. For solving for z from the first and third equations, and equating it to the value found by solving from the second and fourth, we get

$$\frac{a-a'}{m-m'}=\frac{b-b'}{n-n'}.$$

Again, if this condition is satisfied, the four equations are connected by the identical relation

$$(n-n')\{(x-mz-a)-(x-m'z-a')\}=(m-m')\{(y-nz-b)-(y-n'z-b')\},$$

and therefore $(n-n')(x-mz-a)=(m-m')(y-nz-b)$

is the equation of the plane containing both lines.

41. *To find the equations of a line passing through the point $x'y'z'$, and making angles α, β, γ with the axes.*

The projections on the axes, of the distance of $x'y'z'$ from any variable point xyz on the line, are respectively $x-x'$, $y-y'$, $z-z'$; and since these are each equal to that distance multiplied by the cosine of the angle between the line and the axis in question, we have

$$\frac{x-x'}{\cos\alpha} = \frac{y-y'}{\cos\beta} = \frac{z-z'}{\cos\gamma};$$

a form of writing the equations of the line which, although it includes two superfluous constants, yet on account of its symmetry between x, y, z is often used in preference to the form in Art. 39.

Reciprocally, if we desire to find the angles made with the axes by any line, we have only to throw its equation into the form $\frac{x-x'}{A} = \frac{y-y'}{B} = \frac{z-z'}{C}$ when the direction-cosines of the line will be respectively A, B, C, each divided by the square root of the sum of the squares of these three quantities.

Ex. 1. To find the direction-cosines of $x = mz + a$, $y = nz + b$. Writing the equations in the form $\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1}$, the direction-cosines are

$$\frac{m}{\sqrt{(1+m^2+n^2)}}, \quad \frac{n}{\sqrt{(1+m^2+n^2)}}, \quad \frac{1}{\sqrt{(1+m^2+n^2)}}.$$

Ex. 2. To find the direction-cosines of $\frac{x}{l} = \frac{y}{m}$, $z = 0$.

Ans. $\frac{l}{\sqrt{(l^2+m^2)}}, \quad \frac{m}{\sqrt{(l^2+m^2)}}, \quad 0.$

Ex. 3. To find the direction-cosines of

$$Ax + By + Cz + D, \quad A'x + B'y + C'z + D'.$$

Eliminating y and z alternately we reduce them to the preceding form, and the direction-cosines are $\frac{BC' - B'C}{R}$, $\frac{CA' - C'A}{R}$, $\frac{AB' - A'B}{R}$, where R^2 is the sum of the squares of the three numerators.

Ex. 4. To find the equation of the plane through the two intersecting lines

$$\frac{x-x'}{\cos\alpha} = \frac{y-y'}{\cos\beta} = \frac{z-z'}{\cos\gamma}; \quad \frac{x-x'}{\cos\alpha'} = \frac{y-y'}{\cos\beta'} = \frac{z-z'}{\cos\gamma'}.$$

The required plane passes through $x'y'z'$ and its perpendicular is perpendicular to two lines whose direction-cosines are given; therefore, (Art. 15) the required equation is

$$(x-x')(\cos\beta\cos\gamma' - \cos\gamma\cos\beta') + (y-y')(\cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha)$$
$$+ (z-z')(\cos\alpha\cos\beta' - \cos\alpha'\cos\beta) = 0.$$

Ex. 5. To find the equation of the plane passing through the two parallel lines

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma};\ \frac{x-x''}{\cos\alpha}=\frac{y-y''}{\cos\beta}=\frac{z-z''}{\cos\gamma}.$$

The required plane contains the line joining the given points, whose direction-cosines are proportional to $x'-x''$, $y'-y''$, $z'-z''$; the direction-cosines of the perpendicular to the plane are therefore proportional to

$$(y'-y'')\cos\gamma-(z'-z'')\cos\beta,\ (z'-z'')\cos\alpha-(x'-x'')\cos\gamma,$$
$$(x'-x'')\cos\beta-(y'-y'')\cos\alpha.$$

These may therefore be taken as the coefficients of x, y, z, in the required equation, while the absolute term determined by substituting $x'y'z'$ for xyz in the equation is

$$(y'z''-y''z')\cos\alpha+(z'x''-z''x')\cos\beta+(x'y''-x''y')\cos\gamma.$$

42. *To find the equations of the perpendicular from $x'y'z'$ on the plane $Ax+By+Cz+D$.* The direction-cosines of the perpendicular on the plane (Art. 27) are proportional to A, B, C; hence the equations required are

$$\frac{x-x'}{A}=\frac{y-y'}{B}=\frac{z-z'}{C}.$$

43. *To find the direction-cosines of the bisector of the angle between two given lines.*

As we are only concerned with *directions* it is of course sufficient to consider lines through the origin. If we take points $x'y'z'$, $x''y''z''$ one on each line, equidistant from the origin, then the middle point of the line joining these points is evidently a point on the bisector, whose equation would therefore be

$$\frac{x}{x'+x''}=\frac{y}{y'+y''}=\frac{z}{z'+z''},$$

and whose direction-cosines are therefore proportional to

$$x'+x'',\ y'+y'',\ z'+z'';$$

but since $x', y', z', x'', y'', z''$ are evidently proportional to the direction-cosines of the given lines, the direction-cosines of the bisector are

$$\cos\alpha'+\cos\alpha'',\ \cos\beta'+\cos\beta'',\ \cos\gamma'+\cos\gamma'',$$

each divided by the square root of the sum of the squares of these three quantities.

The bisector of the supplemental angle between the lines would be got by substituting for the point $x''y''z''$ a point equidistant from the origin measured in the opposite direction, whose co-ordinates are $-x''$, $-y''$, $-z''$; and therefore the direction-cosines of this bisector are respectively proportional to

$$\cos\alpha' - \cos\alpha'', \quad \cos\beta' - \cos\beta'', \quad \cos\gamma' - \cos\gamma''.$$

N.B. The equation of the *plane* bisecting the angle between two given *planes* is found precisely as at *Conics*, p. 35, and is

$$(x\cos\alpha + y\cos\beta + z\cos\gamma - p) = \pm(x\cos\alpha' + y\cos\beta' + z\cos\gamma' - p').$$

44. *To find the angle made with each other by two lines*

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \quad \frac{x-a}{l'} = \frac{y-b}{m'} = \frac{z-c}{n'}.$$

Evidently (Arts. 13, 41),

$$\cos\theta = \frac{ll' + mm' + nn'}{\sqrt{(l^2 + m^2 + n^2)}\sqrt{(l'^2 + m'^2 + n'^2)}}.$$

COR. The lines are at right angles to each other if

$$ll' + mm' + nn' = 0.$$

Ex. To find the angle between the lines $\frac{x}{2} = \frac{y}{\sqrt{(3)}} = \frac{z}{\sqrt{(2)}}$; $\frac{x}{\sqrt{(3)}} = y$, $z = 0$.

Ans. 30°.

45. *To find the angle between the plane* $Ax + By + Cz + D$, *and the line* $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$.

The angle between the line and the plane is the complement of the angle between the line and the perpendicular on the plane, and we have therefore

$$\sin\theta = \frac{Al + Bm + Cn}{\sqrt{(l^2 + m^2 + n^2)}\sqrt{(A^2 + B^2 + C^2)}}.$$

COR. When $Al + Bm + Cn = 0$, the line is parallel to the plane, for it is then perpendicular to a perpendicular on the plane.

46. *To find the conditions that a line* $x = mz + a$, $y = nz + b$ *should be altogether in a plane* $Ax + By + Cz + D$. Substitute

for x and y in the equation of the plane, and solve for z, when we have

$$z = -\frac{Aa + Bb + D}{Am + Bn + C},$$

and if both numerator and denominator vanish, the value of z is indeterminate and the line is altogether in the plane. We have just seen that the vanishing of the denominator expresses the condition that the line should be parallel to the plane; while the vanishing of the numerator expresses that one of the points of the line is *in* the plane, viz. the point ab where the line meets the plane of xy.

In like manner in order to find the conditions that a right line should lie altogether in any surface, we should substitute for x and y in the equation of the surface, and then equate to zero the coefficient of *every* power of z in the resulting equation. It is plain that the number of conditions thus resulting is one more than the degree of the surface.*

47. *To find the equation of the plane drawn through a given line perpendicular to a given plane.*

Let the line be given by the equations

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0,$$

and let the plane be

$$A''x + B''y + C''z + D'' = 0.$$

Then any plane through the line will be of the form

$$\lambda (Ax + By + Cz + D) + \mu (A'x + B'y + C'z + D') = 0,$$

and in order that it should be perpendicular to the plane we must have

$$(\lambda A + \mu A') A'' + (\lambda B + \mu B') B'' + (\lambda C + \mu C') C'' = 0.$$

* Since the equations of a right line contain four constants, a right line can be determined which shall satisfy any four conditions. Hence any surface of the second degree must contain an infinity of right lines, since we have only three conditions to satisfy and have four constants at our disposal. Every surface of the third degree must contain a finite number of right lines since the number of conditions to be satisfied is equal to the number of disposable constants. A surface of higher degree will not necessarily contain any right line lying altogether in the surface.

This equation determines $\lambda : \mu$, and the equation of the required plane is

$$(A'A''+B'B''+C'C'')(Ax+By+Cz+D)$$
$$=(AA''+BB''+CC'')(A'x+B'y+C'z+D')$$

When the equations of the given plane and line are given in the form

$$x\cos\alpha+y\cos\beta+z\cos\gamma=p;\ \frac{x-x'}{\cos\alpha'}=\frac{y-y'}{\cos\beta'}=\frac{z-z'}{\cos\gamma'};$$

we can otherwise easily determine the equation of the required plane. For it is to contain the given line whose direction-angles are α', β', γ'; and it is also to contain a perpendicular to the given plane whose direction-angles are α, β, γ. Hence (Art. 15) the direction-cosines of a perpendicular to the required plane are proportional to

$$\cos\beta'\cos\gamma-\cos\beta\cos\gamma',\ \cos\gamma'\cos\alpha-\cos\gamma\cos\alpha',\ \cos\alpha'\cos\beta-\cos\alpha\cos\beta'$$

and since the required plane is also to pass through $x'y'z'$, its equation is

$$(x-x')(\cos\beta\cos\gamma'-\cos\beta'\cos\gamma)+(y-y')(\cos\gamma\cos\alpha'-\cos\gamma'\cos\alpha)$$
$$+(z-z')(\cos\alpha\cos\beta'-\cos\alpha'\cos\beta)=0.$$

48. *Given two lines to find the equation of a plane drawn through either parallel to the other.*

First, let the given lines be the intersections of the planes L, M; N, P whose equations are given in the most general form. Then proceeding exactly as in Art. 36, we obtain the identical relation

$$L(A'B''C''')-M(A''B'''C)+N(A'''BC')-P(AB'C'')=(AB'C''D''')$$

the right-hand side of the equation being the determinant, whose vanishing expresses that the four planes meet in a point. It is evident then that the equations

$$L(A'B''C''')-M(A''B'''C)=0,\ N(A'''BC')-P(AB'C'')=0$$

represent parallel planes since they only differ by a constant quantity; but these planes pass each through one of the given lines.

Secondly, let the lines be given by equations of the form

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma};\ \frac{x-x''}{\cos\alpha'}=\frac{y-y''}{\cos\beta'}=\frac{z-z''}{\cos\gamma''}.$$

Then since a perpendicular to the sought plane is perpendicular to the direction of each of the given lines, its direction-cosines (Art. 15) are the same as those given in the last example, and the equations of the sought parallel planes are

$$(x-x')(\cos\beta\cos\gamma'-\cos\beta'\cos\gamma)+(y-y')(\cos\gamma\cos\alpha'-\cos\gamma'\cos\alpha)$$
$$+(z-z')(\cos\alpha\cos\beta'-\cos\alpha'\cos\beta)=0,$$
$$(x-x'')(\cos\beta\cos\gamma'-\cos\beta'\cos\gamma)+(y-y'')(\cos\gamma\cos\alpha'-\cos\gamma'\cos\alpha)$$
$$+(z-z'')(\cos\alpha\cos\beta'-\cos\alpha'\cos\beta)=0.$$

The perpendicular distance between two parallel planes is equal to the difference between the perpendiculars let fall on them from the origin, and is therefore equal to the difference between their absolute terms, divided by the square root of the sum of the squares of the common coefficients of x, y, z. Thus the perpendicular distance between the planes last found is

$$(x'-x'')(\cos\beta\cos\gamma'-\cos\beta'\cos\gamma)+(y'-y'')(\cos\gamma\cos\alpha'-\cos\gamma'\cos\alpha)$$
$$+(z'-z'')(\cos\alpha\cos\beta'-\cos\alpha'\cos\beta)\text{ divided by }\sin\theta,$$

where θ (see Art. 14) is the angle between the directions of the given lines. It is evident that the perpendicular distance here found is shorter than any other line which can be drawn from any point of the one plane to any point of the other.

49. *To find the equations and the magnitude of the shortest distance between two non-intersecting lines.*

The shortest distance between two lines is a line perpendicular to both, and which can be found as follows: Draw through each of the lines, by Art. 47, a plane perpendicular to either of the parallel planes determined by Art. 48; then the intersection of the two planes so drawn will be perpendicular to the parallel planes, and therefore to the given lines which lie in these planes. From the construction it is evident that the line so determined meets both the given lines. Its magnitude is plainly that determined in the last article. Working by Art. 47 the equation of a plane passing through a line whose direction-angles are α, β, γ, and perpendicular to a plane whose direction-cosines are proportional to

$$\cos\beta'\cos\gamma-\cos\beta\cos\gamma',\ \cos\gamma'\cos\alpha-\cos\gamma\cos\alpha',\ \cos\alpha'\cos\beta-\cos\alpha\cos\beta',$$

we find that the line sought is the intersection of the two planes

$$(x-x')(\cos\alpha'-\cos\theta\cos\alpha)+(y-y')(\cos\beta'-\cos\theta\cos\beta)$$
$$+(z-z')(\cos\gamma'-\cos\theta\cos\gamma)=0,$$
$$(x-x'')(\cos\alpha-\cos\theta\cos\alpha')+(y-y'')(\cos\beta-\cos\theta\cos\beta')$$
$$+(z-z'')(\cos\gamma-\cos\theta\cos\gamma')=0.$$

The direction-cosines of the shortest distance must plainly be proportional to

$$\cos\beta'\cos\gamma-\cos\beta\cos\gamma',\ \cos\gamma'\cos\alpha-\cos\gamma\cos\alpha',\ \cos\alpha'\cos\beta-\cos\alpha\cos\beta'.$$

NOTE ON THE PROPERTIES OF TETRAHEDRA.

50. We add as an appendix to the preceding chapters some properties of tetrahedra which, though not obtained by the method of co-ordinates, are worth being set down.

To find the relation between the six lines joining any four points in a plane.

Let a, b, c be the sides of the triangle formed by any three of them ABC, and let d, e, f be the lines joining the fourth point D to these three. Let the angles subtended at D by a, b, c be α, β, γ; then we have $\cos\alpha=\cos(\beta\pm\gamma)$, whence

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma=1.$$

This relation will be true whatever be the position of D, either within or without the triangle ABC. But

$$\cos\alpha=\frac{e^2+f^2-a^2}{2ef},\quad \cos\beta=\frac{f^2+d^2-b^2}{2fd},\quad \cos\gamma=\frac{d^2+e^2-c^2}{2de}.$$

Substituting these values and reducing, we find for the required relation

$$a^2(d^2-e^2)(d^2-f^2)+b^2(e^2-f^2)(e^2-d^2)+c^2(f^2-d^2)(f^2-e^2)$$
$$+a^2d^2(a^2-b^2-c^2)+b^2e^2(b^2-a^2-c^2)+c^2f^2(c^2-a^2-b^2)+a^2b^2c^2=0.$$

51. *To express the volume of a tetrahedron in terms of its six edges.*

Let the sides of the triangle formed by any face ABC be a, b, c; the perpendicular on that face from the remaining vertex be p, and the distances of the foot of that perpendicular from A, B, C be d', e', f'. Then a, b, c, d', e', f' are connected by the relation given in the last article. But if d, e, f be the remaining edges $d^2=d'^2+p^2$, $e^2=e'^2+p^2$, $f^2=f'^2+p^2$; whence $d^2-e^2=d'^2-e'^2$, &c. and putting in these values, we get

$$-F=p^2(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4),$$

where F is the quantity on the left-hand side of the equation in the last article. Now the quantity multiplying p^2 is 16 times the square of the area of the triangle ABC, and since p multiplied by this area is three times the volume of the pyramid, we have $F=-144V^2$.

52. *To find the relation between the six arcs joining four points on the surface of a sphere.*

We proceed precisely as in Art. 50, only substituting for the formulæ there used the corresponding formulæ for spherical triangles, and if α, β, γ, δ, ϵ, ϕ represent the *cosines* of the six arcs in question, we get

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\epsilon^2+\phi^2-\alpha^2\delta^2-\beta^2\epsilon^2-\gamma^2\phi^2+2\alpha\beta\delta\epsilon+2\beta\gamma\epsilon\phi+2\gamma\alpha\delta\phi$$
$$-2\alpha\beta\gamma-2\alpha\epsilon\phi-2\beta\delta\phi-2\gamma\delta\epsilon=1.$$

This relation may be otherwise proved as follows: Let the direction-cosines of the radii to the four points be

$$\begin{array}{lll}\cos\alpha, & \cos\beta, & \cos\gamma,\\ \cos\alpha', & \cos\beta', & \cos\gamma',\\ \cos\alpha'', & \cos\beta'', & \cos\gamma'',\\ \cos\alpha''', & \cos\beta''', & \cos\gamma'''.\end{array}$$

Now from this matrix we can form (by the method of *Lessons on Higher Algebra*, Art. 20) a determinant which shall vanish identically, and which (substituting $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma' = \cos ab$, &c.) is

$$\begin{vmatrix} 1, & \cos ab, & \cos ac, & \cos ad \\ \cos ba, & 1, & \cos bc, & \cos bd \\ \cos ca, & \cos cb, & 1, & \cos cd \\ \cos da, & \cos db, & \cos dc, & 1, \end{vmatrix} = 0,$$

which expanded has the value written above.

53. *To find the radius of the sphere circumscribing a tetrahedron.*

Since any side a of the tetrahedron is the chord of the arc whose cosine is α, we have $\alpha = 1 - \frac{a^2}{2r^2}$, with similar expressions for β, γ, &c.; and making these substitutions, the formula of the last example becomes

$$\frac{F}{4r^6} + \frac{2a^2d^2b^2e^2 + 2b^2e^2c^2f^2 + 2c^2f^2a^2d^2 - a^4d^4 - b^4e^4 - c^4f^4}{16r^8} = 0,$$

whence if $\qquad ad + be + cf = 2S,$

we have $\qquad r^2 = \frac{S(S-ad)(S-be)(S-cf)}{36V^2}.$

The reader may exercise himself in proving that the shortest distance between two opposite sides of the tetrahedron is equal to six times the volume divided by the product of those sides multiplied by the sine of their angle of inclination to each other, which may be expressed in terms of the sides by the help of the relation $2ad \cos\theta = b^2 + e^2 - c^2 - f^2$.

CHAPTER IV.

*PROPERTIES COMMON TO ALL SURFACES OF THE SECOND DEGREE.

54. We shall write the general equation of the second degree

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy + 2px + 2qy + 2rz + d = 0.$$

This equation contains ten terms, and since its signification is not altered if by division we make one of the coefficients unity, it appears that nine conditions are sufficient to determine a surface of the second degree, or as we shall call it for shortness, a quadric surface. Thus if we were given nine points on the surface, by substituting successively the co-ordinates of each in the general equation, we obtain nine equations which are sufficient to determine the nine unknown quantities $\frac{b}{a}$, $\frac{c}{a}$, &c. And in like manner the number of conditions necessary to determine a surface of the n^{th} degree is one less than the number of terms in the general equation.

The equation of a quadric may also (see Art. 37) be expressed as a homogeneous function of the equations of four given planes x, y, z, ω,

$$ax^2 + by^2 + cz^2 + d\omega^2 + 2lyz + 2mzx + 2nxy + 2px\omega + 2qy\omega + 2rz\omega = 0.$$

For the nine independent constants in the equation last written may be so determined that the surface shall pass through nine given points, and therefore may coincide with any given quadric. In like manner (see *Conics*, p. 58) any ordinary x, y, z equations may be made homogeneous by the introduction of the

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* The reader will compare the corresponding discussion of the equation of the second degree (*Conics*, p. 119) and observe the identity of the methods now pursued and of many of the results obtained.

linear unit (which we shall call ω); and we shall frequently employ equations written in this form for the sake of greater symmetry in the results. We shall however for simplicity commence with x, y, z co-ordinates.

55. The co-ordinates are transformed to any parallel axes drawn through a point $x'y'z'$, by writing $x+x'$, $y+y'$, $z+z'$ for x, y, z respectively (Art. 16). The result of this substitution will be that the coefficients of the highest powers of the variables (a, b, c, l, m, n) will remain unaltered, that the new absolute term will be U' (where U' is the result of substituting x', y', z' for x, y, z in the given equation), that the new coefficient of x will be $2(ax'+ny'+mz'+p)$ or $\frac{dU'}{dx'}$, and in like manner that the new coefficients of y and z will be $\frac{dU'}{dy'}$ and $\frac{dU'}{dz'}$.

56. We can transform the general equation to polar co-ordinates by writing $x = A\rho$, $y = B\rho$, $z = C\rho$ (where, if the axes be rectangular, A, B, C are equal to $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, and if they are oblique (see note, p. 7) A, B, C are still quantities depending only on the angles the line makes with the axes) when the equation becomes

$$\rho^2 (aA^2 + bB^2 + cC^2 + 2lBC + 2mCA + 2nAB) + 2\rho (pA + qB + rC) + d = 0.$$

This being a quadratic gives two values for the length of the radius vector corresponding to any given direction; and since any point may be taken for origin it proves that *every right line meets a quadric in two points*, as was proved already (Art. 22).

57. Let us consider first the case where the origin is on the surface (and therefore $d = 0$), in which case one of the roots of the above quadratic is $\rho = 0$; and let us seek the condition that the radius vector should touch the surface at the origin. In this case obviously the second root of the quadratic will also vanish, and the required condition is therefore $pA + qB + rC = 0$.

If we multiply by ρ and replace $A\rho$, $B\rho$, $C\rho$ by x, y, z, this becomes

$$px + qy + rz = 0,$$

and evidently expresses that the radius vector lies in a certain fixed plane. And since A, B, C are subject to no restriction but that already written, *every* radius vector through the origin drawn in this plane touches the surface.

Hence we learn that at a given point on a quadric an infinity of tangent *lines* can be drawn, that these lie all in one plane which is called the *tangent plane* at that point; and that if the equation of the surface be written in the form $u_2 + u_1 = 0$, then $u_1 = 0$ is the equation of the tangent plane at the origin.

58. We can find by transformation of co-ordinates the equation of the tangent plane at any point $x'y'z'$ on the surface. For when we transform to this point as origin the absolute term vanishes, and the equation of the tangent plane is (Art. 56)

$$x\frac{dU'}{dx'} + y\frac{dU'}{dy'} + z\frac{dU'}{dz'} = 0,$$

or, transforming back to the old axes,

$$(x - x')\frac{dU'}{dx'} + (y - y')\frac{dU'}{dy'} + (z - z')\frac{dU'}{dz'} = 0.$$

This may be written in a more symmetrical form by the introduction of the linear unit ω, when, since it is now a homogeneous function, and since $x'y'z'$ is to satisfy the equation of the surface, we have

$$x'\frac{dU'}{dx'} + y'\frac{dU'}{dy'} + z'\frac{dU'}{dz'} + \omega'\frac{dU'}{d\omega'} = 2U' = 0.$$

Adding this to the equation last found, we have the equation of the tangent plane in the form

$$x\frac{dU'}{dx'} + y\frac{dU'}{dy'} + z\frac{dU'}{dz'} + \omega\frac{dU'}{d\omega'} = 0,$$

or, writing at full length,

$$x(ax' + ny' + mz' + p) + y(nx' + by' + lz' + q)$$
$$+ z(mx' + ly' + cz' + r) + px' + qy' + rz' + d = 0.$$

This equation, it will be observed, is symmetrical between xyz and $x'y'z'$, and may likewise be written

$$x'\frac{dU}{dx}+y'\frac{dU}{dy}+z'\frac{dU}{dz}+\omega'\frac{dU}{d\omega}=0.$$

59. *To find the point of contact of a tangent line or plane drawn through a given point $x'y'z'$ not on the surface.*

The equation last found expresses a relation between $xyz\omega$, the co-ordinates of any point on the tangent plane, and $x'y'z'\omega'$ its point of contact; and since now we wish to indicate that the former co-ordinates are given and the latter sought, we have only to remove the accents from the former and accentuate the latter co-ordinates, when we find that the point of contact must lie in the plane

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\omega\frac{dU'}{d\omega'}=0,$$

which is called the *polar plane* of the given point. Since the point of contact need satisfy no other condition, if we take *any* of the points where the polar plane meets the surface, the tangent plane at that point will pass through the given point; and the line joining the point of contact to the given point will be a tangent line to the surface. If all the points of intersection of the polar plane and the surface be joined to the given point, we shall have all the lines which can be drawn through that point to touch the surface, and the assemblage of these lines forms what is called the *tangent cone* through the given point.

N.B. In general a surface generated by right lines which all pass through the same point is called a *cone*, and the point through which the lines pass is called its *vertex*. A cylinder (see p. 15) is the limiting case of a cone when the vertex is infinitely distant.

60. The polar plane may be also defined as the locus of harmonic means of radii passing through the pole. In fact let us examine the locus of points of harmonic section of radii passing through the origin; then if ρ', ρ'' be the roots of the

quadratic of Art. 56, and ρ the radius vector of the locus, we are to have

$$\frac{2}{\rho} = \frac{1}{\rho'} + \frac{1}{\rho''} = -\frac{2(Ap + Bq + Cr)}{d},$$

or, returning to x, y, z co-ordinates,

$$px + qy + rz + d = 0;$$

but this is exactly the polar plane of the origin, as may be seen by making x', y', z' all $= 0$ in the equation written in full (Art. 58).

From this definition of the polar plane, it is evident that if a section of a surface be made by a plane passing through any point, the polar of that point with regard to the section will be the intersection of the plane of section with the polar plane of the given point. For the locus of harmonic means of *all* radii passing through the point, must include the locus of harmonic means of the radii which lie in the plane of section.

61. If the polar plane of any point A pass through B, then the polar plane of B will pass through A.

For since the equation of the polar plane is symmetrical with respect to xyz, $x'y'z'$, we get the same result whether we substitute the co-ordinates of the second point in the equation of the polar plane of the first, or *vice versâ.*

The intersection of the polar planes of A and of B will be a line which we shall call the polar line, with respect to the surface, of the line AB.

It is easy to see that the polar line of the line AB is the locus of the poles of all planes which can be drawn through the line AB.

62. If in the original equation we had not only $d = 0$, but also p, q, r each $= 0$, then the equation of the tangent plane found (Art. 58) becomes illusory, since every term vanishes and no single plane can be called the tangent plane at the origin. In fact the coefficient of ρ (Art. 56) vanishes whatever be the direction of ρ, and therefore *every* line drawn through the origin meets the surface in two consecutive points, and the origin is said to be a double point on the surface.

In the present case, the equation denotes a cone whose vertex is the origin, as in fact does every homogeneous equation in x, y, z. For if such an equation be satisfied by any co-ordinates x', y', z', it will also be satisfied by the co-ordinates Rx', Ry', Rz' (where R is any constant), that is to say, by the co-ordinates of every point on the line joining $x'y'z'$ to the origin. This line then lies wholly in the surface which must therefore consist of a series of right lines drawn through the origin.

The equation of the tangent plane at any point of the cone now under consideration may be written in either of the forms

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}=0,\quad x'\frac{dU}{dx}+y'\frac{dU}{dy}+z'\frac{dU}{dz}=0.$$

The former form (wanting an absolute term) shews that the tangent plane at every point on the cone passes through the origin; the latter form shews that the tangent plane at any point $x'y'z'$ touches the surface at every point of the line joining $x'y'z'$ to the vertex; for the equation will represent the same plane if we substitute Rx', Ry', Rz' for x', y', z'.

When the point $x'y'z'$ is not on the surface, the equation we have been last discussing represents the polar of that point, and it appears in like manner that the polar plane of every point passes through the vertex of the cone, and also that all points which lie on the same line passing through the vertex of a cone have the same polar plane.

To find the polar plane of any point with regard to a cone we need only take any section through that point, and take the polar line of the point with regard to that section; then the plane joining this polar line to the vertex will be the polar plane required. For it was proved (Art. 60) that the polar plane must contain the polar line, and it is now proved that the polar plane must contain the vertex.

63. We can easily find the condition that the general equation of the second degree should represent a cone. For if it does it will be possible by transformation of co-ordinates to

make the new p, q, r, d vanish. The co-ordinates of the new vertex must therefore (Art. 55) satisfy the conditions

$$\frac{dU'}{dx'}=0,\ \frac{dU'}{dy'}=0,\ \frac{dU'}{dz'}=0,\ U'=0,$$

which last combined with the others is equivalent to $\frac{dU'}{d\omega'}=0$. And if we eliminate x', y', z' from the four equations

$$ax'+ny'+mz'+p=0,$$
$$nx'+by'+lz'+q=0,$$
$$mx'+ly'+cz'+r=0,$$
$$px'+qy'+rz'+d=0,$$

we obtain the required condition in the form of the determinant

$$\begin{vmatrix} a, & n, & m, & p \\ n, & b, & l, & q \\ m, & l, & c, & r \\ p, & q, & r, & d \end{vmatrix}=0,$$

which, written at full length, is

$$l^2p^2+m^2q^2+n^2r^2-2mqnr-2nrlp-2lpmq+abcd+2alqr+2bmpr$$
$$+2cnpq+2dlmn-bcp^2-caq^2-abr^2-adl^2-bdm^2-cdn^2=0,$$

which is the *discriminant* of the given equation (see *Lessons on Higher Algebra*, p. 44).

64. Let us return now to the quadratic of Art. 56, in which d is not supposed to vanish, and let us examine the condition that the radius vector should be bisected at the origin. It is obviously necessary and sufficient that the coefficient of ρ in that quadratic should vanish, since we should then get for ρ values equal with opposite signs. The condition required then is

$$pA+qB+rC=0,$$

which multiplied by ρ shews that the radius vector must lie in the plane $px+qy+rz=0$. Hence (Art. 60) *every right line drawn through the origin in a plane parallel to its polar plane is bisected at the origin.*

65. If however we had $p=0$, $q=0$, $r=0$, then *every* line drawn through the origin would be bisected and the origin

would be called the *centre* of the surface. *Every quadric has in general one and but one centre.* For if we seek by transformation of co-ordinates to make the new $p, q, r = 0$, we obtain three equations, viz.

$$\frac{dU'}{dx'} = ax' + ny' + mz' + p = 0,$$

$$\frac{dU'}{dy'} = nx' + by' + lz' + q = 0,$$

$$\frac{dU'}{dz'} = mx' + ly' + cz' + r = 0,$$

which are sufficient to determine the three unknowns x', y', z'. The resulting values are $x' = \frac{\alpha}{\delta}$, $y' = \frac{\beta}{\delta}$, $z' = \frac{\gamma}{\delta}$, where

$$\alpha = p(l^2 - bc) + q(cn - lm) + r(bm - ln),$$

$$\beta = p(cn - lm) + q(m^2 - ca) + r(al - mn),$$

$$\gamma = p(bm - ln) + q(al - mn) + r(n^2 - ab),$$

$$\delta = abc + 2lmn - al^2 - bm^2 - cn^2,$$

or, if Δ be the discriminant,

$$2\alpha = \frac{d\Delta}{dp},\quad 2\beta = \frac{d\Delta}{dq},\quad 2\gamma = \frac{d\Delta}{dr},\quad \delta = \frac{d\Delta}{dd}.$$

If however $\delta = 0$ the co-ordinates of the centre become infinite and the surface has no finite centre. If we write the original equation $u_2 + u_1 + u_0 = 0$, it is evident that δ is the discriminant of u_2.*

* It is possible that the numerators of these fractions might vanish at the same time with the denominator, in which case the co-ordinates of the centre would become indeterminate, and the surface would have an infinity of centres. Thus if the three planes $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$ all pass through the same line, any point on this line will be a centre. The conditions that this should be the case may be written

$$\left\| \begin{matrix} a, & n, & m, & p \\ n, & b, & l, & q \\ m, & l, & c, & r \end{matrix} \right\| = 0,$$

the notation indicating that all the four determinants must = 0, which are got by erasing any of the vertical lines. We shall reserve the fuller discussion of these cases for the next chapter.

66. *To find the locus of the middle points of chords parallel to a given line* $\frac{x}{A} = \frac{y}{B} = \frac{z}{C}$.

If we transform the equation to any point on the locus as origin, the new p, q, r must fulfil the condition (Art. 64) $pA + qB + rC = 0$, and therefore (Art. 55) the equation of the locus is

$$A\frac{dU}{dx} + B\frac{dU}{dy} + C\frac{dU}{dz} = 0.$$

This denotes a plane through the intersection of the planes $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$, that is to say, through the centre of the surface. It is called the diametral plane conjugate to the given direction of the chords.

If $x'y'z'$ be any point on the radius vector drawn through the origin parallel to the given direction, the equation of the diametral plane may be written

$$x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} = 0.$$

If now we take the equation of the polar plane of Rx', Ry', Rz',

$$Rx'\frac{dU}{dx} + Ry'\frac{dU}{dy} + Rz'\frac{dU}{dz} + \frac{dU}{d\omega} = 0,$$

divide it by R, and then make R infinite, we see that the diametral plane is the polar of the point at infinity on a line drawn in the given direction, as we might also have inferred from geometrical considerations (see *Conics*, p. 272). In like manner, the centre is the pole of the plane at infinity, for if the origin be the centre its polar plane (Art. 60) is $d = 0$, which (Art. 29) represents a plane situated at an infinite distance.

In the case where the given surface is a cone, it is evident that the plane which bisects chords parallel to any line drawn through the vertex is the same as the polar plane of any point in that line. In fact it was proved that all points on the line have the same polar plane, therefore the polar of the point at infinity on that line is the same as the polar plane of any other point in it.

67. The plane which bisects chords parallel to the axis of x is found by making $B=0$, $C=0$ in the equation of Art. 66, to be

$$\frac{dU}{dx}=0, \text{ or } ax+ny+mz+p=0,^*$$

and this will be parallel to the axis of y, if $n=0$. But this is also the condition that the plane conjugate to the axis of y should be parallel to the axis of x. Hence *if the plane conjugate to a given direction be parallel to a second given line, the plane conjugate to the latter will be parallel to the former.*

When $n=0$ the axes of x and y are evidently parallel to a pair of conjugate diameters of the section by the plane of xy; and it is otherwise evident that the plane conjugate to each of two conjugate diameters of a section passes through the other. For the locus of middle points of *all* chords of the surface parallel to a given line must include the locus of the middle points of all such chords which are contained in a given plane.

Three diametral planes are said to be conjugate when each is conjugate to the intersection of the other two, and three diameters are said to be conjugate when each is conjugate to the plane of the other two. Thus we should obtain a system of three conjugate diameters by taking two conjugate diameters of any central section together with the diameter conjugate to the plane of that section. If we had in the equation $l=0$, $m=0$, $n=0$, it appears from the commencement of this article that the co-ordinate planes are parallel to three conjugate diametral planes.

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* It follows that the plane $x=0$ will bisect chords parallel to the axis of x, if $n=0$, $m=0$, $p=0$; or, in other words, if the original equation do not contain any odd power of x. But it is otherwise evident that this must be the case in order that for any assigned values of y and z we may obtain equal and opposite values of x.

When the surface is a cone it is evident from what was said (Arts. 62, 66) that a system of three conjugate diameters meets any plane section in points such that each is the pole with respect to the section of the line joining the other two.

68. A diametral plane is said to be principal if it be perpendicular to the chords to which it is conjugate.

The axes being rectangular, and A, B, C the direction-cosines of a chord, we have seen (Art. 66) that the corresponding diametral plane is

$$A\ (ax+ny+mz+p)+B\,(nx+by+lz+q)+C(mx+ly+cz+r)=0,$$

and this will be perpendicular to the chord, if (Art. 42) the coefficients of x, y, z be respectively proportional to A, B, C. This gives us the three equations

$$Aa+Bn+Cm=RA,\ An+Bb+Cl=RB,\ Am+Bl+Cc=RC.$$

From these equations which are linear in A, B, C, we can eliminate A, B, C, when we obtain the determinant

$$\begin{vmatrix} a-R, & n, & m \\ n, & b-R, & l \\ m, & l, & c-R \end{vmatrix}=0,$$

which expanded gives a cubic for the determination of R, viz.

$$R^3-R^2(a+b+c)+R\,(ab+bc+ca-l^2-m^2-n^2)$$
$$-(abc+2lmn-al^2-bm^2-cn^2)=0.$$

And the three values hence found for R being successively substituted in the preceding equations enable us to determine the corresponding values of A, B, C. Hence *a quadric has in general three principal diametral planes*, the three diameters perpendicular to which are called the *axes* of the surface. We shall discuss this equation more fully in the next chapter.

Ex. To find the principal planes of

$$7x^2+6y^2+5z^2-4xy-4yz=6.$$

The cubic for R is

$$R^3-18R^2+99R-162=0,$$

whose roots are 3, 6, 9. Now our three equations are

$$7A-2B=RA,\ -2A+6B-2C=RB,\ -2B+5C=RC.$$

If in these we substitute $R = 3$, we find $2A = B = C$. Multiplying by ρ, and substituting x for $A\rho$, &c., we get for the equations of one of the axes $2x = y = z$. And the plane drawn through the origin, (which is the centre) perpendicular to this line, is $x + 2y + 2z = 0$. In like manner the other two principal planes are $2x - 2y + z = 0$, $2x + y - 2z = 0$.*

69. *The sections of a quadric by parallel planes are similar to each other.*

Since any plane may be taken for the plane of xy, it is sufficient to consider the section made by it, which is found by putting $z = 0$ in the equation of the surface. But the section by any parallel plane is found by transforming the equation to parallel axes through any new origin, and then making $z = 0$. And since the coefficients of the highest terms are unaltered by such transformation, we must obtain in every case the same coefficients for x^2, xy, and y^2, and the curves are therefore similar.

If we retain the planes yz and zx, and transform the plane xy parallel to itself, the section by this plane is got at once by writing $z = c$ in the equation of the surface, since it is evident that it is the same thing whether we write $z + c$ for z, and then make $z = 0$, or whether we write at once $z = c$.

It is easy to prove algebraically, that the locus of centres of parallel sections is the diameter conjugate to their plane, as is geometrically evident.

70. If ρ', ρ'' be the roots of the quadratic of Art. 56, their product $\rho'\rho''$ is $= d$ divided by the coefficient of ρ^2. But if we transform to parallel axes, and consider a radius vector

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* It is proved (*Lessons on Higher Algebra*, p. 112) that if U denote the terms of highest degree in the equation, and S denote

$$(bc - l^2)\,x^2 + (ca - m^2)\,y^2 + (ab - n^2)\,z^2 + 2\,(ef - al)\,yz + 2(fd - bm)\,zx + 2\,(de - cn)\,xy,$$

then the equation of the three principal planes, the centre being origin, is denoted by the determinant

$$\begin{vmatrix} x, & y, & z \\ \frac{dU}{dx}, & \frac{dU}{dy}, & \frac{dU}{dz} \\ \frac{dS}{dx}, & \frac{dS}{dy}, & \frac{dS}{dz} \end{vmatrix} = 0.$$

drawn parallel to the first direction, the coefficient of ρ^2 remains unchanged, and the product is proportional to the new d. Hence if through two given points A, B, any parallel chords drawn meeting the surface in points R, R'; S, S', then the products $RA.AR'$, $SB.BS'$ are to each other in a constant ratio, namely, $U' : U''$ where U', U'' are the results of substituting the co-ordinates of A and of B in the given equation.

71. We shall conclude this chapter by shewing how the theorems already deduced from the discussion of lines passing through the origin might have been derived by a more general process, such as that employed (*Conics*, Art. 150). For symmetry we use homogeneous equations with four variables.

To find the points where a given quadric is met by the line joining two given points $x'y'z'\omega'$, $x''y''z''\omega''$.

Let us take as our unknown quantity the ratio $l : m$, in which the joining line is cut at the point where it meets the quadric, then (Art. 8) the co-ordinates of that point are proportional to

$$mx' + lx'', \quad my' + ly'', \quad mz' + lz'', \quad m\omega' + l\omega'';$$

and if we substitute these values in the equation of the surface, we get for the determination of $l : m$, a *quadratic*

$$m^2 U' + lmP + l^2 U'' = 0.$$

The coefficients of l^2 and m^2 are easily seen to be the results of substituting in the equation of the surface the co-ordinates of each of the points, while the coefficient of lm may be seen (by Taylor's theorem, or otherwise) to be capable of being written in either of the forms

$$x' \frac{dU''}{dx''} + y' \frac{dU''}{dy''} + z' \frac{dU''}{dz''} + \omega' \frac{dU''}{d\omega''},$$

or

$$x'' \frac{dU'}{dx'} + y'' \frac{dU'}{dy'} + z'' \frac{dU'}{dz'} + \omega'' \frac{dU'}{d\omega'}.$$

Having found from this quadratic the values of $l : m$, substituting each of them in the values $\frac{mx' + lx''}{l+m}$ &c., we find the co-ordinates of the points where the quadric is met by the given line.

72. If $x'y'z'\omega'$ be on the surface, then $U'=0$, and one of the roots of the last quadratic is $l=0$, which corresponds to the point $x'y'z'\omega'$, as evidently ought to be the case. In order that the second root should also be $l=0$, we must have $P=0$. If then the line joining $x'y'z'\omega'$ to $x''y''z''\omega''$ touch the surface at the former point, the co-ordinates of the latter must satisfy the equation

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\omega\frac{dU'}{d\omega'}=0,$$

and since $x''y''z''\omega''$ may be *any* point on *any* tangent line through $x'y'z'\omega'$ it follows that every such tangent lies in the plane whose equation has been just written.

73. If $x'y'z'\omega'$ be *not* on the surface, and yet the relation $P=0$ be satisfied, the quadratic of Art. 71 takes the form $m^2U'+l^2U''=0$, which gives values of $l:m$, equal with opposite signs. Hence the line joining the given points is cut by the surface externally and internally in the same ratio; that is to say, is cut harmonically. It follows then that the locus of points of harmonic section of radii drawn through $x'y'z'\omega'$ is the polar plane

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\omega\frac{dU'}{d\omega'}=0.$$

74. In general if the line joining the two points touch the surface, the quadratic of Art. 71 must have equal roots, and the co-ordinates of the two points must be connected by the relation $4U'U''=P^2$. If the point $x'y'z'\omega'$ be fixed; this relation ought to be fulfilled if the other point lie on any of the tangent lines which can be drawn through it. Hence the cone generated by all these tangent lines will have for its equation $4UU'=P^2$, where

$$P=x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\omega\frac{dU'}{d\omega'}.$$

Ex. To find the equation of the tangent cone from the point $x'y'z'$ to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Ans. $\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1\right)^2.$

75. *To find the condition that the plane $\alpha x+\beta y+\gamma z+\delta\omega$ should touch the surface given by the general equation.*

If x, y, z, ω be the co-ordinates of the point of contact, and λ an indeterminate multiplier, we have (Art. 58)

$$\begin{aligned}\lambda\alpha &= ax + ny + mz + p\omega,\\ \lambda\beta &= nx + by + lz + q\omega,\\ \lambda\gamma &= mx + ly + cz + r\omega,\\ \lambda\delta &= px + qy + rz + d\omega,\end{aligned}$$

from which equations, together with $\alpha x+\beta y+\gamma z+\delta\omega=0$, we have to eliminate x, y, z, ω. But solving for x, y, z, ω from these equations, we have (*Higher Algebra*, p. 15)*

$$\begin{aligned}\frac{\Delta}{\lambda}x &= \frac{d\Delta}{da}\alpha + \tfrac{1}{2}\frac{d\Delta}{dn}\beta + \tfrac{1}{2}\frac{d\Delta}{dm}\gamma + \tfrac{1}{2}\frac{d\Delta}{dp}\delta,\\ \frac{\Delta}{\lambda}y &= \tfrac{1}{2}\frac{d\Delta}{dn}\alpha + \frac{d\Delta}{db}\beta + \tfrac{1}{2}\frac{d\Delta}{dl}\gamma + \tfrac{1}{2}\frac{d\Delta}{dq}\delta,\\ \frac{\Delta}{\lambda}z &= \tfrac{1}{2}\frac{d\Delta}{dm}\alpha + \tfrac{1}{2}\frac{d\Delta}{dl}\beta + \frac{d\Delta}{dc}\gamma + \tfrac{1}{2}\frac{d\Delta}{dr}\delta,\\ \frac{\Delta}{\lambda}\omega &= \tfrac{1}{2}\frac{d\Delta}{dp}\alpha + \tfrac{1}{2}\frac{d\Delta}{dq}\beta + \tfrac{1}{2}\frac{d\Delta}{dr}\gamma + \frac{d\Delta}{dd}\delta,\end{aligned}$$

where Δ is the discriminant. Substituting which values in $\alpha x+\beta y+\gamma z+d\omega=0$, we get

$$\alpha^2\frac{d\Delta}{da}+\beta^2\frac{d\Delta}{db}+\gamma^2\frac{d\Delta}{dc}+\delta^2\frac{d\Delta}{dd}+\beta\gamma\frac{d\Delta}{dl}+\gamma\alpha\frac{d\Delta}{dm}+\alpha\beta\frac{d\Delta}{dn}$$
$$+\alpha\delta\frac{d\Delta}{dp}+\beta\delta\frac{d\Delta}{dq}+\gamma\delta\frac{d\Delta}{dr}=0,$$

which is the required relation.

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* It is there proved that the coefficient of β, for example, is the differential of Δ with regard to n on the supposition that the constituents of the determinant Δ are all different. But it is easy to see that the true differential is double this, since the determinant has two symmetrical constituents each $= n$.

This condition may also be written

$$\begin{vmatrix} a, & n, & m, & p, & \alpha \\ n, & b, & l, & q, & \beta \\ m, & l, & c, & r, & \gamma \\ p, & q, & r, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta & \end{vmatrix} = 0.$$

76. The condition that the surface should be touched by any line

$$\alpha x + \beta y + \gamma z + \delta\omega = 0, \quad \alpha' x + \beta' y + \gamma' z + \delta'\omega = 0,$$

is found by eliminating two of the variables between the equations of the line and of the quadric, and forming the condition that the resulting quadratic should have equal roots. The result contains the coefficients of the quadric in the second degree, and is also a quadratic function of the determinants $(\alpha\beta' - \beta\alpha')$, $(\alpha\gamma' - \gamma\alpha')$, &c. Writing these $(\alpha\beta')$, $(\alpha\gamma')$, &c. the result is found to be

$$\Sigma (ab - n^2)(\gamma\delta')^2 + 2\Sigma (mn - al)(\beta\delta')(\gamma\delta')$$
$$+ 2\Sigma nr \{(\alpha\delta')(\gamma\beta') - (\alpha\gamma')(\beta\delta')\},$$

where the sum includes all terms of like form obtained by symmetrical interchange of letters. This condition may also be written

$$\begin{vmatrix} a, & n, & m, & p, & \alpha, & \alpha' \\ n, & b, & l, & q, & \beta, & \beta' \\ m, & l, & c, & r, & \gamma, & \gamma' \\ p, & q, & r, & d, & \delta, & \delta' \\ \alpha, & \beta, & \gamma, & \delta & & \\ \alpha', & \beta', & \gamma', & \delta' & & \end{vmatrix} = 0.$$

If in the condition of the last article we write $\alpha + \lambda\alpha'$ for α, &c., and then form the condition that the equation in λ should have equal roots, the result will be the condition of this article multiplied by the discriminant. For the two planes which can be drawn through a given line to touch a quadric, will coincide either if the line touches the quadric or if the surface has a double point.

CHAPTER V.

CLASSIFICATION OF QUADRICS.

77. Our object in this chapter is the reduction of the general equation of the second degree to the simplest form of which it is susceptible, and the classification of the different surfaces which it is capable of representing.

Let us commence by supposing the quantity which we called δ (Art. 65) *not* to be $=0$. By transforming the equation to parallel axes through the centre, the coefficients p, q, r are made to vanish, and the equation becomes

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy + d' = 0,$$

where d' is the result of substituting the co-ordinates of the centre in the equation of the surface. Remembering that

$$2U' = x'\frac{dU'}{dx'} + y'\frac{dU'}{dy'} + z'\frac{dU'}{dz'} + \omega'\frac{dU'}{d\omega'} = 0,$$

and that the co-ordinates of the centre make the first three of the latter terms to vanish, it is easy to calculate that

$$d' = \frac{p\alpha + q\beta + r\gamma}{\delta} + d = \frac{\Delta}{\delta},$$

where Δ is the discriminant of the equation.

78. Having by transformation to parallel axes made the coefficients of x, y, z to vanish, we can next make the coefficients of yz, zx, and xy vanish by changing the direction of the axes, retaining the new origin; and so reduce the equation to the form

$$Ax^2 + By^2 + Cz^2 = D.*$$

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* D is of course $= \frac{-\Delta}{\delta}$. I suppose in what follows that D is positive. If it were $= 0$, the surface would represent a cone (Art. 63). If it were negative, we should only have to change all the signs in the equation.

It is easy to shew from Art. 17 that we have constants enough at our disposal to effect this reduction, but the method we shall follow is the same as that adopted, *Conics*, p. 141, namely, to prove that there are certain functions of the coefficients which remain unaltered when we transform from one rectangular system to another, and by the help of these relations to obtain the actual values of the new A, B, C.

Let us suppose that by using the most general transformation which is of the form

$$x = \lambda \underline{x} + \mu \underline{y} + \nu \underline{z},\quad y = \lambda' \underline{x} + \mu' \underline{y} + \nu' \underline{z},\quad z = \lambda'' \underline{x} + \mu'' \underline{y} + \nu'' \underline{z},$$

that $\qquad ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy$

becomes $\qquad a'\underline{x}^2 + b'\underline{y}^2 + c'\underline{z}^2 + 2l'\underline{yz} + 2m'\underline{zx} + 2n'\underline{xy},$

which we write for shortness $U = \underline{U}$. And if both systems of co-ordinates be rectangular, we must have

$$x^2 + y^2 + z^2 = \underline{x}^2 + \underline{y}^2 + \underline{z}^2,$$

which we write for shortness $S = \underline{S}$. Then if R be any constant, we must have $U + RS = \underline{U} + R\underline{S}$. And if the first side be resolvable into factors, so must also the second. The discriminants of $U + RS$ and of $\underline{U} + R\underline{S}$ must therefore vanish for the same values of R. But the first discriminant is

$$R^3 - R^2(a + b + c) + R(ab + bc + ca - l^2 - m^2 - n^2) - (abc + 2lmn - al^2 - bm^2 - cn^2).$$

Equating then the coefficients of the different powers of R to the corresponding coefficients in the second, we learn that if the equation be transformed from one set of rectangular axes to another, we must have

$$a + b + c = a' + b' + c',$$

$$bc + ca + ab - l^2 - m^2 - n^2 = b'c' + c'a' + a'b' - l'^2 - m'^2 - n'^2,$$

$$abc + 2lmn - al^2 - bm^2 - cn^2 = a'b'c' + 2l'm'n' - a'l'^2 - b'm'^2 - c'n'^2.\text{*}$$

* There is no difficulty in forming the corresponding equations for oblique co-ordinates We should then substitute for S (see Art. 18),

$$x^2 + y^2 + z^2 - 2yz\cos\lambda - 2zx\cos\mu - 2xy\cos\nu,$$

and proceeding exactly as in the text, we should form a cubic in R, the coefficients of which would bear to each other ratios unaltered by transformation.

79. The above three equations at once enable us to transform the equation so that the new l, m, n shall vanish, since they determine the coefficients of the cubic equation whose roots are the new a, b, c. This cubic is then

$$*A^3 - (a+b+c)A^2 + (bc+ca+ab-l^2-m^2-n^2)A \\ - (abc+2lmn-al^2-bm^2-cn^2) = 0,$$

which may also be written

$$(A-a)(A-b)(A-c)-l^2(A-a)-m^2(A-b)-n^2(A-c)-2lmn=0.$$

We give here Cauchy's proof that the roots of this equation are all real. The proof of a more general theorem, in which this is included, will be found in *Lessons on Higher Algebra*, Lesson XV.

Let the cubic be written in the form

$$(A-a)\{(A-b)(A-c)-l^2\}-m^2(A-b)-n^2(A-c)-2lmn=0.$$

Let α, β be the values of A which make $(A-b)(A-c)-l^2=0$, and it is easy to see that the greater of these roots α is greater than either b or c, and that the less root β is less than either.† Then if we substitute in the given cubic $A=\alpha$, it reduces to

$$-\{(\alpha-b)m^2+2lmn+(\alpha-c)n^2\},$$

and since the quantity within the brackets is a perfect square in virtue of the relation $(\alpha-b)(\alpha-c)=l^2$, the result of substitution is essentially negative. But if we substitute $A=\beta$, the result is

$$(b-\beta)m^2-2lmn+(c-\beta)n^2,$$

which is also a perfect square, and positive. Since then, if we substitute $A=\infty$, $A=\alpha$, $A=\beta$, $A=-\infty$, the results are alternately positive and negative, the equation has three real roots lying within the limits just assigned. The three roots are the coefficients of x^2, y^2, z^2 in the transformed equation, but

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* This is the same cubic as that found, Art. 68, as the reader will easily see ought to be the case.

† We may see this either by actually solving the equation, or by substituting successively $A=\infty$, $A=b$, $A=c$, $A=-\infty$, when we get results $+$, $-$, $-$, $+$, shewing that one root is greater than b, and the other less than c.

it is of course arbitrary which shall be the coefficient of x^2 or of y^2, since we may call whichever axis we please the axis of x.

80. Quadrics are classified according to the signs of the roots of the preceding cubic.

I. First, let all the roots be positive, and the equation can be transformed to

$$Ax^2 + By^2 + Cz^2 = D.$$

The surface makes real intercepts on each of the three axes, and if the intercepts be a, b, c, it is easy to see that the equation of the surface may be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

As it is arbitrary which axis we take for the axis of x, we suppose the axes so taken that a the intercept on the axis of x may be the longest, and c the intercept on the axis of z may be the shortest.

The equation transformed to polar co-ordinates is

$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} + \frac{\cos^2\gamma}{c^2},$$

which (remembering that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$) may be written in either of the forms

$$\frac{1}{\rho^2} = \frac{1}{a^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos^2\beta + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\cos^2\gamma$$

$$= \frac{1}{c^2} - \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\cos^2\alpha - \left(\frac{1}{c^2} - \frac{1}{b^2}\right)\cos^2\beta,$$

from which it is easy to see that a is the maximum and c the minimum value of the radius vector. The surface is consequently limited in every direction, and is called an *ellipsoid*.

Every section of it is therefore necessarily also an ellipse. Thus the section by any plane $z = R$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{R^2}{c^2}$, and we shall obviously cease to have any real section when R is greater than c. The surface therefore lies altogether within the planes $z = \pm c$. Similarly for the other axes.

If two of the coefficients be equal (for instance, $a = b$), then all sections by planes parallel to the plane of xy are circles, and the surface is one *of revolution*, generated by the revolution of an ellipse round its axis major or axis minor, according as it is the two greater or the two less coefficients which are equal. These surfaces are also sometimes called the *prolate* and the *oblate* spheroid.

If all three coefficients be equal, the surface is a sphere.

81. II. Secondly, let one root of the cubic be negative. We may then write the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

where a is supposed greater than b, and where the axis of z evidently does not meet the surface in real points. Using the polar equation

$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{a^2} + \frac{\cos^2\beta}{b^2} - \frac{\cos^2\gamma}{c^2},$$

it is evident that the radius vector meets the surface or not according as the right-hand side of the equation is positive or negative; and that putting it $= 0$, (which corresponds to $\rho = \infty$) we obtain a system of radii which separate the diameters which meet the surface from those that do not. We obtain thus the equation of the *asymptotic cone*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Sections of the surface parallel to the plane of xy are ellipses; those parallel to either of the other two principal planes are hyperbolas. The equation of the elliptic section by the plane $z = R$ being $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{R^2}{c^2}$, we see that we get a real section whatever be the value of R, and therefore that the surface is continuous. It is called the *Hyperboloid of one sheet*.

If $a = b$, it is a surface of revolution.

82. III. Thirdly, let two of the roots be negative, and the equation may be written

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The sections parallel to two principal planes are hyperbolas, while that parallel to the plane of yz is an ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{R^2}{a^2} - 1.$$

It is evident that this will not be real as long as R is within the limits $\pm a$, but that any plane $x = R$ will meet the surface in a real section provided that R is outside these limits. No portion of the surface will then lie between the planes $x = \pm a$, but the surface will consist of two separate portions outside these boundary planes. This surface is called the *Hyperboloid of two sheets*. It is of revolution if $b = c$.

By considering the surfaces of revolution, the reader can easily form an idea of the distinction between the two kinds of hyperboloids. Thus if a common hyperbola revolve round its transverse axis the surface generated will evidently consist of two separate portions; but if it revolve round the conjugate axis it will consist but of one portion, and will be a case of the hyperboloid of one sheet.

IV. If the three roots of the cubic be negative, the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$$

can evidently be satisfied by no real values of the co-ordinates.

V. When the absolute term vanishes, we have the cone as a limiting case of the above. Forms I. and IV. then become

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0,$$

which can be satisfied by no real values of the co-ordinates, while forms II. and III. give the equation of the cone in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

The forms already enumerated exhaust all the varieties of central surfaces.

Ex. 1. $$7x^2 + 6y^2 + 5z^2 - 4yz - 4xy = 6.$$

The discriminating cubic is

$$A^3 - 18A^2 + 99A - 162 = 0,$$

and the transformed equation

$$x^2 + 2y^2 + 3z^2 = 2,$$

an ellipsoid.

Ex. 2. $$11x^2 + 10y^2 + 6z^2 - 12xy - 8yz + 4zx = 12.$$

Discriminating cubic

$$A^3 - 27A^2 + 180A - 324 = 0.$$

Transformed equation

$$x^2 + 2y^2 + 6z^2 = 4,$$

an ellipsoid.

Ex. 3. $$7x^2 - 13y^2 + 6z^2 + 24xy + 12yz - 12zx = \pm 84.$$

Discriminating cubic

$$A^3 - 343A - 2058 = 0.$$

Transformed equation

$$x^2 + 2y^2 - 3z^2 = \pm 12,$$

a hyperboloid of one or of two sheets, according to the sign of the last term.

Ex. 4. $$2x^2 + 3y^2 + 4z^2 + 6xy + 4yz + 8zx = 8.$$

Discriminating cubic is

$$A^3 - 9A^2 - 3A + 20 = 0.$$

By Des Cartes's rule of signs this equation has two positive and one negative root, and therefore represents a hyperboloid of one sheet.

83. Let us proceed now to the case where we have $\delta = 0$. In this case we have seen (Art. 65) that it is generally impossible by any change of origin to make the terms of the first degree in the equation to vanish. But it is in general quite indifferent whether we commence, as in Art. 65, by transforming to a new origin, and so remove the coefficients of x, y, z, or whether we first, as in this chapter, transform to new axes retaining the same origin, and so reduce the terms of highest degree to the form $Ax^2 + By^2 + Cz^2$. When $\delta = 0$, the first transformation being impossible we must commence with the latter. And since the absolute term of the cubic of Art. 79 is δ, one of its roots, that is to say, one of the three quantities A, B, C must in this case $= 0$. The terms of the second degree are therefore reducible to the form $Ax^2 \pm By^2$. This is otherwise evident from the consideration that $\delta = 0$ is the condition that the terms of highest degree should be

resolvable into two real or imaginary factors, in which case they may obviously be also expressed as the difference or sum of two squares. In this way the equation is reduced to the form

$$Ax^2 \pm By^2 + 2p'x + 2q'y + 2r'z + d = 0.$$

We can then, by transforming to a new origin, make the coefficients of x and y to vanish, but not that of z, and the equation takes the form

$$Ax^2 \pm By^2 + 2r'z + d' = 0.$$

I. Let $r' = 0$. The equation then does not contain z, and therefore (Art. 24) represents a cylinder which is elliptic or hyperbolic, according as A and B have the same or different signs. Since the terms of the first degree are absent from the equation the origin is a centre, but so is also equally every other point on the axis of z, which is called the axis of the cylinder. The possibility of the surface having a line of centres is indicated by both numerator and denominator vanishing in the co-ordinates of the centre, Art. 65 (see note p. 42).

If it happened that not only r' but also $d' = 0$, the surface would reduce to two intersecting planes.

II. If r' be not $= 0$, we can by a change of origin make the absolute term vanish, and reduce the equation to the form

$$Ax^2 \pm By^2 + 2r'z' = 0.$$

Let us first suppose the sign of B to be positive. In this case while the sections by planes parallel to the planes of xz or yz are parabolas, those parallel to the plane of xy are ellipses, and the surface is called the *Elliptic Paraboloid*. It evidently extends only in one direction, since the section by any plane $z = c$ is $Ax^2 + By^2 = -2cr'$, and will not be real unless the right-hand side of the equation is positive. When therefore r' is positive, the surface lies altogether on the negative side of the plane of xy, and when r' is negative, on the positive side.

III. If the sign of B be negative, the sections by planes parallel to that of xy are hyperbolas, and the surface is called a *Hyperbolic Paraboloid*. This surface extends indefinitely in both directions. The section by the plane of xy is a pair of right lines.

IV. If $B=0$, that is, if *two* roots of the discriminating cubic vanish, the equation takes the form

$$Ax^2+2q'y+2r'z+d=0,$$

but by changing the axes y and z in their own plane, and taking for new co-ordinate planes the plane $q'y+r'z$ and a plane perpendicular to it through the axis of x, the equation is brought to the form

$$Ax^2+q^2y+d=0,$$

which (Art. 24) represents a cylinder whose base is a parabola.

V. If we have also $q'=0$, $r'=0$, the equation $Ax^2+d=0$ being resolvable into factors would evidently denote a pair of parallel planes.

84. The actual work of reducing the equation of a paraboloid to the form $Ax^2+By^2+2Rz=0$ is shortened by observing that the discriminant is an invariant; that is to say, a function of the coefficients which is not altered by transformation of co-ordinates (*Higher Algebra*, p. 51). Now the discriminant of Ax^2+By^2+2Rz is simply ABR^2, which is therefore equal to the discriminant of the given equation. And as A and B are known, being the two roots of the discriminating cubic which do not vanish, R is also known. The calculation of the discriminant is facilitated by observing that it is in this case a perfect square (*Higher Algebra*, p. 124). Thus let us take the example

$$5x^2-y^2+z^2+6zx+4xy+2x+4y+6z=8.$$

Then the discriminating cubic is $\lambda^3-5\lambda^2-14\lambda=0$ whose roots are 0, 7, and -2. We have therefore $A=7$, $B=-2$. The discriminant in this case is $(p+2q-3r)^2$, or putting in the actual values $p=1$, $q=2$, $r=3$ is 16. Hence we have $14R^2=16$, $R=\frac{4}{\sqrt{(14)}}$, and the reduced equation is $7x^2-2y^2=\frac{8z}{\sqrt{(14)}}$.

If we had not availed ourselves of the discriminant, we should have proceeded as in Art. 68 to find the principal planes answering to the roots 0, 7, -2 of the discriminating cubic, and should have found

$$x+2y-3z=0,\quad 4x+y+2z=0,\quad x-2y-z=0.$$

Since the new co-ordinates are the perpendiculars on these planes, we are to take

$$4x+y+2z=X\sqrt{(21)},\quad x-2y-z=Y\sqrt{(6)},\quad x+2y-3z=Z\sqrt{(14)},$$

from which we can express x, y, z in terms of the new co-ordinates, and the transformed equation becomes

$$7x^2-2y^2+\frac{24x}{\sqrt{(21)}}-2\sqrt{(6)}\,y-\frac{8}{\sqrt{(14)}}\,z=8,$$

which finally transformed to parallel axes through a new origin gives the same reduced equation as before.

If in the preceding example the coefficients p, q, r had been so taken as to fulfil the relation $p+2q-3r=0$, the discriminant would then vanish, but the reduction could be effected with even greater facility as the terms in x, y, z could then be expressed in the form

$$(4x+y+2z)+\lambda\,(x-2y-z).$$

Thus the equation

$$5x^2-y^2+z^2+6zx+4xy+2x+2y+2z=8$$

may be written in the form

$$(4x+y+2z)^2-(x-2y-z)^2+2\,(4x+y+2z)-2\,(x-2y-z)=24,$$

which transformed as before becomes

$$21x^2-6y^2+2x\sqrt{(21)}-2y\sqrt{(6)}=24,$$

and the remainder of the reduction presents no difficulty.

CHAPTER VI.

PROPERTIES OF QUADRICS DEDUCED FROM SPECIAL FORMS OF THEIR EQUATIONS.

CENTRAL SURFACES.

85. WE proceed now to give some properties of central quadrics derived from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. This will include properties of the hyperboloids as well as of the ellipsoids if we suppose the signs of b^2 and of c^2 to be indeterminate.

The equation of the polar plane of the point $x'y'z'$ (or of the tangent plane, if that point be on the surface) is (Art. 59)

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

The perpendicular from the origin on the tangent plane is therefore (Art. 32) given by the equation

$$\frac{1}{p^2} = \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}.$$

And the angles α, β, γ which the perpendicular makes with the axes are given by the equations

$$\cos\alpha = \frac{px'}{a^2}, \quad \cos\beta = \frac{py'}{b^2}, \quad \cos\gamma = \frac{pz'}{c^2},$$

as is evident by multiplying the equation of the tangent plane by p, and comparing it with the form

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p.$$

From the preceding equations we can also immediately get an expression for the perpendicular in terms of the angles it makes with the axes, viz.

$$p^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma.$$

86. *To find the condition that the plane* $\alpha x + \beta y + \gamma z + \delta = 0$ *should touch the surface.*

Comparing this with the equation $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$, we have at once

$$\frac{x'}{a}=-\frac{a\alpha}{\delta},\quad \frac{y'}{b}=-\frac{b\beta}{\delta},\quad \frac{z'}{c}=-\frac{c\gamma}{\delta},$$

and the required condition is

$$a^2\alpha^2+b^2\beta^2+c^2\gamma^2=\delta^2.$$

In the same way, the condition that the plane $\alpha x+\beta y+\gamma z$ should touch the cone $\frac{x^2}{c^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ is

$$a^2\alpha^2+b^2\beta^2-c^2\gamma^2=0.$$

These might also be deduced as particular cases of Art. 75.

87. The normal is a perpendicular to the tangent plane erected at the point of contact. Its equations are obviously

$$\frac{a^2}{x'}(x-x')=\frac{b^2}{y'}(y-y')=\frac{c^2}{z'}(z-z').$$

Let the common value of these be R, then we have

$$x-x'=\frac{Rx'}{a^2},\quad y-y'=\frac{Ry'}{b^2},\quad z-z'=\frac{Rz'}{c^2}.$$

Squaring and adding we find that the length of the normal between $x'y'z'$, and any point on it xyz is $\frac{R}{p}$. But if xyz be taken as the point where the normal meets the plane of xy, we have $z=0$, and the last of the three preceding equations gives $R=c^2$. Hence the length of the intercept on the normal between the point of contact and the plane of xy is $\frac{c^2}{p}$.

88. The sum of the squares of the reciprocals of any three rectangular diameters is constant. This follows immediately from adding the equations

$$\frac{1}{\rho^2}=\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2},$$

$$\frac{1}{\rho'^2}=\frac{\cos^2\alpha'}{a^2}+\frac{\cos^2\beta'}{b^2}+\frac{\cos^2\gamma'}{c^2},$$

$$\frac{1}{\rho''^2}=\frac{\cos^2\alpha''}{a^2}+\frac{\cos^2\beta''}{b^2}+\frac{\cos^2\gamma''}{c^2},$$

whence since $\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$, &c., we have

$$\frac{1}{\rho^2} + \frac{1}{\rho'^2} + \frac{1}{\rho''^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

89. In like manner the sum of the squares of three perpendiculars on tangent planes, mutually at right angles, is constant, as appears from adding the equations

$$p^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma,$$
$$p'^2 = a^2\cos^2\alpha' + b^2\cos^2\beta' + c^2\cos^2\gamma',$$
$$p''^2 = a^2\cos^2\alpha'' + b^2\cos^2\beta'' + c^2\cos^2\gamma''.$$

Hence the locus of the intersection of three tangent planes which cut at right angles in a sphere; since the square of its distance from the centre of the surface is equal to the sum of the squares of the three perpendiculars and therefore to $a^2 + b^2 + c^2$.

CONJUGATE DIAMETERS.

90. The equation of the diametral plane conjugate to the diameter drawn to the point $x'y'z'$ on the surface is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0, \text{ (Art. 68).}$$

It is therefore parallel to the tangent plane at that point. Since any diameter in the diametral plane is conjugate to that drawn to the point $x'y'z'$, it is manifest that when two diameters are conjugate to each other, their direction-cosines are connected by the relation

$$\frac{\cos\alpha\cos\alpha'}{a^2} + \frac{\cos\beta\cos\beta'}{b^2} + \frac{\cos\gamma\cos\gamma'}{c^2} = 0.$$

Since the equation of condition here given is not altered if we write Ka^2, Kb^2, Kc^2 for a^2, b^2, c^2, it is evident that two lines which are conjugate diameters for any surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ are also conjugate diameters for any similar surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = K.$$

And by making $K=0$ we see in particular that any surface and its asymptotic cone have common systems of conjugate diameters.

Following the analogy of methods employed in the case of conics we may denote the co-ordinates of any point on the ellipsoid, by $a\cos\lambda$, $b\cos\mu$, $c\cos\nu$, where λ, μ, ν are the direction-angles of some line; that is to say, are such that $\cos^2\lambda+\cos^2\mu+\cos^2\nu=1$. In this method the two lines answering to two conjugate diameters are at right angles to each other; for writing $\cos\alpha=a\cos\lambda$, $\cos\alpha'=a\cos\lambda'$, &c., the relation last written becomes

$$\cos\lambda\cos\lambda'+\cos\mu\cos\mu'+\cos\nu\cos\nu'=0.$$

91. *The sum of the squares of a system of three conjugate semi-diameters is constant.*

For the square of the length of any semi-diameter $x'^2+y'^2+z'^2$ is, when expressed in terms of λ, μ, ν,

$$a^2\cos^2\lambda\ +b^2\cos^2\mu\ +c^2\cos^2\nu,$$

which when added to

$$a^2\cos^2\lambda'\ +b^2\cos^2\mu'\ +c^2\cos^2\nu',$$
$$a^2\cos^2\lambda''+b^2\cos^2\mu''+c^2\cos^2\nu''$$

is equal to $a^2+b^2+c^2$; since λ, μ, ν, &c. are the direction-angles of three lines mutually at right angles.

92. *The parallelopiped whose edges are three conjugate semi-diameters has a constant volume.*

For if $x'y'z'$, $x''y''z''$, &c. be the extremities of the diameters the volume is (Art. 35)

$$\begin{vmatrix} x', & y', & z' \\ x'', & y'', & z'' \\ x''', & y''', & z''' \end{vmatrix},$$

or
$$abc\begin{vmatrix} \cos\lambda, & \cos\mu, & \cos\nu \\ \cos\lambda', & \cos\mu', & \cos\nu' \\ \cos\lambda'', & \cos\mu'', & \cos\nu'' \end{vmatrix},$$

but the value of the last determinant is unity (see note, p. 20); hence the volume of the parallelopiped is *abc*.

If the axes of any central plane section be a', b', and p the perpendicular on the parallel tangent plane, then $a'b'p = abc$. For if c' be the semi-diameter to the point of contact, and θ the angle it makes with p, the volume of the parallelopiped under the conjugate diameters a', b', c' is $a'b'c' \cos\theta$, but $c' \cos\theta = p$.

93. The theorems just given may also with ease be deduced from the corresponding theorems for conics.

For consider any three conjugate diameters a', b', c', and let the plane of $a'b'$ meet the plane of xy in a diameter A, and let C be the diameter conjugate to A in the section $a'b'$, then we have $A^2 + C^2 = a'^2 + b'^2$; therefore $a'^2 + b'^2 + c'^2 = A^2 + C^2 + c'^2$. Again, since A is in the plane xy, then if B is the diameter conjugate to A in the section by that plane, the plane conjugate to A will be the plane containing B and containing the axis c, and C, c' are therefore conjugate diameters of the same section as B, c. Hence we have $A^2 + C^2 + c'^2 = A^2 + B^2 + c^2$; and since, finally, $A^2 + B^2 = a^2 + b^2$, the theorem is proved. Precisely similar reasoning proves the theorem about the parallelopipeds.

We might further prove these theorems by obtaining, as in the note, p. 52, the relations which exist when the quantity $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2}$ in oblique co-ordinates is transformed to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ in rectangular co-ordinates. These relations are found to be

$$a^2 + b^2 + c^2 = a'^2 + b'^2 + c'^2,$$

$$b^2c^2 + c^2a^2 + a^2b^2 = b'^2c'^2 \sin^2\lambda + c'^2a'^2 \sin^2\mu + a'^2b'^2 \sin^2\nu,$$

$$a^2b^2c^2 = a'^2b'^2c'^2 (1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda \cos\mu \cos\nu).$$

The first and last equations give the properties already obtained. The second expresses that the sum of the squares of the parallelograms formed by three conjugate diameters, taken two by two, is constant.

94. *The sum of the squares of the projections of three conjugate diameters on any line is constant.*

Let the line make angles α, β, γ with the axes, then the projection on it of the semi-diameter terminating in the point $x'y'z'$ is $x' \cos\alpha + y' \cos\beta + z' \cos\gamma$, or, by Art. 90, is

$$a \cos\lambda \cos\alpha + b \cos\mu \cos\beta + c \cos\nu \cos\gamma.$$

Similarly, the others are

$$a \cos\lambda' \cos\alpha + b \cos\mu' \cos\beta + c \cos\nu' \cos\gamma,$$

$$a \cos\lambda'' \cos\alpha + b \cos\mu'' \cos\beta + c \cos\nu'' \cos\gamma,$$

and squaring and adding, we get the sum of the squares

$$a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma.$$

95. *The sum of the squares of the projections of three conjugate diameters on any plane is constant.*

If d, d', d'' be the three diameters, θ, θ', θ'' the angles made by them with the perpendicular on the plane, the sum of the squares of the three projections is $d^2 \sin^2\theta + d'^2 \sin^2\theta' + d''^2 \sin^2\theta''$, which is constant, since $d^2 \cos^2\theta + d'^2 \cos^2\theta' + d''^2 \cos^2\theta''$ is constant by the last article; and $d^2 + d'^2 + d''^2$ by Art. 91.

96. *To find the locus of the intersection of three tangent planes at the extremities of three conjugate diameters.*

The equations of the three tangent planes are

$$\frac{x \cos\lambda}{a} + \frac{y \cos\mu}{b} + \frac{z \cos\nu}{c} = 1,$$

$$\frac{x \cos\lambda'}{a} + \frac{y \cos\mu'}{b} + \frac{z \cos\nu'}{c} = 1,$$

$$\frac{x \cos\lambda''}{a} + \frac{y \cos\mu''}{b} + \frac{z \cos\nu''}{c} = 1.$$

Squaring and adding, we get for the equation of the locus,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

97. *To find the lengths of the axes of the section made by any plane passing through the centre.*

We can readily form the quadratic, whose roots are the reciprocals of the squares of the axes, since we are given the sum and the product of these quantities. Let α, β, γ be the angles which a perpendicular to the given plane makes with the axes, R the intercept by the surface on this perpendicular; then we have (Art. 88)

$$\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{R^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

whence $\frac{1}{a'^2}+\frac{1}{b'^2}=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{\cos^2\alpha}{a^2}-\frac{\cos^2\beta}{b^2}-\frac{\cos^2\gamma}{c^2}\right)$,

while (Art. 92) $\frac{1}{a'^2b'^2}=\frac{p^2}{a^2b^2c^2}=\frac{\cos^2\alpha}{b^2c^2}+\frac{\cos^2\beta}{c^2a^2}+\frac{\cos^2\gamma}{a^2b^2}$.

The quadratic required is therefore

$$\frac{1}{r^4}-\frac{1}{r^2}\left(\frac{\sin^2\alpha}{a^2}+\frac{\sin^2\beta}{b^2}+\frac{\sin^2\gamma}{c^2}\right)+\frac{\cos^2\alpha}{b^2c^2}+\frac{\cos^2\beta}{c^2a^2}+\frac{\cos^2\gamma}{a^2b^2}=0.$$

This quadratic may also be written in the form

$$\frac{a^2\cos^2\alpha}{a^2-r^2}+\frac{b^2\cos^2\beta}{b^2-r^2}+\frac{c^2\cos^2\gamma}{c^2-r^2}=0.$$

This equation may be otherwise obtained from the principles explained in the next article.

98. *Through a given radius OR of a central quadric we can in general draw one section of which OR shall be an axis.*

Describe a sphere with OR as radius, and let a cone be drawn having the centre as vertex and passing through the intersection of the surface and the sphere, and let a tangent plane to the cone be drawn through the radius OR, then OR will be an axis of the section by that plane. For in it OR is equal to the next consecutive radius (both being radii of the same sphere) and is therefore a maximum or minimum; while the tangent line at R to the section is perpendicular to OR, since it is also in the tangent plane to the sphere. OR is therefore an axis of the section.

The equation of the cone can at once be formed by subtracting one from the other, the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=1,$$

when we get

$$x^2\left(\frac{1}{a^2}-\frac{1}{r^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{r^2}\right)=0.$$

If then any plane $x\cos\alpha+y\cos\beta+z\cos\gamma$ have an axis in length $=r$, it must touch this cone, and the condition that it should touch it, is (Art. 86)

$$\frac{a^2\cos^2\alpha}{a^2-r^2}+\frac{b^2\cos^2\beta}{b^2-r^2}+\frac{c^2\cos^2\gamma}{c^2-r^2}=0,$$

which is the equation found in the last article.

In like manner we can find the axes of any section of a quadric given by an equation of the form

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy = 1.$$

The cone of intersection of this quadric with any sphere

$$\lambda (x^2 + y^2 + z^2) = 1$$

is $(a - \lambda) x^2 + (b - \lambda) y^2 + (c - \lambda) z^2 + 2lyz + 2mzx + 2nxy = 0$,

and we see as before, that if λ be the reciprocal of the square of an axis of the section by the plane $x \cos\alpha + y \cos\beta + z \cos\gamma$, this plane must touch the cone whose equation has just been given. The condition that the plane should touch this cone (Art. 75) may be written

$$\begin{vmatrix} a-\lambda, & n, & m, & \cos\alpha \\ n, & b-\lambda, & l, & \cos\beta \\ m, & l, & c-\lambda, & \cos\gamma \\ \cos\alpha, & \cos\beta, & \cos\gamma & \end{vmatrix} = 0,$$

which expanded is

$$\begin{aligned} \lambda^2 - \lambda \{(b+c) \cos^2\alpha + (c+a) \cos^2\beta + (a+b) \cos^2\gamma \\ - 2l \cos\beta \cos\gamma - 2m \cos\gamma \cos\alpha - 2n \cos\alpha \cos\beta\} \\ + (bc - l^2) \cos^2\alpha + (ca - m^2) \cos^2\beta + (ab - n^2) \cos^2\gamma \\ + 2 (mn - al) \cos\beta \cos\gamma + 2 (nl - bm) \cos\gamma \cos\alpha \\ + 2 (lm - cn) \cos\alpha \cos\beta = 0. \end{aligned}$$

CIRCULAR SECTIONS.

99. We proceed to investigate whether it is possible to draw a plane which shall cut a given ellipsoid in a circle. As it has been already proved (Art. 69) that all parallel sections are similar curves, it is sufficient to consider sections made by planes through the centre. Imagine that any central section is a circle with radius r, and conceive a concentric sphere described with the same radius. Then we have just seen that

$$x^2 \left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2 \left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2 \left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0$$

represents a cone having the centre for its vertex and passing through the intersection of the quadric and the sphere. But if the surfaces have a plane section common, this equation must necessarily represent two planes, which cannot take place unless the coefficient of either x^2, y^2, or z^2 vanish. The plane section must therefore pass through one or other of the three axes. Suppose for example we take $r = b$, the coefficient of y vanishes, and there remains

$$x^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = 0,$$

which represents two planes of circular section passing through the axis of y.

The two planes are easily constructed by drawing in the plane of xz a semi-diameter equal to b. Then the plane containing the axis of y, and either of the semi-diameters which can be so drawn, is a plane of circular section.

In like manner two planes can be drawn through each of the other axes, but in the case of the ellipsoid these planes will be imaginary; since we evidently cannot draw in the plane of xy a semi-diameter $= c$, the least semi-diameter in that section being $= b$; nor, again, in the plane of yz a semi-diameter $= a$, the greatest in that section being $= b$.

In the case of the hyperboloid of one sheet c^2 is negative, and the sections through a are those which are real. In the hyperboloid of two sheets where both b^2 and c^2 are negative, if we take $r^2 = -c^2$ (b^2 being less than c^2), we get the two real sections,

$$x^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + y^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) = 0.$$

These two real planes through the centre do not meet the surface, but parallel planes do meet it in circles. In all cases it will be observed that we have only two real central planes of circular section, the series of planes parallel to each of which afford two different systems of circular sections.

100. Any two surfaces whose coefficients of x^2, y^2, z^2, differ only by a constant, have the same circular sections. Thus $Ax^2 + By^2 + Cz^2 = 1$, and $(A + H)x^2 + (B + H)y^2 + (C + H)z^2 = 1$

have the same circular sections as easily appears from the formula in the last article.

The same thing appears by throwing the two equations into the form

$$\frac{1}{\rho^2} = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma,$$

$$\frac{1}{\rho'^2} = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma + H,$$

from which it appears that the difference of the squares of the reciprocals is constant of the corresponding radii vectores of the two surfaces. If then in any section the radius vector be constant, so must also the radius vector of the other. The same consideration shews that any plane cuts both in sections having the same axes, since the maximum or minimum value of the radius vector will in each correspond to the same values of α, β, γ.

Circular sections of a cone are the same as those of a hyperboloid to which it is asymptotic.

101. *Any two circular sections of opposite systems lie on the same sphere.*

The equations of the two planes of section are parallel each to one of the planes represented by

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0.$$

Now since the equation of two planes agrees with the equation of two parallel planes as far as terms of the second degree are concerned, the equation of the two planes must be of the form

$$x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) + u_1 = 0,$$

where u_1 represents some plane. If then we subtract this from the equation of the surface, which every point on the section must also satisfy, we get

$$\frac{1}{r^2}(x^2 + y^2 + z^2) - u_1 = 1,$$

which represents a sphere.

102. All parallel sections are as we have seen similar. If now we draw a series of planes parallel to circular sections the extreme one will be the parallel tangent plane which must meet the surface in an infinitely small circle. Its point of contact is called an *umbilic*. Some properties of these points will be mentioned afterwards. The co-ordinates of the real umbilics are easily found. We are to draw in the section, whose axes are a and c, a semi-diameter $=b$, and to find the co-ordinates of the extremity of its conjugate. Now the formula for conics $b'^2 = a^2 - e^2x^2$ applied to this case gives us

$$b^2 = a^2 - \frac{a^2-c^2}{a^2}x^2,$$

whence $$\frac{x^2}{a^2} = \frac{a^2-b^2}{a^2-c^2}; \text{ similarly } \frac{z^2}{c^2} = \frac{b^2-c^2}{a^2-c^2}.$$

There are accordingly in the case of the ellipsoid four real umbilics in the plane of xz, and four imaginary in each of the other principal planes.

103. It is convenient to add in this place how in like manner we are able to determine the circular sections of the paraboloid given by the equation

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \frac{2z}{c}.$$

Considering a circular section through the origin, whose radius is r, we can see, as in Art. 99, that it must lie in the sphere

$$x^2 + y^2 + z^2 = 2rz.$$

And the cone of intersection of this sphere with the paraboloid is

$$x^2\left(1 - \frac{cr}{a^2}\right) + y^2\left(1 \mp \frac{cr}{b^2}\right) + z^2 = 0.$$

This will represent two planes if one of the terms vanishes. It will represent two real planes, in the case of the elliptic paraboloid, if we take $\frac{cr}{a^2} = 1$, for the equation then becomes $b^2z^2 = (a^2 - b^2)y^2$. But in the case of the hyperbolic paraboloid there is no real circular section, since the same substitution would make the equation of the two planes take the imaginary form $b^2z^2 + (a^2 + b^2)y^2 = 0$.

Indeed, it can be proved in general that no section of the hyperbolic paraboloid can be a closed curve, for if we take its intersection with any plane $z = lx + my + n$, the projection on the plane of xy is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2(lx + my + n)}{c}$ which is necessarily a hyperbola.

RECTILINEAR GENERATORS.

104. We have seen that when the central section is an ellipse all parallel sections are similar ellipses, and the section by a tangent plane is an infinitely small similar ellipse. In like manner when the central section is a hyperbola, the section by any parallel plane is a similar hyperbola, and that by the tangent plane reduces itself to a pair of right lines parallel to the asymptotes of the central hyperbola. Thus if the equation referred to any conjugate diameters be

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - \frac{z^2}{c'^2} = 1,$$

and we consider the section made by any plane parallel to the plane of xz ($y = \beta$), its equation is

$$\frac{x^2}{a'^2} - \frac{z^2}{c'^2} = 1 - \frac{\beta^2}{b'^2}.$$

And it is evident that the value $\beta = b'$ reduces the section to a pair of right lines. Such right lines can only exist on the hyperboloid of one sheet, since if we had the equation

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 + \frac{z'^2}{c'^2},$$

the right-hand side of the equation could not vanish for any value of z. It is also geometrically evident that a right line cannot exist either on an ellipsoid, which is a closed surface; nor on a hyperboloid of two sheets, no part of which, as we saw, lies in the space included between several systems of two parallel planes, while any right line will of course in general intersect them all.

105. Throwing the equation of the hyperboloid of one sheet into the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2},$$

it is evident that the intersection of the two planes

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right), \quad \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)$$

lies on the surface, and by giving different values to λ we get a system of right lines lying in the surface; while, again, we get another system by considering the intersection of the planes

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 + \frac{y}{b}\right), \quad \lambda\left(\frac{x}{a} + \frac{z}{c}\right) = 1 - \frac{y}{b}.$$

What has been just said may be stated more generally as follows: If α, β, γ, δ represent four planes, then the equation $\alpha\gamma = \beta\delta$ represents a hyperboloid of one sheet, which may be generated as the locus of the system of right lines

$$\alpha = \lambda\beta, \quad \lambda\gamma = \delta,$$

or

$$\alpha = \lambda\delta, \quad \lambda\gamma = \beta.$$

In the case of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

the lines may be also expressed by the equations

$$\frac{x}{a} = \frac{z}{c}\cos\theta \mp \sin\theta, \quad \frac{y}{b} = \frac{z}{c}\sin\theta \pm \cos\theta.$$

106. *Any two lines belonging to opposite systems lie in the same plane.*

Consider the two lines

$$\alpha - \lambda\beta, \quad \lambda\gamma - \delta,$$
$$\alpha - \lambda'\delta, \quad \lambda'\gamma - \beta.$$

Then it is evident that the plane $\alpha - \lambda\beta + \lambda\lambda'\gamma - \lambda'\delta$ contains both, since it can be written in either of the forms

$$(\alpha - \lambda\beta) + \lambda'(\lambda\gamma - \delta), \quad \alpha - \lambda'\delta + \lambda(\lambda'\gamma - \beta).$$

It is evident in like manner that *no two lines belonging to the same system lie in the same plane.* Since no plane of the form $(\alpha-\lambda\beta)+R(\lambda\gamma-\delta)$ can ever be identical with $(\alpha-\lambda'\beta)+R'(\lambda'\gamma-\delta)$ if λ and λ' are different. In the same way we see that both the lines

$$\frac{x}{a}=\frac{z}{c}\cos\theta-\sin\theta,\quad \frac{y}{b}=\frac{z}{c}\sin\theta+\cos\theta,$$

$$\frac{x}{a}=\frac{z}{c}\cos\phi+\sin\phi,\quad \frac{y}{b}=\frac{z}{c}\sin\phi-\cos\phi,$$

which belong to different systems lie in the plane

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\frac{z}{c}\cos\tfrac{1}{2}(\theta-\phi)-\sin\tfrac{1}{2}(\theta-\phi).$$

Now this plane is parallel to the second line of the first system

$$\frac{x}{a}=\frac{z}{c}\cos\phi-\sin\phi,\quad \frac{y}{b}=\frac{z}{c}\sin\phi+\cos\phi,$$

but it does not pass through it, for the equation of a parallel plane through this line will be found to be

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\frac{z}{c}\cos\tfrac{1}{2}(\theta-\phi)+\sin\tfrac{1}{2}(\theta-\phi),$$

which differs in the absolute term from the equation of the plane through the first line.

107. We have seen that any tangent plane to the hyperboloid meets the surface in two right lines intersecting in the point of contact, and of course touches the surface in no other point. If through one of these right lines we draw any *other* plane, we have just seen that it will meet the surface in a new right line, and this new plane will touch the surface in the point where these two lines intersect. Conversely, the tangent plane to the surface at any point on a given right line in the surface will contain the right line, but the tangent plane will in general be different for every point of the right line. Thus, take the surface $x\phi=y\psi$, where the line xy lies on the surface, and ϕ and ψ represent planes (though the demonstration would equally hold if they were functions of any higher degree). Then using the equation of the tangent plane

$$(x-x')\frac{dU'}{dx'}+(y-y')\frac{dU'}{dy'}+(z-z')\frac{dU'}{dz'}=0,$$

and seeking the tangent at the point $x=0$, $y=0$, $z=z'$, we find $x\phi'+y\psi'=0$, where ϕ' and ψ' are what ϕ and ψ become on substituting these co-ordinates. And this plane will vary as z' varies.

All this is different in the case of the cone. Here every tangent plane meets the surface in two coincident right lines. The tangent plane then at every point of this right line is the same, and the plane touches the surface along the whole length of the line.

And generally, if the equation of a surface be of the form

$$x\phi+y^2\psi=0,$$

it is seen precisely, as above, that the tangent plane at every point of the line xy is $x=0$.

108. It was proved (Art. 104) that the two lines in which the tangent plane cuts a hyperboloid are parallel to the asymptotes of the parallel central section; but these asymptotes are evidently edges of the asymptotic cone to the surface. Hence every right line which can lie on a hyperboloid is parallel to some one of the edges of the asymptotic cone. It follows also that no three of them can be parallel to the same plane, since, if they were, a parallel plane would cut the asymptotic cone in three edges, which is impossible, the cone being only of the second degree.

109. We have seen that any line of the first system meets all the lines of the second system. Conversely, the surface may be conceived as generated by the motion of a right line which always meets a certain number of fixed right lines.*

Let us remark in the first place, that when we are seeking the surface generated by the motion of a right line, it is necessary that the motion of the right line should be regulated by *three* conditions. In fact, since the equations of a right

* A surface generated by the motion of a right line is called a *ruled* surface. If every generating line is intersected by the next consecutive one, the surface is called a *developable*. If not, it is called a *skew* surface. The hyperboloid of one sheet belongs to the latter class; the cone to the former.

line include four constants, four conditions would absolutely determine the position of a right line. When we are given one condition less, the position of the line is not determined, but it is so far limited that the line will always lie on a certain surface-locus, whose equation can be found as follows: Write down the general equations of a right line $x = mz + p$, $y = nz + q$; then the conditions of the problem establish three relations between the constants m, n, p, q. And combining these three relations with the two equations of the right line, we have five equations from which we can eliminate the four quantities m, n, p, q, and the resulting equation in x, y, z will be the equation of the locus required. Or, again, we may write the equations of the line in the form

$$\frac{x - x'}{\cos\alpha} = \frac{y - y'}{\cos\beta} = \frac{z - z'}{\cos\gamma},$$

then the three conditions give three relations between the constants $x', y', z', \alpha, \beta, \gamma$, and if between these we eliminate α, β, γ, the resulting equation in x', y', z' is the equation of the required locus, since $x'y'z'$ may be any point on the line.

We see then that it is a determinate problem to find the surface generated by a right line which moves so as always to meet *three* fixed right lines.* For expressing, by Art. 40, the condition that the moveable right line shall meet each of the fixed lines, we obtain the three necessary relations between m, n, p, q. Geometrically also we can see that the motion of the line is completely regulated by the given conditions. For a line would be completely determined if it were constrained to pass through a given point and to meet two fixed lines, since we need only draw planes through the given point and each of the fixed lines, when the intersection of these planes would determine the line required. If then the point through which the line is to pass, itself moves along a third fixed line, we have a determinate series of right lines, the assemblage of which forms a surface-locus.

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* Or three fixed curves of any kind.

110. Let us then solve the problem suggested by the last article, viz. to find the surface generated by a right line which always meets three fixed right lines. In order that the work may be shortened as much as possible, let us first examine what choice of axes we must make in order to give the equations of the fixed right lines the simplest form.

And it occurs at once that we ought to take the axes, one parallel to each of the three given right lines.* The only question then is where the origin can most symmetrically be placed. Suppose now that through each of the three right lines we draw planes parallel to the other two, we get thus three pairs of parallel planes forming a parallelopiped, of which the given lines will be edges. And if through the centre of this parallelopiped we draw lines parallel to these edges, we shall have the most symmetrical axes. Let then the equations of the three pairs of planes be

$$x = \pm a, \quad y = \pm b, \quad z = \pm c,$$

then the equations of the three fixed right lines will be

$$y = b, \; z = -c; \quad z = c, \; x = -a; \quad x = a, \; y = -b.$$

The equations of any line meeting the first two fixed lines are

$$z + c = \lambda (y - b); \quad z - c = \mu (x + a),$$

which will intersect the third if

$$c + \mu a + \lambda b = 0,$$

or replacing for λ and μ their values,

$$c(x + a)(y - b) + a(z - c)(y - b) + b(z + c)(x + a),$$

which reduced is

$$ayz + bzx + cxy + abc = 0.$$

On applying the criterion of p. 57 this is found to represent a hyperboloid of one sheet, as is otherwise evident, since

* We could not do this indeed if the three given right lines happened to be all parallel to the same plane. This case will be considered in the next Article. It will not occur when the locus is a hyperboloid of one sheet, see Art. 108.

it represents a central quadric and is known to be a ruled surface. The problem might otherwise be solved thus:

Assume for the equations of the moveable line

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma},$$

the three conditions obtained by expressing that this intersects each of the fixed lines are

$$\frac{y'-b}{\cos\beta}=\frac{z'+c}{\cos\gamma},$$

$$\frac{z'-c}{\cos\gamma}=\frac{x'+a}{\cos\alpha},$$

$$\frac{x'-a}{\cos\alpha}=\frac{y'+b}{\cos\beta}.$$

We can eliminate α, β, γ by multiplying the equations together, and get for the equation of the locus,

$$(x-a)(y-b)(z-c)=(x+a)(y+b)(z+c),$$

or reducing

$$ayz+bzx+cxy+abc=0,$$

the same equation as before.

The following is another general solution of the same problem: Let the first two lines be the intersection of the planes α, β; γ, δ; then the equations of the third can be expressed in the form $\alpha=A\gamma+B\delta$, $\beta=C\gamma+D\delta$. The moveable line, since it meets the first two lines, can be expressed by two equations of the form $\alpha=\lambda\beta$, $\gamma=\mu\delta$. Substituting these values in the equations of the third line we find the condition that it and the moveable line should intersect, viz.

$$A\mu+B=\lambda(C\mu+D).$$

And eliminating λ and μ between this and the equations of the moveable line, we get for the equation of the locus,

$$\beta(A\gamma+B\delta)=\alpha(C\gamma+D\delta).$$

111. From the general theory explained in Art. 105, it is plain that the hyperbolic paraboloid may also have right lines

lying altogether in the surface. For the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ (Art. 83) is included in the general form $\alpha\gamma = \beta\delta$; and the surface contains the two systems of right lines

$$\frac{x}{a} \pm \frac{y}{b} = \lambda, \quad \lambda\left(\frac{x}{a} \mp \frac{y}{b}\right) = \frac{z}{c}.$$

The first equation shews that every line on the surface must be parallel to one or other of the two fixed planes $\frac{x}{a} \pm \frac{y}{b} = 0$; and in this respect is the fundamental difference between right lines on the paraboloid and on the hyperboloid (see Art. 108).

It is proved, as in Art. 106, that any line of one system meets every line of the other system, while no two lines of the same system can intersect.

We give now the investigation of the converse problem, viz. to find the surface generated by a right line which always meets three fixed lines which are all parallel to the same plane. Let the plane to which all are parallel be taken for the plane of xy, any line which meets all three for the axis of z, and let the axes of x and y be taken parallel to two of the fixed lines. Then their equations are

$$x = 0, \; z = a; \quad y = 0, \; z = b; \quad x = my, \; z = c.$$

The equations of any line meeting the first two fixed lines are

$$x = \lambda (z - a), \quad y = \mu (z - b),$$

which will intersect the third if

$$\lambda (c - a) = m\mu (c - b),$$

and the equation of the locus is

$$(a - c)\, x\, (z - b) = (b - c)\, y\, (z - a),$$

which represents a hyperbolic paraboloid since the terms of highest degree break up into two real factors.

In like manner we might investigate the surface generated by a right line which meets *two* fixed lines and is always parallel to a fixed plane. Let it meet the lines

$$x = 0, \; z = a; \quad y = 0, \; z = -a,$$

and be parallel to the plane

$$x \cos\alpha + y \cos\beta + z \cos\gamma = p.$$

Then the equations of the line are

$$x = \lambda (z - a), \quad y = \mu (z + a),$$

which will be parallel to the given plane of

$$\cos\gamma + \lambda \cos\alpha + \mu \cos\beta = 0.$$

The equation of the required locus is therefore

$$\cos\gamma (z^2 - a^2) + x \cos\alpha (z + a) + y \cos\beta (z - a) = 0,$$

which is a hyperbolic paraboloid since the terms of the second degree break up at two real factors.

A hyperbolic paraboloid is the limit of the hyperboloid of one sheet, when the generator in one of its positions may lie altogether at infinity.

We have seen (Art. 104) that a plane is a tangent to a surface of the second degree when it meets it in two real or imaginary lines; and (Art. 83) that a paraboloid is met by the plane at infinity in two real or imaginary lines. Hence a paraboloid is always touched by the plane at infinity.

112. *Four right lines belonging to one system cut all lines belonging to the other system in a constant anharmonic ratio.*

For through the four lines and through any line which meets them all we can draw four planes; and therefore any other line which meets the four lines will be divided in a constant anharmonic ratio (Art. 38).

Conversely, if two non-intersecting lines are divided *homographically* in a series of points, that is to say, so that the anharmonic ratio of any four points on one line is equal to that of the corresponding points on the other; then the lines joining corresponding points will be generators of a hyperboloid of one sheet.

Let the two given lines be α, β; γ, δ. Let any fixed line which meets them both be $\alpha = \lambda'\beta$, $\gamma = \mu'\delta$; then, in order that any other line $\alpha = \lambda\beta$, $\gamma = \mu\delta$ should divide them homographically, we must have (*Conics*, Art. 55) $\frac{\lambda}{\lambda'} = \frac{\mu}{\mu'}$, and if we eliminate λ between the equations $\alpha = \lambda\beta$, $\lambda'\gamma = \mu'\lambda\delta$, the result is $\lambda'\beta\gamma = \mu'\alpha\delta$.

113. In the case of the hyperbolic paraboloid any three right lines of one system cut all the right lines of the other in a constant ratio. For since the generators are all parallel to the same plane, we can draw through any three generators parallels to that plane, and all right lines which meet three parallel planes are cut by them in a constant ratio.

Conversely, if two finite non-intersecting lines be divided, each into the same number of equal parts, the lines joining corresponding points will be generators of a hyperbolic paraboloid. By doing this with threads, the form of this surface can be readily exhibited to the eye.

To prove this directly, let the line which joins two corresponding extremities of the given lines be the axis of z; let the axes of x and y be taken parallel to the given lines, and let the plane of xy be half-way between them. Let the lengths of the given lines be a and b, then the co-ordinates of two corresponding points are

$$z = c, \quad x = \mu a, \quad y = 0,$$
$$z = -c, \quad x = 0, \quad y = \mu b,$$

and the equations of the line joining these points are

$$\frac{x}{a} + \frac{y}{b} = \mu, \quad 2cx - \mu az = \mu ac,$$

whence, eliminating μ, the equation of the locus is

$$2cx = a(z+c)\left(\frac{x}{a} + \frac{y}{b}\right),$$

which represents a hyperbolic paraboloid.

SURFACES OF REVOLUTION.

114. Let it be required to find the conditions that the general equation should represent a surface of revolution. In this case the equation can be reduced (see p. 55), if the surface be central, to the form $\frac{x^2}{a^2} + \frac{y^2}{a^2} \pm \frac{z^2}{c^2} = \pm 1$, and if the surface

be non-central to the form $\frac{x^2}{a^2}+\frac{y^2}{a^2}=\frac{2z}{c}$. In either case then when the highest terms are transformed so as to become the sum of squares of three rectangular co-ordinates, the coefficients of two of those squares are equal. It would appear then that the required condition would be at once obtained by forming the condition that the discriminating cubic should have equal roots. Since however the roots of the discriminating cubic are always positive, its discriminant can be expressed as the sum of squares (see *Higher Algebra*, p. 134), and will not vanish (the coefficients of the given equation being supposed to be real) unless *two* conditions are fulfilled which can be obtained more easily by the following process. We want to find whether it is possible so to transform the equation as to have

$$ax^2+by^2+cz^2+2lyz+2mzx+2nxy=A\,(X^2+Y^2)+CZ^2,$$

but we have (p. 52)

$$x^2+y^2+z^2=X^2+Y^2+Z^2.$$

It is manifest then that by taking $\lambda=A$, we should have the following quantity a perfect square:

$$(ax^2+by^2+cz^2+2lyz+2mzx+2nxy)-\lambda\,(x^2+y^2+z^2),$$

and it is required to find the conditions that this should be possible.

Now it is easy to see that when

$$Ax^2+By^2+Cz^2+2Lyz+2Mzx+2Nxy$$

is a perfect square, the six following conditions are fulfilled:*

$$AB=N^2,\quad BC=L^2,\quad CA=M^2,$$
$$AL=MN,\quad BM=NL,\quad CN=LM;$$

the three former of which are included in the three latter. In the present case then these latter three equations are

$$(a-\lambda)\,l=mn,\quad (b-\lambda)\,m=nl,\quad (c-\lambda)\,n=lm.$$

Solving for λ from each of these equations we see that the reduction is impossible unless the coefficients of the given equation be connected by the two relations

$$a-\frac{mn}{l}=b-\frac{nl}{m}=c-\frac{lm}{n}.$$

* That is to say, the reciprocal equation vanishes identically.

If these relations be fulfilled and if we substitute any of these common values for λ in the function

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2lyz+2mzx+2nxy,$$

it becomes, as it ought, a perfect square, viz.

$$lmn\left(\frac{x}{l}+\frac{y}{m}+\frac{z}{n}\right)^2=(C-A)Z^2,$$

and since the plane $Z=0$ represents a plane perpendicular to the axis of revolution of the surface, it follows that $\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0$ represents a plane perpendicular to that axis.

In the special case where the common values vanish which have been just found for λ, the highest terms in the given equation form a perfect square, and the equation represents either a parabolic cylinder or two parallel planes (see IV. and V., p. 59). These are limiting cases of surfaces of revolution, the axis of revolution in the latter case being any line perpendicular to both planes. The parabolic cylinder is the limit of the surface generated by the revolution of an ellipse round its transverse axis, when that axis passes to infinity.

115. If one of the quantities l, m, n vanish, the surface cannot be of revolution unless a second also vanish. Suppose that we have l and m both $=0$, the preceding conditions become

$$a-n\frac{m}{l}=b-n\frac{l}{m}=c,$$

from which, eliminating the indeterminate $\frac{l}{m}$, we get

$$(a-c)(b-c)=n^2.$$

This condition might also have been obtained at once by expressing that

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2nxy$$

should be a perfect square, and it is plain that we must have

$$\lambda=c;\quad (a-c)(b-c)=n^2.$$

116. The preceding theory might also be obtained from the consideration that in a surface of revolution the problem of

finding the principal planes becomes indefinite. For since every section perpendicular to the axis of revolution is a circle, any system of parallel chords of one of these circles is bisected by the plane passing through the axis of revolution, and through the diameter of the circle perpendicular to the chords, a plane which is perpendicular to the chords. It follows that *every* plane through the axis of revolution is a principal plane. Now the chords which are perpendicular to these diametral planes are given (see p. 45) by the equations

$$(a-R)x+ny+mz=0,\ nx+(b-R)y+lz=0,\ mx+ly+(c-R)z=0,$$

which when R is one of the roots of the discriminating cubic represent three planes meeting in one of the right lines required. The problem then will not become indeterminate unless these equations all represent the same plane, for which we have the conditions

$$\frac{a-R}{n}=\frac{n}{b-R}=\frac{m}{l};\quad \frac{a-R}{m}=\frac{n}{l}=\frac{m}{c-R},$$

which expanded are the same as the conditions found already.

LOCI.

117. We shall conclude this chapter by a few examples of the application of Algebraic Geometry to the *investigation of Loci.*

Ex. 1. To find the locus of a point whose shortest distances from two given non-intersecting right lines are equal.

If the equations of the lines are written in their general form, the solution of this is obtained immediately by the formula of Art. 14. We may get the result in a simple form by taking for the axis of z the shortest distance between the two lines, and choosing for the other axes the lines bisecting the angle between the projections on their plane of the given lines; then their equations are of the form

$$z=c,\ y=mx;\quad z=-c,\ y=-mx,$$

and the conditions of the problem give

$$(z-c)^2+\frac{(y-mx)^2}{1+m^2}=(z+c)^2+\frac{(y+mx)^2}{1+m^2},$$

or

$$cz(1+m^2)+mxy=0.$$

The locus is therefore a hyperbolic paraboloid.

If the shortest distances had been to each other in a given ratio, the locus would have been

$$\{(1+\lambda)z+(1-\lambda)c\}\{(1-\lambda)z+(1+\lambda)c\}$$
$$+\frac{1}{1+m^2}\{(1+\lambda)y+(1-\lambda)mx\}\{(1-\lambda)y+(1+\lambda)mx\}=0,$$

which represents a hyperboloid of one sheet.

Ex. 2. To find the locus of the middle points of all lines parallel to a fixed plane and terminated by two non-intersecting lines.

Take the plane $x=0$ parallel to the fixed plane, and the plane $z=0$, as in the last example, parallel to the two lines and equidistant from them; then the equations of the lines are

$$z=c,\ y=mx+n;\quad z=-c,\ y=m'x+n'.$$

The locus is then evidently the right line which is the intersection of the planes

$$z=0,\ 2y=(m+m')x+(n+n').$$

Ex. 3. To find the surface of revolution generated by a right line turning round a fixed axis which it does not intersect.

Let the fixed line be the axis of z, and let any position of the other be $x=mz+n$, $y=m'z+n'$. Then since any point of the revolving line describes a circle in a plane parallel to that of xy, it follows that the value of x^2+y^2 is the same for every point in such a plane section, and it is plain that the constant value expressed in terms of z is $(mz+n)^2+(m'z+n')^2$. Hence the equation of the required surface is

$$x^2+y^2=(mz+n)^2+(m'z+n')^2,$$

which represents a hyperboloid of revolution of one sheet.

Ex. 4. Two lines passing through the origin move each in a fixed plane, remaining perpendicular to each other, to find the surface (necessarily a cone) generated by a right line, also passing through the origin perpendicular to the other two.

Let the direction-angles of the perpendiculars to the fixed planes be a, b, c; a', b', c', and let those of the variable line be α, β, γ; then the direction-cosines of the intersections with the fixed planes, of a plane perpendicular to the variable line, will be proportional to (Art. 15)

$$\cos\beta\cos c-\cos\gamma\cos b,\quad \cos\gamma\cos a-\cos\alpha\cos c,\quad \cos\alpha\cos b-\cos\beta\cos a.$$
$$\cos\beta\cos c'-\cos\gamma\cos b',\quad \cos\gamma\cos a'-\cos\alpha\cos c',\quad \cos\alpha\cos b'-\cos\beta\cos a',$$

and the condition that these should be perpendicular to each other is

$$(\cos\beta\cos c-\cos\gamma\cos b)(\cos\beta\cos c'-\cos\gamma\cos b')$$
$$+(\cos\gamma\cos a-\cos\alpha\cos c)(\cos\gamma\cos a'-\cos\alpha\cos c')$$
$$+(\cos\alpha\cos b-\cos\beta\cos a)(\cos\alpha\cos b'-\cos\beta\cos a')=0,$$

which represents a cone of the second degree.

Ex. 5. Two planes mutually perpendicular pass each through a fixed line: to find the surface generated by their line of intersection.

Take the axes as in Ex. 1. Then the equations of the planes are

$$\lambda(z - c) + y - mx; \quad \lambda'(z + c) + y + mx = 0,$$

which will be at right angles if $\lambda\lambda' + 1 - m^2 = 0$; and putting in for λ, λ' their values from the pair of equations, we get

$$y^2 - m^2x^2 + (1 - m^2)(z^2 - c^2) = 0,$$

which represents a hyperboloid of one sheet.

If the lines intersect, in which case $c = 0$, the locus reduces to a cone.

Ex. 6. To find the locus of a point, whence three tangent lines, mutually at right angles, can be drawn to the quadric $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

If the equation were transformed so that these lines should become the axes of co-ordinates, the equation of the tangent cone would take the form $Ayz + Bzx + Cxy = 0$, since these three lines are edges of the cone. But the untransformed equation of the tangent cone is, see Art. 74,

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} - 1\right)^2.$$

And we have seen (Art. 78) that if this equation be transformed to any rectangular system of axes, the sum of the coefficients of x^2, y^2, and z^2 will be constant. We have only then to express the condition that this sum should vanish, when we obtain the equation of the required locus, viz.

$$\frac{x^2}{a^2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{y^2}{b^2}\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + \frac{z^2}{c^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Ex. 7. To find the equation of the cone whose vertex is $x'y'z'$ and which stands on the conic in the plane of xy, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The equations of the line joining any point $\alpha\beta$ of the base to the vertex are

$$\alpha(z' - z) = z'x - x'z, \quad \beta(z' - z) = z'y - y'z.$$

Substituting these values in the equation of the base, we get for the required cone

$$\frac{(z'x - x'z)^2}{a^2} + \frac{(z'y - y'z)^2}{b^2} = (z' - z)^2.$$

The following method may be used in general to find the equation of the cone whose vertex is $x'y'z'w'$, and base the intersection of any two surfaces U, V. Substitute in each equation for x, $x + \lambda x'$; for y, $y + \lambda y'$, &c., and let the results be

$$U + \lambda\delta U + \frac{\lambda^2}{1.2}\delta^2 U + \&c.,$$

$$V + \lambda\delta V + \frac{\lambda^2}{1.2}\delta^2 V + \&c.,$$

then the result of eliminating λ between these equations will be the equation of the required cone. For the points where the line joining $x'y'z'w'$ to $xyzw$ meets the surface U are got from the first of these two equations; those where the same line meets the surface V are got from the second; and when the eliminant of the two equations vanishes they have a common root, or the point $xyzw$ lies on a line passing through $x'y'z'w'$ and meeting the intersection of the surfaces.

Ex. 8. To find the equation of the cone whose vertex is the centre of an ellipsoid and base the section made by the polar of any point $x'y'z'$.

$$\textit{Ans.}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}\right)^2.$$

Ex. 9. To find the locus of points on the quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the normals at which intersect the normal at the point $x'y'z'$.

Ans. The points required are the intersection of the surface with the cone

$$a^2(y'z-z'y)(x-x')+b^2(z'x-x'z)(y-y')+c^2(x'y-y'x)(z-z')=0.$$

Ex. 10. To find the locus of the poles of the tangent planes of one quadric with respect to another.

We have only to express the condition that the polar of $x'y'z'w'$, with regard to the second quadric, should touch the first, and have therefore only to substitute $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}, \frac{du}{dw}$ for $\alpha, \beta, \gamma, \delta$ in the condition given Art. 75. The locus is therefore a quadric.

Ex. 11. To find the cone generated by perpendiculars erected at the vertex of a given cone to its several tangent planes.

Let the cone be $Lx^2+My^2+Nz^2=0$, and any tangent plane is $Lx'x+My'y+Nz'z=0$, the perpendicular to which through the origin is $\frac{x}{Lx'}=\frac{y}{My'}=\frac{z}{Nz'}$. If then we call the common value ρ, we have $x'=\frac{x}{L\rho}$, $y'=\frac{y}{M\rho}$, $z'=\frac{z}{N\rho}$, substituting which values in $Lx'^2+My'^2+Nz'^2=0$, ρ^2 disappears, and we have $\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}=0$. The form of the equation shews that the relation between the cones is reciprocal, and that the edges of the first are perpendicular to the tangent plane to the second. It can easily be seen that this is a particular case of the last example.

If the equation of the cone be given in the form

$$ax^2+by^2+cz^2+2dyz+2ezx+2fxy=0,$$

the equation of the reciprocal cone will be the same as that of the reciprocal curve in plane geometry, viz.

$$(bc-d^2)x^2+(ca-e^2)y^2+(ab-f^2)z^2$$
$$+2(ef-ad)yz+2(fd-be)zx+2(de-cf)xy=0.$$

Ex. 12. A line moves about so that three fixed points on it move on fixed planes: to find the locus of any other point on it.

Let the co-ordinates of the locus point P be α, β, γ; and let the three fixed planes be taken for co-ordinate planes meeting the line in points A, B, C. Then it is easy to see that the co-ordinates of A are $0, \frac{AB}{PB}\beta, \frac{AC}{PC}\gamma$, where the ratios $AB:PB$, $AC:PC$ are known. Expressing then, by Art. 10, that the distance PA is constant, the locus is at once found to be an ellipsoid.

Ex. 13. A and O are two fixed points, the latter being on the surface of a sphere. Let the line joining any other point D on the sphere to A meet the sphere again in D'. Then if on OD a portion OP be taken $= AD'$, find the locus of P. [Sir W. R. Hamilton].

We have $AD^2 = AO^2 + OD^2 - 2AO.OD \cos AOD$. But AD varies inversely as the radius vector of the locus, and OD is given, by the equation of the sphere, in terms of the angles it makes with fixed axes. Thus the locus is easily seen to be a quadric of which O is the centre.

Ex. 14. A plane passes through a fixed line, and the lines in which it meets two fixed planes are joined by planes each to a fixed point; find the surface generated by the line of intersection of the latter two planes.

Ex. 15. The four faces of a tetrahedron pass each through a fixed point. Find the locus of the vertex if the three edges which do not pass through it move each in a fixed plane.

The locus is in general a surface of the third degree having the intersection of the three planes for a double point. It reduces to a cone of the second degree when the four fixed points lie in one plane.

Ex. 16. Find the locus of the vertex of a tetrahedron, if the three edges which pass through that vertex each pass through a fixed point, if the opposite face also pass through a fixed point and the three other vertices move in fixed planes.

Ex. 17. A plane passes through a fixed point, and the points where it meets three fixed lines are joined by planes, each to one of three other fixed lines; find the locus of the intersection of the joining planes.

Ex. 18. The sides of a polygon in space pass through fixed points, and all the vertices but one move in fixed planes; find the curve locus of the remaining vertex.

Ex. 19. All the sides of a polygon but one pass through fixed points, the extremities of the free side move on fixed lines, and all the other vertices on fixed planes, find the surface generated by the free side.

CHAPTER VII.

METHODS OF ABRIDGED NOTATION.

118. WE shall in this chapter give an account of some of those properties of quadrics which are most simply derived by methods analogous to those explained in Chap. XIV. of the *Treatise on Conics*. In order to economize space we shall occasionally suppress such details as we think ought to present no difficulty to an intelligent reader. In particular we leave it to the reader to show that the whole theory of Reciprocal Polars, as explained in Chap. XV. of the *Conics*, applies equally to space of three dimensions, the polars being taken with respect to any quadric. We shall thus dispense with the necessity of giving separate proofs of a theorem and of its reciprocal. In the method of Reciprocal Polars it will be observed that a point corresponds to a plane and *vice versâ*, and that to a line (joining two points) corresponds a line (the intersection of two planes). In order to show what corresponds to a curve in space we shall anticipate a little of the theory of curves of double curvature to be explained hereafter.

119. A curve in space may be considered as a series of points in space 1, 2, 3, &c. arranged according to a certain law. If each point be joined to its next consecutive, we shall have a series of lines 12, 23, 34, &c., each line being a tangent to the given curve. The assemblage of these lines forms a surface, and a *developable* surface (see note, p. 75) since any line 12 intersects the consecutive line 23. Again, if we consider the planes 123, 234, 345, &c. containing every three consecutive points, we shall have a series of planes which are called the *osculating* planes of the given curve, and which are tangent planes to the developable generated by its tangents. Now when we reciprocate, it is plain that to the series of points, lines, and planes, will correspond a series of planes, lines, and

points, and thus that the reciprocal of a series of points forming a curve in space will be a series of planes touching a developable. If the curve in space lies all in one plane, the reciprocal planes will all pass through one point, and will be tangent planes to a *cone*.

Thus the series of points common to two surfaces forms a curve. Reciprocally the series of tangent planes common to two surfaces touches a developable which envelopes both surfaces.

The degree of any surface being measured by the number of points in which an arbitrary line meets it, the degree of the surface reciprocal to a given one is the same as the number of tangent planes which can be drawn to the original surface through an arbitrary right line. The reciprocal of a quadric is a quadric, since it may be easily deduced, from Art. 75, that but two tangent planes can be drawn to the quadric through an arbitrary line. The same theorem is proved by forming, as at p. 87, the actual equation of the locus of the polar with respect to the quadric of the tangent planes to another, which equation is at once proved to be of the second degree.

120. Let now U and V represent any two quadrics, then $U+\lambda V$ represents a quadric passing through *every* point common to U and V, and if λ be indeterminate it represents a series of quadrics having a common curve of intersection. Since nine points determine a quadric (Art. 54), $U+\lambda V$ is the most general equation of the quadric passing through eight given points (see *Higher Plane Curves*, p. 21). For if U and V be two quadrics, each passing through the eight points, $U+\lambda V$ represents a quadric also passing through the eight points, and the constant λ can be so determined that the surface shall pass through any ninth point, and can in this way be made to coincide with any given quadric through the eight points. It follows then that all quadrics which pass through eight points have besides a whole series of common points, forming a common curve of intersection; and reciprocally, that all quadrics which touch eight given planes have a whole series of common tangent planes determining a fixed developable which envelopes the whole series of surfaces touching the eight fixed planes.

It is evident also that the problem to describe a quadric through nine points may become indeterminate. For if the ninth point lie any where on the curve which, as we have just seen, is determined by the eighth fixed point, then *every* quadric passing through the eight fixed points will pass through the ninth point, and it is necessary that we should be given a ninth point, *not* on this curve, in order to be able to determine the surface. Thus if U and V be two quadrics through the eight points, we determine the surface by substituting the co-ordinates of the ninth point in $U + \lambda V = 0$; but if these co-ordinates make $U = 0$, $V = 0$, this substitution does not enable us to determine λ.

121. Given seven points [or tangent planes] common to a series of quadrics, then an eighth point [or tangent plane] common to the whole system is determined.

For let U, V, W be three quadrics, each of which passes through the seven points, then $U + \lambda V + \mu W$ may represent *any* quadric which passes through them; for the constants λ, μ may be so determined that the surface shall pass through any two other points, and may in this way be made to coincide with any given quadric through the seven points. But $U + \lambda V + \mu W$ represents a surface passing through *all* points common to U, V, W, and since these intersect in eight points, it follows that there is a point, in addition to the seven given, which is common to the whole system of surfaces.

We see thus that though it was proved in the last article that eight points *in general* determine a curve of double curvature common to a system of quadrics, it is *possible* that they may not. For we have just seen that there is a particular case in which to be given eight points is only equivalent to being given seven. When we say therefore that a quadric is determined by nine points, and that the intersection of two quadrics is determined by eight points, it is assumed that the nine or eight points are perfectly unrestricted in position.*

* The reader who has studied *Higher Plane Curves*, Arts. 22-27, will have no difficulty in developing the corresponding theory for surfaces of any degree. Thus if we are given one less than the number of points

122. If a system of quadrics have a common curve of intersection, that is to say, if they have eight points in common, the polar plane of any fixed point passes through a fixed right line.

If a system of quadrics be inscribed in the same developable, that is to say, if they have eight common tangent planes, the locus of the pole of a fixed plane is a right line.

For if P and Q be the polar planes of a fixed point with regard to U and V respectively, then $P+\lambda Q$ is the polar of the same point with respect to $U+\lambda V$.

In particular, the locus of the centres of all quadrics inscribed in the same developable, or touching the same eight planes, is a right line.

123. If a system of quadrics pass through a common curve of intersection [or be inscribed in a common developable], the polars of a fixed line generate a hyperboloid of one sheet.

Let the polars of two points in the line be $P+\lambda Q$, $P'+\lambda Q'$, then it is evident that their intersection lies on the hyperboloid $PQ'=P'Q$.

124. If a system pass through a common curve, the locus of the pole of a fixed plane is a curve in space of the third degree. For eliminating λ between $P+\lambda Q, P'+\lambda Q', P''+\lambda Q''$ we get the system of determinants

$$\left\|\begin{matrix} P, & P', & P'' \\ Q, & Q', & Q'' \end{matrix}\right\|$$

which represents a curve of the third degree. For the intersection of the surfaces represented by $PQ'=P'Q$, $PQ''=P''Q$, is a curve of the fourth degree, but this includes the right line PQ, which is not part of the intersection of $PQ''=P''Q$,

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necessary to determine a surface of the n^{th} degree, we are given a series of points forming a curve through which the surface must pass; and if we are given two less than the number of points necessary to determine the surface, then we are given a certain number of other points [namely as many as will make the entire number up to n^3] through which the surface must also pass.

$P'Q''=P''Q'$. There is therefore only a curve of the third degree common to all three.

Reciprocally, if a system be inscribed in the same developable, the polar of a fixed point envelopes the developable which is the reciprocal of a curve of the third degree.

125. Given seven points in a quadric, the polar plane of a fixed point passes through a fixed point.

Given seven tangent planes to a quadric, the pole of a fixed plane moves in a fixed plane.

For evidently the polar of a fixed point with regard to $U+\lambda V+\mu W$ will be of the form $P+\lambda Q+\mu R$, and will therefore pass through a fixed point.*

126. Since the discriminant contains the coefficients in the fourth degree, it follows that we have a biquadratic equation to solve to determine λ, in order that $U+\lambda V$ may represent a cone, and therefore *that through the intersection of two quadrics four cones may be described.* The vertices of these cones are determined by the intersection of the four planes,

$$\frac{dU}{dx}+\lambda'\frac{dV}{dx},\quad \frac{dU}{dy}+\lambda'\frac{dV}{dy},\quad \frac{dU}{dz}+\lambda'\frac{dV}{dz},\quad \frac{dU}{d\omega}+\lambda'\frac{dV}{d\omega},$$

where λ' is one of the roots of the biquadratic just referred to; and they are given as the four points common to the series of determinants,

$$\left\|\begin{matrix}\dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz}, & \dfrac{dU}{d\omega}\\[2ex] \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz}, & \dfrac{dV}{d\omega}\end{matrix}\right\|=0.$$

There are four points whose polars are the same with respect to all quadrics passing through a common curve of intersection,

* Dr. Hesse has derived from this theorem a construction for the quadric passing through nine given points. *Crelle*, Vol. XXIV. p. 36. *Cambridge and Dublin Mathematical Journal*, Vol. IV. p. 44. See also some further developments of the same problem by Mr. Townsend, *ib.*, Vol. IV. p. 241.

namely, the vertices of the four cones just referred to. For to express the conditions that

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\omega\frac{dU'}{d\omega'},$$

$$x\frac{dV'}{dx'}+y\frac{dV'}{dy'}+z\frac{dV'}{dz'}+\omega\frac{dV'}{d\omega'},$$

should represent the same plane, we find the very same set of determinants. In like manner there are four planes whose poles are the same with respect to a set of quadrics inscribed in the same developable.

127. As in the case of *Conics* (see Art. 298), if the two quadrics U and V touch each other, the biquadratic in λ will have equal roots. This may be most easily proved by taking the origin at the point of contact, and the tangent plane for the co-ordinate plane z. Then for both the quadrics we shall have $d=0$, $p=0$, $q=0$, and substituting these values in the discriminant, p. 41, the biquadratic becomes

$$(r+\lambda r')^2\{(n+\lambda n')^2-(a+\lambda a')(b+\lambda b')\}=0,$$

which has two equal roots. The condition then that two quadrics should touch is obtained by forming the discriminant of the biquadratic in λ.

In general, it is evident that the ratios of the coefficients of the biquadratic in λ will be invariants with regard to the pair of quadrics.

128. It is to be remarked that when two surfaces touch, the point of contact is a double point on their curve of intersection.

In general, two surfaces of the m^{th} and n^{th} degrees respectively intersect in a curve of the mn^{th} degree; for any plane meets the surfaces in two curves which intersect in mn points. And at each point of the curve of intersection there is a single tangent line, namely, the intersection of the tangent planes at that point to the two surfaces. For any plane drawn through this line meets the surfaces in two curves which touch: such a plane therefore passes through two coincident points of the curve of intersection.

But if the surfaces touch, then *every* plane through the point of contact meets them in two curves which touch, and *every* such plane therefore passes through two coincident points of the curve of intersection. The point of contact is therefore a double point on this curve.

And we can show that, as in plane curves, there are two tangents at the double point. For there are two directions in the common tangent plane to the surfaces, any plane through either of which meets the surfaces in curves having three points in common.

Take the tangent plane for the plane of xy, and let the equations of the surfaces be

$$z + ax^2 + 2nxy + by^2 + \&c.,$$
$$z + a'x^2 + 2n'xy + b'y^2 + \&c.,$$

then any plane $y = \mu x$ cuts the surfaces in curves which osculate (see *Conics*, p. 206), if

$$a + 2n\mu + b\mu^2 = a' + 2n'\mu + b'\mu^2.$$

The two required directions then are given by the equation

$$(a - a')\,x^2 + 2\,(n - n')\,xy + (b - b')\,y^2 = 0.$$

The same may be otherwise proved thus. It will be proved hereafter precisely as at *Higher Plane Curves*, p. 27, that if the equation of a surface be $u_1 + u_2 + u_3 + \&c. = 0$, the origin will be on the surface, and u_1 will include all the right lines which meet the surface in two consecutive points at the origin, while if u_1 is identically 0, the surface has the origin for a double point, and u_2 includes all the right lines which meet the surface in three consecutive points. Now in the case we are considering, by subtracting one equation from the other, we get a surface through the curve of intersection, viz.

$$(a - a')\,x^2 + 2\,(n - n')\,xy + (b - b')\,y^2 + \&c.,$$

in which surface the origin is a double point, and the two lines just written are two lines which meet the surface in three consecutive points.

129. When these lines coincide there is a cusp or stationary point (see *Higher Plane Curves*, p. 28) on the curve of inter-

section. We shall call the contact in this case stationary contact. The condition that this should be the case, the axes being assumed as above, is

$$(a-a')(b-b')=(n-n')^2.$$

Now if we compare the biquadratic for λ, given Art. 127, remembering also that in the form we are now working with, we have $r=r'$, we shall see that when this condition is fulfilled, three roots of the biquadratic become equal to -1. The conditions then for stationary contact are found by forming the conditions that the biquadratic should have three equal roots, viz., $S=0$, $T=0$, S and T being the two invariants of the biquadratic.

130. Since the condition that a quadric should touch a plane (Art. 75) involves the coefficients in the third degree, it follows that of a system of quadrics passing through a common curve, three can be drawn to touch a given plane, and reciprocally, that of a system inscribed in the same developable, three can be described through a given point.

It is obvious that in the former case one can be described through a given point, and in the latter, one to touch a given plane.

In either case, two can be described to touch a given line; for the condition that a quadric should touch a right line (Art. 76) involves the coefficients of the quadric in the second degree.

131. It is also evident geometrically, that only three quadrics of a system having a common curve can be drawn to touch a given plane. For this plane meets the common curve in four points, through which the section by that plane of every surface of the system must pass. Now, since a tangent plane meets a quadric in two right lines, real or imaginary, (Art. 104) these right lines in this case can be only some one of the *three* pairs of right lines which can be drawn through the four points. The points of contact which are the points where the lines of each pair intersect, are (*Conics*, p. 133) each the pole of the line joining the other two with regard to any conic passing

through the four points. Hence, (p. 45) if the vertices of one of the four cones of the system be joined to the three points, the joining lines are conjugate diameters of this cone.

132. A system of surfaces having the same centre and common circular sections may be regarded as a particular case of a system having a common curve; for their equation has been proved (Art. 100) to be of the form $S + \lambda (x^2 + y^2 + z^2)$. And since $x^2 + y^2 + z^2$ represents a cone, it appears that the common centre is one of the four vertices of cones of the system. Moreover, any three conjugate diameters of the imaginary cone $x^2 + y^2 + z^2 = 0$ are at right angles to each other, since this equation represents an infinitely small sphere. *Hence three concentric and concyclic quadrics can be described to touch a given plane, and the lines joining the three points of contact to the centre are mutually at right angles.*

133. If two quadrics touch in two points, their curve of intersection, which in the general case is a curve of double curvature of the fourth degree, breaks up into two plane conics. For if we draw any plane through the two points of contact and through any point of their intersection, this plane will meet the quadrics in sections having three points common, and having common also the two tangents at the points of contact; these sections must therefore be identical. The equations of the quadrics will then be of the form $S = 0$, $S + LM = 0$, where L and M represent the planes of section. It is proved in like manner that the surfaces are enveloped by two common cones of the second degree. For take the point where the intersection of the two given common tangent planes is cut by any other common tangent plane; then the cones having this point for vertex, and enveloping each surface, have common three tangent planes and two edges, and are therefore identical. The reciprocals of a pair of quadrics having double contact will manifestly be a pair of quadrics having double contact, and the two planes of intersection of the one pair will correspond to the vertices of common tangent cones to the other pair. Any point on the line LM will have the same polar with

regard to all surfaces of the system $S+\lambda LM$. For if P be the polar of S, the polar of $S+\lambda LM$ will in general be $P+\lambda(L'M+LM')$, which reduces to P when $L'=0$, $M'=0$. It thus appears again that at the two points where LM meets S, all the surfaces have the same tangent plane.

There are two other points whose polars with regard to all the quadrics are the same, which will be vertices of cones containing both the curves of section. It is easy to see geometrically that these two points lie on the polar of the line LM with regard to the surface S (that is to say, on the intersection of the common tangent planes at the points where LM meets S), and that these points are the foci of the involution determined by the pairs of points where that polar meets S and where it meets L and M.

134. If two surfaces each intersect a third in the same plane curve and in two other plane curves they will also intersect each other again in a plane curve whose plane passes through the line of intersection of the two latter planes.

For evidently two surfaces $S+LM$, $S+LN$ have for their mutual intersection two plane sections made by L, $M-N$.

135. Similar quadrics belong to the class now under discussion. Two quadrics are similar and similarly placed when the terms of the second degree are the same in both (see *Conics*, p. 201). Their equations then are of the form $S=0$, $S+cL=0$. We see then that two such quadrics intersect in general in one plane curve, the other plane of intersection being at infinity. If there be three similar quadrics, their three finite planes of intersection pass through the same right line.

Spheres are all similar quadrics, and therefore are to be considered as having a common section at infinity, which section will of course be an imaginary circle.

A plane section of a quadric will be a circle if it passes through the two points in which its plane meets this imaginary circle at infinity. We may see thus immediately of how many solutions the problem of finding the circular sections of a quadric is susceptible. For the section of the quadric by the plane at infinity meets the section of a sphere by the same plane in four points, which can be joined by six right lines, the planes passing through any one of which meet the quadric in a circle. The six right lines may be divided into three pairs, each pair intersecting in one of the three points whose polars are the same with respect to the section of the quadric and of the sphere. And it is easy to see that these three points determine the direction of the axes of the quadric.

A surface of revolution is one which has double contact with a sphere at infinity. For an equation of the form $x^2+y^2+az^2=b$ can be written in the form

$$(x^2+y^2+z^2-r^2)+\{(a-1)\,z^2-(b-r^2)\}=0,$$

and the latter part represents two planes. It is easy to see then why in this case there is but one direction of real circular sections, determined by the line joining the points of contact of the sections at infinity of a sphere and of the quadric.

FOCI.

136. When S represents a sphere, the equation of the quadric having double contact with it, $S=LM$ expresses as at *Conics*, p. 216, that the square of the tangent from any point on the quadric to the sphere is in a constant ratio to the rectangle under the distances of the same point from two fixed planes. The planes L and M are evidently parallel to the planes of circular section of the quadric since they are planes of its intersection with a sphere. We have seen (*Conics*, p. 217) that the focus of a conic may be considered as an infinitely small circle having double contact with the conic, the chord of contact being the directrix. In like manner we may define a focus of a quadric as an infinitely small sphere having double contact with the quadric, the chord of contact being then the corresponding directrix. That is to say, the point $\alpha\beta\gamma$ is a focus if the equation of the quadric can be expressed in the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\phi,$$

where ϕ is the product of the equations of two planes. We must discuss separately however the two cases, where these planes are real and where they are imaginary. In the one case the equation is of the form $S = LM$, in the other $S = L^2 + M^2$. In the first case the directrix (the line LM) is parallel to that axis of the surface through which real planes of circular section can be drawn. Thus, for example, if the surface be an ellipsoid, the line LM must be parallel to the mean axis. In the second case the line LM must be parallel to one of the other axes.

In either case the section of the quadric by a plane through a focus and the corresponding directrix will be a conic having the same point and line for focus and directrix. For if we take the axes x and y in any plane through LM and then make $z = 0$, the equation reduces to $(x-\alpha)^2 + (y-\beta)^2 = lm$, or else $= l^2 + m^2$ where l, m are the sections of L, M by the plane $z = 0$. But if this plane pass through LM, these sections coincide, and the equation reduces itself to $(x-\alpha)^2 + (y-\beta)^2 = l^2$, which represents a conic having $\alpha\beta$ for focus and l for directrix. This is only the algebraical statement of the fact that the section in question is touched by the infinitely small circle which is the section of S, l being the chord of contact.

137. Let us now examine whether a given quadric necessarily has a focus and whether it has more than one; that is to say, whether the equation of a given quadric can be expressed in the form $S = L^2 \pm M^2$, where S is a point-sphere. Now if the co-ordinate planes x and y were any two planes mutually at right angles passing through LM, the quantity $L^2 \pm M^2$ would be expressed in the form $ax^2 + 2bxy + cy^2$, which by moving round these co-ordinate planes could be made to take the form $Ax^2 \pm By^2$. And if now the origin were moved to any point in the plane through the focus perpendicular to the directrix, the equation $S = L^2 \pm M^2$ would take the form

$$(x-\alpha)^2 + (y-\beta)^2 + z^2 = A(x-\gamma)^2 + B(y-\delta)^2,$$

where α, β are the x and y of the focus, γ, δ those of the foot of the directrix, and where, when A and B have opposite signs, the planes of contact of the focus with the quadric are real, while they are imaginary when A and B have the same sign. Our co-ordinate planes have manifestly been so chosen as to be parallel to the principal planes of the surface, and we now want to find whether by a proper choice of the constants α, β, γ, δ, A, B, the form just written can be made identical with a given equation

$$\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}=1.$$

First, in order that the origin may be the centre, we must have $\alpha=A\gamma$, $\beta=B\delta$, by the help of which equations eliminating γ, δ the form written above becomes

$$(1-A)\,x^2+(1-B)\,y^2+z^2=\frac{1-A}{A}\,\alpha^2+\frac{1-B}{B}\,\beta^2,$$

whence $1-A=\dfrac{N}{L}$, $A=\dfrac{L-N}{L}$; $1-B=\dfrac{N}{M}$, $B=\dfrac{M-N}{M}$;

$$\frac{1-A}{A}\,\alpha^2+\frac{1-B}{B}\,\beta^2=N,$$

or

$$\frac{\alpha^2}{L-N}+\frac{\beta^2}{M-N}=1.$$

Thus it appears that the surface being given the constants A and B are determined, but that the focus may lie anywhere on the *conic*

$$\frac{\alpha^2}{L-N}+\frac{\beta^2}{M-N}=1,$$

which accordingly is called a *focal conic* of the surface.

Since we have purposely said nothing as to either the signs or the relative magnitudes of the quantities L, M, N, it follows that there is a focal conic in *each* of the three principal planes, and also that this conic is confocal with the corresponding principal section of the surface; the conics

$$\frac{\alpha^2}{L}+\frac{\beta^2}{M}=1,\quad \frac{\alpha^2}{L-N}+\frac{\beta^2}{M-N}=1$$

being plainly confocal. Any point $\alpha'\beta'$ on a focal conic being taken for focus, the corresponding directrix is a perpendicular to the plane of the conic drawn through the point

$$\gamma'=\frac{\alpha'}{A},\quad \delta'=\frac{\beta'}{B},\quad \text{or } \gamma'=\frac{L\alpha'}{L-N},\quad \delta'=\frac{M\beta'}{M-N}.$$

These values may be interpreted geometrically by saying that the foot of the directrix is the pole, with respect to the principal section of the surface, of the tangent to the focal conic at the point $\alpha'\beta'$. For this tangent is

$$\frac{\alpha\alpha'}{L-N}+\frac{\beta\beta'}{M-N}=1, \text{ or } \frac{\alpha\gamma'}{L}+\frac{\beta\delta'}{M}=1,$$

which is manifestly the polar of $\gamma'\delta'$ with regard to $\frac{\alpha^2}{L}+\frac{\beta^2}{M}=1$.

Hence, from the theory of plane confocal conics, the line joining any focus to the foot of the corresponding directrix is normal to the focal conic.* The feet of the directrices must evidently lie on that conic which is the locus of the poles of the tangents of the focal conic with regard to the corresponding principal section of the quadric. The equation of this conic is

$$\gamma^2\frac{L-N}{L^2}+\delta^2\frac{M-N}{M^2}=1,$$

as appears by eliminating α', β' from the equation of the focal conic and the equations connecting α', β', γ', δ'. The directrices themselves form a cylinder of which the conic just written is the base.

138. Let us now examine in detail the different classes of central surfaces, in order to investigate the nature of their focal conics and to find to which of the two different kinds of foci the points on each belong. Now it is plain that the equation

$$\frac{\alpha^2}{L-N}+\frac{\beta^2}{M-N}=1$$

will represent an ellipse when N is algebraically the least of the three quantities L, M, N; an hyperbola when N is the middle, and will become imaginary when N is the greatest.

Of the three focal conics therefore of a central quadric, one is always an ellipse, one a hyperbola, and one imaginary. In

* It was proved that the plane joining any focus to the corresponding directrix meets the surface in a section of which that point is the focus. It appears now that this may be stated as a property of any plane normal to a focal conic.

the case of the ellipsoid, for example, the equations of the focal ellipse and focal hyperbola are respectively

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1, \quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1.$$

The corresponding equations for the hyperboloid of one sheet are found by changing the sign of c^2, and those for the hyperboloid of two sheets by changing the sign both of b^2 and c^2.

Further, we have seen that foci belong to the class whose planes of contact are imaginary or are real, according as A and B have the same or opposite signs, and that $A=\frac{L-N}{L}$, $B=\frac{M-N}{M}$. Now if N be the least of the three, both numerators are positive, and the denominators are also positive in the case of the ellipsoid and hyperboloid of one sheet, but in the case of the hyperboloid of two sheets one of the denominators is negative. Hence, the points on the focal ellipse are foci of the class whose planes of contact are imaginary in the cases of the ellipsoid and of the hyperboloid of one sheet, but of the opposite class in the case of the hyperboloid of two sheets. Next, let N be the middle of the three quantities; then the two numerators have opposite signs, and the denominators have the same sign in the case of the ellipsoid, but opposite in the case of either hyperboloid. Hence the points of the focal hyperbola belong to the class whose planes of contact are real in the case of the ellipsoid, and to the opposite class in the case of either hyperboloid. It will be observed then that *all* the foci of the hyperboloid of one sheet belong to the class whose planes of contact are imaginary; but that the focal conics of the other two surfaces contain foci of opposite kinds, the ellipse of the ellipsoid and the hyperbola of the hyperboloid being those whose planes of contact are imaginary. This is equivalent to what appeared (Art. 136) that foci of the other kind can only lie in planes perpendicular to that axis of a quadric through which real planes of circular section can be drawn.

139. Focal conics with real planes of contact intersect the surface, while those of the other kind do not. In fact, if the

equation of a surface can be thrown into the form $S=L^2+M^2$, and if the co-ordinates of any point on the surface make $S=0$, they must also make $L=0$, $M=0$; that is to say, the focus must lie on the directrix. But in this case the surface could only be a cone. For taking the origin at the focus, the equation $x^2+y^2+z^2=L^2+M^2$, where L and M each pass through the origin, would contain no terms except those of the highest degree in the variables, and would therefore represent a cone (p. 40).

The focal conic on the other hand, which includes foci of the first kind, passes through the umbilics. For if the equation of the surface can be thrown into the form $S=LM$, and the co-ordinates of a point on the surface make $S=0$, they must also make either L or $M=0$. But since the surface passes through the intersection of S, L; if the point S lies on L, the plane L intersects the surface in an infinitely small circle; that is to say, is a tangent at an umbilic. From this property Professor Mac Cullagh called focal conics of this latter kind *umbilicar* focal conics.

140. If the given quadric were a cone $\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}=0$, the reduction of the equation to the form $S=L^2\pm M^2$ proceeds exactly as before, and it is proved that the co-ordinates of the focus must fulfil the condition $\frac{\alpha^2}{L-N}+\frac{\beta^2}{M-N}=0$, which represents either two right lines or an infinitely small ellipse according as $L-N$ and $M-N$ have opposite or the same signs. In other words, in this case the focal hyperbola becomes two right lines, while the focal ellipse contracts to the vertex of the cone. For the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, the equation of the focal lines is $\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2+c^2}=0$.

The focal lines of the cone, asymptotic to any hyperboloid, are plainly the asymptotes to the focal hyperbola of the surface.

The foci on the focal lines are all of the class whose planes of contact are imaginary; but the vertex itself, besides being in two ways a focus of this kind, may also be a focus of the other kind, for the equation of the cone can be written in any of the three forms

$$x^2+y^2+z^2=\frac{a^2+c^2}{a^2}x^2+\frac{b^2+c^2}{b^2}y^2,$$

$$\text{or }=\frac{a^2-b^2}{a^2}x^2+\frac{b^2+c^2}{c^2}z^2,\quad \text{or }=\frac{b^2-a^2}{b^2}y^2+\frac{a^2+c^2}{c^2}z^2.$$

The directrix, which corresponds to the vertex considered as a focus, passes through it.

The line joining any point on a focal line to the foot of the corresponding directrix is perpendicular to that focal line. This follows as a particular case of what has been already proved for the focal conics in general, but may also be proved directly. The co-ordinates of the foot of the directrix have been proved to be $\gamma'=\frac{L\alpha'}{L-N}$, $\delta'=\frac{M\beta'}{M-N}$, the equations of the line joining this point to $\alpha'\beta'$ are

$$\frac{\beta'}{M-N}\alpha-\frac{\alpha'}{L-N}\beta=\alpha'\beta'\left(\frac{1}{M-N}-\frac{1}{L-N}\right),$$

and the condition that this should be perpendicular to the focal line $\beta'\alpha=\alpha'\beta$ is $\frac{\alpha'^2}{L-N}+\frac{\beta'^2}{M-N}=0$, which we have already seen is satisfied.

In like manner, as a particular case of Art. 136, the section of a cone by a plane perpendicular to either of its focal lines is a conic of which the point in the focal line is a focus.

141. *The focal lines of a cone are perpendicular to the circular sections of the reciprocal cone* (see Ex. 11, p. 87).

For the circular sections of the cone $Lx^2+My^2+Nz^2=0$, are (see Art. 99)

$$(L-N)\,x^2+(M-N)\,y^2=0,$$

and the corresponding focal lines of the reciprocal cone $\frac{x^2}{L}+\frac{y^2}{M}+\frac{z^2}{N}=0$ are as we have just seen $\frac{x^2}{L-N}+\frac{y^2}{M-N}=0$, and the lines represented by the latter equation are evidently perpendicular to the planes represented by the former.

The theorem of this article is a particular case of the following more general:—*The sections of two reciprocal cones by any plane are polar reciprocals with regard to the foot of the perpendicular on that plane from the common vertex.* For let the plane meet an edge of one cone in a point P, and the perpendicular tangent plane to the other in the line QR, let M be the foot of the perpendicular on the plane from the vertex O, then it is easy to see that the line PM is perpendicular to QR, and if it meet it in S, then since the triangle POS is right-angled, the rectangle $PM.MS$ is equal to the constant OP^2. The curve therefore which is the locus of the points P is the same as that got by letting fall from M perpendiculars on the tangents QR, and taking on each perpendicular a portion inversely as its length. When therefore the section of one cone is a circle, that of the other will be a conic of which M is a focus. We shall discuss with more detail the properties of cones when we treat of sphero-conics.

142. The investigation of the foci of the other species of quadrics proceeds in like manner. Thus for the paraboloids included in the equation $\frac{x^2}{L} + \frac{y^2}{M} = 2z$. This equation can be written in either of the forms

$$(x-\alpha)^2 + y^2 + (z-\gamma)^2 = \frac{L-M}{L}\left(x - \frac{L}{L-M}\alpha\right)^2 + (z-\gamma+M)^2,$$

where $$\frac{\alpha^2}{L-M} = 2\gamma - M,$$

or $$x^2 + (y-\beta)^2 + (z-\gamma)^2 = \frac{M-L}{M}\left(y - \frac{M}{M-L}\beta\right)^2 + (z-\gamma+L)^2,$$

where $$\frac{\beta^2}{M-L} = 2\gamma - L.$$

It thus appears that a paraboloid has two focal parabolæ, which may easily be seen to be each confocal with the corresponding principal section. The focus belongs to one or other of the two kinds already discussed, according to the sign of the fraction $\frac{L-M}{L}$. In the case of the elliptic paraboloid therefore where both L and M are positive, if L be the greater, then the foci in the plane xz are of the class whose planes of contact are imaginary, while those in the plane yz are of the opposite class. But since if we change the sign either of L or of M the quantity $\frac{L-M}{L}$ remains positive, we see that *all* the foci of the hyperbolic paraboloid belong to the former class, a property we have already seen to be true of the hyperboloid of one sheet.

It remains true that the line joining any focus to the foot of the corresponding directrix is normal to the focal curve, and that the foot of the directrix is the pole with regard to the principal section of the tangent to the focal conic. The feet of the directrices lie on a parabola and the directrices themselves generate a parabolic cylinder.

To complete the discussion it remains to notice the foci of the different kinds of cylinders, but it is found without the slightest difficulty that when the base of the cylinder is an ellipse or hyperbola there are two focal lines; namely, lines drawn through the foci of the base parallel to the generators of the cylinder, while, if the base of the cylinder is a parabola, there is one focal line passing in like manner through the focus of the base.

143. The geometrical interpretation of the equation $S = LM$ has been already given. We learn from it this property of foci whose planes of contact are real, that *the square of the distance of any point on a quadric from such a focus is in a constant ratio to the product of the perpendiculars let fall from the point on the quadric, on two planes drawn through the corresponding directrix, parallel to the planes of circular section.* The corresponding property of foci of the other kind, which is less obvious, was discovered by Professor Mac Cullagh. It is, that *the distance of any point on the quadric from such a focus is in a constant ratio to its distance from the corresponding directrix, the latter distance being measured parallel to either of the planes of circular section.*

Suppose, in fact, we try to express the distance of the point

$x'y'z'$ from a directrix parallel to the axis of z and passing through the point whose x and y are γ, δ, the distance being measured parallel to a directive plane $z = mx$. Then a parallel plane through $x'y'z'$, viz. $z - z' = m(x - x')$, meets the directrix in a point whose x and y of course are γ, δ, while its z is given by the equation $z - z' = m(\gamma - x')$. The square of the distance required is therefore

$$(x' - \gamma)^2 + (y' - \delta)^2 + m^2 (x' - \gamma)^2 = (y' - \delta)^2 + (1 + m^2)(x' - \gamma)^2.$$

In the equation then, of Art. 137,

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = A(x - \gamma)^2 + B(y - \delta)^2,$$

where A and B are both positive and A is supposed greater than B, the right-hand side denotes B times the square of the distance of the point on the quadric from the directrix, the distance being measured parallel to the plane $z = mx$ where $m^2 = \dfrac{A - B}{B}$. By putting in the values of A and B, given in Art. 137, it may be seen that this is a plane of circular section, but it is evident geometrically that this must be the case. For consider the section of the quadric by any plane parallel to the directive plane, and since evidently the distances of every point in such a section are measured from the same point on the directrix, the distance therefore of every point in the section from this fixed point is in a constant ratio to its distance from the focus. But when the distances of a variable point from two fixed points have to each other a constant ratio, the locus is a sphere. The section therefore is the intersection of a plane and a sphere; that is, a circle.

An exception occurs when the distance from the focus is to be *equal* to the distance from the directrix. Since the locus of a point equidistant from two fixed points is a plane, it appears as before, that in this case the sections parallel to the directive plane are right lines. By referring to the previous articles it will be seen (see Art. 142) that the ratio we are considering is one of equality ($B = 1$) only in the case of the hyperbolic paraboloid, a surface which the directive plane could not meet in circular sections, seeing that it has not got any. Professor Mac Cullagh calls the ratio of the focal distance to that from the directrix, the modulus of the surface, and the foci having imaginary planes of contact he calls modular foci.*

144. It was observed (Art. 133) that all quadrics of the form $S-LM$ are enveloped by two cones, and when S represents a sphere, these cones must be of revolution as every cone enveloping a sphere must be. Further, when S reduces to a point-sphere, these cones coincide in a single one, having that point for its vertex; and we may therefore infer that the cone enveloping a quadric and having any focus for its vertex is one of revolution.

This theorem being of importance we give a direct algebraical proof of it. First, it will be observed that any equation of the form $x^2+y^2+z^2=(ax+by+cz)^2$ represents a right cone. For if the axes be transformed, remaining rectangular, but so that the plane denoted by $ax+by+cz$ may become one of the co-ordinate planes, the equation of the cone will become $X^2+Y^2+Z^2=\lambda X^2$, which denotes a cone of revolution since the coefficients of Y^2 and Z^2 are equal.

But now if we form, by the rule of Art. 74, the cone whose vertex is the origin and circumscribing $x^2+y^2+z^2-L^2-M^2$, where

$$L=ax+by+cz+d,\quad M=a'x+b'y+c'z+d',$$

it is found to be

$$(d^2+d'^2)(x^2+y^2+z^2-L^2-M^2)+(dL+d'M)^2=0,$$

* In the year 1836 Professor Mac Cullagh published this modular method of generation of quadrics. In 1842 I published the supplementary property possessed by the non-modular foci. Not long after M. Amyot independently noticed the same property, but owing to his not being acquainted with Professor Mac Cullagh's method of generation M. Amyot failed to obtain the complete theory of the foci. Professor Mac Cullagh has published a detailed account of the focal properties of quadrics, which will be found in the *Proceedings of the Royal Irish Academy*, Vol. II. p. 446. Mr. Townsend also has published a valuable paper (*Cambridge and Dublin Mathematical Journal*, Vol. III., pp. 1, 97, 148) in which the properties of foci, considered as the limits of spheres having double contact with a quadric, are very fully investigated.

or $$(d^2+d'^2)(x^2+y^2+z^2)-(d'L-dM)^2=0,$$

which we have seen represents a right cone.

A few additional properties of foci easily deduced from the principles laid down are left as an exercise to the reader.

Ex. 1. The polar of any directrix is the tangent to the focal conic at the corresponding focus.

Ex. 2. The polar plane of any point on a directrix is perpendicular to the line joining that point to the corresponding focus.

Ex. 3. If a line be drawn through a fixed point O cutting any directrix of a quadric, and meeting the quadric in the points A, B, then if F be the corresponding focus, $\tan\frac{1}{2}AFO.\tan\frac{1}{2}BFO$ is constant. This is proved as the corresponding theorem for plane conics. *Conics*, p. 191.

Ex. 4. This remains true if the point O move on any other quadric having the same focus. directrix and planes of circular section.

Ex. 5. If two such quadrics be cut by any line passing through the common directrix, the angles subtended at the focus by the intercepts are equal.

Ex. 6. If a line through a directrix touch one of the quadrics, the chord intercepted on the other subtends a constant angle at the focus.*

145. Having now considered the most remarkable cases of quadrics included in the equation $S-LM$,† let us pass on to the equation $S-L^2=0$, which denotes a surface touching S all along the section of S by the plane L. It is easily shewn from geometrical considerations, as at Art. 133, that two quadrics cannot touch in three points without thus touching all along a plane curve. The equation of the tangent cone to a surface, given p. 48, is a particular case of this equation $S=L^2$. Also two concentric and similar quadrics are to be regarded as

* In this section an account has been given of the relations which each focus of a quadric considered separately bears to the surface. In the next chapter we shall give an account of the properties of those conics which are the assemblage of foci, and of confocal surfaces. These properties were first studied in detail by M. Chasles and by Professor Mac Cullagh who about the same time independently arrived at the principal of them. M. Chasles's results will be found in the notes to his *Aperçu Historique*, published in 1837.

† The case where S breaks up into two planes has been discussed p. 78.

enveloping each other, the plane of contact being at infinity. Any plane obviously cuts the surfaces S and $S-L^2$ in two conics having double contact with each other, and if the section of one reduce to a point-circle, that point must plainly be the focus of the other. Hence *when one quadric envelopes another the tangent plane at the umbilic of one cuts the other in a conic of which the umbilic is the focus;* or if one surface be a sphere every tangent plane to the sphere meets the other surface in a section of which the point of contact is the focus.

Or these things may be seen by taking the origin at the umbilic and the tangent plane the plane of xy, when on making $z=0$, the quantity $S-L^2$ reduces to $x^2+y^2-l^2$, and denotes a conic of which the origin is the focus, and l the directrix.

Two quadrics enveloped by the same third intersect each other in plane curves. Obviously $S-L^2$, $S-M^2$, have the planes $L-M$, $L+M$ for their planes of intersection.

146. The equation $aL^2+bM^2+cN^2+dP^2$, where L, M, N, P represent planes, denotes a quadric such that any one of these four planes is polar of the intersection of the other three. For $aL^2+bM^2+cN^2$ denotes a cone having the point LMN for its vertex, and the equation of the quadric shews that this cone touches the quadric, P being the plane of contact. The four planes form what I shall call a *self-conjugate* tetrahedron with regard to the surface. It has been proved (Art. 126) that given two quadrics there are always four planes whose poles with regard to both are the same. If these be taken for the planes L, M, N, P, the equations of both can be transformed to the forms

$$aL^2+bM^2+cN^2+dP^2=0,\quad a'L^2+b'M^2+c'N^2+d'P^2=0.$$

It might also have been seen, *a priori*, that this is a form to which it must be possible to bring the system of equations of two quadrics. For L, M, N, P involve implicitly three constants each; and the equations written above involve explicitly three independent constants each. The system therefore includes eighteen constants, and is therefore sufficiently general to express the equations of any two quadrics.

147. *To find the condition that one quadric should pass through the vertices of a self-conjugate tetrahedron with regard to another.*

If x, y, z, w denote the faces of such a tetrahedron, then the equation of the one quadric expressed in terms of these assumes the form $ax^2 + by^2 + cz^2 + dw^2 = 0$, while in the equation of the other, the coefficients of x^2, y^2, z^2, w^2 vanish. Now if we form the discriminant of $U + \lambda V$, which we shall write

$$\Delta + \lambda\Theta + \lambda^2\Phi + \lambda^3\Theta' + \lambda^4\Delta' = 0,$$

it will be seen that if all the terms in U except a, b, c, d vanish, then Θ becomes $a'bcd + b'cda + c'dab + d'abc$, which vanishes when a', b', c', d' vanish, and since the coefficients Δ, Θ, &c. are invariants, Θ will be $=0$, no matter how the axes are transformed, if V pass through the vertices of a self-conjugate tetrahedron with regard to U.

When U reduces to $ax^2 + by^2 + cz^2 + dw^2$, the quantity Θ' is

$$a\frac{d\Delta'}{da'} + b\frac{d\Delta'}{db'} + c\frac{d\Delta'}{dc'} + d\frac{d\Delta'}{dd'},$$

but $\dfrac{d\Delta'}{da'} = 0$ is the condition that the plane x should touch the surface V. Hence $\Theta' = 0$ is the condition that the faces of a self-conjugate tetrahedron with regard to U should touch the surface V, as well as the condition that the vertices of a self-conjugate tetrahedron with regard to V should lie on the surface U. If, therefore, one of these things be the case, the other must also. $\Phi = 0$ will be fulfilled if the edges of a self-conjugate tetrahedron with regard to either all touch the other.

Ex. 1. If a sphere be circumscribed about a self-conjugate tetrahedron, the length of the tangent to it from the centre of the quadric is constant.

For when V is a sphere whose centre is α, β, γ and radius r, and U is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, then we have

$$\Delta = -\frac{1}{a^2b^2c^2}, \quad \Delta' = -r^2,$$

$$\Theta = \frac{1}{a^2b^2c^2}\{\alpha^2 + \beta^2 + \gamma^2 - (a^2 + b^2 + c^2 + r^2)\},$$

$$\Theta' = \frac{\alpha^2 - r^2}{a^2} + \frac{\beta^2 - r^2}{b^2} + \frac{\gamma^2 - r^2}{c^2} - 1,$$

$$\Phi = \frac{1}{b^2c^2}(\beta^2 + \gamma^2 - r^2) + \frac{1}{c^2a^2}(\gamma^2 + \alpha^2 - r^2) + \frac{1}{c^2b^2}(\alpha^2 + \beta^2 - r^2) - \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

The equation $\Theta = 0$ contains the theorem enunciated. The corresponding theorem for conics is due to M. Faure.

Ex. 2. If a hyperboloid be such that $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 0$, then the centre of a sphere inscribed in a self-conjugate tetrahedron lies on the surface.

Ex. 3. The locus of the centre of a sphere circumscribing a self-conjugate tetrahedron with regard to a paraboloid is a plane.

148. The vertices of two self-conjugate tetrahedra with regard to a quadric, form a system of eight points, such that every quadric through seven will pass through the eighth. Hesse, *Crelle*, t. xx. p. 297.

For, if x, y, z, w, X, Y, Z, W be the faces of the two tetrahedra, the quadric can be expressed in either of the forms

$$x^2 + y^2 + z^2 + w^2 = 0 = X^2 + Y^2 + Z^2 + W^2,$$

x, y, &c. being supposed to contain constant multipliers implicitly. Now if any quadric given by the general equation in x, y, z, w were transformed to a function of X, Y, Z, W, we find, from the invariance of the function Θ,

$$a + b + c + d = A + B + C + D,$$

and consequently, if seven of these quantities vanish so must the eighth. In like manner any quadric which touches seven faces will touch the eighth.

149. *The lines joining the vertices of any tetrahedron to the corresponding vertices of its polar tetrahedron with regard to a quadric belong to the same system of generators of a hyperboloid of one sheet, and the intersections of corresponding faces of the two tetrahedra possess the same property.*

The result of substituting the co-ordinates of any point 1, in the polar of another point 2, is the same as that of substituting the co-ordinates of 2 in the polar of 1. Let this result be called [1, 2]. Let the polar of 1 be called P_1. Then it is easy to see that the line joining the point 1, to the intersection of P_2, P_3, P_4, is

$$\frac{P_2}{[1, 2]} = \frac{P_3}{[1, 3]} = \frac{P_4}{[1, 4]}.$$

For this denotes a right line passing through the intersection of P_2, P_3, P_4, and whose equation is satisfied by the co-ordinates of 1. The notation will be more compact if we call the four polar planes x, y, z, w, and denote the quantities [1, 2], [1, 3], [1, 4] by n, m, p, that is to say, by the same letters by which we have expressed the coefficients of xy, xz, xw in the general equation of a quadric. Then the equations of the four lines we are considering are

$$\frac{y}{n} = \frac{z}{m} = \frac{w}{p},$$

$$\frac{z}{l} = \frac{w}{q} = \frac{x}{n},$$

$$\frac{w}{r} = \frac{x}{m} = \frac{y}{l},$$

$$\frac{x}{p} = \frac{y}{q} = \frac{z}{r}.$$

Now the condition that any line

$$ax + by + cz + dw = 0, \quad a'x + b'y + c'z + d'w = 0$$

should intersect the first, is, by eliminating x between the last two equations, found to be

$$n(ab' - ba') + m(ac' - ca') + p(ad' - da') = 0,$$

and the conditions that it should intersect each of the other three, are in like manner found to be

$$n(ba' - b'a) + l(bc' - b'c) + q(bd' - b'd) = 0,$$

$$m(ca' - c'a) + l(cb' - c'b) + r(cd' - c'd) = 0,$$

$$p(da' - d'a) + q(db' - d'b) + r(dc' - d'c) = 0.$$

But these four conditions added together vanish identically. Any right line therefore which intersects the first three will intersect the fourth, which is, in other words, the thing to be proved.*

* This theorem is due to M. Chasles. The proof here given is by Mr. Ferrers, *Quarterly Journal of Mathematics*, (Vol. I. p. 241).

The equation of the hyperboloid itself is found by the methods of (p. 78) in the form

$$(lw - qz)(mw - rx)(nw - py) = (lw - ry)(mw - pz)(nw - qx),$$

or $(nr - mq)(lwx + pyz) + (mq - pl)(nwz + rxy)$

$$+ (pl - nr)(mwy + qzx) = 0.$$

150. The second part of the theorem is only the polar reciprocal of the first, but, as an exercise, we give a separate proof of it.

Let [1, 1], [1, 2], &c. have the same signification as before, viz. the result of substituting the co-ordinates of 1, in the polars of 1, 2, &c. Form the determinant

$$\begin{matrix} [1,1], & [1,2], & [1,3], & [1,4], \\ [2,1], & [2,2], & [2,3], & [2,4], \\ [3,1], & [3,2], & [3,3], & [3,4], \\ [4,1], & [4,2], & [4,3], & [4,4], \end{matrix}$$

and let any minor of this determinant, for example, that got by suppressing the second row and third column, be denoted by (2, 3). Then the equation of the plane containing the three points 1, 2, 3, is easily seen to be

$$x(1, 4) + y(2, 4) + z(3, 4) + w(4, 4) = 0.$$

And if, for compactness, we substitute for (1, 4), &c., P, &c. as before, the equations of the four lines are

$$\begin{aligned} x &= 0, \quad Ny + Mz + Pw = 0, \\ y &= 0, \quad Nx + Lz + Qw = 0, \\ z &= 0, \quad Mx + Ly + Rw = 0, \\ w &= 0, \quad Px + Qy + Rz = 0. \end{aligned}$$

Now the conditions that any line

$$ax + by + cz + dw = 0, \quad a'x + b'y + c'z + d'w = 0$$

should intersect each of these are found to be

$$\begin{aligned} N(cd' - c'd) + M(db' - d'b) + P(bc' - b'c) &= 0, \\ N(dc' - d'c) + Q(ca' - c'a) + L(ad' - da') &= 0, \\ M(bd' - b'd) + L(da' - d'a) + R(ab' - a'b) &= 0, \\ P(b'c - bc') + Q(ac' - ca') + R(ba' - b'a) &= 0, \end{aligned}$$

and, as before, the theorem is proved by the fact that these conditions when added vanish identically. The equation of the hyperboloid is found to be

$$x^2MNP + y^2LNQ + z^2LMR + w^2PQR$$
$$+ xyN(PL + QM) + yzL(QM + RN) + zxM(PL + RN)$$
$$+ xwP(MQ + RN) + ywQ(LP + NR) + zwR(LP + QM).$$

As a particular case of these theorems the lines joining each vertex of a circumscribing tetrahedron to the point of contact of the opposite face are generators of the same hyperboloid.

151. Pascal's theorem for conics may be stated as follows: "The sides of any triangle intersect a conic in six points lying in pairs on three lines which intersect each the opposite side of the triangle in three points lying in one right line." M. Chasles has stated the following as the analogous theorem for space of three dimensions: "The sides of a tetrahedron intersect a quadric in twelve points, through which can be drawn four planes, each containing three points lying on edges passing through the same angle of the tetrahedron, then the lines of intersection of each such plane with the opposite face of the tetrahedron, are generators of the same system of a certain hyperboloid."

Let the faces of the tetrahedron be x, y, z, w, and the quadric

$$x^2 + y^2 + z^2 + w^2 - \left(l + \frac{1}{l}\right)yz - \left(m + \frac{1}{m}\right)zx - \left(n + \frac{1}{n}\right)xy$$
$$- \left(p + \frac{1}{p}\right)xw - \left(q + \frac{1}{q}\right)yw - \left(r + \frac{1}{r}\right)zw,$$

then the four planes may be written

$$x = ny + mz + pw,$$
$$y = nx + lz + qw,$$
$$z = qx + ly + rw,$$
$$w = px + qy + rz,$$

whose intersections with the planes x, y, z, w, respectively are a system of lines proved in the last article to be generators of the same hyperboloid.

152. As a further illustration of the use of the invariants, in finding the conditions which express the permanent relations of two quadrics to each other, we investigate the condition that two quadrics shall be such that a tetrahedron may have two pairs of opposite edges on the surface of one while its four faces touch the other.* The one quadric then can be made to assume the form $Pxw + Lyz = 0$. If the four planes x, y, z, w touch a quadric its equation will be found to be of the form

$$x^2 + y^2 + z^2 + w^2 + 2l(xw + yz) + 2m(yw + xz) + 2n(zw + xy) = 0,$$

where $1 + 2lmn = l^2 + m^2 + n^2$, and if l, m, n be each less than unity, we may write for them $-\cos A$, $-\cos B$, $-\cos C$, where A, B, C are the angles of a plane triangle. It will be found then that

$$\Delta = L^2P^2,\ \Delta' = -4\sin^2 A \sin^2 B \sin^2 C,$$
$$\Theta = -2LP(L+P)\cos A,\ \Theta' = 4(L+P)\sin^2 A \sin B \sin C,$$
$$\Phi = -(L+P)^2 \sin^2 A + 4LP\sin B \sin C \cos A,$$

eliminating between which, the required condition is obtained, viz.

$$4\Phi\Delta'\Theta' = \Theta'^3 + 8\Delta'^2\Theta.$$

If the discriminant of $U + \lambda V$ had been written in the form

$$A + 4\lambda B + 6\lambda^2 C + 4\lambda^3 D + E,$$

then the relation in question would be $3CDE = 2D^3 + BE^2$.

153. *To find the equation of the sphere circumscribing a tetrahedron.*

Let the four faces be α, β, γ, δ. Let the four perpendiculars on each face from the opposite vertex be p, p', p'', p'''. Now the equation of the circle circumscribing any triangle abc may be written in the form

$$\frac{(bc)^2\beta\gamma}{p'p''} + \frac{(ca)^2\gamma\alpha}{p''p} + \frac{(ab)^2\alpha\beta}{pp'} = 0,$$

* This appears to be the problem which corresponds to the plane problem of finding the condition that a triangle shall be inscribed in one conic and circumscribed about another.

where α, p, &c., denote perpendiculars on the sides of the triangle, the lengths of which are (bc), &c. But it is evident that for any point in the face δ, the ratio $\alpha : p$ is the same whether α and p denote perpendiculars on the plane α, or perpendiculars on the *line* $\alpha\delta$. We are thus led to the equation required, viz.

$$\frac{(bc)^2\beta\gamma}{p'p''}+\frac{(ca)^2\gamma\alpha}{p''p}+\frac{(ab)^2\alpha\beta}{pp'}+\frac{(ad)^2\alpha\delta}{pp'''}+\frac{(bd)^2\beta\delta}{p'p'''}+\frac{(cd)^2\gamma\delta}{p''p'''}=0.$$

For this is a quadric whose intersection with each of the four faces is the circle circumscribing the triangle of which that face consists.

It will be found, that when the equation of the sphere is written in the above form, the coefficient of x^2, y^2, z^2 is -1. Hence the square of the distance between the centres of the inscribed and circumscribing spheres is

$$D^2=R^2-r^2\left\{\frac{(bc)^2}{p'p''}+\frac{(ca)^2}{p''p}+\frac{(ab)^2}{pp'}+\frac{(ad)^2}{pp'''}+\frac{(bd)^2}{p'p'''}+\frac{(cd)^2}{p''p'''}\right\}.$$

154. From the preceding equation we can deduce the conditions that the general equation should represent a sphere. For the equation of any other sphere can only differ from the preceding by terms of the first degree, which will be of the form $(E\alpha+F\beta+G\gamma+H\delta)\left(\frac{\alpha}{p}+\frac{\beta}{p'}+\frac{\gamma}{p''}+\frac{\delta}{p'''}\right)$, the second factor denoting the plane at infinity. If then we add to the equation of the last article the product of these two factors, identify with the general equation of the second degree, and eliminate the indeterminate constants, the resulting conditions are found to be

$$\frac{Ap^2+Bp'^2-2Npp'}{(ab)^2}=\frac{Bp'^2+Cp''^2-2Lp'p''}{(bc)^2}=\frac{Cp''^2+Ap^2-2Mp''p}{(ca)^2}$$
$$=\frac{Ap^2+Dp'''^2-2Ppp'''}{(ad)^2}=\frac{Bp'^2+Dp'''^2-2Qp'p'''}{(bd)^2}$$
$$=\frac{Cp''^2+Dp'''^2-2Rp''p'''}{(cd)^2}.$$

155. Given two quadrics U and V there are two other principal covariant quadrics in terms of which together with U and V and with the invariants, all other covariant quadrics can be expressed. We shall choose as these two covariants, S the locus of the poles with respect to U of all the tangent planes to V, and S' the locus of the poles with respect to V of all the tangent planes to U, (see Ex. 10, p. 87). Thus if

$$U = ax^2 + by^2 + cz^2 + dw^2, \quad V = a'x^2 + b'y^2 + c'z^2 + d'w^2,$$

it is easily found that

$$S = bcda'^2x^2 + cdab'^2y^2 + dabc'^2z^2 + abcd'^2w^2,$$

$$S' = b'c'd'a^2x^2 + c'd'a'b^2y^2 + d'a'b'c^2z^2 + a'b'c'd^2w^2.$$

These quadrics it will be observed, as well as U and V, have x, y, z, w for the faces of a self-conjugate tetrahedron. Hence we can solve the problem, given two quadrics U and V, to find the equation which denotes the four planes x, y, z, w whose poles with regard to both are the same. For we form the covariants S and S' and then we have only to form the Jacobian of the four functions U, S, V, S', that is to say, the determinant whose four rows are

$$\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}, \frac{dU}{dw},$$

$$\frac{dV}{dx}, \frac{dV}{dy}, \text{\&c.},$$

when we have a function denoting the four planes in question.

156. The condition that $\alpha x + \beta y + \gamma z + \delta w$ should touch U is a contravariant of the third order in the coefficients. If we substitute for each coefficient a, $a + \lambda a'$, &c. we shall get the condition that $\alpha x + \beta y + \gamma z + \delta w$ shall touch the surface $U + \lambda V$, a condition which will be of the form $\sigma + \lambda\tau + \lambda^2\tau' + \lambda^3\sigma' = 0$. The functions $\sigma, \sigma', \tau, \tau'$ each contain α, β, &c. in the second degree, and the coefficients of U and V in the third degree. In terms of these functions we can express the conditions that the sections of U and V by the plane $\alpha x + \beta y + \gamma z + \delta w$ shall have any permanent relation to each other, such as can be expressed in terms of the coefficients of the discriminant of $U + \lambda V$ when

U and V are two plane curves. For instance, the condition that $\alpha x + \beta y + \gamma z + \delta w$ should meet the two surfaces in sections which touch, is got by forming the discriminant with respect to λ of $\sigma + \lambda\tau + \lambda^2\tau' + \lambda^3\sigma' = 0$; or, in other words, this expresses the condition that the plane $\alpha x + \beta y + \gamma z + \delta w$ should pass through a tangent line of the curve of intersection of U and V. This condition will be of the eighth order in α, β, γ, δ, and of the sixth order in the coefficients of *each* of the surfaces.

157. The condition that $\alpha x + \beta y + \gamma z + \delta w$ should touch U, may also be regarded as the equation of the surface reciprocal to U with regard to $x^2 + y^2 + z^2 + w^2$, (see *Conics*, p. 268). And in like manner $\sigma + \lambda\tau + \lambda^2\tau' + \lambda^3\sigma'$ is the equation of the surface reciprocal to $U + \lambda V$. Since if λ varies, $U + \lambda V$ denotes a series of quadrics passing through a common curve, the reciprocal system touches a common developable, the equation of which is found by forming the discriminant of $\sigma + \lambda\tau + \lambda^2\tau' + \lambda^3\sigma'$ with respect to λ. The equation therefore of the developable reciprocal to the curve of intersection of U and V is, as has been noticed in the last article, of the eighth degree in the new variables, and of the sixth degree in the coefficients of each of the surfaces.

By the same method we can form the equation of the developable which touches both U and V. For let $\overline{U}$ and $\overline{V}$ be the surfaces reciprocal to U and V, then the reciprocal of $\overline{U} + \lambda\overline{V}$ will be a surface inscribed in the same developable as U and V, and the discriminant with respect to λ of its equation will be the equation of the required developable.

158. By the help of the canonical form $ax^2 + by^2 + cz^2 + dw^2$ we can readily express the circumscribing developable in terms of U, V and the two covariants S and S'. Let $\overline{U}$ the reciprocal of U be $A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2$, then $A = bcd$, &c., and the reciprocal of $\overline{U}$ will be $BCDx^2 + \&c.$, that is to say, $\Delta^2 U$. Again, the coefficient of λ is $(BCD' + CDB' + DBC')\,x^2 + \&c.$, which is $= \Delta\, aa'\,(b'c'd + c'd'b + d'b'c)\,x^2 + \&c.$, while the quantity multiplied by Δ is

$$(b'c'd'a + c'd'a'b + d'a'b'c + a'b'c'd)\,(ax^2 + \&c.) - (b'c'd'a^2x^2 + \&c.).$$

The developable then is the discriminant of

$$\Delta^2 U + \Delta\lambda\,(\Theta' U - S') + \Delta'\lambda^2\,(\Theta V - S) + \Delta'^2\lambda^3 V.$$

The discriminant being cleared of the irrelevant factor $\Delta^2\Delta'^2$ the result remains of the tenth degree in the coefficientsof each equation. S' and S evidently pass through the curves of contact of the developable with U and V, while the developable meets U again in the curve of intersection of U with $(\Theta V - S)^2 + 4\Delta VS'$, (see Art 160, *infra*).

159. *To find the condition that a given line should pass through the curve of intersection of two quadrics U and V.*

Suppose that we have found by Art. 76 the condition $\rho = 0$, that the line should touch U, and that we substitute in it for each coefficient a, $a + \lambda a'$ the condition becomes $\rho + \lambda\sigma + \lambda^2\rho' = 0$; and if the line have any arbitrary position, we can by solving this quadratic for λ determine two surfaces passing through the curve of intersection UV and touching the given line. But if the line itself pass through UV, then it is easy to see that both these surfaces must coincide, for that the line cannot in general be touched by a surface of the system anywhere but in the point where it meets UV. The condition therefore which we are seeking is $\sigma^2 = 4\rho\rho'$. It is of the second order in the coefficients of each of the surfaces, and of the fourth in the coefficients of each of the planes determining the right line.

The condition $\sigma = 0$ will be fulfilled if the right line is cut harmonically by the two surfaces. In the case where the quadrics are $ax^2 + by^2 + cz^2 + dw^2$, $a'x^2 + b'y^2 + c'z^2 + d'w^2$, and the right line is $\alpha x + \beta y + \gamma z + \delta w$, $\alpha' x + \beta' y + \gamma' z + \delta' w$, the quantity ρ is (see Art. 76) $\Sigma\, ab\,(\gamma\delta' - \gamma'\delta)^2$, by which notation we mean to express the sum of the six terms of like form such as $cd\,(\alpha\beta' - \alpha'\beta)^2$ &c. Then σ will be $\Sigma\,(ab' + ba')(\gamma\delta' - \gamma'\delta)^2$, and $\sigma^2 - 4\rho\rho'$ is

$$\Sigma\,(ab' - ba')^2(\gamma\delta' - \gamma'\delta)^4 + 2\Sigma\,(ab' - ba')(ac' - ca')(\gamma\delta' - \gamma'\delta)^2(\beta\delta' - \beta'\delta)^2$$
$$+ 2\Sigma\,\{(ad' - da')(cb' - bc') + (ac' - ca')(db' - bd')\}(\alpha\beta' - \beta\alpha')^2(\gamma\delta' - \gamma'\delta)^2.$$

160. *To find the equation of the developable generated by the tangent lines of the curve common to U and V.*

If we consider any point on any tangent to this curve, the polar plane of this point with regard to either U or V passes evidently through the point of contact of the tangent on which it lies. The intersection therefore of the two polar planes meets the curve U, V. We find therefore the equation of the developable required by substituting in the condition of the last article for α, β, &c., α', β', &c., $\frac{dU}{dx}$, $\frac{dU}{dy}$, &c., $\frac{dV}{dx}$, $\frac{dV}{dy}$, &c. This developable will then be of the eighth degree in the variables and of the sixth in the coefficients of each surface. When we use the canonical form of the quadrics, it then easily appears that the result is

$$\begin{aligned}&\Sigma\,(ab'-ba')^2\,(cd'-c'd)^4\,z^4w^4\\&+2\Sigma\,(ab'-ba')\,(ac'-ca')\,(cd'-c'd)^2\,(bd'-b'd)^2\,y^2z^2w^4\\&+2x^2y^2z^2w^2\,\{(ab'-ba')\,(cd'-c'd)-(ad'-da')\,(bc'-b'c)\}\\&\times\{(ad'-da')\,(bc'-b'c)-(bd'-db')\,(ca'-c'a)\}\\&\times\{(bd'-db')\,(ca'-c'a)-(ab'-ba')\,(cd'-c'd)\}.\end{aligned}$$

When we make in the above equation $w=0$ we obtain a perfect square, hence each of the four planes x, y, z, w meets the developable in plane curves of the fourth degree which are double lines on the surface.* This is, *a priori* evident, since it is plain from the symmetry of the figure, that through any point in one of these four planes through which one tangent line of the curve UV passes, a second tangent can also be drawn.

By the help of the canonical form the previous result can be expressed in terms of the covariant quadrics when the developable is found to be

$$4(SU-\Delta V^2)(S'V-\Delta' U^2)=(S'U+SV-\Theta' U^2-\Theta V^2+\Phi UV)^2.$$

The curve UV is manifestly a double line† on the locus repre-

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* See *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 171, where, though only the geometrical proof is given, I had arrived at the result by actual formation of the equation of the developable. See *ibid.* Vol. II. p. 68. The equations were also worked out by Mr. Cayley, *ibid.* Vol. V. pp. 50, 55.

† It is proved, as at *Higher Plane Curves*, p. 39, (see also p. 75 of this volume), that when the equation of a surface is $U^2\phi + UV\psi + V^2\chi$, then

sented by this equation, as we otherwise know it to be, and the locus meets U again in the line of the eighth order determined by the intersection of U with

$$(S-\Theta V)^2+4\Delta S'V.$$

This is precisely the same equation as that found in Art. 158, and one can see geometrically that the line of the eighth order is in fact the eight tangents to UV at the points where UV meets S.

RECIPROCAL SURFACES.

161. Although we have made free use already in this chapter of the method of reciprocation, we wish now to enter into a little more detail on the theory of reciprocal surfaces.

To the section of a surface by any plane corresponds the tangent cone which can be drawn to the reciprocal surface through the corresponding point; and in particular to the section of the one by the plane at infinity corresponds the tangent cone which can be drawn to the other through the origin. Hence the asymptotic cone of the one surface is *reciprocal* to the tangent cone which can be drawn to the other from the origin, in the sense that each edge of the one cone is perpendicular to a tangent plane of the other.

Hence also when the origin is *without* a quadric, that is to say, is such that real tangents can be drawn from it to the surface, the reciprocal is a hyperboloid; when it is inside it is an ellipsoid; when the origin is *on* the surface, the tangent plane at infinity touches the reciprocal surface, that is to say, the reciprocal is a paraboloid.

UV is a double line on the surface, the two tangents at any point of it being given by the equation $u^2\phi' + uv\psi' + v^2\chi'$, where u, v are the tangent planes at that point to U and V, and ϕ' is the result of substituting in ϕ the coordinates of that point. Applying this to the above equation it is immediately found that the two tangents are given by the equation $(S'u - Sv)^2 = 0$, where in S', S the co-ordinates of the point are supposed to be substituted. Thus the two tangent planes at every point on the double curve coincide, and the curve is accordingly called a cuspidal curve on the surface.

The reciprocal of a *ruled* surface (that is to say, of a surface generated by the motion of a right line) is a ruled surface. For to a right line corresponds a right line, and to the surface generated by the motion of one right line will correspond the surface generated by the motion of the reciprocal line.* Hence to a hyperboloid of one sheet always corresponds a hyperboloid of one sheet unless the origin be *on* the surface when the reciprocal is a hyperbolic paraboloid.

It was proved (Art. 144) that the tangent cone whose vertex is a focus is one of revolution, hence the *reciprocal of a quadric with respect to a point on a focal conic is a surface of revolution.*

162. The equation of the reciprocal of a quadric given by the general equation is given in Art. 75. The reciprocal of a central surface with regard to any point may also be found as at *Conics*, Art. 320. For the length of the perpendicular from any point on the tangent plane is (see Art. 85)

$$p=\frac{K^2}{\rho}=\sqrt{(a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma)}-(x'\cos\alpha+y'\cos\beta+z'\cos\gamma),$$

and the reciprocal is therefore

$$(xx'+yy'+zz'+K^2)^2=a^2x^2+b^2y^2+c^2z^2.$$

163. The reciprocal of a sphere with regard to any point is a surface of revolution round the transverse axis. This may be proved as at *Conics*, p. 259. It is easily proved that if we have any two points A and B, the distances of these two points from the origin are in the same ratio as the perpendicular from

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* Mr. Cayley has remarked that *the degree of any ruled surface is equal to the degree of its reciprocal.* The degree of the reciprocal is equal to the number of tangent planes which can be drawn through an arbitrary right line. Now it will be formally proved hereafter, but is sufficiently evident in itself, that the tangent plane at any point on a ruled surface contains the generating line which passes through that point. The degree of the reciprocal is therefore equal to the number of generating lines which meet an arbitrary right line. But this is exactly the number of points in which the arbitrary line meets the surface, since every point in a generating line is a point on the surface.

each on the plane corresponding to the other (*Conics*, Art. 98). Now the distance of the centre of a fixed sphere from the origin, and the perpendicular from that centre on any tangent plane to the sphere are both constant. Hence, any point on the reciprocal surface is such that its distance from the origin is in a constant ratio to the perpendicular let fall from it on a fixed plane; namely, the plane corresponding to the centre of the sphere. And this locus is manifestly a surface of revolution of which the origin is a focus.

By reciprocating properties of the sphere we thus get properties of surfaces of revolution round the transverse axis. The left-hand column contains properties of the sphere, the right-hand those of the surfaces of revolution.

Ex. 1. Any tangent plane to a sphere is perpendicular to the line joining its point of contact to the centre.	The line joining focus to any point on the surface is perpendicular to the plane through focus and the intersection with the directrix plane of the tangent plane at the point.
Ex. 2. Every tangent cone to a sphere is a right cone, the tangent planes all making equal angles with the plane of contact.	The cone whose vertex is the focus and base any plane section is a right cone, whose axis is the line joining the focus to the pole of the plane of section.

A particular case of Ex. 2. is "Every plane section of a paraboloid of revolution is projected into a circle on the tangent plane at the vertex."

Ex. 3. Any plane through the centre is perpendicular to the conjugate diameter.	Any plane through the focus is perpendicular to the line joining the focus to its pole.
Ex. 4. The cone whose base is any section of a sphere has its circular sections parallel to the plane of section.	Any tangent cone has for its focal lines the lines joining the vertex of the cone to the two foci.
Ex. 5. Any plane is perpendicular to the line joining centre to its pole.	The line joining any point to the focus is perpendicular to the plane joining the focus to the intersection with the directrix plane of the polar plane of the point.

Ex. 6. Every cylinder enveloping a sphere is right.	Every section passing through the focus has this focus for a focus.
Ex. 7. Any two conjugate right lines are mutually perpendicular.	Any two conjugate lines are such that the planes joining them to the focus are at right angles.
Ex. 8. Any quadric enveloping a sphere is a surface of revolution.	If a quadric envelope a surface of revolution, the cone enveloping the former, whose vertex is a focus of the latter is a cone of revolution.

164. The product of the perpendiculars from the two foci of a surface of revolution round the transverse axis on any tangent plane, is evidently constant. Now if we reciprocate this property with regard to any point, by the method used in Art. 163, we learn that the square of the distance from the origin of any point on the reciprocal surface is in a constant ratio to the product of the distances of the point from two fixed planes.

It appears from Ex. 4, of the last article, that the two planes are planes of circular section of the asymptotic cone to the new surface; that is to say, that they are planes of circular section of the new surface. The intersection of the two planes is the reciprocal of the line joining the two foci; that is to say, of the axis of the surface of revolution. The property just proved* belongs, as we know (Art. 143), to every point on the umbilicar focal conic, hence the reciprocal of any quadric with regard to an umbilicar focus is a surface of revolution round the transverse axis, but with regard to a modular focus is a surface of revolution round the conjugate axis. By reciprocating properties of surfaces of revolution, we obtain properties of any quadric with regard to focus and corresponding directrix. It is to be noted that in either case the axis of the figure of revolution is the reciprocal of the directrix corresponding to the given focus.

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* It was in this way I was first led to this property, and to observe the distinction between the two kinds of foci.

The axis of the figure of revolution is parallel to the tangent to the focal conic at the given focus (see Art. 137).

The left-hand column contains properties of surfaces of revolution, the right-hand of quadrics in general.

Ex. 1. The tangent cone whose vertex is any point on the axis is a right cone whose tangent planes make a constant angle with the plane of contact, which plane is perpendicular to the axis.

The cone whose vertex is a focus and base any section whose plane passes through the corresponding directrix, is a right cone, whose axis is the line joining the focus to the pole of the plane of section, and this right line is perpendicular to the plane through focus and directrix.

Ex. 2. Any tangent plane is at right angles with the plane through the point of contact and the axis.

The line joining a focus to any point on the surface is at right angles to the line joining the focus to the point where the corresponding tangent plane meets the directrix.

Ex. 3. The polar plane of any point is at right angles to the plane containing that point and the axis.

The line joining a focus to any point is at right angles to the line joining the focus to the point where the polar plane meets the directrix.

Ex. 4. Any two conjugate lines are such that the planes joining them to the focus are at right angles.

Any two conjugate lines pierce a plane through a directrix parallel to circular sections, in two points which subtend a right angle at the corresponding focus.

Ex. 5. If a cone circumscribe surface of revolution, one principal plane is plane of vertex and axis, and another is parallel to plane of contact.

The cone whose base is *any* plane section of a quadric and vertex any focus has for one axis the line joining to the focus the pole of the plane, and for another the line joining focus to the point where the plane meets the directrix.

Ex. 6. The cone whose vertex is a focus and base any plane section is a right cone.

The cone is a right cone whose vertex is a focus and base the section made by any tangent cone on a plane through the corresponding directrix parallel to those of the circular sections.

Ex. 7. Locus of intersection of three tangent planes to a paraboloid, mutually at right angles, is a plane.

If through any point on a quadric be drawn three lines mutually at right angles, the plane joining their other extremities passes through a fixed point.

If the point be not on the quadric the plane envelopes a surface of revolution.

Ex. 8. If a quadric envelope a surface of revolution, the axis of the latter is parallel to a principal plane of the former.

If two quadrics envelope each other, the cone, whose vertex is any focus of one and which envelopes the other, has for one axis the line joining that focus to the point where the plane of contact meets the corresponding directrix.

CHAPTER VIII.

CONFOCAL SURFACES.

165. We shall in this chapter give an account of those properties of surfaces which are analogous to those properties of conics which are connected with their foci. And we commence by pointing out a method by which we should be led to the consideration of the focal conics of a quadric, independently of the method followed (Arts. 136, &c.).

Two concentric and coaxal conics are said to be confocal when the difference of the squares of the axes is the same for both. Thus given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, any conic is confocal with it whose equation is of the form

$$\frac{x^2}{a^2 \pm \lambda^2} + \frac{y^2}{b^2 \pm \lambda^2} = 1.$$

If we give the positive sign to λ^2, the confocal conic will be an ellipse; it will also be an ellipse when λ^2 is negative as long as it is less than b^2. When λ^2 is between b^2 and a^2 the confocal curve is a hyperbola, and when λ^2 is greater than a^2 the curve is imaginary. If $\lambda^2 = b^2$ the equation reducing itself to $y^2 = 0$, the axis of x itself is the limit which separates confocal ellipses from hyperbolas. But the two foci belong to this limit in a special sense. In fact through a given point $x'y'$ can in general be drawn two conics confocal to a given one, since we have a quadratic to determine λ^2, viz.

$$\frac{x'^2}{a^2 - \lambda^2} + \frac{y'^2}{b^2 - \lambda^2} = 1,$$

or $$\lambda^4 - \lambda^2 (a^2 + b^2 - x'^2 - y'^2) + a^2b^2 - b^2x'^2 - a^2y'^2 = 0.$$

When $y' = 0$ this quadratic becomes $(\lambda^2 - b^2)(\lambda^2 - a^2 + x'^2) = 0$, and *one* of its roots is $\lambda^2 = b^2$: but if we have also $x'^2 = a^2 - b^2$, the second root is also $\lambda^2 = b^2$, and therefore the two foci are in a special sense points corresponding to the value $\lambda^2 = b^2$. If in the equation $\frac{x^2}{a^2-\lambda^2} + \frac{y^2}{b^2-\lambda^2} = 1$, we make $\lambda^2 = b^2$, $\frac{y^2}{b^2-\lambda^2} = 0$, we get the equation of the two foci $\frac{x^2}{a^2-b^2} = 1$.

166. Now in like manner two quadrics are said to be confocal if the differences of the squares of the axes be the same for both. Thus given the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ any surface is confocal whose equation is of the form

$$\frac{x^2}{a^2 \pm \lambda^2} + \frac{y^2}{b^2 \pm \lambda^2} + \frac{z^2}{c^2 \pm \lambda^2} = 1.$$

If we give λ^2 the positive sign, or if we take it negative and less than c^2 the surface is an ellipsoid. A sphere of infinite radius is the limit of all ellipsoids of the system, being what the equation represents when $\lambda^2 = \infty$. When λ^2 is between c^2 and b^2 the surface is a hyperboloid of one sheet. When it is between b^2 and a^2 it is a hyperboloid of two sheets. When $\lambda^2 = c^2$ the surface reduces itself to the plane $z = 0$, but if we make in the equation $\lambda^2 = c^2$, $\frac{z^2}{\lambda^2 - c^2} = 0$, the points on the conic thus found, viz. $\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1$, belong in a special sense to the limit separating ellipsoids and hyperboloids. In fact, in general through any point $x'y'z'$ can be drawn three surfaces confocal to a given one; for regarding λ^2 as the unknown quantity, we have evidently a cubic for the determination of it; namely,

$$\frac{x'^2}{a^2-\lambda^2} + \frac{y'^2}{b^2-\lambda^2} + \frac{z'^2}{c^2-\lambda^2} = 1,$$

or $$x'^2(b^2-\lambda^2)(c^2-\lambda^2) + y'^2(c^2-\lambda^2)(a^2-\lambda^2) + z'^2(a^2-\lambda^2)(b^2-\lambda^2)$$
$$= (a^2-\lambda^2)(b^2-\lambda^2)(c^2-\lambda^2).$$

If $z' = 0$, one of the roots of this cubic is $\lambda^2 = c^2$, the other two being given by the equation

$$x'^2(b^2-\lambda^2) + y'^2(a^2-\lambda^2) = (a^2-\lambda^2)(b^2-\lambda^2),$$

and a root of *this* equation will also be $\lambda^2 = c^2$, if

$$\frac{x'^2}{a^2 - c^2} + \frac{y'^2}{b^2 - c^2} = 1.$$

The points on the focal ellipse therefore belong in a special sense to the value $\lambda^2 = c^2$. In like manner the plane $y = 0$ separates hyperboloids of one sheet from those of two, and to this limit belongs in a special sense the hyperbola in that plane $\frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2 - b^2} = 1$. The focal conic in the third principal plane is imaginary.

167. *The three quadrics which can be drawn through a given point confocal to a given one are respectively an ellipsoid, a hyperboloid of one sheet, and one of two.* For if we substitute in the cubic of the last article successively

$$\lambda^2 = a^2,\ \lambda^2 = b^2,\ \lambda^2 = c^2,\ \lambda^2 = -\infty,$$

we get results successively $+ - + -$ which proves that the equation has always three real roots, one of which is less than c^2, the second between c^2 and b^2, and the third between b^2 and a^2, and it was shown in the last article that the surfaces corresponding to these values of λ^2 are respectively an ellipsoid, a hyperboloid of one sheet, and one of two.

168. Another convenient way of solving the problem to describe through a given point quadrics confocal to a given one, is to take for the unknown quantity the primary axis of the sought confocal surface. Then since we are given $a'^2 - b'^2$ and $a'^2 - c'^2$ which we shall call h^2 and k^2, we have the equation

$$\frac{x'^2}{a'^2} + \frac{y'^2}{a'^2 - h^2} + \frac{z'^2}{a'^2 - k^2} = 1,$$

or

$$a'^6 - a'^4 (h^2 + k^2 + x'^2 + y'^2 + z'^2)$$
$$+ a'^2 \{h^2k^2 + x'^2 (h^2 + k^2) + y'^2k^2 + z'^2h^2\} - x'^2h^2k^2 = 0.$$

From this equation we can at once express the co-ordinates of the intersection of three confocal surfaces in terms of their axes. Thus if a'^2, a''^2, a'''^2 be the roots of the above equation, the last term of it gives us at once $x'^2h^2k^2 = a'^2a''^2a'''^2$, or

$$x'^2 = \frac{a'^2a''^2a'''^2}{(a^2-b^2)(a^2-c^2)}.$$

And by parity of reasoning, since we might have taken b^2 or c^2 for our unknown, we have

$$y'^2 = \frac{b'^2b''^2b'''^2}{(b^2-a^2)(b^2-c^2)},\quad z'^2 = \frac{c'^2c''^2c'''^2}{(c^2-a^2)(c^2-b^2)}.\text{*}$$

N.B. In the above we suppose b'^2, b''^2, &c. to involve their signs implicitly. Thus c''^2 belonging to a hyperboloid of one sheet is essentially negative, as are also b'''^2 and c'''^2.

169. The preceding cubic also enables us to express the radius vector to the point of intersection in terms of the axes. For the second term of it gives us

$$x'^2 + y'^2 + z'^2 + (a^2-b^2) + (a^2-c^2) = a'^2 + a''^2 + a'''^2,$$

or

$$x'^2 + y'^2 + z'^2 = a'^2 + b''^2 + c'''^2.$$

This expression might also have been worked out directly from the values given for x'^2, y'^2, z'^2 in the last article, by a process which may be employed in reducing other symmetrical functions of these co-ordinates. For on substituting the preceding values and reducing to a common denominator, $x'^2 + y'^2 + z'^2$ becomes

$$\frac{a'^2a''^2a'''^2(b^2-c^2) + b'^2b''^2b'''^2(c^2-a^2) + c'^2c''^2c'''^2(a^2-b^2)}{(b^2-c^2)(c^2-a^2)(a^2-b^2)}.$$

But the numerator obviously vanishes if we suppose either $b^2=c^2$, $c^2=a^2$, $a^2=b^2$. It is therefore divisible by the denominator. The division then is performed as follows: Any term, for example $a'^2a''^2a'''^2c^2$, when divided by a^2-b^2 (or by its equal $a'^2-b'^2$) gives a quotient $a''^2a'''^2c^2$, and a remainder $b'^2a''^2a'''^2c^2$. This remainder divided by $a''^2-b''^2$ gives a quotient

* These expressions enable us easily to remember the co-ordinates of the umbilics. The umbilics are the points (Art. 139) where the focal hyperbola meets the surface. But for the focal hyperbola $a''^2 = a'''^2 = a^2 - b^2$. The co-ordinates are therefore

$$x^2 = a^2\frac{a^2-b^2}{a^2-c^2},\quad y=0,\quad z^2 = c^2\frac{b^2-c^2}{a^2-c^2}.$$

$b'^2a'''^2c^2$ and a remainder $b'^2b''^2a'''^2c^2$, which divided in like manner by $a'''^2 - b'''^2$ gives a quotient $b'^2b''^2c^2$ and a remainder $b'^2b''^2b'''^2c^2$, which is destroyed by another term in the dividend. Proceeding step by step in this manner we get the result already obtained.

170. *Two confocal surfaces cut each other everywhere at right angles.*

Let $x'y'z'$ be any point common to the two surfaces, p' and p'' the lengths of the perpendicular from the centre on the tangent plane to each at that point, then (Art. 85) the direction-cosines of these two perpendiculars are

$$\frac{p'x'}{a'^2},\ \frac{p'y'}{b'^2},\ \frac{p'z'}{c'^2};\ \frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2}.$$

And the condition that the two should be at right angles, is, (Art. 13)

$$p'p''\left\{\frac{x'^2}{a'^2a''^2} + \frac{y'^2}{b'^2b''^2} + \frac{z'^2}{c'^2c''^2}\right\} = 0.$$

But since the co-ordinates $x'y'z'$ satisfy the equations of both surfaces we have

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} = 1,\quad \frac{x'^2}{a''^2} + \frac{y'^2}{b''^2} + \frac{z'^2}{c''^2} = 1.$$

And if we subtract one of these equations from the other, and remember that $a''^2 - a'^2 = b''^2 - b'^2 = c''^2 - c'^2$, the remainder is

$$(a''^2 - a'^2)\left\{\frac{x'^2}{a'^2a''^2} + \frac{y'^2}{b'^2b''^2} + \frac{z'^2}{c'^2c''^2}\right\} = 0,$$

which was to be proved.

At the point therefore where three confocals intersect, each tangent plane cuts the other two perpendicularly, and the tangent plane to any one contains the normals to the other two.

171. *If a plane be drawn through the centre parallel to any tangent plane to a quadric, the axes of the section made by that plane are parallel to the normals to the two confocals through the point of contact.*

It has been proved that the parallels to the normals are at right angles to each other, and it only remains to be proved that they are conjugate diameters in their section. But (Art. 90) the condition that two lines should be conjugate diameters is

$$\frac{\cos\alpha\cos\alpha'}{a'^2}+\frac{\cos\beta\cos\beta'}{b'^2}+\frac{\cos\gamma\cos\gamma'}{c'^2}=0.$$

The direction-cosines then of the normals being

$$\frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2};\quad \frac{p'''x'}{a'''^2},\ \frac{p'''y'}{b'''^2},\ \frac{p'''z'}{c'''^2},$$

we have to prove that

$$p''p'''\left\{\frac{x'^2}{a'^2a''^2a'''^2}+\frac{y'^2}{b'^2b''^2b'''^2}+\frac{z'^2}{c'^2c''^2c'''^2}\right\}=0.$$

But the truth of this equation appears at once on subtracting one from the other the equations which have been proved in the last article,

$$\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}=0,\quad \frac{x'^2}{a'^2a'''^2}+\frac{y'^2}{b'^2b'''^2}+\frac{z'^2}{c'^2c'''^2}=0.$$

172. *To find the lengths of the axes of the central section of a quadric by a plane parallel to the tangent plane at the point $x'y'z'$.*

From the equation of the surface the length of a central radius vector whose direction-angles are α, β, γ is given by the equation

$$\frac{1}{\rho^2}=\frac{\cos^2\alpha}{a'^2}+\frac{\cos^2\beta}{b'^2}+\frac{\cos^2\gamma}{c'^2}.$$

Put for α, β, γ the values given in the last article, and we find for the length of one of these axes,

$$\frac{1}{\rho^2}=p''^2\left\{\frac{x'^2}{a'^2a''^4}+\frac{y'^2}{b'^2b''^4}+\frac{z'^2}{c'^2c''^4}\right\}.$$

Now we have the equations,

$$\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}=0,$$

$$\frac{x'^2}{a''^4}+\frac{y'^2}{b''^4}+\frac{z'^2}{c''^4}=\frac{1}{p''^2}.$$

Subtracting we have

$$\frac{x'^2}{a'^2a''^4}+\frac{y'^2}{b'^2b''^4}+\frac{z'^2}{c'^2c''^4}=\frac{1}{p''^2(a'^2-a''^2)}.$$

And substituting this value in the expression already found for ρ^2 we get $\rho^2 = a'^2 - a''^2$. In like manner the square of the other axis is $a'^2 - a'''^2$.

Hence, if two confocal quadrics intersect, and a radius of one be drawn parallel to the normal to the other at any point of their curve of intersection, this radius is of constant length.

173. Since the product of the axes of a central section by the perpendicular on a parallel tangent plane is equal to abc (Art. 54), we get immediately expressions for the lengths p', p'', p'''. We have

$$p'^2 = \frac{a'^2b'^2c'^2}{(a'^2 - a''^2)(a'^2 - a'''^2)}, \quad p''^2 = \frac{a''^2b''^2c''^2}{(a''^2 - a'^2)(a''^2 - a'''^2)},$$

$$p'''^2 = \frac{a'''^2b'''^2c'''^2}{(a'''^2 - a'^2)(a'''^2 - a''^2)}.$$

These values might have been also obtained by substituting in the equation

$$\frac{1}{p'^2} = \frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4},$$

the values already found for x'^2, y'^2, z'^2 and reducing the resulting value for p'^2 by the method of Art. 169.

The reader will observe the symmetry which exists between these values for p'^2, p''^2, p'''^2, and the values already found for x'^2, y'^2, z'^2. If the three tangent planes had been taken as co-ordinate planes, p', p'', p''' would be the co-ordinates of the centre of the surface. The analogy then between the values for $p'p''p'''$ and those for $x'y'z'$ may be stated as follows: With the point $x'y'z'$ as centre three confocals may be described having the three tangent planes for principal planes and intersecting in the centre of the original system of surfaces. The axes of the new system of confocals are a', a'', a'''; b', b'', b'''; c', c'', c'''. The three tangent planes to the new system are the three principal planes of the original system.

If a central section be parallel to one of these principal planes (the plane of xy for instance) in the surface to which it is a tangent, it appears from Art. 172 that the squares of the axes are $a^2 - b^2$, $a^2 - c^2$. In other words, that the section is precisely equal to the focal ellipse, no matter where the point $x'y'z'$ be situated. In like manner the section parallel to the plane of xz is equal to the focal hyperbola.

174. If D be the diameter of a quadric parallel to the tangent line at any point of its intersection with a confocal, and p the perpendicular on the tangent plane at that point, then pD is constant for every point on that curve of intersection. For the tangent line at any point of the curve of intersection of two surfaces is the intersection of their tangent planes at that point, which in this case (Art. 170) is normal to the third confocal through the point. Hence (Art. 172) $D^2 = a''^2 - a'''^2$, and therefore (Art. 173) $p^2D^2 = \frac{a'^2b'^2c'^2}{a'^2 - a''^2}$ which is constant if a', a'' be given.

175. *To find the locus of the pole of a given plane with regard to a system of confocal surfaces.*

Let the given plane be $Ax + By + Cz = 1$, and its pole $\xi\eta\zeta$; then we must identify the given equation with

$$\frac{x\xi}{a^2-\lambda^2} + \frac{y\eta}{b^2-\lambda^2} + \frac{z\zeta}{c^2-\lambda^2} = 1,$$

whence $$\frac{\xi}{a^2-\lambda^2} = A, \quad \frac{\eta}{b^2-\lambda^2} = B, \quad \frac{\zeta}{c^2-\lambda^2} = C.$$

Eliminating λ^2 between these equations we find, for the equations of the locus,

$$\frac{x}{A} - a^2 = \frac{y}{B} - b^2 = \frac{z}{C} - c^2.$$

The locus is therefore a right line perpendicular to the given plane.

The theorem just proved, implicitly contains the solution of the problem, "to describe a surface confocal to a given one to touch a given plane." For since the pole of a tangent plane to a surface is its point of contact, it is evident that but one surface can be described to touch the given plane, its point of contact being the point where the locus line just determined meets the plane. The theorem of this article may also be stated—"The locus of the pole of the tangent plane to any quadric, with regard to any confocal, is the normal to the first surface."

176. *To find an expression for the distance between the point of contact of any tangent plane, and its pole with regard to any confocal surface.*

Let $x'y'z'$ be the point of contact of a tangent plane to the surface whose axes are a, b, c; ξ, η, ζ the pole of the same plane with regard to the surface whose axes are a', b', c'. Then, as in the last article, we have

$$\frac{x'}{a^2} = \frac{\xi}{a'^2}, \quad \frac{y'}{b^2} = \frac{\eta}{b'^2}, \quad \frac{z'}{c^2} = \frac{\zeta}{c'^2},$$

whence $\xi - x' = \dfrac{a'^2 - a^2}{a^2} x'$, $\eta - y' = \dfrac{b'^2 - b^2}{b^2} y'$, $\zeta - z' = \dfrac{c'^2 - c^2}{c^2} z'$,

squaring and adding

$$D^2 = (a'^2 - a^2)^2 \left\{\frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}\right\},$$

whence $D = \dfrac{a'^2 - a^2}{p}$ where p is the perpendicular from the centre on the plane.

177. *The axes of any tangent cone to a quadric are the normals to the three confocals which can be drawn through the vertex of the cone.*

Consider the tangent plane to one of these three surfaces which pass through the vertex $x'y'z'$; then the pole of that plane with regard to the original surface lies (Art. 61) on the polar plane of $x'y'z'$, and (Art. 175) on the normal to the exterior surface. It is therefore the point where that normal meets the polar plane of $x'y'z'$, that is to say, the plane of contact of the cone.

It follows then (Art. 60) that the three normals meet this plane of contact in three points, such that each is the pole of the line joining the other two with respect to the section of the surface by that plane. But since this is also a section of the cone, it follows (Art. 67) that the three normals are a system of conjugate diameters of the cone, and since they are mutually at right angles they are its axes.

178. If at any point on a quadric a line be drawn touching the surface and through that line two tangent planes to any confocal, these two planes will make equal angles with the tangent plane at the given point on the first quadric. For by the last article that tangent plane is a principal plane of the cone touching the confocal surface and having the given point for its vertex, and the two tangent planes will be tangent planes of that cone. But two tangent planes to any cone drawn through a line in a principal plane make equal angles with that plane.

The *focal cones* (that is to say, the cones whose vertices are any points and which stand on the focal conics) are limiting cases of cones enveloping confocal surfaces, and it is still true that the two tangent planes to a focal cone drawn through any tangent line on a surface make equal angles with the tangent plane in which that tangent line lies. If the surface be a cone its focal conic reduces to two right lines, and the theorem just stated in this case becomes, that any tangent plane to a cone makes equal angles with the planes containing its edge of contact and each of the focal lines. This theorem, however, will be proved independently in Chap. IX.

179. It follows, from Art. 177, that if the three normals be made the axes of co-ordinates, the equation of the cone must take the form $Ax^2 + By^2 + Cz^2 = 0$. To verify this by actual transformation will give us an independent proof of the theorem of Art. 177, and a knowledge of the actual values of A, B, C will be useful to us afterwards.

The equation of the tangent cone given, Art. 74, is

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} - 1\right)^2.$$

If the axes be transformed to parallel axes passing through the vertex of the cone, this equation becomes, as is easily seen,

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right)^2.$$

Now to transform to the three normals as axes, we have to substitute the direction-cosines of these lines in the formulæ of Art. 17, and we see that we have to substitute

$$\text{for } x,\ \frac{p'x'}{a'^2}x + \frac{p''x'}{a''^2}y + \frac{p'''x'}{a'''^2}z,$$

$$\text{for } y,\ \frac{p'y'}{b'^2}x + \frac{p''y'}{b''^2}y + \frac{p'''y'}{b'''^2}z,$$

$$\text{for } z,\ \frac{p'z'}{c'^2}x + \frac{p''z'}{c''^2}y + \frac{p'''z'}{c'''^2}z.$$

180. In order more easily to see the result of this substitution the following preliminary formulæ will be useful:

Let $$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1 = S,*$$

then since $$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} - 1 = 0,$$

we have $$\frac{x'^2}{a^2a'^2} + \frac{y'^2}{b^2b'^2} + \frac{z'^2}{c^2c'^2} = \frac{S}{a'^2 - a^2}.$$

In like manner $$\frac{x'^2}{a^2a''^2} + \frac{y'^2}{b^2b''^2} + \frac{z'^2}{c^2c''^2} = \frac{S}{a''^2 - a^2},$$

and hence $$\frac{x'^2}{a^2a'^2a''^2} + \frac{y'^2}{b^2b'^2b''^2} + \frac{z'^2}{c^2c'^2c''^2} = \frac{S}{(a'^2 - a^2)(a''^2 - a^2)}.$$

Lastly, since $$\frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4} = \frac{1}{p'^2},$$

and $$\frac{x'^2}{a'^2a^2} + \frac{y'^2}{b'^2b^2} + \frac{z'^2}{c'^2c^2} = \frac{S}{a'^2 - a^2},$$

we have $$\frac{x'^2}{a'^4a^2} + \frac{y'^2}{b'^4b^2} + \frac{z'^2}{c'^4c^2} = \frac{S}{(a'^2 - a^2)^2} - \frac{1}{p'^2(a'^2 - a^2)}.$$

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* It may be observed that this quantity S is equal to

$$\frac{(a'^2 - a^2)(a''^2 - a^2)(a'''^2 - a^2)}{a^2b^2c^2},$$

for $a'^2 - a^2$, $a''^2 - a^2$, $a'''^2 - a^2$ are the roots of the cubic of Art. 166, whose absolute term is $a^2b^2c^2S$.

181. When now we make the transformations directed, in the left-hand side of the equation of Art. 179, the coefficient of x^2 is found to be

$$p'^2 S\left\{\frac{x'^2}{a'^4a^2}+\frac{y'^2}{b'^4b^2}+\frac{z'^2}{c'^4c^2}\right\},$$

and that of xy is

$$2p'p''S\left\{\frac{x'^2}{a^2a'^2a''^2}+\frac{y'^2}{b^2b'^2b''^2}+\frac{z'^2}{c^2c'^2c''^2}\right\}.$$

The left-hand side therefore of the transformed equation is

$$S^2\left(\frac{p'x}{a'^2-a^2}+\frac{p''y}{a''^2-a^2}+\frac{p'''z}{a'''^2-a^2}\right)^2-S\left\{\frac{x^2}{a'^2-a^2}+\frac{y^2}{a''^2-a^2}+\frac{z^2}{a'''^2-a^2}\right\}.$$

But the quantity $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}$ treated in like manner becomes

$$S\left(\frac{p'x}{a'^2-a^2}+\frac{p''y}{a''^2-a^2}+\frac{p'''z}{a'''^2-a^2}\right).$$

Its square therefore destroys the first group of terms on the other side of the equation, and the equation of the cone becomes

$$\frac{x^2}{a'^2-a^2}+\frac{y^2}{a''^2-a^2}+\frac{z^2}{a'''^2-a^2}=0,$$

which is the required transformed equation of the tangent cone.

182. As a particular case of the preceding may be found the equations of the focal cones (Art. 178); that is to say, the cone whose vertex is any point $x'y'z'$ and which stands on the focal ellipse or focal hyperbola. These answer to the values a^2-c^2, a^2-b^2 for the square of the primary axis: the equations therefore are

$$\frac{x^2}{c'^2}+\frac{y^2}{c''^2}+\frac{z^2}{c'''^2}=0,$$

$$\frac{x^2}{b'^2}+\frac{y^2}{b''^2}+\frac{z^2}{b'''^2}=0.$$

These equations might also have been found, by forming, as at p. 86, the equations of the focal cones, and then transforming them as in the last articles.

It may be seen without difficulty that any normal and the corresponding tangent plane meet any of the principal planes in a point and line which are pole and polar with regard to the focal conic in that plane. This is a particular case of Art. 177.

183. Having all the necessary formulæ at hand, we give also in this place the transformation of the equation of the quadric itself to the three normals through any point $x'y'z'$ as axes. The equation transformed to parallel axes becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + S + 2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right) = 0.$$

But the transformations of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ and of $\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}$ are given in Art. 181. The transformed equation is therefore at once found to be

$$S\left(\frac{p'x}{a'^2-a^2} + \frac{p''y}{a''^2-a^2} + \frac{p'''z}{a'''^2-a^2} + 1\right)^2 = \frac{x^2}{a'^2-a^2} + \frac{y^2}{a''^2-a^2} + \frac{z^2}{a'''^2-a^2},$$

and the quantity under the brackets on the left-hand side of the equation is evidently the transformed equation of the polar plane of the point.

This equation is somewhat modified if the point $x'y'z'$ is on the surface. The equation transformed to parallel axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 2\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2}\right) = 0.$$

When we transform as before, the coefficient of x^2 becomes

$$p^2\left\{\frac{x'^2}{a^6} + \frac{y'^2}{b^6} + \frac{z'^2}{c^6}\right\},$$

which we write $=\frac{1}{\gamma^2}$: that of y^2 becomes

$$p'^2\left(\frac{x'^2}{a'^4a^2} + \frac{y'^2}{b'^4b^2} + \frac{z'^2}{c'^4c^2}\right) = \frac{1}{a^2-a'^2},$$

that of

$$xy = 2pp'\left(\frac{x'^2}{a^4a'^2} + \frac{y'^2}{b^4b'^2} + \frac{z'^2}{c^4c'^2}\right) = \frac{2p'}{p\,(a'^2-a^2)};$$

the coefficient of yz vanishes while the terms of the first degree reduce to $\frac{2x}{p}$. The transformed equation is therefore

$$\frac{x^2}{\gamma^2}+\frac{y^2}{a^2-a'^2}+\frac{z^2}{a^2-a''^2}-\frac{2p'xy}{p\,(a^2-a'^2)}-\frac{2p''xz}{p\,(a^2-a''^2)}+\frac{2x}{p}=0.$$

184. We give in this place also the transformation of the equation of the reciprocal surface with regard to any point to the three normals through the point. The equation is (Art. 162)

$$(xx'+yy'+zz'+k^2)^2=a^2x^2+b^2y^2+c^2z^2.$$

Now using the formulæ of Art. 179, the quantity $xx'+yy'+zz'+k^2$ is immediately transformed into $(p'x+p''y+p'''z+k^2)$. Again, when $a^2x^2+b^2y^2+c^2z^2$ is transformed, the coefficient of x^2 is

$$p'^2\left(\frac{a^2x'^2}{a'^4}+\frac{b^2y'^2}{b'^4}+\frac{c^2z'^2}{c'^4}\right)$$

$$=(a^2-a'^2)\,p'^2\left(\frac{x'^2}{a'^4}+\frac{y'^2}{b'^4}+\frac{z'^2}{c'^4}\right)+p'^2\left(\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}+\frac{z'^2}{c'^2}\right)$$

$$=a^2-a'^2+p'^2.$$

While the coefficient of xy is

$$2p'p''\left\{\frac{a^2x'^2}{a'^2a''^2}+\frac{b^2y'^2}{b'^2b''^2}+\frac{c^2z'^2}{c'^2c''^2}\right\}.$$

But since $$(a^2-a'^2)\left\{\frac{x'^2}{a'^2a''^2}+\frac{y'^2}{b'^2b''^2}+\frac{z'^2}{c'^2c''^2}\right\}=0,$$

and $$\frac{x'^2}{a''^2}+\frac{y'^2}{b''^2}+\frac{z'^2}{c''^2}=1,$$

we have $$\frac{a^2x'^2}{a'^2a''^2}+\frac{b^2y'^2}{b'^2b''^2}+\frac{c^2z'^2}{c'^2c''^2}=1,$$

and the transformed equation is therefore

$$(a'^2-a^2)\,x^2+(a''^2-a^2)\,y^2+(a'''^2-a^2)\,z^2+2k^2(p'x+p''y+p'''z)+k^4=0.$$

185. To return to the equation of the tangent cone (Art. 181). Its form proves that all cones having a common vertex and circumscribing a series of confocal surfaces are coaxal and confocal. For the three normals through the common vertex are axes to every one of the system of cones; and the form of the equation shows that the differences of the squares of the axes are inde-

pendent of a^2. The equations of the common focal lines of the cones are (Art. 140)

$$\frac{x^2}{a'^2 - a''^2} = \frac{z^2}{a''^2 - a'''^2}; \; y^2 = 0.$$

But it was proved (Art. 172) that the central section of the hyperboloid of one sheet which passes through $x'y'z'$ is

$$\frac{x^2}{a'^2 - a''^2} - \frac{z^2}{a''^2 - a'''^2} = 1,$$

and the section of the hyperboloid by the tangent plane itself is similar to this, or is also

$$\frac{x^2}{a'^2 - a''^2} - \frac{z^2}{a''^2 - a'''^2} = 0.$$

Hence the *focal lines of the system of cones are the generating lines of the hyperboloid which passes through the point*—a theorem due to Jacobi (*Crelle*, Vol. XII. p. 137).

This may also be proved thus: Take any edge of one of the system of cones, and through it draw a tangent plane to that cone and also planes containing the generating lines of the hyperboloid; these latter planes are tangent planes to the hyperboloid, and therefore (Art. 178) make equal angles with the tangent plane to the cone. The two generators are therefore such that the planes drawn through them and through any edge of the cone make equal angles with the tangent plane to the cone; but this is a property of the focal lines (Art. 178).

COR. 1. The reciprocals of a system of confocals, with regard to any point, have the same circular sections. For the reciprocals of the tangent cones from that point have the same circular sections (Art. 141), and these reciprocals are the asymptotic cones of the reciprocal surfaces.

COR. 2. If a system of confocals be projected orthogonally on any plane, the projections are confocal conics. The projections are the sections by that plane of cylinders perpendicular to it, and enveloping the quadrics. And these cylinders may be considered as a system of enveloping cones whose vertex is the point at infinity on the common direction of their generators.

186. *Two confocal surfaces can be drawn to touch a given line.*

Take on the line any point $x'y'z'$; let the axes of the three surfaces passing through it be a', a'', a''', and the angles the line makes with these axes α, β, γ. Then it appears, from Art. 181, that a is determined by the *quadratic*

$$\frac{\cos^2\alpha}{a'^2-a^2}+\frac{\cos^2\beta}{a''^2-a^2}+\frac{\cos^2\gamma}{a'''^2-a^2}=0.$$

If a and a′ be the roots of this quadratic, the two cones

$$\frac{x^2}{a'^2-\text{a}^2}+\frac{y^2}{a''^2-\text{a}^2}+\frac{z^2}{a'''^2-\text{a}^2}=0,\quad \frac{x^2}{a'^2-\text{a}'^2}+\frac{y^2}{a''^2-\text{a}'^2}+\frac{z^2}{a'''^2-\text{a}'^2}=0$$

have the given line as a common edge, and it is proved, precisely as at Art. 170, that the tangent planes to the cones through this line are at right angles to each other. And since the tangent planes to a tangent cone to a surface, by definition touch that surface, it follows that *the tangent planes drawn through any right line to the two confocals which it touches, are at right angles to each other.*

The property that the tangent cones from any point to two intersecting confocals cut each other at right angles, is sometimes expressed as follows: *two confocals seen from any point appear to intersect everywhere at right angles.*

187. *If through a given line tangent planes be drawn to a system of confocals, the corresponding normals generate a hyperbolic paraboloid.*

The normals are evidently parallel to one plane; namely, the plane perpendicular to the given line; and if we consider any one of the confocals, then, by Art. 174, the normal to any plane through the line contains the pole of that plane with regard to the assumed confocal, which pole is a point on the polar line of the given line with regard to that confocal. Hence, every normal meets the polar line of the given line with regard to any confocal. The surface generated by the normals is therefore a hyperbolic paraboloid (Art. 111). It is evident that the surface generated by the polar lines, just referred to, is the same paraboloid, of which they form the other system of generators.

The points in which this paraboloid meets the given line are the two points where this line touches confocals.

A special case occurs when the given line is itself a normal to a surface S of the system. The normal corresponding to any plane drawn through that line is found by letting fall a perpendicular on that plane from the pole of the same plane with regard to S (Art. 175), but it is evident that both pole and perpendicular must lie in the tangent plane to S to which the given line is normal. Hence in this case all the normals lie in the same plane.

From the principle that the anharmonic ratio of four planes passing through a line is the same as that of their four poles with regard to any quadric, it is found at once that any four normals divide homographically all the polar lines corresponding to the given line with respect to the system of surfaces. In the special case, now under consideration, the normals will therefore envelope a conic, which conic will be a parabola, since the normal in one of its positions may lie at infinity; namely, when the surface is an infinite sphere (Art. 166). The point where the given line meets the surface to which it is normal lies on the directrix of this parabola.

188. If α, β, γ be the direction-angles, referred to the three normals through the vertex, of the perpendicular to a tangent plane of the cone of Arts. 179, &c., since this perpendicular lies on the reciprocal cone, α, β, γ must satisfy the relation

$$(a'^2 - a^2)\cos^2\alpha + (a''^2 - a^2)\cos^2\beta + (a'''^2 - a^2)\cos^2\gamma = 0,$$

or

$$a'^2\cos^2\alpha + a''^2\cos^2\beta + a'''^2\cos^2\gamma = a^2.$$

This relation enables us at once to determine the axis of the surface which touches any plane, for if we take any point on the plane, we know a', a'', a''' for that point, as also the angles which the three normals through the point make with the plane, and therefore a^2 is known.

189. If the relation of the last article were proved independently, we should, by reversing the steps of the demonstration, obtain a proof without transformation of co-ordinates

of the equation of the tangent cone (Art. 181). The following proof is due to M. Chasles: The quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is the sum of the squares of the projections on a perpendicular to the given plane of the lines a', a'', a'''. We have seen (Art. 173) that these are the axes of a surface having $x'y'z'$ for its centre and passing through the original centre. And it was proved in the same article that three other conjugate diameters of the same surface are the radius vector from the centre to $x'y'z'$, together with two lines equal and parallel to the axes of the focal ellipse. It was also proved (Art. 74) that the sum of the squares of the projections on any line of three conjugate diameters of a quadric is equal to that of any other three conjugate diameters. It follows then that the quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is equal to the sum of the squares of the projections on the perpendicular from the centre on the given plane, of the radius vector, and of two lines equal and parallel to the axes of the focal ellipse. The last two lines are constant in magnitude and direction, and their projections are therefore constant, while the projection of the radius vector is the perpendicular itself which is constant if $x'y'z'$ belong to the given plane. It is proved then that the quantity

$$a'^2 \cos^2\alpha + a''^2 \cos^2\beta + a'''^2 \cos^2\gamma$$

is constant while the point $x'y'z'$ moves in a given plane; and it is evident that the constant value is the a^2 of the surface which touches the given plane, since for it we have

$$\cos\alpha = 1, \quad \cos\beta = 0, \quad \cos\gamma = 0.$$

190. *The locus of the intersection of three planes mutually at right angles, each of which touches one of three confocals is a sphere.*

This is proved as in Art. 89.

Add together

$$\begin{aligned} p^2 &= a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma, \\ p'^2 &= a'^2 \cos^2\alpha' + b'^2 \cos^2\beta' + c'^2 \cos^2\gamma', \\ p''^2 &= a''^2 \cos^2\alpha'' + b''^2 \cos^2\beta'' + c''^2 \cos^2\gamma'', \end{aligned}$$

when we get $\rho^2 = a^2 + b^2 + c^2 + (a'^2 - a^2) + (a''^2 - a^2)$,
where ρ is the distance from the centre of the intersection of the planes.

Again, by subtracting one from the other, the two equations
$p^2 = a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma,\ p'^2 = a'^2\cos^2\alpha + b'^2\cos^2\beta + c'^2\cos^2\gamma$,
we learn that the difference of the squares of the perpendiculars on two parallel tangent planes to two confocals is constant and equal $a^2 - a'^2$.

191. *Two cones having a common vertex envelope two confocals; to find the length of the intercept made on one of their common edges by a plane through the centre parallel to the tangent plane to one of the confocals through the vertex.* The intercepts made on the four common edges are of course all equal since the edges are equally inclined to the plane of section which is parallel to a common principal plane of both cones.

Let there be any two confocal cones

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 0,\quad \frac{x^2}{\alpha'^2} + \frac{y^2}{\beta'^2} + \frac{z^2}{\gamma'^2} = 0,$$

then for their intersection, we have

$$\frac{x^2}{\alpha^2\alpha'^2(\beta^2-\gamma^2)} = \frac{y^2}{\beta^2\beta'^2(\gamma^2-\alpha^2)} = \frac{z^2}{\gamma^2\gamma'^2(\alpha^2-\beta^2)},$$

and if the common value of these be λ^2, we have

$$x^2 + y^2 + z^2 = \lambda^2(\alpha^2-\beta^2)(\beta^2-\gamma^2)(\alpha^2-\gamma^2).$$

Putting in the values of α^2, β^2, γ^2 from the equations of the tangent cones (Art. 186), and remembering that the x^2 of the plane through the centre is $\dfrac{a'^2b'^2c'^2}{(a'^2-a''^2)(a'^2-a'''^2)}$ (Art. 173), we get for the square of the required intercept

$$\frac{a'^2b'^2c'^2}{(a'^2-\mathrm{a}^2)(a'^2-\mathrm{a}'^2)}.$$

If then the surfaces be all of different kinds this value shews that the intercept is equal to the perpendicular from the centre on the tangent plane at their intersection.

In the particular case where the two cones considered are the cones standing on the focal ellipse, and on the focal hyperbola we have $\mathrm{a}^2 = a^2 - c^2$, $\mathrm{a}''^2 = a^2 - b^2$, and the intercept reduces to a'. Hence, *if through any point on an ellipsoid be drawn a chord meeting both focal conics, the intercept on this chord by a plane through the centre parallel to the tangent plane at the point will be equal to the axis-major of the surface.* This theorem, due to Prof. Mac Cullagh, is analogous to the theorem for plane curves, that a line through the centre parallel to a tangent to an ellipse cuts off on the focal radii portions equal to the axis-major.

192. M. Chasles has used the principles just established to solve the problem to determine the magnitude and direction of the axes of a central quadric being given a system of three conjugate diameters.

Consider first the plane of any two of the conjugate diameters, and we can by plane geometry determine in magnitude and direction the axes of the section by that plane. The tangent plane at P, the extremity of the remaining diameter, will be parallel to the same plane. Now it was proved (Art. 173) that the centre of the given quadric is the point of intersection of three confocals, having the point P for their centre. If now we could construct the focal conics of this new system of confocals, then the two focal cones, whose common vertex is the centre of the original quadric, determine by their mutual intersection four right lines. The six planes containing these four right lines intersect two by two in the directions of the required axes, while (Art. 191) the three tangent planes through the point P cut off on these four lines parts equal in length to the axes.

The focal conics required are immediately constructed. We know the planes in which they lie and the direction of their axes. The lengths of their axes are to be $a^2 - a''^2$, $a'^2 - a''^2$; $a^2 - a'^2$, $a'^2 - a''^2$. But now the lengths of the axes of the given section are $a^2 - a'^2$, $a^2 - a''^2$ (Art. 172), and these latter axes being known, the axes of the focal conics are immediately found.

193. If through any point P on a quadric a chord be drawn, as in Art. 191, touching two confocals, we can find

an expression for the length of that chord. Draw a parallel semi-diameter through the centre, the length of which we shall call R. And if through P there be drawn a plane conjugate to this diameter, and a tangent plane, they will intercept (counting from the centre) portions on the diameter whose product $=R^2$. But the portion intercepted by the conjugate plane is half the chord required, and the portion intercepted by the tangent plane is the intercept found (Art. 191). Hence

$$C=\frac{2R^2\sqrt{\{(a'^2-\mathrm{a}^2)(a'^2-\mathrm{a}'^2)\}}}{a'b'c'}.$$

When the chord is that which meets the two focal conics; $\mathrm{a}^2=a'^2-b'^2$, $\mathrm{a}'^2=a'^2-c'^2$, and $C=\dfrac{2R^2}{a'}$.

194. *To find the locus of the vertices of right cones which can envelope a given surface.*

In order that the equation $\dfrac{x^2}{a'^2-a^2}+\dfrac{y^2}{a''^2-a^2}+\dfrac{z^2}{a'''^2-a^2}=0$ may represent a right cone, two of the coefficients must be equal; that is to say, $a''=a'$, or $a''=a'''$, or in other words, for the point $x'y'z'$ the equation of Art. 166 must have two equal roots, but from what was proved as to the limits within which the roots lie, it is evident that we cannot have equal roots except when λ is equal to one of the principal axes, or when $x'y'z'$ is on one of the focal conics. This agrees with what was proved (Art. 144).

It appears, hence, that the reciprocal of a surface, with regard to a point on a focal conic, is a surface of revolution; and that the reciprocal, with regard to an umbilic, is a paraboloid of revolution. For an umbilic is a point on a focal conic (Art. 139), and since it is on the surface the reciprocal with regard to it is a paraboloid.

Another particular case of this theorem is that two right cylinders can be circumscribed to a central quadric, the edges of the cylinders being parallel to the asymptotes of the focal hyperbola. For a cone whose vertex is at infinity is a cylinder.

As a particular case of the theorem of this article, the cone standing on the focal ellipse will be a right cone only when its vertex is on the focal hyperbola and *vice versâ*. This theorem of course may be stated without any reference to the quadrics of which the two conics are focal conics; that *the locus of the vertices of right cones which stand on a given conic is a conic of opposite species in a perpendicular plane.* If the equation of one conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that of the other will be $\frac{x^2}{a^2 - b^2} - \frac{z^2}{b^2} = 1$.

It was proved (p. 126) that if a quadric circumscribe a surface of revolution, the cone enveloping the former whose vertex is a focus of the latter is of revolution. From this article then we see that the focal conics of a quadric are the locus of the foci of all possible surfaces of revolution which can circumscribe that quadric.

195. The following examples will serve further to illustrate the principles which have been laid down:

Ex. 1. To find the locus of the intersection of generators to a hyperboloid which cuts at right angles.

The section parallel to the tangent plane which contains the generators must be an equilateral hyperbola, so that (Art. 172) $(a''^2 - a^2) + (a''^2 - a'''^2) = 0$. But (Art. 169) the square of the radius vector to the point is

$$a'^2 + b''^2 + c'^2 - (a''^2 - a^2) - (a''^2 - a'''^2).$$

We have, therefore, the locus a sphere, the square of whose radius is equal to $a'^2 + b'^2 + c'^2$. Otherwise thus: If two generators are at right angles, their plane together with the plane of each and of the normal at the point, are a system of three tangent planes to the surface, mutually at right angles, whose intersection lies on the sphere $r^2 = a'^2 + b'^2 + c'^2$ (Art. 89).

Ex. 2. To find the locus of the intersection of three tangent lines to a quadric mutually at right angles (see p. 86).

Let α, β, γ be the angles made by one of these tangents with the normals through the locus point, and since each of these tangents lies on the tangent cone through that point, we have the conditions

$$\frac{\cos^2\alpha}{a'^2 - a^2} + \frac{\cos^2\beta}{a''^2 - a^2} + \frac{\cos^2\gamma}{a'''^2 - a^2} = 0,$$

$$\frac{\cos^2\alpha'}{a'^2 - a^2} + \frac{\cos^2\beta'}{a''^2 - a^2} + \frac{\cos^2\gamma'}{a'''^2 - a^2} = 0,$$

$$\frac{\cos^2\alpha''}{a'^2 - a^2} + \frac{\cos^2\beta''}{a''^2 - a^2} + \frac{\cos^2\gamma''}{a'''^2 - a^2} = 0.$$

Adding, we have

$$\frac{1}{a'^2 - a^2} + \frac{1}{a''^2 - a^2} + \frac{1}{a'''^2 - a^2} = 0.$$

But $a'^2 - a^2$, $a''^2 - a^2$, $a'''^2 - a^2$ are the three roots of the cubic of Art. 166, which arranged in terms of λ^2 is

$$\lambda^6 + \lambda^4 (x^2 + y^2 + z^2 - a^2 - b^2 - c^2)$$
$$- \lambda^2 \{(b^2 + c^2) x^2 + (c^2 + a^2) y^2 + (a^2 + b^2) z^2 - b^2c^2 - c^2a^2 - a^2b^2\}$$
$$+ b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2 - a^2b^2c^2 = 0.$$

And the sum of the reciprocals of the roots will vanish when the coefficient of $\lambda^2 = 0$. This, therefore, gives us the equation of the locus required.

Ex. 3. The section of an ellipsoid by the tangent plane to the asymptotic cone of a confocal hyperboloid is of constant area.

The area (Art. 92) is inversely proportional to the perpendicular on a parallel tangent plane, and we have

$$p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma.$$

But since the perpendicular is an edge of the cone reciprocal to the asymptotic cone of the hyperboloid, we have

$$0 = a'^2 \cos^2 \alpha + b'^2 \cos^2 \beta + c'^2 \cos^2 \gamma,$$

whence $$p^2 = a^2 - a'^2.$$

Ex. 4. To find the length of the perpendicular from the centre on the polar plane of $x'y'z'$ in terms of the axes of the confocals which pass through that point.

Ans. If $a'^2 - a^2 = h^2$, $a''^2 - a^2 = k^2$, $a'''^2 - a^2 = l^2$,

$$\frac{1}{p^2} = \frac{h^2k^2l^2}{a^2b^2c^2} \left\{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{h^2} + \frac{1}{k^2} + \frac{1}{l^2}\right\}.$$

196. Two points, one on each of two confocal ellipsoids, are said to correspond if

$$\frac{x}{a} = \frac{X}{A}, \quad \frac{y}{b} = \frac{Y}{B}, \quad \frac{z}{c} = \frac{Z}{C}.$$

It is evident that the intersection of two confocal hyperboloids pierces a system of ellipsoids in corresponding points, for from the value (Art. 168) $x^2 = \frac{a^2a'^2a''^2}{(a^2 - b^2)(a^2 - c^2)}$, the quantity $\frac{x^2}{a^2}$ is constant as long as the hyperboloids, having a'^2, a''^2 for axes, are constant.

It will be observed that, the principal planes being limits of confocal surfaces, points on the principal planes determined by equations of the form $\frac{x'^2}{a^2} = \frac{X^2}{a^2 - c^2}$, $\frac{y'^2}{b^2} = \frac{Y^2}{b^2 - c^2}$, $Z^2 = 0$, correspond to any point $x'y'z'$ on a surface, and when $x'y'z'$ is in the principal plane, the corresponding point is on the focal conic.

197. The points on the plane of xy, which correspond to the intersection of an ellipsoid with a series of confocal surfaces, form a series of confocal conics, of which the points corresponding to the umbilics are the common foci.

Eliminating z^2 between the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1,$$

we find

$$\frac{(a^2 - c^2)\, x^2}{a^2 a'^2} + \frac{(b^2 - c^2)\, y^2}{b^2 b'^2} = 1,$$

whence the corresponding points are connected by the relation

$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1.$$

This is evidently an ellipse for the intersections with hyperboloids of one sheet, and a hyperbola for the intersections with hyperboloids of two.

The coordinates of the umbilics are

$$x^2 = a^2 \frac{a^2 - b^2}{a^2 - c^2}, \quad y^2 = 0,$$

the points corresponding to which are

$$X^2 = a^2 - b^2, \quad Y = 0,$$

which are therefore the foci of the system of confocal conics.

Curves on the ellipsoid are sometimes expressed by what are called elliptic co-ordinates; that is to say, by an equation of the form $\phi(a', a'') = 0$, expressing a relation between the axes of the confocal hyperboloids which can be drawn through the point. Now since it appears from this article that a' is half the sum and a'' half the difference of the distances of the points corresponding to the points of the locus from the points which correspond to the umbilics, we can from the equation $\phi(a', a'')=0$ obtain an equation $\phi(\rho+\rho', \rho-\rho')=0$, from which we can form the equation of the curve on the principal plane which corresponds to the given locus.

198. If the intersection of a sphere and an ellipsoid be projected on either plane of circular section by lines parallel to the least (or greatest) axis, the projection will be a circle.

This theorem is only a particular case of the following: that "if any two quadrics have common circular sections, any quadric through their intersection will have the same;" a theorem which is evident, since if by making $z=0$ in U and in V the result in each case represents a circle, making $z=0$ in $U+kV$, must also represent a circle.

It will be useful, however, to investigate this particular theorem directly. If we take as axes the axis of y which is a line in the plane of circular section and a perpendicular to it in that plane, the y will remain unaltered, and the new $x^2=$ the old x^2+z^2. But by the equation of the plane of circular section $z^2=\frac{c^2}{a^2}\cdot\frac{a^2-b^2}{b^2-c^2}x^2$, the new $x^2=\frac{b^2}{a^2}\cdot\frac{a^2-c^2}{b^2-c^2}x^2$.

But for the intersection of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad x^2+y^2+z^2=r^2,$$

we have
$$\frac{a^2-c^2}{a^2}x^2+\frac{b^2-c^2}{b^2}y^2=r^2-c^2,$$

which, on substituting for x^2,

$$\frac{b^2-c^2}{a^2-c^2}\cdot\frac{a^2}{b^2}x^2 \text{ becomes } \frac{b^2-c^2}{b^2}(x^2+y^2)=r^2-c^2.$$

It will be observed that to obtain the projection on the planes of circular sections we left y unaltered, and substituted for x^2, $\frac{b^2-c^2}{a^2-c^2}\cdot\frac{a^2}{b^2}x^2$. But to obtain the points corresponding to any point, as in the last article, we substitute for x^2, $\frac{a^2}{a^2-c^2}x^2$, and for y^2, $\frac{b^2}{b^2-c^2}y^2$. Now the squares of the former co-ordinates have to those of the latter the constant ratio $\frac{b^2-c^2}{b^2}$. Hence we may immediately infer from the last article that the projection of the intersection of two confocal quadrics on a plane of circular section of one of them is a conic whose foci are the similar projections of the umbilics; and, again, that given any curve $\phi(a', a'')$ on the ellipsoid we can obtain the algebraic equation of the projection of that curve on the plane of circular section.

199. *The distance between two points, one on each of two confocal ellipsoids is equal to the distance between the two corresponding points.*

We have

$$(x-X)^2+(y-Y)^2+(z-Z)^2$$
$$=x^2+y^2+z^2+X^2+Y^2+Z^2-2(xX+yY+zZ).$$

Now (Art. 169)

$$x^2+y^2+z^2=a^2+b''^2+c''^2, \quad X^2+Y^2+Z^2=A^2+B''^2+C''^2.$$

But for the corresponding points

$$X'^2+Y'^2+Z'^2=A^2+b''^2+c''^2, \quad x'^2+y'^2+z'^2=a^2+B''^2+C''^2.$$

The sum of the squares therefore of the central radii to the two points is the same as that for the two corresponding points. But the quantities xX, yY, zZ are evidently respectively equal to $x'X'$, $y'Y'$, $z'Z'$, since $X'=\frac{Ax}{a}$, $x'=\frac{aX}{A}$, &c. The theorem of this article, due to Sir J. Ivory, is of use in the theory of attractions.

200. In order to obtain a property of quadrics analogous to the property of conics that the sum of the focal distances is constant, Jacobi states the latter property as follows: Take the two points C and C' on the ellipse at the extremity of the axis-major, then the same relation $\rho+\rho'=2a$ which connects the distances from C and C' of any point on the line joining these points, connects also the distances from the foci of any point on the ellipse. Now, in like manner, if we take on the

principal section of an ellipsoid the three points which correspond in the sense explained (Art. 196) to any three points on the focal ellipse, the same relation which connects the distances from the former points of any point in their plane will also connect the distances from the latter points of any point on the surface. In fact, by Art. 198, the distances of the points on the confocal conic from a point on the surface will be equal to the distances of the point on the principal plane which *corresponds* to the point on the surface, from the three points in the principal section.*

201. Conversely, let it be required to find the locus of a point whose distances from three fixed points are connected by the same relation as that which connects the distances from the vertices of a triangle, whose sides are a, b, c, to any point in its plane. Let ρ, ρ', ρ'' be the three distances, then (Art. 50) the relation which connects them is

$$a^2(\rho^2-\rho'^2)(\rho^2-\rho''^2)+b^2(\rho'^2-\rho^2)(\rho'^2-\rho''^2)+c^2(\rho''^2-\rho^2)(\rho''^2-\rho'^2)$$
$$-a^2(b^2+c^2-a^2)\rho^2-b^2(c^2+a^2-b^2)\rho'^2-c^2(a^2+b^2-c^2)\rho''^2+a^2b^2c^2=0.$$

But $\rho^2-\rho'^2$, &c. being only functions of the co-ordinates of the first degree, the locus is manifestly only of the second degree.

That any of the points from which the distances are measured is a focus is proved by shewing that this equation

* Mr. Townsend has shewed from geometrical considerations (*Cambridge and Dublin Mathematical Journal*, Vol. III., p. 154) that this property only belongs to points on the *modular* focal conics, and in fact the points in the plane y which correspond to any point $x'y'z'$ on an ellipsoid are imaginary as easily appears from the formula of Art. 196. Mr. Townsend easily derives Jacobi's mode of generation from Mac Cullagh's modular property. For if through any point on the surface we draw a plane parallel to a circular section, it will cut the directrices corresponding to the three fixed foci in a triangle of invariable magnitude and figure, and the distances of the point on the surface from the three foci will be in a constant ratio to its distances from the vertices of this triangle. And a similar triangle can be formed with its sides increased or diminished in a fixed ratio, the distances from the vertices of which to the point $x'y'z'$ shall be equal to its distances from the foci.

is of the form $S+LM$, where S is the infinitely small sphere whose centre is this point. In other words, it is required to prove that the result of making $\rho^2=0$ in the preceding equation is the product of two equations of the first degree. But that result is

$$a^2(\rho'^2-c^2)(\rho''^2-b^2)+(b^2\rho'^2-c^2\rho''^2)(\rho'^2-\rho''^2+b^2-c^2).$$

Let now the planes represented by $\rho'^2-\rho^2-c^2$, $\rho''^2-\rho^2-b^2$ be L and M, then the result of making $\rho^2=0$ in the equation is

$$a^2LM+(b^2L-c^2M)(L-M),$$

or
$$b^2L^2-2bcLM\cos A+c^2M^2,$$

where A is the angle opposite a in the triangle abc. But this breaks up into two imaginary factors, shewing that the point we are discussing is a focus of the modular kind.

202. *If several parallel tangent planes touch a series of confocals, the locus of their points of contact is a hyperbola.*

Let α, β, γ be the direction-angles of the perpendicular on the tangent planes. Then the direction-cosines of the radius vector to any point of contact are $\frac{a^2\cos\alpha}{rp}$, $\frac{b^2\cos\beta}{rp}$, $\frac{c^2\cos\gamma}{rp}$; as easily appears by substituting in the formula (Art. 85) $\cos\alpha=\frac{px'}{a^2}$, $r\cos\alpha'$ for x' and solving for $\cos\alpha'$. Forming then by Art. 15, the direction-cosines of the perpendicular to the plane of the radius vector and the perpendicular on the tangent plane, we find them to be

$$\frac{(b^2-c^2)\cos\beta\cos\gamma}{rp\sin\phi},\quad \frac{(c^2-a^2)\cos\gamma\cos\alpha}{rp\sin\phi},\quad \frac{(a^2-b^2)\cos\alpha\cos\beta}{rp\sin\phi},$$

where ϕ is the angle between the radius vector and the perpendicular. Now the denominator is double the area of the triangle of which the radius vector and perpendicular are sides. Double the projections, therefore, of this triangle on the co-ordinate planes are

$$(b^2-c^2)\cos\beta\cos\gamma,\quad (c^2-a^2)\cos\gamma\cos\alpha,\quad (a^2-b^2)\cos\alpha\cos\beta.$$

Now these projections being constant for a system of confocal surfaces, we learn that for such a system, both the plane of the triangle and its magnitude is constant. If then CM be the perpendicular on the series of parallel tangent planes and PM the perpendicular on that line from any point of contact P, we have proved that the plane and the magnitude of the triangle CPM are constant, and therefore the locus of P is a hyperbola of which CM is an asymptote.

203. The reciprocal of a system of confocal surfaces

$$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}+\frac{z^2}{c^2-\lambda^2}=1$$

is the system of concyclic surfaces

$$(a^2-\lambda^2)\,x^2+(b^2-\lambda^2)\,y^2+(c^2-\lambda^2)\,z^2=R^4.$$

Now the latter equation denotes a system of quadrics passing through a common curve, one quadric of the system being the point sphere $x^2+y^2+z^2=0$. The reciprocal system is therefore inscribed in a common developable. Many of the properties proved in this chapter for confocal surfaces can be derived as particular cases of properties of surfaces inscribed in a common developable. Compare Arts. 132, 170, and Arts. 122, 175.*

Since the tangent cone from any point on a focal conic is one of revolution; that is to say, one which has double contact with the imaginary circle at infinity (Art. 135), it follows that through any point on a focal conic can be drawn two imaginary planes which will touch every confocal surface, and we thus see geometrically the existence of this developable, the tangent planes to which touch all the confocals. And we can also see that it is the same as the developable generated by the tangent planes to the surface which pass through the tangents to the imaginary circle at infinity. The actual equation of the developable is obtained by forming the discriminant with regard to λ^2 of the equation of the confocals. The imaginary circle at infinity and the focal conics are all double lines on this surface.

* See also Chasles' *Hist. Geom.*, p. 397, and *Quarterly Journal of Mathematics*, Vol. III., p. 155.

CURVATURE OF QUADRICS.

204. The general theory of the curvature of surfaces will be explained in Chap. x., but it will be convenient to state here some theorems on the curvature of quadrics which are immediately connected with the subject of this chapter.

If a normal section be made at any point on a quadric its radius of curvature at that point is equal to $\frac{\beta^2}{p}$, *where* β *is the semi-diameter parallel to the trace of the section on the tangent plane, and* p *is the perpendicular from the centre on the tangent plane.*

We repeat the following proof by the method of infinitesimals from *Conics*, p. 296, which see.

Let P, Q be any two points on a quadric; let a plane through Q parallel to the tangent plane at P meet the central radius CP in R, and the normal at P in S, then the radius of a circle through the points P, Q having its centre on PS is $\frac{PQ^2}{2PS}$. But if the point Q approach indefinitely near to P, QP is in the limit equal to QR; and if we denote CP and the central radius parallel to QR by a' and β, and if P' be the other extremity of the diameter CP, then (Art. 70)

$$\beta^2 : a'^2 :: QR^2 : PR.RP' \; (=2a'.PR);$$

therefore $QR^2 = \frac{2\beta^2.PR}{a'}$ and the radius of curvature $= \frac{\beta^2}{a'} \cdot \frac{PR}{PS}$.

But if from the centre we let fall a perpendicular CM on the tangent plane, the right-angled triangle CMP is similar to PRS and $PR : PS :: a' : p$. And the radius of curvature is therefore $\frac{\beta^2}{a'} \cdot \frac{a'}{p} = \frac{\beta^2}{p}$; which was to be proved.

If the circle through PQ have its centre not on PS but on any line PS' making an angle θ with PS, the only change is that the radius of the circle is $\frac{PQ^2}{2PS'}$, S' being still on the plane drawn through Q parallel to the tangent plane at P.

But PS evidently $= PS' \cos\theta$. The radius of curvature is therefore $\frac{PQ^2}{2PS}\cos A$, or *the value for the radius of curvature of an oblique section is the radius of curvature of the normal section through PQ, multiplied by* $\cos\theta$.

205. These theorems may also easily be proved analytically. It is proved (*Conics*, p. 206) that if $Ax^2 + 2Bxy + Cy^2 + 2Ey = 0$ be the equation of any conic, the radius of curvature at the origin is $= \frac{E}{A}$. If then the equation of any quadric, the plane of xy being a tangent plane, be

$$Ax^2 + 2Bxy + Cy^2 + 2Lxz + 2Myz + Nz^2 + 2Ez = 0,$$

then the radii of curvature by the sections $y = 0$, $x = 0$ are respectively $\frac{E}{A}$, $\frac{E}{C}$. But if the equation be transformed to parallel axes through the centre, the terms of highest degree remain unaltered, and the equation becomes

$$Ax^2 + 2Bxy + Cy^2 + 2Lxz + 2Myz + Nz^2 = H.$$

The squares of the intercepts on the axis of x and y are $\frac{H}{A}$, $\frac{H}{C}$. This proves that the radii of curvature are proportional to the squares of the parallel semi-diameters of a central section. And since, by the theory of conics, the radius of curvature of that section which contains the perpendicular on the tangent plane is $\frac{\beta^2}{p}$, the same is the form of the radius of every other section.

The same may be proved by using the equation of the quadric transformed to any normal and the normals to two confocals as axes (Art. 183), viz.

$$\frac{x^2}{\gamma^2} + \frac{y^2}{a^2 - a'^2} + \frac{z^2}{a^2 - a''^2} - \frac{2p'xy}{p(a^2 - a'^2)} - \frac{2p''xz}{a^2 - a''^2} + \frac{2x}{p} = 0.$$

The radii of curvature of the sections by the planes $z = 0$, $y = 0$ are respectively $\frac{a^2 - a'^2}{p}$, $\frac{a^2 - a''^2}{p}$. The numerators are the squares of the semi-axes of the section by a plane parallel to the tangent plane (Art. 172). The equation of the section made by a plane making an angle θ with the plane of y is found by first turning the axes of co-ordinates round through an angle θ by substituting $y\cos\theta - z\sin\theta$, $y\sin\theta + z\cos\theta$ for y and z, and then making the new $z = 0$. The coefficient of y^2 will then become

$$\frac{\cos^2\theta}{a^2 - a'^2} + \frac{\sin^2\theta}{a^2 - a''^2},$$

and the radius of curvature is

$$\frac{1}{p}\left(\frac{\cos^2\theta}{a^2 - a'^2} + \frac{\sin^2\theta}{a^2 - a''^2}\right).$$

But this coefficient of y^2 is evidently the square of that semi-diameter of the central section, which makes an angle θ with the axis y.

206. It follows from the theorem enunciated in Art. 204, that *at any point on a central quadric the radius of curvature of a normal section has a maximum and minimum value, the directions of the section for these values being parallel to the axis-major and axis-minor of the central section by a plane parallel to the tangent plane.*

These maximum and minimum values are called the *principal radii* of curvature for that point, and the sections to which they belong are called the principal sections. It appears from (Art. 171) that the principal sections contain each the normal to one of the confocals through the point. The intersection of a quadric with a confocal is a curve such that at every point of it the tangent to the curve is one of the principal directions of curvature. Such a curve is called a *line of curvature* on the surface.

In the case of the hyperboloid of one sheet the central section is a hyperbola, and the sections whose traces on the tangent plane are parallel to the asymptotes of that hyperbola will have their radii of curvature infinite; that is to say, they will be right lines, as we know already. In passing through one of those sections the radius of curvature changes sign; that is to say, the direction of the convexity of sections on one side of one of those lines is opposite to that of those on the other.

207. *The two principal centres of curvature are the two poles of the tangent plane with regard to the two confocal surfaces which pass through the point of contact.* For these poles lie on the normal to that plane (Art. 175), and at distances from it $= \dfrac{a^2 - a'^2}{p}$ and $\dfrac{a^2 - a''^2}{p}$ (Art. 176), but these have been just proved to be the lengths of the principal radii of curvature.

We can also hence find, by Art. 176, the co-ordinates of the centres of the two principal circles of curvature, viz.

$$x = \frac{a'^2 x'}{a^2},\quad y = \frac{b'^2 y'}{b^2},\quad z = \frac{c'^2 z'}{c^2};\quad x = \frac{a''^2 x'}{a^2},\quad y = \frac{b''^2 y'}{b^2},\quad z = \frac{c''^2 z'}{c^2}.$$

208. If at each point of a quadric we take the two principal centres of curvature, the locus of all these centres is a surface of two sheets which is called the *surface of centres*. To find its equation, we observe that the co-ordinates x', y', z' satisfy the equations

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1,\quad \frac{x'^2}{a^2 a'^2} + \frac{y'^2}{b^2 b'^2} + \frac{z'^2}{c^2 c'^2} = 0.$$

Substituting for x' in terms of x by the help of the last article, and writing for a'^2, $a^2 - h^2$, &c., we obtain the following two equations:

$$\frac{a^2 x^2}{(a^2 - h^2)^2} + \frac{b^2 y^2}{(b^2 - h^2)^2} + \frac{c^2 z^2}{(c^2 - h^2)^2} = 1,$$

$$\frac{a^2 x^2}{(a^2 - h^2)^3} + \frac{b^2 y^2}{(b^2 - h^2)^3} + \frac{c^2 z^2}{(c^2 - h^2)^3} = 0.$$

These equations express that all the centres which correspond to points on a line of curvature on the given surface, for which h is constant, lie on the intersection of two quadrics. If we eliminate h^2 between these two equations, we get the equation of the surface of centres. I have performed this elimination and given the result, *Quarterly Journal of Mathematics*, Vol. III., p. 218.* The surface is one of the twelfth degree.

* It may be worth while to state the process by which the elimination was effected:

209. We can see *à priori* the nature of the section of the surface by the principal planes. In fact, one of the principal radii of curvature at any point on a principal section is the radius of curvature of the section itself, and the locus of the centres corresponding is evidently the evolute of that section. The other radius of curvature corresponding to any point in the section by the plane of xy is $\frac{c^2}{p}$, as appears from the formula of Art. 204, since c is an axis in every section drawn through the axis of z. From the formulæ of Art. 207 the co-ordinates of the corresponding centre are $\frac{a^2-c^2}{a^2}x'$, $\frac{b^2-c^2}{b^2}y'$; that is to say, they are the poles with regard to the focal conic of the tangent at the point $x'y'$ to the principal section. The locus of the centres will be the reciprocal of the principal section, taken with regard to the focal conic, viz.

$$\frac{a^2x^2}{(a^2-c^2)^2}+\frac{b^2y^2}{(b^2-c^2)^2}=1.$$

The section then by a principal plane of the surface (which is of the twelfth degree) consists of the evolute of a conic, which is of the sixth degree, and of a conic (it will be found) three times over, this conic being a double line on the surface. The section by the plane at infinity is also of a similar nature.

Substitute in the co-ordinates of the centre of curvature (Art. 207) the values for x'^2, y'^2, z'^2 (Art. 168), and we have

$$a^2x^2(a^2-b^2)(a^2-c^2)=a'^6a''^2,\quad b^2y^2(b^2-a^2)(b^2-c^2)=b'^6b''^2,$$
$$c^2z^2(c^2-a^2)(c^2-b^2)=c'^6c''^2.$$

Now if we write $a'^2=a^2-h^2$, $a''^2=a^2-k^2$, the first equation may be thrown into the form

$$a^2x^2(a^2-b^2)(a^2-c^2)=a^8-Pa^6+Qa^4-Ra^2+S,$$

while the right-hand sides of the other two equations are got by writing b^2 and c^2, in turn, instead of a^2 in this last equation. We thus get three linear relations between P, Q, R, S. But further, since these quantities are coefficients of a biquadratic equation which has three roots equal, those coefficients are connected by two relations, one of the second, the other of the third degree. The elimination is thus reduced to elimination between a cubic and a quadratic equation, which is practicable.

210. *The reciprocal of the surface of centres is a surface of the fourth degree.*

It will appear from the general theory of the curvature of surfaces, to be explained in the next chapter, that the tangent plane to either of the confocal surfaces through $x'y'z'$ is also a tangent plane to the surface of centres. The reciprocals of the intercepts which the tangent plane makes on the axes are given by the equation

$$\xi = \frac{x'}{a'^2}, \quad \eta = \frac{y'}{b'^2}, \quad \zeta = \frac{z'}{c'^2}.$$

The relation

$$\frac{x'^2}{a^2a'^2} + \frac{y'^2}{b^2b'^2} + \frac{z'^2}{c^2c'^2} = 0$$

gives between ξ, η, ζ the relation

$$(\xi^2 + \eta^2 + \zeta^2) = (a^2 - a'^2)\left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right),$$

and the relation

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} + \frac{z'^2}{c'^2} = 1$$

gives $\quad (a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1) = (a^2 - a'^2)(\xi^2 + \eta^2 + \zeta^2).$

Eliminating $a^2 - a'^2$, we have

$$(\xi^2 + \eta^2 + \zeta^2)^2 = \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2}\right)(a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - 1).*$$

But it is evident (as at *Higher Plane Curves*, p. 14) that ξ, η, ζ may be understood to be co-ordinates of the reciprocal surface; since, if ξ, η, ζ be the co-ordinates of the pole of the tangent plane with regard to the sphere $x^2 + y^2 + z^2 = 1$, the equation $x\xi + y\eta + z\zeta = 1$ being identical with that of the tangent plane, ξ, η, ζ will be also the reciprocals of the intercepts made by the tangent plane on the axis.

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* This equation was first given, as far as I am aware, by Dr. Booth, *Tangential Co-ordinates*, Dublin, 1840.

CHAPTER IX.

CONES AND SPHERO-CONICS.

211. If a cone of any degree be cut by any sphere, whose centre is the vertex of the cone, the curve of section will evidently be such that the angle between two edges of the cone is measured by the arc joining the two corresponding points on the sphere. When the cone is of the second degree, the curve of section is called a *sphero-conic*. By stating many of the properties of cones of the second degree as properties of sphero-conics, the analogy between them and corresponding properties of conics becomes more striking.*

Strictly speaking, the intersection of a sphere with a cone of the n^{th} degree is a curve of the $2n^{th}$ degree: but when the cone is concentric with the sphere, the curve of intersection may be divided, in an infinity of ways, into two symmetrical and equal portions, either of which may be regarded as analogous to a plane curve of the n^{th} degree. For if we consider the points of the curve of intersection which lie in any hemisphere, the points diametrically opposite evidently trace out a perfectly symmetrical curve in the opposite hemisphere.

Thus then a sphero-conic may be regarded as analogous either to an ellipse or to a hyperbola. A cone of the second degree evidently intersects a concentric sphere in two similar closed curves diametrically opposite to each other. One of the principal planes of the cone meets neither curve, and if we look at either of the hemispheres into which this plane divides

* See M. Chasles's Memoir on Sphero-conics (published in the Sixth Volume of the *Transactions of the Royal Academy of Brussels*, and translated by Professor Graves, Dublin, 1837), from which the enunciations of many of the theorems in this chapter are taken.

the sphere, we see a closed curve analogous to an ellipse. But if we look at one of the hemispheres into which the sphere is divided by a principal plane meeting both the opposite curves, we see a curve consisting of two opposite branches like a hyperbola.

The curve of intersection of any quadric with a concentric sphere is evidently a sphero-conic.

212. The properties of spherical curves have been studied by means of systems of spherical co-ordinates formed on the model of Cartesian co-ordinates. Choose for axes of co-ordinates any two great circles OX, OY intersecting at right angles, and on them let fall perpendiculars PM, PN from any point on the sphere P. These perpendiculars are not, as in plane co-ordinates, equal to the opposite sides of the quadrilateral $OMPN$; and therefore it would seem that there is a certain latitude admissible in our selection of spherical co-ordinates, according as we choose for co-ordinates the perpendiculars PM, PN, or the intercepts OM, ON which they make on the axes.

M. Gudermann of Cleves has chosen for co-ordinates the tangents of the intercepts OM, ON (see Crelle's *Journal*, Vol. VI., p. 240), and the reader will find an elaborate discussion of this system of co-ordinates in the appendix to Dr. Graves's translation of Chasles's Memoir on Sphero-conics. It is easy to see however that if we draw a tangent plane to the sphere at the point O, and if the lines joining the centre to the points M, N, P, meet that plane in points m, n, p; then Om, On will be the Cartesian co-ordinates of the point p. But Om, On are the tangents of the arcs OM, ON. Hence the equation of a spherical curve in Gudermann's system of co-ordinates is in reality nothing but the ordinary equation of the plane curve in which the cone joining the spherical curve to the centre of the sphere is met by the tangent plane at the point O.

So again, if we choose for co-ordinates the sines of the perpendiculars PM, PN, it is easy to see in like manner that the equation of a spherical curve in such co-ordinates is only the equation of the orthogonal projection of that curve on a plane parallel to the tangent plane at the point O.

It seems, however, to us that the properties of spherical curves are obtained more simply and directly from the equations of the cones which join them to the centre, than from the equations of any of the plane curves into which they can be projected.

213. Let the co-ordinates of any point P on the sphere be substituted in the equation of any plane passing through the centre (which we take for origin of co-ordinates), and meeting the sphere in a great circle AB, the result will be the length of the perpendicular from P on that plane; which is the sine of the spherical arc let fall perpendicular from P on the great circle AB. By the help of this principle the equations of cones are interpreted so as to yield properties of spherical curves in a manner precisely corresponding to that used in interpreting the equations of plane curves.

Thus, let α, β be the equations of any two planes through the centre, which may also be regarded as the equations of the great circles in which they meet the sphere, then (as at *Conics*, p. 52) $\alpha - k\beta$ denotes a great circle such that the sine of the perpendicular arc from any point of it on α is in a constant ratio to the sine of the perpendicular on β; that is to say, a great circle dividing the angle between α and β into parts whose sines are in the same ratio.

Thus, again, $\alpha - k\beta$, $\alpha - k'\beta$ denote arcs forming with α and β a pencil whose anharmonic ratio is $\frac{k}{k'}$. And $\alpha - k\beta$, $\alpha + k\beta$ denote arcs forming with α, β a harmonic pencil.

It may be noted here that if A' be the middle point of an arc AB, then B', the fourth harmonic to A', A and B, is a point distant from A' by 90°. For if we join these points to the centre C, CA' is the internal bisector of the angle ACB, and therefore CB' must be the external bisector. Conversely, if two corresponding points of a harmonic system are distant from each other by 90°, each is equidistant from the other two points of the system.

It is convenient also to mention here that if $x'y'z'$ be the co-ordinates of any point on the sphere, then $xx' + yy' + zz'$ denotes the great circle having $x'y'z'$ for its pole. It is in fact the equation of the plane perpendicular to the line joining the centre to the point $x'y'z'$.

214. We can now immediately apply to spherical triangles the methods used for plane triangles (*Conics*, p. 54, &c.). Thus if α, β, γ denote the three sides, then, as in plane triangles, $l\alpha = m\beta = n\gamma$ denote three lines meeting in a point, one of which passes through each of the vertices: while

$$m\beta + n\gamma - l\alpha, \quad n\gamma + l\alpha - m\beta, \quad l\alpha + m\beta - n\gamma$$

are the sides of the triangle formed by connecting the points where each of these joining lines meets the opposite sides of the given triangle; and $l\alpha + m\beta + n\gamma$ passes through the intersections of corresponding sides of this new triangle and of the given triangle.

The equations $\alpha = \beta = \gamma$ evidently represent the three bisectors of the angles of the triangle. And if A, B, C be the angles of the triangle, it is easily proved that as in plane triangles $\alpha \cos A = \beta \cos B = \gamma \cos C$ denote the three perpendiculars. It remains true, as at *Conics*, p. 54, that if the perpendiculars from the vertices of one triangle on the sides of another meet in a point, so will the perpendiculars from the vertices of the second on the sides of the first.

The three bisectors of sides are $\alpha \sin A = \beta \sin B = \gamma \sin C$. The arc $\alpha \sin A + \beta \sin B + \gamma \sin C$ passes through the three points where each side is met by the arc joining the middle points of the other two; or, again, it passes through the point on each side 90° distant from its middle point, for $\alpha \sin A \pm \beta \sin B$ meet γ in two points which are harmonic conjugates with the points in which α, β meet them, and since one is the middle point the other must be 90° distant from it (Art. 213).

It follows from what has been just said that the point where $\alpha \sin A + \beta \sin B + \gamma \sin C$ meets any side is the pole of the great circle perpendicular to that side, and passing through its middle point, and hence that the intersection of the three such perpendiculars; that is to say, the centre of the circumscribing circle is the pole of the great circle $\alpha \sin A + \beta \sin B + \gamma \sin C$.

215. The condition that two great circles $ax + by + cz$, $a'x + b'y + c'z$ should be perpendicular is manifestly

$$aa' + bb' + cc' = 0.$$

The condition that $a\alpha + b\beta + c\gamma$, $a'\alpha + b'\beta + c'\gamma$ should be perpendicular is easily found from this by substituting for α, β, γ their expressions in terms of x, y, z. The result is exactly the same as for the corresponding case in the plane, viz.

$$aa' + bb' + cc' - (bc' + b'c) \cos A - (ca' + c'a) \cos B - (ab' + ba') \cos C = 0.$$

In like manner the sine of the arc perpendicular to $a\alpha + b\beta + c\gamma$, and passing through a given point is found by substituting the co-ordinates of that point in $a\alpha + b\beta + c\gamma$ and dividing by the square root of

$$a^2 + b^2 + c^2 - 2bc \cos A - 2ca \cos B - 2ab \cos C.$$

216. Passing now to equations of the second degree, we may consider the equation $\alpha\gamma = m\beta^2$ either as denoting a cone having α and γ for tangent planes, while β passes through the edges of contact, or as denoting a sphero-conic, having α and γ for tangents, and β for their arc of contact. The equation plainly asserts that the product of the sines of perpendiculars from any point of a sphero-conic on two of its tangents is in a constant ratio to the square of the sine of the perpendicular from the same point on the arc of contact.

In like manner the equation $\alpha\gamma = k\beta\delta$ asserts (see *Conics*, p. 215) that the product of the sines of the perpendiculars from any point of a sphero-conic on two sides of an inscribed quadrilateral is in a constant ratio to the product of sines of perpendiculars on the other two sides. And from this property again may be deduced, precisely as at *Conics*, p. 216, that the anharmonic ratio of the four arcs joining four fixed points on a sphero-conic to any other point on the curve is constant. In like manner almost all the proofs of theorems respecting plane conics (given *Conics*, chaps. XIV., XV.) apply equally to sphero-conics.

217. If α, β represent the planes of circular section (or *cyclic planes*) of a cone, the equation of the cone is of the form $x^2+y^2+z^2=k\alpha\beta$ (Art. 99), which interpreted, as in the last article, shews that the product of the sines of perpendiculars from any point of a sphero-conic on the two cyclic arcs is constant. Or, again, that, "Given the base of a spherical triangle and the product of cosines of sides, the locus of vertex is a sphero-conic, the cyclic arcs of which are the great circles having for their poles the extremities of the given base." The form of the equation shews that the cyclic arcs of sphero-conics are analogous to the asymptotes of plane conics.

Every property of a sphero-conic can be doubled by considering the sphero-conic formed by the cone reciprocal to the given one. Thus (Art. 141) it was proved that the cyclic planes of one cone are perpendicular to the focal lines of the reciprocal cone. If then the points in which the focal lines meet the sphere be called the foci of the sphero-conic, the property established in this article proves that the product of the sines of the perpendiculars let fall from the two foci on any tangent to a sphero-conic is constant.

218. If any great circle meet a sphero-conic in two points P, Q, and the cyclic arcs in points A, B, then $AP=BQ$.

This is deduced from the property of the last article in the same way as the corresponding property of the plane hyperbola is proved. The ratio of the sines of the perpendiculars from P and Q on α is equal to the ratio of the sines of perpendiculars from Q and P on β. But the sines of the perpendiculars from P and Q on α are in the ratio $\sin AP : \sin AQ$, and therefore we have

$$\sin AP : \sin AQ :: \sin BQ : \sin BP,$$

whence it may easily be inferred that $AP=BQ$.

Reciprocally, the two tangents from any point to a sphero-conic make equal angles with the arcs joining that point to the two foci.

219. As a particular case of the theorem of Art. 218 we learn that *the portion of any tangent to a sphero-conic intercepted between the two cyclic arcs is bisected at the chord of contact.* This theorem may also be obtained by the method of infinitesimals from that of Art. 217; or it may be obtained directly from the equation of a tangent, viz.

$$2(xx' + yy' + zz') = k(\alpha'\beta + \alpha\beta').$$

The form of this equation shews that the tangent at any point is constructed by joining that point to the intersection of its polar ($xx' + yy' + zz'$, see Art. 213) with $\alpha'\beta + \beta\alpha'$ which is the fourth harmonic to the cyclic arcs α, β, and the line joining the given point to their intersection. Since then the given point is 90° distant from its harmonic conjugate in respect of the two points where the tangent at that point meets the cyclic arcs, it is equidistant from these points (Art. 213).

Reciprocally, the lines joining any point on a sphero-conic to the two foci make equal angles with the tangent at that point.

220. From the fact that the intercept by the cyclic arcs on any tangent is bisected at the point of contact, it may at once be inferred by the method of infinitesimals (see *Conics*, p. 294) that *every tangent to a sphero-conic forms with the cyclic arcs a triangle of constant area*, or a triangle the sum of whose base angles is constant. This may also be inferred trigonometrically from the fact that the product of sines of perpendiculars on the cyclic arcs is constant. For if we call the intercept of the tangent c, and the angles it makes with the cyclic arcs A and B, the sines of the perpendiculars on α and β are respectively $\sin\frac{1}{2}c \sin A$, $\sin\frac{1}{2}c \sin B$. But considering the triangle of which c is the base and A and B the base angles, then by spherical trigonometry,

$$\sin^2\tfrac{1}{2}c \sin A \sin B = -\cos S \cos(S - C).$$

But C is given, therefore S, the half sum of the angles, is given.

Reciprocally, *the sum of the arcs joining the two foci to any point on a sphero-conic is constant.* Or the same may be deduced by the method of infinitesimals (see *Conics*, p. 297) from the theorem that the focal radii make equal angles with the tangent at any point.*

221. Conversely, again, we can find the locus of a point on a sphere, such that the sum of its distances from two fixed points on the sphere may be constant. The equation $\cos(\rho + \rho') = \cos a$ may be written

$$\cos^2 \rho + \cos^2 \rho' - 2 \cos \rho \cos \rho' \cos a = \sin^2 a.$$

If then α and β denote the planes which are the polars of the two given points, since we have $\alpha = \cos \rho$, the equation of the locus is

$$\alpha^2 + \beta^2 - 2\alpha\beta \cos a = \sin^2 a \, (x^2 + y^2 + z^2).$$

In order to prove that the planes α and β are perpendicular to focal lines of this cone, it is only necessary to shew that sections parallel to either plane have a focus on the line perpendicular to it. Thus let α', α'' be two planes perpendicular to each other and to α, and therefore passing through the line which we want to prove a focal line. Then since

$$x^2 + y^2 + z^2 = \alpha^2 + \alpha'^2 + \alpha''^2,$$

the equation of the locus becomes

$$\sin^2 a \, (\alpha'^2 + \alpha''^2) = (\beta - \alpha \cos a)^2.$$

If then this locus be cut by any plane parallel to α, $\alpha'^2 + \alpha''^2$ is the square of the distance of a point on the section from the intersection of $\alpha'\alpha''$, and we see that this distance is in a constant ratio to the distance from the line in which $\beta - \alpha \cos a$

* Here again we can see that a sphero-conic may be regarded either as an ellipse or hyperbola. The focal lines each evidently meet the sphere in two diametrically opposite points. If we choose for foci two points within one of the closed curves in which the cone meets the sphere, then the *sum* of the focal distances is constant. But if we substitute for one of the focal distances FP, the focal distance from the diametrically opposite point, then since $F'P = 180° - FP$, we should have the *difference* of the focal distances constant.

In like manner we may say that a variable tangent makes with the cyclic arcs angles whose difference is constant, if we substitute its supplement for one of the angles at the beginning of this article.

is cut by the same plane. This line is therefore the directrix of the section, the point $\alpha'\alpha''$ being the focus.

We see thus also that the general equation of a cone having the line xy for a focal line is of the form $x^2 + y^2 = (ax + by + cz)^2$; whence again it follows that *the sine of the distance of any point on a sphero-conic from a focus is in a constant ratio to the sine of the distance of the same point from a certain directrix arc.*

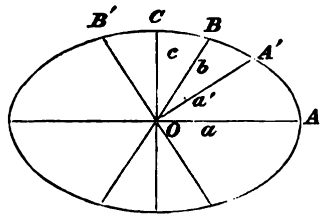
222. *Any two variable tangents meet the cyclic arcs in four points which lie on a circle.* For if L, M be two tangents and R the chord of contact, the equation of the sphero-conic may be written in the form $LM = R^2$; but this must be identical with $\alpha\beta = x^2 + y^2 + z^2$. Hence $\alpha\beta - LM$ is identical with $x^2 + y^2 + z^2 - R^2$. The latter quantity represents a small circle, having the same pole as R, and the form of the other shews that that circle circumscribes the quadrilateral $\alpha\beta LM$.

Reciprocally, the focal radii to any two points on a sphero-conic form a spherical quadrilateral in which a small circle can be inscribed. From this property again may be deduced the theorem that the sum or difference of the focal radii is constant, since the difference or sum of two opposite sides of such a quadrilateral is equal to the difference or sum of the remaining two.

223. From the properties just proved for cones can be deduced properties of quadrics in general. Thus *the product of the sines of the angles that any generator of a hyperboloid makes with the planes of circular section is constant.* For the generator is parallel to an edge of the asymptotic cone whose circular sections are the same as those of the surface. Again, since the focal lines of the asymptotic cone are the asymptotes of the focal hyperbola, it follows from Art. 220 that the sum or difference is constant of the angles which any generator of a hyperboloid makes with the asymptotes to the focal hyperbola. Again, *given one axis of a central section of a quadric, the sum or difference is given of the angles which its plane makes with the planes of circular section.* For (Art. 98) given one axis of a central section its plane touches a cone concyclic

with the given quadric, and therefore the present theorem follows at once from Art. 220.

We get an expression for the sum or difference of the angles, in terms of the given axis, by considering the principal section containing the greatest and least axes of the quadric. We obtain the cyclic planes by inflecting in that section semi-diameters OB, OB' each $= b$. Then the planes containing these lines and perpendicular to the plane of the figure are the cyclic planes. Now if we draw any semi-diameter a' making an angle α with OC, we have



$$\frac{1}{a'^2} = \frac{\cos^2\alpha}{c^2} + \frac{\sin^2\alpha}{a^2}.$$

But a' is obviously an axis of the section which passes through it and is perpendicular to the plane of the figure, and (if a' be greater than b) α is evidently half the sum of the angles BOA', $B'OA'$ which the plane of the section makes with the cyclic planes. If a' be less than b, OA' falls between OB, OB', and α is half the difference of BOA', $B'OA'$. But this sum or difference is the same for all sections having the same axis. Hence, if a', b' be the axes of any central section, making angles θ, θ' with the cyclic planes, we have

$$\frac{1}{b'^2} = \frac{\cos^2\frac{1}{2}(\theta-\theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta-\theta')}{a^2},$$

$$\frac{1}{a'^2} = \frac{\cos^2\frac{1}{2}(\theta+\theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta+\theta')}{a^2}.$$

Subtracting, we have

$$\frac{1}{b'^2} - \frac{1}{a'^2} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\sin\theta\sin\theta',$$

or, *the difference of the squares of the reciprocals of the axes of a central section is proportional to the product of the sines of the angles it makes with the cyclic planes.*

224. We saw (Art. 218) that given two sphero-conics having the same cyclic arcs, the intercept made by the outer

on any tangent to the inner is bisected at the point of contact; and hence, by the method of infinitesimals, that tangent cuts off from the outer a segment of constant area (*Conics*, p. 294).

Again, if two sphero-conics have the same foci, and if tangents be drawn to the inner from any point on the outer, these tangents are equally inclined to the tangent to the outer at that point. Hence, by infinitesimals, (see *Conics*, p. 297) the excess of the sum of the two tangents over the included arc of the inner conic is constant. This theorem is the reciprocal of the first theorem of this article, and it is so that it was obtained by Dr. Graves (see his translation of Chasles's Memoir, p. 77).

225. *To find the locus of the intersection of two tangents to a sphero-conic which cut at right angles.* This is in other words to find the cone generated by the intersection of two rectangular tangent planes to a given cone $\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=0$. Let the direction-angles of the perpendiculars to the two tangent planes be $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$; then they fulfil the relations

$$A\cos^2\alpha'+B\cos^2\beta'+C\cos^2\gamma'=0,\ A\cos^2\alpha''+B\cos^2\beta''+C\cos^2\gamma''=0.$$

But if α, β, γ be the direction-cosines of the line perpendicular to both, we have $\cos^2\alpha=1-\cos^2\alpha'-\cos^2\alpha''$, &c. Therefore adding the two preceding equations, we have for the equation of the locus,

$$Ax^2+By^2+Cz^2=(A+B+C)(x^2+y^2+z^2),$$

a cone concyclic with the reciprocal of the given cone. Reciprocally, the envelope of a chord 90° in length is a sphero-conic, confocal with the reciprocal of the given cone.

226. *To find the locus of the foot of the perpendicular from the focus of a sphero-conic on the tangent.* The work of this question is precisely the same as that of the corresponding problem in plane conics, and the only difference is in the interpretation of the result. Let the equation of the sphero-conic (Art. 221) be $x^2+y^2=t^2$ where $t=ax+by+cz$, then the equation of the tangent is

$$xx'+yy'=tt',$$

and of a perpendicular to it through the point xy is

$$(x' - at')\, y - (y' - bt')\, x = 0.$$

Solving for x', y', and t' from these two equations, and substituting in $x'^2 + y'^2 = t'^2$, we get for the locus required,

$$(x^2 + y^2)\,\{(a^2 + b^2 - 1)\,(x^2 + y^2) + 2cz\,(ax + by) + c^2z^2\} = 0.$$

The quantity within the brackets denotes a cone whose circular sections are parallel to the plane z.

227. It was proved (Art. 214) that the relation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0$$

is not, as *in plano*, an identical relation satisfied by the perpendiculars from any point. It remains then to ask how the three perpendiculars from any point on three fixed great circles are connected. But this question we have implicitly answered already, for the three perpendiculars are each the complement of one of the three distances from the three poles of the sides of the triangle of reference. If then a, b, c be the sides; A, B, C the angles of the triangle of reference, then α, β, γ the sines of the perpendiculars on the sides from any point are connected by the following relation, which is only a transformation of that of Art. 52,

$$\alpha^2 \sin^2 A + \beta^2 \sin^2 B + \gamma^2 \sin^2 C$$

$$+ 2\beta\gamma \sin B \sin C \cos a + 2\gamma\alpha \sin C \sin A \cos b + 2\alpha\beta \sin A \sin B \cos c$$

$$= 1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C.$$

The equation in this form represents a relation between the sines of the arcs represented by α, β, γ. If we want to get a relation between the perpendiculars from any point of the sphere on the planes represented by α, β, γ, we have evidently only to multiply the right-hand side of the preceding equation by r^2, and that equation in α, β, γ will be the transformation of the equation $x^2 + y^2 + z^2 = r^2$.

Hence, it appears that if we equate the left-hand side of the preceding equation to zero, the equation will be the same as $x^2 + y^2 + z^2 = 0$, and therefore denotes the imaginary circle which is the intersection of two concentric spheres; that is to say, the imaginary circle at infinity (see Art. 135).

228. This equation enables us to find the equation of the sphere inscribed in a given tetrahedron, whose faces are α, β, γ, δ. If through the centre three planes be drawn parallel to α, β, γ, the perpendiculars on them from any point will be $\alpha - r$, $\beta - r$, $\gamma - r$. The equation of the sphere is therefore

$$(\alpha - r)^2 \sin^2 A + (\beta - r)^2 \sin^2 B + \&c.$$
$$= r^2 (1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C).$$

But if L, M, N, P denote the areas of the four faces, we have

$$L\alpha + M\beta + N\gamma + P\delta = (L + M + N + P)\, r.$$

Hence, we can eliminate r, and the result is most conveniently written as follows: Let

$$l = \cos^2 \tfrac{1}{2}(bc), \quad m = \cos^2 \tfrac{1}{2}(ca), \quad n = \cos^2 \tfrac{1}{2}(ab),$$
$$p = \cos^2 \tfrac{1}{2}(ad), \quad q = \cos^2 \tfrac{1}{2}(bd), \quad r = \cos^2 \tfrac{1}{2}(cd),$$

where (ad) is the angle made with each other by the planes α, δ. Then the equation of the inscribed sphere is

$$lrq\alpha^2 + mpr\beta^2 + npq\gamma^2 + lmn\delta^2$$
$$+ (lp - mq - nr)(l\alpha\delta + p\beta\gamma) + (mq - nr - lp)(m\beta\delta + q\alpha\gamma)$$
$$+ (nr - lp - mq)(n\gamma\delta + r\alpha\beta) = 0.$$

229. The equation of a small circle (or right cone) is easily expressed. The sine of the distance of any point of the circle from the polar of the centre is constant. Hence, if α be that polar the equation of the circle is $\alpha^2 = \cos^2 \rho\, (x^2 + y^2 + z^2)$.

All small circles then being given by equations of the form $S = \alpha^2$, their properties are all cases of those of conics having double contact with the same conic.

The theory of invariants may be applied to small circles. Let two circles S, S' be

$$x^2 + y^2 + z^2 - \alpha^2 \sec^2 \rho, \quad x^2 + y^2 + z^2 - \beta^2 \sec^2 \rho',$$

and let us form the condition that $\lambda S + S'$ should break up into factors. This cubic being

$$\lambda^3 \Delta + \lambda^2 \Theta + \lambda \Theta' + \Delta' = 0,$$

we have $\Delta = -\tan^2\rho, \quad \Delta' = -\tan^2\rho',$

$$\Theta = \sec^2\rho \sec^2\rho' \sin^2 D - 2\tan^2\rho - \tan^2\rho',$$

$$\Theta' = \sec^2\rho \sec^2\rho' \sin^2 D - 2\tan^2\rho' - \tan^2\rho,$$

where D is the distance between the centres.

Now the corresponding values for two circles in a plane are

$$\Delta = -r^2, \quad \Delta' = -r'^2, \quad \Theta = D^2 - 2r^2 - r'^2, \quad \Theta' = D^2 - 2r'^2 - r^2.$$

Hence, if any invariant relation between two circles in a plane is expressed as a function of the radii and of the distance between their centres, the corresponding relation for circles on a sphere is obtained by substituting for r, r', D; $\tan r, \tan r'$, and $\sec r \sec r' \sin D$.

Thus the condition that two circles in a plane should touch is obtained by forming the discriminant of the cubic equation, and is either $D = 0$ or $D = r \pm r'$. The corresponding equation therefore for two circles on a sphere is

$$\tan r \pm \tan r' = \sec r \sec r' \sin D, \text{ or } \sin D = \sin(r \pm r').$$

Again, if two circles in a plane be the one inscribed in, the other circumscribed about the same triangle, the invariant relation is fulfilled $\Theta'^2 = 4\Delta\Theta'$, which gives for the distance between their centres the expression $D^2 = R^2 - 2Rr$.

The distance therefore between the centres of the inscribed and circumscribed circles of a spherical triangle is given by the formula

$$\sec^2 R \sec^2 r \sin^2 D = \tan^2 R - 2\tan R \tan r.$$

So, in like manner, we can get the relation between two circles inscribed in, and circumscribed about the same spherical polygon.

230. The equation of any small circle (or right cone) in trilinear co-ordinates must (Art. 227) be of the form

$$\alpha^2 \sin^2 A + \beta^2 \sin^2 B + \gamma^2 \sin^2 C$$
$$+ 2\beta\gamma \sin B \sin C \cos a + 2\gamma\alpha \sin C \sin A \cos b + 2\alpha\beta \sin A \sin B \cos c$$
$$= (l\alpha + m\beta + n\gamma)^2.$$

If now the small circle circumscribe the triangle $\alpha\beta\gamma$, the coefficients of α^2, β^2, and γ^2 must vanish, and we must therefore have $l\alpha + m\beta + n\gamma = \alpha \sin A + \beta \sin B + \gamma \sin C$. Hence, as was proved before, this represents the polar of the centre of the circumscribing circle. Substituting this value, the equation of the small circle becomes

$$\beta\gamma \tan \tfrac{1}{2}a + \gamma\alpha \tan \tfrac{1}{2}b + \alpha\beta \tan \tfrac{1}{2}c = 0.$$

The equation of the inscribed circle turns out to be of exactly the same form as in the case of plane triangles, viz.

$$\cos \tfrac{1}{2}A \sqrt{(\alpha)} + \cos \tfrac{1}{2}B \sqrt{(\beta)} + \cos \tfrac{1}{2}C \sqrt{(\gamma)} = 0.$$

CHAPTER X.

GENERAL THEORY OF SURFACES.

INTRODUCTORY CHAPTER.

231. RESERVING for a future chapter a more detailed examination of the properties of surfaces in general, we shall in this chapter give an account of such parts of the general theory as can be obtained with least trouble.

Let the general equation of a surface be written in the form,

$$\begin{gathered} A \\ + Bx + Cy + Dz \\ + Ex^2 + Fy^2 + Gz^2 + 2Hyz + 2Kzx + 2Lxy \\ + \&c. = 0, \end{gathered}$$

or, as we shall write it often for shortness,

$$u_0 + u_1 + u_2 + u_3 + \&c. = 0,$$

where u_2 means the aggregate of terms of the second degree, &c. Then it is evident that u_0 consists of one term, u_1 of three, u_2 of six, &c. The total number of terms in the equation is therefore the sum of $n+1$ terms of the series 1, 3, 6, 10, &c., that is to say, $\frac{(n+1\ (n+2)\ (n+3)}{1.2.3}$.

The number of conditions necessary to determine a surface of the n^{th} degree is one less than this, or $= \frac{n\,(n^2+6n+11)}{6}$.

The equation above written can be thrown into the form of a polar equation by writing $\rho\cos\alpha$, $\rho\cos\beta$, $\rho\cos\gamma$, for x, y, z, when we obviously obtain an equation of the n^{th} degree, which will determine n values of the radius vector answering to any assigned values of the direction-angles α, β, γ.

232. If now the origin be on the surface, we have $u_0 = 0$, and one of the roots of the equation is always $\rho = 0$. But a second root of the equation will be $\rho = 0$ if α, β, γ be connected by the relation

$$B \cos\alpha + C \cos\beta + D \cos\gamma = 0.$$

Now multiplying this equation by ρ it becomes $Bx + Cy + Dz = 0$, and we see that it expresses merely that the radius vector must lie in the plane $u_1 = 0$. No other condition is necessary in order that the radius should meet the surface in two coincident points. Thus we see that in general *through an assumed point on a surface we can draw an infinity of radii vectores which will there meet the surface in two coincident points; that is to say, an infinity of tangent lines to the surface; and these lines lie all in one plane, called the tangent plane, determined by the equation* $u_1 = 0$.

233. *The section of any surface made by a tangent plane is a curve having the point of contact for a double point.**

Every radius vector to the surface, which lies in the tangent plane, is of course also a radius vector to the section made by that plane; and since every such radius vector (Art. 232) meets the section at the origin in two coincident points, the origin is, by definition, a double point (see *Higher Plane Curves*, p. 27).

We have already had an illustration of this in the case of hyperboloids of one sheet, which are met by any tangent plane in a conic having a double point, that is to say, in two right lines. And the point of contact of the tangent plane to a quadric of any other species is equally to be considered as the intersection of two imaginary right lines.

From this article it follows conversely, that any plane meeting a surface in a curve having a double point touches the surface, the double point being the point of contact. If the section have two double points, the plane will be a double tangent plane; and if it have three double points, the plane

* This remark, I believe, was first made by Mr. Cayley: Gregory's *Solid Geometry*, p. 132.

will be a triple tangent plane. Since the equation of a plane contains three constants, it is possible to determine a plane which will satisfy any three conditions, and therefore a finite number of planes can in general be determined which will meet a given surface in a curve having three double points: that is to say, *a surface has in general a determinate number of triple tangent planes.* It will also have an infinity of double tangent planes, the points of contact lying on a certain curve locus on the surface. The degree of this curve, and the number of triple tangent planes will be subjects of investigation hereafter.

234. *Through an assumed point on a surface it is generally possible to draw two lines which shall there meet the surface in three coincident points.*

In order that the radius vector may meet the surface in three coincident points, we must not only, as in Art. 232, have the condition fulfilled

$$B\cos\alpha + C\cos\beta + D\cos\gamma = 0,$$

but also $E\cos^2\alpha + F\cos^2\beta + G\cos^2\gamma$

$$+ 2H\cos\beta\cos\gamma + 2K\cos\gamma\cos\alpha + 2L\cos\alpha\cos\beta = 0.$$

For if these conditions were fulfilled, A being already supposed to vanish, the equation of the n^{th} degree which determines ρ, becomes divisible by ρ^3, and has therefore three roots $= 0$. The first condition expresses that the radius vector must lie in the tangent plane u_1. The second expresses that the radius vector must lie in the surface $u_2 = 0$, or

$$Ex^2 + Fy^2 + Gz^2 + 2Hyz + 2Kzx + 2Lxy = 0.$$

This surface is a cone of the second degree (Art. 62) and since every such cone is met by a plane passing through its vertex in two right lines, two right lines can be found to fulfil the required conditions.

Every plane (beside the tangent plane) drawn through either of these lines, meets the surface in a section having the point of contact for a point of inflexion. For a point of inflexion is a point, the tangent at which meets the curve in three coincident points (*Higher Plane Curves*, p. 35). On this account we shall call the two lines which meet the surface in three coincident points, the *inflexional* tangents at the point.

The existence of these two lines may be otherwise perceived thus. We have proved that the point of contact is a double point in the section made by the tangent plane. And it has been proved (*Higher Plane Curves*, p. 28) that at a double point can always be drawn two lines meeting the section (and therefore the surface) in three coincident points.

235. A double point may be one of three different kinds according as the tangents at it are real, coincident, or imaginary. Accordingly the contact of a plane with a surface may be of three kinds according as the tangent plane meets it in a section having a node, a cusp, or a conjugate point; or in other words according as the inflexional tangents are real, coincident, or imaginary.

Dupin, who first noticed* the difference between these three kinds of contact, stated the matter as follows: Suppose that we confine our attention to points so near the origin that all powers of the co-ordinates above the second may be neglected, then the tangent plane (or a very near plane parallel to it) meets any surface $u_1 + u_2 + u_3 + \&c.$ in the same section in which it meets the quadric $u_1 + u_2$. And according as the sections of this quadric by planes parallel to the tangent plane are ellipses, hyperbolas, or parabolas, so the section made by the tangent plane is to be considered as an infinitely small ellipse, hyperbola, or parabola. This infinitely small section Dupin calls the *indicatrix* at the point of contact, and he divides the points of the surface, according to the nature of the indicatrix into elliptic, hyperbolic, and parabolic points. We shall presently show that there will be in general on every surface a number of parabolic points forming a curve locus, this curve separating the elliptic from the hyperbolic points.

* See Dupin's *Développements de Géométrie*, p. 48.

If the tangent plane be made the plane of xy, and the equation of the surface be

$$z + Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + \&c. = 0,$$

it is manifest that the origin will be an elliptic, hyperbolic, or parabolic point according as B^2 is less, greater than, or equal to AC.*

236. Knowing the equation of the tangent plane when the origin is on the surface, we can, by transformation of co-ordinates, find the equation of the tangent plane at any point. It is proved precisely as at (Art. 58) that this equation may be written in either of the forms

$$(x - x')\frac{dU'}{dx'} + (y - y')\frac{dU'}{dy'} + (z - z')\frac{dU'}{dz'} = 0,$$

or

$$x\frac{dU'}{dx'} + y\frac{dU'}{dy'} + z\frac{dU'}{dz'} + w\frac{dU'}{dw'} = 0.$$

237. Let it be required now to find the tangent plane at a point, indefinitely near the origin, on the surface

$$z + Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + \&c. = 0.$$

We have to suppose x', y' so small that their squares may be neglected; while, since the consecutive point is on the tangent plane, we have $z' = 0$: or, more accurately, the equation of the surface shows that z' is a quantity of the same order as the squares of x' and y'. Then, either by the formula of the last article, or else directly by putting $x + x'$, $y + y'$ for x and y, and taking the linear part of the transformed equation, the equation of a consecutive tangent plane is found to be

$$z + 2(Ax' + By')x + 2(Bx' + Cy')y = 0.$$

* This is sometimes expressed as follows: When the plane of xy is the tangent plane, and the equation of the surface is expressed in the form $z = \phi(x, y)$, we have an elliptic, hyperbolic, or parabolic point according as $\left(\frac{d^2z}{dx\,dy}\right)^2$ is less, greater than, or equal to $\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right)$. It will be found that this is equivalent to the statement in the text; but we do not enter into details because we shall have seldom occasion in practice to deal with equations where z is given explicitly as a function of x and y.

Now (see *Conics*, Art. 141) $(Ax' + By')\,x + (Bx' + Cy')\,y$ denotes the diameter of the conic $Ax^2 + 2Bxy + Cy^2 = F$, which is conjugate to that to the point $x'y'$. Hence *any tangent plane is intersected by a consecutive tangent plane in the diameter of the indicatrix which is conjugate to the direction to which the consecutive point is taken.*

This in fact is geometrically evident from Dupin's point of view. For if we admit that the points consecutive to the given one lie on an infinitely small conic, we see that the tangent plane at any of them will pass through the tangent line to that conic; and this tangent line ultimately coincides with the diameter conjugate to that drawn to the point of contact: for the tangent line is parallel to this conjugate diameter and infinitely close to it.

Thus then all the tangent lines which can be drawn at a point on a surface may be distributed into pairs such that the tangent plane at a consecutive point on either will pass through the other. Two tangent lines so related are called *conjugate tangents.*

In the case where the two inflexional tangents are real, the relation between two conjugate tangents may be otherwise stated. Take the inflexional tangents for the axes of x and y, which is equivalent to making A and $C = 0$ in the preceding equation: then the equation of a consecutive tangent plane is $z + 2B(x'y + y'x) = 0$. And since the lines x, y, $x'y + y'x$, $x'y - y'x$ form a harmonic pencil, we learn that *a pair of conjugate tangents form, with the inflexional tangents, a harmonic pencil.*

238. In the case where the origin is a parabolic point, the equation of the surface can be thrown into the form $z + Ay^2 + \&c. = 0$, and the equation of a consecutive tangent plane will be $z + 2Ay'y = 0$. Hence the tangent plane at *every* point consecutive to a parabolic point passes through the inflexional tangent; and if the consecutive point be taken in this direction so as to have $y' = 0$, then the consecutive tangent plane coincides with the given one. Hence *the tangent plane at a parabolic point is to be considered as a double tangent*

plane, since it touches the surface in two consecutive points.* In this way parabolic points on surfaces may be considered as analogous to points of inflexion on plane curves: for we have proved (*Higher Plane Curves*, p. 35) that the tangent line at a point of inflexion is in like manner to be regarded as a double tangent. A further analogy between parabolic points and points of inflexion will be afterwards stated.

It is convenient to have a name to distinguish double tangent planes which touch in two distinct points, from those now under consideration where the two points of contact coincide. We shall therefore call the latter *stationary* tangent planes, the word expressing that the tangent plane being supposed to move round as we pass from one point of the surface to another, in this case it remains for an instant in the same position. For the same reason we have called the tangent lines at points of inflexion in plane curves, stationary tangents.

239. If on transforming the equation to any point on a surface as origin we have not only $u_0 = 0$ but also all the terms in $u_1 = 0$, so that the equation takes the form

$$Ex^2 + Fy^2 + Gz^2 + 2Hyz + 2Kzx + 2Lxy + u_3 + \&c. = 0,$$

then it is easy to see in like manner that *every* line through the origin meets the curve in two coincident points; and the origin is then called a *double* point. It is easy to see also that a line through the origin there meets the surface in *three* coincident points, provided that its direction-cosines satisfy the equation

$$E\cos^2\alpha + F\cos^2\beta + G\cos^2\gamma$$
$$+ 2H\cos\beta\cos\gamma + 2K\cos\gamma\cos\alpha + 2L\cos\alpha\cos\beta = 0.$$

In other words, *through a double point on a surface can be drawn an infinity of lines which will meet the surface in three coincident points, and these will all lie on a cone of the second degree* whose equation is $u_2 = 0$. Further, of these lines six will

* I believe this was first pointed out, *Cambridge and Dublin Mathematical Journal*, Vol. III. p. 45.

meet the surface in four coincident points; namely, the lines of intersection of the cone u_2 with the cone of the third degree $u_3 = 0$.

Double points on surfaces might be classified according to the number of these lines which are real, or according as two or more of them coincide, but we shall not enter into these details. The only special case which it is important to mention is when the cone u_2 resolves itself into two planes; and this again includes the still more special case when these two planes coincide; that is to say, when u_2 is a perfect square.

240. Every plane drawn through a double point may in one sense be regarded as a tangent plane to the surface, since it meets the surface in a section having a double point, but in a special sense the tangent planes to the cone u_2 are to be regarded as tangent planes to the surface, and the sections of the surface by these planes will each have the origin as a cusp. To a double point then on a surface (which is a point through which can be drawn an infinity of tangent planes), will in general correspond on the reciprocal surface a plane touching the surface in an infinity of points, which will in general lie on a conic. If however the double point be of the special kind noticed at the end of the last article, there will correspond to it on the reciprocal surface a double tangent plane having two points of contact.

241. The results obtained in the preceding articles by taking as our origin the point we are discussing, we shall now extend to the case where the point has any position whatever. Let us first remind the reader (see p. 29) that since the equations of a right line contain four constants, a finite number of right lines can be determined to fulfil four conditions (as, for instance, to touch a surface four times); while an infinity of lines can be found to satisfy three conditions (as, for instance, to touch a surface three times), those right lines generating a certain surface, and their points of contact lying on a certain locus. In a subsequent chapter we shall return to the problem to determine in general the number of solutions when four con-

ditions are given, and to determine the degree of the surface generated, and of the locus of points of contact, when three conditions are given. In this chapter we confine ourselves to the case when the right line is required to pass through a given point, whether on the surface or not. This is equivalent to two conditions; and an infinity of right lines (forming a cone) can be drawn to satisfy one other condition; while a finite number of right lines can be drawn to satisfy two other conditions.

We use Joachimstal's method employed, *Conics*, pp. 81, 134; *Higher Plane Curves*, p. 61; and at p. 47 of this volume. If the quadriplanar co-ordinates of two points be $x'y'z'w'$, $x''y''z''w''$, then the points in which the line joining them is cut by the surface are found by substituting in the equation of the surface, for x, $\lambda x' + \mu x''$, for y, $\lambda y' + \mu y''$, &c. The result will give an equation of the n^{th} degree in $\lambda : \mu$, whose roots will be the ratios of the segments in which the line joining the two given points is cut by the surface at any of the points where it meets it. And the co-ordinates of any of the points of meeting are $\lambda' x' + \mu' x''$, $\lambda' y' + \mu' y''$, $\lambda' z' + \mu' z''$, $\lambda' w' + \mu' w''$, where $\lambda' : \mu'$ is one of the roots of the equation of the n^{th} degree. All this will present no difficulty to any reader who has mastered the corresponding theory for plane curves. And, as in plane curves, the result of the substitution in question may be written

$$\lambda^n U' + \lambda^{n-1}\mu\Delta U' + \frac{1}{1.2}\lambda^{n-2}\mu^2\Delta^2 U' + \&c. = 0,$$

where Δ represents the operation

$$x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'} + w\frac{d}{dw'}.$$

Following the analogy of plane curves we shall call the surface represented by

$$x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} + w'\frac{dU}{dw} = 0,$$

the first polar of the point $x'y'z'w'$. We shall call

$$\left(x'\frac{d}{dx} + y'\frac{d}{dy} + z'\frac{d}{dz} + w'\frac{d}{dw}\right)^2 U = 0$$

the second polar, and so on: the polar plane of the same point being

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+w\frac{dU'}{dw'}=0.$$

Each polar surface is manifestly also a polar of the point $x'y'z'w'$ with regard to all the other polars of higher degree.

If a point be on a surface all its polars touch the tangent plane at that point: for the polar plane with regard to the surface is the tangent plane; and this must also be the polar plane with regard to the several polar surfaces. This may also be seen by taking the polar of the origin with regard to

$$u_0w^n+u_1w^{n-1}+u_2w^{n-2}+\&c.,$$

where we have made the equation homogeneous by the introduction of a new variable w. The polar surfaces are got by differentiating with regard to this new variable. Thus the first polar is

$$nu_0w^{n-1}+(n-1)\,u_1w^{n-2}+(n-2)\,u_2w^{n-3}+\&c.,$$

and if $u_0=0$, the terms of the first degree, both in the surface and in the polar, will be u_1.

242. If now the point $x'y'z'w'$ be *on* the surface, U' vanishes, and one of the roots of the equation in $\lambda:\mu$, will be $\mu=0$. A second root of that equation will be $\mu=0$, and the line will meet the surface in two coincident points at the point $x'y'z'w'$, provided that the coefficient of $\lambda^{n-1}\mu$ vanish in the equation referred to. And in order that this should be the case, it is manifestly sufficient that $x''y''z''w''$ should satisfy the equation of the plane

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+w\frac{dU'}{dw'}=0.$$

It is proved then that all the tangent lines to a surface which can be drawn at a given point lie in a plane whose equation is that just written. By subtracting from this equation, the identity

$$x'\frac{dU'}{dx'}+y'\frac{dU'}{dy'}+z'\frac{dU'}{dz'}+w'\frac{dU'}{dw'}=0,$$

we get the ordinary Cartesian equation of the tangent plane, viz.

$$(x-x')\frac{dU'}{dx'}+(y-y')\frac{dU'}{dy'}+(z-z')\frac{dU'}{dz'}=0.$$

Hence again by Art. 42, can immediately be deduced the equations of the normal, viz.

$$\frac{x-x'}{\frac{dU'}{dx'}}=\frac{y-y'}{\frac{dU'}{dy'}}=\frac{z-z'}{\frac{dU'}{dz'}}.$$

243. The right line will meet the surface in three consecutive points, or the equation we are considering will have for three of its roots $\mu=0$, if not only the coefficients of λ^n and $\lambda^{n-1}\mu$ vanish, but also that of $\lambda^{n-1}\mu^2$: that is to say, if the line we are considering not only lies in the tangent plane, but also in the polar quadric

$$\left(x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}+w\frac{d}{dw'}\right)^2 U'=0.$$

Now (Art. 241) when a point is on a surface all its polars touch the surface. The tangent plane therefore, touching the polar quadric, meets it in two right lines, real or imaginary, which are the two inflexional tangents to the surface. (Art. 234.)

244. *Through a point on a surface can be drawn* $(n+2)(n-3)$ *tangents which will also touch the surface elsewhere.*

In order that the line should touch at the point $x'y'z'w'$, we must, as before, have the coefficients of λ^n and $\lambda^{n-1}\mu=0$; in consequence of which the equation we are considering becomes one of the $(n-2)^{\text{th}}$ degree, and if the line touch the surface a second time this reduced equation must have equal roots. The condition that this should be the case involves the coefficients of that equation in the degree $n-3$; one term, for instance, being $(\Delta^2U'.U)^{n-3}$. By considering that term we see that this discriminant involves the co-ordinates $x'y'z'w'$ in the degree $(n-2)(n-3)$, and $xyzw$ in the degree $(n+2)(n-3)$. When therefore $x'y'z'w'$ is fixed, it denotes a surface which is met by the tangent plane in $(n+2)(n-3)$ right lines.

Thus then we have proved that at any point on a surface an infinity of tangent lines can be drawn: that these in general lie in a plane; that two of them pass through three consecutive points, and $(n+2)(n-3)$ of them touch the surface again.

245. Let us proceed next to consider the case of tangents drawn through a point not on the surface. Since we have in the preceding articles established relations which connect the co-ordinates of any point on a tangent with those of the point of contact, we can, by an interchange of accented and unaccented letters, express that it is the former point which is now supposed to be known, and the latter sought.

Thus for example, making this interchange in the equation of Art. 242, we see that the points of contact of all tangent lines (or of all tangent planes) which can be drawn through $x'y'z'w'$, lie on the first polar, which is of the degree $(n-1)$: viz.

$$x'\frac{dU}{dx}+y'\frac{dU}{dy}+z'\frac{dU}{dz}+w'\frac{dU}{dw}=0.$$

And since the points of contact lie also on the given surface, their locus is the curve of the degree $n(n-1)$, which is the intersection of the surface with the polar.

246. The assemblage of the tangent lines which can be drawn through $x'y'z'w'$ form a cone, the tangent planes to which are also tangent planes to the surface. The equation of this cone is found by forming the discriminant of the equation of the n^{th} degree in λ (Art. 241). For this discriminant expresses that the line joining the fixed point to $xyzw$ meets the surface in two coincident points; and therefore $xyzw$ may be a point on any tangent line through $x'y'z'w'$. The discriminant is easily seen to be of the degree $n(n-1)$, and it is otherwise evident that this must be the degree of the tangent cone. For its degree is the same as the number of lines in which any plane through the vertex cuts it. But such a plane meets the surface in a curve to which $n(n-1)$ tangents can be drawn through the fixed point, and these tangents are also the tangent lines which can be drawn to the surface through the given point.

247. *Through a point not on the surface can in general be drawn* $n(n-1)(n-2)$ *inflexional tangents.* We have seen, Art. 243, that the co-ordinates of any point on an inflexional tangent are connected with those of its point of contact by the relations $U'=0$, $\Delta U'=0$, $\Delta^2 U'=0$. If then we consider the $xyzw$ of any point on the tangent as known; its point of contact is determined as one of the intersections of the given surface U, which is of the n^{th} degree, with its first polar ΔU, which is of the $(n-1)^{\text{th}}$, and with the second polar $\Delta^2 U$, which is of the $(n-2)^{\text{th}}$. There are therefore $n(n-1)(n-2)$ such intersections.

248. *Through a point not on the surface can in general be drawn* $\frac{1}{2}n(n-1)(n-3)$ *double tangents to it.* The points of contact of such lines are proved by Art. 244, to be the intersections of the given surface, of the first polar, and of the surface represented by the discriminant discussed in Art. 244, and which we there saw contained the co-ordinates of the point of contact in the degree $(n-2)(n-3)$. There are therefore $n(n-1)(n-2)(n-3)$ points of contact: and since there are two points of contact on each double tangent, there are half this number of double tangents.

Thus then we have completed the discussion of tangent lines which pass through a given point. We have shown that their points of contact lie on the intersection of the surface with one of the degree $n-1$, that their assemblage forms a cone of the degree $n(n-1)$: that $n(n-1)(n-2)$ of them are inflexional, and $\frac{1}{2}n(n-1)(n-2)(n-3)$ of them are double.

These latter double tangents are also plainly double edges of the tangent cone, since they belong to the cone in virtue of either contact. Along such an edge can be drawn two tangent planes to the cone, namely, the tangent planes to the surface at the two contacts.

The inflexional tangents, however, are also to be regarded as double tangents to the surface: since the line passing through three consecutive points is a double tangent in virtue of joining the first and second, and also of joining the second and third. The inflexional tangents are therefore double tangents whose points of contact coincide. They are therefore double edges of the tangent cone; but the two tangent planes along any such edge coincide. They are therefore cuspidal edges of the cone. We have proved then that *the tangent cone which is of the degree* $n(n-1)$ has $n(n-1)(n-2)$ *cuspidal edges, and* $\frac{1}{2}n(n-1)(n-2)(n-3)$ *double edges;* that is to say, any plane meets the cone in a section having such a number of cusps and such a number of double points.

249. It is proved precisely as for plane curves (*Higher Plane Curves*, page 57), that if we take on each radius vector a length whose reciprocal is the n^{th} part of the sum of the reciprocals of the n radii vectores to the surface, then the locus of the extremity will be the polar plane of the point: that if the point be on the surface, the locus of the extremity of the mean between the reciprocals of the $n-1$ radii vectores will be the polar quadric, &c.

By interchanging accented and unaccented letters in the equation of the polar plane, it is plain that the locus of the poles of all planes which pass through a given point is the first polar of that point. The locus of the pole of a plane which passes through two fixed points is hence seen to be a curve of the $(n-1)^2$ degree, namely, the intersection of the two first polars of these points. We see also that the first polar of every point on the line joining these two points must pass through the same curve. And in like manner the first polars of any three points on a plane determine by their intersection $(n-1)^3$ points, any one of which is a pole of the plane, and through which points the first polar of every other point on the plane must pass.

250. From the theory of tangent lines drawn through a point we can in two ways derive the degree of the reciprocal surface. First; the number of points in which an arbitrary line meets the reciprocal is equal to the number of tangent planes which can be drawn to the given surface through a given line. Consider now any two points A and B on that line, and let C be the point of contact of any tangent plane

passing through AB. Then since the line AC touches the surface, C lies on the first polar of A; and for the same reason it lies on the first polar of B. The points of contact therefore are the intersection of the given surface, which is of the n^{th} degree with the two polar surfaces, which are each of the degree $(n-1)$. The number of points of contact, and therefore *the degree of the reciprocal, is* $n(n-1)^2$.

251. Otherwise thus: let a tangent cone be drawn to the surface having the point A for its vertex; then since every tangent plane to the surface drawn through A touches this cone, the problem is, to find how many tangent planes to the cone can be drawn through any line AB; or if we cut the cone by any plane through B, the problem is to find how many tangent lines can be drawn through B to the section of the cone. But the class of a curve whose degree is $n(n-1)$, which has $n(n-1)(n-2)$ cusps, and $\frac{1}{2}n(n-1)(n-2)(n-3)$ double points is

$$n(n-1)\{n(n-1)-1\}-3n(n-1)(n-2)$$
$$-n(n-1)(n-2)(n-3)=n(n-1)^2.$$

Generally the section of the reciprocal surface by any plane corresponds to the tangent cone to the original surface through any point. And it is easy to see that the degree of the tangent cone to the reciprocal surface (as well as to the original surface) through any point is of degree $n(n-1)$.

252. Returning to the condition that a line should touch a surface

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+w\frac{dU'}{dw'}=0,$$

we see that if all four differentials be made to vanish by the co-ordinates of any point, then every line through the point meets the surface in two coincident points; and the point is therefore a double point. The condition that a given surface may have a double point is obtained by eliminating the variables between the four equations $\frac{dU}{dx}=0$, &c., and is called the discriminant of the given quantic (*Lessons on Higher Algebra*, page 43). The discriminant being the result of elimination between four equations, each of the degree $n-1$, contains the coefficients of each in the degree $(n-1)^3$, and is therefore of the degree $4(n-1)^3$ in the coefficients of the original equation.

It is obvious from what has been said, that when a surface has a double point, the first polar of every point passes through the double point.

The surfaces represented by $\frac{dU}{dx}$, &c., may happen not merely to have points in common, but to have a whole curve common to all four surfaces. This curve will then be a double curve on the surface U, and every point of it will be a double point. Now we saw (Art. 233) that the surface represented by the general Cartesian equation of the n^{th} degree will, in general, have an infinity of double tangent planes; the reciprocal surface therefore will, in general, have an infinity of double points, which will be ranged on a certain curve. The existence then of these double curves is to be regarded among the "ordinary singularities" of surfaces (see *Higher Plane Curves*, page 47).

When the point $x'y'z'w'$ is a double point, U' and $\Delta U'$ vanish identically; and any line through the double point meets the surface in three consecutive points if it satisfies the equation $\Delta^2 U' = 0$, which represents a cone of the second degree.

253. *The polar quadric of a parabolic point on a surface is a cone.*

The polar quadric of the origin with regard to any surface

$$u_0 w^n + u_1 w^{n-1} + u_2 w^{n-2} + \&c. = 0,$$

(where, as in Art. 241, we have introduced w so as to make the equation homogeneous) is found by differentiating $n-2$ times with respect to w. Dividing out by $(n-2)(n-3)\ldots 3$, the polar quadric is

$$n(n-1)u_0 + 2(n-1)u_1 + 2u_2 = 0.$$

Now the origin being a parabolic point, we have seen, Art. 235, that the equation is of the form

$$z + Cy^2 + 2Dzx + 2Ezy + Fz^2 + \&c.,$$

[or, in other words, $u_0 = 0$, and u_2 is of the form $u_1v_1 + w_1^2$]. The polar quadric then is

$$z(n-1+2Dx+2Ey+Fz)+Cy^2=0.$$

But we have seen (page 40) that any equation represents a cone when it is a homogeneous function of three quantities, each of the first degree. The equation just written therefore represents a cone whose vertex is the intersection of the three planes, z, $n-1+2Dx+2Ey+Fz$, and y. The two former planes are tangent planes to this cone, and y the plane of contact.

254. It follows from the last article that if we form the locus of points whose polar quadrics represent a cone, this will meet the surface in the parabolic points. This locus is found by writing down the discriminant of $\Delta^2 U' = 0$. If a, b, &c., denote the second differential coefficients $\frac{d^2U'}{dx'^2}$, $\frac{d^2U'}{dy'^2}$, &c., the discriminant will be (page 41)

$$abcd + 2(alqr + bmpr + cnpq + dlmn)$$
$$-(adl^2 + bdm^2 + cdn^2 + bcp^2 + caq^2 + abr^2)$$
$$+ l^2p^2 + m^2q^2 + n^2r^2 - 2mnqr - 2nlrp - 2lmpq = 0.$$

This denotes a surface of the degree $4(n-2)$, which we shall call the Hessian of the given surface. In the same manner then as the intersection of a plane curve with its Hessian determines the points of inflexion, so the intersection of a surface with its Hessian determines a curve of the degree $4n(n-2)$, which is the locus of parabolic points (see Art. 238).

255. It follows from what has been just proved that through a given point can be drawn $4n(n-1)(n-2)$ *stationary* tangent planes (see Art. 238). For since the tangent plane passes through a fixed point, its point of contact lies on the polar surface, whose degree is $n-1$, and the intersection of this surface with the surface U, and the surface determined in the last article as the locus of points of contact of stationary tangent planes, determine $4n(n-1)(n-2)$ points.

Otherwise thus; the stationary tangent planes to the surface through any point are also stationary tangent planes to the tangent cone through that point, and if the cone be cut by any plane, these planes meet it in the tangents at the points of inflexion of the section. But the number of points of inflexion on a plane curve are determined by the formula (*Higher Plane Curves*, page 91)

$$\iota - \kappa = 3(\nu - \mu).$$

But in this case, Art. 248, we have $\nu = n(n-1)^2$, $\mu = n(n-1)$; therefore $\nu - \mu = n(n-1)(n-2)$, $\kappa = 3n(n-1)(n-2)$. Hence, as before, $\iota = 4n(n-1)(n-2)$.

The number of double tangent planes to the cone are determined by the formula

$$2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9),$$

and $2\delta = n(n-1)(n-2)(n-3)$; $(\nu + \mu - 9) = n^3 - n^2 - 9$.

Hence $\qquad 2\tau = n(n-1)(n-2)(n^3 - n^2 + n - 12)$.

It follows then that through any point can be drawn τ double tangent planes to the surface, where τ is the number just determined. It will be proved hereafter, that the points of contact of double tangent planes lie on the intersection of the surface with one whose degree is $(n-2)(n^3 - n^2 + n - 12)$.

256. *If a right line lie altogether in a surface it will touch the Hessian and therefore the parabolic curve*, (*Cambridge and Dublin Mathematical Journal*, Vol. IV., page 255).

Let the equation of the surface be $x\phi + y\psi = 0$, and let us seek the result of making x and $y = 0$ in the equation of the Hessian, so as thus to find the points where the line meets that surface. Now evidently $\frac{d^2U}{dz^2}$, $\frac{d^2U}{dw^2}$, $\frac{d^2U}{dz\,dw}$, all contain x or y as a factor, and therefore vanish on this supposition. And if we make $a = 0$, $d = 0$, $r = 0$ in the equation of the Hessian, it becomes a perfect square $(lp - mq)^2$, showing that the right line touches the Hessian. If we make $x = 0$, $y = 0$ in $lp - mq$, it reduces to $\frac{d\phi}{dz}\frac{d\psi}{dw} - \frac{d\phi}{dw}\frac{d\psi}{dz}$. It is evident that when the tangent plane touches all along any line, straight or curved, this line lies altogether in the Hessian. The reader can verify this without difficulty, with regard to the surface $x\phi + y^2\psi$.

CURVATURE OF SURFACES.

257. We proceed next to investigate the curvature at any point on a surface of the various sections which can be made by planes passing through that point.

In the first place let it be premised that if the equation of a curve be $u_1 + u_2 + u_3 + \&c. = 0$, the radius of curvature at the origin is the same as for the conic $u_1 + u_2$. For it will be remembered that the ordinary expression for the radius of curvature includes only the co-ordinates of the point and the values of the first and second differential coefficients for that point. But if we differentiate the equation not more than twice, the terms got from differentiating u_3, u_4, &c. contain powers of x and y, and will therefore vanish for $x = 0$, $y = 0$. The values therefore of the differential coefficients for the origin are the same as if they were obtained from the equation $u_1 + u_2 = 0$.

It follows hence that the radius of curvature at the origin (the axes being rectangular) of $y + ax^2 + 2bxy + cy^2 + \&c. = 0$ is $\frac{1}{2a}$ (see *Conics*, p. 206); or this value can easily be found directly from the ordinary expression for the radius of curvature (*Higher Plane Curves*, p. 108).

258. Let now the equation of a surface referred to any tangent plane as plane of xy and the corresponding normal as axis of z, be

$$z + Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + \&c. = 0,$$

and let us investigate the curvature of any normal section, that is of the section by any plane passing through the axis of z. Thus, to find the radius of curvature of the section by the plane xz, we have only to make $y = 0$ in the equation, and we get a curve whose radius of curvature is $\frac{1}{2A}$. In like manner the section by the plane yz has its radius of curvature $= \frac{1}{2C}$. And in order to find the radius of curvature of any section whose plane makes an angle θ with the plane xz, we have only to turn the axes of x and y through an angle θ (by substituting $x\cos\theta - y\sin\theta$ for x, and $x\sin\theta + y\cos\theta$ for y, *Conics*, p. 7); and by then putting $y=0$ it appears as before that the radius of curvature is half the reciprocal of the new coefficient of x^2; that is to say,

$$\frac{1}{2R} = A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta.$$

259. The reader will not fail to observe that this expression for the radius of curvature of a normal section is identical in form with the expression for the square of the diameter of a central conic in terms of the angles which it makes with the axes of co-ordinates. Thus if ρ be the semi-diameter answering to an angle θ of the conic $Ax^2 + 2Bxy + Cy^2 = \frac{1}{2}$, we have $R = \rho^2$.

It may be seen otherwise that the radii of curvature are connected with their directions in the same manner as the squares of the diameters of a central conic. For we have seen that the radii of curvature depend only on the terms in u_1 and u_2. The radii of curvature therefore of all the sections of $u_1 + u_2 + u_3 + \&c.$ are the same as those of the sections of the quadric $u_1 + u_2$; and it was proved (p. 158) that these are all proportional to the squares of the diameters of the central section parallel to the tangent plane.

It is plain that the conic, the squares of whose radii are proportional to the radii of curvature, is similar to the indicatrix.

260. We can now at once apply to the theory of these radii of curvature all the results that we have obtained for the diameters of central conics. Thus we know that the quantity $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ admits of a maximum and minimum value; that the values of θ which correspond to the maximum and minimum are always real, and belong to directions at right angles to each other; and that those values of θ are given by the equation (see *Conics*, p. 140)

$$B\cos^2\theta - (A-C)\cos\theta\sin\theta - B\sin^2\theta = 0.$$

Hence, at any point on a surface there are among the normal sections, one for which the value of the radius of curvature is a maximum and one for which it is a minimum; the directions of these sections are at right angles to each other; and they are the directions of the axes of the indicatrix. They plainly bisect the angles between the two inflexional tangents. We shall call these the *principal* sections, and the corresponding radii of curvature the *principal radii.*

If we turn round the axes of x and y so as to coincide with the directions of maximum and minimum curvature just determined, it is known that the quantity $Ax^2 + 2Bxy + Cy^2$ will take the form $A'x^2 + B'y^2$. Now the formula of the last article, when the coefficient of xy vanishes, gives the following expression for any radius of curvature $\frac{1}{2R} = A' \cos^2\theta + B' \sin^2\theta$. But evidently A' and B' are the values of $\frac{1}{2}R$ corresponding to $\theta = 0$, and $\theta = 90°$. Hence any radius of curvature is expressed in terms of the two principal radii ρ and ρ', and of the angle which the direction of its plane makes with the principal planes, by the formula

$$\frac{1}{R} = \frac{\cos^2\theta}{\rho} + \frac{\sin^2\theta}{\rho'}.*$$

It is plain (as in *Conics*, p. 142) that A' and B', or $\frac{1}{2\rho}$, $\frac{1}{2\rho'}$ are given by a quadratic equation, the sum of these quantities being $A + C$ and their product $AC - B^2$.

When $\rho = \rho'$, all the other radii of curvature are also $= \rho$. The form of the equation then is $z + A(x^2 + y^2) + \&c. = 0$, or the indicatrix is a circle. The origin is then an *umbilic.*

From the expressions in this article we deduce at once, as in the theory of central conics, that *the sum of the reciprocals of the radii of curvature of two normal sections at right angles to each other is constant;* and again, *if normal sections be made through a pair of conjugate tangents* (see Art. 237), *the sum of their radii of curvature is constant.*

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* This formula (with the inferences drawn from it) is due to Euler.

261. It will be observed that the radius of curvature, being proportional to the *square* of the diameter of a central conic, does not become imaginary, but only changes sign, if the quantity $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ becomes negative. Now if radii of curvature directed on one side of the tangent plane are considered as positive, those turned the other way must be considered as negative; and the sign changes when the direction is changed in which the concavity of the curve is turned.

At an elliptic point on a surface; that is to say, when B^2 is less than AC, the sign of $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ remains the same for all values of θ; and therefore at such a point the concavity of every section through it is turned in the same direction.

At a hyperbolic point, that is to say, when B^2 is greater than AC, the radius of curvature twice changes sign and the concavity of some sections is turned in an opposite direction to that of others. The surface in fact cuts the tangent plane in the neighbourhood of the point, and the inflexional tangents mark the directions in which the surface crosses the tangent plane and divide the sections whose concavity is turned one way from those which are turned the other way.* And when we have chosen a hyperbola the squares of whose diameters are proportional to one set of radii, then the other set of radii are proportional to the squares of the diameters of the conjugate hyperbola.

262. Having shewn how to find the radius of curvature of any normal section, we shall next show how to express, in terms of this, the radius of curvature of any oblique section, inclined at an angle ϕ to the normal section, but meeting the tangent plane in the same line. Thus we have seen that the radius of curvature of the normal section made by the plane

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* The illustration of the summit of a mountain pass will enable the reader to conceive how a surface may in two directions sink below the tangent plane, and on the other sides rise above it. The shape of a saddle affords another familiar illustration of the same thing.

$y=0$ is $\frac{1}{2A}$. Now let us turn the axes of y and z round in their plane through an angle ϕ (which is done by substituting $z\cos\phi - y\sin\phi$ for z and $z\sin\phi + y\cos\phi$ for y). If we now make the new $y=0$, we shall get the equation (still to rectangular axes) of the section by a plane making an angle ϕ with the old plane $y=0$, but still passing through the old axis of x; and this equation will plainly be

$$z\cos\phi + Ax^2 + 2Bxz\sin\phi + Cz^2\sin^2\phi + 2Dxz + 2Ez^2\sin\phi + Fz^2 + \&c.,$$

and by the same method as before the radius of curvature is found to be $\frac{\cos\phi}{2A}$, or is $=R\cos\phi$, where R is the radius of curvature of the corresponding normal section. This is MEUNIER'S THEOREM, that *the radius of curvature of an oblique section is equal to the projection on the plane of this section of the radius of curvature of a normal section passing through the same tangent line.* Thus we see that of all sections which can be made through any line drawn in the tangent plane, the normal section is that whose radius of curvature is greatest; that is to say, the normal section is that which is least curved and which approaches most nearly to a straight line.

Meunier's theorem has been already proved in the case of a quadric (see p. 159), and we might therefore, if we had chosen, have dispensed with giving a new proof now; for we have seen that the radius of curvature of any section of $u_1 + u_2 + u_3 + \&c.$ is the same as that of the corresponding section of the quadric $u_1 + u_2$.

263. Every sphere whose centre is on a normal to a surface, and which passes through the point where the normal meets the surface, of course touches the surface. But the contact will be of the kind called *stationary* contact (Art. 129) when the length of the radius of the sphere is equal to one of the principal radii. For if the equations of two surfaces which touch be

$$z + Ax^2 + 2Bxy + Cy^2 + \&c. = 0, \quad z + A'x^2 + 2B'xy + C'y^2 + \&c. = 0,$$

then $\qquad (A-A')\,x^2 + 2\,(B-B')\,xy + (C-C')\,y^2 + \&c.$

passes through their curve of intersection, and it was proved (Art. 128) that the three terms just written represent the tangents to the curve of intersection of the surfaces. These tangents coincide, or there is stationary contact (Art. 129) when $(A-A')(C-C')=(B-B')^2$. When $B=B'=0$, this condition implies either $A=A'$ or $C=C'$. The surface then $z+Ax^2+Cy^2+\&c.=0$ will have stationary contact with the sphere $2rz+x^2+y^2+z^2=0$, if $2r=\frac{1}{A}$ or $=\frac{1}{C}$. But these are the values of the principal radii.

264. The principles laid down in the last article enable us to find an expression for the values of the principal radii at any point; the axes of co-ordinates having any position. It will be observed that what we have proved is, that if $u_1+u_2+\&c.$, $u_1+v_2+\&c.$ represent two surfaces which touch, then the intersection of the plane u_1 with the cone u_2-v_2 gives the two tangents to their curve of intersection: and there is stationary contact when the plane touches the cone.

Now if we transform the equation to any point $x'y'z'$ on the surface as origin, it becomes

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+\frac{1}{1.2}\left(x\frac{d}{dx'}+y\frac{d}{dy'}+z\frac{d}{dz'}\right)^2U'+\&c.,$$

or if we denote the first differential coefficients by L, M, N, P, and the second by a, b, c, &c. as before

$$2(Lx+My+Nz)+ax^2+by^2+cz^2+2lyz+2mzx+2nxy+\&c.=0.$$

The equation then of any sphere having the same tangent plane is

$$2(Lx+My+Nz)+\lambda(x^2+y^2+z^2)=0,$$

and the sphere will have stationary contact with the quadric if λ be determined so as to satisfy the condition that $Lx+My+Nz$ shall touch

$$(a-\lambda)x^2+(b-\lambda)y^2+(c-\lambda)z^2+2lyz+2mzx+2nxy.$$

This condition is

$$\begin{vmatrix} a-\lambda, & n, & m \\ n, & b-\lambda, & l \\ m, & l, & c-\lambda \end{vmatrix}=0,$$

which expanded is

$$\{(b-\lambda)(c-\lambda)-l^2\}L^2+\{(c-\lambda)(a-\lambda)-m^2\}M^2+\{(a-\lambda)(b-\lambda)-n^2\}N^2$$
$$+2\{mn-(a-\lambda)l\}MN+2\{nl-(b-\lambda)m\}NL+2\{lm-(c-\lambda)n\}LM=0,$$

or λ is given by the quadratic

$$(L^2+M^2+N^2)\lambda^2-\{(b+c)L^2+(c+a)M^2+(a+b)N^2$$
$$-2lMN-2mNL-2nLM\}\lambda$$
$$+(bc-l^2)L^2+(ca-m^2)M^2+(ab-n^2)N^2$$
$$+2(mn-al)MN+2(nl-bm)NL+2(lm-cn)LM=0.$$

Now if r be the radius of the sphere

$$\lambda(x^2+y^2+z^2)+2(Lx+My+Nz),$$

we have $r^2=\dfrac{L^2+M^2+N^2}{\lambda^2}$. We therefore find the principal radii by substituting $\dfrac{\sqrt{(L^2+M^2+N^2)}}{r}$ for λ in the preceding quadratic.

The absolute term in the equation for λ may be simplified by writing for L, M, N their values from the equations

$$(n-1)L=ax+ny+lz+pw, \text{ \&c.},$$

when the absolute term reduces to $-\dfrac{Hw^2}{(n-1)^2}$ where H is the Hessian, written at full length Art. 63. We might have seen *a priori* that for any point on the Hessian, the absolute term must vanish. For since the directions of the principal sections bisect the angles between the inflexional tangents; when the inflexional tangents coincide, one of the principal sections coincides with their common direction, and the radius of curvature of this section is infinite, since three consecutive points are on a right line. Hence one of the values of λ (which is the reciprocal of r) must vanish. By equating to nothing the coefficient of λ in the preceding quadratic, we obtain the equation of a surface of the degree $3n-4$, which intersects the given surface in all the points where the principal radii are equal and opposite: that is to say, where the indicatrix is an equilateral hyperbola.

The quadratic of this article might also have been found at once by Art. 98, which gives the axes of a section of the quadric

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy = 1$$

made parallel to the plane $Lx + My + Nz = 0$.

265. From the equations of the last article we can find the radius of curvature of any normal section meeting the tangent plane in a line whose direction-angles are given.

For the centre of curvature lies on the normal, and if we describe a sphere with this centre, and radius equal to the radius of curvature, it must touch the surface, and its equation is of the form

$$2(Lx + My + Nz) + \lambda(x^2 + y^2 + z^2) = 0.$$

The consecutive point on that section of the surface which we are considering satisfies this equation, and also the equation $u_1 + u_2 = 0$,

$$2(Lx + My + Nz) + ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy = 0.$$

Subtracting, we find

$$\lambda = \frac{ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy}{x^2 + y^2 + z^2}.$$

And since this equation is homogeneous, we may write for x, y, z the direction-cosines of the line joining the consecutive point to the origin. As in the last article $\lambda = \frac{\sqrt{(L^2 + M^2 + N^2)}}{r}$. Hence

$$r = \frac{\sqrt{(L^2 + M^2 + N^2)}}{a\cos^2\alpha + b\cos^2\beta + c\cos^2\gamma + 2l\cos\beta\cos\gamma + 2m\cos\gamma\cos\alpha + 2n\cos\alpha\cos\beta}.$$

The problem to find the maximum and minimum radius of curvature is therefore to make the quantity

$$ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy$$

a maximum or minimum subject to the relations

$$Lx + My + Nz = 0, \quad x^2 + y^2 + z^2 = 1.$$

And thus we see again that this is exactly the same problem as that of finding the axes of the central section of a quadric by a plane $Lx + My + Nz$.

266. In like manner the problem to find the *directions* of the principal sections at any point is the same as to find the directions of the axes of the section by the plane $Lx + My + Nz$ of the quadric $ax^2 + by^2 + cz^2 + 2lyz + 2mzx + 2nxy = 1$.

Now given any diameter of a quadric, one section can be drawn through it having that diameter for an axis; the other axis being plainly the intersection of the plane perpendicular to the given diameter with the plane conjugate to it. Thus if the central quadric be $U = 1$, and the given diameter pass through $x'y'z'$, then the diameter perpendicular and conjugate is the intersection of the planes

$$xx' + yy' + zz' = 0, \quad x'\frac{dU}{dx} + y'\frac{dU}{dy} + z'\frac{dU}{dz} = 0.$$

If the former diameter lie in a plane $Lx' + My' + Nz'$, the latter diameter traces out the cone which is represented by the determinant obtained on eliminating $x'y'z'$ from the three preceding equations: viz.

$$(Mz - Ny)\frac{dU}{dx} + (Nx - Lz)\frac{dU}{dy} + (Ly - Mx)\frac{dU}{dz} = 0.$$

And this cone must evidently meet the plane $Lx + My + Nz$ in the axes of the section by that plane. Thus then the directions of the principal sections are determined as the intersection of the tangent plane $Lx + My + Nz$ with the cone

$$(Mz - Ny)(ax + ny + mz) + (Nx - Lz)(nx + by + lz)$$
$$+ (Ly - Mx)(mx + ly + cz) = 0,$$

or $(Mm - Nn)x^2 + (Nn - Ll)y^2 + (Ll - Mm)z^2$

$$+ \{L(b - c) - nM + nN\}yz + \{Ln + M(c - a) - Nl\}zx$$
$$+ \{-Lm + Ml + N(a - b)\}xy.$$

267. The methods used in Art. 264 enable us also easily to find the conditions for an umbilic.* If the plane of xy be

* We might find the condition for an umbilic by forming the condition that the quadratic of Art. 264 should have equal roots. But, as at p. 32, this quadratic having its roots always real is one of the class discussed, *Higher Algebra*, p. 134; whose discriminant can be expressed as the

the tangent plane at an umbilic the equation of the surface is of the form

$$z + A(x^2+y^2) + 2Dxz + 2Eyz + Fz^2 + \&c. = 0;$$

and if we subtract from it the equation of any touching sphere, viz.

$$z + \lambda(x^2+y^2+z^2) = 0,$$

it is evidently possible so to choose λ (namely, by taking it $=A$) that all the terms in the remainder shall be divisible by z. We see thus that if $u_1 + u_2 + \&c.$ represent the surface, and $u_1 + \lambda v_2$ any touching sphere, it is possible, when the origin is an umbilic, so to choose λ that $u_2 - \lambda v_2$ may contain u_1 as a factor. We see then by transformation of co-ordinates as in Art. 264, that any point $x'y'z'$ will be an umbilic if it is possible so to choose λ that

$$(a-\lambda)\,x^2 + (b-\lambda)\,y^2 + (c-\lambda)\,z^2 + 2lyz + 2mzx + 2nxy$$

may contain as a factor $Lx + My + Nz$. If so, the other factor must be

$$\frac{a-\lambda}{L}\,x + \frac{b-\lambda}{M}\,y + \frac{c-\lambda}{N}\,z.$$

Multiplying out and comparing the coefficients of yz, zx, xy, we get the conditions

$$(b-\lambda)\frac{N}{M} + (a-\lambda)\frac{M}{N} = 2l,\quad (c-\lambda)\frac{L}{N} + (a-\lambda)\frac{N}{L} = 2m,$$

$$(a-\lambda)\frac{M}{L} + (b-\lambda)\frac{L}{M} = 2n.$$

Eliminating λ between these equations we obtain for an umbilic the two conditions

$$\frac{bN^2 + cM^2 - 2lMN}{N^2+M^2} = \frac{cL^2 + aN^2 - 2mLN}{L^2+N^2} = \frac{aM^2 + bL^2 - 2nLM}{M^2+L^2}.$$

Since there are only two conditions to be satisfied, a surface of the n^{th} degree has in general a determinate number of umbilics; for the two conditions, each of which represents a

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sum of squares. If therefore we only consider real umbilics, the result of equating the discriminant to nothing is equivalent to two conditions, which can be more easily obtained as in the text.

surface, combined with the equation of the given surface determine a certain number of points. It may happen however that the surfaces represented by the two conditions intersect in a curve which lies (either wholly or in part) on the given surface. In such a case there would be on the given surface a line, every point of which would be an umbilic. Such a line is called *a line of spherical curvature.*

268. There is one case in which the conditions of the last article are not applicable in the form in which we have written them. They appear to be satisfied by making $L=0$, $a=\frac{bN^2+cM^2-2lMN}{N^2+M^2}$; whence we might conclude that the surface $L=0$ must always pass through umbilics on the given surface. Now it is easy to see geometrically that this is not the case, for L $\left(\text{or } \frac{dU}{dx}\right)$ is the polar of the point yzw with respect to the surface, so that if L necessarily passed through umbilics it would follow by transformation of co-ordinates that the first polar of *every* point passes through umbilics. On referring to the last article, however, it will be seen that the investigation tacitly assumes that none of the quantities L, M, N vanish; since, if so, some of the equations which we have used would contain infinite terms. Supposing then L to vanish, we must examine directly the condition that $My+Nz$ may be a factor in

$$(a-\lambda)\,x^2+(b-\lambda)\,y^2+(c-\lambda)\,z^2+2lyz+2mzx+2nxy.$$

We must evidently have $\lambda=a$, and it is then easily seen that we must, as before, have $a=\frac{bN^2+cM^2-2lMN}{N^2+M^2}$, while in addition since the terms $2mzx+2nxy$ must be divisible by $My+Nz$, we must have $Mm=Nn$. Combining then with the two conditions here found, $L=0$, and the equation of the surface, there are four conditions which, except in special cases, cannot be satisfied by the co-ordinates of any points.

If we clear of fractions the conditions given in the last article, it will be found that they each contain either L, M, or N as a factor. And what we have proved in this article is that these factors may be suppressed as irrelevant to the question of umbilics.*

We now proceed to draw some other inferences from what was proved (Art. 263); namely, that the two principal spheres have stationary contact with the surface.

269. *When two surfaces have stationary contact, they touch in two consecutive points.*

The equations of the two surfaces being written as in Art. 263, the tangent planes at a consecutive point are (Art. 237)

$$z+2(Ax'+By')x+2(Bx'+Cy')y=0,$$
$$z+2(A'x'+B'y')x+2(B'x'+C'y')y=0.$$

That these may be identical, we must have

$$Ax'+By'=A'x'+B'y',\quad Bx'+Cy'=B'x'+C'y',$$

and eliminating $x':y'$ between these equations, we have

$$(A-A')(C-C')=(B-B')^2,$$

which is the condition for stationary contact.

The sphere, therefore, whose radius is equal to one of the principal radii touches the surfaces in two consecutive points;

* From what has been said we can infer the number of umbilics which a surface of the n^{th} degree will in general possess. We have seen that the umbilics are determined as the intersection of the given surface with a curve whose equations are of the form $\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}$. Now if A, B, C be of the degree l, and A', B', C' of the degree m, then $AB'-BA'$, $AC'-CA'$ are each of the degree $l+m$, and intersect in a curve of the degree $(l+m)^2$. But the intersection of these two surfaces includes the curve AA' of the degree lm which does not lie on the surface $BC'-CB'$. The degree therefore of the curve in question is l^2+lm+m^2. In the present case $l=3n-4$, $m=2n-2$ and the degree of the curve would seem to be $19n^2-46n+28$. But we have seen that the system we are discussing includes three curves such as

$$L,\ a(M^2+N^2)-(bN^2+cM^2-2lMN)$$

which do not pass through umbilics. Subtracting therefore from the number just found $3(n-1)(3n-4)$, we see that the umbilics are determined as the intersection of the given surface with a curve of the degree $(10n^2-25n+16)$, and therefore that the number of umbilics is in general $n(10n^2-25n+16)$.

or two consecutive normals to the surface are also normals to the sphere, and consequently intersect in its centre. Now we know that in plane curves the centre of the circle of curvature may be regarded as the intersection of two consecutive normals to the curve. In surfaces the normal at any point will not meet the normal at a consecutive point taken arbitrarily. But we see here that if the consecutive point be taken in the direction of either of the principal sections, the two consecutive normals will intersect, and their common length will be the corresponding principal radius. On account of the importance of this theorem we give a direct investigation of it.

270. *To find in what cases the normal at any point on a surface is intersected by a consecutive normal.* Take the tangent plane for the plane of xy, and let the equation of the surface be

$$z + Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 + \&c. = 0.$$

Then we have seen (Art. 237) that the equation of a consecutive tangent plane is

$$z + 2(Ax' + By')\,x + 2(Bx' + Cy')\,y = 0,$$

and a perpendicular to this through the point $x'y'$ will be

$$\frac{x - x'}{Ax' + By'} = \frac{y - y'}{Bx' + Cy'} = 2z.$$

This will meet the axis of z (which was the original normal) if

$$\frac{x'}{Ax' + By'} = \frac{y'}{Bx' + Cy'}.$$

The direction therefore of a consecutive point whose normal meets the given normal is determined by the equation

$$Bx'^2 + (C - A)\,x'y' - By'^2 = 0.$$

But this is the same equation (Art. 260) which determines the directions of maximum and minimum curvature. At any point on a surface therefore there are two directions, at right angles to each other, such that the normal at a consecutive point taken on either, intersects the original normal. And these directions are those of the two principal sections at the point. Taking for greater simplicity the directions of the principal sections as axes of co-ordinates; that is to say, making $B = 0$

in the preceding equations, the equation of a consecutive normal becomes $\frac{x-x'}{Ax'} = \frac{y-y'}{Cy'} = 2z$, whence it is easy to see that the normals corresponding to the points $y'=0$, $x'=0$ intersect the axis of z at distances respectively $z=\frac{1}{2}A$, $z=\frac{1}{2}C$. The intercepts therefore on a normal by the two consecutive ones which intersect it are equal to the principal radii.*

271. We may also arrive at the same conclusions by seeking the locus of points on a surface, the normals at which meet a fixed normal which we take for axis of z. Making $x=0$, $y=0$ in the equation of any other normal we see that the point where it meets the surface must satisfy the condition $\frac{x}{\frac{dU}{dx}} = \frac{y}{\frac{dU}{dy}}$. The curve where this surface meets the given surface has the extremity of the given normal for a double point, the two tangents to which are the two principal tangents to the surface at that point.

The special case where the fixed normal is one at an umbilic deserves notice. The equation of the surface being of the form $z + A(x^2+y^2) + \&c. = 0$, the lowest terms in the equation $x\frac{dU}{dy} = y\frac{dU}{dx}$, when we make $z=0$, will be of the third degree, and the umbilic is a triple point on the curve locus. Thus while every normal immediately consecutive to the normal

* M. Bertrand, in his theory of the curvature of surfaces, calculates the angle made by the consecutive normal with the plane containing the original normal and the consecutive point $x'y'$. Supposing still the directions of the principal sections to be axes of co-ordinates, the direction-cosines of the consecutive normal are proportional to $2Ax'$, $2Cy'$, while those of a tangent line perpendicular to the radius vector are proportional to $-y'$, x', 0. Hence the cosine of the angle between these two lines, or the sine of the angle which the consecutive normal makes with the normal section, is proportional to $(C-A)x'y'$; or, if α be the angle which the direction of the consecutive point makes with one of the principal tangents, is proportional to $(C-A)\sin 2\alpha$. When $\alpha = 0$ or $= 90°$, this angle vanishes and the consecutive normal is in the plane of the original normal.

at the umbilic meets the latter normal, there are three directions along any of which the next following normal will also meet the normal at the umbilic.

272. A *line of curvature** on a surface is a line traced on it such that the normals at any two consecutive points of it intersect. Thus starting with any point M on a surface, we may go on to either of the two consecutive points N, N' whose normals were proved to intersect the normal at M. The normal at N, again, is intersected by the consecutive normals at two points P, P', the element NP being a continuation of the element MN while the element NP' is approximately perpendicular to it. In like manner we might pass from the point P to another consecutive point Q and so have a line of curvature $MNPQ$. But we might evidently have pursued the same process had we started in the direction MN'. Hence, at any point M on a surface can be drawn two lines of curvature; these cut at right angles and are touched by the two "principal tangents" at M. A line of curvature will ordinarily not be a plane curve, and even in the special case where it is plane it need not coincide with a principal section at M, though it must touch such a section. For the principal section must be normal to the surface, and the line of curvature may be oblique.

A very good illustration of lines of curvature is afforded by the case of the surfaces generated by the revolution of any plane curve round an axis in its plane. At any point P of such a surface one line of curvature is the plane section passing through P and through the axis, or, in other words, is the generating curve which passes through P. For all the normals to this curve are also normals to the surface, and being in one plane, they intersect. The corresponding principal radius at P is evidently the radius of curvature of the plane section at the same point. The other line of curvature at P is the

* The whole theory of lines of curvature, umbilics, &c. is due to Monge. See his "Application de l'Analyse à la Géométrie," p. 124, Liouville's Edition.

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circle which is the section made by a plane drawn through P perpendicular to the axis of the surface; for the normals at all the points of this section evidently intersect the axis of the surface at the same point, and therefore intersect each other. The intercept on the normal between P and the axis is plainly the second principal radius of the surface.

The generating curve which passes through P is a principal section of the surface, since it contains the normal and touches a line of curvature; but the section perpendicular to the axis is not a principal section because it does not contain the normal at P. The second principal section at that point would be the plane section drawn through the normal at P and through the tangent to the circle described by P. The example chosen serves also to illustrate Meunier's theorem; for the radius of the circle described by P (which, as we have seen, is an oblique section of the surface) is the projection on that plane of the intercept on the normal between P and the axis, and we have just proved that this intercept is the radius of curvature of the corresponding normal section.

273. It was proved (Art. 266) that the direction-cosines of the tangent line to a principal section fulfil the relation

$$\begin{aligned}&(M\cos\gamma - N\cos\beta)\ (a\ \cos\alpha + n\ \cos\beta + m\ \cos\gamma)\\ &+ (N\cos\alpha - L\ \cos\gamma)\ (n\ \cos\alpha + b\cos\beta + l\ \cos\gamma)\\ &+ (L\ \cos\beta - M\cos\alpha)\ (m\ \cos\alpha + l\ \cos\beta + c\ \cos\gamma) = 0.\end{aligned}$$

Now the tangent line to a principal section is also the tangent to the line of curvature; while, if ds be the element of the arc of any curve, the projections of that element upon the three axes being dx, dy, dz, it is evident that the cosines of the angles which ds makes with the axes are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$. The differential equation of the line of curvature is therefore got by writing dx, dy, dz for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in the preceding formula.

This equation may also be found directly as follows (see Gregory's *Solid Geometry*, p. 256): Let α, β, γ be the coordinates of a point common to two consecutive normals.

Then, if xyz be the point where the first normal meets the surface, by the equations of the normal, we have $\frac{\alpha - x}{L} = \frac{\beta - y}{M} = \frac{\gamma - z}{N}$: or if we call the common value of these fractions θ, we have

$$\alpha = x + L\theta, \quad \beta = y + M\theta, \quad \gamma = z + N\theta.$$

But if the second normal meet the surface in a point $x + dx$, $y + dy$, $z + dz$, then expressing that $\alpha\beta\gamma$ satisfies the equations of the second normal, we get the same results as if we differentiate the preceding equations, considering $\alpha\beta\gamma$ as constant, or

$$dx + Ld\theta + \theta dL = 0, \quad dy + Md\theta + \theta dM = 0, \quad dz + Nd\theta + \theta dN = 0,$$

from which equations eliminating θ, $d\theta$, we have the same determinant as in Art. 266, viz.

$$\begin{vmatrix} dx, & dy, & dz \\ L, & M, & N \\ dL, & dM, & dN \end{vmatrix} = 0.$$

Of course

$$dL = adx + ndy + mdz, \quad dM = ndx + bdy + ldz, \quad dN = mdx + ldy + cdz.$$

Ex. To find the differential equation of the lines of curvature of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here we have

$$L = \frac{x}{a^2}, \quad M = \frac{y}{b^2}, \quad N = \frac{z}{c^2}; \quad dL = \frac{dx}{a^2}, \quad dM = \frac{dy}{b^2}, \quad dN = \frac{dz}{c^2}.$$

Substituting these values in the preceding equation it becomes, when expanded,

$$(b^2 - c^2)\, xdydz + (c^2 - a^2)\, ydzdx + (a^2 - b^2)\, zdxdy = 0.$$

Knowing as we do that the lines of curvature are the intersections of the ellipsoid with a system of concentric quadrics (Art. 206), it would be easy to assume for the integral of this equation $Ax^2 + By^2 + Cz^2 = 0$, and to determine the constants by actual substitution. If we assume nothing as to the form of the integral we can eliminate z and dz by the help of the equation of the surface, and so get a differential equation in two variables which is the equation of the projection of the lines of curvature on the plane of xy. Thus, in the present case, multiplying by $\frac{z}{c^2}$ and reducing by the equation of the ellipsoid and its differential, we have

$$\{(b^2-c^2)\,xdy+(c^2-a^2)\,ydx\}\left\{\frac{xdx}{a^2}+\frac{ydy}{b^2}\right\}=(a^2-b^2)\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}dxdy,$$

or writing $\qquad \frac{a^2\,(b^2-c^2)}{b^2\,(a^2-c^2)}=A,\quad \frac{a^2\,(a^2-b^2)}{a^2-c^2}=B,$

$$Axy\left(\frac{dy}{dx}\right)^2+(x^2-Ay^2-B)\,\frac{dy}{dx}-xy=0,$$

the integral of which (see Boole's *Differential Equations*, Ex. 3, p. 135) is

$$\frac{x^2}{B}-\frac{y^2}{BC}=\frac{1}{AC+1},$$

or the lines of curvature are projected on the principal plane into a series of conics whose axes a', b' are connected by the relation

$$\frac{a'^2\,(a^2-c^2)}{a^2\,(a^2-b^2)}+\frac{b'^2\,(b^2-c^2)}{b^2\,(b^2-a^2)}=1.$$

It is not difficult to see that this coincides with the account given of the lines of curvature in Art. 206.

274. The theorem that confocal quadrics intersect in lines of curvature is a particular case of a theorem due to Dupin,* which we shall state as follows: *If three surfaces intersect at right angles, and if each pair also intersect at right angles at their next consecutive common point, then the directions of the intersections are the directions of the lines of curvature on each.*

Take the point common to all three surfaces as origin, and the three rectangular tangent planes as co-ordinate planes; then the equations of the surfaces are of the form

$$\begin{aligned} x+ay^2+2byz+cz^2+2dzx+\&c. &= 0,\\ y+a'z^2+2b'zx+c'x^2+2d'zy+\&c. &= 0,\\ z+a''x^2+2b''xy+c''y^2+\&c. &= 0. \end{aligned}$$

* Développements de Géométrie, cinquième Mémoire. The demonstration here given is by Professor W. Thomson: see Gregory's *Solid Geometry*, p. 263. *Cambridge Mathematical Journal*, Vol. IV., p. 62.

At a consecutive point common to the first and second surfaces, we must have $x = 0$, $y = 0$, $z = z'$ where z' is very small. The consecutive tangent planes are

$$(1 + 2dz')\,x + \quad 2bz'y \quad + 2cz'z = 0,$$
$$2b'z'x + (1 + 2d'z')\,y + 2a'z'z = 0.$$

Forming the condition that these should be at right angles and only attending to the terms where z' is of the first degree, we have $b + b' = 0$.

In like manner, in order that the other pairs of surfaces may cut at right angles at a consecutive point, we must have $b' + b'' = 0$, $b'' + b = 0$, and the three equations cannot be fulfilled unless we have b, b', b'' each separately $= 0$; in which case the form of the equations shows (Art. 260) that the axes are the directions of the lines of curvature on each. Hence follows the theorem in the form given by Dupin; namely, that *if there be three systems of surfaces, such that every surface of one system is cut at right angles by all the surfaces of the other two systems, then the intersection of two surfaces belonging to different systems is a line of curvature on each.* For, at each point of it, it is, by hypothesis, possible to draw a third surface cutting both at right angles.

275. *If two surfaces cut at right angles,* and if their intersection is a line of curvature on one, it is also a line of curvature on the other.*

Proceeding as in the last article, and taking the origin at any point of their intersection, we must, in order that they may cut at right angles, have $b + b' = 0$, whence if $b = 0$, $b' = 0$.

Otherwise thus: the direction-cosines of the tangent planes of the two surfaces being proportional to L, M, N; L', M', N'; the direction-cosines of their line of intersection are proportional to $MN' - M'N$, $NL' - N'L$, $LM' - L'M$; and in order that this intersection should be the direction of a line of curva-

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* This is also true if they cut at any constant angle.

ture on the first surface, we must have the condition fulfilled (Art. 273)

$$\begin{vmatrix} MN' - M'N, & NL' - N'L, & LM' - ML' \\ L, & M, & N \\ dL, & dM, & dN \end{vmatrix} = 0,$$

which expanded is

$$(LL' + MM' + NN')(LdL + MdM + NdN) \\ - (L^2 + M^2 + N^2)(L'dL + M'dM + N'dN) = 0.$$

If the two surfaces are at right angles, we have

$$LL' + MM' + NN' = 0,$$

and the condition just written reduces to

$$L'dL + M'dM + N'dN = 0,$$

from which two equations we infer

$$LdL' + MdM' + NdN' = 0;$$

but this is the condition that the line of intersection should be a line of curvature on the second surface.

276. A line of curvature is, by definition, such that the normals to the surface at two consecutive points of it intersect each other. If then we consider the surface generated by all the normals along a line of curvature, this will be a developable surface (Note, p. 75) since two consecutive generating lines intersect. The developable generated by the normals along a line of curvature manifestly cuts the given surface at right angles.

The locus of points where two consecutive generators of a developable intersect is a curve whose properties will be more fully explained in the next chapter, and which is called the *cuspidal edge* of that developable. Each generator is a tangent to this curve, for it joins two consecutive points of the curve; namely, the points where the generator in question is met by the preceding and by the succeeding generator (see Art. 119).

Consider now the normal at any point M of a surface; through that point can be drawn two lines of curvature

$MNPQ$, &c., $MN'P'Q'$, &c.: let the normals at the points M, N, P, Q, &c. intersect in C, D, E, &c., and those at M, N', P', Q' in C', D', E'; then it is evident that the curve CDE, &c. is the cuspidal edge of the developable generated by the normals along the first line of curvature while $C'D'E'$ is the cuspidal edge of the developable generated by the normals along the second. The normal at M, as has just been explained, touches these curves at the points C, C' which are the two centres of curvature corresponding to the point M.

What has been proved may be stated as follows: The cuspidal edge of the developable generated by the normals along a line of curvature, is the locus of one of the systems of centres of curvature corresponding to all the points of that line.

277. The assemblage of the centres of curvature C, C' answering to all the points of a surface is a surface of two sheets called the *surface of centres* (see Art. 208). The curve CDE lies on one sheet while $C'D'E'$ lies on the other sheet. Every normal to the given surface touches both sheets of the surface of centres: for it has been proved that the normal at M touches the two curves CDE, $C'D'E'$, and every tangent line to a curve traced on a surface is also a tangent to the surface.

Now if from a point, not on a surface, be drawn two consecutive tangent lines to a surface, the plane of those lines is manifestly a tangent plane to the surface; for it is a tangent plane to the cone which is drawn from the point touching the surface. But if two consecutive tangent lines intersect on the surface, it cannot be inferred that their plane touches the surface. For if we cut the surface by any plane whatever, any two consecutive tangents to the curve of section (which, of course, are also tangent lines to the surface) intersect on the curve, and yet the plane of these lines is supposed not to touch the surface.

Consider now the two consecutive normals at the points M, N, these are both tangents to both sheets of the surface of centres. And since the point C in which they intersect is on the first sheet but not necessarily on the second, the plane of the two normals is the tangent plane to the second sheet of the surface of centres.

The plane of the normals at the points M, N' is the tangent plane to the other sheet of the surface of centres. But because the two lines of curvature through M are at right angles to each other, it follows that these two planes are at right angles to each other. Hence, *the tangent planes to the surface of centres at the two points* C, C', *where any normal meets it, cut each other at right angles.*

It is manifest that for every umbilic on the given surface, the two sheets of the surface of centres have a point common; or, in other words, the surface of centres has a double point; and if the original surface have a line of spherical curvature, the surface of centres will have a double line. The two sheets will cut at right angles every where along this double line.

278. It is convenient to define here a *geodesic line* on a surface, and to establish the fundamental property of such a line; namely, that its osculating plane (see Art. 119) at any point is normal to the surface. A geodesic line is the form assumed by a strained thread lying on a surface and joining any two points on the surface. It is plain that the geodesic is ordinarily the shortest line on the surface by which the two points can be joined, since, by pulling at the ends of the thread, we must shorten it as much as the interposition of the surface will permit. Now the resultant of the tensions along two consecutive elements of the curve, formed by the thread, lies in the plane of those elements, and since it must be destroyed by the resistance of the surface, it is normal to the surface; hence, *the plane of two consecutive elements of the geodesic contains the normal to the surface.**

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* I have followed Monge in giving this proof, the mechanical principles which it involves being so elementary that it seems pedantic to object to the introduction of them. For the benefit of those who would prefer a purely geometrical proof I add one or two in the text. For readers familiar with the theory of maxima and minima it is scarcely necessary to add that

The same thing may also be proved geometrically. In the first place, if two points A, C in different planes be connected by joining each to a point B in the intersection of the two planes, the sum of AB and BC will be less than the sum of any other joining lines AB', $B'C$, if AB and BC make equal angles with TT', the intersection of the planes. For if one plane be made to revolve about TT' until it coincide with the other, AB and BC become one right line since the angle TBA is supposed to be equal to $T'BC$; and the right line AC is the shortest by which the points A and C can be joined.

It follows then that if AB and BC be consecutive elements of a curve traced on a surface, that curve will be the shortest line connecting A and C when AB and BC make equal angles with BT, the intersection of the tangent planes at A and C.

We see then that AB (or its production) and BC are consecutive edges of a right cone having BT for its axis. Now the plane containing two consecutive edges is a tangent plane to the cone; and since every tangent plane to a right cone is perpendicular to the plane containing the axis and the line of contact, it follows that the plane ABC (the osculating plane to the geodesic) is perpendicular to the plane AB, BT which is the tangent plane at A. The theorem of this article is thus established.

M. Bertrand has remarked (*Liouville*, t. XIII., p. 73, cited by Cayley, *Quarterly Journal*, Vol. I., p. 186) that this fundamental property of geodesics follows at once from Meunier's theorem (see Art. 262). For it is evident, that for an indefinitely small arc the chord of which is given, the excess in length over the chord is so much the less as the radius of curvature is greater. The shortest arc therefore joining two

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a geodesic need not be the absolutely shortest line by which two points on the surface may be joined. Thus, if we consider two points on a sphere joined by a great circle, the remaining portion of that great circle, exceeding 180° is a geodesic though not the shortest line connecting the points. The geodesic however will always be the shortest line if the two points considered be taken sufficiently near.

indefinitely near points A, B, on a surface is that which has the greatest radius of curvature, and we have seen that this is the normal section.

279. Returning now to the surface of centres, I say that the curve CDE (Art. 277) which is the locus of points of intersection of consecutive normals along a line of curvature is a geodesic on the sheet of the surface of centres on which it lies. For we saw (Art. 277) that the plane of two consecutive normals to the surface (that is to say, the plane of two consecutive tangents to this curve) is the tangent plane to the second sheet of the surface of centres and is perpendicular to the tangent plane at C to that sheet of the surface of centres on which C lies. Since then the osculating plane of the curve CDE is always normal to the surface of centres, the curve is a geodesic on that surface.

280. We have given the equations connected with lines of curvature on the supposition that the equation of the surface has been given, as it ordinarily is, in the form $\phi(x, y, z) = 0$. As it is convenient, however, that the reader should be able to find here the formulæ which have been commonly employed, we shall conclude this chapter by giving the principal equations in the form given by Monge and by most subsequent writers, viz. when the equation of the surface is in the form $z = \phi(x, y)$. We use the ordinary notations

$$dz = pdx + qdy, \quad dp = rdx + sdy, \quad dq = sdx + tdy.$$

We might derive the results in this form from those found already; for since we have $U = \phi(x, y) - z = 0$, we have

$$\frac{dU}{dx} = p, \quad \frac{dU}{dy} = q, \quad \frac{dU}{dz} = -1,$$

with corresponding expressions for their second differential coefficients. We shall, however, repeat the investigations for this form as they are usually given.

The equation of a tangent plane is

$$z - z' = p(x - x') + q(y - y'),$$

and the equations of the normal are

$$(x-x')+p(z-z')=0, \quad y-y'+q(z-z')=0.$$

If then $\alpha\beta\gamma$ be any point on the normal and xyz the point where it meets the surface, we have

$$(\alpha-x)+p(\gamma-z)=0, \quad (\beta-y)+q(\gamma-z)=0.$$

And if $\alpha\beta\gamma$ also satisfy the equations of a second normal, the differentials of these equations must vanish, or

$$dx+pdz=(\gamma-z)\,dp, \quad dy+qdz=(\gamma-z)\,dq;$$

whence, eliminating $(\gamma-z)$, we have the equation of condition

$$(dx+pdz)\,dq=(dy+qdz)\,dp.$$

Putting in for dz, dp, dq their values already given, and arranging, we have

$$\frac{dy^2}{dx^2}\{(1+q^2)\,s-pqt\}+\frac{dy}{dx}\{(1+q^2)\,r-(1+p^2)\,t\}-\{(1+p^2)\,s-pqr\}=0.$$

This equation determines the projections on the plane of xy of the two directions in which consecutive normals can be drawn so as to intersect the given normal.

281. From the equations of the preceding article we can also find the lengths of the principal radii. The equations

$$dx+pdz=(\gamma-z)\,dp, \quad dy+qdz=(\gamma-z)\,dq,$$

when transformed as above become

$$\{1+p^2-(\gamma-z)\,r\}\;dx+\{pq-(\gamma-z)\,s\}\;dy=0,$$

$$\{1+q^2-(\gamma-z)\,t\}\;dy+\{pq-(\gamma-z)\,s\}\;dx=0,$$

whence eliminating $dx:dy$, we have

$$(\gamma-z)^2\,(rt-s^2)-(\gamma-z)\,\{(1+q^2)\,r-2pqs+(1+p^2)\,t\}+(1+p^2+q^2)=0.$$

Now $\gamma-z$ is the projection of the radius of curvature on the axis of z; and the cosine of the angle the normal makes with that radius being $\dfrac{1}{\sqrt{(1+p^2+q^2)}}$, we have

$$R=(\gamma-z)\,\sqrt{(1+p^2+q^2)}.$$

Eliminating then $\gamma - z$ by the help of the last equation, R is given by the equation

$$R^2(rt-s^2) - R\{(1+q^2)r - 2pqs + (1+p^2)t\}\sqrt{(1+p^2+q^2)} + (1+p^2+q^2)^2 = 0.$$

282. From the preceding theorems can be deduced Joachimsthal's theorem (see *Crelle*, Vol. XXX., p. 347) that if a line of curvature be a plane curve, its plane makes a constant angle with the tangent plane to the surface at any of the points where it meets it. Let the plane be $z=0$, then the equation of Art. 278

$$(dx+pdz)\,dq = (dy+qdz)\,dp$$

becomes $dxdq = dydp$. But we have also $pdx + qdy = 0$, consequently $pdp + qdq = 0$; $p^2+q^2 =$ constant. But p^2+q^2 is the square of the tangent of the angle which the tangent plane makes with the plane xy; since $\cos\gamma = \dfrac{1}{\sqrt{(1+p^2+q^2)}}$.

Otherwise thus (see Liouville, Vol. XI., p. 87): Let MM', $M'M''$ be two consecutive and equal elements of a line of curvature, then the two consecutive normals are two perpendiculars to these lines passing through their middle points I, I', and C the point of meeting of the normals is equidistant from the lines MM', $M'M''$. But if from C we let fall a perpendicular CO on the plane $MM'M''$, O will be also equidistant from the same elements; and therefore the angle $CIO = CI'O$. It is proved then that the inclination of the normal to the plane of the line of curvature remains unchanged as we pass from point to point of that line.

More generally let the line of curvature not be plane. Then as before the tangent planes through MM' and through $M'M''$ make equal angles with the plane $MM'M''$. And evidently the angle which the second tangent plane makes with a second osculating plane $M'M''M'''$ differs from the angle which it makes with the first by the angle between the two osculating planes. Thus we have Lancret's theorem, that *along a line of curvature the variation in the angle between the tangent plane to the surface and the osculating plane to the curve is equal to the angle between the two osculating planes.*

For example, *if a line of curvature be a geodesic it must be plane.* For then the angle between the tangent plane and osculating plane does not vary, being always right: therefore the osculating plane itself does not vary. From the same principles we obtain a simple proof of the theorem of Art. 275.

283. Finally, to obtain the radius of curvature of any normal section. Since the centre of curvature $\alpha\beta\gamma$ lies on the normal, we have

$$(\alpha - x) + p(\gamma - z) = 0, \quad (\beta - y) + q(\gamma - z) = 0.$$

Further, we have

$$(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2 = R^2.$$

And since this relation holds for three consecutive points of the section which is osculated by the circle we are considering, we have

$$(\alpha - x)\, dx + (\beta - y)\, dy + (\gamma - z)\, dz = 0,$$

$$(\alpha - x)\, d^2x + (\beta - y)\, d^2y + (\gamma - z)\, d^2z = dx^2 + dy^2 + dz^2.$$

Combining this last with the preceding equations, we have

$$\frac{\alpha - x}{p} = \frac{\beta - y}{q} = -\frac{\gamma - z}{1} = \frac{R}{\sqrt{(1 + p^2 + q^2)}} = \frac{dx^2 + dy^2 + dz^2}{pd^2x + qd^2y - d^2z}.$$

But differentiating the equation $dz = pdx + qdy$, we have

$$d^2z - pd^2x - qd^2y = rdx^2 + 2sdxdy + tdy^2,$$

whence $R = \pm\sqrt{(1 + p^2 + q^2)}\, \dfrac{dx^2 + dy^2 + (pdx + qdy)^2}{rdx^2 + 2sdxdy + tdy^2}.$

The radius of curvature therefore of a section whose projection on the plane of xy is parallel to $y = mx$ is

$$\pm\sqrt{(1 + p^2 + q^2)}\, \frac{(1 + p^2) + 2pqm + (1 + q^2)\, m^2}{r + 2sm + tm^2}.$$

The conditions for an umbilic are got by expressing that this value is independent of m, and are

$$\frac{1 + p^2}{r} = \frac{pq}{s} = \frac{1 + q^2}{t}.$$

CHAPTER XI.

CURVES AND DEVELOPABLES.

SECTION I. PROJECTIVE PROPERTIES.

284. It was proved (p. 13) that two equations represent a curve in space. Thus the equations $U=0$, $V=0$ represent the curve of intersection of the surfaces U, V.

The degree of a curve in space is measured by the number of points in which it is met by any plane. Thus, if U, V be of the m^{th} and n^{th} degrees respectively, the surfaces which they represent are met by any plane in curves of the same degrees, which intersect in m, n points. The curve UV is therefore of the mn^{th} degree.

By eliminating the variables alternately between the two given equations, we obtain three equations

$$\phi(y, z)=0, \quad \psi(z, x)=0, \quad \chi(x, y)=0,$$

which are the equations of the projections of the curve on the three co-ordinate planes. Any one of the equations taken separately represents the cylinder whose edges are parallel to one of the axes, and which passes through the curve (Art. 24). The theory of elimination shows that the equation $\phi(y, z)=0$ obtained by eliminating x between the given equations is of the mn^{th} degree. And it is also geometrically evident that any cone or cylinder* standing on a curve of the r^{th} degree is of the r^{th} degree. For if we draw any plane through the vertex of the cone [or parallel to the generators of the cylinder] this plane meets the cone in r lines; namely, the lines joining the vertex to the r points where the plane meets the curve.

* A cylinder is plainly the limiting case of a cone, whose vertex is at infinity.

285. Now, conversely, if we are given any curve in space and desire to represent it by equations, we need only take the three plane curves which are the projections of the curve on the three co-ordinate planes; then any two of the equations $\phi(y, z)=0$, $\psi(z, x)=0$, $\chi(x, y)=0$ will represent the given curve. But ordinarily these will not form the simplest system of equations by which the curve can be represented. For if r be the degree of the curve, these cylinders being each of the r^{th} degree, any two intersect in a curve of r^2 degree; that is to say, not merely in the curve we are considering but in an extraneous curve of the degree r^2-r. And if we wish not merely to obtain a system of equations satisfied by the points of the given curve, but also to exclude all extraneous points, we must preserve the system of three projections; for the projection on the third plane of the extraneous curve in which the first two cylinders intersect will be different from the projection of the given curve.

It *may* be possible by combining the equations of the three projections to arrive at two equations $U=0$, $V=0$, which shall be satisfied for the points of the given curve, and for no other. But it is not generally true that *every* curve in space is the complete intersection of two surfaces. To take the simplest example, consider two quadrics having a right line common, as, for example, two cones having a common edge. The intersection of these surfaces, which is in general of the fourth degree, must consist of the common right line, and of a curve of the third degree. Now since the only factors of 3 are 1 and 3, a curve of the third degree cannot be the complete intersection of two surfaces unless it be a plane curve; but the curve we are considering cannot be a plane curve,* for if so any arbitrary line in its plane would meet it in three points, but such a line could not meet either quadric in more

* Curves in space which are not plane curves have commonly been called "curves of double curvature." In what follows, I use the word "curve" to denote a curve in space, which ordinarily is not a plane curve, and I add the adjective "twisted" when I want to state expressly that the curve is not a plane curve.

than two, and therefore could not pass through three points of their curve of intersection.

286. If a curve be either the complete or partial intersection of two surfaces U, V, the tangent to the curve at any point is evidently the intersection of the tangent planes to the two surfaces, and is represented by the equations

$$x\frac{dU'}{dx'}+y\frac{dU'}{dy'}+z\frac{dU'}{dz'}+w\frac{dU'}{dw'}=0,$$

$$x\frac{dV'}{dx'}+y\frac{dV'}{dy'}+z\frac{dV'}{dz'}+w\frac{dV'}{dw'}=0.$$

The direction-cosines of the tangent are plainly proportional to $MN'-M'N$, $NL'-N'L$, $LM'-L'M$, where L, M, &c. are the first differential coefficients.

An exceptional case arises when the two surfaces touch, in which case the point of contact is a double point on their curve of intersection. All this has been explained before (see Art. 128). As a particular case of the above, the projection of the tangent line to any curve is the tangent to its projection; and when the curve is given as the intersection of the two cylinders $y=\phi(z)$, $x=\psi(z)$, the equations of the tangent are

$$y-y'=\frac{d\phi}{dz}(z-z'),\quad x-x'=\frac{d\psi}{dz}(z-z').$$

This may be otherwise expressed as follows: Consider any element of the curve ds; it is projected on the axes of coordinates into dx, dy, dz. The direction-cosines of this element are therefore $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, and the equations of the tangent are

$$\frac{x-x'}{\frac{dx}{ds}}=\frac{y-y'}{\frac{dy}{ds}}=\frac{z-z'}{\frac{dz}{ds}}.$$

Since the sum of the squares of the three cosines are equal to unity, we have $ds^2=dx^2+dy^2+dz^2$.

We shall postpone to another section the theory of normals, radii of curvature, and in short everything which involves the consideration of angles, and in this section we shall only consider what may be called the projective properties of curves.

287. The theory of curves is in a great measure identical with that of developables on which account it is necessary to enter more fully into the latter theory. In fact it was proved (Art. 119) that the reciprocal of a series of points forming a curve is a series of planes enveloping a developable. We there showed that the points of a curve regarded as a system of points 1, 2, 3, &c. give rise to a system of lines; namely, the lines 12, 23, 34, &c. joining each point to its next consecutive, these lines being the tangents to the curve: and that they also give rise to a system of planes, viz. the planes 123, 234, &c. containing every three consecutive points of the system, these planes being the osculating planes of the curve. The assemblage of the lines of the system forms a surface whose equation can be found when the equation of the curve is given. For the two equations of the tangent line to the curve involve the three co-ordinates x', y', z', which being connected by two relations are reducible to a single parameter; and by the elimination of this parameter from the two equations, we obtain the equation of the surface. Or, in other words, we must eliminate $x'y'z'$ between the two equations of the tangent and the two equations of the curve. We have said (Art. 119) that the surface generated by the tangents is a developable since every two consecutive positions of the generating line intersect each other. The name given to this kind of surface is derived from the property that it can be unfolded into a plane without crumpling or tearing. Thus imagine any series of lines Aa, Bb, Cc, Dd, &c. (which for the moment we take at a finite distance from each other) and such that each intersects the consecutive in the points a, b, c, &c.; and suppose a surface to be made up of the faces AaB, BbC, CcD, &c., then it is evident that such a surface could be developed into a plane by turning the face AaB round aB as a hinge until it formed a continuation of BbC; by turning the two, which we had thus made into one face, round cC until they formed

a continuation of the next face and so on. In the limit when the lines Aa, Bb, &c. are indefinitely near, the assemblage of plane elements forms a developable which, as just explained, can be unfolded into one plane.

The reader will find no difficulty in conceiving this from the examples of developables with which he is most familiar, viz. a cone or a cylinder. There is no difficulty in folding a sheet of paper into the form of either surface and in unfolding it again into a plane. But it will easily be seen to be impossible to fold a sheet of paper into the form of a sphere (which is not a developable surface); or, conversely, if we cut a sphere in two it is impossible to make the portions of the surface lie smooth in one plane.

288. The plane AaB containing two consecutive generating lines is evidently, in the limit, a tangent plane to the developable. It is plain that we might consider the surface as generated by the motion of the plane AaB according to some assigned law, the envelope of this plane in all its positions being the developable. Now if we consider the developable generated by the tangent lines of a curve in space, the equations of the tangent at any point $x'y'z'$ are plainly functions of those co-ordinates, and the equation of the plane containing any tangent and the next consecutive (in other words, the equation of the osculating plane at any point $x'y'z'$) is also a function of these co-ordinates. But since $x'y'z'$ are connected by two relations, namely, the equations of the curve; we can eliminate any two of them, and so arrive at this result, that *a developable is the envelope of a plane whose equation contains a single variable parameter.* To make this statement better understood we shall point out an important difference between the cases when a plane curve is considered as the envelope of a moveable line, and when a surface in general is considered as the envelope of a moveable plane.

289. The equation of the tangent to a plane curve is a function of the co-ordinates of the point of contact; and these two co-ordinates being connected by the equation of the curve,

we can either eliminate one of them, or else express both in terms of a third variable so as to obtain the equation of the tangent as a function of a single variable parameter. The converse problem to obtain the envelope of a right line whose equation includes a variable parameter has been discussed, *Higher Plane Curves*, p. 93. Let the equation of any tangent line be $u=0$, where u is of the first degree in x and y, and the constants are functions of a parameter α. Then the line answering to the value of the parameter $\alpha+h$ is $u+\frac{du}{d\alpha}\frac{h}{1}+\frac{d^2u}{d\alpha^2}\frac{h^2}{1.2}+\&c.$; and the point of intersection of these two lines is given by the equations $u=0$, $\frac{du}{d\alpha}+\frac{h}{1.2}\frac{d^2u}{d\alpha^2}+\&c.=0$. And, in the limit, the point of intersection of a line with the next consecutive (or, in other words, the point of contact of any line with its envelope) is given by the equations $u=0$, $\frac{du}{d\alpha}=0$. If from these two equations we eliminate α we obtain the locus of the points of intersection of each line of the system with the next consecutive; that is to say, the equation of the envelope of all these lines. It is easy to prove that the result of this elimination represents a curve to which u is a tangent. For if in u we replace α by its value, in terms of x and y, derived from the equation $\frac{du}{d\alpha}=0$, we have $\frac{du}{dx}=\left(\frac{du}{dx}\right)+\frac{du}{d\alpha}\frac{d\alpha}{dx}$ and $\frac{du}{dy}=\left(\frac{du}{dy}\right)+\frac{du}{d\alpha}\frac{d\alpha}{dy}$, where $\left(\frac{du}{dx}\right)$, $\left(\frac{du}{dy}\right)$ are the differentials of u on the supposition that α is constant. And since $\frac{du}{d\alpha}=0$ it is evident that $\frac{du}{dx}$, $\frac{du}{dy}$ are the same as on the supposition that α is constant. It follows that the eliminant in question denotes a curve touched by u.

If it be required to draw a tangent to this curve through any point, we have only to substitute the co-ordinates of that point in the equation $u=0$, and determine α so as to satisfy that equation. This problem will have a definite number of solutions, and the number will plainly be the number of tangents which can be drawn to the curve from an arbitrary point; that is to say, the class of the curve. For example, the envelope of the line

$$a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0,$$

where a, b, c, d, are linear functions of the co-ordinates, is plainly a curve of the third class.

290. Now let us proceed in like manner with a surface. The equation of the tangent plane to a surface is a function of the three co-ordinates, which being connected by only one relation (viz. the equation of the surface), the equation of the tangent plane, when most simplified, contains two variable parameters. The converse problem is to find the envelope of a plane whose equation $u = 0$ contains two variable parameters α, β. The equation of any other plane answering to the values $\alpha + h$, $\beta + k$ will be

$$u + \left(h\frac{du}{d\alpha} + k\frac{du}{d\beta}\right) + \frac{1}{1.2}\left(h^2\frac{d^2u}{dx^2} + \&c.\right) = 0.$$

Now in the limit, when h and k are taken indefinitely small, they may preserve any finite ratio to each other $k = \lambda h$. We see thus that the intersection of any plane by a consecutive one is not a definite line, but may be any line represented by the equations $u = 0$, $\frac{du}{d\alpha} + \lambda\frac{du}{d\beta} = 0$, where λ is indeterminate. But we see also that all planes consecutive to u pass through the *point* given by the equations $u = 0$, $\frac{du}{d\alpha} = 0$, $\frac{du}{d\beta} = 0$.

From these three equations we can eliminate the parameters α, β, and so find the locus of all those points where a plane of the system is met by the series of consecutive planes. It is proved, as in the last article, that the surface represented by this eliminant is touched by u. If it be required to draw a tangent plane to this surface through any point, we have only to substitute the co-ordinates of that point in the equation $u = 0$. The equation then containing two indeterminates α and β can be satisfied in an infinity of ways; or, as we know, through a given point an infinity of tangent planes can be drawn to the surface, these planes enveloping a cone.

Suppose, however, that we either consider β as constant, or as any definite function of α, the equation of the tangent plane is reduced to contain a single parameter, and the envelope of those particular tangent planes which satisfy the assumed condition is a developable. Thus, again, we may see the analogy between a developable and a curve. When a surface is considered as the locus of a number of points connected by a given relation, if we add another relation connecting the points we obtain a curve traced on the given surface. So if we consider a surface as the envelope of a series of planes connected by a single relation, if we add another relation connecting the planes we obtain a developable enveloping the given surface.

291. Let us now see what properties of developables are to be deduced from considering the developable as the envelope of a plane whose equation contains a single variable parameter. In the first place it appears that through any assumed point can be drawn, not as before an infinity of planes of the system, forming a cone; but a definite number of planes. Thus if it be required to find the envelope of $a\alpha^3 + 3b\alpha^2 + 3c\alpha + d$, where a, b, c, d represent planes, it is obvious that only three planes of the system can be drawn through a given point, since on substituting the co-ordinates of any point we get a cubic for α. Again, any plane of the system is cut by a consecutive plane in a definite line; namely, the line $u=0$, $\frac{du}{d\alpha}=0$; and if we eliminate α between these two equations we obtain the surface generated by all those lines, which is the required developable.

It is proved, as at Art. 289, that the plane u touches the developable at every point which satisfies the equations $u=0$, $\frac{du}{d\alpha}=0$; or, in other words, touches along the whole of the line of the system corresponding to u. It was proved (Art. 107) that in general when a surface contains a right line the tangent plane at each point of the right line is different. But in the case of the developable the tangent plane at every point is the same. If x be the plane which touches all along the line

xy, the equation of the surface can be thrown into the form $x\phi + y^2\psi = 0$ (see p. 75).*

292. Let us now consider three consecutive planes of the system, and it is evident as before that their intersection satisfies the equations $u = 0$, $\frac{du}{d\alpha} = 0$, $\frac{d^2u}{d\alpha^2} = 0$. For any value of α, the point is thus determined where any line of the system is met by the next consecutive. The locus of these points is got by eliminating α between these equations. We thus obtain two equations in x, y, z, one of them being the equation of the developable. These two equations represent a curve traced on the developable. Thus it is evident that starting with the definition of a developable as the envelope of a moveable plane, we are led back to its generation as the locus of tangents to a curve. For the consecutive intersections of the planes form a series of lines, and the consecutive intersection of the lines are a series of points forming a curve to which the lines are tangents. We shall presently show that the curve is a cuspidal edge† on the developable.

* It seems unnecessary to enter more fully into the subject of envelopes in general, since what is said in the text applies equally if u, instead of representing a plane, denote any surface whose equation includes a variable parameter. Monge calls the curve $u = 0$, $\frac{du}{d\alpha} = 0$, in which any surface of the system is intersected by the consecutive, the *characteristic* of the envelope. For the nature of this curve depends only on the manner in which the variables x, y, z enter into the function u, and not on the manner in which the constants depend on the parameter. Thus when u represents a plane, the characteristic is always a right line, and the envelope is the locus of a system of right lines. When u represents a sphere, the characteristic, being the intersection of two consecutive spheres, is a circle and the envelope is the locus of a system of circles. And so envelopes in general may be divided into families according to the nature of the characteristic.

† Monge has called this the "arête de rebroussement," or "edge of regression" of the developable. There is a similar curve on every envelope, namely, the locus of points in which each "characteristic" is met by the next consecutive. The part of the characteristic on one side of this curve generates one sheet of the envelope, and that on the other side generates another sheet. The two sheets touch along this curve which is their

293. Four consecutive planes of the system will not meet in a point unless the four conditions be fulfilled $u=0$, $\frac{du}{da}=0$, $\frac{d^2u}{da^2}=0$, $\frac{d^3u}{da^3}=0$. It is in general possible to find certain values of a, for which this condition will be satisfied. For if we eliminate x, y, z, we get the condition that the four planes, whose equations have been just written, shall meet in a point. This condition is a function of a; and by equating this function to nothing, we shall in general get a determinate number of values of a for which the condition is satisfied. There are therefore in general a certain number of points of the system through which four planes of the system pass; or, in other words, a certain number of points in which three consecutive lines of the system intersect. We shall call these, as at *Higher Plane Curves*, p. 28, the *stationary points* of the system; since in this case the point determined as the intersection of two consecutive lines, coincides with that determined as the intersection of the next consecutive pair.

Reciprocally, there will be in general a certain number of planes of the system which may be called *stationary planes*. These are the planes which contain four consecutive points of the system; for in such a case the planes 123, 234 evidently coincide.

294. We shall now show how, from Plücker's equations connecting the ordinary singularities of plane curves,* Mr. Cayley†

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common limit, and is a cuspidal edge of the envelope. Thus in the case of a cone the parts of the generating lines on opposite sides of the vertex generate opposite sheets of the cone, and the cuspidal edge in this case reduces itself to a single point, namely, the vertex.

* These equations are as follows: see *Higher Plane Curves*, p. 91. Let μ be the degree of a curve, ν its class, δ the number of its double points, τ that of its double tangents, κ the number of its cusps, ι that of its points of inflexion; then

$$\nu = \mu(\mu-1) - 2\delta - 3\kappa; \quad \mu = \nu(\nu-1) - 2\tau - 3\iota,$$
$$\iota = 3\mu(\mu-2) - 6\delta - 8\kappa; \quad \kappa = 3\nu(\nu-2) - 6\tau - 8\iota.$$

Whence also $\iota - \kappa = 3(\nu - \mu)$; $2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9)$.

† See Liouville's *Journal*, Vol. x., p. 245; *Cambridge and Dublin Mathematical Journal*, Vol. v., p. 18.

has deduced equations connecting the ordinary singularities of developables. We shall first make an enumeration of these singularities. We speak of the "points of the system," the "lines of the system," and the "planes of the system" as explained (Art. 119).

Let m be the number of points of the system which lie in any plane; or, in other words, the *degree* of the curve which generates the developable.

Let n be the number of planes of the system which can be drawn through an arbitrary point. We have proved (Art. 291) that the number of such planes is definite. We shall call this number the *class* of the system.

Let r be the number of lines of the system which intersect an arbitrary right line. It is plain that if we form the condition that u, $\frac{du}{d\alpha}$ and any assumed right line may intersect, the result will be a function of α, which being equated to nothing gives a definite number of values of α. Let r be the number of solutions of this equation. We shall call this number the *rank* of the system, and we shall show that all other singularities of the system can be expressed in terms of the three just enumerated.

Let α be the number of stationary planes, and β the number of stationary points (Art. 293).

Two non-consecutive lines of the system may intersect. When this happens we call the point of meeting a "point on two lines," and their plane a "plane through two lines." Let x be the number of "points on two lines" which lie in a given plane, and y the number of "planes through two lines" which pass through a given point.

In like manner we shall call the line joining any two points of the system a "line through two points," and the intersection of any two planes a "line in two planes." Let g be the number of "lines in two planes" which lie in a given plane, and h the number of "lines through two points" which pass through a given point.

The developable has other singularities which will be determined in a subsequent chapter, but these are the singularities which Plücker's equations (note, p. 233) enable us to determine.

295. Consider now the section of the developable by any plane. It is obvious that the points of this curve are the traces on its plane of the "lines of the system," while the tangent lines of the section are the traces on its plane of the "planes of the system." The degree of the section is therefore r, since it is equal to the number of points in which an arbitrary line drawn in its plane meets the section, and we have such a point whenever the line meets a "line of the system."

The class of the section is plainly n. For the number of tangent lines to the section drawn through an arbitrary point is evidently the same as the number of "planes of the system" drawn through the same point.

A double point on the section will arise whenever two "lines of the system" meet the plane of section in the same point. The number of such points by definition is x. The tangent lines at such a double point are usually distinct because the two planes of the system corresponding to the lines of the system intersecting in any of the points x are commonly different.

The number of double tangents to the section is in like manner g; since a double tangent arises whenever two planes of the system meet the plane of section in the same line.

The m points of the system which lie in the plane of section are cusps of the section. For they are double points as being the intersection of two lines of the system; and the tangent planes at these points coincide, since the two consecutive lines, intersecting in one of the points m, lie in the same plane of the system. This proves, what we have already stated, that the curve whose tangents generate the developable is a cuspidal edge on the developable; for it is such that every plane meets that surface in a section which has as cusps the points where the same plane meets the curve.

Lastly, we get a point of inflexion (or a stationary tangent) wherever two consecutive planes of the system coincide. The number of points of inflexion is therefore α.

We are to substitute then in the formulæ, note p. 233,

$$\mu=r,\quad \nu=n,\quad \delta=x,\quad \tau=g,\quad \kappa=m,\quad \iota=\alpha.$$

And we have

$$n = r(r-1) - 2x - 3m; \quad r = n(n-1) - 2g - 3a,$$

$$a = 3r(r-2) - 6x - 8m; \quad m = 3n(n-2) - 6g - 8a,$$

whence also

$$m - a = 3(r-n); \quad 2(x-g) = (r-n)(r+n-9).$$

296. Another system of equations is found by considering the cone whose vertex is any point and which stands on the given curve. It appears at once by considering the section of a cone by any plane that the same equations connect the double points, double tangent planes, &c. of cones, which connect the double points, double tangents, &c. of plane curves.

The edges of the cone which we are now considering are the lines joining the vertex to all the points of the system; and the tangent planes to the cone are the planes connecting the vertex with the lines of the system, for evidently the plane containing two consecutive edges of the cone must contain the line joining two consecutive points of the system.

The degree of the cone is plainly the same as the degree of the curve and is therefore m.

The class of the cone is the same as the number of tangent planes to the cone which pass through an arbitrary line drawn through the vertex. Now since each tangent plane contains a line of the system, it follows that we have as many tangent planes passing through the arbitrary line as there are lines of the system which meet that line. The number sought is therefore r.*

A double edge of the cone arises when the same edge of the cone passes through two points of the system, or $\delta = h$. The tangent planes along that edge are the planes joining the vertex to the lines of the system which correspond to each of these points.

* It is easy to see that the class of this cone is the same as the degree of the developable which is the reciprocal of the points of the given system. Hence, *the degree of the developable generated by the tangents to any curve is the same as the degree of the developable which is the reciprocal of the points of that curve*, see note, p. 124.

A double tangent plane will arise when the same plane through the vertex contains two lines of the system; or $\tau = y$.

A stationary or cuspidal edge of the cone will only exist when there is a stationary point in the system; or $\kappa = \beta$.

Lastly, a stationary tangent plane will exist when a plane containing two consecutive lines of the system passes through the vertex; or $\iota = n$.

Thus we have $\mu = m$, $\nu = r$, $\delta = h$, $\tau = y$, $\kappa = \beta$, $\iota = n$. Hence by the formulæ (note p. 233)

$$r = m(m-1) - 2h - 3\beta; \quad m = r(r-1) - 2y - 3n,$$
$$n = 3m(m-2) - 6h - 8\beta; \quad \beta = 3r(r-2) - 6y - 8n.$$

Whence also

$$(n - \beta) = 3(r - m); \quad 2(y - h) = (r - m)(r + m - 9).$$

And combining these equations with those found in the last article, we have also

$$\alpha - \beta = 2(n - m); \; x - y = n - m; \; 2(g - h) = (n - m)(n + m - 7).$$

Plücker's equations enable us, when three of the singularities of a plane curve are given, to determine all the rest. Now three quantities r, m, n are common to the equations of this and of the last article. Hence, *when any three of the singularities which we have enumerated, of a curve in space, are given, all the rest can be found.*

297. To illustrate this theory, let us take the developable which is the envelope of the plane

$$at^k + kbt^{k-1} + \frac{k(k-1)}{1.2} ct^{k-2} + \&c. = 0,$$

where t is a variable parameter, a, b, c, &c. represent planes, and k is any integer.

The class of this system is obviously k, and the equation of the developable being the discriminant of the preceding equation, its degree is $2(k-1)$; hence $r = 2(k-1)$.

Also it is easy to see that this developable can have no stationary planes. For in general if we compare coefficients in the equations of two planes, three conditions must be satisfied in order that the two planes may be identical. If then we attempt to determine t so that any plane may be identical with the consecutive one, we find that we have three conditions to satisfy, and only one constant t at our disposal.

Having then $n=k$, $r=2(k-1)$, $\alpha=0$, the equations of the last two articles enable us to determine the remaining singularities. The result is

$$m=3(k-2);\quad \beta=4(k-3);\quad x=2(k-2)(k-3);$$

$$y=2(k-1)(k-3);\quad g=\frac{(k-1)(k-2)}{2};\quad h=\frac{9k^2-53k+80}{2}.$$

The greater part of these values can be obtained independently as at *Higher Plane Curves*, p. 94. But in order to economize space we do not enter into details.

298. The case considered in the last article, which is that when the variable parameter enters only rationally into the equation, enables us to verify easily many properties of developables. Since the system $u=0$, $\frac{du}{dt}=0$ is obviously reducible to

$$at^{k-1}+(k-1)\,bt^{k-2}+\&c.=0,\quad bt^{k-1}+(k-1)\,ct^{k-2}+\&c.=0,$$

and the system $u=0$, $\frac{du}{dt}=0$, $\frac{d^2u}{dt^2}=0$ is reducible to

$$at^{k-2}+(k-2)\,bt^{k-3}+\&c.=0,\quad bt^{k-2}+(k-2)\,ct^{k-3}+\&c.=0,$$
$$ct^{k-2}+(k-2)\,dt^{k-3}+\&c.=0;$$

it follows that a is itself a plane of the system (namely, that corresponding to the value $t=\infty$), ab is the corresponding line, and abc the corresponding point. Now we know from the theory of discriminants (see *Higher Algebra*, p. 47) that the equation of the developable is of the form $a\phi+b^2\psi=0$, where ψ is the discriminant of u when in it a is made $=0$. Thus we verify what was stated (Art. 291) that a touches the developable along the whole length of the line ab. Further, ψ is itself of the form $b\phi'+c^2\psi'$. If now we consider the section of the developable by one of the planes of the system (or, in other words, if we make $a=0$ in the equation of the developable), the section consists of the line ab twice and of a curve of the degree $r-2$; and this curve (as the form of the equation shows) touches the line ab at the point abc, and consequently meets it in $r-4$ other points. These are all "points on two lines," being the points where the line ab meets other lines of the system. And it is generally true that if r be the rank of a developable *each line of the system meets* $r-4$ *other lines of the system.* The locus of these points forms a double curve on the developable, the degree of which is x, and the other properties of which will be given in a subsequent chapter, where we shall also determine certain other singularities of the developable.

We add here a table of the singularities of some special sections of the developable. The reader, who may care to examine the subject, will find no great difficulty in establishing them. I have given the proof of the greater part of them, *Cambridge and Dublin Mathematical Journal*, Vol. V., p. 24.

Section by a plane of the system

$$\mu=r-2,\ \nu=n-1,\ \iota=\alpha,\ \kappa=m-3,\ \tau=g-n+2,\ \delta=x-2r+8.$$

Cone whose vertex is a point of the system

$$\mu=m-1,\ \nu=r-2,\ \iota=n-3,\ \kappa=\beta,\ \tau=y-2r+8,\ \delta=h-m+2.$$

Section by plane passing through a line of the system

$$\mu=r-1,\ \nu=n,\ \iota=\alpha+1,\ \kappa=m-2,\ \tau=g-1,\ \delta=x-r+4.$$

Cone whose vertex is on a line of the system

$$\mu=m,\ \nu=r-1,\ \iota=n-2,\ \kappa=\beta+1,\ \tau=y-r+4,\ \delta=h-1.$$

Section by plane through two lines

$$\mu=r-2,\ \nu=n,\ \iota=\alpha+2,\ \kappa=m-4,\ \tau=g-2,\ \delta=x-2r+9.$$

Cone whose vertex is a point on two lines

$$\mu=m,\ \nu=r-2,\ \iota=n-4,\ \kappa=\beta+2,\ \tau=y-2r+9,\ \delta=h-2.$$

Section by a stationary plane

$$\mu=r-3,\ \nu=n-2,\ \iota=\alpha-1,\ \kappa=m-4,\ \tau=g-2n+6,\ \delta=x-3r+13.$$

Cone whose vertex is a stationary point

$$\mu=m-2,\ \nu=r-3,\ \iota=n-4,\ \kappa=\beta-1,\ \tau=y-3r+13,\ \delta=h-2m+6.$$

SECTION II. CLASSIFICATION OF CURVES.

299. The following enumeration rests on the principle that a curve of the degree r meets a surface of the degree p in pr points. This is evident when the curve is the complete intersection of two surfaces whose degrees are m and n. For then we have $r=mn$ and the three surfaces intersect in mnp points. It is true also by definition when the surface breaks up into p planes. We shall assume that, in virtue of the law of continuity, the principle is generally true.

The use we make of the principle is this. Suppose that we take on a curve of the degree r, as many points as are sufficient to determine a surface of the degree p; then if the number of points so assumed be greater than pr, the surface described through the points must altogether contain the curve; for otherwise the principle would be violated.

We assume in this that the curve is a *proper* curve of the degree r, for if we took two curves of the degrees m and n (where $m+n=r$), the two together might be regarded as a complex curve of the degree r, and if *either* lay altogether on any surface of the degree p, of course we could take on that curve any number of points common to the curve and surface. All this will be sufficiently illustrated by the examples which follow.

300. *There is no line of the first degree but the right line.* For through any two points of a line of the first degree and any assumed point we can describe a plane which must altogether contain the line, since otherwise we should have a line of the first degree meeting the plane in more points than one. In like manner we can draw a second plane containing the line, which must therefore be the intersection of two planes; that is to say, a right line.

There is no proper line of the second degree but a conic. Through any three points of the line we can draw a plane, which the preceding reasoning shows must altogether contain the line. The line must therefore be a plane curve of the second degree.

The exception noted at the end of the last article would occur if the line of the second degree consisted of two right lines not in the same plane; for then the plane through three points of the system would only contain *one* of the right lines. In what follows we shall not think it necessary to notice this again, but shall speak only of proper curves of their respective orders.

301. *A curve of the third degree must either be a plane cubic or the partial intersection of two quadrics*, as explained, Art. 285.*

For through seven points of the curve and any two other points describe a quadric; and as before, it must altogether contain the curve. If the quadric break up into two planes, the curve may be a plane curve lying in one of the planes. As we may evidently have plane curves of any degree we shall not think it necessary to notice these in subsequent cases. If then the quadric do not break up into planes, we can draw a second quadric through the seven points, and the intersection of the two quadrics includes the given cubic. The complete intersection being of the fourth degree, it must be the cubic together with a right line; it is proved therefore that the only non-plane cubic is that explained, Art. 285.

302. The cone containing a curve of the m^{th} degree and whose vertex is a point on the curve, is of the degree $m-1$; hence the cone containing a cubic and whose vertex is on the curve is of the second degree.† *We can thus describe a twisted*

* Non-plane curves of the third degree appear to have been first noticed by Möbius in his *Barycentric Calculus*, 1827. Some of their most important properties are given by M. Chasles in Note XXXIII. to his *Aperçu Historique*, 1837, and in a paper in Liouville's *Journal* for 1857, p. 397. More recently the properties of these curves have been treated of by M. Schröter, *Crelle*, Vol. LVI., and by Professor Cremona of Milan, *Crelle*, Vol. LVIII., p. 138. Considerable use has been made of the latter paper in the articles which immediately follow.

† M. Chasles hastily said that conversely the locus of the vertex of a cone of the second degree passing through six points, is the cubic through these points. But as Mr. Weddle pointed out, *Cambridge and Dublin*

cubic through six given points. For we can describe a cone of the second degree of which the vertex and five edges are given, since evidently we are thus given five points in the section of the cone by any plane, and can thus determine that section. If then we are given six points a, b, c, d, e, f, we can describe a cone having the point a for vertex, and the lines ab, ac, ad, ae, af for edges; and in like manner a cone having b for vertex and the lines ba, bc, bd, be, bf for edges. The intersection of these cones consists of the common edge ab and of a cubic which is the required curve passing through the six points.

The theorem that the lines joining six points of a cubic to any seventh are edges of a quadric cone, leads at once to the following by Pascal's theorem: "The lines of intersection of the planes 712, 745; 723, 756; 734, 761 lie in one plane." Or in other words, "the points where the planes of three consecutive angles 567, 671, 712 meet the opposite sides lie in one plane passing through the vertex 7."* Conversely if this be true for two vertices of a heptagon it is true for all the rest: for then these two vertices are vertices of cones of the second degree containing the other points, which must therefore lie on the cubic which is the intersection of the cones.

303. *A cubic traced on a hyperboloid of one sheet meets all its generators of one system once, and those of the other system twice.*

Any generator of a quadric meets in two points its curve of intersection with any other quadric, namely, in the two points where the generator meets the other quadric. Now when the

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Mathematical Journal, Vol. v., p. 69, the locus of the vertex is not a curve but a surface, namely, that obtained by eliminating λ, μ, ν between the four differentials of $S + \lambda U + \mu V + \nu W$, where S, U, V, W are any surfaces through the six points.

The locus of the vertex of a cone of the second order which passes through seven points *is* a curve and is of the sixth order. When eight points are given four cones can be described through them. See appendix "on the order of systems of equations."

* M. Cremona adds that when the six points are fixed and the seventh variable, this plane passes through a fixed chord of the cubic.

intersection consists of a right line and a cubic, it is evident that the generators of the same system as the line, since they do not meet the line, must meet the cubic in the two points; while the generators of the opposite system, since they meet the line in one point, only meet the cubic in one other point.

Conversely we can describe a system of hyperboloids through a cubic and any chord which meets it twice. For take seven points on the curve, and an eighth on the chord joining any two of them; then through these eight points an infinity of quadrics can be described. But since three of these points are on a right line, that line must be common to all the quadrics, as must also the cubic on which the seven points lie.

304. The question to find the envelope of $at^3 - 3bt^2 + 3ct - d$ (where a, b, c, d represent planes and t is a variable parameter) is a particular case of that discussed, Art. 297. We have

$$r = 4, \quad m = n = 3, \quad \alpha = \beta = 0, \quad x = y = 0, \quad g = h = 1.$$

Thus the system is of the same nature as the reciprocal system, and all theorems respecting it are consequently two-fold. The system being of the third degree must be of the kind we are considering; and this also appears from the equation of the envelope

$$(ad - bc)^2 = 4(b^2 - ac)(c^2 - bd),$$

for it is easy to see that any pair of the surfaces $ad - bc$, $b^2 - ac$, $c^2 - bd$, have a right line common, while there is a cubic common to all three, which is a double line on the envelope.

It appears from the table just given that every plane contains one "line in two planes"; or that the section of the developable by any plane has one double tangent; while reciprocally through any point can be drawn one line to meet the cubic twice; the cone therefore, whose vertex is that point, and which stands on the curve has one double point; or in other words, *the cubic is projected on any plane into a cubic having a double point.*

The three points of inflexion of a plane cubic are in one right line. Now it was proved (Art. 296) that the points of inflexion correspond to the three planes of the system which can be drawn through the vertex of the cone. Hence the three points of the system which correspond to the three planes which can be drawn through any point O, lie in one plane passing through that point.*

Further it is known that when a plane cubic has a conjugate point, its three points of inflexion are real; but that when the cubic has a double point, the tangents at which are real, then two of the points of inflexion are imaginary. Hence if the chord which can be drawn through any point O meet the cubic in two real points, then two of the planes of the system which can be drawn through O are imaginary. Reciprocally, if through any line two real planes of the system can be drawn, then any plane through that line meets the curve in two imaginary points, and only one real one.†

305. These theorems can also be easily established algebraically; for the point of contact of the plane $at^3 - 3bt^2 + 3ct - d$, being given by the equations $at = b$, $bt = c$, $ct = d$, may be denoted by the co-ordinates $a = 1$, $b = t$, $c = t^2$, $d = t^3$. Now the three values of t answering to planes passing through any point are given by the cubic $a't^3 - 3b't^2 + 3c't - d' = 0$, whence it is evident from the values just found, that the points of contact lie in the plane $a'd - 3b'c + 3c'b - d'a = 0$. But this plane passes through the given point. Hence *the intersection of three planes of the system lies in the plane of the corresponding points.* The equation just written is unaltered if we interchange accented and unaccented letters. Hence *if a point A be in the plane corresponding to a point B, B will be in the plane corresponding to A.* And again, the planes which correspond to all the points of a line AB pass through a fixed right line, namely the intersection of the planes corresponding to A and B. The relation between the lines is plainly reciprocal. To any plane of the system will correspond in this sense the corresponding point of the system; and to a line in two planes corresponds a chord joining two points.

The three points where any plane $Aa + Bb + Cc + Dd$ meets the curve have their t's given by the equation

* Chasles, *Liouville*, 1857. Schröter, *Crelle*, Vol. LVI.

† Joachimsthal, *Crelle*, Vol. LVI., p. 45. Cremona, *Crelle*, Vol. LVIII., p. 146.

$Dt^3 + Ct^2 + Bt + A = 0$, and when this is a perfect cube, the plane is a plane of the system. From this it follows at once, as Joachimsthal has remarked, that any plane drawn through the intersection of two real planes of the system meets the curve in but one real point. For in such a case the cubic just written is the sum of two cubes and has but one real factor.

306. We have seen (Art. 124) that a twisted cubic is the locus of the poles of a fixed plane with regard to a system of quadrics having a common curve. More generally such a curve is expressed by the result of the elimination of λ between the system of equations $\lambda a = a'$, $\lambda b = b'$, $\lambda c = c'$. Now since the anharmonic ratio of four planes whose equations are of the form $\lambda a = a'$, $\lambda' a = a'$, &c. depends only on the coefficients λ, λ', &c. (see *Conics*, Art. 56), this mode of obtaining the equation of the cubic may be interpreted as follows: Let there be a system of planes through any line aa', a homographic system through any other line bb', and a third through cc', then the locus of the intersection of three corresponding planes of the systems is a twisted cubic. The lines aa', bb', cc' are evidently lines through two points, or chords of the cubic. Reciprocally, if three right lines be homographically divided, the plane of three corresponding points envelopes the developable generated by a twisted cubic, and the three right lines are "lines in two planes" of the system.

The line joining two corresponding points of two homographically divided lines, touches a conic when the lines are in one plane, and generates a hyperboloid when they are not. Hence given a series of points on a right line and a homographic series either of tangents to a conic or of generators of a hyperboloid, the planes joining each point to the corresponding line envelope a developable as above stated.

Ex. If the four faces of a tetrahedron pass through fixed lines, and three vertices move in fixed lines, the locus of the remaining vertex is a twisted cubic. Any number of positions of the base form a system of planes which divide homographically the three lines on which the corners of the base move, whence it follows that the three planes which intersect in the vertex are corresponding planes of three homographic systems.

307. From the theorems of the last article it follows conversely that "the planes joining four fixed points of the system to any variable line through two points form a constant anharmonic system" and "four fixed planes of the system divide any 'line in two planes' in a constant anharmonic ratio." It is very easy to prove these theorems independently. Thus we know that the section of the developable by any plane A* of the system, consists of the corresponding line a of the system twice, together with a conic to which all other planes of the system are tangents. Thus then the anharmonic property of the tangents to a conic shows at once that four planes cut any two lines in two planes, AB, AC in the same anharmonic ratio; and in like manner AC is cut in the same ratio as CD.

As a particular case of these theorems, since the lines of the system are both lines in two planes and lines through two points; *four fixed planes of the system cut all the lines of the system in the same anharmonic ratio;* and *the planes joining four fixed points of the system to all the lines of the system are a constant anharmonic system.*

Many particular inferences may be drawn from these theorems as at *Conics*, p. 273, which see.

Thus consider four points α, β, γ, δ; and let us express that the planes joining them to the lines a, b, and $\alpha\beta$, cut the line $\gamma\delta$ homographically. Let the planes A, B meet $\gamma\delta$ in points t, t'. Let the planes joining the line a to β, and the line b to α meet $\gamma\delta$ in k, k'. Then we have

$$\{tk\gamma\delta\} = \{k't'\gamma\delta\} = \{kk'\gamma\delta\}.$$

If the points t, k' coincide, it follows from the first equation that the points k, t' coincide, and from the second that the points t, t', γ, δ are a harmonic system. Thus we obtain Prof. Cremona's theorem, that if a series of chords meet the line of intersection of any plane A with the line joining the corresponding point α to any line b of the system, then they

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* It is often convenient to denote the planes of the system by capital letters, the corresponding lines by italics, and the corresponding points by Greek letters.

will also meet the line of intersection of the plane B with the line joining β to a; and will be cut harmonically where they meet these two lines and where they meet the curve.

The reader will have no difficulty in seeing when it will happen that one of these lines passes to infinity, in which case the other line becomes a diameter.

308. We have seen that the sections of the developable by the planes of the system are conics. We may therefore investigate the locus of the centres of these conics, or more generally the locus of the poles with respect to these conics of the intersections of their planes with a fixed plane. Since in every plane we can draw a "line in two planes" we may suppose that the fixed plane passes through the intersection of two planes of the system A, B.

Now consider the section by any other plane C, the traces on that plane of A and B are tangents to that section, and the pole of any line through their intersection lies on their chord of contact, that is to say, lies on the line joining the points where the lines of the system a, b, meet C. But since all planes of the system cut the lines a, b homographically, the joining lines generate a hyperboloid of one sheet, of which a and b are generators. However then the plane be drawn through the line AB, the locus of poles is this hyperboloid. But further, it is evident that the pole of any plane through the intersection of A, B lies in the plane which is the harmonic conjugate of that plane with respect to those tangent planes. The locus therefore which we seek is a plane conic. It is plain also from the construction that since the poles when any plane $A + \lambda B$ is taken for the fixed plane, lie on a conic in the plane $A - \lambda B$; conversely the locus when the latter is taken for fixed plane is a conic in the former plane.*

309. In conclusion, it is obvious enough that cubics may be divided into four species according to the different sections of the curve by the plane at infinity. Thus that plane may

* The theorems of this article are taken from Prof. Cremona's paper.

either meet the curve in three real points; in one real and two imaginary points; in one real and two coincident points, that is to say, a line of the system may be at infinity; or lastly, in three coincident points, that is to say, a plane of the system may be altogether at infinity. These species have been called the cubical hyperbola, cubical ellipse, cubical hyperbolic parabola, and cubical parabola. It is plain that when the curve has real points at infinity, it has branches proceeding to infinity, the lines of the system corresponding to the points at infinity being asymptotes to the curve. But when the line of the system is itself at infinity as in the third and fourth cases, the branches of the curve are of a parabolic form proceeding to infinity without tending to approach to any finite asymptote. Since the quadric cones which contain the curve become cylinders when their vertex passes to infinity, it is plain that three quadric cylinders can be described containing the curve, the edges of the cylinders being parallel to the asymptotes. Of course in the case of the cubical ellipse two of these cylinders are imaginary: in the case of the hyperbolic parabola there are only two cylinders, one of which is parabolic, and in the case of the cubical parabola there is but one cylinder which is parabolic.

It follows from Art. 304 that in the case of the cubical ellipse the plane at infinity contains a real line in two planes, which is imaginary in the case of the cubical hyperbola. That is to say, in the former case, but not in the latter, two planes of the system can be parallel. From the anharmonic property we infer that in the case of the cubical parabola three planes of the system divide in a constant ratio all the lines of the system. In this case all the planes of the system cut the developable in parabolas. The system may be regarded as the envelope of $xt^3 - 3yt^2 + 3zt - d$ where d is constant. For further details we refer to Prof. Cremona's Memoir.

310. We proceed now to the classification of curves of higher orders. We have proved (Art. 299) that through any curve can be described two surfaces, the lowest values of whose degrees in each case there is no difficulty in determining. It

is evident then on the other hand that if commencing with the simplest values of μ and ν we discuss all the different cases of the intersection of two surfaces whose degrees are μ and ν, we shall include all possible curves up to the r^{th} order, the value of this limit r being in each case easy to find when μ and ν are given. With a view to such a discussion we commence by investigating the characteristics of the curve of intersection of two surfaces.* We have obviously $m = \mu\nu$, and if the surfaces do not touch, as we shall suppose they do not, their curve of intersection has no multiple points (p. 95), and therefore $\beta = 0$. In order to determine completely the character of the system, it is necessary to know one more of its singularities, and we choose to seek for r, the degree of the developable generated by the tangents. Now this developable is got by eliminating $x'y'z'$ between the four equations

$$U' = 0,\ V' = 0,\ Lx + My + Nz + Pw = 0,\ L'x + M'y + N'z + P'w = 0,$$

where L, M, &c. are the first differential coefficients. These equations are respectively of the degrees μ, ν, $\mu - 1$, $\nu - 1$: and since only the last two contain xyz, these variables enter into the result in the degree

$$\mu\nu(\nu - 1) + \mu\nu(\mu - 1) = \mu\nu(\mu + \nu - 2).$$

Otherwise thus: the condition that a line of the system should intersect the arbitrary line

$$\alpha x + \beta y + \gamma z + \delta w,\ \alpha' x + \beta' y + \gamma' z + \delta' w$$

is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ L, & M, & N, & P \\ L', & M', & N', & P' \end{vmatrix} = 0,$$

which is evidently of the degree $\mu + \nu - 2$. This denotes a surface which is the locus of the points, the intersection of whose polar planes with respect to U and V meet the arbitrary line. And the points where this locus meets the curve UV

* The theory explained in the remainder of this section is taken from a paper dated July, 1849, which I published in the *Cambridge and Dublin Mathematical Journal*, Vol. v., p. 23.

are the points for which the tangents to that curve meet the arbitrary line.

Having then $m = \mu\nu$, $\beta = 0$, $r = \mu\nu(\mu + \nu - 2)$, we find, by Art. 296,

$$n = 3\mu\nu(\mu + \nu - 3),\ \alpha = 2\mu\nu(3\mu + 3\nu - 10),\ 2h = \mu\nu(\mu - 1)(\nu - 1),$$

$$2g = \mu\nu\{(3\mu + 3\nu - 9)^2 - 22(\mu + \nu) + 71\},$$

$$2x = \mu\nu\{(\mu + \nu - 2)^2 - 4(\mu + \nu) + 8\},$$

$$2y = \mu\nu\{\mu\nu(\mu + \nu - 2)^2 - 10(\mu + \nu) + 28\}.$$

311. We verify this result by determining independently h the number of "lines through two points" which can pass through a given point, that is to say, the number of lines which can be drawn through a given point so as to pass through two points of the intersection of U and V. For this purpose it is necessary to remind the reader of the method employed at the foot of p. 86 in order to find the equation of the cone whose vertex is any point and which passes through the intersection of U and V. Let us suppose that the vertex of the cone is taken on the curve so as to have both U and $V = 0$ for the co-ordinates of the vertex. Then it appears from p. 86 that the equation of the cone is the result of eliminating λ between

$$\delta U + \frac{\lambda}{1.2}\delta^2 U + \frac{\lambda^2}{1.2.3}\delta^3 U + \&c. = 0,$$

$$\delta V + \frac{\lambda}{1.2}\delta^2 V + \frac{\lambda^2}{1.2.3}\delta^3 V + \&c. = 0.$$

These equations in λ are of the degrees $\mu - 1$, $\nu - 1$; δU, $\delta^2 U$, &c. contain the co-ordinates $x'y'z'$, xyz in the degrees $\mu - 1, 1$; $\mu - 2, 2$, &c. A specimen term of the result is $(\delta U)^{\nu-1} V^{\mu-1}$. Thus it appears that the result contains the variables xyz in the degree $\nu - 1 + \nu(\mu - 1) = \mu\nu - 1$; while it contains $x'y'z'$ in the degree $(\mu - 1)(\nu - 1)$. Every edge of this cone of the degree $\mu\nu - 1$, whose vertex is a point on the curve, is of course a "line through two points." If now in this cone we consider the co-ordinates of any point xyz on the cone as known and $x'y'z'$ as sought, this equation of the degree $(\mu - 1)(\nu - 1)$ combined with the equations U and V determine the "points" belonging to all the "lines through two points" which can pass through the assumed point. The total number of such points is therefore $\mu\nu(\mu-1)(\nu-1)$, and the number of lines through two points is of course half this.

The number determined in this article, I call the number of *apparent* double points in the intersection of two surfaces, for to an eye placed at any point two branches of a curve appear to intersect if any line drawn through the eye meet both branches.

312. Let us now consider the case when the curve UV has also *actual* double points; that is to say, when the two surfaces touch in one or more points. Now in this case, the number of *apparent* double points remains precisely the same as in the last article, and the cone, standing on the curve of intersection and whose vertex is any point, has as double edges the lines joining the vertex to the points of contact in *addition* to the number determined in the last article. It is easy to see that the investigation of the last article does not include the lines joining an arbitrary point to the points of contact. That investigation determines the number of cases when the radius vector from any point has two values the same for both surfaces, but the radius vector to a point of contact has only one value the same for both, since the point of contact is not a double point on either surface. Every point of contact then adds one to the number of double edges on the cone, and therefore diminishes the degree of the developable by two. This might also be deduced from Art. 310 since the surface generated by the tangents to the curve of intersection must include as a factor the tangent plane at a point of contact, since every tangent line in that plane touches the curve of intersection.

If the surfaces have stationary contact at any point (Art. 129) the line joining this point to the vertex of the cone is a cuspidal edge of that cone. If then the surfaces touch in t points of ordinary contact and in β of stationary contact, we have

$$m=\mu\nu, \quad \beta=\beta, \quad 2h=\mu\nu(\mu-1)(\nu-1)+2t,$$
$$r=\mu\nu(\mu+\nu-2)-2t-3\beta,$$

and the reader can calculate without difficulty how the other numbers in Art. 310 are to be modified.

We can hence obtain a limit to the number of points at which two surfaces can touch if their intersection do not break up into curves of lower order; for we have only to subtract the number of apparent double points from the maximum number of double points which a curve of the degree $\mu\nu$ can have (*Higher Plane Curves*, p. 31).

313. We shall now show that when the curve of intersection of two surfaces breaks up into two simpler curves, the characteristics of these curves are so connected that when those of the one are known those of the other can be found. It was proved (Art. 311) that the points belonging to the "lines through two points" which pass through a given point are the intersection of the curve UV with a surface whose degree is $(\mu - 1)(\nu - 1)$. Suppose now that the curve of intersection breaks up into two whose degrees are m and m', where $m + m' = \mu\nu$, then evidently the "two points" on any of these lines must either lie both on the curve m, both on the curve m', or one on one curve and the other on the other. Let the number of lines through two points of the first curve be h, those for the second curve h', and let H be the number of lines which pass through a point on each curve, or, in other words, the number of *apparent intersections* of the curves. Considering then the points where each of the curves meet the surface of the degree $(\mu - 1)(\nu - 1)$, we have obviously the equations

$$m(\mu - 1)(\nu - 1) = 2h + H, \quad m'(\mu - 1)(\nu - 1) = 2h' + H,$$

whence $$2(h - h') = (m - m')(\mu - 1)(\nu - 1).$$

Thus when m and h are known m' and h' can be found. To take an example which we have already discussed, let the intersection of two quadrics consist in part of a right line (for which $m' = 1$, $h' = 0$), then the remaining intersection must be of the third degree $m = 3$, and the equation above written determines $h = 1$.

314. In like manner it was proved (Art. 310) that the locus of points, the intersection of whose polar planes with

regard to U and V meets an arbitrary line, is a surface of the degree $\mu+\nu-2$. The first curve meets this surface in the t points where the curves m and m' intersect (since U and V touch at these points) and in the r points for which the tangent to the curve meets the arbitrary line. Thus then

$$m(\mu+\nu-2)=r+t, \quad m'(\mu+\nu-2)=r'+t,$$

$$(m-m')(\mu+\nu-2)=r-r',$$

an equation which can easily be proved to follow from that in the last article.

The intersection of the cones which stand on the curves m, m' consists of the t lines to the points of actual meeting of the curves and of the H lines of apparent intersection; and the equation $H+t=mm'$ is easily verified by using the values just found for H and t, remembering also that $m'=\mu\nu-m$, $r=m(m-1)-2h$.

315. Having now established the principles which we shall have occasion to employ, we resume our enumeration of the different species of curves of the fourth order. *Every quartic curve lies on a quadric.* For the quadric determined by nine points on the curve must altogether contain the curve (Art. 299). It is not generally true that a second quadric can be described through the curve; there are therefore two principal families of quartics, viz. those which are the intersection of two quadrics, and those through which only one quadric can pass.* We commence with the curves of the first family. The characteristics of the intersection of two quadrics which do not touch are (Art. 310)

$$m=4,\ n=12,\ r=8,\ \alpha=16,\ \beta=0,\ x=16,\ y=8,\ g=38,\ h=2.$$

Several of these results can be established independently. Thus we have given (Art. 160) the equation of the developable generated by the tangents to the curve which is of the eighth degree. It is there proved also that the developable has in each of the four principal planes a double line of the fourth

* The existence of this second family of quartics was, I believe, first pointed out in the Memoir already referred to.

order, whence $x=16$.* Again, it is shown, p. 123, that the equation of the osculating plane is $S'U=SV$, which contains the co-ordinates of the point of contact in the third degree. If then it be required to draw an osculating plane through any assumed point, the points of contact are determined as the intersections of the curve UV with a surface of the third degree, and the problem therefore admits of twelve solutions; $n=12$. Lastly, every generator of a quadric containing the curve is evidently a "line through two points" (Art. 303). Since then we can describe through any assumed point a quadric of the form $U+\lambda V$, the two generators of that quadric which pass through the point are two lines through two points, or $h=2$. The lines through two points may be otherwise found by the following construction, the truth of which it is easy to see: Draw a plane through the assumed point O, and through the intersection of its polar planes with respect to the two quadrics, this plane meets the quadrics in four points which lie on two right lines intersecting in O.

A quartic of this species is determined by eight points (Art. 120).

316. Secondly, let the two quadrics touch: then (Art. 312) the cone standing on the curve has a double edge more than in the former case, and the developable is of a degree less by two. Hence

$$m=4,\ n=6,\ r=6;\ g=6,\ h=3;\ \alpha=4,\ \beta=0;\ x=6,\ y=4.$$

Thirdly, the quadrics may touch at a stationary point, when we have

$$m=4,\ n=4,\ r=5;\ g=2,\ h=2;\ \alpha=1,\ \beta=1;\ x=2,\ y=2.$$

This system† may be expressed as the envelope of

$$at^4+6ct^2+4dt+e,$$

where t is a variable parameter. The envelope is

$$(ae+3c^2)^3=27\,(ace-ad^2-c^3)^2,$$

* It ought to have been stated also that the developable circumscribing two quadrics has, as double lines, a conic in each of the principal planes, see Art. 158. The number $y=8$ is thus accounted for.

† I owe this remark to Mr. Cayley.

which expanded contains a as a factor and so reduces to the fifth degree. The cuspidal edge is the intersection of $ae+3c^2$, $4ce-3d^2$.

Since a cone of the fourth degree cannot have more than three double edges, two quadrics cannot touch in more points than one, unless their curve of intersection break up into simpler curves. If two quadrics touch at two points on the same generator, this right line is common to the surfaces, and the intersection breaks up into a right line and a cubic. If they touch at two points not on the same generator, the intersection breaks up into two plane conics whose planes intersect in the line joining the points.

317. If a quartic curve be not the intersection of two quadrics it must be the partial intersection of a quadric and a cubic. We have already seen that the curve must lie on a quadric, and if through thirteen points on it, and six others which are not in the same plane,* we describe a cubic, it must contain the given curve. The intersection of this cubic with the quadric already found must be the given quartic together with a line of the second degree, and the apparent double points of the two curves are connected by the relation $h-h'=2$, as appears on substituting in the formula of Art. 313 the values $m=4$, $m'=2$, $\mu=3$, $\nu=2$. When the line of the second degree is a plane curve (whether conic or two right lines), we have $h'=0$; therefore $h=2$, or the quartic is one of the species already examined having two apparent double points. It is easy to see otherwise that if a cubic and quadric have a plane curve common, through their remaining intersection a second quadric can be drawn; for the equations of the quadric and cubic are of the form $zw=u_2$, $zv_2=u_2x$, which intersect on $v_2=xw$. If, however, the cubic and quadric have common two right lines not in the same plane, this is a system having one apparent double point, since through any point can be

* This limitation is necessary, otherwise the cubic might consist of the quadric and of a plane. Thus if a curve of the fifth order lie in a quadric it cannot be proved that a cubic distant from the quadric can contain the given curve; see *Cambridge and Dublin Mathematical Journal*, Vol. v., p. 27.

drawn a transversal meeting both lines. Since then $h' = 1$, $h = 3$ or these quartics have three apparent double points, and are therefore essentially distinct from those already discussed which cannot have more than two. The numerical characteristics of these curves are precisely the same as those of the first species in Art. 316, the cone standing on either curve having three double edges, and the difference being that one of the double edges in one case proceeds from an actual double point while in the other they all proceed from apparent double points.

This system of quartics is the reciprocal of that given by the envelope of $at^4 + 4bt^3 + 6ct^2 + 4dt + e$. Moreover, this latter system has, in addition to its cuspidal curve of the sixth order, a nodal curve of the fourth which is of the kind now treated of.

It is proved, as in Art. 303, that these quartics are met in three points by all the generators of the quadric on which they lie, which are of the same system as the lines common to the cubic and quadric, and are met once by the generators of the opposite system. The cone standing on the curve, whose vertex is any point of it is then a cubic having a double edge, that double edge being one of the generators passing through the vertex of the quadric which contains the curve. Thus while any cubic may be the projection of the intersection of two quadrics, quartics of this second family can only be projected into cubics having a double point. The quadric may be considered as the surface generated by all the "lines through three points" of the curve. It is plain from what has been stated, that *every quartic, having three apparent double points, may be considered as the intersection of a quadric with a cone of the third order having one of the generators of the quadric as a double edge.*

318. Mr. Cayley has remarked that it is possible to describe through eight points a quartic of this second family. We want to describe through the eight points a cone of the third degree having its vertex at one of them, and having a double edge, which edge shall be a generator of a quadric

through the eight points. Now it was proved (Art. 315) that if a system of quadrics be described through eight points all the generators at any one of them lie on a cone of the third degree, which passes through the quartic curve of the first family determined by the eight points. Further, if S, S', S'' be three cubical cones having a common vertex and passing through seven other points, $\lambda S + \mu S' + \nu S''$ is the general equation of a cone fulfilling the same conditions; and if it have a double edge $\lambda \frac{dS}{dx} + \mu \frac{dS'}{dx} + \nu \frac{dS''}{dx}$ passes through that edge. Eliminating then λ, μ, ν between the three differentials, the locus of double edges is the cone of the sixth order

$$\frac{dS}{dx}\left(\frac{dS'}{dy}\cdot\frac{dS''}{dz} - \frac{dS''}{dy}\cdot\frac{dS'}{dz}\right) + \&c. = 0.$$

The intersection then of this cone of the sixth degree with the other of the third determines right lines, through any of which can be described a quadric and a cubic cone fulfilling the given conditions. It is to be observed, however, that the lines connecting the assumed vertex with the seven other points are simple edges on one of these cones and double edges on the other, and these (equivalent to fourteen intersections) are irrelevant to the solution of the problem. *Four quartics therefore can be described through the points.*

319. There is no difficulty in carrying on this enumeration to curves of higher orders. The reader will find, in the Memoir already cited, a classification of curves of the fifth order, which consist of three families having four, five, or six apparent double points; the first of which may have in addition one or two, and the second one, actual double or cuspidal points. We shall conclude this section by applying some of the results already obtained in it, to the solution of a problem which occasionally presents itself. "Three surfaces whose degrees are μ, ν, ρ have a certain curve common to all three; how many of their $\mu\nu\rho$ points of intersection are absorbed by the curve? In other words, in how many points do the surfaces intersect in addition to this common curve?" Now let the first two surfaces intersect in the given curve, whose

degree is m, and in a complementary curve $\mu\nu - m$, then the points of intersection not on the first curve must be included in the $(\mu\nu - m)\,\rho$ intersections of the latter curve with the third surface. But some of these intersections are on the curve m, since it was proved (Art. 314) that the latter curve intersects the complementary curve in $m(\mu + \nu - 2) - r$ points. Deducting this number from $(\mu\nu - m)\,\rho$ we find that the surfaces intersect in $\mu\nu\rho - m(\mu + \nu + \rho - 2) + r$ points which are not on the curve m; or that the common curve absorbs $m(\mu + \nu + \rho - 2) - r$ points of intersection.

In precisely the same way we solve the corresponding question if the common curve be a double curve on the surface ρ. We have then to subtract from the number $(\mu\nu - m)\,\rho$, $2\{m(\mu + \nu - 2) - r\}$ points, and we find that the common curve diminishes the intersections by $m(\rho + 2\mu + 2\nu - 4) - 2r$ points.

These numbers expressed in terms of the apparent double points of the curve m are

$$m(\mu + \nu + \rho - m - 1) + 2h \text{ and } m(\rho + 2\mu + 2\nu - 2m - 2) + 4h.$$

320. The last article enables us to answer the question: "If the intersection of two surfaces is in part a curve of order m which is a double curve on one of the surfaces; in how many points does it meet the complementary curve of intersection?" Thus, in the example last considered, the surfaces μ, ρ intersect in a double curve m and a complementary curve $\mu\rho - 2m$; and the points of intersection of the three surfaces are got by subtracting from $(\mu\rho - 2m)\,\nu$ the number of intersections of the double curve with the complementary. Hence

$$(\mu\rho - 2m)\,\nu - \iota = \mu\nu\rho - m(\rho + 2\mu + 2\nu - 4) + 2r,$$

whence $$\iota = m(\rho + 2\mu - 4) - 2r.$$

We can verify this formula when the curve m is the complete intersection of two surfaces U, V whose degrees are k and l. Then ρ is of the form $AU^2 + BUV + CV^2$ where A is of the degree $\rho - 2k$, &c., and μ is of the form $DU + EV$ where D is of the degree $\mu - k$. The intersections of the double curve with the complementary are the points for which one of the tangent planes to one surface at a point on the double curve coincide with the tangent plane to the other surface. They are therefore the intersection of the curve UV with the surface $AE^2 - BDE + CD^2$ which is of the degree $\rho + 2\mu - 2(k+l)$. The number of intersections is $kl\{\rho + 2\mu - 2(k+l)\}$ which coincides with the formula already obtained on putting $kl = m$, $kl(k+l-2) = r$.

321. From the preceding article we can show how, when two surfaces partially intersect in a curve which is a double curve on one of them, the singularities of this curve and its complementary are connected. The first equation of Art. 314 ceases to be applicable because the surface $\mu + \nu - 2$ altogether contains the double curve, but the second equation gives us

$$m'(\mu + \nu - 2) = 2\iota + r' = r' + 2m(\mu + 2\nu - 4) - 4r,$$

whence $\quad 4r - r' = (2m - m')(\mu + \nu - 2) + 4m(\nu - 2).$

In like manner we find that the apparent double points of the two curves are connected by the relation

$$8h - 2h' = (2m - m')(\mu - 1)(\nu - 1) - 2m(\nu - 1).$$

Thus when a quadric passes through a double line on a cubic the remaining intersection is of the fourth degree, of the sixth rank, and has three apparent double points.

SECTION III. NON-PROJECTIVE PROPERTIES OF CURVES.

322. As we shall more than once in this section have occasion to consider lines indefinitely close to each other, it is convenient to commence by showing how some of the formulæ obtained in the first chapter are modified when the lines considered are indefinitely near. We proved (Art. 14) that the angle of inclination of two lines is given by the formula

$$\sin^2\theta = (\cos\beta\cos\gamma' - \cos\beta'\cos\gamma)^2 + (\cos\gamma\cos\alpha' - \cos\gamma'\cos\alpha)^2 + (\cos\alpha\cos\beta' - \cos\alpha'\cos\beta)^2.$$

When the lines are indefinitely near we may substitute for $\cos\alpha'$, $\cos\alpha + \delta\cos\alpha$, &c., and put $\sin\theta = \delta\theta$, when we have

$$\delta\theta^2 = (\cos\beta\,\delta\cos\gamma - \cos\gamma\,\delta\cos\beta)^2 + (\cos\gamma\,\delta\cos\alpha - \cos\alpha\,\delta\cos\gamma)^2 + (\cos\alpha\,\delta\cos\beta - \cos\beta\,\delta\cos\alpha)^2.$$

If the direction-cosines of any line be $\frac{l}{r}, \frac{m}{r}, \frac{n}{r}$ where $l^2+m^2+n^2=r^2$, the preceding formula gives

$$r^4\delta\theta^2=(m\delta n-n\delta m)^2+(n\delta l-l\delta n)^2+(l\delta m-m\delta l)^2.$$

Since we have

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1;$$

$$\cos\alpha\ \delta\cos\alpha+\cos\beta\ \delta\cos\beta+\cos\gamma\ \delta\cos\gamma=0,$$

if we square the latter equation and add it to the expression for $\delta\theta^2$, we get another useful form

$$\delta\theta^2=(\delta\cos\alpha)^2+(\delta\cos\beta)^2+(\delta\cos\gamma)^2.$$

It was proved (Art. 15) that $\cos\beta\cos\gamma'-\cos\beta'\cos\gamma$, &c. are proportional to the direction-cosines of the perpendicular to the plane of the two lines. It follows then that the direction-cosines of the perpendicular to the plane of the consecutive lines just considered are proportional to $m\delta n-n\delta m$, $n\delta l-l\delta n$, $l\delta m-m\delta l$, the common divisor being $r^2\delta\theta$.

Again, it was proved (Art. 43) that the direction-cosines of the line bisecting the obtuse angle made with each other by two lines are proportional to

$$\cos\alpha-\cos\alpha',\ \cos\beta-\cos\beta',\ \cos\gamma-\cos\gamma',\ \&c.$$

Hence when two lines are indefinitely near, the direction-cosines of a line drawn in their plane, and perpendicular to their common direction are proportional to $\delta\cos\alpha$, $\delta\cos\beta$, $\delta\cos\gamma$, the common divisor being $\delta\theta$.

323. We proved (Art. 286) that the direction-cosines of a tangent to a curve are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, while, if the curve be given as the intersection of two surfaces, these cosines are proportional to $MN'-M'N$, $NL'-N'L$, $LM'-L'M$, where L, M, &c. denote the first differential coefficients.

An infinity of normal lines can evidently be drawn at any point of the curve. Of these two have been distinguished by special names; viz. the normal which lies in the osculating plane which is commonly called the *principal normal;* and the normal perpendicular to that plane, which being normal to two consecutive elements of the curve has been called by M. Saint-Venant the Binormal.

All the normals lie in the plane perpendicular to the tangent line, viz.

$$(x-x')\,dx+(y-y')\,dy+(z-z')\,dz=0$$

in the one notation; or in the other

$$(MN'-M'N)(x-x')+(NL'-N'L)(y-y')$$
$$+(LM'-L'M)(z-z')=0.$$

324. Let us consider now the equation of the osculating plane. Since it contains two consecutive tangents of the curve, its direction-cosines (Art. 322) are proportional to

$$dy\,d^2z-dz\,d^2y,\quad dz\,d^2x-dx\,d^2z,\quad dx\,d^2y-dy\,d^2x,$$

quantities which for brevity we shall call X, Y, Z. The equation of the osculating plane is therefore

$$X(x-x')+Y(y-y')+Z(z-z')=0.$$

The same equation might have been obtained (by Art. 30) by forming the equation of the plane joining the three consecutive points

$$x'y'z';\quad x'+dx',\ y'+dy',\ z'+dz';$$
$$x'+2dx'+d^2x',\ y'+2dy'+d^2y',\ z'+2dz'+d^2z'.$$

In applying this formula we may simplify it by taking one of the co-ordinates at pleasure as the independent variable, and so making d^2x, d^2y or $d^2z=0$.

325. In order to be able to illustrate by an example the application of the formulæ of this section, it is convenient here to form the equations and state some of the properties of the *helix* or curve formed by the thread of a screw. The helix may be defined as the form assumed by a right line traced in any plane when that plane is wrapped round the surface of a right cylinder.* From this definition the equations of the helix are

* Conversely a helix becomes a right line when the cylinder on which it is traced is developed into a plane, and is therefore a geodesic on the cylinder (Art. 278).

easily obtained. The equation of any right line $y = mx$ expresses that the ordinate is proportional to the intercept which that ordinate makes on the axis of x. If now the plane of the right line be wrapped round a right cylinder so that the axis of x may coincide with the circular base, the right line will become a helix, and the ordinate of any point of the curve will be proportional to the intercept, measured along the circle, which that ordinate makes on the circular base, counting from any fixed point on it. Thus the co-ordinates of the projection on the plane of the base, of any point of the helix are of the form $x = a\cos\theta$, $y = a\sin\theta$, where a is the radius of the circular base. But the height z has been just proved to be proportional to the arc θ. Hence the equations of the helix are

$$x = a\cos\frac{z}{h}, \quad y = a\sin\frac{z}{h}, \quad \text{whence also } x^2 + y^2 = a^2.$$

We plainly get the same values for x and y when the arc increases by 2π, or when z increases by $2\pi h$; hence the interval between the threads of the screw is $2\pi h$.

Since we have

$$dx = -\frac{a}{h}\sin\frac{z}{h}\,dz = -\frac{y}{h}\,dz, \quad dy = \frac{a}{h}\cos\frac{z}{h}\,dz = \frac{x}{h}\,dz,$$

we have $ds^2 = \dfrac{a^2 + h^2}{h^2}\,dz^2$. It follows that $\dfrac{dz}{ds}$ is constant, or the angle made by the tangent to the helix with the axis of z (which is the direction of the generators of the cylinder) is constant. It is easy to see that this is the same as the angle made with the generators by the line into which the helix is developed when the cylinder is developed into a plane.

The length of the arc of the curve is evidently in a constant ratio to the height ascended.

The equations of the tangent are (Art. 286)

$$\frac{x - x'}{y'} = -\frac{y - y'}{x'} = -\frac{z - z'}{h}.$$

If then x and y be the co-ordinates of the point where the tangent pierces the plane of the base, we have from the preceding equations

$$(x-x')^2+(y-y')^2=(x'^2+y'^2)\frac{z'^2}{h^2}=a^2\frac{z'^2}{h^2},$$

or the distance between the foot of the tangent and the projection of the point of contact is equal to the arc which measures the distance along the circle of that projection from the initial point. This also can be proved geometrically, for if we imagine the cylinder developed out on the tangent plane, the helix will coincide with the tangent line, and the line joining the foot of the tangent to the projection of the point of contact will be the arc of the circle developed into a right line. Thus then the locus of the points where the tangent meets the base is the involute of the circle.

The equation of the normal plane is

$$y'x-x'y=h(z-z').$$

To find the equation of the osculating plane, we have

$$d^2x=-\frac{1}{h^2}xdz^2,\quad d^2y=-\frac{1}{h^2}ydz^2,\quad d^2z=0,$$

whence the equation of the osculating plane is

$$h(y'x-x'y)=a^2(z-z').$$

The form of the equation shows that the osculating plane makes a constant angle with the plane of the base. We leave it as an exercise to the reader to find the tangent, normal plane, and osculating plane of the intersection of two central quadrics.

326. We can give the equation of the osculating plane a form more convenient in practice when the curve is given as the intersection of two surfaces U, V. Since the osculating plane passes through the tangent line, its equation must be of the form

$$\lambda(Lx+My+Nz+Pw)=\mu(L'x+M'y+N'z+P'w),$$

where $Lx+$&c. is the tangent plane to the first surface. This equation is identically satisfied by the co-ordinates of a point common to the two surfaces, and by those of a consecutive point; and on substituting the co-ordinates of a second consecutive point, we get

$\mu = Ld^2x + Md^2y + Nd^2z + Pd^2w,\ \lambda = L'd^2x + M'd^2y + N'd^2z + P'd^2w.$

But differentiating the equation

$$Ldx + Mdy + Ndz + Pdw = 0,$$

we get $$Ld^2x + Md^2y + Nd^2z + Pd^2w = -U',$$

where $U' = adx^2 + bdy^2 + cdz^2 + ddw^2$

$$+ 2ldy\,dz + 2mdz\,dx + 2ndx\,dy + 2pdx\,dw + 2qdy\,dw + 2rdz\,dw,$$

where a, b, &c. are the second differential coefficients. Now dx, &c. satisfy the equations

$$Ldx + Mdy + Ndz + Pdw = 0,\quad L'dx + M'dy + N'dz + P'dw = 0;$$

and since we may either, as in ordinary Cartesian equations, take w as constant; or else x, or y, or z; or more generally may take any linear function of these co-ordinates as constant; we may therefore add to the two preceding equations the arbitrary equation

$$\alpha dx + \beta dy + \gamma dz + \delta dw = 0.$$

Now it can easily be verified that if we substitute in any quadric the intersection of three planes

$$Lx + My + Nz + Pw,\ L'x + M'y + N'z + P'w,\ \alpha x + \beta y + \gamma z + \delta w,$$

the result U' will be proportional to the determinant (see p. 50)

$$\begin{vmatrix} a, & n, & m, & p, & L, & L', & \alpha \\ n, & b, & l, & q, & M, & M', & \beta \\ m, & l, & c, & r, & N, & N', & \gamma \\ p, & q, & r, & d, & P, & P', & \delta \\ L, & M, & N, & P & & & \\ L', & M', & N', & P' & & & \\ \alpha, & \beta, & \gamma, & \delta & & & \end{vmatrix}.$$

Now this determinant may be reduced by subtracting from the fifth column multiplied by $(m-1)$ the sum of the first four columns, multiplied respectively by x, y, z, w; when the whole of the fifth column vanishes except the last row which becomes

$-(\alpha x+\beta y+\gamma z+\delta w)$. In like manner we may then subtract from the fifth row multiplied by $(m-1)$ the sum of the first four rows multiplied respectively by x, y, z, w, when in like manner the whole of the fifth row vanishes except the fifth column which is $-(\alpha x+\beta y+\gamma z+\delta w)$. Thus the determinant reduces to

$$\frac{(\alpha x+\beta y+\gamma z+\delta w)^2}{(m-1)^2}\begin{vmatrix} a, & n, & m, & p, & L' \\ n, & b, & l, & q, & M' \\ m, & l, & c, & r, & N' \\ p, & q, & r, & d, & P' \\ L', & M', & N', & P' & \end{vmatrix}.$$

If we call the determinant last written S and the corresponding determinant for the other equation S', the equation of the osculating plane is

$$\frac{S'}{(n-1)^2}(Lx+My+Nz+Pw)=\frac{S}{(m-1)^2}(L'x+M'y+N'z+P'w).*$$

This equation has been verified in the case of two quadrics, see note, p. 123.

Ex. 1. To find the osculating plane of

$$ax^2+by^2+cz^2+dw^2, \quad a'x^2+b'y^2+c'z^2+d'w^2.$$

Ans. $(ab'-ba')(ac'-ca')(ad'-da')x'^3x+(ba'-b'a)(bc'-b'c)(bd'-b'd)y'^3y$
$+(ca'-c'a)(cb'-c'b)(cd'-c'd)z'^3z+(da'-d'a)(db'-d'b)(dc'-d'c)w'^3w=0.$

Ex. 2. To find the osculating plane of the line of curvature

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1, \quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}-1.$$

$$\textit{Ans.}\ \frac{a''^2xx'}{a^2a'^2}+\frac{b''^2yy'}{b^2b'^2}+\frac{c''^2zz'}{c^2c'^2}=1.$$

327. The condition that four points should lie in one plane, or in other words, that a point on the curve should be the point of contact of a stationary plane, is got by substituting in the equation of the plane through three consecutive points, the coordinates of a fourth consecutive point. Thus from the equation of Art. 324 the condition required is the determinant

$$d^3x(dy\,d^2z-dz\,d^2y)+d^3y(dz\,d^2x-dx\,d^2z)+d^3z(dx\,d^2y-dy\,d^2x)=0.$$

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* This equation is due to M. Hesse, see Crelle's *Journal*, Vol. XLI.

If a curve in space be a plane curve, this condition must be fulfilled by the co-ordinates of every point of it.*

328. We shall next consider the circle determined by three consecutive points of the curve, which, as in plane curves, is called the circle of curvature. It obviously lies in the osculating plane: its centre is the intersection of the traces on that plane, by two consecutive normal planes; and its radius is commonly called the radius of *absolute* curvature, to distinguish it from the radius of *spherical* curvature, which is the radius of the sphere determined by four consecutive points on the curve, and which will be investigated presently. If through the centre of a circle a line be drawn perpendicular to its plane, any point on this line is equidistant from all the points of the circle, and may be called a pole of the circle. Now the intersection of two consecutive normal planes, evidently passes through the centre of the circle of curvature, and is perpendicular to its plane. Monge has therefore called the lines of intersection of two consecutive normal planes, the *polar* lines of the surface. It is evident that all the normal planes envelope a developable of which these polar lines are the generators, and which accordingly has been called the polar surface. We shall presently state some properties of this surface. The polar line is evidently parallel to the line called the Binormal (Art. 323).

329. In order to obtain the radius of curvature we shall first calculate the *angle of contact*, that is to say, the angle made with each other by two consecutive tangents to the

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* I have not succeeded in completing the reduction of the corresponding condition when the curve is given as the intersection of two surfaces U, V. M. Bischoff (*Crelle*, Vol. LVIII.) gives as the resulting condition the Jacobian of the four surfaces U, V, S, S' (see Art. 155); but M. Bischoff's reasoning is unsound, and his result is only correct in the case where the surfaces are quadrics. The condition in general is of the degree $6m + 6n - 20$ in the coefficients, as might be inferred from the value of α, Art. 310. It is the sum of two terms, one of which is the Jacobian, and the other is the same function of the first and second differential coefficients as the Jacobian is when the surfaces are quadrics.

curve. The direction-cosines of the tangent being $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, it follows from Art. 322 that $d\theta$ the angle between two consecutive tangents is given by either of the formulæ

$$d\theta^2 = \left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2,^*$$

or
$$ds^4 d\theta^2 = X^2 + Y^2 + Z^2,$$
where
$$X = dy\,d^2z - dz\,d^2y, \text{ \&c.}$$

The truth of the latter formula may be seen geometrically: for the right-hand side of the equation denotes the square of double the triangle formed by three consecutive points (Art. 31); but two sides of this triangle are each ds, and the angle between them is $d\theta$, hence double the area is $ds^2 d\theta$.

If now ds be the element of the arc, the tangents at the extremities of which make with each other the angle $d\theta$, then since the angle made with each other by two tangents to a circle is equal to the angle that their points of contact subtend at its centre, we have $\rho d\theta = ds$. And the element of the arc and the two tangents being common to the curve and the circle of curvature, the radius of curvature is given by the formula

$$\rho = \frac{ds}{d\theta}; \text{ whence } \rho^2 = \frac{ds^2}{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2};$$

or
$$\rho^2 = \frac{ds^6}{X^2 + Y^2 + Z^2}.$$

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* By performing the differentiations indicated, another value for $d\theta^2$ is found without difficulty,

$$ds^2 d\theta^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2.$$

This formula may also be proved geometrically. Let AB, BC be two consecutive elements of the curve; AD a line parallel and equal to BC; then since the projections of BC on the axes are $dx + d^2x$, $dy + d^2y$, $dz + d^2z$, it is plain that the projections on the axes of the diagonal CD are d^2x, d^2y, d^2z, whence $CD^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2$. But CD projected on the element of the arc is d^2s, and on a line perpendicular to it is $ds\,d\theta$: whence

$$(d^2s)^2 + (ds\,d\theta)^2 = (d^2x)^2 + (d^2y)^2 + (d^2z)^2.$$

Ex. To find the radius of curvature of the helix. Using the formulæ of Art. 325, we find $\rho = \frac{a^2 + h^2}{a}$; or the radius of curvature is constant.

330. Having thus determined the magnitude of the radius of curvature, we are enabled by the formulæ of Art. 322 also to determine its position. For the direction-cosines of a line drawn in the plane of two consecutive tangents, and perpendicular to their common direction are by that article,

$$\frac{1}{d\theta}\, d\frac{dx}{ds},\ \frac{1}{d\theta}\, d\frac{dy}{ds},\ \frac{1}{d\theta}\, d\frac{dz}{ds};\ \text{ or } \rho\frac{d\frac{dx}{ds}}{ds},\ \rho\frac{d\frac{dy}{ds}}{ds},\ \rho\frac{d\frac{dz}{ds}}{ds}.$$

If x', y', z' be the co-ordinates of a point on the curve, and x, y, z those of the centre of curvature, then the projections of the radius of curvature on the axes are $x'-x$, $y'-y$, $z'-z$; but they are also $\rho\cos\alpha$, $\rho\cos\beta$, $\rho\cos\gamma$. Putting in then for $\cos\alpha$, $\cos\beta$, $\cos\gamma$ their values just found, the co-ordinates of the centre of curvature are determined by the equations

$$x'-x=\rho^2\frac{d\frac{dx}{ds}}{ds},\ \ y'-y=\rho^2\frac{d\frac{dy}{ds}}{ds},\ \ z'-z=\rho^2\frac{d\frac{dz}{ds}}{ds}.$$

331. When a curve is given as the intersection of two surfaces which cut at right angles, an expression for the radius of curvature can be easily obtained. Let r and r' be the radii of curvature of the normal sections of the two surfaces, the sections being made along the tangent to the curve; and let ϕ be the angle which the osculating plane makes with the first normal plane: then by Meunier's theorem, we have $\rho = r\cos\phi$ and also $\rho = r'\sin\phi$, whence $\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r'^2}$.

The same equations determine the osculating plane by the formula $\tan\phi = \frac{r}{r'}$.

If the angle which the surfaces make with each other be ω, the corresponding formula is

$$\frac{\sin^2\omega}{\rho^2} = \frac{1}{r^2} + \frac{1}{r'^2} - \frac{2\cos\omega}{rr'}.$$

We can hence obtain an expression for the radius of curvature of a curve given as the intersection of two surfaces. We may write $L^2+M^2+N^2=R^2$, $L'^2+M'^2+N'^2=R'^2$; and we have

$$\cos\omega=\frac{LL'+MM'+NN'}{RR'},$$

$$\sin^2\omega=\frac{(MN'-M'N)^2+(NL'-N'L)^2+(LM'-L'M)^2}{R^2R'^2}.$$

We must then substitute in the formula of Art. 265,

$$\cos\alpha=\frac{MN'-M'N}{RR'\sin\omega},\ \cos\beta=\frac{NL'-N'L}{RR'\sin\omega},\ \cos\gamma=\frac{LM'-L'M}{RR'\sin\omega}.$$

The denominator of that formula becomes

$$\begin{vmatrix} a, & n, & m, & L, & L' \\ n, & b, & l, & M, & M' \\ m, & l, & c, & N, & N' \\ L, & M, & N & & \\ L', & M', & N' & & \end{vmatrix}$$

which reduced, as in Art. 326, becomes $\frac{1}{(m-1)^2}S$. And we have

$$r=\frac{(m-1)^2R^2R'^2\sin^2\omega}{S}.$$

In like manner $$r'=\frac{(n-1)^2R^2R'^2\sin^2\omega}{S'}.$$

Whence $$\frac{1}{\rho^2}=\frac{S^2}{(m-1)^4R^6R'^4\sin^6\omega}$$

$$+\frac{S'^2}{(n-1)^4R^4R'^6\sin^6\omega}-\frac{2SS'\cos\omega}{(m-1)^2(n-1)^2R^5R'^5\sin^6\omega}.$$

332. Let us now consider the angle made with each other by two consecutive osculating planes, which we shall call the *angle of torsion*, and denote by $d\eta$. The direction-cosines of the osculating plane being proportional to X, Y, Z; the second formula of Art. 322 gives

$$(X^2+Y^2+Z^2)^2d\eta^2=(YdZ-ZdY)^2+(ZdX-XdZ)^2+(XdY-YdX)^2.$$

Now $$Y = dzd^2x - dxd^2z, \quad Z = dxd^2y - dyd^2x,$$
$$dY = dzd^3x - dxd^3z, \quad dZ = dxd^3y - dyd^3x.$$

Therefore (*Lessons on Higher Algebra*, p. 16)
$$YdZ - ZdY = Mdx,$$
where M is the determinant
$$Xd^3x + Yd^3y + Zd^3z.$$
Hence $$(X^2 + Y^2 + Z^2)^2 d\eta^2 = M^2ds^2,$$
$$d\eta = \frac{Mds}{X^2 + Y^2 + Z^2}.$$

This formula may be also proved geometrically. For M denotes six times the volume of the pyramid made by four consecutive points, while $X^2 + Y^2 + Z^2$ denotes four times the square of the area of the triangle formed by three consecutive points. Now if A be the triangular base of a pyramid, A' an adjacent face making an angle η with the base, s the side common to the two faces, and p the perpendicular from the vertex on s, so that $2A' = sp$: then for the volume of the pyramid we have $3V = Ap \sin\eta$ and $6Vs = 2Aps \sin\eta = 4AA' \sin\eta$. Now in the case considered, the common side is ds, and in the limit $A = A'$; hence $6Vds = 4A^2d\eta$. Q.E.D.

Following the analogy of the radius of curvature which is $\frac{ds}{d\theta}$, the later French writers denote the quantity* $\frac{ds}{d\eta}$ by the letter r, and call it the *radius of torsion*; but the reader will observe that this is not, like the radius of curvature, the radius of a real circle intimately connected with the curve.

333. In the same manner, however, as we have considered an osculating circle determined by three consecutive points of the system, we may consider an osculating right cone determined by three consecutive planes of the system. Imagine that a sphere is described having as centre the point of the system in which the three planes intersect; let the lines of the system

* The quantity $\frac{d\eta}{ds}$ is also sometimes called the "second curvature" of the curve.

passing through that point meet the sphere in A and B; and let the corresponding planes meet the same sphere in AT, BT; then if we describe a small circle of the same sphere passing through A and B, and touched by AT, BT, the cone whose vertex is the centre, and which stands on that small circle will evidently osculate the given curve. The problem then is, being given $d\eta$ the angle between two consecutive tangents to a small circle of a sphere, and $d\theta$ the corresponding arc of the circle to find H its radius.

Let C be the centre of the circle, and from the right-angled triangle CAT we have $\sin AT = \frac{\tan AC}{\tan ATC}$. If then ϕ be the external angle between two tangents to a circle, s the length of the two tangents; H the radius of the circle is given by the formula $\tan H = \frac{\sin \frac{1}{2}s}{\tan \frac{1}{2}\phi}$. In the limit s is the element of the arc of the circle, and $\tan H = \frac{ds}{d\phi}$, or according to the notation used, $\tan H = \frac{d\theta}{d\eta} = \frac{r}{\rho}$.*

334. Imagine that through every line of the system there is drawn a plane perpendicular to the corresponding osculating plane, the assemblage of these planes generates a developable which is called the *rectifying* developable. The reason of the name is, that the given curve is obviously a geodesic on this developable, since its osculating plane is, by construction, every where normal to the surface. If therefore the developable be developed into a plane, the given curve will become a right line.

The intersection of two consecutive planes of the rectifying developable is the *rectifying line.* Now since the plane passing through the edge of a right cone perpendicular to its tangent plane passes through its axis, it follows that the rectifying plane passes through the axis of the osculating cone considered

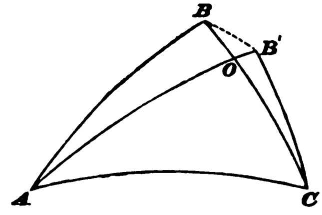
* It has been proved by M. Bertrand that when the ratio $r : \rho$ is constant, the curve must be a helix traced on a cylinder: and by Puiseux, that when r and ρ are both constant, the cylinder has a circular base.

in the last article; and therefore that *the rectifying line is the axis of that osculating cone.* The rectifying line may be therefore constructed by drawing in the rectifying plane a line making with the tangent line an angle H, where H has the value determined in the last article.

The rectifying surface is the surface of centres of the original developable. In fact it was proved (Art. 277) that the normal planes to the original surface along the two principal tangents touch the surface of centres; but the generating line itself is in every point of it one of the principal tangents; the rectifying plane therefore touches the surface of centres which is the envelope of all these rectifying planes. The centre of curvature at any point on a developable of the other principal section, namely, that perpendicular to the generating line, is the point where its plane meets the corresponding rectifying line, for evidently the traces on this plane of two consecutive rectifying planes are two consecutive normals to the section. Hence if l be the distance of any point on the developable from the cuspidal edge measured along the generator, the radius of curvature of the transverse section is $l \tan H$. When l vanishes, this radius of curvature vanishes as it ought, the point being a cusp.

In the case of the helix the rectifying surface is obviously the cylinder on which the curve is traced.

335. *To find the angle between two successive radii of curvature.*



Let AB, BC be traces on any sphere with radius unity, of planes parallel to the osculating and normal planes, then the central radius to B is the direction of the radius of curvature. If AB', $B'C$ be consecutive positions of the osculating and normal planes, B' is in the direction of the consecutive radius of curvature, and BB' measures the angle between them. Now the triangle BOB' being a very small right-angled triangle, we have

$$BB'^2 = BO^2 + OB'^2.$$

But since the angle ABC is right, BO measures BAB', which is $d\eta$, the angle between two consecutive osculating planes, and OB' measures OCB', which is $d\theta$, the angle between two consecutive normal planes. The required angle is therefore given by the formula $BB''^2 = d\eta^2 + d\theta^2$; where $d\eta$ and $d\theta$ have the values already found. The series of radii of curvature at all the points of a curve generate a surface on the properties of which we have not space to dwell. It is evidently a skew surface (see note, p. 75), since two consecutive radii do not in general intersect (see Art. 338, *infra*).

Ex. 1. To find the equation of the surface of the radii of curvature in the case of the helix.

The radius of curvature being the intersection of the osculating and normal planes has for its equations (Art. 325) $x'y = y'x$, $z = z'$, from which we are to eliminate $x'y'z'$ by the help of the equations of the curve. And writing the equations of the helix $x = a \cos nz$, $y = a \sin nz$, the required surface is $y \cos nz = x \sin nz$.

Ex. 2. To find the equation of the developable generated by the tangents of a helix. The equations of the tangent being

$$(x - a \cos nz') = - na \sin nz' (z - z'), \quad y - a \sin nz' = na \cos nz' (z - z'),$$

the result of eliminating z' is found to be

$$x \cos \left\{nz \pm \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} + y \sin \left\{nz \pm \frac{(x^2 + y^2 - a^2)^{\frac{1}{2}}}{a}\right\} = a.$$

Since this equation becomes impossible when $x^2 + y^2 < a^2$, it is plain that no part of the surface lies within the cylinder on which the helix is traced.

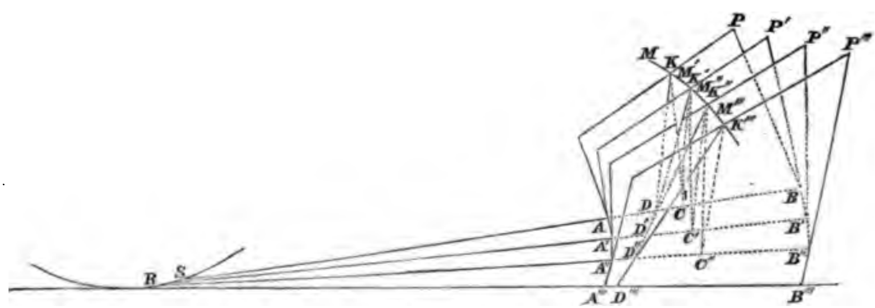
336. We shall now speak of the *polar developable* generated by the normal planes to the given curve. Fourier has remarked, that the "angle of torsion" of the one system is equal to the "angle of contact" of the other, as is sufficiently obvious since the planes of this new system are perpendicular to the lines of the original system, and *vice versâ*. The reader will observe however that it does not follow that the $\frac{d\theta}{ds}$ of one system is equal to the $\frac{d\eta}{ds}$ of the other, because the ds is not the same for both.

Since the intersection of the normal planes at two consecutive points K, K' of the curve is the axis of a circle of which K and K' are points (Art. 328), it follows that if any point D on that line be joined to K and K', the joining lines are equal and make equal angles with that axis.

It is plain that three consecutive normal planes intersect in the centre of the osculating sphere; hence *the cuspidal edge of the polar developable is the locus of centres of spherical curvature.*

In the case of a plane curve this polar developable reduces to a cylinder standing on the evolute of the curve.

337. *Every curve has an infinity of evolutes lying on the polar developable;** that is to say, the given curve may be generated in an infinity of ways by the unrolling of a string wound round a curve traced on that developable. Let MM', $M'M''$, &c. denote the successive elements of the curve, K, K', &c. the middle points of these elements, then the planes drawn through the points K perpendicular to the elements are the normal planes. The lines AB, $A'B'$, &c. are the lines in which each normal plane is intersected by the consecutive; these lines being the generators of the polar developable, and



hence tangents to the cuspidal edge RS of that surface. Draw now at pleasure† any line KD in the first normal plane, meeting the first generator in D; join DK' which being in the second normal plane will meet the second generator $A'B'$, say in D'. In like manner, let $K''D'$ meet $A''B''$ in D''. We

* See Monge, p. 396.

† This figure is taken from Leroy's *Geometry of Three Dimensions.*

get thus a curve $DD'D''$ traced on the polar developable which is an evolute of the given curve. For the lines DK, $D'K'$, &c. the tangents to the curve $DD'D''$, are normals to the curve $KK'K''$, and the lengths $DK = DK'$, $D'K' = D'K''$, &c. (see Art. 336). If therefore DK be a part of a thread wound round $DD'D''$, it is plain that as the thread is unwound the point K will move along the given curve.

Since the first line DK was arbitrary the curve has an infinity of evolutes. A plane curve has thus an infinity of evolutes lying on the cylinder whose base is the evolute in the plane of the curve. For example, in the special case where the evolute reduces to a point; that is, when the curve is a circle, the circle can be described by moving round a thread of constant length fastened to any point on the axis passing through the centre of the circle.

In the general case, *all the evolute curves $DD'D''$, &c. are geodesics on the polar developable.*

For we have seen (p. 219) that a curve is a geodesic when two successive tangents to it make equal angles with the intersection of the corresponding tangent planes of the surface; and it has just been proved (Art. 336) that DK, DK' which are two successive tangents to the evolute make equal angles with AB which is the intersection of two consecutive tangent planes of the developable. An evolute may then be found by drawing a thread as tangent from K to the polar developable, and winding the continuation of that tangent freely round the developable.

338. The locus of centres of curvature is a curve on the polar developable, but is *not* one of the system of evolutes. Let the first osculating plane $MM'M''$ meet the first two normal planes in KC, $K'C$, then C is the first centre of curvature: and in like manner the second centre is C', the point of intersection of $K'C'$, $K''C'$, the lines in which the second osculating plane $M'M''M'''$ is met by the second and third normal planes. Now the radii $K'C$, $K'C'$ are distinct, since they are the intersections of the same normal plane by two different osculating planes, $K'C'$ will therefore meet the line AB in a

point I which is distinct from C. Consequently the two radii of curvature KC, $K'C'$ situated in the planes P, P' have no common point in AB the intersection of these planes; two consecutive radii therefore do not intersect, unless in the case where two consecutive osculating planes coincide.

The centres of curvature then not being given by the successive intersections of consecutive radii; these radii are not tangents to the locus of centres. Any radius therefore KC would not be the continuation of a thread wound round $CC'C''$, and the unwinding of such a thread would not give the curve $KK'K''$, except in the case where the latter is a plane curve.*

339. *To find the radius of the sphere through four consecutive points.* Let R be the radius of any sphere, ρ the radius of a section by a plane making an angle η with the normal plane at any point; then, by Meunier's theorem, $R\cos\eta = \rho$; and for a consecutive plane making an angle $\eta + \delta\eta$, we have $\delta\rho = -R\sin\eta\,\delta\eta$. Hence $R^2 = \rho^2 + \left(\frac{d\rho}{d\eta}\right)^2$.

We have then only to give in this expression to ρ and $d\eta$ the values already found (Arts. 330, 332).

$\frac{d\rho}{d\eta}$ is obviously the length of the perpendicular distance from the centre of the sphere to the plane of the circle of curvature.

340. *To find the co-ordinates of the centre of the osculating sphere.*

Let the equation of any normal plane be

$$(\alpha - x)\,dx + (\beta - y)\,dy + (\gamma - z)\,dz = 0,$$

where xyz is the point on the curve, and $\alpha\beta\gamma$ any point on the plane; then the equation of a consecutive normal plane combined with the preceding gives

$$(\alpha - x)\,d^2x + (\beta - y)\,d^2y + (\gamma - z)\,d^2z = ds^2.$$

* The characteristics of the polar developable may be investigated by arguments similar to those used *Higher Plane Curves*, Art. 116; thus it is easy to see that the class of that developable is $m + r$, where m and r have the same meaning as at p. 234.

And the equation of the third plane gives

$$(\alpha - x)\, d^3x + (\beta - y)\, d^3y + (\gamma - z)\, d^3x = 3ds\, d^2s.$$

Let us denote as before $dy d^2z - dz d^2y$, &c. by X, Y, Z; $dy d^3z - dz d^3y$, &c. by X', Y', Z', and the determinant $Xd^3x + Yd^3y + Zd^3z$ by M. Then solving the preceding equations, we have

$$M(\alpha - x) = -X'ds^3 + 3Xds d^2s,\quad M(\beta - y) = -Y'ds^3 + 3Yds d^2s,$$
$$M(\gamma - z) = -Z'ds^3 + 3Zds d^2s.$$

By squaring and adding these equations we obtain another expression for R^2, which is what the value in the last article would become when for ρ and $\frac{d\rho}{d\eta}$ we substitute their values.

We add a few other expressions, the greater part of which admit of simple geometrical proofs, the details of which want of space obliges us to omit.

Ex. 1. If σ be the arc of the curve which is the locus of centres of absolute curvature,

$$d\sigma^2 = d\rho^2 + \rho^2 d\eta^2; \quad \text{or} \quad d\sigma = Rd\eta.$$

Ex. 2. If Σ be the length of the arc of the locus of centres of spherical curvature $d\Sigma = \frac{RdR}{\delta}$; where $\delta = \frac{d\rho}{d\eta}$ is the distance between the centres of the osculating circle and osculating sphere. From this expression we immediately get values for the radii of curvature and of torsion of this locus, remembering that the angle of torsion is the angle of contact of the original and vice versâ.

Ex. 3. The angle between two consecutive rectifying lines is dH.

Ex. 4. The angle ψ between two successive R's is given by the formula

$$R^2\psi^2 = ds^2 + d\Sigma^2 - dR^2.*$$

* The reader will find further details on the subjects treated of in this section in a Memoir by M. de Saint-Venant, *Journal de l'Ecole Polytechnique, Cahier* XXX., who has also collected into a table about a hundred formulæ for the transformation and reduction of calculations relative to the theory of non-plane curves; and in a paper by M. Frenet, Liouville, Vol. XVII., p. 437. I abridge the following historical sketch from M. de Saint-Venant's Memoir: "Curve lines not contained in the same plane have been successively studied by Clairaut (*Recherches sur les courbes à double courbure*, 1731), who has brought into use the title by which they have been commonly known (previously, however, employed by Pitot) and who

SECTION IV. CURVES TRACED ON SURFACES.

341. It remains to say something of the properties of curves considered as belonging to a particular surface. Thus the sphere we know has a geometry of its own, where great circles take the place of lines in a plane; and in like manner each surface has a geometry of its own, the geodesics on that surface answering to right lines.

We have already by anticipation given the fundamental property of a geodesic (Art. 278). The differential equation is immediately obtained from the property there proved, that the normal lies in the plane of two successive elements of the curve and bisects the angle between them; hence L, M, N which are proportional to the direction-cosines of the normal must be proportional to $d\frac{dx}{ds}$, $d\frac{dy}{ds}$, $d\frac{dz}{ds}$, which are the direction-cosines of the bisector (Art. 322). Thus "if the tangents to a geodesic make a constant angle with a fixed line, the normals along it will be parallel to a fixed plane," and *vice versâ* (Dickson, *Cambridge and Dublin Mathematical Journal*, Vol. v., p. 168). For from the equation

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = \text{constant},$$

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has given expressions for the projections of these curves, for their tangents, normals, arc, &c.; by Monge (*Mémoire sur les développées, &c.* presented in 1771, and inserted in Vol. x., 1785, of the '*Savants étrangers*,' as well as in his '*Application de l'Analyse à la Géométrie*') who gave expressions for the normal plane, centre and radius of curvature, evolutes, polar lines and polar developable, centre of osculating sphere, for the criterion for 'points of simple inflexion' where four consecutive points are in a plane, and for 'points of double inflexion' where three consecutive points are in a right line; by Tinseau (*Solution de quelques problèmes*, &c. presented in 1774, *Savants étrangers*, Vol. IX., 1780) who was the first to consider the osculating plane and the developable generated by the tangents; by Lacroix (*Calcul Differentiel*) who was the first to render the formulæ symmetrical by introducing the differentials of the three co-ordinates; and by Lancret (*Mémoire sur les courbes à double courbure*, read 1802, and inserted Vol. I., 1805, of *Savants étrangers* de l'Institut) who calculated the angle of torsion, and introduced the consideration of the rectifying lines and rectifying surface."

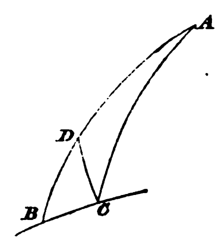
which denotes that the tangents make a constant angle with a fixed line, we can deduce

$$aL + bM + cN = 0,$$

which denotes that the normals are parallel to a fixed plane.

342. *If through any point on a surface there be drawn two indefinitely near and equal geodesics, the line joining their extremities is at right angles to both.**

Let $AB = AC$ and let us suppose the angle at B not to be right, but to be $= \theta$. Take $BD = \frac{BC}{\cos\theta}$, and then because all the sides of the triangle BCD are infinitely small it may be treated as a plane triangle and the angle DCB is a right angle. We have therefore $DC < DB$, $AD + DC < AB$, and therefore $< AC$. It follows that AC is not the shortest path from A to C, contrary to hypothesis. Or the proof may be stated thus: The shortest line from a point A to any curve on a surface meets that curve perpendicularly. For if not, take a point D on the radius vector from A and indefinitely near to the curve; and from this point let fall a perpendicular on the curve [which we can do by taking along BC a portion $= BD \cos\theta$ and joining the point so found to D]. We can pass then from D to the curve more shortly by going along the perpendicular than by travelling along the assumed radius vector which is therefore not the shortest path.



Hence, if every geodesic through A meet the curve perpendicularly, the length of that geodesic is constant. It is also evident mechanically that the circle described on any surface by a strained cord from a fixed point is every where perpendicular to the direction of the cord.

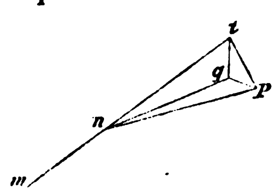
343. The theorem just proved is the fundamental theorem of the method of infinitesimals, applied to right lines (*Conics*,

* This theorem is due to Gauss, who also proves it by the Calculus of Variations; see the Appendix to Liouville's Edition of Monge, p. 528.

pp. 289, &c.). All the theorems therefore which are there proved by means of this principle will be true if instead of right lines we consider geodesics traced on any surface. For example, "if we construct on any surface the curve answering to an ellipse or hyperbola; that is to say, the locus of a point the sum or difference of whose geodesic distances from two fixed points on the surface is constant; then the tangent at any point of the locus bisects the angle between the geodesics joining the point of contact to the fixed points." The converse of this theorem is also true. Again, "if two geodesic tangents to a curve, through any point P, make equal angles with the tangent to a curve along which P moves, then the difference between the sum of these tangents and the intercepted arc of the curve which they touch is constant" (see *Conics*, Art. 356). Again, "if equal portions be taken on the geodesic normals to a curve, the line joining their extremities cuts all at right angles," or "if two different curves both cut at right angles a system of geodesics they intercept a constant length on each vector of the series." We shall presently apply these principles to the case of geodesics traced on quadrics.

344. As the curvature of a plane curve is measured by the ratio which the angle between two consecutive tangents bears to the element of the arc; so the *geodesic curvature* of a curve on a surface is measured by the ratio borne to the element of the arc by the angle between two consecutive geodesic tangents. The following calculation of the radius of geodesic curvature, due to M. Liouville,* gives at the same time a proof of Meunier's theorem.

Let mn, np be two consecutive and equal elements of the curve. Produce $nt = mn$, and let fall the perpendicular tq on the plane mnp. If now θ be the angle of contact $tp = \theta ds$. Now nq is the second element of the normal section: let $tnq = \theta'$, then θ' is the angle of contact



* Appendix to Monge, p. 576.

of the normal section, and $tq = \theta' ds$. Now the angle $qtp\,(=\phi)$ is the angle between the osculating plane of the curve and the plane of normal section, and since $tq = tp \cos\phi$ we have $\theta' = \theta \cos\phi$ and $\frac{1}{R} = \frac{\cos\phi}{\rho}$ which is Meunier's theorem; R being the radius of curvature of the normal section and ρ that of the given curve.

Now, in like manner, pnq being θ'' the geodesic angle of contact, we have $pq = \theta'' ds$ and $pq = tp \sin\phi$, or $\frac{1}{r} = \frac{\sin\phi}{\rho}$. The geodesic* radius of curvature is therefore $\frac{\rho}{\sin\phi}$. It is easy to see that this geodesic radius is the absolute radius of curvature of the plane curve into which the given curve would be transformed, by circumscribing a developable to the given surface along the given curve, and unfolding that developable into a plane.

345. The theory of geodesics traced on quadrics may be said to depend on Joachimsthal's fundamental theorem that at every point on such a curve pD is constant where, as at Art. 174, p is the perpendicular on the tangent plane at the point, and D is the diameter of the quadric parallel to the tangent to the curve at the same point. This may be proved by the help of the two following principles: (1) If from any point two tangent lines be drawn to a quadric, their lengths are proportional to the parallel diameters. This is evident from Art. 70; and (2) If from each of two points A, B on the quadric perpendiculars be let fall on the tangent plane at the other, these perpendiculars will be proportional to the perpendiculars from the centre on the same planes. For the length of the perpendicular from $x''y''z''$ on the tangent plane at $x'y'z'$ is $p\left(\frac{x'x''}{a^2} + \frac{y'y''}{b^2} + \frac{z'z''}{c^2} - 1\right)$, and the perpendicular

* I have not adopted the name "second geodesic curvature" introduced by M. Bonnet. It is intended to express the ratio borne to the element of the arc by the angle which the normal at one extremity makes with the plane containing the element and the normal at the other extremity.

from $x'y'z'$ on the tangent plane at $x''y''z''$ is

$$p'\left(\frac{x'x''}{a^2}+\frac{y'y''}{b^2}+\frac{z'z''}{c^2}-1\right).$$

If now from the points A, B there be drawn lines AT, BT to any point T on the intersection of the tangent planes at A and B, and if AT make an angle i with the intersection of the planes, the angle between the planes being ω; then the perpendicular from A to the intersection of the planes is $AT\sin i$ and from A on the other plane is $AT\sin i\sin\omega$. In like manner the perpendicular from B on the tangent plane at A is $BT\sin i'\sin\omega$. If now the lines AT, BT make equal angles with the intersection of the planes, the lines AT, BT are proportional to the perpendiculars from A and B on the two planes. But AT and BT are proportional to D and D', and the perpendiculars are as the perpendiculars from the centre p' and p. Hence $Dp=D'p'$. But it was proved (Art. 278) that if AT, TB be successive elements of a geodesic they make equal angles with the intersection of the tangent planes at A and B. Hence the quantity pD remains unchanged as we pass from point to point of the geodesic. Q.E.D.*

346. On account of the importance of the preceding theorem we wish also to show how it may be deduced from the differential equations of a geodesic.† Differentiating the equation

$$\frac{L^2}{R^2}+\frac{M^2}{R^2}+\frac{N^2}{R^2}=1,$$

(where L, M, N are the differential coefficients and $R^2=L^2+M^2+N^2$), and then substituting for L, &c., $d\dfrac{dx}{ds}$, &c. (Art. 341), we get

$$d\left(\frac{dx}{ds}\right)d\left(\frac{L}{R}\right)+d\left(\frac{dy}{ds}\right)d\left(\frac{M}{R}\right)+d\left(\frac{dz}{ds}\right)d\left(\frac{N}{R}\right)=0.$$

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* This proof is by Dr. Graves, *Crelle*, Vol. XLII., p. 279.

† See Joachimsthal, *Crelle*, Vol. XXVI., p. 155; Bonnet, *Journal de l'Ecole Polytechnique*, Vol. XIX., p. 138; Dickson, *Cambridge and Dublin Mathematical Journal*, Vol. V., p. 168.

It is to be remarked that this equation is also true for a line of curvature; for since $\frac{L}{R}$, &c. are the direction-cosines of the normal, the direction-cosines of a line in the same plane with two consecutive normals and perpendicular to them are (Art. 322) proportional to $d\left(\frac{L}{R}\right)$, &c. Hence the $\frac{dx}{ds}$, &c. of a line of curvature are proportional to $d\left(\frac{L}{R}\right)$. But if now we differentiate

$$\frac{dx^2}{ds^2}+\frac{dy^2}{ds^2}+\frac{dz^2}{ds^2}=1,$$

and substitute for $\frac{dx}{ds}$ the value just given we have again the equation

$$d\left(\frac{dx}{ds}\right)d\left(\frac{L}{R}\right)+d\left(\frac{dy}{ds}\right)d\left(\frac{M}{R}\right)+d\left(\frac{dz}{ds}\right)d\left(\frac{N}{R}\right)=0.$$

If we actually perform the differentiations, and reduce the result by the differential equation of the surface $Ldx+Mdy+Ndz=0$, and its consequence

$$dLdx+dMdy+dNdz=-(Ld^2x+Md^2y+Nd^2z),$$

we get

$$(dLdx+dMdy+dNdz)(dRds-Rd^2s)$$
$$+(dLd^2x+dMd^2y+dNd^2z)\,Rds=0,$$

or
$$\frac{dLd^2x+dMd^2y+dNd^2z}{dLdx+dMdy+dNdz}+\frac{dR}{R}-\frac{d^2s}{ds}=0.$$

347. The preceding equation is true for a geodesic or line of curvature on any surface, but when the surface is only of the second degree, a first integral of the equation can be found. In fact we have

$$dLd^2x+dMd^2y+dNd^2z=\tfrac{1}{2}d\,(dLdx+dMdy+dNdz).$$

This may be easily verified by using the general equation of a quadric, or more simply by using the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

when $L=\frac{x}{a^2}$, $M=\frac{y}{b^2}$, $N=\frac{z}{c^2}$; $dL=\frac{dx}{a^2}$, $dM=\frac{dy}{b^2}$, $dN=\frac{dz}{c^2}$;

by substituting which values the equation is at once established.

The equation of the last article then consists of terms each separately integrable. Integrating we have

$$R^2\,(dL\,dx+dM\,dy+dN\,dz)=C\,ds^2.$$

Now from the preceding values

$$R^2=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{1}{p^2},$$

and $$\frac{dL}{ds}\frac{dx}{ds}+\frac{dM}{ds}\frac{dy}{ds}+\frac{dN}{ds}\frac{dz}{ds}=\frac{1}{a^2}\frac{dx^2}{ds^2}+\frac{1}{b^2}\frac{dy^2}{ds^2}+\frac{1}{c^2}\frac{dz^2}{ds^2}.$$

But the right-hand side of the equation denotes the reciprocal of the square of a central radius whose direction-cosines are $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$.

The geometric meaning therefore of the integral we have found is $pD=$ constant.*

348. *The constant pD has the same value for all geodesics which pass through an umbilic.* For at the umbilic the p is of course common to all, being $=\frac{ac}{b}$; and since the central section parallel to the tangent plane at the umbilic is a circle, the diameter parallel to the tangent line to the geodesic is constant; being always equal to the mean axis b. Hence for a geodesic passing through an umbilic, we have $pD=ac$.

* Dr. Hart proves the same theorem as follows: Consider any plane section of an ellipsoid, let ϖ be the perpendicular from the centre of the section on the tangent line, d the diameter of the section parallel to that tangent, i the angle the plane of the section makes with the tangent plane at any point. Then along the section ϖd is constant, and it is evident that pD is in a fixed ratio to $\varpi d \sin i$. Hence along the section pD varies as $\sin i$ and will be a maximum where the plane meets the surface perpendicularly. But a geodesic osculates a series of normal sections; therefore, for such a line pD is constant, its differential always vanishing. *Cambridge and Dublin Mathematical Journal*, Vol. IV., p. 84.

Let now any point on a quadric be joined by geodesics to two umbilics, since we have just proved that pD is the same for both geodesics, and since at the point of meeting the p is the same for both, the D for that point must also have the same value for both; that is to say, the diameters are equal which are drawn parallel to the tangents to the geodesics at their point of meeting. But two equal diameters of a conic make equal angles with its axes; and we know that the axes of the central section of a quadric parallel to the tangent plane at any point are parallel to the directions of the lines of curvature at that point. Hence, *the geodesics joining any point on a quadric to two umbilics make equal angles with the lines of curvature through that point.**

It follows that the geodesics joining any point to the two opposite umbilics, which lie on the same diameter, are continuations of each other; since the vertically opposite angles are equal which these geodesics make with either line of curvature through the point.

It follows also (see Art. 343) that *the sum or difference is constant of the geodesic distances of all the points on the same line of curvature from two umbilics.* The sum is constant when the two umbilics chosen are interior with respect to the line of curvature; the difference when for one of these umbilics we substitute that diametrically opposite so that one of the umbilics is interior, the other exterior to the line of curvature.

If A, A' be two opposite umbilics, and B another umbilic, since the sum $PA + PB$ is constant and also the difference $PA' - PB$; it follows that $PA + PA'$ is constant; that is to say, *all the geodesics which connect two opposite umbilics are of equal length.* In fact, it is evident that two indefinitely near geodesics connecting the same two points on any surface must be equal to each other.

349. *The constant pD has the same value for all geodesics which touch the same line of curvature.*

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* This theorem and its consequences developed in the following articles are due to Mr. Michael Roberts, Liouville, Vol. XI., p. 1.

It was proved (Art. 174) that pD has a constant value all along a line of curvature; but at the points where either geodesic touches the line of curvature both p and D have the same value for the geodesic and the line of curvature.

Hence then a system of lines of curvature has properties completely analogous to those of a system of confocal conics in a plane; the umbilics answering to the foci. For example, *two geodesic tangents drawn to one from any point on another make equal angles with the tangent at that point.* Dr. Graves's theorem for plane conics holds also for lines of curvature, viz. that the excess of the sum of two tangents to a line of curvature over the intercepted arc is constant, while the intersection moves along another line of curvature of the same species (see *Conics*, p. 297).

350. The equation $pD =$ constant has been written in another convenient form.* Let a', a'' be the primary semi-axes of two confocal surfaces through any point on the curve, and let i be the angle which the tangent to the geodesic makes with one of the principal tangents. Then since $a^2 - a'^2$, $a^2 - a''^2$ (Art. 172) are the semi-axes of the central section parallel to the tangent plane, any other semi-diameter of that section is given by the equation

$$\frac{1}{D^2} = \frac{\cos^2 i}{a^2 - a''^2} + \frac{\sin^2 i}{a^2 - a'^2},$$

while, again, $\dfrac{1}{p^2} = \dfrac{(a^2 - a'^2)(a^2 - a''^2)}{a^2 b^2 c^2}$ (Art. 173).

The equation therefore $pD =$ constant is equivalent to

$$(a^2 - a'^2)\cos^2 i + (a^2 - a''^2)\sin^2 i = \text{constant},$$

or to

$$a'^2 \cos^2 i + a''^2 \sin^2 i = \text{constant}.$$

351. *The locus of the intersection of two geodesic tangents to a line of curvature, which cut at right angles, is a sphero-conic.*

This is proved as the corresponding theorem for plane conics. If a', a'' belong to the point of intersection, we have

$$a'^2 \cos^2 i + a''^2 \sin^2 i = \text{constant}, \quad a'^2 \sin^2 i + a''^2 \cos^2 i = \text{constant},$$

hence

$$a'^2 + a''^2 = \text{constant};$$

* By Liouville, Vol. IX., p. 401.

and therefore (Art. 169) the distance of the point of intersection from the centre of the quadric is constant. The locus of intersection is therefore the intersection of the given quadric with a concentric sphere. The demonstration holds if the geodesics are tangents to different lines of curvature; and, as a particular case, the locus of the foot of the geodesic perpendicular from an umbilic on the tangent to a line of curvature is a sphero-conic.

352. *To find the locus of intersection of geodesic tangents to a line of curvature which cut at a given angle* (Besge, Liouville, XIV. p. 247).

The tangents from any point whose a', a'' are given, to a given line of curvature are determined by the equation $a'^2 \cos^2 i + a''^2 \sin^2 i = \beta$; and since they make equal angles with either of the principal radii through that point, i the angle they make with one of these radii is half the angle they make with each other. We have therefore

$$\tan \tfrac{1}{2}\theta = \frac{\sqrt{(\beta - a''^2)}}{\sqrt{(a'^2 - \beta)}}; \quad \tan \theta = \frac{2\sqrt{(\beta - a''^2)}\sqrt{(a'^2 - \beta)}}{a'^2 + a''^2 - 2\beta},$$

$$(a'^2 + a''^2 - 2\beta)^2 \tan^2 \theta = 4\beta\,(a'^2 + a''^2) - 4a'^2 a''^2 - 4\beta^2.$$

This is reduced to ordinary co-ordinates by the equations (Arts. 168, 169)

$$a'^2 + a''^2 = x^2 + y^2 + z^2 + b^2 + c^2 - a^2; \quad a'^2 a''^2 = \frac{x^2 (a^2 - b^2)(a^2 - c^2)}{a^2},$$

whence it appears that the locus required is the intersection of the quadric with a surface of the fourth degree.*

353. It was proved (Art. 186) that two confocals can be drawn to touch a given line; that if the axes of the three surfaces passing through any point on the line be a, a', a'' and the angle the line makes with the three normals at the

* Mr. Michael Roberts has proved (Liouville, Vol. XV., p. 291) by the method of Art. 197, that the projection of this curve on the plane of circular sections is the locus of the intersection of tangents at a constant angle to the conic into which the line of curvature is projected.

point be α, β, γ; then the axis-major of the touched confocal is determined by the quadratic

$$\frac{\cos^2\alpha}{a^2-\mathrm{a}^2}+\frac{\cos^2\beta}{a'^2-\mathrm{a}^2}+\frac{\cos^2\gamma}{a''^2-\mathrm{a}^2}=0.$$

Let us suppose now that the given line is a tangent to the quadric whose axis is a, we have then $\cos\alpha=0$, since the line is of course at right angles to the normal to the first surface; and we have $\cos\beta=\sin\gamma$, since the tangent plane to the surface a contains both the line and the other two normals. The angle γ is what we have called i in the articles immediately preceding. The axis then of the second confocal touched by the given line is determined by the equation

$$\frac{\sin^2 i}{a'^2-\mathrm{a}^2}+\frac{\cos^2 i}{a''^2-\mathrm{a}^2}=0, \quad \text{or } a'^2\cos^2 i+a''^2\sin^2 i=\mathrm{a}^2.$$

If then we write the equation of a geodesic (Art. 351) $a'^2\cos^2 i+a''^2\sin^2 i=\mathrm{a}^2$, we see from this article that that equation expresses that *all the tangent lines along the same geodesic touch the confocal surface whose primary axis is* a.*

The geodesic itself will touch the line of curvature in which this confocal intersects the original surface; for the tangent to the geodesic at the point where the geodesic meets the confocal is, as we have just proved, also the tangent to the confocal at that point. The geodesic therefore and the intersection of the confocal and the given surface have a common tangent.

The osculating planes of the geodesic are plainly tangent planes to the same confocal; since they are the planes of two consecutive tangent lines to that confocal.

The value of pD for a geodesic passing through an umbilic is ac (Art. 348); and the corresponding equation is therefore $a'^2\cos^2 i+a''^2\sin^2 i=a^2-b^2$. Now the confocal, whose primary axis is $\sqrt{(a^2-b^2)}$, reduces to the umbilicar focal conic. Hence, as a particular case of the theorems just proved,

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* The theorems of this article are taken from M. Chasles's Memoir, Liouville, Vol. XI., p. 5.

all tangent lines to a geodesic which passes through an umbilic, intersect the umbilicar focal conic.

Conversely, if from any point O on that focal conic rectilinear tangents be drawn to a quadric and those tangents produced geodetically on the surface, the lines so produced will pass through the opposite umbilic; the whole lengths from O to the umbilic being equal.

354. From the fact (proved p. 144) that tangent planes drawn through any line to the two confocals which touch it are at right angles to each other, we might have inferred directly, precisely as at Art. 279, that tangent lines to a geodesic touch a confocal. For the plane of two consecutive tangents to a geodesic being normal to the surface is tangent to the confocal touched by the first tangent. The second tangent to the geodesic therefore touches the same confocal; as, in like manner, do all the succeeding tangents. Having thus established the theorem of the last article, we could, by reversing the steps of the proof, obtain an independent demonstration of the theorem $pD = \text{constant}$.

355. *The developable circumscribed to a quadric along a geodesic has its cuspidal edge on another quadric, which is the same for all geodesics touching the same line of curvature.*

For any point on the cuspidal edge is the intersection of three consecutive tangent planes to the given quadric, and the three points of contact, by hypothesis determine an osculating plane of a geodesic which (Art. 353) touches a fixed confocal. The point on the cuspidal edge is the pole of this plane with respect to the given quadric; but the pole with respect to one quadric of a tangent plane to another lies on a third fixed quadric.

356. M. Chasles has given the following generalization of Mr. Roberts's theorem, Art. 348. *If a thread fastened at two fixed points on one quadric A be strained by a pencil moving along a confocal B* (so that the thread of course lies in geodesics where it is in contact with the quadrics and in right lines in the space between them), *then the pencil will trace*

a line of curvature on the quadric A. For the two geodesics on the surface B, which meet in the locus point P, evidently make equal angles with the locus of P; but these geodesics have as tangents the rectilinear parts of the thread which both touch the same confocal; therefore (Art. 353) the pD is the same for both geodesics, and hence the line bisecting the angle between them is a line of curvature.

A particular case of this theorem is that the focal ellipse of a quadric can be described by means of a thread fastened to two fixed points on opposite branches of the focal hyperbola.

357. *Elliptic Co-ordinates.* The method used (Arts. 351, 352) in which the position of a point on the ellipsoid is defined by the primary axes of the two hyperboloids intersecting in that point, is called the method of Elliptic Co-ordinates (see p. 152 and *Higher Plane Curves*, p. 276). It being more convenient to work with unaccented letters, I follow M. Liouville* in denoting the quantities which we have hitherto called a', a'' by the letters μ, ν; and in this notation the equation of the lines of curvature of one system would be of the form $\mu =$ constant, and those of the other $\nu =$ constant. The equation of a geodesic (Art. 350) would be written $\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2$; and when the geodesic passes through an umbilic, we have $\mu'^2 = a^2 - b^2 = h^2$. It will be remembered (Art. 166) that μ lies between the limits h and k, and ν between the limits k and 0.

Throwing the equation of a geodesic into the form

$$\mu^2 + \nu^2 \tan^2 i = \mu'^2 (1 + \tan^2 i);$$

we see that it is satisfied (whatever be μ') by the values $\mu^2 = \nu^2$, $\tan^2 i = -1$. Whence it follows that the same pair of imaginary tangents, drawn from an umbilic, touch all the lines of curvature,† a further analogy to the foci of plane conics.

358. *To express in elliptic co-ordinates the element of the arc of any curve on the surface.* Let us consider first the

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* I cannot, however, bring myself to imitate him in calling the axis of the ellipsoid ρ; and his denoting the quantities $a^2 - b^2$, $a^2 - c^2$ (which we call h^2, k^2) by the letters b^2, c^2, seems likely to confuse.

† Mr. Roberts, Liouville, Vol. xv., p. 289.

element of any line of curvature, μ = constant. Let that line be met by the two consecutive hyperboloids, whose axes are ν and $\nu + d\nu$; then, since it cuts them perpendicularly, the intercept between them is equal to the difference between the central perpendiculars on the tangent planes to the two hyperboloids. But (Art. 190) $(p'' + dp'')^2 - p''^2 = (\nu + d\nu)^2 - \nu^2$ or $p''dp'' = \nu d\nu$. Now we have proved that $dp'' = d\sigma$, the element of the arc we are seeking, and

$$p''^2 = \frac{a''^2 b''^2 c''^2}{(a^2 - a''^2)(a'^2 - a''^2)} = \frac{\nu^2 (h^2 - \nu^2)(k^2 - \nu^2)}{(a^2 - \nu^2)(\mu^2 - \nu^2)}.$$

Hence
$$d\sigma^2 = \frac{(a^2 - \nu^2)(\mu^2 - \nu^2)}{(h^2 - \nu^2)(k^2 - \nu^2)} d\nu^2.$$

In like manner the element of the arc of the line of curvature ν = constant is given by the formula

$$d\sigma'^2 = \frac{(a^2 - \mu^2)(\mu^2 - \nu^2)}{(\mu^2 - h^2)(k^2 - \mu^2)} d\mu^2.$$

Now if through the extremities of the element of the arc ds of any curve, we draw lines of curvature of both systems, we form an elementary rectangle of which $d\sigma$, $d\sigma'$ are the sides and ds the diagonal. Hence

$$ds^2 = \frac{(a^2 - \mu^2)(\mu^2 - \nu^2)}{(\mu^2 - h^2)(k^2 - \mu^2)} d\mu^2 + \frac{(a^2 - \nu^2)(\mu^2 - \nu^2)}{(h^2 - \nu^2)(k^2 - \nu^2)} d\nu^2.$$

359. In like manner we can express the area of any portion of the surface bounded by four lines of curvature; two lines μ_1, μ_2, and two ν_1, ν_2. For the element of the area is

$$d\sigma_1 d\sigma_2 = \frac{(\mu^2 - \nu^2)\sqrt{\{(a^2 - \mu^2)(a^2 - \nu^2)\}}}{\sqrt{\{(\mu^2 - h^2)(k^2 - \mu^2)(h^2 - \nu^2)(k^2 - \nu^2)\}}} d\mu d\nu,$$

the integral of which is

$$\int_{\mu_2}^{\mu_1} \frac{\mu^2 \sqrt{(a^2 - \mu^2)}\, d\mu}{\sqrt{\{(\mu^2 - h^2)(k^2 - \mu^2)\}}} \int_{\nu_2}^{\nu_1} \frac{\sqrt{(a^2 - \nu^2)}\, d\nu}{\sqrt{\{(h^2 - \nu^2)(k^2 - \nu^2)\}}}$$
$$- \int_{\mu_2}^{\mu_1} \frac{\sqrt{(a^2 - \mu^2)}\, d\mu}{\sqrt{\{(\mu^2 - h^2)(k^2 - \mu^2)\}}} \int_{\nu_2}^{\nu_1} \frac{\nu^2 \sqrt{(a^2 - \nu^2)}\, d\nu}{\sqrt{\{(h^2 - \nu^2)(k^2 - \nu^2)\}}}.*$$

So, in like manner, we can find the differential equation of the

* The area of the surface of the ellipsoid was thus first expressed by Legendre, *Traité des Fonctions Elliptiques*, Vol. I., p. 352.

orthogonal trajectory of a curve whose differential equation is $Md\mu + Nd\nu$. For the orthogonal trajectory to $Pd\sigma + Qd\sigma'$ is plainly $\frac{d\sigma}{P} - \frac{d\sigma'}{Q}$; since $d\sigma$, $d\sigma'$ are a system of rectangular co-ordinates. But $Md\mu + Nd\nu$ can be thrown without difficulty into the form $Pd\sigma + Qd\sigma'$ by the equations of the last article. The equation of the orthogonal trajectory is thus found to be

$$\frac{a^2 - \mu^2}{(\mu^2 - h^2)(k^2 - \mu^2)} \frac{d\mu}{M} - \frac{a^2 - \nu^2}{(h^2 - \nu^2)(k^2 - \nu^2)} \frac{d\nu}{N} = 0.$$

360. The first integral of a geodesic $\mu^2 \cos^2 i + \nu^2 \sin^2 i = \mu'^2$ can be thrown into a form in which the variables are separated and the second integral can be obtained. That equation gives

$$\tan i = \sqrt{\left(\frac{\mu^2 - \mu'^2}{\mu'^2 - \nu^2}\right)}.$$

But $$\tan i = \frac{d\sigma'}{d\sigma} = \frac{\sqrt{(a^2 - \mu^2)(h^2 - \nu^2)(k^2 - \nu^2)}}{\sqrt{(a^2 - \nu^2)(\mu^2 - h^2)(k^2 - \mu^2)}} \frac{d\mu}{d\nu},$$

whence equating, we have

$$\frac{\sqrt{(a^2 - \mu^2)}\, d\mu}{\sqrt{(\mu^2 - \mu'^2)(\mu^2 - h^2)(k^2 - \mu^2)}} \pm \frac{\sqrt{(a^2 - \nu^2)}\, d\nu}{\sqrt{\{(\mu'^2 - \nu^2)(h^2 - \nu^2)(k^2 - \nu^2)\}}} = 0,$$

the terms of which can be integrated separately.*

If the geodesic passes through the umbilics, we have $\mu'^2 = h^2$ (Art. 357), and the equation of the geodesic is

$$\frac{\sqrt{(a^2 - \mu^2)}}{(\mu^2 - h^2)\sqrt{(k^2 - \mu^2)}} d\mu \pm \frac{\sqrt{(a^2 - \nu^2)}}{(h^2 - \nu^2)\sqrt{(k^2 - \nu^2)}} d\nu = 0.$$

361. *To find an expression for the length of any portion of a geodesic.* The element of the geodesic is the hypotenuse of a right-angled triangle of which $d\sigma$, $d\sigma'$ are the sides and whose base angle is i. Hence we have $ds = \sin i\, d\sigma' \pm \cos i\, d\sigma$; and putting in $\sin i = \frac{\sqrt{(\mu^2 - \mu'^2)}}{\sqrt{(\mu^2 - \nu^2)}}$, $\cos i = \frac{\sqrt{(\mu'^2 - \nu^2)}}{\sqrt{(\mu^2 - \nu^2)}}$, and giving $d\sigma$, $d\sigma'$ the values of Art. 358, we have

$$ds = d\mu \sqrt{\left\{\frac{(\mu^2 - \mu'^2)(a^2 - \mu^2)}{(\mu^2 - h^2)(k^2 - \mu^2)}\right\}} \pm d\nu \sqrt{\left\{\frac{(\mu'^2 - \nu^2)(a^2 - \nu^2)}{(h^2 - \nu^2)(k^2 - \nu^2)}\right\}}.$$

* The equation of a geodesic was first integrated by Jacobi, *Crelle*, Vol. XIX., p. 309.

If ρ be the element of a line through the umbilics, we have

$$d\rho = d\mu \sqrt{\left(\frac{a^2-\mu^2}{k^2-\mu^2}\right)} \pm d\nu \sqrt{\left(\frac{a^2-\nu^2}{k^2-\nu^2}\right)}.$$

It is to be noted that when we give to the radical in the last article the sign + we must give that in this article the sign −. This appears by forming (Art. 359) the differential equation of the orthogonal trajectory to a geodesic through an umbilic, an equation which must be equivalent to $d\rho = 0$ (Art. 342).

362. In place of denoting the position of any point on an ellipsoid by the elliptic co-ordinates μ, ν, we might use geodesic polar co-ordinates and denote a point by ρ its geodesic distance from an umbilic, and by ω the angle which the radius vector makes with the line joining the umbilics. Now the equation (Art. 360) of a geodesic passing through an umbilic gives the sum of two integrals equal to a constant. This constant cannot be a function of ρ since it remains the same as we go along the same geodesic: it must therefore be a function of ω only; and if we pass from any point to an indefinitely near one, *not* on the same geodesic radius vector, we shall have

$$\frac{\sqrt{(a^2-\mu^2)}\,d\mu}{(\mu^2-h^2)\sqrt{(k^2-\mu^2)}} \pm \frac{\sqrt{(a^2-\nu^2)}\,d\nu}{(h^2-\nu^2)\sqrt{(k^2-\nu^2)}} = \phi'(\omega)\,d\omega.$$

We shall determine the form of the function by calculating its value for a point indefinitely near the umbilic, for which $\mu = \nu = h$. The left-hand side of the equation then becomes $\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)} \times$ limit of $\left(\frac{d\mu}{\mu^2-h^2} + \frac{d\nu}{h^2-\nu^2}\right)$. Now if we put $\mu = h+\eta$, $\nu = h - \varepsilon$, the quantity whose limit we want to find is $\frac{d\eta}{2h\eta+\eta^2} - \frac{d\varepsilon}{2h\varepsilon - \varepsilon^2}$, which, as η and ε tend to vanish, becomes the limit of $\frac{1}{2h}\left(\frac{d\eta}{\eta} - \frac{d\varepsilon}{\varepsilon}\right)$ or of $\frac{1}{2h}\,d\log\frac{\eta}{\varepsilon}$.

Now since the angle external to the vertical angle of the triangle formed by the line joining any point to two umbilics, is bisected by the direction of the line of curvature, that external angle is double the angle i in the formula $\mu^2\cos^2 i + \nu^2 \sin^2 i = h^2$. In the limit when the vertex of the triangle approaches the umbilic, the external angle of the triangle becomes ω, and we have at the umbilic

$$(h+\eta)^2 \cos^2 \tfrac{1}{2}\omega + (h-\varepsilon)^2 \sin^2 \tfrac{1}{2}\omega = h^2,$$

and in the limit

$$\tan^2 \tfrac{1}{2}\omega = \frac{\eta}{\varepsilon}.$$

Using this value, the limit of the left-hand side of the equation is

$$\frac{1}{2h} \sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)}\, d\left(\log \tan^2 \tfrac{1}{2}\omega\right).$$

We have therefore

$$\frac{\sqrt{(a^2-\mu^2)}\, d\mu}{(\mu^2-h^2)\sqrt{(k^2-\mu^2)}} + \frac{\sqrt{(a^2-\nu^2)}\, d\nu}{(h^2-\nu^2)\sqrt{(k^2-\nu^2)}} = \frac{1}{h}\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)} \frac{d\omega}{\sin\omega}.$$

And the constant which occurs in the integrated equation of a geodesic through an umbilic is of the form

$$\frac{1}{2h} \sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)} \log \tan^2 \tfrac{1}{2}\omega + C.$$

363. If P, Q be two consecutive points on a curve, and if PP' be drawn perpendicular to the geodesic radius vector OQ, it is evident that $PQ^2 = PP'^2 + P'Q^2$. Now since (Art. 342) $OP = OP'$, we have $P'Q = d\rho$, while PP' being the element of an arc of a geodesic circle, for which ρ is constant (or $d\rho = 0$), must be of the form $Pd\omega$. Hence the element of the arc of a curve on any surface can be expressed by a formula $ds^2 = d\rho^2 + P^2 d\omega^2$. We propose now to examine the form of the function P for the case of radii vectores drawn through an umbilic of an ellipsoid. Let us consider the line of curvature $\mu = \mu'$. We have then (Art. 361)

$$ds^2 = d\nu^2 \frac{(\mu'^2-\nu^2)(a^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}.$$

And by the same article

$$d\rho^2 = d\nu^2 \frac{a^2-\nu^2}{k^2-\nu^2},$$

whence

$$P^2 d\omega^2 = \frac{(\mu'^2-h^2)(a^2-\nu^2)}{(h^2-\nu^2)(k^2-\nu^2)}\, d\nu^2.$$

But (Art. 362), when μ is constant,

$$\frac{\sqrt{(a^2-\nu^2)}\, d\nu}{(h^2-\nu^2)\sqrt{(k^2-\nu^2)}} = \frac{1}{h}\sqrt{\left(\frac{a^2-h^2}{k^2-h^2}\right)} \frac{d\omega}{\sin\omega}.$$

Putting in this value for dv, we have

$$P^2 = \frac{(a^2-h^2)(h^2-\nu^2)(\mu'^2-h^2)}{h^2(k^2-h^2)\sin^2\omega} = \frac{b^2b'^2b''^2}{(b^2-a^2)(b^2-c^2)\sin^2\omega} = \frac{y^2}{\sin^2\omega}$$

(Art. 168); therefore

$$P = \frac{y}{\sin\omega}.$$

In this investigation it is not necessary to assume the result of the last article. If we substitute for the right-hand side of the equation in the last article an undetermined function of ω, it is proved in like manner that $P = y\phi(\omega)$. We determine then the form of the function by remembering that in the neighbourhood of the umbilic the surface approaches to the form of a sphere. Now on a sphere the formula of rectification is $ds^2 = d\rho^2 + \sin^2\rho\, d\omega^2$. Hence $P = \sin\rho$. But in the sphere $y = \sin\rho\sin\omega$. The function therefore which multiples y is $\frac{1}{\sin\omega}$.

364. Consider now the triangle formed by joining any point P to the two umbilics O, O'. Then for the arc OP we have the function $P = \frac{y}{\sin\omega}$ and for the arc $O'P$, connecting P with the other umbilic, we have the function $P' = \frac{y}{\sin\omega'}$, and $P : P' :: \sin\omega : \sin\omega'$, an equation analogous to that which expresses that the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles; since P and P' in the rectification of arcs on the ellipsoid answer to $\sin\rho$, $\sin\rho'$ on the sphere.

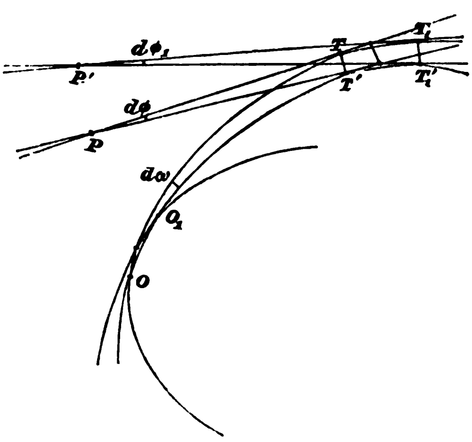
365. Again, if P be any point on a line of curvature we know (Art. 348) $d\rho \pm d\rho' = 0$, where ρ and ρ' are the distances from the two umbilics. Now if θ be the angle which the radius vector OP makes with the tangent, the perpendicular element $Pd\omega$ is evidently $d\rho\tan\theta$. But the radius vector $O'P$ makes also the angle θ with the tangent. Hence, we have

$$Pd\omega \pm P'd\omega' = 0, \quad \text{or} \quad \frac{d\omega}{\sin\omega} \pm \frac{d\omega'}{\sin\omega'} = 0,$$

whence $\tan\frac{1}{2}\omega \tan\frac{1}{2}\omega'$ is constant when the sum of sides of the triangle is given; and $\tan\frac{1}{2}\omega$ is to $\tan\frac{1}{2}\omega'$ in a given ratio when the difference of sides of the triangle is given. Thus then the distance between two umbilics being taken as the base of a triangle, when either the product or the ratio of the tangents of the halves of the base angles is given; the locus of vertex is a line of curvature.*

From this theorem follow many corollaries: for instance, "If a geodesic through an umbilic O meet a line of curvature in points P, P', then (according to the species of the line of curvature) either the product or the ratio of $\tan\frac{1}{2}PO'O$, $\tan\frac{1}{2}P'O'O$ is constant." Again, "if the geodesics joining to the umbilics any point P on a line of curvature meet the curve again in P', P'', the locus of the intersection of the transverse geodesics $O'P'$, OP'' will be a line of curvature of the same species."

366. Mr. Roberts's expression for the element of an arc perpendicular to an umbilical geodesic has been extended as follows by Dr. Hart: Let OT, OT' be two consecutive geodesics touching the line of curvature formed by the intersection of the surface with a confocal B, $d\omega$ the angle at which they intersect; then the tangent at any point T of either geodesic touches B in a point P (Art. 353); and if TT' be taken conjugate to TP, the tangent plane at T' passes through TP



* This theorem, as well as those on which its proof depends, (Art. 362, &c.) is due to Mr. M. Roberts, to whom this department of Geometry owes so much (Liouville, Vols. XIII., p. 1, and XV., p. 275).

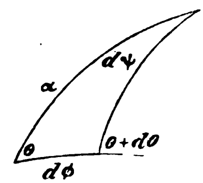
(Art. 237) and the tangent line to the geodesic at T' touches the confocal B in the same point P. We want now to express in the form $Pd\omega$ the perpendicular distance from T' to TP. Let the tangents at consecutive points, one on each geodesic, intersect in P' and make with each other an angle $d\phi'$. Let normals at the points T_1, T_1' meet the tangents PT, PT' at the points T_2, T_2', then since the difference between $T_1'T_1'$, T_2T_2' is infinitely small of the third order, $PT_2d\phi$ and $P'T_1d\phi'$ are equal to the same degree of approximation. But PT_1, $P'T_2$ are proportional to D and D' the diameters of the surface B drawn parallel to the two successive tangents to the geodesic. Hence $Dd\phi = D'd\phi'$. This quantity therefore remains invariable as we proceed along the geodesic; but at the point O, $d\phi = d\omega$; if therefore D_0 be the diameter of B parallel to the tangent at O to the geodesic, $Dd\phi = D_0d\omega$; and therefore the distance we want to express $PTd\phi = \frac{D_0}{D} td\omega$, where $t (= PT)$ is the length of the tangent from T to the confocal B; or $\frac{D_0}{D} t$ is a mean between the segments of a chord of B drawn through T parallel to the tangent at O. When the geodesic passes through an umbilic, the surface B reduces to the plane of the umbilics, and $\frac{D_0}{D} t$ becomes the line drawn through T to meet the plane of the umbilics parallel to the tangent at O; which is Mr. Roberts's expression.

Hence, *if a geodesic polygon circumscribe a line of curvature, and if all the angles but one move on lines of curvature, this also will move on a line of curvature, and the perimeter of the polygon will be constant when the lines of curvature are of the same species.* The proof is identical with that given for the corresponding property of plane conics (*Conics*, Art. 358).

367. If a geodesic joining any umbilic to that diametrically opposite, and making an angle ω with the plane of the umbilics, be continued so as to return to the first umbilic, it will not, as in the case of the sphere, return on its former path, but on its return will make with the plane of the umbilics an angle different from ω. In order to prove this we

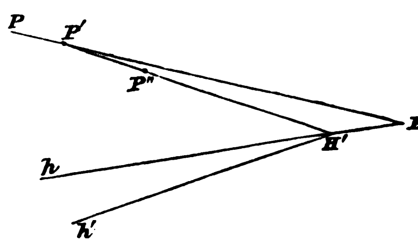
shall investigate an expression for θ, the angle made with the plane of the umbilics by the osculating plane at any point of that geodesic.

It is convenient to prefix the following lemma. In a spherical triangle let one side and the adjacent angle remain finite while the base diminishes indefinitely, it is required to find the limit of the ratio of the base to the difference of the base angles measured in the same direction. The formula of spherical trigonometry $\cos\frac{1}{2}(A+B) = \sin\frac{1}{2}C\,\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}$ gives us in the limit $d\theta = \cos a d\psi$. But evidently $\sin a d\psi = \sin\theta d\phi$. Hence $\frac{d\theta}{\sin\theta} = \frac{d\phi}{\tan\alpha}$.



Now we know (Art. 353) that the tangent line at any point of a geodesic passing through an umbilic, if produced goes to meet the plane of the umbilics in a point on the focal hyperbola; and the osculating plane of the geodesic at that point will be the plane joining the point to the corresponding tangent of the focal hyperbola. We know also (Art. 194) that the cone circumscribing an ellipsoid and whose vertex is any point on the focal hyperbola is a right cone.

Let now PP' be an element of an umbilical geodesic produced to meet the focal hyperbola in H. Let $P'P''$ be the consecutive element meeting the focal hyperbola in H'; then if Hh, $H'h'$ be two consecutive tangents to the focal hyperbola; PHh, $P'H'h'$ will be two consecutive osculating planes. Imagine now a sphere round H', and consider the spherical triangle formed by radii to the points h, h', P'. Then if $d\phi$ be the angle $hH'h'$, the angle of contact of the focal hyperbola; θ the angle between the osculating plane and $hH'h'$ the plane of the umbilics, while $hH'P'$ is α the semi-angle of the cone;



then the spherical triangle is that considered in our lemma, and we have $\frac{d\theta}{\sin\theta} = \frac{d\phi}{\tan\alpha}$.

In order to integrate this equation we must express $d\phi$ in terms of α; and this we may regard as a problem in plane geometry, for α is half the angle included between the tangents from H to the principal section in the plane of the umbilics, while $d\phi$ is the angle of contact of the focal hyperbola at the same point. Now if a', b'; a'', b'' be the axes of an ellipse and hyperbola passing through H, confocal to an ellipse whose axes are a, b; and if 2α be the angle included between the tangents from H to the latter ellipse, we have (see *Conics*, Ex. 10, p. 192) $\tan^2\alpha = \frac{a^2 - a''^2}{a'^2 - a^2}$. Differentiating, regarding a'' as constant (since we proceed to a consecutive point along the same confocal hyperbola), we have $d\alpha = -\tan\alpha \frac{a'da'}{a'^2 - a''^2}$. But if p, p' be the central perpendiculars on the tangents at H to the ellipse and hyperbola, we have $a'da' = pdp$ (Art. 358). Now dp is the element of the arc of the focal hyperbola, and if ρ be the radius of curvature at the same point, $dp = \rho d\phi$. But $\rho = \frac{a'^2 - a''^2}{p'}$. Hence $d\alpha = -\tan\alpha \frac{pd\phi}{p'}$ or $d\alpha = \tan\alpha \frac{a'b'd\phi}{a''b''}$.

But $a'^2 = a^2 + (a^2 - a''^2)\cot^2\alpha$, $b'^2 = b^2 + (a^2 - a''^2)\cot^2\alpha$.

Hence $\frac{d\phi}{\tan\alpha} = \frac{a''b''d\alpha}{\sqrt{(a^2 - a''^2 + a^2\tan^2\alpha)}\sqrt{(a^2 - a''^2 + b^2\tan^2\alpha)}}$.

In the case under consideration the axes of the touched ellipse are a, c; while the squares of the axes of the confocal hyperbola are $a^2 - b^2$, $b^2 - c^2$. Hence we have the equation

$$\frac{d\theta}{\sin\theta} = \frac{\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}\,d\alpha}{\sqrt{(b^2 + a^2\tan^2\alpha)}\sqrt{(b^2 + c^2\tan^2\alpha)}}.$$

Integrating this, and taking one limit of the integral at the umbilic where we have $\theta = \omega$, and $\alpha = \frac{1}{2}\pi$; we have

$$\log\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\omega} = \int_{\frac{1}{2}\pi}^{\alpha} \frac{\sqrt{(a^2 - b^2)}\sqrt{(b^2 - c^2)}\,d\alpha}{\sqrt{(b^2 + a^2\tan^2\alpha)}\sqrt{(b^2 + c^2\tan^2\alpha)}}.$$

If then I be the value of this integral; we have $\tan\frac{1}{2}\theta = k \tan\frac{1}{2}\omega$, where $k = e^I$.

Now this integral obviously does not change sign between the limits $\pm\frac{1}{2}\pi$, that is to say, in passing from one umbilic to the other. If then ω' be the value of θ for the umbilic opposite to that from which we set out; at this limit I has a value different from zero, and k a value different from unity; and we have $\tan\frac{1}{2}\omega' = k' \tan\frac{1}{2}\omega$; ω' is therefore always different from ω. And in like manner the geodesic returns to the original umbilic, making an angle ω'' such that $\tan\frac{1}{2}\omega'' = k^2 \tan\frac{1}{2}\omega$, and so it will pass and repass for ever making a series of angles the tangents of whose halves are in continued proportion.*

368. If we consider edges belonging to the same tangent cone, whose vertex is any point H on the focal hyperbola, α (and therefore k) is constant; and the equation $\tan\frac{1}{2}\theta = k\tan\frac{1}{2}\omega$ gives $\frac{d\theta}{\sin\theta} = \frac{d\omega}{\sin\omega}$. Now since the osculating plane of the geodesic is normal to the surface, and therefore also normal to the tangent cone, it passes through the axis of that cone. If then we cut the cone by a plane perpendicular to the axis, the section is evidently a circle whose radius is $\frac{y}{\sin\theta}$, and the element of the arc is $\frac{yd\theta}{\sin\theta}$, or $\frac{yd\omega}{\sin\omega}$. Now this element, being the distance, at their point of contact, of two consecutive sides of the circumscribing cone, is what we have called (Art. 363) $Pd\omega$, and we have thus from the investigation of the last article an independent proof of the value found for P (Art. 363).

369. *Lines of level.* The inequalities of level of a country can be represented on a map by a series of curves marking the points which are on the same level. If a series of such curves be drawn, corresponding to equi-different heights, the

* The theorems of this article are Dr. Hart's, *Cambridge and Dublin Mathematical Journal*, Vol. IV., p. 82; but in the mode of proof I have followed Mr. William Roberts, Liouville 1857, p. 213.

places where the curves lie closest together evidently indicate the places where the level of the country changes most rapidly. Generally, the curves of level of any surface are the sections of that surface by a series of horizontal planes, which we may suppose all parallel to the plane of xy. The equations of the horizontal projections of such a series are got by putting $z = c$ in the equation of the surface; and a differential equation common to all these projections is got by putting $dz = 0$ in the differential equation of the surface, when we have

$$\frac{dU}{dx}\,dx + \frac{dU}{dy}\,dy = 0.$$

We can make this a function of x and y only, by eliminating the z which may enter into the differential coefficients, by the help of the equation of the surface.

Lines of greatest slope. The line of greatest slope through any point is the line which cuts all the lines of level perpendicularly; and the differential equation of its projection therefore is

$$\frac{dU}{dx}\,dy - \frac{dU}{dy}\,dx = 0.$$

The line of greatest slope is often defined as that, the tangent at every point of which makes the greatest angle with the horizon. Now it is evident that the line in any tangent plane which makes the greatest angle with the horizon is that which is perpendicular to the horizontal trace of that plane. And we get the same equation as before by expressing that the projection of the element of the curve (whose direction-cosines are proportional to dx, dy) is perpendicular to the trace whose equation is

$$\frac{dU}{dx'}(x - x') + \frac{dU}{dy'}(y - y') - \frac{dU}{dz'}z' = 0.*$$

* It is evident that the differential equation of the curve, which is always perpendicular to the intersection of the tangent plane, [whose direction-cosines are as L, M, N] by a fixed plane whose direction-cosines are a, b, c, is

$$\begin{vmatrix} dx, & dy, & dz \\ L, & M, & N \\ a, & b, & c \end{vmatrix} = 0.$$

Ex. To find the line of greatest slope on the quadric $Ax^2 + By^2 + Cz^2 = D$.

The differential equation is $Axdy = Bydx$, which integrated, gives $\left(\frac{x}{x'}\right)^B = \left(\frac{y}{y'}\right)^A$, where the constant has been determined by the condition that the line shall pass through the point $x = x'$, $y = y'$. The line of greatest slope is the intersection of the quadric by the cylinder whose equation has just been written, and will be a curve of double curvature except when $x'y'$ lies in one of the principal planes when the equation just found reduces to $x = 0$ or $y = 0$.

370. We shall conclude this chapter by giving an account of Gauss's theory of the curvature of surfaces.* In plane curves we measure the curvature of an arc of given length by the angle between the tangents, or between the normals, at its extremities; in other words, if we take a circle whose radius is unity, and draw radii parallel to the normals at the extremities of the arc, the ratio of the intercepted arc of the circle to the arc of the curve affords a measure of the curvature of the arc. In like manner if we have a portion of a surface bounded by any closed curve, and if we draw radii of a unit sphere parallel to the normals at every point of the bounding curve, the area of the corresponding portion of the sphere is called by Gauss the *total curvature* of the portion of the surface under consideration. And if at any *point* of a surface we divide the total curvature of the superficial element adjacent to the point by the area of the element itself, the quotient is called the *measure of curvature* for that point.

371. We proceed to express the measure of curvature by a formula. Then since the tangent plane at any point on the surface, and at the corresponding point on the unit sphere are by hypothesis parallel; the areas of any elementary portions on each are proportional to their projections on any of the co-ordinate planes. Let us consider then their projections on the plane of xy, and let us suppose the equation of the surface to be given in the form $z = \phi(x, y)$

* The reader will find his paper reprinted in the appendix to Liouville's edition of Monge.

If then x, y, z be the co-ordinates of any point on the surface, X, Y, Z those of the corresponding point on the unit sphere, $x+dx$, $x+\delta x$, $X+dX$, $X+\delta X$, &c., the co-ordinates of two adjacent points on each: then the areas of the two elementary triangles formed by the points considered, are evidently in the ratio

$$dX\delta Y - dY\delta X : dx\delta y - dy\delta x.$$

But dX, dY : δX, δY are connected with dx, dy, &c., by the same linear transformations, viz.,

$$dX = \frac{dX}{dx}dx + \frac{dX}{dy}dy, \quad dY = \frac{dY}{dx}dx + \frac{dY}{dy}dy;$$

$$\delta X = \frac{dX}{dx}\delta x + \frac{dX}{dy}\delta y, \quad \delta Y = \frac{dY}{dx}\delta x + \frac{dY}{dy}\delta y;$$

whence by the theory of linear transformations, or by actual multiplication,

$$dX\delta Y - dY\delta X = (dx\delta y - dy\delta x)\left(\frac{dX}{dx}\frac{dY}{dy} - \frac{dX}{dy}\frac{dY}{dx}\right),$$

and the quantity $\frac{dX}{dx}\frac{dY}{dy} - \frac{dX}{dy}\frac{dY}{dx}$ is the measure of curvature.

Now X, Y, Z being the projections on the axes of a unit line parallel to the normal are proportional to the cosines of the angles which the normal makes with the axes. We have therefore

$$X = \frac{p}{\sqrt{(1+p^2+q^2)}}, \quad Y = \frac{q}{\sqrt{(1+p^2+q^2)}},$$

$$\frac{dX}{dx} = \frac{(1+q^2)r - pqs}{(1+p^2+q^2)^{\frac{3}{2}}}, \quad \frac{dX}{dy} = \frac{(1+q^2)s - pqt}{(1+p^2+q^2)^{\frac{3}{2}}},$$

$$\frac{dY}{dx} = \frac{(1+p^2)s - pqt}{(1+p^2+q^2)^{\frac{3}{2}}}, \quad \frac{dY}{dy} = \frac{(1+p^2)t - pqs}{(1+p^2+q^2)^{\frac{3}{2}}},$$

whence
$$\frac{dX}{dx}\frac{dY}{dy} - \frac{dX}{dy}\frac{dY}{dx} = \frac{(rt-s^2)}{(1+p^2+q^2)^2}.$$

But from the equation of (Art. 281, p. 222) it appears that the value just found for *the measure of curvature is* $\frac{1}{RR'}$, where *R and R' are the two principal radii of curvature at the point.*

372. It is easy to verify geometrically the value thus found. For consider the elementary rectangle whose sides are in the directions of the principal tangents. Let the lengths of the sides be λ, λ', and consequently its area $\lambda\lambda'$. Now the normals at the extremities of λ intersect, and if they make with each other an angle θ, we have $\theta = \frac{\lambda}{R}$ where R is the corresponding radius of curvature. But the corresponding normals of the sphere make with each other, by hypothesis, the same angle; and their length is unity. If therefore μ be the length of the element on the sphere corresponding to λ, we have $\frac{\lambda}{R} = \mu$. In like manner we have $\frac{\lambda'}{R'} = \mu'$; and $\frac{\mu\mu'}{\lambda\lambda'} = \frac{1}{RR'}$: which was to be proved.

373. Gauss has proved that if a surface supposed to be flexible but not extensible be deformed in any way: (that is to say, if the shape of the surface be changed, yet so that the distance between any two points measured along the surface remains the same) then the measure of curvature at every point remains unaltered. We have had an example of such a change in the case of a developable surface which is such a deformation of a plane (Art. 287). And the measure of curvature vanishes for the developable as well as for the plane, one of the principal radii being infinite (Art. 334). To establish the theorem in general, let us suppose that any point on the surface instead of being given by three co-ordinates connected by the equation of the surface is given by two independent co-ordinates. Let

$$dx = adu + a'dv,\ dy = bdu + b'dv,\ dz = cdu + c'dv,$$

then $ds^2 = dx^2 + dy^2 + dz^2 = (a^2 + b^2 + c^2)\, du^2$

$$+ 2\,(aa' + bb' + cc')\, du\,dv + (a'^2 + b'^2 + c'^2)\, dv^2.$$

If we write this equation

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2,$$

what we want to prove is that the measure of curvature, or that the product of the principal radii, is a function of E, F, G.

In fact, let $x'y'z'$ denote the point of the deformed surface corresponding to any point xyz of the given surface. Then x', y', z' are given functions of x, y, z, and can therefore also be expressed in terms of u and v. And the element of any arc of the deformed surface can be expressed in the form

$$ds'^2 = E'du^2 + 2F'dudv + G'dv^2.$$

But the condition that the length of the arc shall be unaltered by transformation, manifestly requires $E=E'$, $F=F'$, $G=G'$. Any function therefore of E, F, G is unaltered by such a deformation as we are considering.

Now it will be remembered (see p. 203) that the principal radii are given by a quadratic, in which the coefficient of λ^2 is $(L^2+M^2+N^2)^2$; and the absolute term is

$$(\mathrm{bc}-\mathrm{l}^2)\,L^2+(\mathrm{ca}-\mathrm{m}^2)\,M^2+(\mathrm{ab}-\mathrm{n}^2)\,N^2$$
$$+\,2\,(\mathrm{mn}-\mathrm{al})\,MN+2\,(\mathrm{nl}-\mathrm{bm})\,NL+2\,(\mathrm{lm}-\mathrm{cn})\,LM.^*$$

We shall separately express each of these quantities in terms of E, F, G.

374. Now if we substitute in the equation of the surface $Ldx+Mdy+Ndz=0$, the values of dx, dy, dz given in the last article, and remember that since u and v are independent variables, the coefficients of du and dv must vanish separately, we have

$$La+Mb+Nc=0,\quad La'+Mb'+Nc'=0.$$

Consequently we have

$$L=\lambda\,(bc'-b'c),\quad M=\lambda\,(ca'-c'a),\quad N=\lambda\,(ab'-a'b),$$

where λ is indeterminate, and

$$L^2+M^2+N^2=\lambda^2\,\{(a^2+b^2+c^2)\,(a'^2+b'^2+c'^2)-(aa'+bb'+cc')^2\},$$
$$=\lambda^2\,(EG-F^2).$$

(See *Lessons on Higher Algebra*, Art. 21).

375. Let us now examine the result of making in the absolute term, given Art. 373, the same substitution, viz.

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* We use Roman letters in order that the a, b, c of p. 203 may not be confounded with a, b, c used in a different sense in this article.

$L=\lambda(bc'-b'c)$, &c. Now an equation which we had occasion to use in the theory of conics (see *Conics*, Ex. 5, p. 269) enables us to write this result in a more simple form. Let us write down the equation of a conic

$$\mathrm{a}x^2+\mathrm{b}y^2+\mathrm{c}z^2+2\mathrm{l}yz+2\mathrm{m}zx+2\mathrm{n}xy=0,$$

and substituting for x, y, z; $a+ka'$, $b+kb'$, $c+kc'$; let us write the result $U+2kV+k^2U'$, then

$$UU'-V^2=(\mathrm{bc}-\mathrm{l}^2)(bc'-cb')^2+(\mathrm{ca}-\mathrm{m}^2)(ca'-c'a)^2+\&c.$$

In fact, either side of this equation, equated to nothing, expresses the condition that the line joining the points abc, $a'b'c'$ should touch the conic. The equation however may be verified by actual multiplication. What we want to calculate then is $\lambda^2(UU'-V^2)$ where

$$U=\mathrm{a}a^2+\mathrm{b}b^2+\mathrm{c}c^2+2\mathrm{l}bc+2\mathrm{m}ca+2\mathrm{n}ab,$$

$$U'=\mathrm{a}a'^2+\mathrm{b}b'^2+\mathrm{c}c'^2+2\mathrm{l}b'c'+2\mathrm{m}c'a'+2\mathrm{n}a'b',$$

$$V=\mathrm{a}aa'+\mathrm{b}bb'+\mathrm{c}cc'+\mathrm{l}(bc'+b'c)+\mathrm{m}(ca'+c'a)+\mathrm{n}(ab'+a'b).$$

Now let us differentiate the equation $Ldx+Mdy+Ndz=0$, and we get

$$Ld^2x+Md^2y+Nd^2z$$
$$=-(\mathrm{a}dx^2+\mathrm{b}dy^2+\mathrm{c}dz^2+2\mathrm{l}dydz+2\mathrm{m}dzdx+2\mathrm{n}dxdy).$$

If now we write

$$d^2x=\alpha du^2+\alpha' dudv+\alpha'' dv^2,$$

$$d^2y=\beta du^2+\beta' dudv+\beta'' dv^2,$$

$$d^2z=\gamma du^2+\gamma' dudv+\gamma'' dv^2,$$

and making these substitutions on the left-hand side of the preceding equation, substitute for dx, dy, dz, from Art. 373, we get, by equating the coefficients of du^2, $dudv$, and dv^2,

$$L\alpha+M\beta+N\gamma=-U,\quad L\alpha'+M\beta'+N\gamma'=-V,$$
$$L\alpha''+M\beta''+N\gamma''=-U',$$

and what we want to calculate is the value of

$$\lambda^2\{(L\alpha+M\beta+N\gamma)(L\alpha''+M\beta''+N\gamma'')-(L\alpha'+M\beta'+N\gamma')^2\},$$

when for L, M, N are substituted the values in Art. 374.

The result is λ^4 multiplied by

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} \times \begin{vmatrix} \alpha'', & \beta'', & \gamma'' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} - \begin{vmatrix} \alpha', & \beta', & \gamma' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}^2.$$

Now if these products be expanded according to the ordinary rule for multiplication of determinants, they give the difference between the two determinants*

$$\begin{vmatrix} \alpha\alpha''+\beta\beta''+\gamma\gamma'', & a\alpha''+b\beta''+c\gamma'', & a'\alpha''+b'\beta''+c'\gamma'' \\ a\alpha+b\beta+c\gamma, & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha+b'\beta+c'\gamma, & aa'+bb'+cc', & a'^2+b'^2+c'^2 \end{vmatrix},$$

$$\begin{vmatrix} \alpha'^2+\beta'^2+\gamma'^2, & a\alpha'+b\beta'+c\gamma', & a'\alpha'+b'\beta'+c'\gamma' \\ a\alpha'+b\beta'+c\gamma', & a^2+b^2+c^2, & aa'+bb'+cc' \\ a'\alpha'+b'\beta'+c'\gamma', & aa'+bb'+cc', & a'^2+b'^2+c'^2 \end{vmatrix}.$$

376. Now it is easy to show that the terms in these determinants are functions of E, F, G and their differentials. Referring to the definitions of a, b, c, α, α', α'', &c. (Arts. 373, 375) it is obvious that

$$\alpha=\frac{da}{du},\quad \alpha'=\frac{da}{dv}=\frac{da'}{du},\quad \alpha''=\frac{da'}{dv},\ \&c.,$$

whence since

$$E=a^2+b^2+c^2,\quad F=aa'+bb'+cc',\quad G=a'^2+b'^2+c'^2,$$

$$a\alpha+b\beta+c\gamma=\tfrac{1}{2}\frac{dE}{du},\quad a\alpha'+b\beta'+c\gamma'=\tfrac{1}{2}\frac{dE}{dv},$$

$$a'\alpha'+b'\beta'+c'\gamma'=\tfrac{1}{2}\frac{dG}{du},\quad a'\alpha''+b'\beta''+c'\gamma''=\tfrac{1}{2}\frac{dG}{dv},$$

$$a\alpha''+b\beta''+c\gamma''=\frac{dF}{dv}-(a'\alpha'+b'\beta'+c'\gamma')=\frac{dF}{dv}-\tfrac{1}{2}\frac{dG}{du},$$

$$a'\alpha+b'\beta+c'\gamma=\frac{dF}{du}-(a\alpha'+b\beta'+c\gamma')=\frac{dF}{du}-\tfrac{1}{2}\frac{dE}{dv}.$$

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* I owe to Mr. Williamson the remark that the application of this rule exhibits the result in a form which manifests the truth of Gauss's theorem.

It will be seen that these equations express in terms of E, F, G every term in the preceding determinants except the leading one in each. To express these, differentiate, with regard to v the equation last written, and we have

$$\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' = \frac{d^2F}{dudv} - \tfrac{1}{2}\frac{d^2E}{dv^2} - \left(a'\frac{d\alpha}{dv} + b'\frac{d\beta}{dv} + c'\frac{d\gamma}{dv}\right).$$

Again, differentiate, with regard to u, the equation

$$a'\alpha' + b'\beta' + c'\gamma' = \tfrac{1}{2}\frac{dG}{du},$$

and we have

$$\alpha'^2 + \beta'^2 + \gamma'^2 = \tfrac{1}{2}\frac{d^2G}{du^2} - \left(a'\frac{d\alpha'}{du} + b'\frac{d\beta'}{du} + c'\frac{d\gamma'}{du}\right).$$

Now because $\frac{d\alpha}{dv} = \frac{d\alpha'}{du}$, &c., the quantities within the bracket in the last two equations are equal. And since the leading term in each determinant is multiplied by the same factor, in subtracting the determinants we are only concerned with the difference of these terms, and the quantity within the brackets disappears from the result. This result is λ^4 multiplied by the difference of the determinants

$$\begin{vmatrix} \frac{d^2F}{dudv} - \frac{1}{2}\frac{d^2E}{dv^2}, & \frac{dF}{dv} - \frac{1}{2}\frac{dG}{du}, & \frac{1}{2}\frac{dG}{dv} \\ \frac{1}{2}\frac{dE}{du}, & E, & F \\ \frac{dF}{du} - \frac{1}{2}\frac{dE}{dv}, & F, & G \end{vmatrix},$$

and

$$\begin{vmatrix} \frac{1}{2}\frac{d^2G}{du^2}, & \frac{1}{2}\frac{dE}{dv}, & \frac{1}{2}\frac{dG}{du} \\ \frac{1}{2}\frac{dE}{dv}, & E, & F \\ \frac{1}{2}\frac{dG}{du}, & F, & G \end{vmatrix}.$$

We get the measure of curvature by dividing the quantity now formed, by $(L^2 + M^2 + N^2)^2$ whose value is given (Art. 374) when the common factor λ^4 disappears and the result is obviously a function of E, F, G and their differentials. Gauss's theorem is therefore proved.

We add the actual expansion of the determinants, though not necessary to the proof. Writing the measure of curvature K, we have

$$4(EG-F^2)^2K = E\left\{\frac{dE}{dv}\frac{dG}{dv} - 2\frac{dF}{du}\frac{dG}{dv} + \left(\frac{dG}{du}\right)^2\right\}$$

$$+F\left\{\frac{dE}{du}\frac{dG}{dv} - \frac{dE}{dv}\frac{dG}{du} - 2\frac{dE}{dv}\frac{dF}{dv} + 4\frac{dF}{du}\frac{dF}{dv} - 2\frac{dF}{du}\frac{dG}{du}\right\}$$

$$+G\left\{\frac{dE}{du}\frac{dG}{du} - 2\frac{dE}{du}\frac{dF}{dv} + \left(\frac{dE}{dv}\right)^2\right\}$$

$$-2(EG-F^2)\left(\frac{d^2E}{dv^2} - 2\frac{d^2F}{du\,dv} + \frac{d^2G}{du^2}\right),$$

(Liouville's Monge, p. 523).*

377. We may consider two systems of curves traced on the surface, for one of which u is constant, and for the other v; so that any point on the surface is the intersection of a curve of each system. The expression then $ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2$ shows that $\sqrt{(E)}\,du$ is the element of the curve, passing through the point, for which v is constant; and $\sqrt{(G)}\,dv$ is the element of the curve for which u is constant. If these two curves intersect at an angle ω, then since ds is the diagonal of a parallelogram of which $\sqrt{(E)}\,du$, $\sqrt{(G)}\,dv$ are the sides, we have $\cos\omega = \frac{F}{\sqrt{(EG)}}$, while the area of the parallelogram being $d\sigma d\sigma' \sin\omega = \sqrt{(EG-F^2)}\,du\,dv$. If the curves of the system u cut at right angles those of the system v, we must have $F=0$.

A particular case of these formulæ is when we use geodesic polar co-ordinates in which case we saw that we always have

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* MM. Bertrand, Diguet, and Puiseux (see Liouville, Vol. XIII., p. 80; Appendix to Monge, p. 583) have established Gauss's theorem by calculating the perimeter and area of a geodesic circle on any surface, whose radius, supposed to be very small, is s. They find for the perimeter $2\pi s - \frac{\pi s^3}{3RR'}$, and for the area $\pi s^2 - \frac{\pi s^4}{12RR'}$. And of course the supposition that these are unaltered by deformation implies that RR' is constant.

an expression of the form $ds^2 = d\rho^2 + P^2 d\omega^2$. Now if in the formulæ of the last article we put $F = 0$, $E =$ constant, it becomes

$$4E^2G^2K = E\left(\frac{dG}{du}\right)^2 - 2EG\frac{d^2G}{du^2},$$

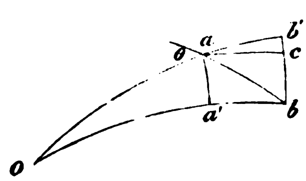
and if we put

$$E = 1,\ G = P^2,\ u = \rho,\ K = \frac{1}{RR'},\ \text{we have } \frac{d^2P}{d\rho^2} + \frac{P}{RR'} = 0,$$

an equation which must be satisfied by the function P on any surface, if $Pd\omega$ expresses the element of the arc of a geodesic circle. Mr. Roberts verifies (*Cambridge and Dublin Mathematical Journal*, Vol. III., p. 161) that this equation is satisfied by the function $\frac{y}{\sin\omega}$ on a quadric.

378. Gauss applies these formulæ to find the total curvature, in his sense of the word, of a geodesic triangle on any surface. The element of the area being $Pd\omega d\rho$, and the measure of curvature being $-\frac{1}{P}\frac{d^2P}{d\rho^2}$; the total curvature is found by twice integrating $-\frac{d^2P}{d\rho^2}d\rho d\omega$. Integrating first with respect to ρ, we get $\left(C - \frac{dP}{d\rho}\right)d\omega$. Now if the radii are measured from one vertex of the given triangle, the integral is plainly to vanish for $\rho = 0$; and it is plain also that for $\rho = 0$ we must have $\frac{dP}{d\rho} = 1$; for as ρ tends to vanish, the length of an element perpendicular to the radius tends to become $\rho d\omega$. Hence the first integral is $d\omega\left(1 - \frac{dP}{d\rho}\right)$.

This may be written in a more convenient form as follows: Let θ be the angle which any radius vector makes with the element of a geodesic ab. Now since $aa' = Pd\omega$, $bb' = (P + dP)d\omega$; and if $cb = aa'$, we have $b'c = dPd\omega$, and the angle $b'ac = \frac{dP}{d\rho}d\omega$. But $b'ac$ is evidently the diminution of the angle



θ in passing to a consecutive point; hence $d\theta = -\frac{dP}{d\rho}\,d\omega$. The integral just found is therefore $d\omega + d\theta$, which integrated a second time is $\omega + \theta' - \theta''$, where ω is the angle between the two extreme radii vectores which we consider, and θ', θ'' are the corresponding values of θ. If we call A, B, C the internal angles of the triangle formed by the two extreme radii and by the base, we have $\omega = A$, $\theta' = B$, $\theta'' = \varpi - C$, and the total curvature is $A + B + C - \pi$. Hence the excess over 180° of the sum of the angles of a geodesic triangle is measured by the area of that portion of a unit sphere which corresponds to the directions of the normals along the sides of the given triangle.

The portion on the unit sphere corresponding to the area enclosed by a geodesic returning upon itself is half the sphere. For if the radius vector travel round so as to return to the point whence it set out the extreme values of θ' and θ'' are equal, while ω has increased by 2π. The measure of curvature is therefore 2π or half the surface of the sphere.*

* For some other interesting theorems, relative to the deformation of surfaces, see Mr. Jellett's paper "On the Properties of Inextensible Surfaces, *Transactions of the Royal Irish Academy*, Vol. XXII. The theory of surfaces applicable to one another was the subject proposed by the French Academy as their Prize Question for 1860, and the report of the Commission to which the decision was referred, gives reason to think that the Memoirs sent in for competition will, when published, add considerably to what had been previously known on the subject.

CHAPTER XII.

FAMILIES OF SURFACES.

379. Let the equations of a curve

$$\phi(x, y, z, c_1, c_2 \dots c_n) = 0, \quad \psi(x, y, z, c_1, c_2 \dots c_n) = 0,$$

include n parameters, or undetermined constants: then it is evident that if n equations connecting these parameters be given, the curve is completely determined. If, however, only $n-1$ relations between the parameters be given, the equations above written may denote an infinity of curves; and the assemblage of all these curves constitutes a surface whose equation is obtained by eliminating the n parameters from the given $n+1$ equations; viz. the $n-1$ relations, and the two equations of the curve. Thus, for example, if the two equations above written denote a variable curve, the motion of which is regulated by the conditions that it shall intersect $n-1$ fixed directing curves, the problem is of the kind now under consideration. For by eliminating x, y, z between the two equations of the variable curve and the two equations of any one of the directing curves, we express the condition that these two curves should intersect, and thus have one relation between the n parameters. And having $n-1$ such relations we find the equation of the surface generated, in the manner just stated. We had (Art. 109) a particular case of this problem.

Those surfaces for which the form of the functions ϕ and ψ is the same, are said to be *of the same family*, though the equations connecting the parameters may be different. Thus if the motion of the same variable curve were regulated by several different sets of directing curves, all the surfaces generated would be said to belong to the same family. In several important cases the equations of all surfaces belonging to the same family can be included in one equation involving one or more arbitrary functions; the equation of any individual surface of the family being then got by particularizing the form of the functions. If we eliminate the arbitrary functions by differentiation, we get a partial differential equation, common to all surfaces of the family, which ordinarily is the expression of some geometrical property common to all surfaces of the family, and which leads more directly than the functional equation to the solution of some classes of problems.

380. The simplest case is when the equations of the variable curve include but two constants.* Solving in turn for each of these constants, we can throw the two given equations into the form $u = c_1$, $v = c_2$; where u and v are known functions of x, y, z. In order that this curve may generate a surface we must be given one relation connecting c_1, c_2, which will be of the form $c_1 = \phi(c_2)$; whence putting for c_1 and c_2 their values, we see that, whatever be the equation of connection, the equation of the surface generated must be of the form $u = \phi(v)$.

We can also in this case readily obtain the partial differential equation which must be satisfied by all surfaces of the family. For if $U = 0$ represents any such surface, U can only differ by a constant multiplier from $u - \phi(v)$. Hence we have $\lambda U = u - \phi(v)$, and differentiating

$$\lambda \frac{dU}{dx} = \frac{du}{dx} - \phi'(v) \frac{dv}{dx},$$

with two similar equations for the differentials with respect to y and z. Eliminating then λ and $\phi'(v)$, we get the required partial differential equation in the form of a determinant

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0,$$

where, for shortness, we write U_1, U_2, U_3, &c. for

$$\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}, \text{ \&c.}$$

* If there were but one constant the elimination of it would give the equation of a definite surface, not of a family of surfaces.

In this case u and v are supposed to be known functions of the co-ordinates; and the equation just written establishes a relation of the first degree between $\frac{dU}{dx}, \frac{dU}{dy}, \frac{dU}{dz}$.

If the equation of the surface were written in the form $z-\phi(x, y)=0$; we should have $\frac{dU}{dz}=1$, $\frac{dU}{dx}=-p$, $\frac{dU}{dy}=-q$, where p and q have the usual signification, and the partial differential equation of the family is of the form $Pp+Qq=R$, where P, Q, R are known functions of the co-ordinates. And conversely the integral of such a partial differential equation, which (see Boole's *Differential Equations*, p. 322) is of the form $u=\phi(v)$, geometrically represents a surface which can be generated by the motion of a curve whose equations are of the form $u=c_1$, $v=c_2$.

The partial differential equation affords the readiest test whether a given surface belongs to any assigned family. We have only to give to U_1, U_2, U_3, their values derived from the equation of the given surface, which values must identically satisfy the partial differential equation of the family if the surface belong to that family.

381. If it be required to determine a particular surface of a given family $u=\phi(v)$, by the condition that the surface shall pass through a given curve, the form of the function in this case can be found by writing down the equations $u=c_1$, $v=c_2$, and eliminating x, y, z between these equations and those of the fixed curve, when we find a relation between c_1 and c_2, or between u and v, which is the equation of the required surface. The geometrical interpretation of this process is that we direct the motion of a variable curve $u=c_1$, $v=c_2$ by the condition that it shall move so as always to intersect the given fixed curve. All the points of the latter are therefore points on the surface generated.

If it be required to find a surface of the family $u=\phi(v)$ which shall envelope a given surface, we know that at every point of the curve of contact U_1, U_2, U_3, &c. have the same value for the fixed surface and for that which envelopes it.

If then in the partial differential equation of the given family, we substitute for U_1, U_2, U_3 their values derived from the equation of the fixed surface, we get an equation which will be satisfied for every point of the curve of contact, and which therefore combined with the equation of the fixed surface determines that curve. The problem is therefore reduced to that considered in the first part of this article; namely, to describe a surface of the given family through a given curve. All this theory will be better understood from the following examples of important families of surfaces belonging to the class here considered; viz. whose equations can be expressed in the form $u = \phi(v)$.

382. *Cylindrical Surfaces.* A cylindrical surface is generated by the motion of a right line, which remains always parallel to itself. Now the equations of a right line include four independent constants; if then the direction of the right line be given, this determines two of the constants, and there remain but two undetermined. The family of cylindrical surfaces belongs to the class considered in the last two articles.

Thus if the equations of a right line be given in the form $x = lz + p$, $y = mz + q$; l and m which determine the directions of the right line are supposed to be given; and if the motion of the right line be regulated by any condition (such as that it shall move along a certain fixed curve, or envelope a certain fixed surface) this establishes a relation between p and q, and the equation of the surface comes out in the form

$$x - lz = \phi(y - mz).$$

More generally, if the right line is to be parallel to the intersection of the two planes $ax + by + cz$, $a'x + b'y + c'z$, its equations must be of the form

$$ax + by + cz = \alpha, \quad a'x + b'y + c'z = \beta,$$

and the equation of the surface generated must be of the form

$$ax + by + cz = \phi(a'x + b'y + c'z).$$

Writing $ax + by + cz$ for u, and $a'x + b'y + c'z$ for v in the equation of Art. 380, we see that the partial differential equation of cylindrical surfaces is

$$(bc'-b'c)\,U_1+(ca'-c'a)\,U_2+(ab'-a'b)\,U_3=0,$$

or (Ex. 3, p. 26) $U_1\cos\alpha+U_2\cos\beta+U_3\cos\gamma=0$, where α, β, γ are the direction-cosines of the generating line. Remembering that U_1, U_2, U_3 are proportional to the direction-cosines of the normal to the surface, it is obvious that the geometrical meaning of this equation is that the tangent plane to the surface is always parallel to the direction of the generating line.

Ex. 1. To find the equation of the cylinder whose edges are parallel to $x = lz$, $y = mz$, and which passes through the plane curve $z = 0$, $\phi(x, y) = 0$.

Ans. $\phi(x - lz,\ y - mz) = 0$.

Ex. 2. To find the equation of the cylinder whose sides are parallel to the intersection of $ax + by + cz$, $a'x + b'y + c'z$, and which passes through the intersection of $\alpha x + \beta y + \gamma z = \delta$, $F(x, y, z) = 0$. Solve for x, y, z between the equations $ax + by + cz = u$, $a'x + b'y + c'z = v$, $\alpha x + \beta y + \gamma z = \delta$, and substitute the resulting values in $F(x, y, z) = 0$.

Ex. 3. To find the equation of a cylinder, the direction-cosines of whose edges are l, m, n, and which passes through the curve $U = 0$, $V = 0$. The elimination may be conveniently performed as follows: If x', y', z' be the co-ordinates of the point where any edge meets the directing curve; x, y, z those of any point on the edge, we have $\dfrac{x-x'}{l}=\dfrac{y-y'}{m}=\dfrac{z-z'}{n}$. Calling the common value of these functions θ, we have

$$x' = x - l\theta,\quad y' = y - m\theta,\quad z' = z - n\theta.$$

Substitute these values in the equations $U = 0$, $V = 0$, which $x'y'z'$ must satisfy; and between the two resulting equations eliminate the unknown θ, the result will be the equation of the cylinder.

Ex. 4. To find the cylinder, the direction-cosines of whose edges are l, m, n, and which envelopes the quadric $Ax^2 + By^2 + Cz^2 = 1$. From the partial differential equation, the curve of contact is the intersection of the quadric with $Alx + Bmy + Cnz = 0$. Proceeding then as in the last example the equation of the cylinder is found to be

$$(Al^2 + Bm^2 + Cn^2)(Ax^2 + By^2 + Cz^2 - 1) = (Alx + Bmy + Cnz)^2.$$

383. *Conical Surfaces.* These are generated by the motion of a right line which constantly passes through a fixed point. Expressing that the co-ordinates of this point satisfy the equa-

tions of the right line, we have two relations connecting the four constants in the general equations of a right line. In this case therefore the equations of the generating curve contain but two undetermined constants, and the problem is of the kind discussed Art. 380.

Let the equations of the generating line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

where α, β, γ are the known co-ordinates of the vertex of the cone, and l, m, n are proportional to the direction-cosines of the generating line; and where the equations, though apparently containing three undetermined constants, actually contain only two, since we are only concerned with the ratios of the quantities l, m, n.

Writing the equations then in the form

$$\frac{x-\alpha}{z-\gamma}=\frac{l}{n},\quad \frac{y-\beta}{z-\gamma}=\frac{m}{n};$$

we see that the conditions of the problem must establish a relation between $\frac{l}{n}$ and $\frac{m}{n}$, and that the equation of the cone must be of the form $\frac{x-\alpha}{z-\gamma}=\phi\left(\frac{y-\beta}{z-\gamma}\right)$.

It is easy to see that this is equivalent to saying that the equation of the cone must be a homogeneous function of the three quantities $x-\alpha$, $y-\beta$, $z-\gamma$; as may also be seen directly from the consideration that the conditions of the problem must establish a relation between the direction-cosines of the generator: that these cosines being $\frac{l}{\sqrt{(l^2+m^2+n^2)}}$, &c. any equation expressing such a relation is a homogeneous function of l, m, n, and therefore of $x-\alpha$, $y-\beta$, $z-\gamma$, which are proportional to l, m, n.

When the vertex of the cone is the origin, its equation is of the form $\frac{x}{z}=\phi\left(\frac{y}{z}\right)$; or, in other words, is a homogeneous function of x, y, z.

The partial differential equation is found by putting

$u=\frac{x-\alpha}{z-\gamma}$, $v=\frac{y-\beta}{z-\gamma}$, in the equation of Art. 380, and when cleared of fractions is

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ z-\gamma, & 0, & -(x-\alpha) \\ 0, & z-\gamma, & -(y-\beta) \end{vmatrix} = 0,$$

or
$$(x-\alpha)\frac{dU}{dx}+(y-\beta)\frac{dU}{dy}+(z-\gamma)\frac{dU}{dz}=0.$$

This equation evidently expresses that the tangent plane at any point of the surface must always pass through the fixed point $\alpha\beta\gamma$.

We have already given in p. 86 the method of forming the equation of the cone standing on a given curve; and p. 190 the method of forming the equation of the cone which envelopes a given surface.

384. *Conoidal Surfaces.* These are generated by the motion of a line which always intersects a fixed axis and remains parallel to a fixed plane. These two conditions leave two of the constants in the equations of the line undetermined, so that these surfaces are of the class considered Art. 380. If the axis is the intersection of the planes α, β, and the generator is to be parallel to the plane γ; the equations of the generator are $\alpha=c_1\beta$, $\gamma=c_2$, and the general equation of conoidal surfaces is obviously $\frac{\alpha}{\beta}=\phi(\gamma)$.*

The partial differential equation is (Art. 380)

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ \beta\alpha_1-\alpha\beta_1, & \beta\alpha_2-\alpha\beta_2, & \beta\alpha_3-\alpha\beta_3 \\ \gamma_1, & \gamma_2, & \gamma_3 \end{vmatrix} = 0,$$

where $\alpha=\alpha_1x+\alpha_2y+\alpha_3z+\alpha_4$, &c. The left-hand side of the equation may be expressed as the difference of two determinants $\beta(U_1\alpha_2\gamma_3)-\alpha(U_1\beta_2\gamma_3)=0$.

* In like manner the equation of any surface generated by the motion of a line meeting two fixed lines $\alpha\beta$, $\gamma\delta$ must be of the form $\frac{\alpha}{\beta}=\phi\left(\frac{\gamma}{\delta}\right)$.

This equation may be derived directly by expressing that the tangent plane at any point on the surface contains the generator: the tangent plane, therefore, the plane drawn through the point on the surface, parallel to the directing plane, and the plane $\alpha'\beta - \alpha\beta'$ joining the same point to the axis, have a common line of intersection. The terms of the determinant just written are the coefficients of x, y, z in the equations of these three planes.

In practice we are almost exclusively concerned with right conoids; that is, where the fixed axis is perpendicular to the directing plane. If that axis be taken as the axis of z, and the plane for plane of xy, the functional equation is $y = x\phi(z)$, and the partial differential equation is $x\dfrac{dU}{dx} + y\dfrac{dU}{dy} = 0$.

The lines of greatest slope (Art. 370) are in this case always projected into circles. For in virtue of the partial differential equation just written, the equation of Art. 370,

$$\frac{dU}{dy}dx - \frac{dU}{dx}dy = 0,$$

transforms itself into $xdx + ydy = 0$, which represents a series of concentric circles. The same thing is evident geometrically: for the lines of level are the generators of the system; and these being projected into a series of radii all passing through the origin, are cut orthogonally by a series of concentric circles.

Ex. 1. To find the equation of the right conoid passing through the axis of z and through a plane curve, whose equations are $x = a$, $F(y, z) = 0$. Eliminating then x, y, z between these equations and $y = c_1x$, $z = c_2$, we get $F(c_1a, c_2) = 0$; or the required equation is $F\left(\dfrac{ay}{x}, z\right) = 0$.

Wallis's cono-cuneus is when the fixed curve is a circle $[x = a,\ y^2 + z^2 = r^2]$. Its equation is therefore $a^2y^2 + x^2z^2 = r^2x^2$.

Ex. 2. Let the directing curve be a helix, the fixed line being the axis of the cylinder on which the helix is traced. The equation is that given Ex. 1, p. 273. This surface is often presented to the eye, being that formed by the under surface of a spiral staircase.

385. *Surfaces of Revolution.* The fundamental property of a surface of revolution is that its section perpendicular to its

axis must always consist of one or more circles whose centres are on the axis. Such a surface may therefore be conceived as generated by a circle of variable radius whose centre moves along a fixed right line or axis, and whose plane is perpendicular to that axis. If the equations of the axis be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$, then the generating circle in any position may be represented as the intersection of the plane perpendicular to the axis $lx+my+nz=c_1$, with the sphere whose centre is any fixed point on the axis

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=c_2.$$

These equations contain but two undetermined constants; the problem therefore is of the class considered (Art. 380) and the equation of the surface must be of the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\phi\,(lx+my+nz).$$

When the axis of z is the axis of revolution we may take the origin as the point $\alpha\beta\gamma$, and the equation becomes

$$x^2+y^2+z^2=\phi\,(z),\ \text{or}\ z=\psi\,(x^2+y^2).$$

The partial differential equation is found by the formula of Art. 380 to be

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ l, & m, & n \\ x-\alpha, & y-\beta, & z-\gamma \end{vmatrix}=0,$$

or $$\{m(z-\gamma)-n(y-\beta)\}\frac{dU}{dx}$$

$$+\{n(x-\alpha)-l(z-\gamma)\}\frac{dU}{dy}+\{l(y-\beta)-m(x-\alpha)\}\frac{dU}{dz}=0.$$

When the axis of z is the axis of revolution this reduces to

$$y\frac{dU}{dx}-x\frac{dU}{dy}=0.$$

The partial differential equation expresses that the normal always meets the axis of revolution. For if we wish to express the condition that the two lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},\quad \frac{x-x'}{U_1}=\frac{y-y'}{U_2}=\frac{z-z'}{U_3}$$

should intersect; we may write the common value of the equal fractions in each case, θ and θ'. Solving then for x, y, z, and equating the values derived from the equations of each line, we have

$$\alpha + l\theta = x' + U_1\theta', \quad \beta + m\theta = y' + U_2\theta', \quad \gamma + n\theta = z' + U_3\theta';$$

whence eliminating θ, θ' the result is the determinant already found

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ l, & m, & n \\ x'-\alpha, & y'-\beta, & z'-\gamma \end{vmatrix} = 0.$$

386. The equation of the surface generated by the revolution of a given curve round a given axis, is found (Art. 381) by eliminating x, y, z between

$$lx + my + nz = u, \quad (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = v,$$

and the two equations of the curve; replacing then u and v by their values. We have already had an example of this (Ex. 3, p. 85) and we take as a further example "to find the surface generated by the revolution of a circle $[y = 0, (x-a)^2 + z^2 = r^2]$ round an axis in its plane [the axis of z]."

Putting $z = u$, $x^2 + y^2 = v$ and eliminating between these equations, and those of the circle, we get

$$\{\sqrt{(v)} - a\}^2 + u^2 = r^2, \text{ or } \{\sqrt{(x^2+y^2)} - a\}^2 + z^2 = r^2,$$

which cleared of radicals is

$$(x^2 + y^2 + z^2 + a^2 - r^2)^2 = 4a^2(x^2 + y^2).$$

It is obvious that when a is greater than r, that is to say, when the revolving circle does not meet the axis, neither can the surface, which will be the form of an anchor ring, the space about the axis being empty. On the other hand, when the revolving circle meets the axis, the segments into which the axis divides the circle generate distinct sheets of the surface, intersecting in points on the axis $z = \sqrt{(r^2 - a^2)}$, which are nodal points on the surface.

The sections of the anchor ring by planes parallel to the axis are found by putting y = constant in the preceding equation. The equation of the section may immediately be thrown into the form $SS' =$ constant, where S and S' represent circles. The sections are lemniscates of various kinds (see fig., *Higher Plane Curves*, p. 204). It is geometrically evident, that as the plane of section moves away from the axis, it continues to cut in two distinct ovals, until it touches the surface [$y = a - r$] when it cuts in a curve having a double point [Bernouilli's Lemniscate]; after which it meets in a continuous curve.

Ex. Verify that $x^3 + y^3 + z^3 - 3xyz = r^3$ is a surface of revolution.

Ans. The axis of revolution is $x = y = z$.

387. The families of surfaces which have been considered are the most interesting of those whose equations can be expressed in the form $u = \phi(v)$. We now proceed to the case when the equations of the generating curve include more than two parameters. By the help of the equations connecting these parameters, we can, in terms of any one of them, express all the rest; and thus put the equations of the generating curve into the form

$$F\{x, y, z, c, \phi(c), \psi(c), \&c.\} = 0, \quad f\{x, y, z, c, \phi(c), \psi(c), \&c.\} = 0.$$

The equation of the surface generated is obtained by eliminating c between these equations; and, as has been already stated, all surfaces are said to be of the same family for which the form of the functions F and f is the same, whatever be the forms of the functions ϕ, ψ, &c. But since evidently the elimination cannot be effected until some definite form has been assigned to the functions ϕ, ψ, &c. it is not generally possible to form a single functional equation including all surfaces of the same family: and we can only represent them, as above written, by a pair of equations from which there remains a constant to be eliminated. We can however eliminate the arbitrary functions by differentiation and obtain a partial differential equation, common to all surfaces of the same family; the order of that equation being, as we shall presently prove, equal to the number of arbitrary functions ϕ, ψ, &c.

It is to be remarked however that in general the order of the partial differential equation obtained by the elimination of a number of arbitrary functions from an equation is higher than the number of functions eliminated. Thus if an equation include two arbitrary functions ϕ, ψ, and if we differentiate with respect to x and y which we take as independent variables, the differentials combined with the original equation form a system of three equations containing four unknown functions ϕ, ψ, ϕ', ψ'. The second differentiation (twice with regard to x, twice with regard to y, and with regard to x and y) gives us three additional equations; but then from the system of six equations it is not generally possible to eliminate the six quantities ϕ, ψ, ϕ', ψ', ϕ'', ψ''. We must therefore proceed to a third differentiation before the elimination can be effected. It is easy to see, in like manner, that to eliminate n arbitrary functions we must differentiate $2n-1$ times. The reason why, in the present case, the order of the differential equation is less, is that the functions eliminated are all functions *of the same quantity*.

388. In order to show this it is convenient to consider first the special case, where a family of surfaces can be expressed by a single functional equation. This will happen when it is possible by combining the equations of the generating curve to separate one of the constants so as to throw the equations into the form $u=c_1$; $F(x, y, z, c_1, c_2 \ldots c_n)=0$. Then expressing, by means of the equations of condition, the other constants in terms of c_1, the result of elimination is plainly of the form

$$F\{x, y, z, u, \phi(u), \psi(u), \&c.\}=0.$$

Now if, as before, we denote by U_1, the differential with respect to x of the equation of the surface, and by F_1, the differential on the supposition that u is constant, we have

$$U_1=F_1+\frac{dF}{du}u_1,$$

$$U_2=F_2+\frac{dF}{du}u_2,$$

$$U_3=F_3+\frac{dF}{du}u_3.$$

Now in these equations, the derived functions ϕ', ψ', &c. only enter in the term $\frac{dF}{du}$; they can therefore be all eliminated together; and we can form the equation, homogeneous in U_1, U_2, U_3,

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ F_1, & F_2, & F_3 \\ u_1, & u_2, & u_3 \end{vmatrix} = 0,$$

which contains only the original functions ϕ, ψ, &c. If we write this equation $V = 0$, we can form from it in like manner the equation

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ V_1, & V_2, & V_3 \\ u_1, & u_2, & u_3 \end{vmatrix} = 0,$$

which still contains no arbitrary functions but the original ϕ, ψ, &c., but which contains the second differential coefficients of U, these entering into V_1, V_2, V_3. From the equation last found we can in like manner form another, and so on; and from the series of equations thus obtained (the last being of the n^{th} order of differentiation) we can eliminate the n functions ϕ, ψ, &c.

If we omit the last of these equations, we can eliminate all but one of the arbitrary functions, and according to our choice of the function to be retained, can obtain n different equations of the order $n - 1$, each containing one arbitrary function. These are the first integrals of the final differential equation of the n^{th} order. In like manner we can form $\frac{n(n-1)}{1 \cdot 2}$ equations of the second order, each containing two arbitrary functions, and so on.

389. If we take x and y as the independent variables, and as usual write $dz = pdx + qdy$, $dp = rdx + sdy$, &c., the process of forming these equations may be more conveniently stated as follows: "Take the total differential of the given equation on the supposition that u is constant,

$$F_1 dx + F_2 dy + F_3 (pdx + qdy) = 0;$$

put $dy = mdx$, and substitute for m its value derived from the differential of $u = 0$, viz.

$$u_1 dx + u_2 dy + u_3 (pdx + qdy) = 0."$$

For if we differentiate the given equation with respect to x and y, we get

$$F_1 + pF_3 + \frac{dF}{du}(u_1 + pu_3) = 0,$$

$$F_2 + qF_3 + \frac{dF}{du}(u_2 + qu_3) = 0,$$

and the result of eliminating $\frac{dF}{du}$ from these two equations is the same as the result of eliminating m between the equations

$$F_1 + pF_3 + m(F_2 + qF_3) = 0, \quad u_1 + pu_3 + m(u_2 + qu_3) = 0.$$

It is convenient in practice to choose for one of the equations representing the generating curve, its projection on the plane of xy; then since this equation does not contain z, the value of m derived from it will not contain p or q, and the first differential equation will be of the form

$$p + qm = R,$$

R being also a function not containing p or q. The only terms then containing r, s, or t in the second differential equation are those derived from differentiating $p + qm$, and that equation will be of the form

$$r + 2sm + tm^2 = S,$$

where S may contain x, y, z, p, q, but not r, s, or t. If now we had only two functions to eliminate, we should solve for these constants from the original functional equation of the surface, and from $p + qm = R$; and then substituting these values in m and in S, the *form* of the final second differential equation would still remain

$$r + 2sm' + tm'^2 = S',$$

where m' and S' might contain x, y, z, p, q. In like manner if we had three functions to eliminate, and if we denote the partial differentials of z of the third order by α, β, γ, δ, the partial differential equation would be of the form

$$\alpha + 3m\beta + 3m^2\gamma + m^3\delta = T.$$

And so on for higher orders. This theory will be illustrated by the examples which follow.

390. *Surfaces generated by lines parallel to a fixed plane.* This is a family of surfaces which includes conoids as a particular case. Let us in the first place take the fixed plane for the plane of xy. Then the equations of the generating line are of the form $z = c_1$, $y = c_2x + c_3$. The functional equation of the surface is got by substituting in the latter equation for c_2, $\phi(z)$, and for c_3, $\psi(z)$. Since in forming the partial differential equation we are to regard z as constant, we may as well leave the equations in the form $z = c_1$, $y = c_2x + c_3$. These give us

$$p + qm = 0, \quad m = c_2.$$

According as we eliminate c_3 or c_2, these equations give us $p + qc_2 = 0$, $px + qy = qc_3$. There are therefore two equations of the first order, each containing one arbitrary function, viz.

$$p + q\phi(z) = 0, \quad px + qy = q\psi(z).$$

To eliminate completely arbitrary functions, differentiate $p + qm = 0$, remembering that since $m = c_2$ it is to be regarded as constant, when we get

$$r + 2sm + tm^2 = 0,$$

and eliminating m by means of $p + qm = 0$, the required equation is

$$q^2r - 2pqs + p^2t = 0.$$

Next let the generating line be parallel to $ax + by + cz$; its equations are

$$ax + by + cz = c_1, \quad y = c_2x + c_3;$$

and the functional equation of the family of surfaces is got by writing for c_2 and c_3, functions of $ax + by + cz$. Differentiating, we have

$$a + cp + m(b + cq) = 0, \quad m = c_2.$$

The equations got by eliminating one arbitrary function are therefore

$$a + cp + (b + cq)\,\phi(ax + by + cz) = 0,$$

$$(a + cp)\,x + (b + cq)\,y = (b + cq)\,\psi(ax + by + cz).$$

Differentiating $a+bm+c(p+mq)$, and remembering that m is to be regarded as constant, we have

$$r+2sm+tm^2=0,$$

and introducing the value of m already found

$$(b+cq)^2 r-2(a+cp)(b+cq)s+(a+cp)^2 t=0.$$

391. This equation may also be arrived at by expressing that the tangent planes at two points on the same generator intersect, as they evidently must, on that generator. Let α, β, γ be the running co-ordinates, x, y, z those of the point of contact; then any generator is the intersection of the tangent plane

$$\gamma-z=p(\alpha-x)+q(\beta-y),$$

with a plane through the point of contact parallel to the fixed plane

$$a(\alpha-x)+b(\beta-y)+c(\gamma-z)=0,$$

whence $$(a+cp)(\alpha-x)+(b+cq)(\beta-y)=0.$$

Now if we pass to the line of intersection of this tangent plane with a consecutive plane, α, β, γ remain the same, while x, y, z, p, q vary. Differentiating the equation of the tangent plane, we have

$$(rdx+sdy)(\alpha-x)+(sdx+tdy)(\beta-y)=0.$$

And eliminating $\alpha-x$, $\beta-y$,

$$(b+cq)(rdx+sdy)=(a+cp)(sdx+tdy).$$

But since the point of contact moves along the generator which is parallel to the fixed plane, we have

$$adx+bdy+cdz=0, \text{ or } (a+cp)\,dx+(b+cq)\,dy=0.$$

Eliminating then dx, dy from the last equation, we have, as before,

$$(b+cq)^2 r-2(a+cp)(b+cq)s+(a+cp)^2 t=0.$$

392. *Surfaces generated by lines which meet a fixed axis.* This class also includes the family of conoids. In the first place let the fixed axis be the axis of z; then the equations of the generating line are of the form $y=c_1x$, $z=c_2x+c_3$; and the equation of the family of surfaces is got by writing in the

latter equation for c_2 and c_3, arbitrary functions of $\frac{y}{x}$. Differentiating, we have $m=c_1$, $p+mq=c_2$, whence

$$px+qy=x\phi\left(\frac{y}{x}\right), \text{ and } z-px-qy=\psi\left(\frac{y}{x}\right).$$

Differentiating again, we have $r+2sm+tm^2=0$, and putting for m its value $=c_1=\frac{y}{x}$, the required differential equation is

$$rx^2+2sxy+ty^2=0.$$

This equation may also be obtained by expressing that two consecutive tangent planes intersect in a generator. As, in Art. 391, we have for the intersection of two consecutive tangent planes

$$(rdx+sdy)(\alpha-x)+(sdx+tdy)(\beta-y)=0.$$

But any generator lies in the plane $\alpha y=\beta x$, or $(\alpha-x)y=(\beta-y)x$. Eliminating therefore

$$x(rdx+sdy)+y(sdx+tdy)=0.$$

But $\frac{dy}{dx}=\frac{\beta}{\alpha}=\frac{y}{x}$. Therefore, as before, $rx^2+2sxy+ty^2=0$.

More generally let the line pass through a fixed axis $\alpha\beta$, where $\alpha=ax+by+cz+d$, $\beta=a'x+b'y+c'z+d'$. Then the equations of the generating line are $\alpha=c_1\beta$, $y=c_2x+c_3$, and the equation of the family of surfaces is $y=x\phi\frac{\alpha}{\beta}+\psi\frac{\alpha}{\beta}$. Differentiating, we have

$$m=c_2,\ a+cp+m(b+cq)=c_1\{a'+c'p+m(b'+c'q)\}.$$

Differentiating again, we have $r+2sm+tm^2=0$, and putting in for m from the last equation, the required partial differential equation is

$$\begin{aligned}\{(a+cp)\beta-(a'+c'p)\alpha\}^2 t \\ -2\{(a+cp)\beta-(a'+c'p)\alpha\}\{(b+cq)\beta-(b'+c'q)\alpha\}s \\ +\{(b+cq)\beta-(b'+c'q)\alpha\}^2 r=0.\end{aligned}$$

393. If the equation of a family of surfaces contain n arbitrary functions of the same quantity, and if it be required

to determine a surface of the family which shall pass through n fixed curves, we write down the equations of the generating curve $u=c_1$, $F(x, y, z, c_1, c_2, \&c.)=0$, and expressing that the generating curve meets each of the fixed curves, we have a sufficient number of equations to eliminate c_1, c_2, &c. Thus to find a surface of the family $x+y\phi(z)+\psi(z)=0$ which shall pass through the fixed curves $y=a$, $F(x, z)=0$; $y=-a$, $F_1(x, z)=0$. The equations of the generating line being $z=c_1$, $x=yc_2+c_3$, we have, by substitution,

$$F(ac_2+c_3, c_1)=0, \quad F_1(c_3-ac_2, c_1)=0,$$

or replacing for c_1, c_3, their values,

$$F\{x+c_2(a-y), z\}=0, \quad F_1\{x-c_2(a+y), z\}=0,$$

by eliminating c_2 between which the required surface is found.

Ex. Let the directing curves be

$$y=a, \quad \frac{x^2}{b^2}+\frac{z^2}{c^2}=1, \quad y=-a, \quad x^2+z^2=c^2,$$

we eliminate c_2 between

$$\frac{\{x+c_2(a-y)\}^2}{b^2}+\frac{z^2}{c^2}=1, \quad \{x-c_2(a+y)\}^2+z^2=c^2.$$

Solving for c_2 from each, we have

$$\frac{\frac{b}{c}\sqrt{(c^2-z^2)}-x}{a-y}=\frac{x-\sqrt{(c^2-z^2)}}{a+y}.$$

The result is apparently of the eighth degree, but is resolvable into two conoids distinguished by giving the radicals the same or opposite signs in the last equation.

394. We have now seen that when the equation of a family of surfaces contains a number of arbitrary functions of the same quantity, it is convenient, in forming the partial differential equation, to substitute for the equation of the surface, the two equations of the generating curve. It is easy to see then that this process is equally applicable when the family of surfaces cannot be expressed by a single functional equation. The arbitrary functions which enter into the equations (Art. 387) are all functions of the same quantity, though the expression of that quantity in terms of the co-ordinates is unknown. If then

differentiating that quantity gives $dy = mdx$, we can eliminate the unknown quantity m, between the total differentials of the two equations of the generating curve, and so obtain the partial differential equation required. In practice it is convenient to choose for one of the equations of the generating curve, its projection on the plane xy.

For example, let it be required to find the general equation of ruled surfaces; that is to say, of surfaces generated by the motion of a right line. The equations of the generating line are $z = c_1x + c_3$, $y = c_2x + c_4$, and the family of surfaces is expressed by substituting for c_2, c_3, c_4 arbitrary functions of c_1. Differentiating, we have $p + mq = c_1$, $m = c_2$. Differentiating the first of these equations, m being proved to be constant by the second, we have $r + 2sm + tm^2 = 0$. As this equation still includes m or c_2, the expression for which, in terms of the co-ordinates is unknown, we must differentiate again, when we have $\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = 0$, where α, β, γ, δ are the third differential coefficients. Eliminating m between the cubic and quadratic just found, we have the required partial differential equation. It evidently resolves itself into the two linear equations of the third order got by substituting in turn for m in the cubic the two roots of the quadratic.

This equation might be got geometrically by expressing that the tangent planes at three consecutive points on a generator pass through that generator. The equation $dx = pdx + qdy$ is a relation between 1, p, q, which are proportional to the direction-cosines of a tangent plane, while dx, dy, dz are proportional to the direction-cosines of any line in that plane passing through the point of contact. If then we pass to a second tangent plane, through a consecutive point on the same line, we are to make p, q vary while the mutual ratios of dx, dy, dz remain constant. This gives $rdx^2 + 2sdxdy + tdy^2 = 0$. To pass to a third tangent plane, we differentiate again, regarding $dx : dy$ constant; and thus have $\alpha dx^3 + 3\beta dx^2dy + 3\gamma dxdy^2 + \delta dy^3 = 0$. Eliminating $dx : dy$ between the last two equations, we have the same equation as before.

The first integrals of this equation are found, as explained (Art. 388), by omitting the last equation and eliminating all but one of the constants. Thus we have the equation $p + mq = c_1$, from which it appears that one of the integrals is $p + mq = \phi(m)$, where m is one of the roots of $r + 2sm + tm^2 = 0$. The other two first integrals are

$$y - mx = \psi(m), \text{ and } z - px - mqx = \chi(m).$$

The three second integrals are got by eliminating m from any pair of these equations.

395. *Envelopes.* If the equation of a surface include n parameters connected by $n-1$ relations, we can in terms of any one express all the rest, and throw the equation into the form

$$z = F\{x, y, c, \phi(c), \psi(c), \text{\&c.}\}.$$

Eliminating c between this equation and $\frac{dF}{dc} = 0$, we find the envelope of all the surfaces obtained by giving different values to c. The envelopes so found are said to be of the same family as long as the form of the function F remains the same, no matter how the forms of the functions ϕ, ψ, &c. vary. The curve of intersection of the given surface with $\frac{dF}{dc}$ is the *characteristic* (see p. 232) or line of intersection of two consecutive surfaces of the system. Considering the characteristic as a moveable curve from the two equations of which c is to be eliminated, it is evident that the problem of envelopes is included in that discussed, Art. 387, &c. If the function F contain n arbitrary functions ϕ, ψ, &c., then since $\frac{dF}{dc}$ contains ϕ', ψ', &c., it would seem, according to the theory previously explained, that the partial differential equation of the family ought to be of the $2n^{\text{th}}$ order. But on examining the manner in which these functions enter, it is easy to see that the order reduces to the n^{th}. In fact, differentiating the equation $z = F$, we get

$$p = F_1 + \frac{dF}{dc}c_1, \quad q = F_2 + \frac{dF}{dc}c_2,$$

but since $\frac{dF}{dc} = 0$, we have $p = F_1$, $q = F_2$, where, since F_1 and F_2

are the differentials on the supposition that c is constant, these quantities only contain the original functions ϕ, ψ and not the derived ϕ', ψ'. From this pair of equations we can form another, as in Art. 394, and so on, until we come to the n^{th} order, when, as easily appears from what follows, we have equations enough to eliminate all the parameters.

396. We need not consider the case when the given equation contains but one parameter, since the elimination of this between the equation and its differential gives rise to the equation of a definite surface and not of a family of surfaces. Let the equation then contain two parameters a, b, connected by an equation giving b as a function of a, then between the three equations $z = F$, $p = F_1$, $q = F_2$, we can eliminate a, b, and the form of the result is evidently $f(x, y, z, p, q) = 0$.

For example, let us examine the envelope of a sphere of fixed radius, whose centre moves along any plane curve in the plane of xy. This is a particular case of the general class of tubular surfaces which we shall consider presently.

Now the equation of such a sphere being

$$(x-\alpha)^2 + (y-\beta)^2 + z^2 = r^2,$$

and the conditions of the problem assigning a locus along which the point $\alpha\beta$ is to move, and therefore determining β in terms of α, the equation of the envelope is got by eliminating α between

$$(x-\alpha)^2 + \{y-\phi(\alpha)\}^2 + z^2 = r^2, \quad (x-\alpha) + \{y-\phi(\alpha)\}\,\phi'(\alpha) = 0.$$

Since the elimination cannot be effected until the form of the function ϕ is assigned, the family of surfaces can only be expressed by the combination of two equations just written. We might also obtain these equations by expressing that the surface is generated by a fixed circle, which moves so that its plane shall be always perpendicular to the path along which its centre moves. For the equation of the tangent to the locus of $\alpha\beta$ is

$$y-\beta = \frac{d\beta}{d\alpha}(x-\alpha) \text{ or } y-\phi(\alpha) = \phi'(\alpha)(x-\alpha).$$

And the plane perpendicular to this is

$$(x-\alpha)+\phi'\alpha\{y-\phi(\alpha)\}=0,$$

as already obtained. To obtain the partial differential equation, differentiate the equation of the sphere, regarding α, β as constant, when we have $x-\alpha+pz=0$, $y-\beta+qz=0$. Solving for $x-\alpha$, $y-\beta$ and substituting in the equation of the sphere, the required equation is

$$z^2(1+p^2+q^2)=r^2.$$

We might have at once obtained this equation as the geometrical expression of the fact that the length of the normal is constant and equal to r, as it obviously is.

397. Before proceeding further we wish to show how the arbitrary functions which occur in the equation of a family of envelopes can be determined by the conditions that the surface in question passes through given curves. The tangent line to one of the given curves at any point of course lies in the tangent plane to the required surface; but since the enveloping surface has at any point the same tangent plane as the enveloped surface which passes through that point, it follows that each of the given curves at every point of it touches the enveloped surface which passes through that point. If then the equation of the enveloped surface be

$$z=F(x, y, c_1, c_2 \ldots c_n),$$

the envelope of this surface can be made to pass through $n-1$ given curves; for by expressing that the surface whose equation has been just written touches each of the given curves, we obtain $n-1$ relations between the constants c_1, c_2, &c., which combined with the two equations of the characteristic enable us to eliminate these constants. For example, the family of surfaces discussed in the last article contains but two constants and one arbitrary function, and can therefore be made to pass through one given curve. Let it then be required to find an envelope of the sphere

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2,$$

which shall pass through the right line $x=mz$, $y=0$. The points of intersection of this line with the sphere being given by the quadratic

$$(mz-\alpha)^2+\beta^2+z^2=r^2, \text{ or } (1+m^2)z^2-2mz\alpha+\alpha^2+\beta^2-r^2=0,$$

the condition that the line should touch the sphere is

$$(1+m^2)(\alpha^2+\beta^2-r^2)=m^2\alpha^2.$$

We see thus that the locus of the centres of spheres touching the given line is an ellipse. The envelope required then is a kind of elliptical anchor ring, whose equation is got by eliminating α, β between

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2, \quad (1+m^2)(\alpha^2+\beta^2-r^2)=m^2\alpha^2,$$

$$(x-\alpha)\,d\alpha+(y-\beta)\,d\beta=0, \quad \alpha d\alpha+(1+m^2)\,\beta d\beta=0,$$

from which last two equations we have

$$(1+m^2)\,\beta\,(x-\alpha)=\alpha\,(y-\beta).$$

The result is a surface of the eighth degree.

398. Again, let it be required to determine the arbitrary function so that the envelope surface may also envelope a given surface. At any point of contact of the required surface with the fixed surface $z=f(x, y)$, the moveable surface $z=F(x, y, c_1, c_2, \&c.)$ which passes through that point, has also the same tangent plane as the fixed surface. The values then of p and q derived from the equations of the fixed surface and of the moveable surface must be the same. Thus we have $f_1=F_1$, $f_2=F_2$, and if between these equations and the two equations $z=F$, $z=f$, which are satisfied for the point of contact, we eliminate x, y, z, the result will give a relation between the parameters. The envelope may thus be made to envelope as many fixed surfaces as there are arbitrary functions in the equation. Thus, for example, let it be required to determine a tubular surface of the kind discussed (Art. 397), which shall touch the sphere $x^2+y^2+z^2=R^2$. This surface must then touch $(x-\alpha)^2+(y-\beta^2)+z^2=r^2$. We have therefore $\dfrac{x}{z}=\dfrac{x-\alpha}{z}$, $\dfrac{y}{z}=\dfrac{y-\beta}{z}$; conditions which imply $z=0$, $\dfrac{x}{y}=\dfrac{x-\alpha}{y-\beta}$ or $\beta x=\alpha y$. Eliminating x and y by the help of

these equations, between the equation of the fixed and moveable sphere, we get $4(\alpha^2+\beta^2)R^2=(R^2-r^2+\alpha^2+\beta^2)^2$. This gives a quadratic for $\alpha^2+\beta^2$, whose roots are $(R\pm r)^2$; showing that the centre of the moveable sphere moves on one or other of two circles, the radius being either $R\pm r$. The surface required is therefore one or other of two anchor rings, the opening of the rings corresponding to the values just assigned.

399. We add one or two more examples of families of envelopes whose equations include but one arbitrary function. To find the envelope of a right cone whose axis is parallel to the axis of z, and whose vertex moves along any assigned curve in the plane of xy. Let the equation of the cone in its original position be $z^2=m^2(x^2+y^2)$; then if the vertex be moved to the point α, β, the equation of the cone becomes $z^2=m^2\{(x-\alpha)^2+(y-\beta)^2\}$, and if we are given a curve along which the vertex moves, β is given in terms of α. Differentiating we have $pz=m^2(x-\alpha)$, $qz=m^2(y-\beta)$; and eliminating we have $p^2+q^2=m^2$. This equation expresses that the tangent plane to the surface makes a constant angle with the plane of xy, as is evident from the mode of generation. It can easily be deduced hence that the area of any portion of the surface is in a constant ratio to its projection on the plane of xy.

400. The families of surfaces, considered (Arts. 396, 399), are both included in the following: "To find the envelope of a surface of any form which moves without rotation, its motion being directed by a curve along which any given point of the surface moves." Let the equation of the surface in its original position be $z=F(x, y)$, then if it be moved without turning so that the point originally at the origin shall pass to the position $\alpha\beta\gamma$, the equation of the surface will evidently be $z-\gamma=F(x-\alpha, y-\beta)$. If we are given a curve along which the point $\alpha\beta\gamma$ is to move, we can express α, β in terms of γ, and the problem is one of the class to be considered in the next article, where the equation of the envelope includes two arbitrary functions. Let it be given however that the directing

curve *is drawn on a certain known surface*, then, of the two equations of the directing curve, one is known and only one arbitrary, so that the equation of the envelope includes but one arbitrary function. Thus if we assume β an arbitrary function of α, the equation of the fixed surface gives γ as a known function of α, β. It is easy to see how to find the partial differential equation in this case. Between the three equations

$$z-\gamma=F(x-\alpha, y-\beta),\ p=F_1(x-\alpha, y-\beta),\ q=F_2(x-\alpha, y-\beta),$$

solve for $x-\alpha$, $y-\beta$, $z-\gamma$, when we find

$$x-\alpha=f(p, q),\ y-\beta={}'f(p, q),\ z-\gamma={}''f(p, q).$$

If then the equation of the surface along which $\alpha\beta\gamma$ is to move be $\Gamma(\alpha, \beta, \gamma)=0$, the required partial differential equation is

$$\Gamma\{x-f(p, q),\ y-{}'f(p, q),\ z-{}''f(p, q)\}=0.$$

The three functions f, $'f$, $''f$, are evidently connected by the relation $d''f=pdf+qd'f$.

It is easy to see that the partial differential equation just found is the expression of the fact that the tangent plane at any point on the envelope, is parallel to that at the corresponding point on the original surface.

Ex. To find the partial differential equation of the envelope of a sphere of constant radius whose centre moves along any curve traced on a fixed equal sphere

$$x^2+y^2+z^2=r^2.$$

The equation of the moveable sphere is $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2$, whence

$$x-\alpha+p(z-\gamma)=0,\quad y-\beta+q(z-\gamma)=0,$$

and we have

$$x-\alpha=\frac{-pr}{(1+p^2+q^2)^{\frac{1}{2}}},\quad y-\beta=\frac{-qr}{(1+p^2+q^2)^{\frac{1}{2}}},\quad z-\gamma=\frac{r}{(1+p^2+q^2)^{\frac{1}{2}}}.$$

If we write $1+p^2+q^2=\rho^2$ it is easy to see, by actual differentiation, that the relation is fulfilled

$$d\frac{1}{\rho}=-pd\left(\frac{p}{\rho}\right)-qd\left(\frac{q}{\rho}\right).$$

The partial differential equation is

$$(x\rho+pr)^2+(y\rho+qr)^2+(z\rho-r)^2=\rho^2r^2,$$

or

$$(x^2+y^2+z^2-r^2)(1+p^2+q^2)^{\frac{1}{2}}+2(px+qy-z)r=0.$$

401. We now proceed to investigate the form of the partial differential equation of the envelope, when the equation of the moveable surface contains three constants connected by two relations. If the equation of the surface be $z = F(x, y, a, b, c)$, then we have $p = F_1$, $q = F_2$. Differentiating again, as in Art. 389, we have

$$r + sm = F_{11} + mF_{12},\quad s + tm = F_{12} + mF_{22};$$

and eliminating m, the required equation* is

$$(r - F_{11})(t - F_{22}) = (s - F_{12})^2.$$

The functions F_{11}, F_{12}, F_{22} contain a, b, c, for which we are to substitute their values in terms of p, q, x, y, z derived from solving the preceding three equations, when we obtain an equation of the form

$$Rr + 2Ss + Tt + U(rt - s^2) = V,$$

where R, S, T, U, V are connected by the relation

$$RT + UV = S^2.$$

402. The following examples are among the most important of the cases where the equation includes three parameters.

Developable Surfaces. These are the envelope of the plane $z = ax + by + c$, where for b and c we may write $\phi(a)$ and $\psi(a)$. Differentiating we have $p = a$, $q = b$, whence $q = \phi(p)$. Any surface therefore is a developable surface if p and q are connected by a relation independent of x, y, z. Thus the family (Art. 399) for which $p^2 + q^2 = m^2$, is a family of developable surfaces. We have also $z - px - qy = \psi(p)$, which is the other first integral of the final differential equation. This last is got by differentiating again the equations $p = a$, $q = b$, when we have $r + sm = 0$, $s + tm = 0$, and eliminating m, $rt - s^2 = 0$, which is the required equation.

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* I owe to Professor Boole my knowledge of the fact that when the equation of the moveable surface contains three parameters, the partial differential equation is of the form stated above. He has kindly allowed me to consult, previous to its publication, a memoir of his in which this theorem is given.

By comparing Arts. 264, 281 it appears that the condition $rt = s^2$ is satisfied at every parabolic point on a surface. The same thing may be shewn directly by transforming the equation $rt - s^2 = 0$ into a function of the differential coefficients of U, by the help of the relations

$$U_1 + pU_3 = 0, \quad U_2 + qU_3 = 0,$$

$$U_{11} + 2U_{13}p + U_{33}p^2 = -rU_3; \quad U_{12} + pU_{23} + qU_{13} + pqU_{33} = -sU_3;$$

$$U_{22} + 2U_{23}q + U_{33}q^2 = -tU_3;$$

when the equation $rt - s^2$ becomes identical with the equation of the Hessian. We see now then that every point on a developable is a parabolic point, as is otherwise evident, for since (Art. 298) the tangent plane at any point meets the surface in two coincident right lines, the two inflexional tangents at that point coincide. The Hessian of a developable must therefore always contain the equation of the surface itself as a factor. The Hessian of any surface being of the degree $4n - 8$, that of a developable consists of the surface itself, and a surface of $3n - 8$ degree which we shall call the Pro-Hessian. We may return to this subject hereafter.

403. *Tubular Surfaces.* Let it be required to find the differential equation of the envelope of a sphere of constant radius, whose centre moves on any curve. We have, as in Art. 400,

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = R^2,$$

$$x - \alpha + p(z - \gamma) = 0, \quad y - \beta + q(z - \gamma) = 0,$$

whence

$$1 + p^2 + (z - \gamma) r + m\{pq + (z - \gamma) s\} = 0,$$

$$pq + (z - \gamma) s + m\{1 + q^2 + (z - \gamma) t\} = 0.$$

And therefore

$$\{1 + p^2 + (z - \gamma) r\}\{1 + q^2 + (z - \gamma) t\} = \{pq + (z - \gamma) s\}^2.$$

Substituting for $z - \gamma$ its value $\dfrac{R}{(1 + p^2 + q^2)^{\frac{1}{2}}}$ (Art. 400) this becomes

$$R^2(rt - s^2) - R\{(1+q^2)r - 2pqs + (1+p^2)t\}\sqrt{(1+p^2+q^2)} + (1+p^2+q^2)^2 = 0,$$

which denotes, Art. 281, that at any point on the required envelope one of the two principal radii of curvature is equal to R as is geometrically evident.

404. We shall briefly show what the form of the differential equation is when the equation of the surface whose envelope is sought contains four constants. We have, as before, in addition to the equation of the surface the three equations $p = F_1$, $q = F_2$, $(r - F_{11})(t - F_{22}) = (s - F_{12})^2$. Let us, for shortness, write the last equation $\rho\tau = \sigma^2$, and let us write $\alpha - F_{111} = A$, $\beta - F_{112} = B$, $\gamma - F_{122} = C$, $\delta - F_{222} = D$; then, differentiating $\rho\tau = \sigma^2$, we have

$$(A + Bm)\,\tau + (C + Dm)\,\rho - 2\,(B + Cm)\,\sigma = 0.$$

Substituting for m from the equation $\sigma + \tau m = 0$, and remembering that $\rho\tau = \sigma^2$, we have

$$A\tau^3 - 3B\sigma\tau^2 + 3\,C\sigma^2\tau - D\sigma^3 = 0,$$

in which equation we are to substitute for the parameters implicitly involved in it, their values derived from the preceding equations. The equation is therefore of the form

$$\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = U,$$

where m and U are functions of x, y, z, p, q, r, s, t. In like manner we can form the differential equation when the equation of the moveable surface includes a greater number of parameters.

405. Having in the preceding articles explained how partial differential equations are formed, we shall next show how from a given partial differential equation can be derived another differential equation satisfied by every characteristic of the family of surfaces to which the given equation belongs (see Monge, p. 53). In the first place, let the given equation be of the first order; that is to say, of the form $f(x, y, z, p, q) = 0$. Now if this equation belong to the envelope of a moveable surface, it will be satisfied not only by the envelope but also by the moveable surface in any of its positions. This follows from the fact that the envelope touches the moveable surface, and therefore that at the point of contact x, y, z, p, q are the same for both. Now if x, y, z be the

co-ordinates of any point on the characteristic, since such a point is the intersection of two consecutive positions of the moveable surface, the equation $f(x, y, z, p, q) = 0$ will be satisfied by these values of x, y, z, whether p and q have the values derived from one position of the moveable surface or from the next consecutive. Consequently, if we differentiate the given equation, regarding p and q as alone variable, then the points of the characteristic must satisfy the equation

$$Pdp + Qdq = 0.$$

Or we might have stated the matter as follows: Let the equation of the moveable surface be $z = F(x, y, \alpha)$, where the constants have all been expressed as functions of a single parameter α. Then (Art. 395) we have $p = F_1(x, y, \alpha)$, $q = F_2(x, y, \alpha)$, which values of p and q may be substituted in the given equation. Now the characteristic is expressed by combining with the given equation its differential with respect to α: and α only enters into the given equation in consequence of its entering into the values for p and q. Hence we have, as before, $P\frac{dp}{d\alpha} + Q\frac{dq}{d\alpha} = 0$.

Now since the tangent line to the characteristic at any point of it, lies in the tangent plane to either of the surfaces which intersect in that point, the equation $dz = pdx + qdy$ is satisfied, whether p and q have the values derived from one position of the moveable surface or from the next consecutive. We have therefore $\frac{dp}{d\alpha}dx + \frac{dq}{d\alpha}dy = 0$. And combining this equation with that previously found, we obtain the differential equation of the characteristic $Pdy - Qdx = 0$.

Thus if the given equation be of the form $Pp + Qq = R$, the characteristic satisfies the equation $Pdy - Qdx = 0$, from which equation combined with the given equation and with $dz = pdx + qdy$, can be deduced $Pdz = Rdx$, $Qdz = Rdy$. The reader is aware (see Boole's *Differential Equations*, p. 322) of the use made of those equations in integrating this class of equations. In fact, if the above system of simultaneous equations integrated give $u = c_1$, $v = c_2$, these are the equations of the characteristic, or generating curve, in any of its positions, while in order that v may be constant whenever u is constant, we must have $u = \phi(v)$.

Ex. Let the equation be that considered (Art. 396), viz. $z^2(1+p^2+q^2)=r^2$, then any characteristic satisfies the equation $pdy = qdx$, which indicates (Art. 370) that the characteristic is always a line of greatest slope on the surface, as is geometrically evident.

406. The equation just found for the characteristic generally includes p and q, but we can eliminate these quantities by combining with the equation just found, the given partial differential equation and the equation $dz = pdx + qdy$. Thus, in the last example, from the equations $z^2(1+p^2+q^2) = r^2$, $qdx = pdy$, we derive

$$z^2(dx^2 + dy^2 + dz^2) = r^2(dx^2 + dy^2).$$

The reader is aware that there are two classes of differential equations of the first order, one derived from the equation of a single surface, as, for instance, by the elimination of any constant from an equation $U=0$, and its differential

$$U_1 dx + U_2 dy + U_3 dz = 0.$$

An equation of this class expresses a relation between the direction-cosines of every tangent line drawn at any point on the surface. The other class is obtained by combining the equations of two surfaces, as, for instance, by eliminating three constants between the equations $U=0$, $V=0$ and their differentials. An equation of this class expresses a relation satisfied by the direction-cosines of the tangent to any of the curves which the system U, V represents for any value of the constants. The equations now under consideration belong to the latter class. Thus the geometrical meaning of the equation chosen for the example is that the tangent to any of the curves denoted by it, makes with the plane of xy an angle whose cosine is $\frac{z}{r}$. This property is true of every circle in a vertical plane whose radius is r; and the equation might be obtained by eliminating the constants α, β, m, between the equations

$$(x-\alpha)^2 + (y-\beta)^2 + z^2 = r^2, \quad x - \alpha + m(y-\beta) = 0.$$

407. The differential equation found, as in the last article, is not only true for every characteristic of a family of surfaces, but since each characteristic touches the cuspidal edge of the surface generated, the ratios $dx : dy :: dz$ are the same for any characteristic and the corresponding cuspidal edge; and consequently the equation now found is satisfied by the cuspidal edge of every surface of the family under consideration. Thus in the example chosen, the geometrical property expressed by the differential equation not only is true for a circle in a vertical plane, but remains true if the circle be wrapped on any vertical cylinder; and the cuspidal edge of the given family of surfaces always belongs to the family of curves thus generated.

Precisely as a partial differential equation in p, q (expressing as it does a relation between the direction-cosines of the tangent plane), is true as well for the envelope as for the particular surfaces enveloped; so the total differential equations here considered are true both for the cuspidal edge and the series of characteristics which that edge touches. The same thing may be stated otherwise as follows: the system of equations $U=0$, $\frac{dU}{d\alpha}=0$ which, when α is regarded as constant, represents the characteristic, represents the cuspidal edge when α is an unknown function of the variables to be eliminated by means of the equation $\frac{d^2U}{d\alpha^2}=0$. But evidently the equations $U=0$, $\frac{dU}{d\alpha}=0$ have the same differentials when α is considered as variable, subject to this condition, as if α were constant.

Thus, in the example of the last article, if in the equations $(x-\alpha)^2+(y-\beta)^2+z^2=r^2$, $(x-\alpha)+m(y-\beta)=0$, we write $\beta=\phi(\alpha)$, $m=\phi'(\alpha)$, and combine with these the equation $1+\phi'(\alpha)^2=(y-\beta)\,\phi''(\alpha)$, the differentials of the first and second equations are the same when α is variable in virtue of the third equation, as if it were constant; and therefore the differential equation obtained by eliminating α, β, m between the first two equations and their differentials on the supposition that these quantities are constant, holds equally when they vary according to the rules here laid down. And we shall obtain the equations of a curve satisfying this differential equation by giving any form we please to $\phi(\alpha)$ and then eliminating α between the equations

$$(x-\alpha)^2 + \{y-\phi(\alpha)\}^2 + z^2 = r^2, \quad x-\alpha+\phi'(\alpha)\{y-\phi(\alpha)\}=0,$$
$$1+\{\phi'(\alpha)\}^2 = \{y-\phi(\alpha)\}\,\phi''(\alpha).*$$

408. In like manner can be found the differential equation of the characteristic, the given equation being of the second order (see Monge, p. 74). In this case we can have two consecutive surfaces, satisfying the given differential equation, and touching each other all along their line of intersection. For instance, if we had a surface generated by a curve moving so as to meet two fixed directing curves, we might conceive a new surface generated by the same curve meeting two new directing curves, and if these latter directing curves touch the former at the points where the generating curve meets them, it is evident that the two surfaces touch along this line. In the case supposed then the two surfaces have x, y, z, p, q common along their line of intersection and can differ only with regard to r, s, t. Differentiate then the given differential equation considering these quantities alone variable, and let

* It is convenient to insert here a remark made by Mr. M. Roberts, viz. that if in the equation of any surface we substitute for x, $x+\lambda dx$, for y, $y+\lambda dy$, for z, $z+\lambda dz$, and then form the discriminant with respect to λ, the result will be the differential equation of the cuspidal edge of any developable enveloping the given surface. In fact it is evident (see Art. 246) that the discriminant expresses the condition that the tangent to the curve represented by it touches the given surface. Thus the general equation of the cuspidal edge of developables circumscribing a sphere is

$$(x^2+y^2+z^2-a^2)(dx^2+dy^2+dz^2) = (xdx+ydy+zdz)^2,$$

or $$(ydz-zdy)^2+(zdx-xdz)^2+(xdy-ydx)^2 = a^2(dx^2+dy^2+dz^2).$$

In the latter form it is evident that the same equation is satisfied by a geodesic traced on any cone whose vertex is the origin. For if the cone be developed into a plane, the geodesic will become a right line, and if the distance of that line from the origin be a, then the area of the triangle formed by joining any element ds to the origin is half ads, but this is evidently the property expressed by the preceding equation.

the result be $Rdr + Sds + Tdt = 0$. But since p and q are constant along this line, we have $drdx + dsdy = 0$, $dsdx + dtdy = 0$. Eliminating then dr, ds, dt, the required equation for the characteristic is

$$Rdy^2 - Sdxdy + Tdy^2 = 0.$$

In the case of any of the equations of the second order, which we have already had, this equation would turn out a perfect square. When it does not so turn out, it breaks up into two factors, which, if rational, belong to two independent characteristics represented by separate equations; and if not, denote two branches of the same curve intersecting on the point of the surface which we are considering.

409. In fact when the motion of a surface is regulated by a single parameter (see Art. 290), the equation of its envelope, as we have seen, contains only functions of a single quantity, and the differential equation belongs to the simpler species just referred to. But if the motion of the surface be regulated by two parameters, its contact with its envelope being not a curve, but a point; then the equation of the envelope will in general contain functions of two quantities, and the differential equation will be of the more general form. As an illustration of the occurrence of the latter class of equations in geometrical investigations, we take the equation of the family of surfaces which has one set of its lines of curvature parallel to a fixed plane, $y = mx$. Putting $dy = mdx$ in the equation of Art. 280, the differential equation of the family is

$$m^2\{(1+q^2)s - pqt\} + m\{(1+q^2)r - (1+p^2)t\} - \{(1+p^2)s - pqr\} = 0.$$

As it does not enter into the plan of this treatise to treat of the integration of such equations, we refer to Monge, p. 161 for a very interesting discussion of this equation. Our object being only to show how such differential equations present themselves in geometry, we shall show that the preceding equation arises from the elimination of α, β between the following equation and its differentials with respect to α and β:

$$(x-\alpha)^2 + (y-\beta)^2 + \{z - \phi(\alpha + m\beta)\}^2 = \psi(\beta - m\alpha)^2.$$

Differentiating with respect to α and β, we have

$$(x-\alpha) + \quad (z-\phi)\,\phi' = m\psi'\psi,$$

$$(y-\beta) + m\,(z-\phi)\,\phi' = -\psi'\psi,$$

whence $\quad (x-\alpha) + m\,(y-\beta) + (1+m^2)\,(z-\phi)\,\phi' = 0.$

But we have also

$$(x-\alpha) + p\,(z-\phi) = 0, \quad (y-\beta) + q\,(z-\phi) = 0,$$

whence $\quad (x-\alpha) + m\,(y-\beta) + (p+mq)\,(z-\phi) = 0.$

And by comparison with the preceding equation, we have $p+mq = (1+m^2)\,\phi'(\alpha+m\beta)$. If then we call $\alpha+m\beta$, γ the problem is reduced to eliminate γ between the equations

$$x+my-\gamma+(p+mq)\,\{z-\phi(\gamma)\}=0, \quad p+mq=(1+m^2)\,\phi'(\gamma).$$

Differentiating with regard to x and y, we have

$$(1+p^2+mpq) + (r+ms)\,\{z-\phi(\gamma)\} - \{1+(p+mq)\,\phi'\}\,\gamma_1,$$

$$\{m\,(1+q^2)+pq\} + (s+mt)\,\{z-\phi(\gamma)\} - \{1+(p+mq)\,\phi'\}\,\gamma_2,$$

but from the second equation

$$r+ms : s+mt :: \gamma_1 : \gamma_2.$$

Hence the result is

$$(1+p^2+mpq)\,(s+mt) = \{m\,(1+q^2)+pq\}\,(r+ms),$$

as was to be proved.

RULED SURFACES.*

410. On account of the importance of ruled surfaces, we add some further details as to this family of surfaces.

The tangent plane at any point on a generator evidently contains that generator, which is one of the inflexional tangents (Art. 234) at that point. Each different point on the generator has a different tangent plane (Art. 107) which may be constructed as follows: We know that through a given point

* The theorems in this section are principally taken from M. Chasles's Memoir, Quetelet's *Correspondance*, t. XI., p. 50, and from Mr. Cayley's paper, *Cambridge and Dublin Mathematical Journal*, Vol. VII., p. 171.

can be drawn a line intersecting two given lines; namely, the intersection of the planes joining the given point to the given lines. Now consider three consecutive generators, and through any point A on one, draw a line meeting the other two. This line, passing through three consecutive points on the surface, will be the second inflexional tangent at A, and therefore the plane of this line and the generator at A is the tangent plane at A. In this construction it is supposed that two consecutive generators do not intersect, which ordinarily they will not do. There may be on the surface, however, singular generators which are intersected by a consecutive generator, and in this case the plane containing the two consecutive generators is a tangent plane at every point on the generator. In special cases also two consecutive generators may coincide, in which case the generator is a double line on the surface.

411. *The anharmonic ratio of four tangent planes passing through a generator is equal to that of their four points of contact.* Let three fixed lines A, B, C be intersected by four transversals in points $aa'a''a'''$, $bb'b''b'''$, $cc'c''c'''$. Then the anharmonic ratio $\{bb'b''b'''\} = \{cc'c''c'''\}$, since either measures the ratio of the four planes drawn through A and the four transversals. In like manner $\{cc'c''c'''\} = \{aa'a''a'''\}$ either measuring the ratio of the four planes through B (see Art. 112). Now let the three fixed lines be three consecutive generators of the ruled surface, then by the last article, the transversals meet any of these generators A in four points, the tangent planes at which are the planes containing A and the transversals. And by this article it has been proved that the anharmonic ratio of the four planes is equal to that of the points where the transversals meet A.

412. Given any generator of a ruled surface, we can describe a hyperboloid of one sheet, which shall have this generator in common with the ruled surface, and which shall also have the same tangent plane with that surface at every point of their common generator. For it is evident from the construction of Art. 410 that the tangent plane at every point

on a generator is fixed, when the two next consecutive generators are given, and consequently that if two ruled surfaces have three consecutive generators in common, they will touch all along the first of these generators. Now any three non-intersecting right lines determine a hyperboloid of one sheet (Art. 76); the hyperboloid then determined by any generator and the two next consecutive will touch the given surface as required.

In order to see the full bearing of the theorem here enunciated, let us suppose that the axis of z lies altogether in any surface of the n^{th} degree, then every term in its equation must contain either x or y; and that equation arranged according to the powers of x and y will be of the form

$$u_{n-1}x + v_{n-1}y + u_{n-2}x^2 + v_{n-2}xy + w_{n-2}y^2 + \&c. = 0,$$

where u_{n-1}, v_{n-1} denote functions of z of the $(n-1)^{\text{th}}$ degree, &c. Then (see Art. 107) the tangent plane at any point on the axis will be $u'_{n-1}x + v'_{n-1}y = 0$, where u'_{n-1} denotes the result of substituting in u_{n-1} the co-ordinates of that point. Conversely, it follows that any plane $y = mx$ touches the surface in $n-1$ points, which are determined by the equation $u_{n-1} + mv_{n-1} = 0$. If however u_{n-1}, v_{n-1} have a common factor u_p, so that the terms of the first degree in x and y may be written $u_p(u_{n-p-1}x + v_{n-p-1}y) = 0$, then the equation of the tangent plane will be $u'_{n-p-1}x + v'_{n-p-1}y = 0$, and evidently in this case any plane $y = mx$ will touch the surface only in $n-p-1$ points. It is easy to see that the points on the axis for which $u_p = 0$ are double points on the surface. Now what is asserted in the theorem of this article is, that when the axis of z is not an isolated right line on a surface, but one of a system of right lines by which the surface is generated, then the form of the equation will be

$$u_{n-2}(ux + vy) + \&c. = 0,$$

so that the tangent plane at any point on the axis will be the same as that of the hyperboloid $ux + vy$, viz. $u'x + v'y = 0$. And any plane $y = mx$ will touch the surface in but one point. The factor u_{n-2} indicates that there are on each generator $n-2$ points which are double points on the surface.

413. We can verify the theorem just stated, for an important class of ruled surfaces, viz., those any generator of which can be expressed by two equations of the form

$$at^m + bt^{m-1} + ct^{m-2} + \&c. = 0, \quad a't^n + b't^{n-1} + c't^{n-2} + \&c. = 0,$$

where a, a', b, b', &c. are linear functions of the co-ordinates, and t a variable parameter. Then the equation of the surface obtained by eliminating t between the equations of the generator (*Higher Algebra*, p. 34), may be written in the form of a determinant, the first row and first column of which are identical, viz., (ab'), (ac'), (ad'), &c. Now the line aa' is a generator, namely, that answering to $t = \infty$; and we have just proved that either a or a' will appear in every term both of the first row and of the first column. Since then every term in the expanded determinant contains a factor from the first row and a factor from the first column, the expanded determinant will be a function of, at least, the second degree in a and a', except that part of it which is multiplied by (ab'), the term common to the first row and first column. But that part of the equation which is only of the first degree in a and a' determines the tangent at any point of aa'; the ruled surface is therefore touched along that generator by the hyperboloid $ab' - ba' = 0$.

If a and b (or a' and b') represent the same plane, then the generator aa' intersects the next consecutive, and the plane a touches along its whole length. If we had $b = ka$, $b' = ka'$, the terms of the first degree in a and a' would vanish, and aa' would be a double line on the surface.

414. Returning to the theory of ruled surfaces in general, it is evident that any plane through a generator meets the surface, in that generator and in a curve of the $(n-1)^{\text{th}}$ degree meeting the generator in $n-1$ points. Each of these points being a double point in the curve of section is (Art. 233) in a certain sense a point of contact of the plane with the surface. But we have seen (Art. 412) that only one of them is properly a point of contact of the plane; the other $n-2$ are fixed points on the generator, not varying as the plane through it is

changed. They are the points where this generator meets other non-consecutive generators, and are points of a double curve on the surface. Thus then *a skew ruled surface in general has a double curve which is met by every generator in* $n-2$ *points.* It may of course happen that two or more of these $n-2$ points may coincide, and that the multiple curve on the surface may be of higher order than the second. In the case considered in the last article it can be proved (see Appendix on the Order of Systems of Equations) that the multiple curve is of the order $\frac{(m+n-1)(m+n-2)}{1.2}$, and that there are on it $\frac{(m+n-2)(m+n-3)(m+n-4)}{1.2.3}$ triple points.

A ruled surface having a double line will in general not have any cuspidal line unless the surface be a developable, and the section by any plane will therefore be a curve having double points but not cusps.

415. Consider now the cone whose vertex is any point, and which envelopes the surface. Since every plane through a generator touches the surface in some point, the tangent planes to the cone are the planes joining the series of generators to the vertex of the cone. The cone will, in general, not have any stationary tangent planes: for such a plane would arise when two consecutive generators lie in the same plane passing through the vertex of the cone. But it is only in special cases that a generator will be intersected by one consecutive; the number of planes through two consecutive generators is therefore finite; and hence one will, in general, not pass through an assumed point. The class of the cone, being equal to the number of tangent planes which can be drawn through any line through the vertex, is equal to the number of generators which can meet that line, that is to say, to the degree of the surface (see note, p. 124). We have proved now that the *class* of the cone is equal to the *degree* of a section of the surface; and that the former has no stationary tangent planes as the latter has no stationary, or cuspidal, points. The equations then which connect any three of the singularities

of a curve prove that the number of double tangent planes to the cone must be equal to the number of double points of a section of the surface; or in other words, that the number of planes containing two generators which can be drawn through an assumed point, is equal to the number of points of intersection of two generators which lie in an assumed plane.*

416. We shall illustrate the preceding theory by an enumeration of some of the singularities of the ruled surface generated by a line meeting three fixed directing curves, the degrees of which are m_1, m_2, m_3.†

The degree of the surface generated is equal to the number of generators which meet an assumed right line; it is therefore equal to the number of intersections of the curve m_1 with the ruled surface having for directing curves the curves m_2, m_3 and the assumed line; that is to say, it is m_1 times the degree of the latter surface. The degree of this again is, in like manner, m_2 times the degree of the ruled surface whose directing curves are two right lines and the curve m_3, while by a repetition of the same argument, the degree of this last is $2m_3$. It follows that the degree of the ruled surface when the generators are curves m_1, m_2, m_3, is $2m_1m_2m_3$.

The three directing curves are multiple lines on the surface, whose orders are respectively m_2m_3, m_3m_1, m_1m_2. For through any point on the first curve pass m_2m_3 generators, the intersections namely of the cones having this point for a common vertex, and resting on the curves m_2, m_3.

417. The order of the ruled surface being $2m_1m_2m_3$, it follows, from Art. 414, that any generator is intersected by $2m_1m_2m_3-2$ other generators. But we have seen that at the points where it meets the directing curves, it meets $(m_2m_3-1)+(m_3m_1-1)+(m_1m_2-1)$ other generators. Conse-

* These theorems are Mr. Cayley's. *Cambridge and Dublin Mathematical Journal*, Vol. VII., p. 171.

† I published a discussion of this surface, *Cambridge and Dublin Mathematical Journal*, Vol. VIII., p. 45.

quently it must meet $2m_1m_2m_3 - (m_2m_3 + m_3m_1 + m_1m_2) + 1$ generators, in points not on the directing curves. We shall establish this result independently by seeking the number of generators which can meet a given generator. Let us commence by determining the degree of the ruled surface whose directing curves are the curves m_1, m_2, and the given generator, which is a line resting on both. In the first place this right line is a multiple line of the order $m_1m_2 - 1$, since obviously, through any point of it can be drawn this number of lines (distinct from the given line itself) meeting the curves m_1, m_2. But the section of the surface by a plane through the given line, will be that line itself $(m_1m_2 - 1)$ times, together with the $(m_1 - 1)(m_2 - 1)$ generators, obtained by joining any of the points where the plane meets the curve m_1 to one of those where it meets the curve m_2. Thus then the degree of the section (and therefore of the surface) is

$$(m_1m_2 - 1) + (m_1 - 1)(m_2 - 1) = 2m_1m_2 - m_1 - m_2.$$

Multiplying this number by m_3, we get the number of points where this new ruled surface is met by the curve m_3. But amongst these will be reckoned $(m_1m_2 - 1)$ times the point where the given generator meets the curve m_3. Subtracting this number then, there remain $2m_1m_2m_3 - m_2m_3 - m_1m_3 - m_1m_2 + 1$ points of the curve m_3, through which can be drawn a line to meet the curves m_1, m_2, and the assumed generator. But this is in other words the thing to be proved.

418. The ruled surface will contain a certain number of double generators, those namely which meet one of the directing curves twice and the other two once. The number of such lines resting twice on the curve m_1 is proved by reasoning similar to that used before, to be m_2m_3 times the degree of the ruled surface generated by a right line resting twice on m_1 and also on an arbitrary line. Now if h_1 be the number of apparent double points of the curve m_1, that is to say, the number of lines which can be drawn through an assumed point to meet that curve twice, it is evident that the assumed right line will on this ruled surface be a multiple line of the

order h_1, and the section of the ruled surface by a plane through that line, will be that line h_1 times together with the $\frac{1}{2}m_1(m_1-1)$ lines joining any pair of the points where the plane cuts the curve m_1. The degree of this ruled surface will then be $h_1+\frac{1}{2}m_1(m-1)$, and the total number of double generators on the original ruled surface is

$$m_2m_3\{h_1+\tfrac{1}{2}m_1(m_1-1)\}+m_3m_1\{h_2+\tfrac{1}{2}m_2(m_2-1)\}+m_1m_2\{h_3+\tfrac{1}{2}m_3(m_3-1)\}.$$

I am unable to give the order of the double curve in general, but in the particular case where one of the directing curves is a right line, and the other two curves of the degree m_1, m_2, it is evident that the section by any plane through the directing right line consists of that right line m_1m_2 times together with m_1m_2 lines intersecting in $\frac{1}{2}m_1m_2(m_1-1)(m_2-1)$ points not on the directing curves. This latter therefore would appear to be in this case the order of the nodal curve, *unless* it intersect the directing line in a certain number of points, which, if so, must be added to the order of the curve. There are, of course, besides, double generators, as determined in the first part of this article.

It is easy to see, in like manner, that the surface generated by a right line resting twice on a curve m and on a right line, will have, besides its double generators, a double curve, whose order is, *at least*, $\frac{1}{8}m(m-1)(m-2)(m-3)$.

419. The degree of the ruled surface, as calculated by Art. 416, will admit of reduction if any pair of the directing curves have points in common. Thus if the curves m_2, m_3 have a point in common, it is evident that the cone whose vertex is this point, and base the curve m_1 will be included in the system, and that the order of the ruled surface proper will be reduced by m_1. And generally if the three pairs made out of the three directing curves have common respectively α, β, γ points, the order of the ruled surface will be reduced by $m_1\alpha+m_2\beta+m_3\gamma$.* Thus if the directing lines be two right

* My attention was called by Mr. Cayley to this reduction which takes place when the directing curves have points in common.

lines and a twisted cubic, the surface is in general of the sixth order, but if each of the lines intersect the cubic the order is only of the fourth. If each intersect it twice the surface is a quadric. If one intersect it twice and the other once, the surface is a skew surface of the third degree on which the former line is a double line.

Again, let the directing curves be any three plane sections of a hyperboloid of one sheet. According to the general theory the surface ought to be of the sixteenth order, and let us see how a reduction takes place. Each pair of directing curves have two points common; namely, the points in which the line of intersection of their planes meets the surface. And the complex surface of the sixteenth order consists of six cones of the second order, together with the original quadric reckoned twice. That it must be reckoned twice, appears from the fact that the four generators which can be drawn through any point on one of the directing curves, are two lines belonging to the cones, and *two* generators of the given hyperboloid.

In general, if we take as directing curves three plane sections of any ruled surface, the equation of the ruled surface generated will have, in addition to the cones and to the original surface, a factor denoting another ruled surface which passes through the given curves. For it will generally be possible to draw lines, meeting all three curves, which are not generators of the original surface.

420. Returning to the case of ruled surfaces in general; we know that a series of planes through any line and a series at right angles to them form a system in involution, the anharmonic ratio of any four being equal to that of their four conjugates. It follows then, from Art. 411, that the system formed by the points of contact of any plane, and of a plane at right angles to it, form a system in involution; or, in other words, the system of points where planes through any generator touch the surface, and where they are normal to the surface, form a system in involution. The centre of the system is the point where the plane which touches the surface at infinity, is normal to the surface; and by the known properties of in-

volution, the distances from this point of the points where any other plane touches and is normal, form a constant rectangle.

421. *The normals to any ruled surface along any generator, generate a hyperbolic paraboloid.* It is evident that they are all parallel to the same plane, namely the plane perpendicular to the generator. We may speak of the anharmonic ratio of four lines parallel to the same plane, meaning thereby that of four parallels to them through any point. Now in this sense the anharmonic ratio of four normals is equal to that of the four corresponding tangent planes, which (Art. 411) is equal to that of their points of contact, which again (Art. 419) is equal to that of the points where the normals meet the generator. But a system of lines parallel to a given plane and meeting a given line generates a hyperbolic paraboloid, if the anharmonic ratio of any four is equal to that of the four points where they meet the line. This proposition follows immediately from its converse, which we can easily establish.

The points where four generators of a hyperbolic paraboloid intersect a generator of the opposite kind, are the points of contact of the four tangent planes which contain these generators, and therefore the anharmonic ratio of the four points is equal to that of the four planes. But the latter ratio is measured by the four lines in which these planes are intersected by a plane parallel to the four generators, and these intersections are lines parallel to these generators.

422. The central points of the involution (Art. 419) are, it is easy to see, the points where each generator is nearest the next consecutive, that is to say, the point where each generator is intersected by the shortest distance between it and its next consecutive. The locus of the points on the generators of a ruled surface, where each is closest to the next consecutive, is called the *line of striction* of the surface. It may be remarked, in order to correct a not unnatural mistake (see *Lacroix*, Vol. III., p. 668), that the shortest distance between two consecutive generators is *not* an element of the

line of striction. In fact if Aa, Bb, Cc be three consecutive generators, ab the shortest distance between the two former, then $b'c$ the shortest distance between the second and third will in general meet Bb in a point b' distinct from b, and the element of the line of striction will be ab' and not ab.

Ex. 1. To find the line of striction of the hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z.$$

Any pair of generators may be expressed by the equations

$$\frac{x}{a} + \frac{y}{b} = \lambda z, \quad \frac{x}{a} - \frac{y}{b} = \frac{1}{\lambda},$$

$$\frac{x}{a} + \frac{y}{b} = \mu z, \quad \frac{x}{a} - \frac{y}{b} = \frac{1}{\mu}.$$

Both being parallel to the plane $\frac{x}{a} - \frac{y}{b}$, their shortest distance is perpendicular to this plane, and therefore lies in the plane

$$(a^2 + b^2)\left\{\frac{x}{a} + \frac{y}{b} - \mu z\right\} + (a^2 - b^2)\left\{\frac{x}{a} - \frac{y}{b} - \frac{1}{\mu}\right\},$$

which intersects the first generator in the point $z = \frac{a^2 - b^2}{a^2 + b^2}\frac{1}{\lambda\mu}$.

When the two generators approach to coincidence, we have for the co-ordinates of the point where either is intersected by their shortest distance

$$z = \frac{a^2 - b^2}{a^2 + b^2}\frac{1}{\lambda^2}, \quad \frac{x}{a} + \frac{y}{b} = \frac{a^2 - b^2}{a^2 + b^2}\frac{1}{\lambda},$$

and hence $(a^2 + b^2)\left(\frac{x}{a} + \frac{y}{b}\right) = (a^2 - b^2)\left(\frac{x}{a} - \frac{y}{b}\right)$, or $\frac{x}{a^3} + \frac{y}{b^3} = 0$.

The line of striction is therefore the parabola in which this plane cuts the surface. The same surface considered as generated by the lines of the other system has another line of striction lying in the plane

$$\frac{x}{a^3} - \frac{y}{b^3} = 0.$$

Ex. 2. To find the line of striction of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Ans. It is the intersection of the surface with

$$\frac{a^2A^2}{x^2} + \frac{b^2B^2}{y^2} = \frac{c^2C^2}{z^2},$$

where $$A = \frac{1}{b^2} + \frac{1}{c^2}, \quad B = \frac{1}{a^2} + \frac{1}{c^2}, \quad C = \frac{1}{b^2} - \frac{1}{a^2}.$$

CHAPTER XIII.

SURFACES DERIVED FROM QUADRICS.

THE WAVE SURFACE.

423. BEFORE proceeding to surfaces of the third degree, we think it more simple to treat of surfaces derived from quadrics, the theory of which is more closely connected with that explained in preceding chapters. The equation of the surface of centres has been already given (Art. 208), and we proceed now to define, and form the equation of, Fresnel's Wave Surface.*

If a perpendicular through the centre be erected to the plane of any central section of a quadric, and on it lengths be taken equal to the axes of the section, the locus of their extremities will be a surface of two sheets which is called the wave surface. Its equation is at once derived from Arts. 97, 98, where the lengths of the axes of any section are expressed in terms of the angles which a perpendicular to its plane makes with the axes of the surface. The same equation then expresses the relation which the length of a radius vector to the wave surface bears to the angles which it makes with the axes. The equation of the Wave Surface is therefore

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

where $r^2=x^2+y^2+z^2$. Or, multiplying out,

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)$$
$$-\{a^2x^2(b^2+c^2)+b^2y^2(c^2+a^2)+c^2z^2(a^2+b^2)\}+a^2b^2c^2=0.$$

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* See Fresnel, *Mémoires de l'Institut*, Vol. VII., p. 136, published 1827.

From the first form it appears at once that the intersection of the wave surface by a concentric sphere, is a sphero-conic.

424. The section by one of the principal planes (*e.g.* the plane z) breaks up into a circle and ellipse

$$(x^2+y^2-c^2)(a^2x^2+b^2y^2-a^2b^2).$$

This is also geometrically evident, since if we consider any section of the generating quadric, through the axis of z, one of the axes of that section is equal to c, while the other axis lies in the plane xy. If then we erect a perpendicular to the plane of section, and on it take portions equal to each of these axes, the extremities of one portion will trace out a circle whose radius is c, while the locus of the extremities of the other portion, will plainly be the principal section of the generating quadric, only turned round through 90°. In each of the principal planes the surface has four double points; namely, the intersection of the circle and ellipse just mentioned. If x', y' be the co-ordinates of one of these intersections, the tangent cone (Art. 239), at this double point, has for its equation

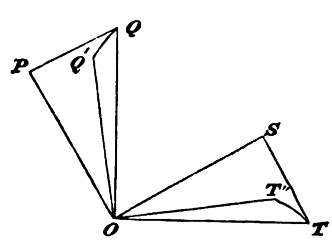
$$4(xx'+yy'-c^2)(a^2xx'+b^2yy'-a^2b^2)+z^2(a^2-c^2)(b^2-c^2)=0.$$

The generating quadric being supposed to be an ellipsoid, it is evident that in the case of the section by the plane z, the circle whose radius is c lies altogether within the ellipse whose axes are a, b: and in the case of the section by the plane x, the circle whose radius is a, lies altogether without the ellipse whose axes are b, c. Real double points occur only in the section by the plane y; they are evidently the points corresponding to the circular sections of the generating ellipsoid.

The section by the plane at infinity also breaks up into factors $x^2+y^2+z^2$, $a^2x^2+b^2y^2+c^2z^2$, and may therefore also be considered as an imaginary circle and ellipse, which in like manner give rise to four imaginary double points of the surface situated at infinity. Thus the surface has in all sixteen nodal points, only four of which are real.

425. The wave surface is one of a class of surfaces which may be called *apsidal surfaces.* Any surface being given, if we assume any point as pole, draw any section through that pole, and on the perpendicular through the pole to the plane of section, take lengths equal to the *apsidal* (that is to say, to the maximum or minimum) radii of that section; then the locus of the extremities of these perpendiculars is the apsidal surface derived from the given one. The equation of the apsidal surface may always be calculated, as in Art. 98. First form the equation of the cone whose vertex is the pole, and which passes through the intersection with the given surface of a sphere of radius r. Each edge of this cone is proved (as at Art. 98) to be an apsidal radius of the section of the surface by the tangent plane to the cone. If then we form the equation of the reciprocal cone, whose edges are perpendicular to the tangent planes to the first cone, we shall obtain all the points on the apsidal surface which correspond to the tangent planes of the assumed cone. And by considering r variable, in the equation of this latter cone, we have the equation of the apsidal surface.

426. If OQ be any radius vector to the generating surface, and OP the perpendicular to the tangent plane at the point Q, then OQ will be an apsidal radius of the section passing through OQ and through OR which is supposed to be perpendicular to the plane of the paper POQ. For the tangent plane at Q passes through PQ and is perpendicular to the plane of the paper; the tangent line to the section QOR lies in the tangent plane and is therefore also perpendicular to the plane of the paper. Since then OQ is perpendicular to the tangent line in the section QOR, it is an apsidal radius of that section.



It follows that OT, the radius of the apsidal surface corresponding to the point Q, lies in the plane POQ and is perpendicular and equal to OQ.

427. *The perpendicular to the tangent plane to the apsidal surface at T lies also in the plane POQ, and is perpendicular and equal to OP.**

Consider first a radius OT' of the apsidal surface, indefinitely near to OT, and lying in the plane TOR, perpendicular to the plane of the paper. Now OT' is by definition equal to an apsidal radius of the section of the original surface by a plane perpendicular to OT', and this plane must pass through OQ. Again an apsidal radius of a section is equal to the next consecutive radius. The apsidal radius therefore of a section passing through OQ, and indefinitely near the plane QOR, will be equal to OQ. It follows then that $OT = OT'$, and therefore that the tangent at T to the section TOR is perpendicular to OT, and therefore perpendicular to the plane of the paper. The perpendicular to the tangent plane at T must therefore lie in the plane of the paper, but this is the first part of the theorem which was to be proved.

Secondly, consider an indefinitely near radius OT'' in the plane of the paper; this will be equal to an apsidal radius of the section ROQ', where OQ' is indefinitely near to OQ. But, as before, this apsidal radius being indefinitely near to OQ' will be equal to it, and therefore OT'' will be equal as well as perpendicular to OQ'. The angle then $T''TO$ is equal to $Q'QO$, and therefore the perpendicular OS is equal and perpendicular to OP.

It follows from the symmetry of the construction that if a surface A is the apsidal of B, then conversely B is the apsidal of A.

428. *The polar reciprocal of an apsidal surface, with respect to the origin O, is the same as the apsidal of the reciprocal, with respect to O, of the given surface.*

For if we take on OP, OQ portions inversely proportional to them, we shall have Op, Oq, a radius vector and corresponding perpendicular on tangent plane of the reciprocal of

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* These theorems are due to Prof. MacCullagh, *Transactions of the Royal Irish Academy*, Vol. XVI.

the given surface. And if we take portions equal to these on the lines OS, OT which lie in their plane, and are respectively perpendicular to them, then by the last article we shall have a radius vector, and corresponding perpendicular on tangent plane, of the apsidal of the reciprocal. But these lengths being inversely as OS, OT are also a radius vector, and perpendicular on tangent plane of the reciprocal of the apsidal. The apsidal of the reciprocal is therefore the same as the reciprocal of the apsidal.

In particular, the reciprocal of the wave surface generated from any ellipsoid, is the wave surface generated from the reciprocal ellipsoid.

We might have otherwise seen that the reciprocal of a wave surface is a surface also of the fourth degree, for the reciprocal of a surface of the fourth degree is in general of the thirty-sixth degree (Art. 250); but it is proved, as for plane curves, that each double point on a surface reduces the degree of its reciprocal by two; and we have proved (Art. 424) that the wave surface has sixteen double points.

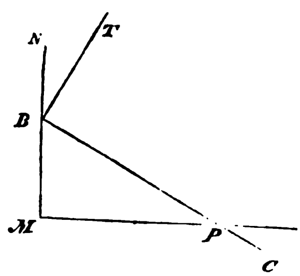
To a nodal point on any surface (which is a point through which can be drawn an infinity of tangent planes, touching a cone of the second degree) answers on the reciprocal surface a tangent plane, having an infinity of points of contact, lying in a conic. From knowing then that a wave surface has four real double points, and that the reciprocal of a wave surface is a wave surface, we infer that the wave surface has four tangent planes which touch all along a conic. We shall now show geometrically that this conic is a circle.*

429. It is convenient to premise the following lemmas:

LEMMA I. "If two lines passing though a fixed point, and at right angles to each other, move each in a fixed plane, the

* Sir W. R. Hamilton first showed that the wave surface has four nodes, the tangent planes at which envelope cones, and that it has four tangent planes which touch along circles, *Transactions of the Royal Irish Academy*, Vol. XVII, p. 132. Dr. Lloyd experimentally verified the optical theorems thence derived, *Ibid*, p. 145. The geometrical investigations which follow are due to Professor Mac Cullagh, p. 248.

plane containing the two lines envelopes a cone whose sections parallel to the fixed planes are parabolas." The plane of the paper is supposed to be parallel to one of the fixed planes, and the other fixed plane is supposed to pass through the line MN. The fixed point O in which the two lines intersect is supposed to be above the paper, P being the foot of the perpendicular from it on the plane of the paper. Now let OB be one position of the line which moves in the plane OMN, then the other line OA which is parallel to the plane of the paper being perpendicular to OB and to OP is perpendicular to the plane OBP. But the plane OAB intersects the plane of the paper in a line BT parallel to OA, and therefore perpendicular to BP. And the envelope of BT is evidently a parabola of which P is the focus and MN the tangent at the vertex.



LEMMA II. "If a line OC be drawn perpendicular to OAB, it will generate a cone whose circular sections are parallel to the fixed planes." (Ex. 4, p. 85). It is proved, as at p. 106, that the locus of C is the polar reciprocal, with respect to P, of the envelope of BT. The locus is therefore a circle passing through P.

LEMMA III. "If a central radius of a quadric moves in a fixed plane, the corresponding perpendicular on tangent plane also moves in a fixed plane." Namely, the plane perpendicular to the diameter conjugate to the first plane, to which the tangent plane must be parallel.

430. Suppose now (see figure, Art. 426) that the plane OQR (where OR is perpendicular to the plane of the paper) is a circular section of a quadric, then OT is the nodal radius of the wave surface, which remains the same while OQ moves in the plane of the circular sections; and we wish to find the cone generated by OS. But OS is perpendicular to OR which moves in the plane of the circular sections and to OP

which moves in a fixed plane by Lemma III., therefore OS generates a cone whose circular sections are parallel to the planes POR, QOR. Now T is a fixed point, and TS is parallel to the plane POR, therefore the locus of the point S is a circle.

The tangent cone at the node is evidently the reciprocal of the cone generated by OS, and is therefore a cone whose sections parallel to the same planes are parabolas.

Secondly, suppose the line OP to be of constant length, which will happen when the plane POR is a transverse section of one of the two right cylinders which circumscribe the ellipsoid, then the point S is fixed, and it is proved precisely as in the first part of this article that the locus of T is a circle.

431. The equations of p. 173 give immediately another form of the equation of the wave surface. It is evident thence, that if θ, θ' be the angles which any radius vector makes with the lines to the nodes, then the lengths of the radius vector are, for one sheet,

$$\frac{1}{\rho^2} = \frac{\cos^2 \frac{1}{2}(\theta - \theta')}{c^2} + \frac{\sin^2 \frac{1}{2}(\theta - \theta')}{a^2},$$

and for the other

$$\frac{1}{\rho'^2} = \frac{\cos^2 \frac{1}{2}(\theta + \theta')}{c^2} + \frac{\sin^2 \frac{1}{2}(\theta + \theta')}{a^2},$$

while
$$\frac{1}{\rho^2} - \frac{1}{\rho'^2} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin\theta \sin\theta'.$$

It follows hence also that the intersections of a wave surface with a series of concentric spheres, are a series of confocal sphero-conics. For in the preceding equations if ρ or ρ' be constant, we have $\theta \pm \theta'$ constant.

432. The equation of the wave surface has also been expressed as follows by Mr. W. Roberts in elliptic co-ordinates. The form of the equation

$$\frac{a^2x^2}{a^2 - r^2} + \frac{b^2y^2}{b^2 - r^2} + \frac{c^2z^2}{c^2 - r^2} = 0,$$

shows that the equation may be got by eliminating r^2 between the equations

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1, \text{ and } x^2+y^2+z^2=r^2.$$

Giving r^2 any series of constant values, the first equation denotes a series of confocal quadrics, the axis of z being the primary axis, and the axis of x the least. Since r^2 is always less than a^2 and greater than c^2, the equation always denotes a hyperboloid, which will be of one or of two sheets according as r^2 is greater or less than b^2. The intersections of the hyperboloids of one sheet with corresponding spheres generate one sheet of the wave surface, and those of two sheets the other.

Now if the surface denote a hyperboloid of one sheet, and if λ, μ, ν denote the primary axes of three confocal surfaces of the system now under consideration which pass through any point, then the equation gives us $r^2-c^2=\mu^2$, but (Art. 169)

$$r^2=\lambda^2+\mu^2+\nu^2-h^2-k^2,$$

whence the equation in elliptic co-ordinates is

$$\lambda^2+\nu^2=c^2+h^2+k^2=a^2+b^2-c^2.$$

In like manner the equation of the other sheet is

$$\lambda^2+\mu^2=a^2+b^2-c^2.$$

The general equation of the wave surface also implies $\mu^2+\nu^2=a^2+b^2-c^2$, but this denotes an imaginary locus.

Since, if λ is constant, μ is constant for one sheet and ν for the other, it follows that if through any point on the surface be drawn an ellipsoid of the same system, it will meet one sheet in a line of curvature of one system, and the other sheet in a line of the other system.

If the equations of two surfaces expressed in terms of λ, μ, ν, when differentiated give

$$Pd\lambda+Qd\mu+Rd\nu=0, \quad P'd\lambda+Q'd\mu+R'd\nu=0,$$

the condition that they should cut at right angles is (Art. 359)

$$\frac{PP'(\lambda^2-h^2)(\lambda^2-k^2)}{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}+\frac{QQ'(\mu^2-h^2)(k^2-\mu^2)}{(\lambda^2-\mu^2)(\mu^2-\nu^2)}+\frac{RR'(h^2-\nu^2)(k^2-\nu^2)}{(\lambda^2-\nu^2)(\mu^2-\nu^2)}=0,$$

which is satisfied if $P=0$, $Q=0$, $R'=0$. Hence any surface

ν = constant cuts at right angles any whose equation is of the form $\phi(\lambda, \mu) = 0$. The hyperboloid therefore, ν = constant, cuts at right angles one sheet of the wave surface, while it meets the other in a line of curvature on the hyperboloid.

433. *The plane of any radius vector of the wave surface and the corresponding perpendicular on the tangent plane, makes equal angles with the planes through the radius vector and the nodal lines.* For the first plane is perpendicular to OR (Art. 426) which is an axis of the section QOR of the generating ellipsoid, and the other two planes are perpendicular to the radii of that section whose lengths are b, the mean axis of the ellipsoid, and these two equal lines make equal angles with the axis. The planes are evidently at right angles to each other, which are drawn through any radius vector, and the perpendiculars on the tangent planes at the points where it meets the two sheets of the surface.

Reciprocating the theorem of this article we see that the plane through any line through the centre and through one of the points where planes perpendicular to that line touch the surface, makes equal angles with the planes through the same line and through perpendiculars from the centre on the planes of circular contact (Art. 430).

434. If the co-ordinates of any point on the generating ellipsoid be $x'y'z'$, and the primary axes of confocals through that point a', a''; then the squares of the axes of the section parallel to the tangent plane are $a^2 - a'^2$, $a^2 - a''^2$, which we shall call ρ^2, ρ'^2. These then give the two values of the radius vector of the wave surface, whose direction-cosines are $\frac{px'}{a^2}$, $\frac{py'}{b^2}$, $\frac{pz'}{c^2}$. We shall now calculate the length and the direction-cosines of the perpendicular on the tangent plane at either of the points where this radius vector meets the surface. It was proved (Art. 427) that the required perpendicular is equal and perpendicular to the perpendicular at the point where the ellipsoid is met by one of the axes of the section; and the direction-cosines of this axis are $\frac{p'x'}{a'^2}$, $\frac{p'y'}{b'^2}$, $\frac{p'z'}{c'^2}$. The

co-ordinates of its extremity are then these several cosines multiplied by ρ, and the direction-cosines of the corresponding perpendicular of the ellipsoid are

$$P\rho\,\frac{p'x'}{a^2a'^2},\quad P\rho\,\frac{p'y'}{b^2b'^2},\quad P\rho\,\frac{p'z'}{c^2c'^2},$$

where $$\frac{1}{P^2}=\rho^2p'^2\left\{\frac{x'^2}{a^4a'^4}+\frac{y'^2}{b^4b'^4}+\frac{z'^2}{c^4c'^4}\right\}.$$

Now if the quantity within the brackets be multiplied by $(a^2-a'^2)^2$, we see at once that it will become $\frac{1}{p^2}+\frac{1}{p'^2}$. Hence $\frac{1}{P^2}=\frac{p^2+p'^2}{p^2\rho^2}$; and $P^2=\frac{p^2\rho^2}{p^2+p'^2}$.

This then gives the length of the perpendicular on the tangent plane at the point on the wave surface which we are considering. Its direction-cosines are obtained from the consideration that it is perpendicular to the two lines whose direction-cosines are respectively

$$\frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2};\quad P\rho\,\frac{p'x'}{a^2a'^2},\ P\rho\,\frac{p'y'}{b^2b'^2},\ P\rho\,\frac{p'z'}{c^2c'^2}.$$

Forming by Art. 15 the direction-cosines of a line perpendicular to these two, we find, after a few reductions,

$$\frac{Px'}{p\rho}\left(1-\frac{p''^2}{a''^2}\right),\quad \frac{Py'}{p\rho}\left(1-\frac{p''^2}{b''^2}\right),\quad \frac{Pz'}{p\rho}\left(1-\frac{p''^2}{c''^2}\right).$$

In fact it is verified without difficulty that the line whose direction-cosines have been just written is perpendicular to the two preceding.

It follows hence also, that the equation of the tangent plane at the same point is

$$xx'\left(1-\frac{p''^2}{a''^2}\right)+yy'\left(1-\frac{p''^2}{b''^2}\right)+zz'\left(1-\frac{p''^2}{c''^2}\right)=p\rho.$$

In like manner the tangent plane at the other point where the same radius vector meets the surface is

$$xx'\left(1-\frac{p'^2}{a'^2}\right)+yy'\left(1-\frac{p'^2}{b'^2}\right)+zz'\left(1-\frac{p'^2}{c'^2}\right)=p\rho'.$$

435. If θ be the angle which the perpendicular on the tangent plane makes with the radius vector, we have $P=\rho\cos\theta$; but we have in the last article proved $P^2=\dfrac{p^2\rho^2}{p^2+p'^2}$. Hence $\cos^2\theta=\dfrac{p^2}{p^2+p'^2}$, $\tan^2\theta=\dfrac{p'^2}{p^2}$. This expression may be transformed by means of the values given for p and p' (Art. 173). We have therefore

$$p^2=\frac{a^2b^2c^2}{\rho^2\rho'^2},\quad p'^2=\frac{(a^2-\rho^2)(b^2-\rho^2)(c^2-\rho^2)}{\rho^2(\rho^2-\rho'^2)}.$$

Whence
$$\tan^2\theta=-\frac{\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho^2}{b^2}\right)\left(1-\frac{\rho^2}{c^2}\right)}{1-\frac{\rho^2}{\rho'^2}}.$$

In this form the expression is analogous to the value for the angle between the normal and central radius vector of a plane ellipse, viz.,

$$\tan^2\theta=-\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho^2}{b^2}\right).$$

In the case of the wave surface it is manifest that $\tan\theta$ vanishes only when $\rho=a$, b, or c, and becomes indeterminate when $\rho=\rho'=b$.

436. The expression $\tan\theta=\dfrac{p'}{p}$ leads to a construction for the perpendiculars on the tangent planes at the points where a given radius vector meets the two sheets of the surface. The perpendiculars must lie in one or other of two fixed planes (Arts. 433, 434), and if a plane be drawn perpendicular to the radius vector at a distance p, it is evident from the expression for $\tan\theta$, that p' is the distance to the radius vector from the point where the perpendicular on the tangent plane meets this plane. Thus we have the construction, "Draw a tangent plane to the generating ellipsoid perpendicular to the given radius vector, from its point of contact let fall perpendiculars on the two planes of Art. 433, then the lines joining to the centre the feet of these perpendiculars, are the perpendiculars required.

We obtain by reciprocation a similar construction, to determine the points where planes parallel to a given one touch the two sheets of the surface.

437. I have sometimes found it convenient to transform the equation of the surface, as at Art. 180, so as to make the radius vector to any point on the surface the axis of z, and the axes of the corresponding section of the generating ellipsoid the axes of x and y. We may write the equation of the surface in the form

$$(a^2x^2 + b^2y^2 + c^2z^2 - b^2c^2 - c^2a^2 - a^2b^2)(x^2 + y^2 + z^2) + a^2b^2c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 1\right) = 0.$$

Now $x^2 + y^2 + z^2$ remains unaltered by transformation, and we have given, Arts. 183, 184, the transformations of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad a^2x^2 + b^2y^2 + c^2z^2.$$

Consequently the transformed equation is

$$\{p^2z^2 + (p'^2 + \rho^2)x^2 + (p''^2 + \rho'^2)y^2 + 2pp'xz + 2pp''yz + 2p'p''xy\}$$
$$\times (x^2 + y^2 + z^2) - p^2z^2(\rho^2 + \rho'^2) - x^2(p^2\rho^2 + p'^2\rho'^2 + p''^2\rho^2 + \rho^2\rho'^2)$$
$$- y^2(p^2\rho'^2 + p'^2\rho'^2 + p''^2\rho^2 + \rho^2\rho'^2) - 2pp'\rho'^2xz - 2pp''\rho^2yz + p^2\rho^2\rho'^2 = 0.$$

In this transformation we have substituted for the quantity called γ^2 (Art. 183), its value derived from the equation

$$\frac{1}{\gamma^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{\rho^2} - \frac{1}{\rho'^2},$$

and have also used the identical equation

$$b^2c^2 + c^2a^2 + a^2b^2 = p^2(\rho^2 + \rho'^2) + p'^2\rho'^2 + p''^2\rho^2 + \rho^2\rho'^2.$$

It is easy to see that if we make x and $y = 0$ in the equation thus transformed, we get for z^2 the values ρ^2 and ρ'^2 as we ought.

If we transform the equation to parallel axes through the point $z = \rho$, the linear part of the equation becomes

$$2p\rho(\rho^2 - \rho'^2)(pz + p'x),$$

from which the results already obtained as to the position of the tangent plane may be independently established.

By expanding the terms of the second degree, the values of the principal radii of curvature, and the directions of curvature can be established, but I have arrived at no results of importance.

438. The equation of the reciprocal of the wave surface is got by writing $\frac{\lambda^2}{a}$ for a, &c., in the equation of the wave surface; and if this be transformed as in the preceding article, it becomes

$$\begin{aligned}(x^2+y^2+z^2)\{p^2\rho'^2x^2+p^2\rho^2y^2-2pp'\rho'^2xz-2pp''\rho^2yz \\ +z^2(p'^2\rho'^2+p''^2\rho^2+\rho^2\rho'^2)\} \\ -\lambda^4(p^2+p''^2+\rho'^2)x^2-\lambda^4(p^2+p'^2+\rho^2)y^2-\lambda^4(p'^2+p''^2+\rho^2+\rho'^2)z^2 \\ +2\lambda^4p'p''xy+2\lambda^4pp'xz+2\lambda^4pp''yz+\lambda^8=0.\end{aligned}$$

We know that the surface is touched by the plane $z=\frac{\lambda^2}{\rho}$, and if we put in this value for z, we find, as we ought, a curve having for a double point the point $y=0$, $x=\frac{p'\lambda^2}{p\rho}$. If in the equation of the curve we make $y=0$, we get

$$\left(px-\frac{p'\lambda^2}{\rho}\right)^2\left\{\rho'^2x^2+\frac{\lambda^4}{\rho^2}(\rho'^2-\rho^2)\right\},$$

from which we learn that that chord of the outer sheet of the wave surface which joins any point on the inner sheet to the foot of the perpendicular from the centre on the tangent plane is bisected at the point on the inner sheet. The inflexional tangents are parallel to

$$\{p'^2\rho'^2+p^2(\rho'^2-\rho^2)\}x^2-2p'p''\rho^2xy+\{p'^2\rho^2+\rho^2(\rho'^2-\rho^2)\}y^2,$$

a result of which I do not see any geometrical interpretation.*

* I have no space for a discussion what the lines of curvature on the wave surface are *not*, though a hasty assertion on this subject in Crelle's Journal has led to interesting investigations by M. Bertrand, *Comptes Rendus*, Nov. 1858; Combescure and Brioschi, Tortolini's *Annali di Matematica*, Vol. II., pp. 135, 278. It is worth while to cite an observation of Brioschi, that if in the plane $lx+my+nz=\phi$; l, m, n, ϕ be functions

439. We shall next consider the surface *parallel* to a given quadric, that is to say, the surface which may either be defined as the envelope of planes parallel to the tangent planes of the quadric, and at a given distance from them; or else as the locus of the points taken on the normals at a fixed distance from the surface, (*Higher Plane Curves*, p. 273). It is evident that the sphere whose centre is any point on the parallel surface, and radius the given distance, will touch the original quadric. We can then most easily form the equation of the parallel surface by expressing (Art. 127) the condition that the given quadric $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$, may be touched by the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=k^2.$$

This is done by forming the discriminant with respect to t, of a biquadratic whose coefficients are given p. 112, but which may be written in the form

$$\frac{t\alpha^2}{a^2+t}+\frac{t\beta^2}{b^2+t}+\frac{t\gamma^2}{c^2+t}=t+k^2.$$

The result represents a surface of the twelfth degree, but which, when we make $k=0$, reduces to the quadric taken twice, together with the imaginary developable (Art. 203) which envelopes all quadrics confocal to the given one. This readily appears from the form in which the equation of the biquadratic has been written.

440. The locus of the feet of perpendiculars let fall, from any fixed point, on the tangent planes of a surface is a de-

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of two variables u, v, as in Art. 373, then the plane will envelope a surface in which curves of the families u = constant, v = constant, will, at their intersection, be touched by conjugate tangents of the surface, if the condition be fulfilled,

$$\begin{vmatrix} l, & m, & n, & \phi \\ l_1, & m_1, & n_1, & \phi_1 \\ l_2, & m_2, & n_2, & \phi_2 \\ l_{12}, & m_{12}, & n_{12}, & \phi_{12} \end{vmatrix}=0.$$

where the suffixes 1, 2, denote differentiation with respect to u and v respectively: while the curves will cut at right angles if

$$(l^2+m^2+n^2)(l_1l_2+m_1m_2+n_1n_2)=(ll_1+mm_1+nn_1)(ll_2+mm_2+nn_2).$$

rived surface to which French mathematicians have of late thought it worth while to give a distinctive name, "podaire," which we shall translate as the *pedal* of the given surface. From the pedal may, in like manner, be derived a new surface, and from this another, &c. forming a series of second, third, &c. pedals. Again, the envelope of planes drawn perpendicular to the radii vectores of a surface, at their extremities is a surface of which the given surface is the pedal, and which we may call the first negative pedal. The surface derived in like manner from this is the second negative, and so on. Pedal curves and surfaces have been studied in particular by Mr. W. Roberts, *Liouville*, Vols. X. and XII., by M. Tortolini, and by Mr. Hirst, Tortolini's *Annali*, Vol. II., p. 95. We shall here give some of their results, but must omit the greater part of them, which relate to problems concerning rectification, quadrature, &c., which, on account of want of space, cannot be included in this treatise. If Q be the foot of the perpendicular from O on the tangent plane at any point P, it is easy to see that the sphere described on the diameter OP touches the locus of Q; and consequently the normal at any point Q of the pedal passes through the middle point of the corresponding radius vector OP. It immediately follows hence that the perpendicular OR on the tangent plane at Q lies in the plane POQ, and makes the angle $QOR = POQ$, so that the right-angled triangle QOR is similar to POQ; and if we call the angle QOR, α, so that the first perpendicular OQ is connected with the radius vector by the equation $p = \rho \cos\alpha$, then the second perpendicular OR will be $\rho \cos^2\alpha$, and so on.*

It is obvious that if we form the polar reciprocals of a curve or surface A and its pedal B, we shall have a surface a which will be the pedal of b; hence if we take a surface S and its successive pedals S_1, S_2, $\ldots S_n$, the reciprocals will be

* Thus the radius vector to the n^{th} pedal is of length $\rho \cos^n \alpha$, and makes with the radius vector to the curve the angle $n\alpha$. Using this definition of the method of derivation Mr. Roberts has considered fractional derived curves and surfaces. Thus for $n = \frac{1}{2}$, the curve derived from the ellipse is Cassini's oval. An analogous surface may be derived from the ellipsoid.

a series S', S'_{-1}, S'_{-2}, $\ldots S'_{-n}$, the derived in the latter case being negative pedals.

It is also obvious that the first pedal is the *inverse* (*Higher Plane Curves*, p. 239) of the polar reciprocal of the given surface (that is to say, the surface derived from it by substituting in its equation, for the radius vector, its reciprocal); and that the inverse of the series S_1, S_2, $\ldots S_n$ will be the series S', S'_{-1}, $\ldots S'_{n-1}$.

441. As we shall not have opportunity to return to the general theory of inversion, we give in this place the following statement (taken from Hirst, *Tortolini*, Vol. II., p. 165) of the principal properties of inverse surfaces.

(1) Three pairs of corresponding points on two inverse surfaces lie on the same sphere, (and two pairs of corresponding points on the same circle) which cuts orthogonally the unit sphere whose centre is the origin.

(2) By the property of a quadrilateral inscribed in a circle the line ab joining any two points on one curve makes the same angle with the radius vector Oa, that the line joining the corresponding points $a'b'$ makes with the radius vector Ob'. In the limit then, if ab be the tangent at any point a, the corresponding tangent on the inverse curve makes the same angle with the radius vector.

(3) In like manner for surfaces, two corresponding tangent planes are equally inclined to the radius vector, the two corresponding normals lying in the same plane with the radius vector, and forming with it an isosceles triangle whose base is the intercepted portion of the radius vector.

(4) It follows immediately from (2) that the angle which two curves make with each other at any point is equal to that which the inverse curves make at the corresponding point.

(5) In like manner it follows from (3) that the angle which two surfaces make with each other at any point is equal to that which the inverse surfaces make at the corresponding point.

(6) The inverse of a line or plane is a circle or sphere passing through the origin.

(7) Any circle may be considered as the intersection of a plane, and a sphere A through the origin. Its inverse therefore is another circle, which is a sub-contrary section of the cone whose vertex is the origin, and which stands on the given circle.

(8) The centre of the second circle lies on the line joining the origin to a the vertex of the cone circumscribing the sphere A along the given circle. For a is evidently the centre of a sphere B which cuts A orthogonally. The plane therefore which is the inverse of A cuts B' the inverse of B orthogonally, that is to say, in a great circle, whose centre is the same as the centre of B'. But the centres of B and of B' lie in a right line through the origin.

(9) To a circle osculating any curve, evidently corresponds a circle osculating the inverse curve.

(10) For inverse surfaces, the centres of curvature of two corresponding normal sections lie in a right line with the origin. To the normal section α at any point m corresponds a curve α' situated on a sphere A passing through the origin; and the osculating circle c' of α' is the inverse of c the osculating circle of α. If now α_1 be the normal section which touches α' at the point m', then by Meunier's theorem, the centre of c' is the projection on its plane of the centre of c_1 the osculating circle of α_1. But the normal $m'c_1$ evidently touches the sphere A at m', so that c_1 is the vertex of the cone circumscribed to A along c', and theorem (10) therefore follows from theorem (8).

(11) To the two normal sections at m whose centres of curvature occupy extreme positions on the normal at m, will evidently correspond two sections enjoying the same property; therefore to the two principal sections on one surface correspond two principal sections on the other, and to a line of curvature on one, a line of curvature on the other.

442. The first pedal of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, being the inverse of the reciprocal ellipsoid, has for its equation

$$a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2.$$

This surface is Fresnel's "Surface of Elasticity." The inverse of a system of confocals cutting at right angles is evidently a system of surfaces of elasticity cutting at right angles; the lines of curvature therefore of the surface of elasticity are determined as the intersection with it of two surfaces of the same nature derived from concyclic quadrics.

The origin is evidently a double point on this surface, and the imaginary circle in which any sphere cuts the plane at infinity is a double line on the surface.

443. Mr. Cayley first obtained the equation of the first negative pedal of a quadric, that is to say, of the envelope of planes drawn perpendicular to the central radii at their extremities. It is evident that if we describe a sphere passing through the centre of the given quadric, and touching it at any point $x'y'z'$, then the point xyz on the derived surface which corresponds to $x'y'z'$, is the extremity of the diameter of this sphere, which passes through the centre of the quadric. We thus easily find the expressions

$$x = x'\left(2 - \frac{t}{a^2}\right), \quad y = y'\left(2 - \frac{t}{b^2}\right), \quad z = z'\left(2 - \frac{t}{c^2}\right);$$

where $$t = x'^2 + y'^2 + z'^2.$$

Solving these equations for x', y', z' and substituting their values in the two equations

$$xx' + yy' + zz' = x'^2 + y'^2 + z'^2, \quad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1,$$

we get $$\frac{x^2}{\left(2 - \frac{t}{a^2}\right)} + \frac{y^2}{\left(2 - \frac{t}{b^2}\right)} + \frac{z^2}{\left(2 - \frac{t}{c^2}\right)} = t,$$

$$\frac{x^2}{a^2\left(2 - \frac{t}{a^2}\right)^2} + \frac{y^2}{b^2\left(2 - \frac{t}{b^2}\right)^2} + \frac{z^2}{c^2\left(2 - \frac{t}{c^2}\right)^2} = 1.$$

Now the second of these equations is the differential, with respect to t, of the first equation; and the required surface is therefore represented by the discriminant of this equation, which we can easily form, the equation being only of the fourth degree. If we write this biquadratic

$$At^4 + 4Bt^3 + 6Ct^2 + 4Dt + E,$$

it will be found that A and B do not contain x, y, z, while C, D, E contain them, each in the second degree. Now the discriminant is of the sixth degree in the coefficients, and is of the form $A\phi + B^2\psi$; consequently it can contain x, y, z only in the tenth degree. This therefore is the degree of the surface required.

Its section by one of the principal planes consists of the first negative pedal of the corresponding principal section of the ellipsoid, which is a curve of the sixth order, together with a conic, counted twice, which is a double curve on the surface. The double points on the principal planes answer to points on the ellipsoid for which $x'^2 + y'^2 + z'^2 = 2a^2$ or $2b^2$ or $2c^2$, as easily appears from the expressions given for x, y, z in the beginning of the article. There is a cuspidal conic at infinity, and besides, a finite cuspidal curve of the sixteenth degree.

The reader will find (*Philosophical Transactions*, 1858, and *Tortolini*, Vol. II., p. 168) a discussion by Mr. Cayley of the different forms assumed by the surface and by the cuspidal and nodal curves according to the different relative values of a^2, b^2, c^2.

444. Mr. W. Roberts has solved the problem discussed in the last article in another way, by proving that the problem to find the negative pedal of a surface, is identical with that of forming the equation of the parallel surface. The former problem is to find the envelope of the plane

$$xx' + yy' + zz' = x'^2 + y'^2 + z'^2$$

where x', y', z' satisfy the equation of the surface. The second problem, being that of finding the envelope of a sphere whose centre is on the surface and radius $= k$, is to find the envelope of

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = k^2,$$

or $$2xx' + 2yy' + 2zz' = x^2 + y^2 + z^2 - k^2 + x'^2 + y'^2 + z'^2.$$

Now in finding this envelope the unaccented letters are treated as constants, and it is evident that both problems are particular cases of the problem to find, under the same conditions, the envelope of

$$ax' + by' + cz' = x'^2 + y'^2 + z'^2 + d.$$

And it is evident that if we have the equation of the parallel surface, we have only to write in it for k^2, $x^2 + y^2 + z^2$, and then $\frac{1}{2}x$, $\frac{1}{2}y$, $\frac{1}{2}z$ for x, y, z; when we have the equation of the negative pedal. Thus having obtained by Art. 439 the equation of the parallel to a quadric, we can find by the substitutions here explained, the equation of the first negative, the origin being anywhere, as easily as when the origin is the centre. Further, if we write for k, $k+k'$, and then make the same substitution for k, we obtain the first negative, the origin being anywhere, of the parallel to the quadric, a problem which it would probably not be easy to solve in any other way.

Having found, as above, the equation of the first negative of a quadric, we have only to form its inverse, when we have the equation of the second positive pedal (Art. 440).

Ex. 1. To find the envelope of planes drawn perpendicularly at the extremities of the radii vectores to the plane $ax + by + cz + d$.

Here the parallel surface consists of a pair of planes, whose equation is $(ax + by + cz + d)^2 = k^2$, that of the envelope is therefore

$$(ax + by + cz + 2d)^2 = x^2 + y^2 + z^2.$$

Ex. 2. To find, in like manner, the first negative of the sphere

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2.$$

The parallel surface consists of the pair of concentric spheres

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = (r \pm k)^2.$$

The envelope is therefore

$$(x - 2\alpha)^2 + (y - 2\beta)^2 + (z - 2\gamma)^2 = \{2r \pm \sqrt{(x^2 + y^2 + z^2)}\}^2,$$

which denotes a quadric of revolution.

CHAPTER XIV.

SURFACES OF THE THIRD DEGREE.

445. The general theory of surfaces, explained p. 190, &c., gives the following results, when applied to cubical surfaces. The tangent cone whose vertex is any point, and which envelopes such a surface is, in general, of the sixth degree, having six cuspidal edges and no ordinary double edge. It is consequently of the twelfth class, having twenty-four stationary, and twenty-seven double tangent planes. Since then through any line twelve tangent planes can be drawn to the surface, any line meets the reciprocal in twelve points; and the reciprocal is, in general, of the twelfth degree. Its equation can be found as at *Higher Plane Curves*, p. 99. The problem is the same as that of finding the condition that the plane

$$\alpha x + \beta y + \gamma z + \delta w$$

should touch the surface. Multiply the equation of the surface by δ^3, and then eliminate δw by the help of the equation of the plane. The result is a homogeneous cubic in x, y, z, containing also α, β, γ, δ in the third degree. The discriminant of this equation is of the twelfth degree in its coefficients, and therefore of the thirty-sixth in $\alpha\beta\gamma\delta$: but this consists of the equation of the reciprocal surface multiplied by the irrelevant factor δ^{24}. The form of the discriminant of a homogeneous cubical function in x, y, z is $64S^3 = T^2$ (*Higher Plane Curves*, p. 190). The same then will be the form of the reciprocal of a surface of the third degree, S being of the fourth, and T of the sixth degree in α, β, γ, δ; (that is to say, S and T are *contravariants* of the given equation of the above degrees). It is easy to see that they are also of the same degree in the coefficients of the given equation.

446. Surfaces may have either multiple points or multiple lines. When a surface has a double line of the degree p; then any plane meets the surface in a section having p double points. There is, therefore, the same limit to the degree of the double curve on a surface of the n^{th} degree, that there is to the number of double points on a curve of the n^{th} degree. Since a curve of the third degree can have only one double point; if a surface of the third degree has a double line, that line must be a right line.* A cubic having a double line is necessarily a ruled surface, for every plane passing through this line meets the surface in the double line, reckoned twice, and in another line; but these other lines form a system of generators resting on the double line as director. If we make the double line the axis of z, the equation of the surface will be of the form

$$(ax^3 + 3bx^2y + 3cxy^2 + dy^3) + z(a'x^2 + 2b'xy + c'y^2) + (a''x^2 + 2b''xy + c''y^2) = 0,$$

which we may write $u_3 + zu_2 + v_2 = 0$. At any point on the double line there will be a pair of tangent planes $z'u_2 + v_2 = 0$. But as z' varies this denotes a system of planes in involution (*Conics*, p. 287). Hence *the pair of tangent planes at any point on the double line, are two conjugate planes of a system in involution.*

There are two values of z', real or imaginary, which will make $z'u_2 + v_2$ a perfect square; there are therefore two points on the double line at which the tangent planes coincide; and any plane through either of which meets the surface in a section having this point for a cusp. If the values of these squares be X^2 and Y^2, it is evident that u_2 and v_2 can each be expressed in the form $lX^2 + mY^2$. If then we turn round the axes so

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* If a surface have a double or other multiple line, the reciprocal formed by the method of the last article would vanish identically; because then *every* plane meets the surface in a curve having a double point, and therefore the plane $\alpha x + \beta y + \gamma z + \delta w$ is to be considered as touching the surface, independently of any relation between α, β, γ, δ. The reciprocal can be formed in this case by eliminating x, y, z, w between $u = 0$, $\alpha = u_1$, $\beta = u_2$, $\gamma = u_3$, $\delta = u_4$.

as to have for co-ordinate planes, the planes X, Y, that is to say, the tangent planes at the cuspidal points; then every term in the equation will be divisible by either x^2 or y^2, and the equation may be reduced to the form $zx^2 = wy^2$.*

In this form it is evident that the surface is generated by lines $y = \lambda x$, $z = \lambda^2 w$; intersecting the two directing lines xy, zw; and the generators join the points of a system on zw to the points of a system in involution on xy, homographic with the first system. Any plane through zw meets the surface in a pair of right lines, and is to be regarded as touching the surface in the two points where these lines meet zw. Thus then as the line xy is a line, every point of which is a double point, so the line zw is a line, every plane through which is a double tangent. The reciprocal of this surface, which is that considered Art. 419, is of like nature with itself.

The tangent cone whose vertex is any point, and which envelopes the surface, consists of the plane joining the point to the double line, reckoned twice, and a proper tangent cone of the fourth order. When the point is on the double line the cone reduces to the second order.

447. There is one case, to which my attention was called by Mr. Cayley, in which the reduction to the form $zx^2 = wy^2$ is not possible. If u_2 and v_2, in the last article, have a common factor, then choosing the plane represented by this for one of the co-ordinate planes, we can easily throw the equation of the surface into the form $y^3 + x(zx + wy) = 0$.

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* It is here supposed that the planes X, Y, the double planes of the system in involution, are real. We can always, however, reduce to the form $w(x^2 \pm y^2) + 2zxy$, the upper sign corresponding to real, and the lower to imaginary, double planes. In the latter case the double line is altogether "really" in the surface, every plane meeting the surface is a section having the point where it meets the line for a real node. In the former case this is only true for a limited portion of the double line, sections which meet it elsewhere having the point of meeting for a conjugate point; the two cuspidal points marking these limits on the double line. A right line, every point of which is a cusp, cannot exist on a cubic unless when the surface is a cone.

The plane x touches the surface along the whole length of the double line, and meets the surface in three coincident right lines. The other tangent plane at any point coincides with the tangent plane to the hyperboloid $zx+wy$. This case may be considered as a limiting case of that considered in the last article; viz., when the double director xy coincides with the single one wz. The following generation of the surface may be given. Take a series of points on xy, and a homographic series of planes through it; then the generator of the cubic through any point on the line, lies in the corresponding plane, and may be completely determined by taking as director any plane cubic having a double point where its plane meets the double line.*

448. The argument which proves that a proper cubic curve cannot have more than one double point does not apply to surfaces. In fact the line joining two double points, since it is to be regarded as meeting the surface in four points, must lie altogether in the surface; but this does not imply that the surface breaks up into others of lower dimensions. The consideration of the tangent cone however supplies a limit to the number of double points on any surface. We have seen (Art. 251) that the tangent cone necessarily has a certain number of double and cuspidal edges, and since every double point on the surface adds a double edge to the tangent cone, there cannot be more double points than will make up the total number of double edges of the tangent cone to the maximum number which such a cone can have. Thus a curve of the sixth degree having six cusps can have only four other double points; therefore since the tangent cone to a cubic is of the sixth order, having six cuspidal edges, the surface can at most have four double points.

When a surface has a double point, the line joining this point to any assumed point is, as has been said, a double edge of the tangent cone from the latter point; and it is easy to

* The reader is referred to an interesting geometrical memoir on cubical ruled surfaces by Cremona, "Atte del Reale Istituto Lombardo," Vol. II., p. 291.

see that the tangent planes along this double edge are the planes drawn through this line to touch the cone generated by the tangents at the double point. If then this cone break up into two planes, it follows that such a point entails a cuspidal edge on the tangent cone through any assumed point. A cubic then can have only three such biplanar double points. The reciprocal of a cubic then having one or more double points may be of any degree from the tenth to the third, each ordinary double point reducing the degree by two, and each biplanar by three.

If the two planes of contact at a biplanar point coincide, the line joining this to any assumed point will be a *triple* edge on the tangent cone through that point, and the degree of the reciprocal will be reduced by six.

Ex. 1. What is the degree of the reciprocal of $xyz = w^3$?

Ans. There are three biplanar points in the plane w, and the reciprocal is a cubic.

Ex. 2. What is the reciprocal of $\frac{l}{x} + \frac{m}{y} + \frac{n}{z} + \frac{p}{w} = 0$?

Ans. This represents a cubic having the vertices of the pyramid $xyzw$ for double points; and the reciprocal must be of the fourth degree.

The equation of the tangent plane at any point $x'y'z'w'$ can be thrown into the form $\frac{lx}{x'^2} + \frac{my}{y'^2} + \frac{nz}{z'^2} + \frac{pw}{w'^2} = 0$, whence it follows that the condition that $\alpha x + \beta y + \gamma z + \delta w$ should be a tangent plane is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} + (p\delta)^{\frac{1}{2}} = 0,$$

an equation which, cleared of radicals, is of the fourth degree. Generally the reciprocal of $ax^n + by^n + cz^n + pw^n$ is of the form

$$A\alpha^{\frac{n}{n-1}} + B\beta^{\frac{n}{n-1}} + C\gamma^{\frac{n}{n-1}} + D\delta^{\frac{n}{n-1}} = 0,$$

(*Higher Plane Curves*, p. 102).

A cubic having four double points is also the envelope of

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta,$$

where a, b, c, l, m, n represent planes; and $\alpha:\gamma$, $\beta:\gamma$ are two variable parameters. It is obvious that the envelope is of the third degree; and it is of the fourth class; since if we substitute the co-ordinates of two points we can determine four planes of the system passing through the line joining these points.

The tangent cone to this surface, whose vertex is any point on the surface, being of the fourth degree, and having four double edges, must break up into two cones of the second degree.

449. The equation of a cubic having no multiple point may be thrown into the form $ax^3 + by^3 + cz^3 + dv^3 + ew^3 = 0$, where x, y, z, v, w represent planes, and where for simplicity we suppose that the constants implicitly involved in x, y, &c. have been so chosen, that the identical relation connecting the equations of any five planes (Art. 37) may be written in the form $x + y + z + v + w = 0$. In fact the general equation of the third degree contains twenty terms and therefore nineteen independent constants, but the form just written contains five terms and therefore four expressed independent constants, while besides the equation of each of the five planes implicitly involves three constants. The form just written therefore contains the same number of constants as the general equation. This form given by Mr. Sylvester in 1851 (*Cambridge and Dublin Mathematical Journal*, Vol. VI., p. 199) is most convenient for the investigation of the properties of cubical surfaces in general.*

450. If we write the equation of the first polar of any point with regard to a surface of the n^{th} order

$$x'L + y'M + z'N + w'P = 0,$$

* It was observed (*Higher Plane Curves*, Art. 18) that two forms may apparently contain the same number of independent constants, and yet that one may be less general than the other. Thus when a form is found to contain the same number of constants as the general equation, it is not absolutely demonstrated that the general equation is reducible to this form; and Clebsch has noticed a remarkable exception in the case of curves of the fourth order. In the present case, though Mr. Sylvester gave his theorem without further demonstration, he states that he was in possession of a proof that the general equation could be reduced to the sum of five cubes and in but a single way. Such a proof has been published by Mr. Clebsch (*Crelle*, Vol. LIX., p. 193). He erroneously ascribes the theorem in the text to Steiner, who gave it in the year 1856 (*Crelle*, Vol. LIII., p. 133). It chanced that surfaces of the third order were studied in this country a few years before German mathematicians turned their attention to this subject; and consequently, though, as might be expected from his ability, M. Steiner's investigations led him to several important results, these had been almost all well known here some years before.

then, if it have a double point, that point will satisfy the equations

$$ax' + ny' + mz' + pw' = 0, \quad nx' + by' + lz' + qw' = 0,$$

$$mx' + ly' + cz' + rw' = 0, \quad px' + qy' + rz' + dw' = 0,$$

where a, b, &c. denote second differential coefficients corresponding to these letters, as we have used them in the general equation of the second degree. Now if between the above equations we eliminate $x'y'z'w'$, we obtain the locus of all points which are double points on first polars. This is of the degree $4(n-2)$ and is in fact the Hessian (Art. 254). If we eliminate the $xyzw$ which occur in a, b, &c., since the four equations are each of the degree $(n-2)$, the resulting equation in $x'y'z'w'$ will be of the degree $4(n-2)^3$, and will represent the locus of points whose first polars have double points. Or, again, H is the locus of points whose polar quadrics are cones, while the second surface, which we shall call J, is the locus of the vertices of such cones. In the case of surfaces of the third degree, it is easy to see that the four equations above written are symmetrical between $xyzw$ and $x'y'z'w'$; and therefore that the surfaces H and J are identical. Thus then *if the polar quadric of any point A with respect to a cubic be a cone whose vertex is B, the polar quadric of B is a cone whose vertex is A.* The points A and B are said to be corresponding points on the Hessian (see *Higher Plane Curves*, p. 154, &c.).

451. *The tangent plane to the Hessian of a cubic at A is the polar plane of B with respect to the cubic.* For if we take any point A' consecutive to A and on the Hessian, the pole of any plane through AA' will be somewhere on the intersection of the first polars of A and A'; but these being consecutive and both cones, it appears (as at *Higher Plane Curves*, p. 155) that B, the vertex of this cone, is a pole of any plane through A, A', and therefore of the tangent plane at A. And the polar plane of any point A on the Hessian of a surface of any degree is the tangent plane of the corresponding point B on the surface J. In particular *the tangent planes to U along the parabolic curve, are tangent planes to the surface J:* that is to say,

in the case of a cubic *the developable circumscribing a cubic along the parabolic curve, also circumscribes the Hessian.* If any line meet the Hessian in two corresponding points A, B, and in two other points C, D, the tangent planes at A, B intersect along the line joining the two points corresponding to C, D.

452. We shall also investigate the preceding theorems by means of the canonical form. The polar quadric of any point with regard to $ax^3+by^3+cz^3+dv^3+ew^3$ is got by substituting for w its value $-(x+y+z+v)$, when we can proceed according to the ordinary rules, the equation being then expressed in terms of four variables. We thus find for the polar quadric $ax'x^2+by'y^2+cz'z^2+dv'v^2+ew'w^2=0$. If we differentiate this equation with respect to x, remembering that $dw=-dx$, we get $ax'x=ew'w$; and since the vertex of the cone must satisfy the four differentials with respect to x, y, z, v, we find that the co-ordinates x', y', z', v', w' of any point A on the Hessian are connected with the co-ordinates x, y, z, v, w of B, the vertex of the corresponding cone, by the relations

$$ax'x=by'y=cz'z=dv'v=ew'w.$$

And since we are only concerned with mutual ratios of co-ordinates, we may take 1 for the common value of these quantities and write the co-ordinates of B, $\frac{1}{ax'}$, $\frac{1}{by'}$, $\frac{1}{cz'}$, $\frac{1}{dv'}$, $\frac{1}{ew'}$. Since the co-ordinates of B must satisfy the identical relation $x+y+z+v+w=0$, we thus get the equation of the Hessian

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}+\frac{1}{dv}+\frac{1}{ew}=0,$$

or $bcdeyzvw+cdeazvwx+deabvwxy+eabcwxyz+abcdxyzv=0$. This form of the equation shows that the line vw lies altogether in the Hessian, and that the point xyz is a double point on the Hessian; and since the five planes x, y, z, v, w give rise to ten combinations, whether taken by twos or by threes we have Mr. Sylvester's theorem that *the five planes form a pentahedron whose ten vertices are double points on the Hessian and whose ten edges lie on the Hessian.* The polar quadric of the point

xyz is $dv'v^2 + ew'w^2$, which resolves itself into two planes intersecting along vw, any point on which line may be regarded as the point B corresponding to xyz; thus then *there are ten points whose polar quadrics break up into pairs of planes; these points are double points on the Hessian, and the intersections of the corresponding pairs of planes are lines on the Hessian.* It is by proving these theorems independently* that the resolution of the given equation into the sum of five cubes can be completely established.

The equation of the tangent plane at any point of the Hessian may be written

$$\frac{x}{ax'^2} + \frac{y}{by'^2} + \frac{z}{cz'^2} + \frac{v}{dv'^2} + \frac{w}{ew'^2} = 0,$$

which, if we substitute for x', $\frac{1}{ax'}$, &c., becomes

$$ax''^2x + by''^2y + cz''^2z + dv''^2v + ew''^2w = 0,$$

but this is the polar plane of the corresponding point with regard to U (Art. 451).

453. If we consider all the points of a fixed plane, their polar planes envelope a surface, which (as at *Higher Plane Curves*, p. 152) is also the locus of points whose polar quadrics touch the given plane. The parameters in the equation of the variable plane enter in the second degree; the problem is therefore that considered (Ex. 2, Art. 448) and the envelope is a cubic surface having four double points. The polar planes of the points of the section by the cubic are the tangent planes at those points, consequently this polar cubic of the given plane is inscribed in the developable formed by the tangent planes to the cubic along the section by the given plane (*Higher*

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* It will appear from the appendix "on the order of systems of equations," that a symmetric determinant of p rows and columns, each constituent of which is a function of the n^{th} order in the variables, represents a surface of the np degree having $\frac{1}{6}p(p^2-1)n^3$ double points; and thus that the Hessian of a surface of the n^{th} degree always has $10(n-2)^3$ double points.

Plane Curves, Art. 161). The polar plane of any point A of the section of the Hessian by the given plane, touches the Hessian (Art. 451) and is therefore a common tangent plane of the Hessian and of the polar cubic now under consideration. But the polar quadric of B, being a cone whose vertex is A, is to be regarded as touching the given plane at A; hence B is also the point of contact of this polar plane with the polar cubic. We thus obtain a theorem of Steiner's that *the polar cubic of any plane touches the Hessian along a certain curve.* This curve is the locus of the points B corresponding to the points of the section of the Hessian by the given plane. Now if points lie in any plane $lx + my + nz + pv + qw$, the corresponding points lie on the surface of the fourth order $\frac{l}{ax} + \frac{m}{by} + \frac{n}{cz} + \frac{p}{dv} + \frac{q}{ew}$. Now the intersection of this surface with the Hessian is of the sixteenth order, and includes the ten right lines xy, zw, &c. The remaining curve of the sixth order is the curve along which the polar cubic of the given plane touches the Hessian. The four double points lie on this curve; they are the points whose polar quadrics are cones touching the given plane.

454. If on the line joining any two points $x'y'z'$, $x''y''z''$, we take any point $x' + \lambda x''$, &c., it is easy to see that its polar plane is of the form $P_{11} + \lambda P_{12} + \lambda^2 P_{22}$, where P_{11}, P_{22} are the polar planes of the two given points, and P_{12} is the polar plane of either point with regard to the polar quadric of the other. The envelope of this plane, considering λ variable, is evidently a quadric cone whose vertex is the intersection of the three planes. This cone is clearly a tangent cone to the polar cubic of any plane through the given line, the vertex of the cone being a point on that cubic. If the two assumed points be corresponding points on the Hessian, P_{12} vanishes identically; for, the equation of the polar plane, with respect to a cone, of its vertex vanishes identically. Hence *the polar plane of any point of the line joining two corresponding points on the Hessian passes through the intersection of the tangent*

*planes to the Hessian at these points.** In any assumed plane we can draw three lines joining corresponding points on the Hessian; for the curve of the sixth degree considered in the last article meets the assumed plane in three pairs of corresponding points. The polar cubic then of the assumed plane will contain three right lines; as will otherwise appear from the theory of right lines on cubics which we shall now explain.

455. We said, note, p. 29, that a cubical surface necessarily contains right lines, and we now enquire how many in general lie on the surface.† In the first place it is to be observed that if a right line lie on the surface, every plane through it is a double tangent plane because it meets the surface in a right line and conic; that is to say, in a section having two double points. The planes then joining any point to the right lines on the surface are double tangent planes to the surface and therefore also double tangent planes to the tangent cone whose vertex is that point. But we have seen (Art. 445) that the number of such double tangent planes is *twenty-seven*.

This result may be otherwise established as follows: let us suppose that a cubic contains one right line, and let us examine in how many ways a plane can be drawn through that right line, such that the conic in which it meets the surface may break up into two right lines. Let the right line be wz; let the equation of the surface be $wU=zV$; let us substitute $w=\mu z$, divide out by z, and then form the discriminant of the resulting quadric in x, y, z. Now in this quadric it is seen without difficulty that the coefficients of x^2, xy, and y^2 only contain μ in the first degree; that those of

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* Steiner says that there are one hundred lines such that the polar plane of any point of one of them passes through a fixed line, but I believe that his theorem ought to be amended as above.

† The theory of right lines on a cubical surface was first studied in the year 1849 in a correspondence between Mr. Cayley and me, the results of which were published, *Cambridge and Dublin Mathematical Journal*, Vol. IV., pp. 118, 252. Mr. Cayley first observed that a definite number of right lines must lie on the surface; the determination of that number as above, and the discussions in Art. 458 were supplied by me.

xz and yz contain μ in the second degree, and that of z^2 in the third degree. It follows hence that the equation obtained by equating the discriminant to nothing is of the fifth degree in μ: and therefore that *through any right line on a cubical surface can be drawn five planes, each of which meets the surface in another pair of right lines;* and consequently *every right line on a cubic is intersected by ten others.* Consider now the section of the surface by one of the planes just referred to. Every line on the surface must meet in some point the section by this plane, and therefore must intersect some one of the three lines in this plane. But each of these lines is intersected by eight in addition to the lines in the plane; there are therefore twenty-four lines on the cubic besides the three in the plane; that is to say, *twenty-seven in all.*

We shall hereafter show how to form the equation of a surface of the ninth order meeting the given cubic in those lines.

456. Since the equation of a plane contains three independent constants, a plane may be made to fulfil any three conditions, and therefore a finite number of planes can be determined which shall touch a surface in three points. We can now determine this number in the case of a cubical surface. We have seen that through each of the twenty-seven lines can be drawn five triple tangent planes: for every plane intersecting in three right lines touches at the vertices of the triangle formed by them, these being double points in the section. The number 5×27 is to be divided by three, since each of the planes contains three right lines; *there are therefore in all forty-five triple tangent planes.*

457. *Every plane through a right line on a cubic is obviously a double tangent plane; and the pairs of points of contact form a system in involution.* Let the axis of z lie on the surface, and let the part of the equation which is of the first degree in x and y be $(az^2 + bz + c)\,x + (a'z^2 + b'z + c')\,y$; then the two points of contact of the plane $y = \mu x$ are determined by the equation

$$(az^2 + bz + c) + \mu\,(a'z^2 + b'z + c') = 0,$$

but this denotes a system in involution (*Conics*, p. 287). It follows hence, from the known properties of involution, that two planes can be drawn through the line to touch the surface in two coincident points: that is to say, which cut it in a line and a conic touching that line. The points of contact are evidently the points where the right line meets the parabolic curve on the surface. It was proved (Art. 256) that the right line touches that curve. The two points then where the line touches the parabolic curve, together with the points of contact of any plane through it, form a harmonic system. Of course the two points where the line touches the parabolic curve may be imaginary.

458. The number of right lines may also be determined thus. The form $ace = bdf$, (where a, b, &c. represent planes) is one which implicitly involves nineteen independent constants, and therefore is one into which the general equation of a cubic may be thrown.* This surface obviously contains nine lines (ab, cd, &c.). Any plane then $a = \mu b$ which meets the surface in right lines meets it in the same lines in which it meets the hyperboloid $\mu ce = df$. The two lines are therefore generators of different species of that hyperboloid. One meets the lines cd, ef; and the other the lines cf, de. And, since μ has three values, there are three lines which meet ab, cd, ef. The same thing follows from the consideration that the hyperboloid determined by these lines must meet the surface in three more lines (Art. 313).

Now there are clearly six hyperboloids, ab, cd, ef; ab, cf, de, &c., which determine eighteen lines in addition to the nine with which we started, that is to say as before, twenty-seven in all.

If we denote each of the eighteen lines by the three which it meets, the twenty-seven lines may be enumerated as follows: there are the original nine ab, ad, af, cb, cd, cf, eb, ed, ef: together with $(ab.cd.ef)_1$, $(ab.cd.ef)_2$, $(ab.cd.ef)_3$, and in like manner three lines of each of the forms $ab.cf.de$, $ad.bc.ef$,

* It will be found in one hundred and twenty ways.

ad.be.cf, af.bc.de, af.be.cd. The five planes which can be drawn through any of the lines *ab* are the planes *a* and *b*, meeting respectively in the pairs of lines *ad*, *af*; *bc*, *be*; and the three planes which meet in $(ab.cd.ef)_1$, $(ab.cf.de)_1$; $(ab.cd.ef)_2$, $(ab.cf.de)_2$; $(ab.cd.ef)_3$, $(ab.cf.de)_3$. The five planes which can be drawn through any of the lines $(ab.cd.ef)_1$, cut in the pairs of lines, *ab*, $(ab.cf.de)_1$; *cd*, $(af.cd.be)_1$; *ef*, $(ad.bc.ef)_1$; and in $(ad.be.cf)_2$, $(af.bc.de)_3$; $(ad.be.cf)_3$, $(af.bc.de)_2$.

459. Prof. Schäfli has made a new arrangement of the lines (*Quarterly Journal of Mathematics*, Vol. II., p. 116) which leads to a simpler notation, and gives a clearer conception how they lie. Writing down the two systems of six non-intersecting lines

$$ab,\ cd,\ ef,\ (ad.be.cf)_1,\ (ad.be.cf)_2,\ (ad.be.cf)_3,$$
$$cf,\ be,\ ad,\ (ab.cd.ef)_1,\ (ab.cd.ef)_2,\ (ab.cd.ef)_3;$$

it is easy to see that each line of one system, does not intersect the line of the other system which is written in the same vertical line, but that it intersects the five other lines of the second system. We may write then these two systems

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,$$
$$b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6,$$

which is what Schäfli calls a "double-six." It is easy to see from the previous notation that the line which lies in the plane of a_1, b_2 is the same as that which lies in the plane of a_2, b_1. Hence the fifteen other lines may be represented by the notation c_{12}, c_{34}, &c., where c_{12} lies in the plane of a_1, b_2, and there are evidently fifteen combinations in pairs of the six numbers 1, 2, &c. The five planes which can be drawn through c_{12} are the two which meet in the pairs of lines a_1b_2, a_2b_1, and those which meet in $c_{34}c_{56}$, $c_{35}c_{46}$, $c_{36}c_{45}$. There are evidently thirty planes which contain a line of each of the systems *a*, *b*, *c*: and fifteen planes which contain three *c* lines. It will be found that out of the twenty-seven lines can be constructed thirty-six "double-sixes."

460. We can now geometrically construct a system of twenty-seven lines which can belong to a cubical surface. We may start by taking arbitrarily any line a_1 and five others which intersect it, b_2, b_3, b_4, b_5, b_6. These determine a cubical surface, for if we describe such a surface through four of the points where a_1 is met by the other lines and through three more points on each of these lines, then the cubic determined by these nineteen points contains all the lines, since each line has four points common with the surface. Now if we are given four non-intersecting lines, we can in general draw two transversals which shall intersect them all; for the hyperboloid determined by any three meets the fourth in two points through which the transversals pass.* Through any four then of the lines b_3, b_4, b_5, b_6 we can draw in addition to the line a_1 another transversal a_2, which must also lie on the surface since it meets it in four points. In this manner we construct the five new lines a_2, a_3, a_4, a_5, a_6. If we then take another transversal meeting the four first of these lines, the theory already explained shows that it will be a line b_1 which will also meet

* If the hyperboloid touches the fourth line, the two transversals reduce to a single one, and it is evident that the hyperboloid determined by any three others of the four lines also touches the remaining one. This remark I believe is Mr. Cayley's. If we denote the condition that two lines should intersect by (12), then the condition that four lines should be met by only one transversal is expressed by equating to nothing the determinant

$$\begin{vmatrix} - & (12), & (13), & (14) \\ (21), & - & (23), & (24) \\ (31), & (32), & - & (34) \\ (41), & (42), & (43), & - \end{vmatrix}.$$

The vanishing of the determinant formed in the same manner from five lines, is the condition that they are all met by a common transversal. The vanishing of the similar determinant for six lines, expresses that they are connected by a relation which has been called the "involution of six lines:" and which will be satisfied when the lines can be the directions of six forces in equilibrium. The reader will find several interesting communications on this subject by Messrs. Sylvester and Cayley, and by M. Chasles, in the *Comptes Rendus* for 1861, *Premier Semestre.*

the fifth. We have thus constructed a "double-six." We can then immediately construct the remaining lines by taking the plane of any pair a_1b_2, which will be met by the lines b_1, a_2 in points which lie on the line c_1c_2.

461. M. Schäfli has made an analysis of the different species of cubics according to the reality of the twenty-seven lines. He finds thus five species: *A.* all the lines and planes real; *B.* fifteen lines and fifteen planes real; *C.* seven lines and five planes real; that is to say, there is one right line through which five real planes can be drawn, only three of which contain real triangles; *D.* three lines and thirteen planes real: namely, there is one real triangle through every side of which pass four other real planes; and, *E.* three lines and seven planes real.

I have also given (*Cambridge and Dublin Mathematical Journal*, Vol. IV., p. 256) an enumeration of the modifications of the theory when the surface has one or more double points. It may be stated generally that the cubic has always twenty-seven right lines and forty-five triple tangent planes, if we count a line or plane through a double point as two, through two double points as four, and a plane through three such points as eight. Thus, if the surface has one double point, there are six lines passing through that point, and fifteen other lines one in the plane of each pair. There are fifteen treble tangent planes not passing through the double point. Thus $2 \times 6 + 15 = 27$; $2 \times 15 + 15 = 45$.

Again, if the surface have four double points, the lines are the six edges of the pyramid formed by the four points (6×4), together with three others lying in the same plane, each of which meets two opposite edges of the pyramid. The planes are the plane of these three lines 1, six planes each through one of these lines and through an edge (6×2), together with the four faces of the pyramid (4×8).

The reader will find the other cases discussed in the paper just referred to.

INVARIANTS AND COVARIANTS OF A CUBIC.

462. We shall in this section give an account of the principal invariants, covariants, &c. that a cubic can have. We only suppose the reader to have learned from the *Lessons on Higher Algebra*, or elsewhere, some of the most elementary properties of these functions. An invariant of the equation of a surface is a function of the coefficients, whose vanishing expresses some permanent property of the surface, as for example that it has a nodal point. A covariant, as for example the Hessian, denotes a surface having to the original surface some relation which is independent of the choice of axes. A contravariant is a relation between α, β, γ, δ, expressing the condition that the plane $\alpha x + \beta y + \gamma z + \delta w$ shall have some permanent relation to the given surface, as for example that it shall touch the surface. The property of which we shall make the most use in this section is that proved (*Lessons on Higher Algebra*, p. 66), viz., that if we substitute in a contravariant for α, β, &c., $\frac{d}{dx}$, $\frac{d}{dy}$, &c., and then operate on either the original function or one of its covariants we shall get a new covariant, which will reduce to an invariant if the variables have disappeared from the result. In like manner if we substitute in any covariant for x, y, &c. $\frac{d}{d\alpha}$, $\frac{d}{d\beta}$, &c., and operate on a contravariant, we get a new contravariant.

Now in discussing the properties of a cubic we mean to use Mr. Sylvester's canonical form in which it is expressed by the sum of five cubes. We have calculated for this form the Hessian (Art. 452), and there would be no difficulty in calculating other covariants for the same form. It remains to show how to calculate contravariants in the same case. Let us suppose that when a function U is expressed in terms of four independent variables, we have got any contravariant in α, β, γ, δ: and let us examine what this becomes when the function is expressed by five variables connected by a linear relation. But obviously we can reduce the function of five variables to one of four, by substituting for the fifth its value in terms of the others: viz. $w = -(x+y+z+v)$. To find then the condition that the plane $\alpha x + \beta y + \gamma z + \delta v + \varepsilon w$ may have any assigned relation to the given surface, is the same problem as to find that the plane $(\alpha - \varepsilon)x + (\beta - \varepsilon)y + (\gamma - \varepsilon)z + (\delta - \varepsilon)v$ may have the same relation to the surface, its equation being expressed in terms of four variables; so that the contravariant in five letters is derived from that in four by substituting $\alpha - \varepsilon$, $\beta - \varepsilon$, $\gamma - \varepsilon$, $\delta - \varepsilon$ respectively for α, β, γ, δ. Every contravariant in five letters is therefore a function of the differences between α, β, γ, δ, ε. This method will be better understood from the following example.

Ex. The equation of a quadric is given in the form

$$ax^2 + by^2 + cz^2 + dv^2 + ew^2 = 0,$$

where $x + y + z + v + w = 0$: to find the condition that $\alpha x + \beta y + \gamma z + \delta v + \varepsilon w$ may touch the surface. If we reduce the equation of the quadric to a function of four variables by substituting for w its value in terms of the others, the coefficients of x^2, y^2, z^2, v^2 are respectively $a + e$, $b + e$, $c + e$, $d + e$ while every other coefficient becomes e. If now we substitute these values in the equation of Art. 75, the condition that the plane $\alpha x + \beta y + \gamma z + \delta v$ touches, becomes

$$\alpha^2 (bcd + bce + cde + dbe) + \beta^2 (cda + cde + dae + ace) + \gamma^2 (dab + dae + abe + bde)$$
$$+ \delta^2 (abc + abe + bce + cae) - 2e (ad\beta\gamma + bd\gamma\alpha + cd\alpha\beta + bc\alpha\delta + ca\beta\delta + ab\gamma\delta) = 0.$$

Lastly, if we write in the above for α, β, &c., $\alpha - \varepsilon$, $\beta - \varepsilon$, &c., it becomes

$$bcd (\alpha - \varepsilon)^2 + cda (\beta - \varepsilon)^2 + dab (\gamma - \varepsilon)^2 + abc (\delta - \varepsilon)^2 + bce (\alpha - \delta)^2 + cae (\beta - \delta)^2$$
$$+ abe (\gamma - \delta)^2 + ade (\beta - \gamma)^2 + bde (\alpha - \gamma)^2 + cde (\alpha - \beta)^2 = 0,$$

a contravariant which may be briefly written $\Sigma cde (\alpha - \beta)^2 = 0$.

463. We have referred to the theorem that when a contravariant in four letters is given, we may substitute for α, β, γ, δ differential symbols with respect to x, y, z, w; and that then by operating with the function so obtained on any covariant we get a new covariant. Suppose now that we operate on a function expressed in terms of five letters x, y, z, v, w. Since x appears in this function both explicitly and also where it is introduced in w, the differential with respect to x is $\frac{d}{dx} + \frac{d}{dw}\frac{dw}{dx}$, or, in virtue of the relation connecting w

with the other variables, $\frac{d}{dx} - \frac{d}{dw}$. Hence a contravariant in four letters is turned into an operating symbol in five by substituting for

$$\alpha, \beta, \gamma, \delta; \frac{d}{dx} - \frac{d}{dw}, \frac{d}{dy} - \frac{d}{dw}, \frac{d}{dz} - \frac{d}{dw}, \frac{d}{dv} - \frac{d}{dw}.$$

But we have seen in the last article that the contravariant in five letters has been obtained from one in four, by writing for α, $\alpha - \varepsilon$, &c. It follows then immediately that *if in any contravariant in five letters we substitute for* $\alpha, \beta, \gamma, \delta, \varepsilon$, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{dv}, \frac{d}{dw}$, *we obtain an operating symbol, with which operating on the original function, or on any covariant, we obtain a new covariant or invariant.* The importance of this is that when we have once found a contravariant of the form in five letters we can obtain a new covariant without the laborious process of recurring to the form in four letters.

Ex. We have seen that $\Sigma cde\,(\alpha - \beta)^2$ is a contravariant of the form

$$ax^2 + by^2 + cz^2 + dv^2 + ew^2.$$

If then we operate on the quadric with $\Sigma cde \left(\frac{d}{dx} - \frac{d}{dy}\right)^2$, the result, which only differs by a numerical factor from

$$bcde + cdea + deab + eabc + abcd$$

is an invariant of the quadric. It is in fact its discriminant, and could have been obtained from the expression Art. 63, by writing as in the last article $a + e$, $b + e$, $c + e$, $d + e$ for a, b, c, d, and putting all the other coefficients equal to e.

464. In like manner it is proved that we may substitute in any covariant function for x, y, z, v, w, differential symbols with regard to $\alpha, \beta, \gamma, \delta, \varepsilon$, and that operating with the function so obtained on any contravariant we get a new contravariant. In fact if we first reduce the function to one of four variables, and then make the differential substitution which we have a right to do, we have substituted for

$$x, y, z, v, w;\ \frac{d}{d\alpha}, \frac{d}{d\beta}, \frac{d}{d\gamma}, \frac{d}{d\delta}, \text{ and } -\left(\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \frac{d}{d\delta}\right).$$

But since the contravariant in five letters was obtained from that in four by writing $\alpha - \epsilon$ for α, &c. it is evident that the differentials of both with regard to α, β, γ, δ are the same, while the differential of that in five letters with respect to ϵ is the negative sum of the differentials of that in four letters with respect to α, β, γ, δ. But this establishes the theorem. By this theorem and that in the last article we can, being given any covariant and contravariant, generate another, which again combined with the former gives rise to new ones without limit.

465. The polar quadric of any point with regard to the cubic $ax^3 + by^3 + cz^3 + dv^3 + ew^3$ is

$$axx'^2 + byy'^2 + czz'^2 + dvv'^2 + eww'^2 = 0.$$

Now the Hessian is the discriminant of the polar quadric. Its equation therefore, by Ex., Art. 463, is $\Sigma bcdeyzvw = 0$, as was already proved, Art. 452. Again, what we have called (Art. 453) the polar cubic of a plane

$$\alpha x + \beta y + \gamma z + \delta v + \epsilon w,$$

being the condition that this plane should touch the polar quadric is (by Ex., Art. 462) $\Sigma cdezvw (\alpha - \beta)^2 = 0$. This is what is called a mixed concomitant, since it contains both sets of variables x, y, &c., and α, β, &c.

If now we substitute in this for α, β, &c., $\frac{d}{dx}$, $\frac{d}{dy}$, &c., and operate on the original cubic, we get the Hessian; but if we operate on the Hessian we get a covariant of the fifth order in the variables, and the seventh in the coefficients to which we shall afterwards refer as Φ,

$$\Phi = abcde \Sigma ab x^2 y^2 z.$$

In order to apply the method indicated (Arts. 463, 464) it is necessary to have a contravariant; and for this purpose I have calculated the contravariant σ which occurs in the equation of the reciprocal surface, which, as we have already seen, is of the form $64\sigma^3 = \tau^2$. The contravariant σ expresses the condition that any plane $\alpha x + \beta y + \&c.$ should meet the surface in a cubic for which Aronhold's invariant S vanishes. It is of the fourth degree both in α, β, &c. and in the coefficients of the cubic. In the case of four variables the leading term is α^4 multiplied by the S of the ternary cubic got by making $x=0$ in the equation of the surface. The remaining terms are calculated from this by means of the differential equation (*Lessons in Higher Algebra*, p. 70). The form being found for four variables, that for five is calculated from it as in Art. 462. I suppress the details of the calculation which though tedious presents no difficulty. The result is

$$\sigma = \Sigma abcd\,(\alpha - \epsilon)\,(\beta - \epsilon)\,(\gamma - \epsilon)\,(\delta - \epsilon)\,..........[1],$$

For facility of reference I mark the contravariants with numbers between brackets and the covariants by numbers between parentheses, the cubic itself and the Hessian being numbered (1) and (2). We can now, as already explained, from any given covariant and contravariant generate a new one, by substituting in that in which the variables are of lowest dimensions, differential symbols for the variables, and then operating on the other. The result is of the difference of their degrees in the variables, and of the sum of their degrees in the coefficients. If both are of equal dimensions, it is indifferent with which we operate. The result in this case is an invariant of the sum of their degrees in the coefficients. The results of this process are given in the next article.

466. (*a*) Combining (1) and [1], we expect to find a contravariant of the first degree in the variables, and the fifth in the coefficients; but this vanishes identically.

(*b*) (2) and [1] gives an invariant to which we shall refer as invariant A,

$$A = \Sigma b^2c^2d^2e^2 - 2abcde\Sigma abc.$$

If A be expressed by the symbolical method explained (*Lessons on Higher Algebra*, p. 77), its expression is

$$(1235)\,(1246)\,(1347)\,(2348)\,(5678)^2.$$

(*c*) Combining [1] with the square of (1) we get a covariant quadric of the sixth order in the coefficients

$$abcde\,(ax^2+by^2+cz^2+dv^2+ew^2)\ldots\ldots\ldots\ldots(3),$$

which expressed symbolically is (1234) (1235) (1456) (2456).

(*d*) (3) and [1] gives a contravariant quadric

$$a^2b^2c^2d^2e^2\Sigma\,(\alpha-\beta)^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots[2].$$

(*e*) (1) and [2] gives a covariant plane of the eleventh order in the coefficients

$$a^2b^2c^2d^2e^2\,(ax+by+cz+dv+ew)\ldots\ldots\ldots\ldots(4).$$

(*f*) (3) and [2] gives an invariant B,

$$a^3b^3c^3d^3e^3\,(a+b+c+d+e).$$

(*g*) Combining with (3) the mixed concomitant (Art. 465) we get a covariant cubic of the ninth order in the coefficients

$$abcde\,\Sigma cde\,(a+b)\,zvw\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

(*h*) Combining (5) and [1] we have a linear contravariant of the thirteenth order, viz.

$$abcde\,\Sigma\,(a-b)\,(\alpha-\beta)\,\{(a+b)\,c^2d^2e^2-abcde\,(cd+de+ec)\}.$$

It seems unnecessary to give further details as to the steps by which particular covariants are found, and we may therefore sum up the principal results.

467. It is easy to see that every invariant is a symmetric function of the quantities a, b, c, d, e. If then we denote the sum of these quantities, of their products in pairs, &c., by p, q, r, s, t; every invariant can be expressed in terms of these five quantities, and therefore in terms of the five following fundamental invariants, which are all obtained by proceeding with the process exemplified in the last article

$$A=s^2-4rt,\ B=t^3p,\ C=t^4s,\ D=t^6q,\ E=t^8;$$

whence also $$C^2-AE=4t^9r.$$

We can, however, form skew invariants which cannot be rationally expressed in terms of the five fundamental invariants, although their squares can be rationally expressed in terms of these quantities. The simplest invariant of this kind is got by expressing in terms of its coefficients the discriminant of the equation whose roots are a, b, c, d, e. This, it will be found, gives in terms of the fundamental invariants A, B, C, D, E, an expression for t^{36} multiplied by the product of the squares of the differences of all the quantities a, b, &c. This invariant being a perfect square, its square root is an invariant F of the one hundredth degree. Its expression in terms of the fundamental invariants is given, *Philosophical Transactions*, 1860, p. 233.

The discriminant can easily be expressed in terms of the fundamental invariants. It is obtained by eliminating the variables between the four differentials with respect to x, y, z, v, that is to say,

$$ax^2 = by^2 = cz^2 = dv^2 = ew^2.$$

Hence x^2, y^2, &c. are proportional to $bcde$, $cdea$, &c. Substituting then in the equation $x + y + z + v + w = 0$, we get the discriminant

$$\sqrt{(bcde)} + \sqrt{(cdea)} + \sqrt{(deab)} + \sqrt{(eabc)} + \sqrt{(abcd)} = 0.$$

Clearing of radicals, the result, expressed in terms of the principal invariants, is

$$(A^2 - 64B)^2 = 16384\,(D + 2AC).$$

468. The cubic has four fundamental covariant planes of the orders 11, 19, 27, 43 in the coefficients, viz.

$$L = t^2\Sigma ax,\ L' = t^3\Sigma bcde x,\ L'' = t^4\Sigma a^2x,\ L''' = t^8\Sigma a^3x.$$

Every other covariant, including the cubic itself, can in general be expressed in terms of these four, the coefficients being invariants. The condition that these four planes should meet in a point, is the invariant F of the one hundredth degree.

There are linear contravariants the simplest of which, of the thirteenth degree, has been already given; the next being of the twenty-first, $t^4\Sigma(a - b)(\alpha - \beta)$; the next of the twenty-ninth, $t^5\Sigma cde\,(a - b)(\alpha - \beta)$, &c.

There are covariant quadrics of the sixth, fourteenth, twenty-second, &c. orders; and contravariants of the tenth, eighteenth, &c. the order increasing by eight.

There are covariant cubics of the ninth order $\Sigma tcde\,(a+b)\,zuv$, and of the seventeenth, $t^3\Sigma a^2x^3$, &c.

If we call the original cubic U, and this last covariant V, since if we form a covariant or invariant of $U+\lambda V$, the coefficients of the several powers of λ are evidently covariants or invariants of the cubic: it follows that given any covariant or invariant of the cubic we are discussing, we can form from it a new one of the degree sixteen higher in the coefficients, by performing on it the operation

$$t^3\left(a^2\frac{d}{da}+b^2\frac{d}{db}+c^2\frac{d}{dc}+d^2\frac{d}{dd}+e^2\frac{d}{de}\right).$$

Of higher covariants we only think it necessary here to mention one of the fifth order, and fifteenth in the coefficients t^3xyzvw which gives the five fundamental planes: and one of the ninth order, Θ the locus of points whose polar planes with respect to the Hessian touch their polar quadrics with respect to U. Its equation is expressed by the determinant at the top of p. 50, if α, β, &c. denote the first differential coefficients with respect to the Hessian, and a, b, &c. the second differentials with respect to the cubic.

The equation of a covariant whose intersection with the given cubic determines the twenty-seven lines is $\Theta=4\text{H}\Phi$, where Φ has the meaning explained, Art. 465. We shall give M. Clebsch's proof of this at the end of the volume. I had verified the form, which had been suggested to me by geometrical considerations, by examining the following form, to which the equation of the cubic can be reduced, by taking for the planes x and y the tangent planes at the two points where any line meets the parabolic curve, and two determinate planes through these points for the planes w, z,

$$z^2y+w^2x+2xyz+2xyw+ax^2y+by^2x+cx^2z+dy^2w+exw^2+fyz^2=0.$$

The part of the Hessian then which does not contain either x or y is z^2w^2: the corresponding part of Φ is $-2\,(cz^5+dw^5)$, and of Θ is $-8w^2z^2\,(cz^5+dw^5)$. The surface $\Theta-4\text{H}\Phi$ has therefore no part which does not contain either x or y, and the line xy lies altogether on the surface, as in like manner do the rest of the twenty-seven lines.*

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* This section is abridged from a paper which I contributed to the *Philosophical Transactions*, 1860, p. 229. Shortly after the reading of my memoir, and before its publication, there appeared two papers in Crelle's *Journal*, Vol. 58, by Professor Clebsch of Carlsruhe, in which some of my results were anticipated: in particular the expression of all the invariants of a cubic in terms of five fundamental: and the expression given above for the surface passing through the twenty-seven lines. The method however which I pursued was different from that of Professor Clebsch, and the discussion of the covariants, as well as the notice of the invariant F, I believe were new. Clebsch has expressed his last four invariants as functions of the coefficients of the Hessian. Thus the second is the invariant $(1234)^4$ of the Hessian, &c.

CHAPTER XV.

GENERAL THEORY OF SURFACES.

469. We shall in this chapter proceed, in continuation of Art. 256, with the general theory of surfaces, and shall first mention a few miscellaneous theorems which are sometimes useful.

The locus of the points whose polar planes with regard to four surfaces M, N, P, Q (whose degrees are m, n, p, q) meet in a point, is a surface of the degree $m+n+p+q-4$. For its equation is evidently got by equating to nothing the determinant whose constituents are the four differential coefficients of each of the four surfaces. If a surface of the form $aM+bN+cP$ touch Q, the point of contact is evidently a point on the locus just considered, and must lie somewhere on the curve of the degree $q(m+n+p+q-4)$ where Q is met by the locus surface. In like manner, $pq(m+n+p+q-4)$ surfaces of the form $aM+bN$, can be drawn so as to touch the curve of intersection of P, Q; for the point of contact must be some one of the points where the curve PQ meets the locus surface.

It follows hence that the condition that two of the mnp points of intersection of three surfaces M, N, P may coincide, contains the coefficients of the first in the degree $np(2m+n+p-4)$; and in like manner for the other two surfaces. For if in this condition we substitute for each coefficient a of M, $a+\lambda a'$, where a' is the corresponding coefficient of another surface M' of the same degree as M, it is evident that the degree of the result in λ, is the same as the number of surfaces of the form $M+\lambda M'$ which can be drawn to touch the curve of intersection of N, P.*

* Moutard, *Terquem's Annales*, Vol. XIX., p. 58.

I had arrived at the same result otherwise thus: (see *Quarterly Journal*, Vol. I., p. 339) Two of the points of intersection coincide if the curve of intersection MN touch the curve MP. At the point of contact then the tangent planes to the three surfaces have a line in common: and these planes therefore have a point in common with any arbitrary plane $ax+by+cz+dw$. The point of contact then satisfies the determinant, one row of which is a, b, c, d: and the other three rows are formed by the four differentials of each of the three surfaces. The condition that this determinant may be satisfied by a point common to the three surfaces is got by eliminating between the determinant and M, N, P. The result will contain a, b, c, d in the degree mnp; and the coefficients of M in the degree $np(m+n+p-3)+mnp$. But this result of elimination contains as a factor the condition that the plane $ax+by+cz+dw$ may pass through one of the points of intersection of M, N, P. And this latter condition contains a, b, c, d in the degree mnp, and the coefficients of M in the degree np. Dividing out this factor, the quotient, as already seen, contains the coefficients of M in the degree

$$np(2m+n+p-4).$$

470. The locus of points whose polar planes with regard to three surfaces have a right line common, is, as may be inferred from the last article, the curve denoted by the system of determinants

$$\left\| \begin{matrix} u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \\ w_1, & w_2, & w_3, & w_4 \end{matrix} \right\| = 0,$$

where u_1, &c. denote the differential coefficients. But this curve (see Appendix) is of the order $(m'^2+n'^2+p'^2+m'n'+n'p'+p'm')$, where m' is the order of u_1, &c., that is to say, $m'=m-1$. If a surface of the form $aM+bN$ touch P, the point of contact is evidently a point on the locus just considered, and therefore the number of such surfaces which can be drawn to touch P, is equal to the number of points in which this locus curve meets P, that is to say, is p times the degree of that curve. Reasoning then as in the last article we see that the condition that two surfaces M and N should touch, contains the coefficients of M in the degree $n(n'^2 + 2m'n' + 3m'^2)$ or

$$n(n^2 + 2mn + 3m^2 - 4n - 8m + 6),$$

and in like manner contains the coefficients of N in the degree $m(m^2 + 2mn + 3n^2 - 4m - 8n + 6)$. Moutard, *Terquem*, Vol. XIX., p. 65.

We add, in the form of examples, a few theorems to which it does not seem worth while to devote a separate article.

Ex. 1. Two surfaces U, V of the degrees m, n intersect; the number of tangents to their curve of intersection which are also inflexional tangents of the first surface, is $mn(3m + 2n - 8)$.

The inflexional tangents at any point on a surface are generating lines of the polar quadric of that point; any plane therefore through either tangent touches that polar quadric. If then we form the condition that the tangent plane to V may touch the polar quadric of U, which condition involves the second differentials of U in the third degree, and the first differentials of V in the second degree, we have the equation of a surface of the degree $(3m + 2n - 8)$ which meets the curve of intersection in the points, the tangents at which are inflexional tangents on U.

Ex. 2. In the same case to find the degree of the surface generated by the inflexional tangents to U at the several points of the curve UV.

This is got by eliminating $x'y'z'w'$, between the equations

$$U' = 0,\ V' = 0,\ \Delta U' = 0,\ \Delta^2 U' = 0,$$

which are in $x'y'z'w'$ of the degrees respectively m, n, $m - 1$, $m - 2$, and in $xyzw$ of the degrees 0, 0, 1, 2. The result is therefore of the degree $mn(3m - 4)$.

Ex. 3. To find the degree of the developable which touches a surface along its intersection with its Hessian. The tangent planes at two consecutive points on the parabolic curve, intersect in an inflexional tangent (Art. 238); and, by the last example, since $n = 4(m - 2)$, the degree of the surface generated by these inflexional tangents is $4m(m - 2)(3m - 4)$. But since at every point of the parabolic curve the two inflexional tangents coincide, and therefore the surfaces generated by each of these tangents coincide, the number just found must be divided by two, and the degree required is $2m(m - 2)(3m - 4)$.

Ex. 4. To find the characteristics, as at p. 239, of the developable which touches a surface along any plane section of a surface whose degree is m. The section of the developable by the given plane is the section of the given surface, together with the tangents at its $3m(m - 2)$ points of inflexion. Hence we easily find

$$\mu = 6m(m - 2),\ \nu = m(m - 1),\ r = m(3m - 5),\ \alpha = 0,\ \beta = 2m(5m - 11),\ \&c.$$

Ex. 5. To find the characteristics of the developable which touches a surface of the degree m along its intersection with a surface of degree n.

Ans. $\nu = mn(m-1)$, $\alpha = 0$, $r = mn(3m+n-6)$ whence the other singularities are found as at p. 239.

Ex. 6. To find the characteristics of the developable touching two given surfaces, neither of which has multiple lines.

Ans. $\nu = mn(m-1)^2(n-1)^2$; $\alpha = 0$, $r = mn(m-1)(n-1)(m+n-2)$.

Ex. 7. To find the characteristics of the curve of intersection of two developables.

The surfaces are of degrees r and r', and since each has a nodal and cuspidal curve of degrees respectively x and m, x' and m', therefore the curve of intersection has $rx' + r'x$ and $rm' + r'm$ actual nodal and cuspidal points. The cone therefore which stands on the curve and whose vertex is any point, has nodal and cuspidal edges in addition to those considered at p. 250; and the formulæ there given must then be modified. We have as there $\mu = rr'$; but the degree of the reciprocal of this cone is

$$\rho = rr'(r + r' - 2) - r(2x' + 3m') - r'(2x + 3m),$$

or by the formulæ of p. 236, $\rho = rn' + nr'$. In like manner

$$\nu = ar' + a'r + 3rr'.$$

Ex. 8. To find the characteristics of the developable generated by a line meeting two given curves. This is the reciprocal of the last example. We have therefore $\nu = rr'$, $\rho = rm' + mr'$, $\mu = \beta r' + \beta' r + 3rr'$.

CONTACT OF LINES WITH SURFACES.

471. We now return to the class of problems proposed in Art. 241, viz., to find the degree of the curve traced on a surface by the points of contact of a line which satisfies three conditions. The cases we shall consider are: (A) to find the curve traced by the points of contact of lines which meet in four consecutive points; (B) when a line is an inflexional tangent at one point and an ordinary tangent at another, to find the degree of the curve formed by the former points; and (C) that of the curve formed by the latter; (D) to find the curve traced by the points of contact of triple tangent lines. To these may be added: (a) to find the degree of the surface formed by the lines A; (b) to find the degree of that formed by the lines considered in (B) and (C); (c) to find the degree of that generated by the triple tangents.

Now to commence with problem A; if a line meet a surface in four consecutive points we must at the point of contact not only have $U' = 0$, but also $\Delta U' = 0$, $\Delta^2 U' = 0$, $\Delta^3 U' = 0$. The tangent line must then be common to the surfaces denoted by the last three equations. We find the condition that this may be possible by the method by which the points of inflexion, and of contact of double tangents, are determined; *Higher Plane Curves*, pp. 77, 86.

472. Let three surfaces U, V, W contain $xwyz$ in the degrees respectively $\lambda, \lambda', \lambda''$; and $x'y'z'w'$ in degrees μ, μ', μ''; and let the $\lambda\lambda'\lambda''$ points of intersection of these surfaces all coincide with $x'y'z'w'$: then it is required to find what further condition must be fulfilled in order that they may have a line in common. When this is the case any arbitrary plane $ax + by + cz + dw$ must be certain to have a point in common with the three surfaces (namely the point where it is met by the common line), and therefore the result of elimination between U, V, W, and the arbitrary plane must vanish. This result is of the degree $\lambda\lambda'\lambda''$ in $abcd$, and $\lambda'\lambda''\mu + \lambda''\lambda\mu' + \lambda\lambda'\mu''$ in $x'y'z'w'$. But since the resultant is obtained by multiplying together the result of substituting in $ax + by + cz + dw$, the co-ordinates of each of the points of intersection of UVW, this result must be of the form

$$\Pi\,(ax' + by' + cz' + dw')^{\lambda\lambda'\lambda''}.$$

Now the condition $ax' + by' + cz' + dw' = 0$, merely indicates that the arbitrary plane passes through $x'y'z'w'$, in which case it passes through a point common to the three surfaces whether they have a common line or not. The condition therefore that they should have a common line is $\Pi = 0$; and this must be of the degree

$$\lambda'\lambda''\mu + \lambda''\lambda\mu' + \lambda\lambda'\mu'' - \lambda\lambda'\lambda''.$$

In the case of the three surfaces $\Delta U'$, $\Delta^2 U'$, $\Delta^3 U'$, we have

$$\lambda = 1,\ \lambda' = 2,\ \lambda'' = 3;\ \mu = n - 1,\ \mu' = n - 2,\ \mu'' = n - 3.$$

Hence, by the formula just given, Π is of the degree $(11n - 24)$. *The points of contact* then of lines which meet the surface in four consecutive points: or (as we may call them) *of double* *inflexional tangents lie on the intersection of the surface with a derived surface S of the degree* $11n-24$.*

473. The equation of the surface generated by the double inflexional tangents is got by eliminating $x'y'z'w'$ between $U'=0$, $\Delta U'=0$, $\Delta^2 U'=0$, $\Delta^3 U'=0$; which result, by the ordinary rule, is of the degree

$$n(n-2)(n-3)+2n(n-1)(n-3)+3n(n-1)(n-2)$$
$$=6n^3-22n^2+18n.$$

Now this result expresses the locus of points whose first, second, and third polars intersect on the surface; and since if a point be anywhere on the surface, its first, second, and third polars intersect in six points on the surface, we infer that the result of elimination must be of the form $U^6M=0$. The degree of M is therefore

$$2n(n-3)(3n-2).$$

474. We can in like manner solve problem *B*. For the point of contact of an inflexional tangent we have $U'=0$, $\Delta U'=0$, $\Delta^2 U'=0$: and if it touch the surface again, we have besides $W'=0$, where W' is the discriminant of the equation of the degree $n-3$ in $\lambda:\mu$, which remains when the first three terms vanish of the equation, p. 187. For W then we have $\lambda''=(n+3)(n-4)$, $\mu''=(n-3)(n-4)$; and having, as

* I gave this theorem in 1849 (*Cambridge and Dublin Journal*, Vol. IV., p. 260). I obtained the equation in an inconvenient form (*Quarterly Journal*, Vol. I., p. 336): and in one more convenient (*Philosophical Transactions*, 1860, p. 229) which I shall presently give. But I substitute for my own investigation the very beautiful piece of analysis by which Professor Clebsch performed the elimination indicated in the text, *Crelle*, Vol. LVIII., p. 93. As the calculation is long, and the method, which is applicable to other problems also, deserves to be studied, I have thought it better to place it by itself in an appendix than to introduce it here. Mr. Cayley has observed that exactly in the same manner as the equation of the Hessian is the transformation of the equation $rt-s^2$ which is satisfied for every point of a developable, so the equation $S=0$ is the transformation of the equation (p. 330) which is satisfied for every point on a ruled surface.

in the last article, $\lambda = 1$, $\mu = n-1$; $\lambda' = 2$, $\mu' = n-2$, we have for the degree of Π

$$2(n-3)(n-4) + (n-2)(n+3)(n-4)$$
$$+ 2(n-1)(n+3)(n-4) - 2(n+3)(n-4).$$

The degree then of the surface which passes through the points B is $(n-4)(3n^2+5n-24)$.

The equation of the surface generated by the lines (b) which are in one place inflexional and in another ordinary tangents is found by eliminating $x'y'z'w'$ between the four equations $U'=0$, $\Delta U'=0$, $\Delta^2 U'=0$, $W'=0$; and from what has been just stated as to the degree of the variables in each of these equations the degree of the resultant is

$$n(n-2)(n-3)(n-4) + 2n(n-1)(n-3)(n-4)$$
$$+ n(n-1)(n-2)(n+3)(n-4) = n(n-4)(n^3+3n^2-20n+18).$$

But it appears, as in the last article, that this resultant contains as a factor, U in the power $2(n+3)(n-4)$. Dividing out this factor the degree of the surface (b) remains

$$n(n-3)(n-4)(n^2+6n-4).$$

475. In order that a tangent at the point $x'y'z'w'$ may elsewhere be an inflexional tangent, we must have $\Delta U'=0$, (an equation for which $\lambda=1$, $\mu=n-1$), and besides we must have satisfied the system of two conditions that the equation of the degree $n-2$ in $\lambda:\mu$, which remains when the first two terms vanish of the equation, p. 187, may have three roots all equal to each other. If then λ', μ'; λ'', μ'' be the degrees in which the variables enter into these two conditions, the order of the surface which passes through the points (C) is, by Art. 472, $\lambda'\mu'' + \lambda''\mu' + (n-2)\lambda'\lambda''$. But (see Appendix on the order of systems of equations)

$$\lambda'\lambda'' = (n-4)(n^2+n+6),\ \lambda'\mu''+\lambda''\mu' = (n-2)(n-4)(n+6).$$

The order of the surface C is therefore

$$(n-2)(n-4)(n^2+2n+12).$$

The locus of the points of contact of triple tangent lines is investigated in like manner, except that for the conditions that the equation just considered should have three roots all equal, we substitute the conditions that the same equation should have two distinct pairs of equal roots. It will be proved in the Appendix that for this system of conditions we have

$$\lambda'\lambda'' = \tfrac{1}{2}(n-4)(n-5)(n^2+3n+6),$$

$$\lambda'\mu'' + \lambda''\mu' = (n-2)(n-4)(n-5)(n+3).$$

The order of the surface which determines the points (D) is, therefore, $\tfrac{1}{2}(n-2)(n-4)(n-5)(n^2+5n+12)$.

To find the surface generated by the triple tangents we are to eliminate $x'y'z'w'$ between $U'=0$, $\Delta U'=0$, and the two conditions, the order of the result being

$$n\mu'\mu'' + n(n-1)(\lambda'\mu'' + \lambda''\mu'):$$

but since this result contains as a factor $U^{\lambda'\lambda''}$: in order to find the order of the surface (C) we are to substract $n\lambda'\lambda''$ from the number just written. Substituting the values first given for $\lambda'\lambda''$, $\lambda'\mu''+\lambda''\mu'$; and for $\mu'\mu''$, $\tfrac{1}{2}(n-2)(n-3)(n-4)(n-5)$, we get for the order of the surface (c),

$$n(n-3)(n-4)(n-5)(n^2+3n-2),$$

a number which probably ought to be divided by three.

476. There remains to be considered another class of problems, viz., the determination of the number of tangents which satisfy four conditions. The following is an enumeration of these problems. To determine: (α) the number of lines which meet in five consecutive points; (β) the number of points at which both the inflexional tangents meet in four consecutive points; (γ) the number of lines which are doubly inflexional tangents in one place, and ordinary tangents in another; (δ) of lines inflexional in two places; (ϵ) inflexional in one place and ordinary tangents in two others; (ζ) of lines which touch in four places. None of these problems has as yet been solved: but we can find equations which determine a major limit to the number of points α, &c.

If a line meet in five consecutive points it touches the surface S (Art. 472), since both at the first and second of these points it is possible to draw a line meeting the surface in four consecutive points. The points α then are points on the curve US, such that the tangent to that curve is one of the inflexional tangents of U. Therefore, by Ex. 1, p. 403, these points lie on a derived surface whose degree is

$$\{3n+2(11n-24)-8\}=25n-56.$$

But the points β also lie on the same surface; for these are evidently double points on the curve US, that is to say, points at which U and S touch each other. At these points also therefore the tangent plane to S passes through an inflexional tangent of U. We get then an equation

$$\alpha+\lambda\beta=n(11n-24)(25n-56),$$

where λ is a numerical multiplier, which I believe to be $=2$, but which possibly may be greater. Another limit to the number of points α and β is obtained from Professor Clebsch's calculation in the appendix.

In like manner the points α, γ are both included in the intersections of the surfaces U, S, and that found as the locus of points B, Art. 474. And other equations of connexion are found in like manner, but not sufficient to determine the number of points.

CONTACT OF PLANES WITH SURFACES.

477. We can discuss the cases of planes which touch a surface, in the same manner as we have done those of touching lines. Every plane which touches a surface meets it in a section having a double point: but since the equation of a plane includes three constants, a determinate number of tangent planes can be found which will fulfil two additional conditions. And if but one additional condition be given, an infinite series of tangent planes can be found which will satisfy it, those planes enveloping a developable, and their points of contact tracing out a curve on the surface. It may be required either to determine the number of solutions when three conditions are given, or to determine the nature of the curves and developables just mentioned, when two conditions are given. Of the latter class of problems we shall consider but

two, viz., the discussion of the case when the plane meets the surface in a section having a cusp; or, when it meets in a section having two double points. Other cases have been considered by anticipation in the last section, as for example, the case when a plane meets in a section having a double point, one of the tangents at which meets in four consecutive points.

478. Let the co-ordinates of three points be $x'y'z'w'$, $x''y''z''w''$, $xyzw$; then those of any point on the plane through the points will be $\lambda x' + \mu x'' + \nu x$, $\lambda y' + \mu y'' + \nu y$, &c.: and if we substitute these values for $xyzw$ in the equation of the surface, we shall have the relation which must be satisfied for every point where this plane meets the surface. Let the result of substitution be $[U] = 0$, then $[U]$ may be written

$$\lambda^n U' + \lambda^{n-1}\mu\Delta_{\prime\prime} U' + \lambda^{n-1}\nu\Delta U' + \frac{\lambda^{n-2}}{1.2}(\mu\Delta_{\prime\prime} + \nu\Delta)^2 U' + \&c. = 0,$$

where $$\Delta_{\prime\prime} = x''\frac{d}{dx} + y''\frac{d}{dy} + z''\frac{d}{dz} + w''\frac{d}{dw};$$

$$\Delta = x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'} + w\frac{d}{dw'}.$$

The plane will touch the surface if the discriminant of this equation in λ, μ, ν vanish. If we suppose two of the points fixed and the third to be variable, then this discriminant will represent all the tangent planes to the surface which can be drawn through the line joining the two fixed points.

We shall suppose the point $x'y'z'w'$ to be on the surface, and the point $x''y''z''w''$ to be taken anywhere on the tangent plane at that point: then we shall have $U' = 0$, $\Delta_{\prime\prime} U' = 0$, and the discriminant will become divisible by the square of $\Delta U'$. For of the tangent planes, which can be drawn to a surface through any tangent line to that surface, two will coincide with the tangent plane at the point of contact of that line. If the tangent plane at $x'y'z'w'$ be a double tangent plane, then the discriminant we are considering, instead of being, as in other cases, only divisible by the square of the equation of the tangent plane, will contain its *cube* as a factor. In order to examine the condition that this may be so, let us for brevity write the equation $[U]$ as follows, the coefficients of λ^n, $\lambda^{n-1}\mu$ being supposed to vanish,

$$T\lambda^{n-1}\nu + \frac{1}{1.2}\lambda^{n-2}(A\mu^2 + 2B\mu\nu + C\nu^2) + \&c. = 0.$$

T represents the tangent plane at the point we are considering, C its polar quadric, while $A = 0$ is the condition that $x''y''z''w''$ should lie on that polar quadric. Now it will be found that the discriminant of $[U]$ is of the form

$$T^2A(B^2 - AC)^2\phi + T^3(\qquad) = 0,$$

where ϕ is the discriminant when T vanishes as well as U and $\Delta_{\prime\prime}U$. In order that the discriminant may be divisible by T^3, some one of the factors which multiply T^2 must either vanish or be divisible by T.

479. First then let A vanish. This only denotes that the point $x''y''z''w''$ lies on the polar quadric of $x'y'z'w'$: or, since it also lies in the tangent plane, that the point $x''y''z''w''$ lies on one of the inflexional tangents at $x'y'z'w'$. Thus we learn that if the class of a surface be p, then of the p tangent planes which can be drawn through an ordinary tangent line, two coincide with the tangent plane at its point of contact, and there can be drawn $p-2$ distinct from that plane: but that if the line be an inflexional tangent, three will coincide with that tangent plane, and there can be drawn only $p-3$ distinct from it. If we suppose that $x''y''z''w''$ has not been taken on an inflexional tangent, A will not vanish, and we may set this factor aside as irrelevant to the present discussion.

We may examine at the same time the conditions that T should be a factor in $B^2 - AC$, and in ϕ.

The problem which arises in both these cases is the following: Suppose that we are given a function V, whose degrees in $x'y'z'w'$, in $x''y''z''w''$, and in $xyzw$ are respectively (λ, μ, μ). Suppose that this represents a surface having as a multiple line of the order μ the line joining the first two points; or, in other words, that it represents a series of planes through that line: to find the condition that one of these planes should be the tangent plane T whose degrees are $(n-1, 0, 1)$. If so any arbitrary line which meets T will meet V, and therefore if we eliminate between the equations $T=0$, $V=0$, and the equations of an arbitrary line

$$ax+by+cz+dw=0, \quad a'x+b'y+c'z+d'w=0,$$

the resultant R must vanish. This is of the degree μ in $abcd$, in $a'b'c'd'$, and in $x''y''z''w''$, and of the degree $\mu(n-1)+\lambda$ in $x'y'z'w'$. But evidently if the assumed right line met the line joining $x'y'z'w'$, $x''y''z''w''$, R would vanish even though T were not a factor in V. The condition $(M=0)$, that the two lines should meet is of the first degree in all the quantities we are considering: and we see now that R is of the form $M^\mu R'$. R' remains a function of $x'y'z'w'$ alone, and is of the degree $\mu(n-2)+\lambda$.

480. To apply this to the case we are considering, since the discriminant of $[U]$ represents a series of planes through $x'y'z'w'$, $x''y''z''w''$, it follows that B^2-AC and ϕ both represent planes through the same line. The first is of the degrees $\{2(n-2), 2, 2\}$, while ϕ is of the degrees $(n-2)(n^2-6)$, n^3-2n^2+n-6, n^3-2n^2+n-6, as appears by subtracting the sum of the degrees of T^4, A, and $(B^2-AC)^2$ from the degrees of the discriminant of $[U]$, which is of the degree $n(n-1)^2$ in all the variables. It follows then from the last article that the condition $(H=0)$ that T should be a factor in B^2-AC is of the degree $4(n-2)$, and the condition $(K=0)$ that T should be a factor in ϕ is of the degree $(n-2)(n^3-n^2+n-12)$. At all points then of the intersection of U and H the tangent plane must be considered double. H is no other than the Hessian; the tangent plane at every point of the curve UH meets the surface in a section having a cusp, and is to be counted as double (Art. 238). The curve UK is the locus of points of contact of planes which touch the surface in two distinct points.

481. Let us consider next the series of tangent planes which touch along the curve UH. They form a developable whose degree is $\rho=2n(n-2)(3n-4)$, Ex. 2, p. 403. The

class of the same developable, or the number of planes of the system which can be drawn through an assigned point, is $\nu = 4n(n-1)(n-2)$. For the points of contact are evidently the intersections of the curve UH with the first polar of the assigned point. We can also determine the number of stationary planes of the system. If the equation of U, the plane z being the tangent plane at any point on the curve UH, be $z + y^2 + u_3 + \&c. = 0$, it is easy to show that the direction of the tangent to UH is in the line $\frac{d^2u_3}{dx^2} = 0$. Now the tangent planes to U are the same at two consecutive points proceeding along the inflexional tangent y. If then u_3 do not contain any term x^3, (that is to say, if the inflexional tangent meet the surface in four consecutive points) the direction of the tangent to the curve UH is the same as that of the inflexional tangent: and the tangent planes at two consecutive points on the curve UH will be the same. The number of stationary tangent planes is then equal to the number of intersections of the curve UH with the surface S. But since the curve touches the surface, as will be shewn in the appendix, we have $\alpha = 2n(n-2)(11n-24)$. From these data all the singularities of the developable which touches along UH can be determined, as at p. 237. We have

$$\mu = n(n-2)(28n-60),\quad \nu = 4n(n-1)(n-2),\quad \rho = 2n(n-2)(3n-4),$$

$$\alpha = 2n(n-2)(11n-24),\quad \beta = n(n-2)(70n-160);$$

$$2g = n(n-2)(16n^4 - 64n^3 + 80n^2 - 108n + 156),$$

$$2h = n(n-2)(784n^4 - 4928n^3 + 10320n^2 - 7444n + 548).$$

The developable here considered answers to a cuspidal line on the reciprocal surface, whose singularities are got by interchanging μ and ν, α and β, &c. in the above formulæ.

The class of the developable touching along UK, which is the degree of a double curve on the reciprocal surface, is seen as above to be $n(n-1)(n-2)(n^3 - n^2 + n - 12)$. Its other singularities will be obtained in the next section, where we shall also determine the number of solutions in some cases where a tangent plane is required to fulfil two other conditions.

THEORY OF RECIPROCAL SURFACES.

482. Understanding by the ordinary singularities of a surface, those which in general exist either on the surface or its reciprocal, we may make the following enumeration of them. A surface may have a double curve of degree b and a cuspidal of degree c. The tangent cone determined as in Art. 246, includes doubly the curve standing on the double curve, and trebly that standing on the cuspidal curve, so that if the degree of the tangent cone proper be a, we have

$$a + 2b + 3c = n(n-1).$$

The class of the cone a is the same as the degree of the reciprocal. Let a have δ double and κ cuspidal edges. Let b have k apparent double points, and t triple points which are also triple points on the surface; and let c have h apparent double points. Let the curves b and c intersect in γ points, which are stationary points on the former, in β which are stationary points on the latter, and in i which are singular points on neither. Let the curve of contact a meet b in ρ points, and c in σ points. Let the same letters accented denote singularities of the reciprocal surface.

483. We saw (Art. 247) that the points where the curve of contact meets $\Delta^2 U$ give rise to cuspidal edges on the tangent cone. But when the line of contact consists of the complex curve $a + 2b + 3c$, and when we want to determine the number of cuspidal edges on the cone a, the points where b and c meet $\Delta^2 U$ are plainly irrelevant to the question. Neither shall we have cuspidal edges answering to all the points where a meets $\Delta^2 U$, since a common edge of the cones a and c is to be regarded as a cuspidal edge of the complex cone, although not so on either cone considered separately. The following formulæ contain an analysis of the intersections of each of the curves a, b, c, with the surface $\Delta^2 U$,

$$\left.\begin{aligned} a(n-2) &= \kappa + \rho + 2\sigma \\ b(n-2) &= \rho + 2\beta + 3\gamma + 3t \\ c(n-2) &= 2\sigma + 4\beta + \gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (A).$$

The reader can see without difficulty that the points indicated in these formulæ are included in the intersections of $\Delta^2 U$ with a, b, c, respectively: but it is not so easy to see the reason for the numerical multipliers which are used in the formulæ. Although it is probably not impossible to account for these constants by *a priori* reasoning, I prefer to explain the method by which I was led to them inductively.*

484. We know that the reciprocal of a cubic is a surface of the twelfth degree which has a cuspidal edge of the twenty-fourth degree, since its equation is of the form $64S^3 = T^2$, where S is of the fourth, and T of the sixth degree (p. 376). Each of the twenty-seven lines on the surface answers to a double line on the reciprocal (p. 378). The proper tangent cone, being the reciprocal of a plane section of the cubic, is of the sixth degree, and has nine cuspidal edges. Thus we have $a' = 6$, $b' = 27$, $c' = 24$, $n' = 12$, $a' + 2b' + 3c' = 12.11$. The intersections of the curves c' and b' with the line of contact of a cone a' through any assumed point, answer to tangent planes to the original cubic, whose points of contact are the intersections of an assumed plane with the parabolic curve UH, and with the twenty-seven lines. Consequently there are twelve points σ', and twenty-seven points ρ'; one of the latter points lying on each of the lines of which the nodal line of the reciprocal surface is made up.

Now the sixty points of intersection of the curve a' with the second polar which is of the tenth degree, consist of the nine points κ', the twenty-seven points ρ', and the twelve points σ'. It is manifest then that the last points must

* The first attempt to explain the effect of nodal and cuspidal lines on the degree of the reciprocal surface, was made in the year 1847 in two papers which I contributed to the *Cambridge and Dublin Mathematical Journal*, Vol. II., p. 65, and IV., p. 188. It was not till the close of the year 1849, however, that the discovery of the twenty-seven right lines on a cubic, by enabling me to form a clear conception of the nature of the reciprocal of a cubic, led me to the theory in the form here explained. Some few additional details will be found in a memoir which I contributed to the *Transactions of the Royal Irish Academy*, Vol. XXIII., p. 461.

count double, since we cannot satisfy an equation of the form $9a + 27b + 12c = 60$, by any integer values of a, b, c except 1, 1, 2. Thus we are led to the first of the equations (A).

Consider now the points where any of the twenty-seven lines b meets the same surface of the tenth order. The points β' answer to the points where the twenty-seven right lines touch the parabolic curve; and there are two such points on each of these lines (Art. 256). There are also five points t on each of these lines (Art. 455), and we have just seen that there is one point ρ. Now since the equation $a + 2b + 5c = 10$, can have only the systems of integer solutions (1, 2, 1) or (3, 1, 1), the ten points of intersection of one of the lines with the second polar must be made up either $\rho' + 2\beta' + t'$, or $3\rho' + \beta' + t'$, and the latter form is manifestly to be rejected. But considering the curve b' as made up of the twenty-seven lines, the points t' occur each on three of these lines: we are then led to the formula $b'(n-2) = \rho' + 2\beta' + 3t'$.

The example we are considering does not enable us to determine the coefficient of γ in the second formula A, because there are no points γ on the reciprocal of a cubic.

Lastly, the two hundred and forty points in which the curve c meets the second polar are made up of the twelve points σ', and the fifty-four points β'. Now the equation $12a + 54b = 240$ only admits of the systems of integer solutions (11, 2), or (2, 4), and the latter is manifestly to be preferred. In this way we are led to assign all the coefficients of the equations (A) except those of γ.

485. Let us now examine in the same way the reciprocal of a surface of the n^{th} order, which has no multiple points. We have then $n' = n(n-1)^2$, $n' - 2 = (n-2)(n^2+1)$, $a' = n(n-1)$; and for the nodal and cuspidal curves we have (Art. 255)

$$b' = \tfrac{1}{2}n(n-1)(n-2)(n^3 - n^2 + n - 12), \quad c' = 4n(n-1)(n-2).$$

The number of cuspidal edges on the tangent cone to the reciprocal, answering to the number of points of inflexion on a plane section of the original, gives us $\kappa' = 3n(n-2)$. The points ρ' and σ', answer to the points of intersection of an assumed plane with the curves UK and UH (Art. 480):

hence $\rho' = n(n-2)(n^3 - n^2 + n - 12)$, $\sigma' = 4n(n-2)$. Substitute these values in the formula $a'(n'-2) = \kappa' + \rho' + 2\sigma'$, and it is satisfied identically, thus verifying the first of formulæ (A).

We shall next apply to the same case the third of the formulæ (A). It was proved (Art. 481) that the number of points β' is $2n(n-2)(11n-24)$. Now the intersections of the nodal and cuspidal curves on the reciprocal surface answer to the planes which touch at the points of meeting of the curves UH, and UK on the original surface. If a plane meet the surface in a section having an ordinary double point and a cusp, since from the mere fact of its touching at the latter point it is a double tangent plane, it belongs in two ways to the system which touches along UK; or, in other words, it is a stationary plane of that system. And since evidently the points β' are to be included in the intersections of the nodal and cuspidal curve, the points U, H, K must either answer to points β' or points γ'. Assuming, as it is natural to do, that the points β count double among the intersections of UHK, we have

$$\gamma' = n\{4(n-2)\}.\{(n-2)(n^3 - n^2 + n - 12)\} - 4n(n-2)(11n-24)$$
$$= 4n(n-2)(n-3)(n^3 + 3n - 16).$$

But if we substitute the values already found for c', n', σ', β', the quantity $c'(n'-2) - 2\sigma' - 4\beta'$ becomes also equal to the value just assigned for γ'. Thus the third of the formulæ A is verified. It would have been sufficient to assume that the points β count μ times among the intersections of UHK, and to have written the third of the formulæ provisionally,

$$c(n'-2) = 2\sigma + 4\beta + \lambda\gamma,$$

when, proceeding as above, it would have been found that the formulæ could not be satisfied unless $\lambda = 1$, $\mu = 2$.

It only remains to examine the second of the formulæ (A). We have just assigned the values of all the quantities involved in it except t'. Substituting then these values, we find that the number of triple tangent planes to a surface of the n^{th} degree is given by the formula

$$6t' = n(n-2)(n^7 - 4n^6 + 7n^5 - 45n^4 + 114n^3 - 111n^2 + 548n - 960).$$

486. It was proved (Art. 248) that the points of contact of those edges of the tangent cone which touch in two distinct points lie on a certain surface of the degree $(n-2)(n-3)$. Now when the tangent cone is, as before, a complex cone $a+2b+3c$, it is evident that among these double tangents will be included those common edges of the cones ab, which meet the curves a, b in distinct points: and similarly for the other pairs of cones. If then we denote by $[ab]$ the number of the apparent intersections of the curves a and b; that is to say, the number of points in which these curves seen from any point of space seem to intersect, though they do not actually do so; the following formulæ will contain an analysis of the intersections of a, b, c, with the surface of the degree $(n-2)(n-3)$:

$$a(n-2)(n-3) = 2\delta + 3[ac] + 2[ab],$$
$$b(n-2)(n-3) = 4k + [ab] + 3[bc],$$
$$c(n-2)(n-3) = 6h + [ac] + 2[bc].$$

Now the number of apparent intersections of two curves is at once deduced from that of their actual intersections. For if cones be described having a common vertex and standing on the two curves, their common edges must answer either to apparent or actual intersections. Hence,

$$^*[ab] = ab - 2\rho, \quad [ac] = ac - 3\sigma, \quad [bc] = bc - 3\beta - 2\gamma - i.$$

Substituting these values, we have

$$\left.\begin{aligned} a(n-2)(n-3) &= 2\delta + 2ab + 3ac - 4\rho - 9\sigma \\ b(n-2)(n-3) &= 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho \\ c(n-2)(n-3) &= 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma \end{aligned}\right\} \ldots (B).$$

The first and third of these equations are satisfied identically if we substitute for β, γ, ρ, σ, &c. the values used in the last

* If the surface have a nodal curve, but no cuspidal, there will still be a determinate number i of cuspidal points on the nodal curve, and the above equation receives the modification $[ab] = ab - 2\rho - i$. In determining however the degree of the reciprocal surface the quantity $[ab]$ is eliminated.

article, to which we are to add $2\delta = n(n-2)(n^2-9)$, $i=0$, and the value of h given (Art. 481), viz.

$$2h = n(n-2)(16n^4 - 64n^3 + 80n^2 - 108n + 156).$$

The second equation enables us to determine k by the equation

$$8k = n(n-2)(n^{10} - 6n^9 + 16n^8 - 54n^7$$
$$+ 164n^6 - 288n^5 + 547n^4 - 1058n^3 + 1068n^2 - 1214n + 1464);$$

from this expression the rank of the developable of which b is the cuspidal edge can be calculated by the formula

$$R = b^2 - b - 2k - 6t - 3\gamma.$$

Putting in the values already obtained for these quantities, we find

$$R = 4n(n-2)(n-3)(n^2 + 2n - 4).$$

This is then the rank of the developable formed by the planes which have double contact with the given surface.*

487. From formulæ A and B we can calculate the diminution in the degree of the reciprocal caused by the singularities on the original surface enumerated Art. 482. If the degree of a cone diminish from m to $m-l$, that of its reciprocal diminishes from $m(m-1)$ to $(m-l)(m-l-1)$; that is to say, is reduced by $l(2m-l-1)$. Now the tangent cone to a surface is in general of the degree $n(n-1)$, and we have seen that when the surface has nodal and cuspidal lines this degree is reduced

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* In order to verify the theory it would be necessary to show that this number R coincides with what may be deduced from Ex. 5, p. 404. In the first place the developable generated by the cuspidal curve on the reciprocal surface corresponds with that which envelopes the given surface along UH, and which, by the example cited, ought to be of the degree $28(n-2)^2$, but if we subtract from this the number β, we get the value already determined. In like manner, if we take the surface enveloping the given surface along UK (Art. 480) and subtract from the degree determined, as in the example cited, $4\gamma + \beta + 6t$, we get not R but $\frac{1}{2}R$. Possibly this may be because all the tangent planes which envelope the developable in question are double tangent planes; but it must be owned that there are points in all this theory which need further explanation.

by $2b+3c$. There is a consequent diminution in the degre of the reciprocal surface

$$D=(2b+3c)(2n^2-2n-2b-3c-1).$$

But the existence of nodal and cuspidal curves on the surfac causes also a diminution in the number of double and cuspid edges in the tangent cone. From the diminution in the degre of the reciprocal surface just given must be subtracted twic the diminution of the number of double edges and three time that of the cuspidal edges. Now from formulæ A, we have

$$\kappa=(a-b-c)(n-2)+6\beta+4\gamma+3t.$$

But since if the surface had no multiple lines the number of cuspidal edges on the tangent cone would be $(a+2b+3c)(n-2$ the diminution of the number of cuspidal edges is

$$K=(3b+4c)(n-2)-6\beta-4\gamma-3t.$$

Again, from the first system of equations (Art. 486), we have

$$(a-2b-3c)(n-2)(n-3)=2\delta-8k-18h-12[bc],$$

and putting for $[bc]$ its value

$$2\delta=(a-2b-3c)(n-2)(n-3)+8k+18h+12bc-36\beta-24\gamma-1$$

But if the surface had no multiple lines 2δ would

$$=(a+2b+3c)(n-2)(n-3).$$

The diminution then in the number of double edges is giv by the formulæ

$$2H=(4b+6c)(n-2)(n-3)-8k-18h-12bc+36\beta+24\gamma+1$$

The entire diminution then in the degree of the reciproc $D-3K-2H$ is, when reduced,

$$n(7b+12c)-4b^2-9c^2-8b-15c+8k+18h-18\beta-12\gamma-12i+$$

488. The formulæ A and B can be thrown into a form mo convenient for use. If we remember that $a+2b+3c=n(n-1$ the first of formulæ B may be written

$$a^2+a(-4n+6)=2\delta-4\rho-9\sigma,$$

or, adding three times the first of formulæ A,

$$a^2-na=2\delta+3\kappa-\rho-3\sigma.$$

But $a^2 - a - 2\delta - 3\kappa$ is n' the degree of the reciprocal surface. Hence

$$(n-1)\,a = n' + \rho + 3\sigma.$$

The truth of this equation may be otherwise seen from the consideration that a, the curve of simple contact from any one point, intersects the first polar of any other point, either in the n' points of contact of tangent planes passing through the line joining the two points, or else in the ρ points where a meets b, or the σ points where it meets c, since every first polar passes through the curves b, c.

Adding the second of formulæ B to four times the second of formulæ A, and giving R the same meaning as in Art. 486, we get, in like manner,

$$2\rho = 2R + \beta + 3i,$$

an equation of which I do not see the geometrical explanation; although evidently the R points on b the tangents at which meet any line are included among the ρ points on b which are points of contact of tangent planes through that line.

If the last of each of the formulæ be treated in like manner, and if we call S the order of the developable generated by the curve c; that is, if we write

$$S = c^2 - c - 2h - 3\beta,$$

we have $$c\,(n-6) = 12S + \gamma + 8i - 18\sigma.$$

489. The effect of multiple lines in diminishing the degree of the reciprocal may be otherwise investigated. The points of contact of tangent planes which can be drawn through a given line are the intersections with the surface of the curve of degree $(n-1)^2$ which is the intersection of the first polars of any two points on the line. Now let us first consider the case when the surface has only an ordinary double curve of degree b. The first polars of the two points pass each through this curve, so that their intersection breaks up into this curve b and a complemental curve d. Now in looking for the points of contact of tangent planes through the given line, in the first place, instead of taking the points where the complex curve $b+d$ meets the surface, we are only to take those in

[illegible] the degree [illegible] all the [illegible] which it meets [illegible] being in number [illegible] system b. [illegible] the curve b, [illegible] which we are [illegible] given surface, and of [illegible] surfaces touch. [illegible] that is [illegible]

[illegible]

DEVELOPABLE SURFACES

[illegible]

developable reduces to nothing. This application of the theory both verifies the theory itself and enables us to determine some singularities of developables not given, p. 239. We use the notation of the section referred to. The tangent cone to a developable consists of n planes; it has therefore no cuspidal edges and $\frac{1}{2}n(n-1)$ double edges. The simple line of contact (a) consists of n lines of the system each of which meets the cuspidal edge m once, and the double line x in $(r-4)$ points. The lines m and x intersect at the α points of contact of the stationary planes of the system; for since there three consecutive lines of the system are in the same plane, the intersection of the first and third gives a point on the line x.*

We have then the following table. The letters on the left-hand side of the equations refer to the notation of this Chapter and those on the right to that of Chapter XI.:

$$n=r,\ a=n,\ b=x,\ c=m;\ \rho=n(r-4),\ \sigma=n,\ \kappa=0,\ \beta=\beta,\ h=h,\ i=\alpha;$$

and the quantities t, γ, k remain to be determined. On substituting these values in formulæ A and B, pp. 414, 418, we get the system of equations

$$\left.\begin{aligned}
n(r-2) &= n\{2+(r-4)\},\\
x(r-2) &= n(r-4)+2\beta+3\gamma+3t,\\
m(r-2) &= 2n+4\beta+\gamma,\\
n(r-2)(r-3) &= n\{(n-1)+2x+3m-4(r-4)-9\},\\
x(r-2)(r-3) &= 4k+nx+3mx-9\beta-6\gamma-3\alpha-2n(r-4),\\
m(r-2)(r-3) &= 6h+mn+2mx-6\beta-4\gamma-2\alpha-3n
\end{aligned}\right\}\ldots(C).$$

The first of these equations is identically true, and the fourth is satisfied by the help of the equation, proved p. 236,

$$r(r-1)=2x+3m+n.$$

If we eliminate γ between the third and sixth equations, we obtain also an equation already proved to be true (Chapter XI.). The three remaining equations determine the three quantities, whose values have not before been given, viz. t the number of "points on three lines" of the system; γ the number of

* It is only on account of their occurrence in this example that I was led to include the points i in the theory.

points of the system through each of which passes another non-consecutive line of the system; and k the number of apparent double points of the nodal line of the developable. These quantities being determined, we can by an interchange of letters write down the reciprocal singularities, viz. the number of "planes through three lines," &c.

From Art. 488 can be deduced R the rank of the developable of which x is the cuspidal edge. For we have

$$2R = 2n(r-4) - \beta - 3a.$$

Ex. 1. Let it be required to apply the preceding theory to the case considered, Art. 297.

Ans. $\gamma = 6(k-3)(k-4)$, $3t = 4(k-3)(k-4)(k-5)$,

$$k = (k-3)(2k^3 - 18k^2 + 57k - 65),\ R = 2(k-1)(k-3).$$

And for the reciprocal singularities

$\gamma' = 2(k-2)(k-3)$, $3t' = 4(k-2)(k-3)(k-4)$,

$$k' = (k-2)(k-3)(2k^2 - 10k + 11),\ R' = 6(k-3)^2.$$

Ex. 2. Two surfaces intersect the sum of whose degrees are p and their products q.

Ans. $\gamma = q(pq - 2q - 6p + 16)$.

This follows from the table, p. 250, but can be proved directly, see *Transactions of the Royal Irish Academy*, Vol. XXIII., p. 470,

$$R = 3q(p-2)\{q(p-3) - 1\}.$$

Ex. 3. To find the singularities of the developable generated by a line resting twice on a given curve. The planes of this system are evidently "planes through two lines" of the original system: the class of the system is therefore y; and the other singularities are the reciprocals of those of the system whose cuspidal edge is x, calculated in this article. Thus the rank of the system, or the order of the developable is given by the formula

$$2R' = 2m(r-4) - a - 3\beta.$$

491. Since the degree of the reciprocal of a ruled surface reduces always to the degree of the original surface (p. 349) the theory of reciprocal surfaces ought to account for this reduction. I have not obtained this explanation for ruled surfaces in general, but some particular cases are examined and accounted for in the Memoir in the *Transactions of the Royal Irish Academy* already cited. I give only one example here.

Let the equation of the surface be derived from the elimination of t between the equations

$$at^k + bt^{k-1} + \&c. = 0, \quad a't^l + b't^{l-1} + \&c. = 0,$$

where a, a', &c. are any linear functions of the co-ordinates. Then if we write $k + l = \mu$, the degree of the surface is μ, having a double line of the order $\frac{1}{2}(\mu - 1)(\mu - 2)$, on which are $\frac{1}{6}(\mu - 2)(\mu - 3)(\mu - 4)$ triple points. For the apparent double points of this double curve, we have

$$2k = \tfrac{1}{4}(\mu - 2)(\mu - 3)(\mu^2 - 5\mu + 8);$$

and the developable generated by that curve is of the order $2(\mu - 2)(\mu - 3)$. It will be found then that we have

$$a = 2(\mu - 1), \quad b = \tfrac{1}{2}(\mu - 1)(\mu - 2), \quad \kappa = 3(\mu - 2), \quad \delta = 2(\mu - 2)(\mu - 3)$$

values which agree with what was proved, Art. 459, viz. that the number of cuspidal edges in the tangent cone is diminished by $3b(\mu - 2) - 3t$; while the double edges are diminished by $2b(n - 2)(n - 3) - 4k$. In verifying the separate formulæ B the remark, note, p. 418, must be attended to.

492. It may be mentioned here that the Hessian of a ruled surface meets the surface only in its multiple lines, and in the generators each of which is intersected by one consecutive. For, p. 347, if xy be any generator, that part of the equation which is only of the first degree in x and y is of the form $(xz + yw)\phi$. Then, Art. 256, the part of the Hessian which does not contain x and y is

$$\left\{\left(\phi + z\frac{d\phi}{dz}\right)\left(\phi + w\frac{d\phi}{dw}\right) - wz\frac{d\phi}{dz}\frac{d\phi}{dw}\right\}^2,$$

which reduces to ϕ^4. But xy intersects ϕ only in the points where it meets multiple lines. But if the equation be of the form $ux + vy^2$ (Art. 256) the Hessian passes through xy. Thus in the case we have considered the number of lines which meet one consecutive are easily seen to be $2(\mu - 2)$; and the curve UH whose order is $4\mu(\mu - 2)$ consists of these lines each counting for two and therefore equivalent to $4(\mu - 2)$ in the intersection; together with the double line equivalent to $4(\mu - 1)(\mu - 2)$. Again, if a surface have a multiple line whose degree is m, and order of multiplicity p, it will be a line of order $4(p-1)$ on the Hessian, and will be equivalent to $4mp(p-1)$ on the curve UH. Now the ruled surface generated by a line resting on two right lines and on a curve m (which is supposed to have no actual multiple point) is of order $2m$, having the right lines as multiples of order m; having $\frac{1}{2}m(m-1)+h$ double generators, and $2r$ generators which meet a consecutive one. Comparing then the order of the curve UH with the sum of the orders of the curves of which it is made up, we have

$$16m(m-1) = 8m(m-1) + 4m(m-1) + 8h + 4r = 0,$$

an equation which is identically true.

If we form the Hessian of the developable $xu + yv^2$, it appears in like manner that we get the developable itself multiplied by a series of terms, in which the part independent of x and y is $v\left\{\frac{d^2u}{dz^2}\frac{d^2u}{dw^2} - \left(\frac{d^2u}{dz\,dw}\right)^2\right\}$. This proves that the Pro-Hessian (see p. 338) meets each generator in the first place where that generator meets v; that is to say, twice in the point on the curve m, and in $r-4$ points on the curve x; and in the second place where the generator meets the Hessian of u; that is to say, in the Hessian of the system formed by those $r-4$ points combined with the point on m taken three times; in which Hessian the latter point will be included four times. The intersection then of the generator with the Pro-Hessian consists of the point on m taken six times, of the $r-4$ points on x, and of $2(r-5)$ other points.

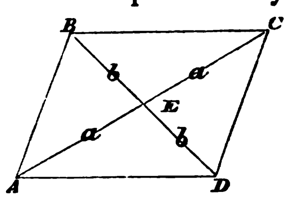
APPENDIX I.

ON THE CALCULUS OF QUATERNIONS.

1. THE Calculus of Quaternions having been successfully employed by its inventor Sir W. R. Hamilton in the deduction of geometrical theorems, it may seem proper to add some account of it to that which has been given in the preceding pages of other methods of investigating the properties of space of three dimensions. Neither the space now at my disposal, nor my knowledge of the subject, allow me to attempt here to teach this calculus; but in the following sketch I hope to give the reader some idea what quaternions are, and how they may be used in geometrical enquiries; referring him for further information to Sir W. R. Hamilton's papers "On Symbolical Geometry" in the *Cambridge and Dublin Mathematical Journal*, to his "Lectures," and to his forthcoming "Elements of Quaternions."

Vectors. In Algebraic Geometry though the symbols x, y, z, &c. are used each with reference to a line measured in a certain assigned direction, yet in the equations employed these symbols denote merely the *magnitudes* of the lines which they represent; and the equations only express that certain arithmetical operations are to be performed on the numbers which express the ratios of each of the lines x, y, z to the linear unit. Thus if we form the sum $x+y+z$ of three known lines, the result is a line of determinate length but of no assigned direction. In the quaternion calculus a symbol denoting a line must always express direction as well as length; and if for instance we form the sum $x+y+z$, it is necessary to assign the direction as well as the length of the line which is the result. In this calculus then the signs + and − are used not with reference to numerical addition or subtraction, but with reference to direction (as we proceed to explain), and denote geometrical, not algebraical, addition and subtraction.

2. Let the line or vector AB be understood to denote the operation of proceeding from the point A to the point B; then BC in like manner would denote the operation of proceeding from B to C. The sign + may naturally be employed to denote the consecutive performance of these two operations; thus $AB+BC$ would denote that we proceed first from A to B, and then from B to C; and since the result is the same as if we had gone direct from A to C, we have $AB+BC=AC$. The sum of two vectors then is the diagonal of the parallelogram of which these lines are adjacent sides. If AB and BC were portions of the same right line, then their sum would be the ordinary algebraic sum of the two lines; and it is easy to see by successive addition that if a denote any vector, and m any arithmetical multiplier, ma denotes a vector coincident in direction with that represented by a, and in length bearing to it the ratio $m:1$. Two vectors are said to be equal if one can be moved without rotation so as to coincide with the other: that is to say, two equal lengths measured on parallel lines are said to be equal. By the help of this convention we can interpret and verify the equation $a+b=b+a$. Let the vector a be represented by either of the equal lines AE, EC, and b by either of the equal lines DE, EB; then if we take a first we have $a+b=AB$, but if we commence with b we have $b+a=DC$; and these results are equal since AB and DC are equal and parallel.



It is evident on interpretation of the equation that

$$(a+b)+c=a+(b+c).$$

Thus we see that the sign + when geometrically interpreted as here proposed, conforms to the ordinary rules of algebraic addition, viz., to the *commutative* law $a+b=b+a$, and the *associative* law $(a+b)+c=a+(b+c)$.

3. Denoting, as before, by AB the operation of going from A to B, $-AB$ naturally denotes the reversing of this operation, viz., that of going from B to A, so that $AB+BA=0$. It can easily be deduced hence that if $a+b=c$, $a=c-b$. Since the addition of lines according to the method just explained corresponds exactly to the composition of mechanical forces acting on a point, we can prove, as in Mechanics, that any line may be resolved into the sum of three lines whose directions are those of three given rectangular axes. If now unit lines measured along the axes of x, y, z respectively be denoted by i, j, k; and if the numerical ratios which the lengths of the co-ordinates of any point P bear to the unit line be denoted, as in algebraic geometry, by x, y, z, then in this calculus these co-ordinates will be denoted by ix, jy, kz respectively, and the vector from the origin to P will be denoted by $ix+jy+kz$. And since any vector is equal to a parallel one through the origin, *there is no vector which may not be expressed in the form* $ix+jy+kz$.

If α, β be any two co-initial vectors it is easy to see that $\dfrac{l\alpha+m\beta}{l+m}$ is a vector drawn from the same origin to the point where the line joining their extremities is cut in the ratio $l:m$, and that $\dfrac{l\alpha+m\beta+n\gamma}{l+m+n}$ denotes a vector terminating in the plane through the extremities of α, β, γ. If α and β be both of unit length, $l\alpha+m\beta$ makes with α and β angles whose sines are in the ratio $l:m$. These principles may be used to establish geometrical theorems. Thus $\frac{1}{4}(\alpha+\beta+\gamma+\delta)$ is the vector to the centre of gravity of the tetrahedron formed by the extremities of α, β, γ, δ; from which form inferences may be deduced as in Ex., p. 6.

4. *Quaternions.* We have now shown how lines considered with respect to their direction as well as to their magnitude may be added and subtracted, and we come next to speak of multiplication and division. It is not obvious what sense we are to attach to the product of two lines, but it is natural to interpret the quotient $\dfrac{\alpha}{\beta}$ as denoting the operation necessary

to change the line β into the line α so that $\frac{\alpha}{\beta}\beta=\alpha$. If the vectors α and β be portions of the same line, it is evident that the quotient is a numerical constant, or, as Sir W. R. Hamilton calls it, a *scalar:* but, when this is not the case, in order to change β into α we have not only suitably to alter its length, but also to turn it through a certain angle in a certain plane. Now we have seen that a vector is reducible to the sum of three distinct terms, and we might have foreseen this because in order to determine a vector we must know three things, viz., its length, and its direction-cosines, equivalent to two more conditions. But to determine a geometrical quotient four things are necessary, viz., the numerical ratio of the lengths of the two lines compared, the angle through which one must be turned in order to coincide with the other, and the direction-cosines of the plane of that angle, equivalent to two more conditions. We shall presently show how to express any such quotient as the sum of four irreducible terms: it is thence called a quaternion. It is agreed on that the four elements just mentioned shall be *sufficient* to determine such a quotient as we are considering: that is to say, that two quotients are said to be equal $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$, first, if the lengths of the lines be proportional, $\alpha:\beta::\gamma:\delta$; secondly, if the angle between α and β be equal to that between γ and δ; and thirdly, if all four lines be parallel to the same plane. In other words the geometrical ratio of two lines is considered unchanged, not only if both be increased or diminished in the same proportion, but also if they be turned round in their plane their mutual inclination being unaltered.

5. Two geometrical fractions having a common denominator are added by adding their numerators: that is to say, we have $\frac{\alpha}{\delta}+\frac{\beta}{\delta}=\frac{\alpha+\beta}{\delta}$ as in common algebra. We can thus reduce any such fraction to one, the two lines in which are at right angles to each other. For if the fraction be, γ divided by δ, we can resolve γ into the sum of two lines $\alpha+\beta$,

one of them in the direction of δ (in fact the projection of γ on δ), and the other perpendicular to it. Now since α is supposed to be in the same direction as δ, their ratio is a mere number or scalar, while the ratio of β to δ is that of two rectangular lines. Thus then we can reduce every quaternion to the form $S+V$, the sum of a scalar part and a vector part, the latter part being so called because we shall presently see that the ratio of two rectangular lines can be adequately represented by a vector perpendicular to their plane.

A quaternion may be resolved in another way; viz. into a numerical factor multiplied by the ratio of two equal lines. We have obviously $\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma}=\frac{\alpha}{\gamma}$; for if we first turn γ into β and then β into α, the result is evidently the turning γ into α. If now β be supposed to be a line equal to γ, and in the direction of α, the ratio of α to β is a mere number; and the ratio α to γ is resolved into the product of this number into the ratio of the equal lines β and γ. Sir W. R. Hamilton calls this the resolution of a quaternion into the product of a *tensor* and a *versor:* the tensor being the number expressing in what ratio the line γ is to be increased or diminished in order to be made equal to β, and the versor expressing through what angle it is to be turned.

Thus suppose that the symbol I denotes the operation of turning a line round through a right angle in a plane perpendicular to the vector i: [in order to fix the ideas we may agree that the direction of the rotation shall be that of the hands of a watch as we look along i:] then mI denotes the operation of turning the line round as before, and at the same time altering the length in the ratio $m:1$.

Thus then if the denominator of a fraction be a line of unit length, and its numerator of length l; if the angle between them be θ, and the unit vector perpendicular to their plane be ρ, we may first resolve l into the portions $l\cos\theta$, $l\sin\theta$ measured in the direction of the denominator and perpendicular to it, and if V denote the operation of turning through a right angle round the axis ρ, without change of length, the given fraction is resolved into the parts $l\cos\theta + l\sin\theta . V$.

If the position of the numerator and denominator had been interchanged, it is easy to see that the operation of turning through the same angle in the opposite direction would have been expressed $l\cos\theta - l\sin\theta . V$.

6. If ρ, α, β be three vectors such that $\rho = \alpha + \beta$, and if V, A, B represent rectangular rotations perpendicular to these vectors as above explained, then $V = A + B$. For (see fig., p. 358) let $\rho = OS$, $\alpha = OT$, $\beta = TS$, and let OP, OQ, QP be equal and perpendicular to these lines, then if OR be a line perpendicular to the plane of the paper equal in length to OS, we have $\frac{OP}{OR} = V$, $\frac{OQ}{OR} = A$, $\frac{QP}{OR} = B$, therefore $V = A + B$.

It follows then that the symbols of rectangular rotation may be resolved in precisely the same way as the vectors in Art. 3; and, therefore, that if I, J, K denote rotations without change of length round the three axes respectively: then a similar rotation round an axis ρ, making with these the angles α, β, γ, may be resolved into the sum $I\cos\alpha + J\cos\beta + K\cos\gamma$. And in like manner the fraction partially resolved in the last article may be completely resolved into the sum

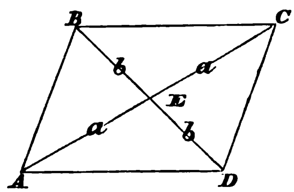
$$l\cos\theta + l\sin\theta\,(I\cos\alpha + J\cos\beta + K\cos\gamma).$$

We see then that the most general expression for a geometrical fraction is of the form $a + bI + cJ + dK$, where a, b, c, d are numerical constants. It is because it can thus be reduced to the sum of four terms that it is called a quaternion.

7. Multiplication of fractions, as already intimated, denotes the successive performance of the operations represented by the factors. Thus $\frac{\alpha}{\beta}.\frac{\beta}{\gamma}$ denotes that we first perform the operation of turning γ into β, and then that of turning β into α, the result being the same as if we turned γ into α. To multiply any two fractions $\frac{\alpha}{\beta}.\frac{\gamma}{\delta}$, it is only necessary to turn $\frac{\gamma}{\delta}$ round in its plane until its numerator coincide with the intersection

of the planes of the two fractions; and $\frac{\alpha}{\beta}$ until its denominator coincide with the same line, when the multiplication is performed as before.

It at once appears hence that when we multiply two quaternions, the order of the factors is not indifferent. Thus, let A, B, C, D represent four points on a sphere of which O is the centre. Then if we first turn OD to OE through an angle b, and then OE to OC through an angle a, the result is the operation of turning OD to OC.



But if we had commenced with the operation of turning through the angle a, which is that of turning OA to OE, and then OE to OB through the angle b, the result is the operation of turning OA to OB. Now though the arc AB is equal to CD, the plane of AB is generally different from that of CD, and therefore the product $\frac{OC}{OE} \cdot \frac{OE}{OD}$ is not equal to $\frac{OB}{OE} \cdot \frac{OE}{OA}$, which is the product of two equal factors taken in opposite order.

If the arcs a and b be each 90°, then indeed the plane of AB will be the same as that of CD, but the direction of the rotations in the two products will be opposite. If then we multiply together two rectangular quaternions A, B, (that is, such that the rotation is through a right angle) we see from Art. 5 that if $A.B$ be of the form $l\cos\theta + l\sin\theta.V$, then $B.A$ will be of the form $l\cos\theta - l\sin\theta.V$. Two quaternions thus related are said to be conjugate quaternions: that is, when one is of the form scalar + vector: and the other, the same scalar − the same vector.

It follows as a particular case of the last, that when $\theta = 90°$, the product of two rectangular quaternions whose planes are at right angles to each other, gives $A.B = -B.A$. As this is a fundamental theorem we shall presently prove it independently.

8. It is seen without difficulty that the multiplication of quaternions is a distributive operation: viz., that the product of the quaternions $\left(\frac{\alpha+\beta+\gamma+\delta+\mu}{\lambda}\right)\frac{\lambda}{\mu}$ is the sum of the

several products $\frac{\alpha}{\lambda}\frac{\lambda}{\mu}$, $\frac{\beta}{\lambda}\frac{\lambda}{\mu}$, &c.; and that the same thing is true if the order of multiplication be reversed. Hence then if we have two quaternions,* each expressed in the form

$$(a+bI+cJ+dK)(a'+b'I+c'J+d'K),$$

the product is the sum of the sixteen terms got by combining each of the first four terms with each of the second four, care however being taken to attend to the order of the multiplication. Let us then examine the meaning of the terms II, IJ, &c., which occur in such a product. Now if we remember that I denotes a rectangular rotation round the axis of x as axis, and that the effect of such a rotation would be to change a line in the direction of the axis of y to that of z, and one in the direction of z into the negative direction of y, we can write down the equations $Ij=k$, $Ik=-j$. In like manner, $Jk=i$, $Ji=-k$; $Ki=j$, $Kj=-i$. Let us now consider the effect of two of these operations performed consecutively. If we first operate on j with I, and then again with I on the result k, we get $I^2j=-j$, or $I^2=-1$. In like manner $J^2=-1$, $K^2=-1$, and since it is evidently true, no matter what line be taken for the axis of rotation, that the effect of twice turning round a right angle is to reverse the position of the line operated on; it follows that the square of every rectangular quaternion may be said to be -1.

Again we have seen that $Ij=k$, $Jk=i$; hence $JIj=i$; but $Kj=-i$; hence $JI=-K$. In like manner, from the equations $Ji=-k$, $Ik=-j$, $Ki=j$, we conclude $IJ=K$. Hence $IJ=K=-JI$. In like manner $JK=I=-KJ$: $KI=J=-IK$.

If now we compare the equations $Ij=k$, $IJ=K$, &c., we shall find that the equations which represent the effect of the operations I, J, K on the lines i, j, k, are exactly the same in form as those which denote the effects of the successive performance of these operations. Now since in the practice of this calculus we are concerned with the laws according to

* It is also true, though it is not to be taken for granted, that when we take the continued product of three quaternions $(qq')q''=q(q'q'')$.

which the symbols combine with each other rather than with their interpretation, it is found unnecessary to keep up the distinction of notation between I, J, K; i, j, k. Whatever propositions are true of the symbols in the one sense, are equally true in the other, and, by interpreting some vectors as lines and others as rotations, we can give a variety of significations to the same equation all of which will be equally true. We shall then understand i to denote at pleasure either a unit line measured along the direction of the axis of x, or a rotation through a right angle round that axis. In like manner a rectangular rotation round any unit vector α is represented by the letter α as already stated in Art. 5. We shall write the general form of a quaternion $a+bi+cj+dk$; and we shall combine these symbols according to the laws $i^2=j^2=k^2=-1$; $ij=k=-ji$; $jk=i=-kj$; $ki=j=-ik$.

In forming the continued product of a number of factors the order must be carefully attended to, except that if a scalar or number is one of the factors its order is indifferent, and it may be brought to the left hand as a multiplier of the whole. Thus, if α, β, γ be any three unit vectors, or rectangular quaternions, and if we multiply $\beta\gamma$ by $\alpha\beta$ the result $\alpha\beta^2\gamma$ is $-\alpha\gamma$, since $\beta^2=-1$.

Ex. 1. To form the square of the unit vector $i\cos\alpha + j\cos\beta + k\cos\gamma$. By actual multiplication, we get

$$i^2\cos^2\alpha + j^2\cos^2\beta + k^2\cos^2\gamma + (jk+kj)\cos\beta\cos\gamma + (ki+ik)\cos\gamma\cos\alpha + (ij+ji)\cos\alpha\cos\beta,$$

which, in virtue of the relations connecting i, j, k, reduces to

$$-(\cos^2\alpha + \cos^2\beta + \cos^2\gamma), \text{ or to } -1,$$

as ought to be the case. If the vector be not of unit length the square of $ix+jy+kz$ is, in like manner, $-(x^2+y^2+z^2)$, or is the negative square of the length of the line which the vector represents. We may express this by saying that the square of any vector is the negative square of the *tensor* of that vector.

Ex. 2. To find the product of two unit vectors

$$(i\cos\alpha + j\cos\beta + k\cos\gamma),\quad (i\cos\alpha' + j\cos\beta' + k\cos\gamma').$$

Ans. $-(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma') + i(\cos\beta\cos\gamma' - \cos\gamma\cos\beta')$
$+ j(\cos\gamma\cos\alpha' - \cos\alpha\cos\gamma') + k(\cos\alpha\cos\beta' - \cos\beta\cos\alpha')$.

If θ be the angle between the two vectors, α'', β'', γ'' the direction-cosines of a perpendicular to their plane, the product may be written

$$-\cos\theta + \sin\theta\,(i\cos\alpha'' + j\cos\beta'' + k\cos\gamma'').$$

(This agrees with Art. 5.) If the vectors were respectively of lengths l, l', this product would evidently be multiplied by ll'.

If the product had been taken in different order the scalar part of the product would still be $-\cos\theta$, but the vector part would change sign. Hence, if we denote by S and V the operation of taking the scalar and vector part of a quaternion, we have $S(\alpha\beta) = S(\beta\alpha) = \cos\theta$, $V(\alpha\beta) = -V(\beta\alpha)$. And again, we have $\alpha\beta + \beta\alpha = 2S(\alpha\beta)$.

If the two vectors be at right angles the scalar part of the product evidently vanishes. Hence the condition that two vectors α, β, may be at right angles is $S(\alpha\beta) = 0$.

Thus then if ρ be a variable vector passing through the origin, and α a fixed vector, the equation $S(\rho\alpha) = 0$ may be taken as the equation of the plane through the origin perpendicular to α, since ρ is evidently limited to that plane.

Let it be required to find the equation of any other plane. Let the perpendicular from the origin on that plane be denoted in length and direction by α, and let the radius vector to any point of the plane be ρ, then $\rho - \alpha$ is the vector joining the extremity of this radius vector to the foot of the perpendicular, and since this line is, by hypothesis, to be perpendicular to α, the equation required is $S(\rho - \alpha)\,\alpha = 0$ or $S(\rho\alpha) = \alpha^2$. But α^2 is a scalar, and we may therefore divide by it under the sign S, and write the equation in the form $S\left(\frac{\rho}{\alpha}\right) = 1$. This equation may also be inferred from what was stated in a previous article, viz., that the scalar part of the above fraction denotes the projection of the line ρ on the line α.

In like manner the equation $S\left(\frac{\alpha}{\rho}\right) = 1$, which expresses that the projection of the fixed line α on the direction ρ is in length equal to ρ, obviously represents the sphere described on the vector α as diameter.

Again, the equation $S\left(\frac{\rho}{\alpha}\right) S\left(\frac{\beta}{\rho}\right) = 1$, in the first place represents a cone because if it is satisfied for any value of ρ, it will also be satisfied for the value $m\rho$, where m is any constant. Secondly, it passes through the intersection of $S\frac{\rho}{\alpha} = 1$, $S\frac{\beta}{\rho} = 1$: it is therefore the cone whose base is the circle represented by the two equations just written.

Ex. 3. To find the product of two quaternions. We have only to multiply out $a + bi + cj + dk$, $a' + b'i + c'j + d'k$. We may form a clearer conception of the result by separating the scalar and vector parts, and writing the two quaternions $S + V$, $S' + V'$, when the product is $SS' + SV' + S'V + VV'$. Now if it be required to find the scalar part of the product (since SV'

and $S'V$ are mere vectors), it is $SS' + S(VV')$, or the scalar of the product is the product of the scalars + the scalar part of the product of the vectors.

Thus let α, β, γ be three radii vectores of a sphere; then it is an identical equation that $\frac{\alpha}{\beta} = \frac{\alpha}{\gamma}\frac{\gamma}{\beta}$. Now if a, b, c be the sides of the spherical triangle formed by the extremities of these vectors; $\cos a$, $\cos b$, $\cos c$ are the scalars of the three quaternions, and the scalar part of the product of the vectors on the right-hand side of the equation is the product of their tensors $\sin a$, $\sin b$, into the cosine of the angle between them, thus we have the fundamental formula of spherical trigonometry

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

9. We can, in like manner, form the product of three vectors. It is found, without difficulty, by actual multiplication, that if $ix+jy+kz$, $ix'+jy'+kz'$, $ix''+jy''+kz''$ be the three vectors, the scalar part of the product is the determinant whose three rows are x, y, z; x', y', z'; x'', y'', z''. *Hence if α, β, γ be the three vectors, the condition that they should lie in one plane is* $S(\alpha\beta\gamma) = 0$ (Note, p. 19).

This is also evident from the consideration that if $S(\alpha\beta\gamma)=0$ then $\alpha\beta\gamma$ is a pure vector, but $\alpha\beta\gamma = \alpha . S(\beta\gamma) + \alpha V(\beta\gamma)$ therefore $\alpha V(\beta\gamma)$ is a pure vector, or α is perpendicular to $V(\beta\gamma)$, and therefore is in the plane of β and γ. Q. E. D.

Thus we can find the equation of the plane passing through the extremity of three vectors α, β, γ. By hypothesis, the lines joining the extremity of any variable vector terminating in the plane, with the extremities of the assumed vectors, lie in the plane. We have, therefore, $S(\rho-\alpha)(\rho-\beta)(\rho-\gamma) = 0$.

In expanding this we may omit such terms as $S\rho^2\gamma$, because ρ^2 is a scalar, and $\rho^2\gamma$ a mere vector whose scalar is nothing. The expanded product is then

$$S(\rho\beta\gamma + \alpha\rho\gamma + \alpha\beta\rho) = S\alpha\beta\gamma,$$

and the vector perpendicular to the plane is $V(\beta\gamma + \gamma\alpha + \alpha\beta)$.

Returning to the product of the three vectors, it is also found by actual multiplication, that

$$V(\alpha\beta\gamma) = \alpha S(\beta\gamma) - \beta S(\gamma\alpha) + \gamma S(\alpha\beta),$$

an equation of great use.

In connection with this, the following identical equation may be given,

$$\delta S(\alpha\beta\gamma) = \alpha S(\beta\gamma\delta) - \beta S(\gamma\alpha\delta) + \gamma S(\alpha\beta\delta),$$

as also that the vector part of the product $V\alpha\beta V\gamma\delta$ may be written in either of the forms

$$\alpha S(\beta\gamma\delta) - \beta S(\gamma\alpha\delta) \text{ or } \gamma S(\alpha\beta\delta) - \delta S(\alpha\beta\gamma).$$

In fact, $V\alpha\beta$ denotes a line perpendicular to α and β; the vector now required must therefore lie in the plane, both of α and β, and of γ and δ.

10. As an example of the method of applying this calculus to a geometrical problem, we shall investigate the problem to find the equation of the surface generated by a line resting on three directing lines. In the first place we may follow a process proceeding after the analogy of the co-ordinate methods. It is seen immediately by substituting $\frac{1}{l+m}(l\alpha + m\alpha')$ for α in the equation of a plane through three points, that the equation of the plane through the extremity of the vector just written, and through a fixed line, *e.g.*, through the extremities of the vectors β, γ, is of the form $lA + mB = 0$, where A denotes the plane through $\alpha\beta\gamma$, and B that through $\alpha'\beta\gamma$. If then we join any assumed point on the vector $\alpha\alpha'$ to the other two lines we get the equation of two planes in the form $lA + mB = 0$, $lA' + mB' = 0$, from which, eliminating l, m, we get the locus in the form $AB' = BA'$.

Otherwise thus, we are to express the condition that, if we join by planes any assumed point on the locus to the three lines, the joining planes have a line common. The vectors perpendicular to these planes will then be co-planar. Let then the first line be parallel to the line α, and pass through the extremity of a vector α'; then the vector perpendicular to the plane through this line being perpendicular to α and to $\alpha' - \rho$ is $V\alpha(\alpha' - \rho)$, and the required equation is

$$S\{V\alpha(\alpha' - \rho) \,.\, V\beta(\beta' - \rho) \,.\, V\gamma(\gamma' - \rho)\} = 0,$$

which is reduced and expanded by the last article.

11. We give one more example to shew how infinitesimals are introduced into this calculus. The equation of any sphere is $\rho^2 = -c^2$.

Now let the line joining the extremity of ρ to an indefinitely near point be $d\rho$, then the next consecutive radius vector is $\rho + d\rho$, and we have

$$(\rho + d\rho)^2 = -c^2,$$

or expanding and neglecting the square of $d\rho$,

$$\rho d\rho + d\rho . \rho = 0,$$

or

$$S(\rho . d\rho) = 0,$$

which indicates that the radius ρ is perpendicular to the tangent line $d\rho$.

Very much more must be said if it were intended to give any complete account of this Calculus, as, for example, the method of finding the equations of tangents and normals, lines of curvature, geodesics, &c. But enough has been said to dispose the reader to give credit to the assertion that there is no geometrical problem to which it may not be applied; that it is very rich in transformations; and that its processes though constantly following the analogies of the co-ordinate methods, are by no means slavishly dependent on that system.

APPENDIX II.

ON SYSTEMS OF ORTHOGONAL SURFACES.*

IT might be thought, from Dupin's theorem, that being given a series of surfaces, involving a parameter, it would be always possible to determine two other systems, each containing a parameter, and cutting the surfaces of the given system at right angles, and along their lines of curvature. This, however, is not the case. In order that a given family of surfaces, with a parameter, may form one of a triple orthogonal system, an equation, or equations, of condition must be satisfied.

M. Serret arrives at the conclusion (see *Liouville*, Vol. XII., p. 241) that in order that the equation $F(x, y, z) = \alpha$, where α is a parameter, may be one of a triple orthogonal system, the function must satisfy two partial differential equations of the sixth order. We give Serret's investigation of the particular case where the given function is the sum of three functions of x, y, z respectively.

Let an equation then be given of the form

$$X + Y + Z = \alpha \quad \text{.......................(1).}$$

It is required to determine the condition, to which these functions must be subject, in order that the surfaces (1) may have a pair of conjugate orthogonal systems. Suppose that

$$F_1(x, y, z) = \beta, \quad F_2(x, y, z) = \gamma,$$

are these systems, and it is evident by the conditions of the problem that we have (X', Y', Z' being the first derived functions of X, Y, Z)

* For the following appendix, on a subject which I had omitted in the preceding treatise, I am almost entirely indebted to a manuscript note kindly placed at my disposal by the Rev. W. Roberts, as well as to his papers published in the *Comptes Rendus.*

$$\left.\begin{aligned} X'\frac{d\beta}{dx}+Y'\frac{d\beta}{dy}+Z'\frac{d\beta}{dz}&=0\\ X'\frac{d\gamma}{dx}+Y'\frac{d\gamma}{dy}+Z'\frac{d\gamma}{dz}&=0\\ \frac{d\beta}{dx}\frac{d\gamma}{dx}+\frac{d\beta}{dy}\frac{d\gamma}{dy}+\frac{d\beta}{dz}\frac{d\gamma}{dz}&=0 \end{aligned}\right\}\ldots\ldots\ldots\ldots(2).$$

Proceeding to integrate the first two of these equations by the ordinary methods of partial differential equations, we find that β and γ are functions of u and v, where

$$u=\int\frac{dx}{X'}-\int\frac{dy}{Y'},\quad v=\int\frac{dx}{X'}-\int\frac{dz}{Z'}.$$

Consequently the third of equations (2) becomes

$$\left(\frac{d\beta}{du}+\frac{d\beta}{dv}\right)\left(\frac{d\gamma}{du}+\frac{d\gamma}{dv}\right)+\frac{X'^2}{Y'^2}\frac{d\beta}{du}\frac{d\gamma}{du}+\frac{X'^2}{Z'^2}\frac{d\beta}{dv}\frac{d\gamma}{dv}=0\ldots\ldots(3).$$

Now u and v being functions of x, y, z, we may regard y and z as functions of u, v, and x. Hence x enters (3) as an indeterminate parameter, and the quantities β and γ must satisfy not only (3) but also the derivatives of (3) obtained by differentiating it on the supposition that x alone is variable. Differentiating (3) with respect to x on this hypothesis, and remembering that

$$\frac{dy}{dx}=\frac{Y'}{X'},\quad \frac{dz}{dx}=\frac{Z'}{X'},$$

we find,

$$\frac{X''-Y''}{Y'^2}\frac{d\beta}{du}\frac{d\gamma}{du}+\frac{X''-Z''}{Z'^2}\frac{d\beta}{dv}\frac{d\gamma}{dv}=0,$$

X'', Y'', Z'' being the second derived functions of X, Y, Z. Differentiating once more, and denoting the third derived functions by X''', Y''', Z''', we get

$$\frac{X'X'''-Y'Y'''-2Y''(X''-Y'')}{Y'^2}\frac{d\beta}{du}\frac{d\gamma}{du}$$
$$+\frac{X'X'''-Z'Z'''-2Z''(X''-Z'')}{Z'^2}\frac{d\beta}{dv}\frac{d\gamma}{dv}=0.$$

Hence at once results the equation of condition sought, namely,

$$X'X'''(Y''-Z'')+Y'Y'''(Z''-X'')+Z'Z'''(X''-Y'')$$
$$+2(X''-Y'')(Y''-Z'')(Z''-X'')=0.$$

This relation expresses the condition that a family of surfaces, of the particular form represented by equation (1), should form one of a triple orthogonal system. It was first given by M. Bouquet, *Liouville*, Vol. XI., p. 446, but the above proof has been taken from M. Serret's memoir.

Even when the equations of condition are satisfied by an assumed equation it does not seem easy to determine the two conjugate systems. Thus M. Bouquet observed that the condition just found is satisfied when the given system is of the form $x^m y^n z^p = a$, but he gave no clue to the discovery of the conjugate systems. This lacuna has been completely supplied by M. Serret, who has shown much ingenuity and analytical power in deducing the equations of the conjugate systems, when the equation of condition is satisfied. The actual results are, however, of a rather complicated character. We must content ourselves with referring the reader to his memoir, only mentioning the simplest case obtained by him, and which there is no difficulty in verifying *a posteriori*. He has shown that the three equations,

$$\frac{yz}{x} = \alpha,$$

$$\sqrt{(x^2+y^2)} + \sqrt{(x^2+z^2)} = \beta,$$

$$\sqrt{(x^2+y^2)} - \sqrt{(x^2+z^2)} = \gamma,$$

represent a triple system of conjugate orthogonal surfaces. The surfaces (α) are hyperbolic paraboloids. The system (β) is composed of the closed portions, and the system (γ) of the infinite sheets, of the surfaces of the fourth order,

$$(z^2-y^2)^2 - 2\beta^2(z^2+y^2+2x^2) + \beta^4 = 0.$$

M. Serret has observed that it follows at once from what has been stated above, that in a hyperbolic paraboloid, of which the principal parabolæ are equal, the sum or difference of the distances of every point of the same line of curvature from two fixed generatrices is constant.

Mr. W. Roberts, expressing in elliptic co-ordinates the condition that two surfaces should cut orthogonally, has sought for systems orthogonal to $L+M+N=\alpha$, where L, M, N are functions of the three elliptic co-ordinates respectively. He has thus added some systems of orthogonal surfaces to those previously known (*Comptes Rendus*, September 23, 1861). Of these perhaps the most interesting, geometrically, is that whose equation in elliptic co-ordinates is $\mu\nu = \alpha\lambda$, and for which he has given the following construction. Let a fixed point in the line of one of the axes of a system of confocal ellipsoids be made the vertex of a series of cones circumscribed to them. The locus of the curves of contact will be a determinate surface, and if we suppose the vertex of the cones to move along the axis, we obtain a family of surfaces involving a parameter. Two other systems are obtained by taking points situated on the other axes as vertices of circumscribing cones. The surfaces belonging to these three systems will intersect, two by two, at right angles.

It may be readily shown that the lines of curvature of the above mentioned surfaces (which are of the third order) are circles, whose planes are perpendicular to the principal planes of the ellipsoids. Let A, B, be two fixed points, taken respectively upon two of the axes of the confocal system. To these points two surfaces intersecting at right angles will correspond. And the curve of their intersection will be the locus of points M on the confocal ellipsoids, the tangent planes at which pass through the line AB. Let P be the point where the normal to one of the ellipsoids at M meets the principal plane containing the line AB, and because P is the pole of AB in reference to the focal conic in this plane, P is a given point. Hence the locus of M, or a line of curvature, is a circle in a plane perpendicular to the principal plane containing AB.

APPENDIX III.

CLEBSCH'S CALCULATION OF THE SURFACE S.*

1. In this appendix we give the calculation referred to p. 406, by which the equation is determined of a surface which meets a given surface at the points of contact of lines which meet it in four consecutive points. It was proved, Art. 472, that in order to obtain this equation it is necessary to eliminate between the equation of an arbitrary plane, and the functions $\Delta U'$, $\Delta^2 U'$, $\Delta^3 U'$. We perform this elimination by solving for the co-ordinates of the two points of intersection of the arbitrary plane, the tangent plane $\Delta U'$, and the polar quadric $\Delta^2 U'$; substituting these co-ordinates successively in $\Delta^3 U'$, and multiplying the results together. I write with M. Clebsch, the four co-ordinates of the point of contact x_1, x_2, x_3, x_4; the running co-ordinates y_1, y_2, y_3, y_4; the differential coefficients u_1, u_2, u_3, u_4; the second and third differential coefficients being denoted in like manner by sub-indices, as u_{12}, u_{123}. Through each of the lines of intersection of $\Delta U'$, $\Delta^2 U'$, we can draw a plane, so that by suitably determining t_1, t_2, t_3, t_4 we can, in an infinity of ways, form an equation identically satisfied

$$\Delta^2 U' + (t_1 y_1 + t_2 y_2 + t_3 y_3 + t_4 y_4)\,\Delta U'$$
$$= (p_1 y_1 + p_2 y_2 + p_3 y_3 + p_4 y_4)\,(q_1 y_1 + q_2 y_2 + q_3 y_3 + q_4 y_4) \ldots (A).$$

We shall suppose this transformation effected; but it is not necessary for us to determine the actual values of t_1, &c., for it will be found that these quantities will disappear from the result. Let the arbitrary plane be $c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$, then it is evident that the co-ordinates of the intersections of

* See Note, p. 406.

the arbitrary plane, the tangent plane $u_1y_1+u_2y_2+u_3y_3+u_4y_4$, and $\Delta^3 U'$, are the four determinants of the two systems

$$\left\|\begin{matrix} c_1, & c_2, & c_3, & c_4 \\ u_1, & u_2, & u_3, & u_4 \\ p_1, & p_2, & p_3, & p_4 \end{matrix}\right\|, \qquad \left\|\begin{matrix} c_1, & c_2, & c_3, & c_4 \\ u_1, & u_2, & u_3, & u_4 \\ q_1, & q_2, & q_3, & q_4 \end{matrix}\right\|.$$

These co-ordinates have now to be substituted in $\Delta^3 U'$, which we write in the symbolical form $(a_1y_1+a_2y_2+a_3y_3+a_4y_4)^3$; where a_1 means $\frac{d}{dx_1}$, &c., so that after expansion we may substitute for any term $a_1a_2a_3y_1y_2y_3$, $u_{123}y_1y_2y_3$, &c. It is evident then that the result of substituting the co-ordinates of the first point in $\Delta^3 U'$, may be written as the cube of the symbolical determinant $\Sigma a_1c_2u_3p_4$, where after cubing we are to substitute third differential coefficients, for the powers of the a's as has been just explained. In like manner we write the result of substituting the co-ordinates of the second point $(\Sigma b_1c_2u_3q_4)^3$, (where b_1 is a symbol used in the same manner as a_1). The eliminant required may therefore be written

$$(\Sigma a_1c_2u_3p_4)^3\,(\Sigma b_1c_2u_3q_4)^3=0.*$$

The above result may be written in the more symmetrical form

$$(\Sigma a_1c_2u_3p_4)^3\,(\Sigma b_1c_2u_3q_4)^3+(\Sigma b_1c_2u_3p_4)^3\,(\Sigma a_1c_2u_3q_4)^3=0.$$

For since the quantities a, b, are after expansion replaced by differentials, it is immaterial whether the symbol used originally were a or b; and the left-hand side of this equation when expanded is merely the double of the last expression. We have now to perform the expansion, and to get rid of p and q by means of equation A. We shall commence by thus banishing p and q.

---

* The reason why we use a different symbol for $\frac{d}{dx_1}$, &c. in the second determinant, is because if we employed the same symbol, the expanded result would evidently contain sixth powers of a, that is to say, sixth differential coefficients. We avoid this by the employment of different symbols, as in Mr. Cayley's "Hyperdeterminant Calculus," (*Lessons on Higher Algebra*, p. 79) with which the method here used is substantially identical.

2. Let us write

$$F = (\Sigma a_1 c_2 u_3 p_4)(\Sigma b_1 c_2 u_3 q_4), \quad G = (\Sigma b_1 c_2 u_3 p_4)(\Sigma a_1 c_2 u_3 q_4).$$

The eliminant is $F^3 + G^3 = 0$, or $(F+G)^3 - 3FG(F+G) = 0$. We shall separately examine $F+G$, and FG, in order to get rid of p and q. If the determinants in F were so far expanded as to separate the p and q which they contain, we should have

$$F = (m_1 p_1 + m_2 p_2 + m_3 p_3 + m_4 p_4)(n_1 q_1 + n_2 q_2 + n_3 q_3 + n_4 q_4),$$
$$G = (n_1 p_1 + n_2 p_2 + n_3 p_3 + n_4 p_4)(m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4),$$

where, for example, m_4 is the determinant $\Sigma a_1 c_2 u_3$, and n_4 is $\Sigma b_1 c_2 u_3$. If then i, j be any two subindices the coefficient of $m_i n_j$ in $F+G$ is $(p_i q_j + p_j q_i)$. And we may write

$$F + G = \Sigma\Sigma m_i n_j (p_i q_j + p_j q_i),$$

where both i and j are to be given every value from 1 to 4. But, by comparing coefficients in equation A, we have

$$p_i q_j + p_j q_i = 2u_{ij} + (t_i u_j + t_j u_i),$$

whence $\quad F + G = 2\Sigma\Sigma m_i n_j u_{ij} + \Sigma\Sigma m_i n_j (t_i u_j + t_j u_i).$

Now it is plain that if for every term of the form $p_i q_j + p_j q_i$ we substitute $t_i u_j + t_j u_i$, the result is the same as if in F and G we everywhere altered p and q into t and u. But if in the determinants $\Sigma a_1 c_2 u_3 q_4$, $\Sigma b_1 c_2 u_3 q_4$ we alter q into u, the determinants would vanish as having two columns the same. The latter set of terms therefore in $F+G$ disappears, and we have $\frac{1}{2}(F+G) = \Sigma\Sigma m_i n_j u_{ij}$.

Now if we remember what is meant by m_i, n_j this double sum may be written in the form of a determinant

$$-\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & a_1, & c_1, & u_1 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & a_2, & c_2, & u_2 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & a_3, & c_3, & u_3 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & a_4, & c_4, & u_4 \\ b_1, & b_2, & b_3, & b_4 & \cdots & \cdots & \cdots \\ c_1, & c_2, & c_3, & c_4 & \cdots & \cdots & \cdots \\ u_1, & u_2, & u_3, & u_4 & \cdots & \cdots & \cdots \end{vmatrix}.$$

For since this determinant must contain a constituent from each of the last three rows and columns it is of the first degree in u_{11}, &c., and the coefficient of any term u_{14} is

$$-\{\Sigma a_2 c_3 u_4 \Sigma b_1 c_2 u_3 + \Sigma a_1 c_2 u_3 \Sigma b_2 c_3 u_4\} \text{ or } -(m_1 n_4 + m_4 n_1).$$

In the determinant just written the matrix of the Hessian is bordered vertically with a, c, u; and horizontally with b, c, u. As we shall have frequently occasion to use determinants of this kind we shall find it convenient to denote them by an abbreviation, and shall write the result that we have just arrived at,

$$F + G = -2\begin{pmatrix} a, & c, & u \\ b, & c, & u \end{pmatrix}.$$

3. The quantity FG is transformed in like manner. It is evidently the product of

$$(m_1 p_1 + m_2 p_2 + m_3 p_3 + m_4 p_4)(m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4),$$

and $\quad (n_1 p_1 + n_2 p_2 + n_3 p_3 + n_4 p_4)(n_1 q_1 + n_2 q_2 + n_3 q_3 + n_4 q_4)$.

Now if the first line be multiplied out, and for every term $(p_1 q_2 + p_2 q_1)$ we substitute its value derived from equation A, it appears, as before, that the terms including t vanish, and it becomes $\Sigma\Sigma m_i n_j u_{ij}$, which, as before, is equivalent to $\begin{pmatrix} a, & c, & u \\ a, & c, & u \end{pmatrix}$, where the notation indicates the determinant formed by bordering the matrix of the Hessian both vertically and horizontally with a, c, u. The second line is transformed in like manner: and we thus find that $(F+G)^3 - 3FG(F+G) = 0$ transforms into

$$\begin{pmatrix} a, & c, & u \\ b, & c, & u \end{pmatrix}\left\{4\begin{pmatrix} a, & c, & u \\ b, & c, & u \end{pmatrix}^2 - 3\begin{pmatrix} a, & c, & u \\ a, & c, & u \end{pmatrix}\begin{pmatrix} b, & c, & u \\ b, & c, & u \end{pmatrix}\right\} = 0.$$

It remains to complete the expansion of this symbolical expression; and to throw it into such a form that we may be able to divide out $c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$. We shall for shortness write a, b, c, instead of $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$, $b_1 x_1 + $&c., $c_1 x_1 + $&c.

4. On inspection of the determinant, p. 446, which we have called $\begin{pmatrix} a, & c, & u \\ b, & c, & u \end{pmatrix}$, it appears that, since

$$u_{11} x_1 + u_{12} x_2 + u_{13} x_3 + u_{14} x_4 = (n-1) u_1, \text{ \&c.},$$

this determinant may be reduced by multiplying the first four columns by x_1, x_2, x_3, x_4, and subtracting their sum from the last column multiplied by $(n-1)$, and similarly for the rows; when it becomes

$$-\frac{1}{(n-1)^2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & a_1, & c_1, & 0 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & a_2, & c_2, & 0 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & a_3, & c_3, & 0 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & a_4, & c_4, & 0 \\ b_1, & b_2, & b_3, & b_4, & 0, & 0, & -b \\ c_1, & c_2, & c_3, & c_4, & 0, & 0, & -c \\ 0, & 0, & 0, & 0, & -a, & -c & \end{vmatrix},$$

which partially expanded is

$$-\frac{1}{(n-1)^2}\left\{c^2\begin{pmatrix}a\\b\end{pmatrix} - ac\begin{pmatrix}c\\b\end{pmatrix} - b^2\begin{pmatrix}c\\a\end{pmatrix} + ab\begin{pmatrix}c\\c\end{pmatrix}\right\},$$

where $\begin{pmatrix}a\\b\end{pmatrix}$ denotes the matrix of the Hessian bordered with a single line, vertically of a's and horizontally of b's.

In like manner we have

$$\begin{pmatrix}a, & c, & u\\a, & c, & u\end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c^2\begin{pmatrix}a\\a\end{pmatrix} - 2ac\begin{pmatrix}a\\c\end{pmatrix} + a^2\begin{pmatrix}c\\c\end{pmatrix}\right\},$$

$$\begin{pmatrix}b, & c, & u\\b, & c, & u\end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c^2\begin{pmatrix}b\\b\end{pmatrix} - 2bc\begin{pmatrix}b\\c\end{pmatrix} + b^2\begin{pmatrix}c\\c\end{pmatrix}\right\}.$$

Now as it will be our first object to get rid of the letter a, we may make these expressions a little more compact by writing $cb_1 - bc_1 = d_1$, &c., when it is easy to see that

$$\begin{pmatrix}d\\d\end{pmatrix} = c^2\begin{pmatrix}b\\b\end{pmatrix} - 2bc\begin{pmatrix}b\\c\end{pmatrix} + b^2\begin{pmatrix}c\\c\end{pmatrix};$$

$$\begin{pmatrix}c\\d\end{pmatrix} = c\begin{pmatrix}b\\c\end{pmatrix} - b\begin{pmatrix}c\\c\end{pmatrix};\ \begin{pmatrix}a\\d\end{pmatrix} = c\begin{pmatrix}a\\b\end{pmatrix} - b\begin{pmatrix}a\\c\end{pmatrix}.$$

Thus

$$\begin{pmatrix}b, & c, & u\\b, & c, & u\end{pmatrix} = -\frac{1}{(n-1)^2}\begin{pmatrix}d\\d\end{pmatrix};\ \begin{pmatrix}a, & c, & u\\b, & c, & u\end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c\begin{pmatrix}a\\d\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix}\right\},$$

and the equation of the surface, as given at the end of Art. 3, may be altered into

$$\left\{c\begin{pmatrix}a\\d\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix}\right\}\left[4\left\{c\begin{pmatrix}a\\d\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix}\right\}^2 - 3\begin{pmatrix}d\\d\end{pmatrix}\left\{c^2\begin{pmatrix}a\\a\end{pmatrix} - 2ac\begin{pmatrix}a\\c\end{pmatrix} + a^2\begin{pmatrix}c\\c\end{pmatrix}\right\}\right].$$

5. We proceed now to expand and substitute for each term $a_1a_2a_3$, &c. the corresponding differential coefficient. Then, in the first place, it is evident that

$$a^3 = n(n-1)(n-2)u = 0;\ a^2a_1 = (n-1)(n-2)u_1, \text{ \&c.}$$

Hence
$$a^2\binom{a}{c} = (n-1)(n-2)\binom{u}{c}.$$

But the last determinant is reduced as in many similar cases, by subtracting the first four columns multiplied respectively by x_1, x_2, x_3, x_4 from the fifth column, and so causing it to vanish except the last row. Thus we have

$$a^2\binom{a}{c} = -(n-2)Hc.$$

Again, $\binom{a}{a}$ is (see *Lessons on Higher Algebra*, p. 124) $= -\Sigma \dfrac{dH}{du_{mn}} a_m a_n$. We have therefore

$$a\binom{a}{a} = -(n-2)\Sigma \frac{dH}{du_{mn}} u_{mn} = -4(n-2)H.$$

Lastly it is necessary to calculate $a\binom{a}{c}\binom{a}{d}$. Now if U_{mn} denote the minor obtained from the matrix of the Hessian by erasing the line and column which contains u_{mn}; it is easy to see that $a\binom{a}{c}\binom{a}{d} = -(n-2)\Sigma U_{mp}U_{np}u_{mn}c_pd_q$, where the numbers m, n, p, q are each to receive in turn all the values 1, 2, 3, 4. But, see *Lessons on Higher Algebra*, Art. 28,

$$U_{mp}U_{nq} = U_{mn}U_{pq} - H\frac{dU_{pq}}{du_{mn}}.$$

Substituting this, and remembering that $\Sigma U_{mn}u_{mn} = 4H$, we have

$$a\binom{a}{c}\binom{a}{d} = -(n-2)H\binom{c}{d}.$$

Making then these substitutions we have

$$\left\{c\binom{a}{d} - a\binom{c}{d}\right\}^3 = c^3\binom{a}{d}^3 + 3(n-2)Hc^2\binom{c}{d}\binom{d}{d} - 3(n-2)Hcd\binom{c}{d}^2.$$

$$\left\{c\binom{a}{d} - a\binom{c}{d}\right\}\left\{c^2\binom{a}{a} - 2ac\binom{a}{c} + a^2\binom{c}{c}\right\}$$
$$= c^3\binom{a}{d}\binom{a}{a} + 4(n-2)Hc^2\binom{c}{d} - (n-2)Hcd\binom{c}{c}.$$

But attending to the meaning of the symbols d_1, &c., we see that d or $d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4$ vanishes identically. If then we substitute in the equation which we are reducing the values just obtained it becomes divisible by c^3, and is then brought to the form

$$4\binom{a}{d}^3 - 3\binom{a}{d}\binom{a}{a}\binom{d}{d} = 0.$$

6. To simplify this further we put for d its value when it becomes

$$4\left\{c\binom{b}{a} - b\binom{c}{a}\right\}^3 - 3\binom{a}{a}\left\{c\binom{b}{a} - b\binom{c}{a}\right\}\left\{c^2\binom{b}{b} - 2bc\binom{b}{c} + b^2\binom{c}{c}\right\}.$$

Now this is exactly the form reduced in the last article, except that we have b instead of a, and a in place of d. We can then write down

$$4\left\{c\binom{b}{a} - b\binom{c}{a}\right\}^3 = 4\left\{c^3\binom{b}{a}^3 + 3(n-2)Hc^2\binom{c}{a}\binom{a}{a} - 3(n-2)Hca\binom{c}{a}^2\right\},$$

while the remaining part of the equation becomes

$$3\binom{a}{a}\left\{c^3\binom{b}{a}\binom{b}{b} + 4(n-2)Hc^2\binom{c}{a} - (n-2)Hca\binom{c}{c}\right\}.$$

But (Art. 5) the last term in both these can be reduced to $12(n-2)^2H^2c\binom{c}{c}$. Subtracting then, the factor c^2 divides out again, and we have the final result cleared of irrelevant factors, expressed in the symbolical form

$$\binom{b}{a}\left\{4\binom{b}{a}^2 - 3\binom{b}{b}\binom{a}{a}\right\} = 0.$$

7. It remains to shew how to express this result in the ordinary notation. In the first place we may transform it by the identity (see Art. 76, and *Lessons on Higher Algebra*, Art. 28)

$$H\binom{a,\ b}{a,\ b} = \binom{a}{a}\binom{b}{b} - \binom{a}{b}^2,$$

whereby the equation becomes

$$\binom{b}{a}\binom{a}{a}\binom{b}{b} - 4H\binom{b}{a}\binom{a,\ b}{a,\ b} = 0.$$

Now $\binom{b}{a}\binom{a}{a}\binom{b}{b}$ expresses the covariant which we have before called Θ. For giving to U_{mn} the same meaning as before, the symbolical expression expanded, may be written $\Sigma U_{mn} U_{pq} U_{rs} u_{mnr} u_{pqs}$, where each of the suffixes is to receive every value from 1 to 4. But the differential coefficient of H with respect to x_r can easily be seen to be $\Sigma U_{mn} u_{mnr}$, so that Θ is $\Sigma U_{rs} \frac{dH}{dx_r} \frac{dH}{dx_s}$, which is, in another notation what we have called Θ, p. 399. The covariant S is then reduced to the form $\Theta - 4H\Phi$, where

$$\Phi = \binom{b}{a}\binom{a,\ b}{a,\ b} = \Sigma U_{mn} U_{pq,rs} u_{mpq} u_{nrs},$$

where $U_{pq,rs}$ denotes a second minor formed by erasing two rows and two columns from the matrix of the Hessian, a form scarcely so convenient for calculation as that in which I had written the equation, *Philosophical Transactions*, 1860, p. 239. For surfaces of the third degree Clebsch has observed that ϕ reduces, as was mentioned before, to $\Sigma U_{mn} H_{mn}$, where H_{mn} denotes a second differential coefficient of H.

8. If at any point on a surface *both* inflexional tangents meet in four coincident points, *either* of the intersections of $\Delta U'$, $\Delta^2 U'$ and an arbitrary plane must satisfy $\Delta^3 U'$. For such points then the expression at the end of Art. 1 must vanish, as soon as we have made the symbolical substitution for a, independently of any supposition as to the value of the b symbols. In the equation then which we found at this stage of our work

$$\binom{a}{d}\left\{4\binom{a}{d}^2 - 3\binom{a}{a}\binom{d}{d}\right\} = 0,$$

we may consider both the b and c symbols which occur in d as arbitrary constants, and the equation just written whose degree in the variables is easily seen to be $10n - 18$, shows that *through the points on a surface where two doubly inflexional tangents can be drawn*, or, in other words, through the points where S and U touch, *an infinity of surfaces can pass of the degree* $10n - 18$. (See Art. 476.)

To find the points on a surface where a line can be drawn to meet in five consecutive points, we have to form the condition that the intersection of $\Delta U'$, $\Delta^2 U'$, and an arbitrary plane should satisfy $\Delta^4 U'$, as well as $\Delta^3 U'$. M. Clebsch has applied to $\Delta^4 U'$ the same symbolical method of elimination which has been here applied to $\Delta^3 U'$. He has succeeded in dividing out the factor c^6 from this result: but in the final form which he has found, and for which I refer to his memoir, there remain c symbols in the second degree, and the result being of the degree $14n - 30$ in the variables, all that can be concluded from it is that through the points which I have called α, (p. 408) an infinity of surfaces can be drawn of the degree $14n - 30$. We can say therefore that the number of such points does not exceed $n(11n - 24)(14n - 30)$.

9. *The surface S touches the surface H along a certain curve.* Since the equation S is of the form $\Theta - 4H\Phi = 0$, it is sufficient to prove that Θ touches H. But since Θ is got by bordering the matrix of the Hessian with the differentials of the Hessian, $\Theta = 0$ is equivalent to the symbolical expression $\binom{H}{H} = 0$. But, by an identical equation already made use of, we have

$$H\begin{pmatrix} c, & H \\ c, & H \end{pmatrix} = \binom{H}{H}\binom{c}{c} - \binom{H}{c}^2.$$

where c is arbitrary. Hence Θ touches H along its intersection with the surface of the degree $7n - 15$, $\binom{H}{c}$. It is proved then that S touches H, and that through the curve of contact an infinity of surfaces can pass of the degree $7n - 15$. We have made use, p. 413, of the theorem that the curves US and UH touch each other.

APPENDIX IV.

ON THE ORDER OF SYSTEMS OF EQUATIONS.*

1. We have showed, p. 250, how to determine the characteristics of a curve given as the intersection of two surfaces; but it has been remarked (p. 225) that there are many curves which cannot be so represented. There is no algebraic curve, however, which may not be represented by means of the equations of a system of surfaces; because (p. 240) by taking m large enough we can always find a number of surfaces of the m^{th} degree each of which shall entirely contain the curve. But any two surfaces of the system will not *define* the curve, for their intersection will in general consist of the curve in question and an extraneous curve besides; so that the curve is usually not the complete intersection of any two, but only that part of the intersection which is common to all the rest. The object of this appendix is to show how, when a system of equations is given denoting surfaces which pass through a common curve, the characteristics of that curve can be determined.

In like manner if we are given r points in space, we can always, by taking m large enough, determine a number of surfaces of the m^{th} degree which shall pass through the given points. But ordinarily the intersection of three such surfaces will consist of the given points and extraneous points besides; and we cannot *define* the given points except by a system of more than three equations, the given points being the only ones which satisfy *all* the equations. Conversely, it is the object of this appendix, when such a system of equations is given, to ascertain the number of points which satisfy all.

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* See *Quarterly Journal of Mathematics*, Vol, I., p. 246.

2. The simplest illustration of this is to take four planes $a+\lambda\alpha$, $b+\lambda\beta$, $c+\lambda\gamma$, $d+\lambda\delta$; where a, α, &c. represent planes, and λ is an indeterminate coefficient; then if we form the condition that these four planes should meet in a point, this condition is known to be of the fourth degree in λ. It follows that four values of λ can be found for which these equations will represent planes meeting in a point. And obviously the four points so found must satisfy any of the six equations (such as $a\beta=b\alpha$), which are got by eliminating λ between any pair of the given equations. Yet these all represent surfaces of the second degree, any three of which intersect in *eight* points. It follows then that the system of equations

$$\left\|\begin{matrix} a, & b, & c, & d \\ \alpha, & \beta, & \gamma, & \delta \end{matrix}\right\| = 0,$$

denotes a system of surfaces having four points in common; but that any three surfaces of the system intersect not only in these four points but in four extraneous points. In general then, suppose we are given $r+3$ equations involving r parameters, it is evident that by elimination of the variables we get a sufficient number of equations to determine systems of values of the parameters for which the equations will denote surfaces having a point in common. It is evident also that such points must satisfy the equations got by eliminating the parameters between any $r+1$ of the given equations. And yet any three of these latter equations will denote surfaces intersecting not only in these points common to all but in certain extraneous points besides.

3. In like manner if we had been given the three planes $a+\lambda\alpha$, $b+\lambda\beta$, $c+\lambda\gamma$, it is obvious that we may give to λ an infinity of values, to every one of which corresponds a point which is the intersection of the three corresponding planes. It is obvious also that the locus of all these points must be a curve common to all the surfaces $a\beta-b\alpha$, $b\gamma-c\beta$, $c\alpha-a\gamma$. But it was proved, p. 241, that though any two of these surfaces intersect in a curve of the fourth degree, there is only a cubic common to all three. And in general if we are

given $r+2$ equations, involving r parameters, an infinity of systems of values of these parameters can be determined for which the equations will denote surfaces having a point in common. The locus of these points will be a curve, which will be common to all the surfaces got by eliminating the parameters between any $r+1$ of the equations. Yet any two such surfaces will intersect not only in this curve but in an extraneous curve. Let us suppose then that we have $r+1$ equations, involving r parameters in the first degree. The elimination of these gives rise to a system of determinants

$$\left\| \begin{matrix} a, & b, & c, & \ldots\ldots \\ a', & b', & c', & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{matrix} \right\| = 0,$$

where the number of horizontal rows is supposed to be r, and vertical $r+1$. We propose to determine the characteristics of the curve which is common to the surfaces represented by all these determinants.

4. To fix the ideas we take the matrix with four rows and five columns

$$\left\| \begin{matrix} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \\ a''', & b''', & c''', & d''', & e''' \end{matrix} \right\|,$$

the method of proof used in this case being generally applicable. We suppose the functions a, b, &c. to be of any degree, but we suppose the degrees of the corresponding functions in either the same row or the same column to be equi-different. Thus, if the letters a, b, c, &c. also indicate the degrees of these functions, we suppose the degrees of a', b', &c. to be $a+\alpha$, $b+\alpha$, &c., of a'', b'', &c. to be $a+\beta$, $b+\beta$, &c., &c. Let P and Q denote the sum of the quantities a, b, &c., and the sum of their products in pairs; and let p and q denote the corresponding sums for the quantities α, β: then we assert that *the order of the curve common to all the surfaces represented by the determinants of the system is* $Q+pP+p^2-q$.

In the first place we observe that if this formula is true it follows that if the orders of the functions b, b', b'', &c. had been $a+\alpha'$, $a'+\alpha'$, $a''+\alpha'$; those of c, c', $a+\beta'$, $a'+\beta'$, &c.; and if we denoted the sum of the quantities a, a', a'', a''', and of their products in pairs by P', Q', and the corresponding sums for α', β', &c. by p', q', then the order of the curve investigated would be $q'+p'P'+P'^2-Q'$. For the terms of the top row being of the degrees a, $a+\alpha'$, $a+\beta'$, $a+\gamma'$, $a+\delta'$; the quantities which we have called P and Q are respectively $5a+p'$, and $10a^2+4ap'+q'$; while, since the differences between the orders of the first and succeeding rows are $a'-a$, $a''-a$, $a'''-a$, it follows that the quantities called p and q are $P'-4a$, $Q'-3P'a+6a^2$, and on substituting these values in $Q+pP+p^2-q$ we get $q'+pP'+P'^2-Q'$. And in general if the number of rows be k we have

$$P=ka+p',\quad Q=\tfrac{1}{2}k(k-1)a^2+(k-1)ap'+q';$$

$$p=P'-(k-1)a',\quad q=Q'-(k-2)aP'+\tfrac{1}{2}(k-1)(k-2)a^2:$$

and the one formula is deduced from the other as before.

5. To establish now the truth of the formula it is sufficient to show that it is true for a system with k rows if it is true for a system with $k-1$. It is easy to see that the curve we are considering is part of the intersection of the surfaces denoted by the two determinants $(ab'c''d''')$, $(ab'c''e''')$; and that these two surfaces both pass through the curve

$$\left\| \begin{matrix} a, & a', & a'', & a''' \\ b, & b', & b'', & b''' \\ c, & c', & c'', & c''' \end{matrix} \right\|,$$

which does not lie on the surfaces represented by the other determinants of the system. Now if the sum and sum of products in pairs of the quantities a, b, c be denoted by P'', Q': it is easy to see that the order of the two determinants is $P''+p+d$, $P''+p+e$. While the order of the irrelevant curve is, by the last article, $q+pP''+P''^2-Q''$. Subtracting this number then from the product of the other two, we get

$$p^2-q+(P''+d+e)p+Q''+P''(d+e)+de,$$

or to $p^2 - q + Pp + Q$. Now the truth of the theorem is easily seen for a matrix of two rows and three columns, therefore it is generally true. It is needless to remark that the formula was at first obtained by commencing with the simpler case and proceeding on to the general one.

When all the rows are of the same order; we have α, β, &c. all $=0$, and therefore p, q both $=0$, and the order of the system is Q.

6. Next let it be required to find the order of the developable generated by the curve considered in the preceding articles. Let R be the sum of the products in threes of the quantities a, b, c, &c. (Art. 4), and r the corresponding sum of the quantities α, β, γ, &c., then we say that the order of the developable in question is

$$(p^2 - q + pP + Q)(P + 2p - 2) - q(p + P) + R + r.$$

In the first place, admitting the truth of this formula it follows that if P', Q', R' had been used with reference to a, a', a'', a''' the orders of the terms in the first vertical row, &c., then the capital and small letters in the formula would simply be interchanged, and the order of the developable would be

$$(P'^2 - Q' + p'P' + q')(p' + 2P' - 2) - Q'(p' + P') + R' + r'.$$

This is proved exactly as in Art. 4.

Again, we know, p. 253, that the ranks of two systems which together make up the intersection of two surfaces are connected by the relation

$$\rho - \rho' = (m - m')(\mu + \nu - 2).$$

We must then substitute in this formula (see Art. 5)

$$\mu = P'' + p + d, \quad \nu = P'' + p + e,$$

$$m = p^2 - q + pP + Q, \quad m' = q + pP'' + P''^2 - Q'',$$

$$\rho' = (P''^2 - Q'' + pP'' + q)(p + 2P'' - 2) - Q''(p + P'') + R'' + r.$$

And if these substitutions be made and the result reduced by the identities

$$P = P'' + d + e, \quad Q = Q'' + (d + e)P'' + de,$$

$$R = R'' + (d + e)Q'' + P''de,$$

we get

$$\rho = (p^2 - q + pP + Q)(P + 2p - 2) - q(p + P) + R + r.$$

It follows then that if the formula be true for a matrix with k rows, it is true for one with $k+1$; and since it is easily proved to be true for three rows, it is generally true.

7. Let us next consider a matrix such as

$$\left\| \begin{matrix} a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \\ a''', & b''', & c''', & d''', & e''', & f''' \end{matrix} \right\|,$$

where the number of columns exceeds the number of rows by two, and let us examine how many points are common to all the surfaces represented by the determinants of the system. Now any three surfaces $(ab'c''d''')$, $(ab'c''e''')$, $(ab'c''f''')$ have common the curve

$$\left\| \begin{matrix} a, & a', & a'', & a''' \\ b, & b', & b'', & b''' \\ c, & c', & c'', & c''' \end{matrix} \right\|,$$

and if m, n, p be the degrees of the surfaces, μ and ρ the degree and rank of the curve, then (see p. 258) the surfaces will intersect in points not on this curve, in number

$$mnp - \mu(m + n + p - 2) + \rho.$$

Then, if we use the same notation as before, we are to substitute

$$m = P'' + p + d,\ n = P'' + p + e,\ p = P'' + p + f,$$

$$\mu = q + pP'' + P''^2 - Q'',$$

$$\rho = (q + pP'' + P''^2 - Q'')(p + 2P'' - 2) - Q''(p + P'') + R'' + r,$$

and the result reduced is

$$R + pQ + (p^2 - q)P + p^3 - 2pq + r.$$

The capital and small letters would be interchanged if we had used P, Q, R in reference to the letters in the first column a, a', a'', a'''. If the several rows had been of the same degrees, that is, if α, β, &c. all $= 0$, then the number of points represented by the system is R.

8. It may be deduced hence that the surface represented by any symmetrical determinant has a determinate number of double points. Let the sum, sum of products in pairs, and sum of products in threes of the degrees of the leading terms a_{11}, a_{22}, a_{33}, &c. be P, Q, R, then the number of such double points is $\frac{1}{2}(PQ-R)$.

Now we have the identical equation (*Lessons on Higher Algebra*, Art. 28) $A_{11}A_{22}-(A_{12})^2=CD$, where A_{11} means the minor obtained by erasing from the given determinant the line and column containing a_{11}, D is the determinant itself, and C is the second minor obtained by erasing the two lines and columns which contain a_{11}, a_{22}. Now it is evident that the surface represented by $A_{11}A_{22}-(A_{12})^2$ has as double points the intersections of A_{11}, A_{22}, A_{12}; and the degrees of these being respectively $P-a$, $P-b$, $P-\frac{1}{2}(a+b)$, the number of double points is the product of these three numbers. Let the sum, and sum of product of pairs, of the terms exclusive of a and b, be denoted by p'', q'', then the product

$$(P-a)(P-b)\{P-\tfrac{1}{2}(a+b)\}$$

is $$\tfrac{1}{2}\{PQ+p''Q+(p''^2-q'')P+p''^3-p''q''\}.$$

These are then double points on the complex system CD; and are therefore either double points on C, double points on D, or points of intersection of C and D. Now if we erase from the matrix the first two rows, all the determinants of the remaining system (of which C is one) have common a number of points, which can be calculated by the formula of the last article, by writing $\frac{1}{2}(c+a)$, $\frac{1}{2}(c+b)$, &c., $\frac{1}{2}(d+a)$, $\frac{1}{2}(d+b)$, &c., for the degrees of the rows. The result is

$$\tfrac{1}{8}(R+p''Q+(p''^2-q'')P+(p''^3-2p''q''+r'').$$

But the points whose number has been just found are points at which A_{11}, A_{22}, A_{12} touch, and they each count for four among the intersections of these surfaces. Subtracting then four times the number just found from the total number of intersections, we get

$$\tfrac{1}{2}(p''q''-r''+PQ-R),$$

whence we learn that if the number of double points on the surface represented by the symmetrical determinant C is $\frac{1}{2}(p''q''-r'')$, that of those on the surface D is $\frac{1}{2}(PQ-R)$, and the first theorem being established in the simplest case the other is generally true.

9. There is still another question which may be proposed concerning the curves, Art. 4. Let there be four surfaces whose degrees are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and whose coefficients contain any new variable in the degrees $\mu_1, \mu_2, \mu_3, \mu_4$, then the eliminant of these four equations contains the new variable in the degree

$$\lambda_1\lambda_2(\mu_3\lambda_4+\mu_4\lambda_3)+\lambda_3\lambda_4(\mu_1\lambda_2+\mu_2\lambda_1).$$

Now $\lambda_1\lambda_2$, $\lambda_3\lambda_4$ are the orders of the curve of intersection of the first and second, and third and fourth surfaces respectively; and if we call $\mu_1\lambda_2+\mu_2\lambda_1$, $\mu_3\lambda_4+\mu_4\lambda_3$ the weights of the same curves, we can assert that the weight of the condition that two curves may intersect is the sum of the products of the weight of each curve by the order of the other. Now we have seen what is the order of the curve denoted by a system of determinants, such as Art. 4; it remains to enquire what is the weight of the same system. It is easy to see that when a curve breaks up into two simpler curves the weight of the complex curve is equal to the sum of the weights of its components. We may therefore proceed as in Art. 4, and the following is the result. Let the functions a, b, c, &c. contain the new variable in the degrees A, B, C, &c.; a', b', &c. in the degrees $A+\alpha'$, $B+\alpha'$, &c. Let P', Q', R' denote the sum, sum of products in pairs, &c. of the quantities A, B, &c. and let p', q', &c. denote the corresponding sums for α', β', &c.: let S denote the sum $\Sigma(aB)$ where each a is multiplied by all the capital letters except A, so that S is also $=PP'-\Sigma(aA)$. Let also $s=\Sigma(\alpha\beta')$, which is in like manner $pp'-\Sigma(\alpha\alpha')$. Then the weight of the system is

$$S+pP'+p'P+2pp'-s,$$

which may also be written

$$(P+p)(P'+p')+\Sigma(\alpha\alpha')-\Sigma(aA).$$

If we had used P to denote the sum of the degrees in the first column instead of in the first row, &c., then the capital and small letters in the preceding formula would be interchanged.

10. We propose next to investigate the order and weight of the system of conditions that the two equations

$$at^m + bt^{m-1} + ct^{m-2} + \&c. = 0, \quad a't^n + b't^{n-1} + c't^{n-2} + \&c. = 0,$$

may have two common roots. It is evident that in order that this should be the case, two conditions must be fulfilled; and if t be a parameter, and a, b, &c. functions of the co-ordinates, these conditions will represent a curve in space. But in point of fact, we obtain not two, but a system of conditions, no two of which suffice to *define* the given curve. These conditions are (*Lessons on Higher Algebra*, Art. 33) the determinants of the system

$$\left\| \begin{array}{cccc} a, & b, & c, & \ldots \\ & a, & b, & \ldots \\ & & a, & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a', & b', & c', & \ldots \\ & a', & b', & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array} \right\|$$

where the first line is repeated $n-1$ times, and the second $m-1$ times; there are $m+n-2$ rows, and $m+n-1$ columns. The problem is then a particular case of that considered, Art. 4. We suppose the degrees of the functions introduced to be equi-different: that is to say, if the degrees of a, a' be λ, μ, we suppose those of b, b' to be $\lambda+\alpha$, $\mu+\alpha$; of c, c' to be $\lambda+2\alpha$, $\mu+2\alpha$, &c. To find the order of the system, we use the formula (Art. 4) $q+pP+P^2-Q$. p is the sum of $m+n-2$ terms of the series α, 2α, 3α, &c., and is therefore, if we write $m+n=k$, $\frac{(k-1)(k-2)}{1.2}\alpha$. In the same case q is the sum of products in pairs of these quantities, and is therefore

$$= \frac{(k-3)(k-2)(k-1)(3k-4)}{1.2.3.4}\alpha^2.$$

Again P is the sum of $n-1$ terms of the series λ, $\lambda - a$, $\lambda - 2a$, &c., and of $m-1$ terms of the series μ, $\mu - a$, $\mu - 2a$, &c. We have then

$$P = (n-1)\lambda + (m-1)\mu - \tfrac{1}{2}a\{(n-1)(n-2) + (m-1)(m-2)\}.$$

In like manner Q is the sum of the products in pairs of the same quantities, and we have

$$\begin{aligned} Q = \tfrac{1}{2}(n-1)(n-2)\lambda^2 + \tfrac{1}{2}(m-1)(m-2)\mu^2 + (m-1)(n-1)\lambda\mu \\ - \tfrac{1}{2}\lambda a\{(n-1)(n-2)^2 + (n-1)(m-1)(m-2)\} \\ - \tfrac{1}{2}\mu a\{(m-1)(m-2)^2 + (m-1)(n-1)(n-2)\} \\ + \tfrac{1}{2}a^2(m-1)(m-2)(n-1)(n-2) \\ + \frac{(m-1)(m-2)(m-3)(3m-4)}{1.2.3.4}a^2 + \frac{(n-1)(n-2)(n-3)(3n-4)}{1.2.3.4}a^2. \end{aligned}$$

Collecting these terms, the order of the required system is

$$\tfrac{1}{2}n(n-1)\lambda^2 + \tfrac{1}{2}m(m-1)\mu^2 + (m-1)(n-1)\lambda\mu + \tfrac{1}{2}n(n-1)(2m-1)\lambda a$$
$$+ \tfrac{1}{2}m(m-1)(2n-1)\mu a + \tfrac{1}{2}mn(m-1)(n-1)a^2.$$

If the eliminant of the equations

$$at^m + bt^{m-1} + \&c. = 0, \quad a't^n + \&c. = 0,$$

represent a surface, the curve here considered is a double curve on that surface.

If all the functions a, b, &c. are of the first degree, the surface generated is a ruled surface; and writing $\lambda = \mu = 1$ and $a = 0$ in the preceding formula, we find that the order of the double curve is $\tfrac{1}{2}(m+n-1)(m+n-2)$.

If the two equations considered are of the same degree, that is to say, if $m = n$, we may write $\lambda + \mu = p$, $\lambda\mu = q$, and the same formula gives for the degree of the double curve

$$\tfrac{1}{2}n(n-1)(p+na)\{p+(n-1)a\} - (n-1)q.$$

11. We can in like manner determine the order of the system of conditions that the equations $at^m + \&c.$, $a't^n + \&c.$ may have three common roots. When geometrically interpreted these conditions represent triple points on the surface represented by the eliminant of the two equations. The conditions are represented by a system of determinants, the matrix for which is formed as in the last article, save that the line

a, b, c is repeated $n-2$ times, and the line a', b', c', $m-2$ times; and the matrix consists of $m+n-2$ columns and $m+n-4$ rows. The order of the system is calculated from Art. 7, and is found to be

$$\frac{n(n-1)(n-2)}{1.2.3}\lambda^3+\frac{m(m-1)(m-2)}{1.2.3}\mu^3+\tfrac{1}{2}(n-1)(n-2)(m-2)\lambda^2\mu$$
$$+\tfrac{1}{2}(m-1)(m-2)(n-2)\lambda\mu^2+\tfrac{1}{2}(m-1)n(n-1)(n-2)\lambda^2\alpha$$
$$+\tfrac{1}{2}(n-1)m(m-1)(m-2)\mu^2\alpha+\tfrac{1}{2}(m-2)(n-2)\{m(n-1)+n(m-1)\}\lambda\mu\alpha$$
$$+\{\tfrac{1}{2}n(n-1)(n-2)m(m-2)+\tfrac{1}{3}n(n-1)(n-2)\}\alpha^2\lambda$$
$$+\{\tfrac{1}{2}m(m-1)(m-2)n(n-2)+\tfrac{1}{3}n(m-1)(m-2)\}\alpha^2\mu$$
$$+\tfrac{1}{6}m(m-1)(m-2)n(n-1)(n-2)\alpha^3.$$

In the case where the surface is a ruled surface, putting $\alpha=0$, $\lambda=\mu=1$, we get for the number of triple points

$$\frac{(m+n-2)(m+n-3)(m+n-4)}{1.2.3}.$$

The order of the developable generated by the double curve (Art. 10) is calculated in like manner by the formula of Art. 8, but the number so found must be reduced by four times the number of triple points just found, which are also triple points on that curve. Thus in the case of the ruled surface the rank of the double curve is $2(m+n-2)(m+n-3)$.

To find the weight of the same system we have only to apply the same method to the formula of Art 8. Let the term a contain the variable to be eliminated in the degree λ and the uneliminated in the degree λ', and let the terms b, c, &c. decrease regularly in the former and increase in the latter; so that their degrees are $\lambda-1$, $\lambda-2$, $\lambda'-1$, $\lambda'-2$, &c., then the weight of the system is

$$n(n-1)\lambda\lambda'+m(m-1)\mu\mu'+(m-1)(n-1)(\lambda\mu'+\lambda'\mu)$$
$$+\tfrac{1}{2}n(n-1)(2m-1)(\lambda-\lambda')+\tfrac{1}{2}m(m-1)(2n-1)(\mu-\mu')-mn(m-1)(n-1).$$

12. The next system we discuss is that formed by the system of conditions that the three equations

$$at^l+bt^{l-1}+\&c.=0,\quad a't^m+b't^{m-1}+\&c.=0,\quad a''t^n+b''t^{n-1}+\&c.=0$$

may have a common factor. The system may be expressed by the three equations obtained by eliminating t in turn between every pair of these equations, a system equivalent to two conditions. Systems of equations of lower degree can be got by multiplying the given equations by t, t^2, &c. until there is a sufficient number to eliminate dialytically all the powers of t. The order of the system may be found by eliminating from the equations x, y, z which enter implicitly into a, b, c, when the order of the resulting equation in t determines the order of the system. Let us suppose that the orders of a, a', a'', are λ, μ, ν respectively and of b, b', b''; $\lambda - 1$, $\mu - 1$, &c., then I found (*Quarterly Journal*) that the order of the system is

$$\lambda\mu\nu - (\lambda - l)(\mu - m)(\nu - n),$$

and its weight

$$l(\mu\nu' + \mu'\nu) + m(\nu\lambda' + \nu'\lambda) + n(\lambda\mu' + \lambda'\mu)$$
$$+ mn(\lambda - \lambda') + nl(\mu - \mu') + lm(\nu - \nu') - 2lmn;$$

the value for the weight however having been only obtained by induction.

13. It is a particular case of the preceding to find the order and weight of the system of conditions that an equation $at^n + bt^{n-1} + \&c.$ may have three equal roots; because these conditions are found by expressing that the three second differential equations may have a common factor. Writing in the preceding for l, m, and n, $n-2$; for μ, $\lambda - 1$; and for ν, $\lambda - 2$, we find for the order of the system

$$3(n-2)\lambda(\lambda - n) + n(n-1)(n-2),$$

and in like manner for its weight

$$6(n-2)\lambda\lambda' + 3n(n-2)(\lambda - \lambda') - 2n(n-1)(n-2).$$

Again, to find the order and weight of the system of conditions that the same equation may have two distinct pairs of equal roots; we form first, by Art. 10, the order and weight of the system of conditions that the two first differentials $at^{n-1} + \&c.$, $bt^{n-1} + \&c.$ may have two common factors. We subtract then the order and weight of the system found in the first part of this article. The result is that the order is

$$2(n-2)(n-3)\lambda(\lambda - n) + \tfrac{1}{2}n(n-1)(n-2)(n-3),$$

and the weight is

$$4(n-2)(n-3)\lambda\lambda'+2n(n-2)(n-3)(\lambda-\lambda')-n(n-1)(n-2)(n-3).$$

The formulæ of this article are those of which use has been made p. 407. It would be desirable to find in like manner the order and weight of the system of conditions that three curves should have two points common, that four curves should meet in a point, that a curve should have a cusp, or two pairs of double points, &c., but these problems have not yet been solved.

THE END.

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