CHAPTER VIII.

GRASSMANN'S SPACE ANALYSIS.

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ART. 1. EXPLANATIONS AND DEFINITIONS.

The algebra with which the student is already familiar deals directly with only one quality of the various geometric and mechanical entities, such as lines, forces, etc., namely, with their magnitude. Such questions as How much? How far? How long? are answered by an algebraic operation or series of operations. Questions of direction and position are dealt with indirectly by means of systems of coordinates of various kinds. In this chapter an algebra* will be developed which deals directly with the three qualities of geometric and mechanical quantities, viz., magnitude, position, and direction. A geometric quantity may possess one, two, or all three of these properties simultaneously; thus a straight line of given length has all three, while a point has only one.

The geometric quantities with which we are to be concerned are the point, the straight line, the plane, the vector, and the plane-vector.

When the word "line" is used by itself, a "straight line" will be always intended. A portion of a given straight line of definite length will be called a "sect"; though when the length

* The algebra of this chapter is a particular case of the very general and comprehensive theory developed by Hermann Grassmann, and published by him in 1844 under the title "Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik." He published also a second treatise on the subject in 1862.

of the sect is a matter of indifference, the word line will frequently be used instead. Similarly, a definite area of a given plane will be called a "plane-sect."

If a point recede to infinity, it has no longer any significance as regards position, but still indicates a direction, since all lines passing through finite points, and also through this point at infinity, are parallel. Similarly, a line wholly at infinity fixes a plane direction, that is, all planes passing through finite points, and also through this line at infinity, are parallel. Thus a point and line at infinity are respectively equivalent to a line direction and a plane direction.

A quantity possessing magnitude only will be termed a "scalar" quantity. Such are the ordinary subjects of algebraic analysis, a, x, $\sin\theta$, $\log z$, etc., and they may evidently be intrinsically either positive or negative.

The letter T prefixed to a letter denoting some geometric quantity will be used to designate its absolute or numerical magnitude, always positive. Thus, if L be a sect, and P a plane-sect, then TL is the length of L, and TP is the area of P. That portion of a geometric quantity whose magnitude is unity will be called its "unit," and will be indicated by prefixing the letter U; thus $UL =$ unit of $L =$ sect one unit long on line L.* Hence we have $TL \,.\, UL = L$.

ART. 2. SUM AND DIFFERENCE OF TWO POINTS.

In geometric addition and subtraction we shall use the ordinary symbols $+$, $-$, $=$, but with modified significance, as will appear in the development of the subject.

Every mathematical, or other, theory rests on certain fundamental assumptions, the justification for these assumptions

* The word "scalar" and the use of the letters T and U, as above, were introduced by Hamilton in his Quaternions. T stands for tensor, i.e., stretcher, and TL is the factor that stretches UL into L. The notation $| L |$ for absolute magnitude is not used, because the sign | has been appropriated by Grassmann to another use.

lying in the harmony and reasonableness of the resulting theory, and its accordance with the ascertained facts of nature.

Our first assumption, then, will be that the associative and commutative laws hold for geometric addition and subtraction, that is, whatever A, B, C may represent, we have

$$A+B+C=(A+B)+C=A+(B+C)$$
$$=A+C+B=(A+C)+B.$$

We shall also assume that we always have $A-A=0$, and that the same quantity may be added to or subtracted from both sides of an equation without affecting the equality.

Now let p_1, p_2 be two points, and consider the equation

$$p_2+p_1-p_1=p_2+(p_1-p_1)=p_2. \tag{1}$$

In this form we have an identity. Write it, however, in the form

$$p_2-p_1+p_1=(p_2-p_1)+p_1=p_2, \tag{2}$$

and it appears that p_2-p_1 is an operator that changes p_1 into p_2 by being added to it. Conceive this change of p_1 into p_2 to take place along the straight line through p_1 and p_2; then the operation is that of moving a point through a definite length or distance in a definite direction, namely, from p_1 to p_2. This operator has been called by Hamilton "a vector,"* that is, a carrier, because it carries p_1 rectilinearly to p_2. Grassmann gives to it the name Strecke, and some writers now use the word "stroke" in the same sense.

Again, p_2-p_1 is the difference of two points, and the only difference that can exist between them is that of position, i.e. a certain distance in a certain direction.

Hence we may regard p_2-p_1 as a directed length, and also as the operator which moves p_1 over this length in this direction. Writing $p_2-p_1=\epsilon$, equation (2) becomes

$$p_1+\epsilon=p_2. \tag{3}$$

* See the first of Hamilton's Lectures on Quaternions, where a very full discussion of equation (2) will be found. Also Grassmann (1862), Art. 227.

Thus the sum of a point and a vector is a point distant from the first by the length of the vector and in its direction.

Since $p_2 - p_1 = -(p_1 - p_2)$, it appears that the negative of a vector is a vector of the same length in the opposite direction.

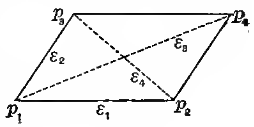
If $p_2 - p_1 = 0$, or $p_2 = p_1$, p_2 must coincide with p_1 because there is now no difference between the two points.

The question arises as to what, if any, effect the operator $p_2 - p_1$ should have on any other point p_3, that is, what is the value of the expression $p_2 - p_1 + p_3$?

We will assume that it is some point p_4, so that we have $p_2 - p_1 + p_3 = p_4$,

or
$$p_2 - p_1 = p_4 - p_3. \tag{4}$$

This implies that the transference from p_3 to p_4 is the same in amount and direction as that from p_1 to p_2, that is, that p_1, p_2, p_4, p_3 are the four corners of a parallelogram taken in order. Thus equal vectors have the same length and direction, and, conversely, vectors having the same length and direction are equal.



Note that parallel vectors of equal length are not necessarily equal, for their directions may be opposite.

Equation (4) may also be written
$$p_2 + p_3 = p_4 + p_1, \tag{5}$$
so that, whatever meaning may be assigned to the sum of two points, if we are to be consistent with assumptions already made, we must have the sum of either pair of opposite corner-points of a parallelogram equal to the sum of the other pair. The sum cannot therefore depend on the actual distances apart of the points forming the pairs, for the ratio of these two distances may be made as large or as small as we please.

If n be a scalar quantity, $n\epsilon$ will denote that the operation ϵ is to be performed n times on a point to which $n\epsilon$ is added, that is, the point will be moved n times the length of ϵ; hence

$n\epsilon$ is a vector n times as long as ϵ, and having the same or the opposite direction according to the sign of n.

In the figure above, let

$$p_2 - p_1 = \epsilon_1, \quad p_3 - p_1 = \epsilon_2, \quad p_4 - p_1 = \epsilon_3, \quad p_3 - p_2 = \epsilon_4.$$

Then

$$\epsilon_1 + \epsilon_2 = p_2 - p_1 + p_3 - p_1 = p_2 - p_1 + p_4 - p_2 = p_4 - p_1 = \epsilon_3, \quad (5)$$

since, by Art. 4, $p_3 - p_1 = p_4 - p_2$.

Also, $$\epsilon_2 - \epsilon_1 = p_3 - p_2 = \epsilon_4. \quad (6)$$

Hence, if two vectors are drawn outwards from a point, and the parallelogram of which these are two adjacent sides is completed, then the two diagonals of this parallelogram will represent respectively the sum and difference of the two vectors, the sum being that diagonal which passes through the origin of the two vectors, and the difference that which passes through their extremities.*

Again, $p_2 - p_1 + p_3 - p_2 + p_1 - p_3 = 0 = \epsilon_1 + \epsilon_4 + (-\epsilon_2)$; hence the sum of three vectors represented by the sides of a triangle taken around in order is zero.

Similarly, if $p_1, p_2, \ldots p_n$ be any n points whatever taken as corners of a closed polygon, we shall have

$$(p_2 - p_1) + (p_3 - p_2) + (p_4 - p_3) + \cdots + (p_n - p_{n-1}) + (p_1 - p_n) = 0;$$

that is, the sum of vectors represented by the sides taken in order about the polygon is zero. By "taken in order" is not meant that any particular order of the points must be observed in forming the polygon, which is evidently unnecessary, but simply that, when the polygon is formed, the vectors will be the operators that will move a point from the starting position along the successive sides back to this position again, so that the final distance from the starting-point will be nothing.

ART. 3. SUM OF TWO WEIGHTED POINTS.†

Consider the sum $m_1p_1 + m_2p_2$, in which m_1 and m_2 are scalars, that is, numbers, positive or negative, and p_1, p_2 are points.

* Grassmann (1844), § 15.

† Grassmann (1844), § 95, and (1862), Art. 227.

The scalars m_1 and m_2 will be regarded as values or weights assigned to the points p_1 and p_2. When any weight is of unit value the figure 1 will be omitted, so that p means $1p$, and is called a unit point. Occasionally, however, a letter may be used to denote a point whose weight is not unity.

To assist his thinking, the reader may consider the weights initially as like or unlike parallel forces acting at the points.

In order to arrive at a meaning for the above expression we shall make two reasonable assumptions, which will prove to be consistent with those already made, viz., first, that the sum is a point, and second, that its weight is the sum of the weights of the two given points. Denoting this sum-point by $\bar{p}$, we write

$$m_1 p_1 + m_2 p_2 = (m_1 + m_2)\bar{p}. \tag{7}$$

Transposing, we have $m_1(p_1 - \bar{p}) = m_2(\bar{p} - p_2)$, or

$$\frac{p_1 - \bar{p}}{m_2} = \frac{\bar{p} - p_2}{m_1}. \tag{8}$$

Both members of (8) are vectors, and, being equal, they must, by Art. 4, be parallel. This requires that $\bar{p}$ shall be collinear with p_1 and p_2. Also, since $p_1 - \bar{p}$ and $\bar{p} - p_2$ are vectors whose lengths are respectively the distances from p_1 to $\bar{p}$ and from $\bar{p}$ to p_2, it follows that these distances are in the ratio of m_2 to m_1. Hence, $\bar{p}$ is a point on the line $p_1 p_2$ whose distances from p_1 and p_2 are inversely proportional to the weights of these points. We shall call $\bar{p}$ the mean point of the two weighted points. If m_1 and m_2 are both positive, (8) shows that $\bar{p}$ must lie between p_1 and p_2; but if one, say m_2, is negative, let $m_2 = -m_2'$. Thus

$$m_1(p_1 - \bar{p}) = m_2'(p_2 - \bar{p}), \tag{9}$$

and $\bar{p}$ is on the same side of each point, that is, its direction from each point is the same. Also, since its distances from the two points are inversely as their weights, $\bar{p}$ must be nearest the point whose weight is greatest.

Case when $m_1 + m_2 = 0$, or $m_2 = -m_1$.*—With this condition equations (7) and (8) become

$$m_1p_1 + m_2p_2 = m_1(p_1 - p_2) = 0 \cdot \bar{p}, \tag{10}$$

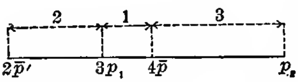
and

$$\bar{p} - p_1 = \bar{p} - p_2. \tag{11}$$

Thus $\bar{p}$ is in the same direction from each point, that is, not between them, and yet is equidistant from them. This requires either that the two points shall coincide, that is, $p_2 = p_1$, which evidently satisfies (10) and (11); or else, p_1 and p_2 being different points, that $\bar{p}$ shall be at an infinite distance. Thus the sum is in this case a point of zero weight at infinity.† Eq. (10) shows that a zero point at infinity is equivalent to a vector, or directed quantity, as stated in Art. 1. It has been shown in Art. 2 that $p_2 = p_1$ is the condition that p_1 and p_2 coincide; let us consider the equality of weighted points in general, say $m_1p_1 = m_2p_2$. Hence, by (7), there is found $m_1p_1 - m_2p_2 = (m_1 - m_2)\bar{p} = 0$; hence, since $\bar{p}$ cannot be zero, $m_1 - m_2 = 0$, or $m_1 = m_2$; and therefore $m_1(p_1 - p_2) = 0$, or, since $m_1 \gtrless 0$, $p_1 - p_2 = 0$, that is, $p_1 = p_2$. Therefore, if any two points are equal, their weights must be the same and their positions identical, that is, they are the same point.

Exercise 1.—To find the sum and difference of the two weighted points $3p_1$ and p_2:

$$3p_1 + p_2 = 4\bar{p}, \qquad 3p_1 - p_2 = 2\bar{p}',$$



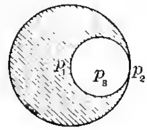
and the mean points are as shown in the figure. The reciprocals of the distances of $\bar{p}$, p_1, and $\bar{p}'$ from p_2, viz., $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, are in arithmetical progression, hence the points form a harmonic range.

Exercise 2.—Given a circular disk with a circular disk of

* Grassmann (1862), Art. 222.

† Compare the case of the resultant of unlike parallel forces of equal magnitude.

half its radius removed, as in the figure; to find the centroid of the remaining portion.



Take p_1 at center of large circle, p_3 at center of small circle, and p_2 at the point of contact; then $p_3 = \frac{1}{2}(p_1 + p_2)$. The areas of the two circles are as 1 : 4; call them 1 and 4. Then it is as if there were a weight 4 at p_1, and a weight -1 at p_3; hence $\bar{p} = [4p_1 - \frac{1}{2}(p_1 + p_2)] \div 3 = (7p_1 - p_2) \div 6$.

Prob. 1. Show that $p_1, p_2, m_1p_1 + m_2p_2$, and $m_1p_1 - m_2p_2$ are four points forming a harmonic range.

Prob. 2. An inscribed right-angled triangle is cut from a circular disk; show that the centroid of the remainder of the disk is at the point

$$\frac{(3\pi - 2 \sin 2\alpha)\, p_1 - p_2 \sin 2\alpha}{3(\pi - \sin 2\alpha)},$$

if p_1 is the center of the circle, p_2 the opposite vertex of the triangle, and α one of its angles.

ART. 4. SUM OF ANY NUMBER OF POINTS.

As in the last article we assume the sum to be a point whose weight is equal to the sum of the weights of the given points; thus,

$$\sum_1^n mp = \bar{p}\sum_1^n m. \tag{12}$$

Let e be some fixed point, and subtract $e\sum_1^n m$ from both sides of (12); thus we have

$$\sum_1^n m(p - e) = (\bar{p} - e)\sum_1^n m, \tag{13}$$

an equation which gives a simple construction for $\bar{p}$.

If $\sum_1^n m = 0$, then $m_1 = -\sum_2^n m$, and

$$\sum_1^n mp = m_1p_1 + \sum_2^n mp = m_1\left(p_1 - \frac{\sum_2^n mp}{\sum_2^n m}\right), \tag{14}$$

so that the sum becomes the difference of two unit points, or a vector whose direction is parallel to the line joining p_1 with the mean of all the other points of the system, and whose length is m_1 times the distance between these points. Since any point of the system may be designated as p_1, it follows that the line joining any point of the system to the mean of all the others is parallel to any other such line. If $\sum_1^n mp = 0$, equation (14) shows that p_1 is the mean of all the other points of the system, and, since any one of the points may be taken as p_1, any point of the system is the mean of all the others.

Let $n = 3$ in (12) and (13); then

$$m_1p_1 + m_2p_2 + m_3p_3 = (m_1 + m_2 + m_3)\bar{p}, \qquad (15)$$

$$m_1(p_1 - e) + m_2(p_2 - e) + m_3(p_3 - e) = (m_1 + m_2 + m_3)(\bar{p} - e), \quad (16)$$

and $\bar{p}$ is on the line joining the point $m_1p_1 + m_2p_2$ with p_3, and therefore inside the triangle $p_1p_2p_3$ if the m's are all positive. If m_3 be negative and numerically less than $m_1 + m_2$, then $\bar{p}$ will have passed across the line p_1p_2 to the outside of the triangle. If m_1 and m_2 are negative and their sum numerically less than m_3, then $\bar{p}$ will have passed outside the triangle through p_3, i.e., it will have crossed p_2p_3 and p_3p_1. The point $\bar{e}$ must evidently always be in the plane $p_1p_2p_3$.

As a numerical example let $m_1 = 3$, $m_2 = 4$, $m_3 = -5$, so that (16) becomes

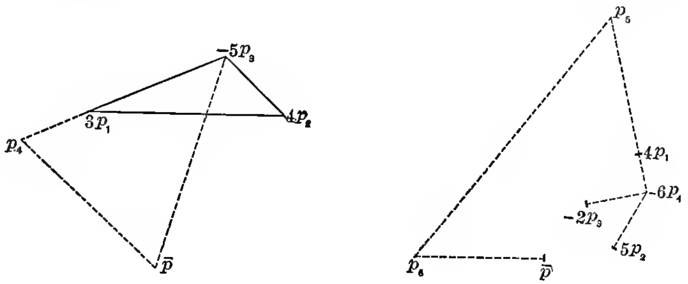
$$\bar{p} - e = \tfrac{3}{2}(p_1 - e) + 2(p_2 - e) - \tfrac{5}{2}(p_3 - e).$$

Now, since e may be any point whatever, put $e = p_3$; then $\bar{p} - p_3 = \tfrac{3}{2}(p_1 - p_3) + 2(p_2 - p_3)$, and the construction is shown in the figure. $p_4 - p_3 = \tfrac{3}{2}(p_1 - p_3)$, and $\bar{p} - p_4 = 2(p_2 - p_3)$.

As another example take $\bar{p} = 4p_1 + 5p_2 - 2p_3 - 6p_4$, or, by (13), making $e = p_4$,

$$\begin{aligned}\bar{p} - p_4 &= 4(p_1 - p_4) + 5(p_2 - p_4) - 2(p_3 - p_4)\\ &= p_5 - p_4 + p_6 - p_5 + \bar{p} - p_6.\end{aligned}$$

When any number of geometric quantities can be connected with each other by an equation of the form $\Sigma mp = 0$, in which the m's are finite and different from zero, then they are said to be mutually dependent, that is, any one can be expressed in terms of the others. If no such relation can exist between the



quantities, they are independent. We obtain from what has preceded the following conditions:

That two points shall concide,

$$m_1p_1 + m_2p_2 = 0. \tag{17}$$

That three points shall be collinear,

$$m_1p_1 + m_2p_2 + m_3p_3 = 0. \tag{18}$$

That four points shall be coplanar,

$$m_1p_1 + m_2p_2 + m_3p_3 + m_4p_4 = 0. \tag{19}$$

It follows that three non-collinear points cannot be connected by an equation like (18) unless each coefficient is separately zero. Similarly four non-coplanar points cannot be connected by an equation like (19) unless each coefficient is separately zero.

The significance of these statements will be presently illustrated.

The following are corresponding equations of condition for vectors:

That two vectors shall be parallel,

$$n_1\epsilon_1 + n_2\epsilon_2 = 0. \tag{20}$$

That three vectors shall be parallel to one plane,

$$n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 = 0. \qquad (21)$$

These conditions follow from the results of Art. 2, or from equations (17) and (18) by regarding the ϵ's as points at infinity. If in addition to (21) we have

$$n_1 + n_2 + n_3 = 0, \qquad (22)$$

the extremities of the three vectors, if radiating from a point, will be collinear: for, let $e_0 \ldots e_3$ be four points so taken that $e_1 - e_0 = \epsilon_1$, $e_2 - e_0 = \epsilon_2$, $e_3 - e_0 = \epsilon_3$; then (21) becomes

$$n_1(e_1 - e_0) + n_2(e_2 - e_0) + n_3(e_3 - e_0) = 0,$$

or by (22) $$n_1 e_1 + n_2 e_2 + n_3 e_3 = 0,$$

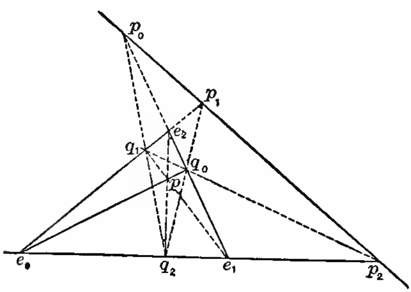
which by (18) requires e_1, e_2, e_3 to be collinear.

It may be shown similarly that

$$\sum_1^4 n\epsilon = \sum_1^4 n = 0 \qquad (23)$$

are the conditions that four vectors radiating from a point shall have their extremities coplanar.

Exercise 3.—Given a triangle $e_0e_1e_2$ and a point p in its plane; pe_0 cuts e_1e_2 in q_0, pe_1 cuts e_2e_0 in q_1, pe_2 cuts e_0e_1 in q_2, q_1q_2 cuts e_1e_2 in p_0, q_2q_0 cuts e_2e_0 in p_1, and q_0q_1 cuts e_0e_1 in p_2: to show that p_0, p_1, and p_2 are collinear.



Let $p = n_0e_0 + n_1e_1 + n_2e_2$; then q_0, q_1, q_2 coincide respectively with $n_1e_1 + n_2e_2$, $n_2e_2 + n_0e_0$, and $n_0e_0 + n_1e_1$ because p lies on the line joining e_0 with q_0, etc. Hence, if x_0, x_1, y_0, y_1 are scalars,

$$p_2 = x_0e_0 + x_1e_1 = y_0(n_1e_1 + n_2e_2) + y_1(n_2e_2 + n_0e_0);$$

hence $$(x_0 - y_1n_0)e_0 + (x_1 - y_0n_1)e_1 - n_2(y_0 + y_1)e_2 = 0.$$

Now the e's are not collinear, and yet are connected by a relation of the form of equation (18); hence, as was there shown, each coefficient must be zero; accordingly

$$x_0 - y_1 n_0 = x_1 - y_0 n_1 = y_0 + y_1 = 0,$$

whence we find $\qquad x_0 : x_1 = n_0 : - n_1.$

hence $\qquad (n_0 - n_1)p_2 = n_0 e_0 - n_1 e_1$, and similarly

$$(n_1 - n_2)p_0 = n_1 e_2 - n_2 e_2, \quad (n_2 - n_0)p_1 = n_2 e_2 - n_0 e_0.$$

Adding, we have

$$(n_1 - n_2)p_0 + (n_2 - n_0)p_1 + (n_0 - n_1)p_2 = 0;$$

therefore, by (18), p_0, p_1, p_2 are collinear.

Exercise 4.—Let $p = \sum_0^2 ne \div \sum_0^2 n$ be any point in the plane of the triangle $e_0e_1e_2$: show that lines through the middle points of the sides e_1e_2, e_2e_0, and e_0e_1 of the triangle parallel to e_0p, e_1p, and e_2p meet in a point

$$p' = [(n_1 + n_2)e_0 + (n_2 + n_0)e_1 + (n_0 + n_1)e_2] \div 2\sum_0^2 n.$$

By the conditions the vector from the middle point of e_1e_2 to p' is a multiple of the vector $e_0 - p$; hence

$$p' - \tfrac{1}{2}(e_1 + e_2) = x(e_0 - p) \quad \text{or}$$

$$p' = \tfrac{1}{2}(e_1 + e_2) + x(e_0 - p) = \tfrac{1}{2}(e_0 + e_1) + y(e_2 - p),$$

or, substituting value of p,

$$p' = \tfrac{1}{2}(e_1 + e_2) + x(e_0 - \Sigma ne \div \Sigma n) = \tfrac{1}{2}(e_0 + e_1) + y(e_2 - \Sigma ne \div \Sigma n).$$

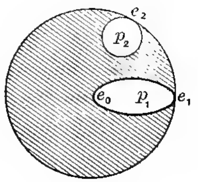
hence $\qquad [(x - \tfrac{1}{2})\Sigma n + n_0(y - x)]e_0 + n_1(y - x)e_1$

$$+ [(\tfrac{1}{2} - y)\Sigma n + n_2(y - x)]e_2 = 0;$$

therefore, as in the previous exercise, each coefficient must be zero, whence $x = y = \frac{1}{2}$, and substituting we find p' as above. It follows also that the distances of p' from the middle points of the sides are the halves of the distances of p from the opposite vertices.

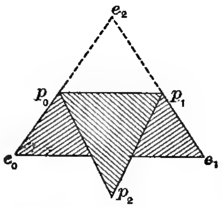
Prob. 3. Show that $\bar{e} = \frac{1}{3}\sum_0^2 e$ is collinear with p and p' of Exercise 4. Also that, by properly choosing p, it follows that $\bar{e}$ is collinear with the common point of the perpendiculars from the vertices on the opposite sides, and the common point of the perpendiculars to the sides at their middle points.

Prob. 4. Given two circles and an ellipse, as in the figure, with centers at e_0, p_2, and p_1. Radii of circles 4 and 1, axes of ellipse 2 and 4, small circle and ellipse touching large circle at e_2 and e_1 respectively, $e_0e_1e_2$ an equilateral triangle: show that the centroid of the remainder of the large circle, after the small areas are removed, will be at



$$\bar{p} = \tfrac{1}{13}(16e_0 - p_2 - 2p_1) = \tfrac{1}{52}(59e_0 - 4e_1 - 3e_2).$$

Prob. 5. If a sheet of tin in the shape of an isosceles triangle be folded over as in the figure, show that its centroid is given by $3\bar{p} = \frac{1}{27}[35(e_0 + e_1) + 11e_2]$.



Prob. 6. If a tetrahedron $e_0e_1e_2e_3$ have a tetrahedron of $\frac{1}{8}$ of its volume cut off by a plane parallel to $e_0e_1e_2$, and one of $\frac{1}{64}$ of its volume cut off by a plane parallel to $e_1e_2e_3$, show that the centroid of the remaining solid is at

$$\bar{p} = \tfrac{1}{880}(227e_0 + 175e_3 + 239(e_1 + e_2)).$$

ART. 5. REFERENCE SYSTEMS.

Let p be any unit point, e_0, e_1, e_2 three fixed unit points, and w, x, y scalars; then, writing

$$p = we_0 + xe_1 + ye_2, \tag{24}$$

we must have also, because p is a unit point,

$$w + x + y = 1, \tag{25}$$

and p is the mean of the weighted points we_0, xe_1, ye_1. The point p may occupy any position whatever in the plane $e_0e_1e_2$; for it is on the line joining $we_0 + xe_1$ with e_2, and by varying y and $w + x$, $\dfrac{w}{x}$ remaining constant, p may be moved along this line from $-\infty$ to $+\infty$; while by varying the ratio $\frac{w}{x}$ the point $we_0 + xe_1$ may be moved from $-\infty$ to $+\infty$ along e_0e_1, and thus the first line will be rotated through 180 degrees, and p may thus be given any position whatever in the plane.

A system of unit points to which the positions of other points may be referred is called a reference system, and the triangle $e_0e_1e_2$ is a reference triangle. For reasons that will appear later, the double area of this triangle will be taken as the unit of measurement of area for a point system in two-dimensional space.

Similarly, in solid space, taking a fourth point e_3, we write

$$p = we_0 + xe_1 + ye_2 + ze_3, \qquad (26)$$

which implies also $\quad w + x + y + z = 1; \qquad (27)$

and p may be shown as above to be capable of occupying any position whatever in space by properly assigning the values of w, x, y, z; so that $e_0, \ldots e_3$ form a reference system for points in three-dimensional space. The tetrahedron $e_0e_1e_2e_3$ is called the reference tetrahedron, and six times its volume will be taken as the unit of volume for a point system in three-dimensional space.

Eliminating w between (24) and (25), we have

$$p = e_0 + x(e_1 - e_0) + y(e_2 - e_0), \qquad (28)$$

from which it may also be easily seen that p may be any point in the plane $e_0e_1e_2$. Writing $p - e_0 = \rho$, $e_1 - e_0 = \epsilon_1$, $e_2 - e_0 = \epsilon_2$, (28) becomes

$$\rho = x\epsilon_1 + y\epsilon_2, \qquad (29)$$

and ϵ_1, ϵ_2 form a plane reference system for vectors.

Similarly, from (26) and (27) we find

$$\rho = x\epsilon_1 + y\epsilon_2 + z\epsilon_3, \qquad (30)$$

and ϵ_1, ϵ_2, ϵ_3 are a reference system for vectors in solid space, any vector whatever being expressible in terms of these three.

If, in equations (25) and (26), the reference vectors are of unit length and mutually perpendicular, we have unit, normal reference systems, and in this case ι_1, ι_2, ι_3 will generally be used instead of ϵ_1, ϵ_2, ϵ_3.

Exercise 5.—To change from one reference system to another, say from e_0, e_1, e_2 to e_0', e_1', e_2'.

The new reference points must be connected with the old ones by equations such as

$$e_0 = l_0e_0' + l_1e_1' + l_2e_2', \quad e_1 = m_0e_0' + m_1e_1' + m_2e_2',$$
$$e_2 = n_0e_0' + n_1e_1' + n_2e_2'.$$

Then any point $p = x_0e_0 + x_1e_1 + x_2e_2$ will be expressed in terms of the new reference points by substituting the values of e_0, etc., as given. If e_0', e_1', e_2' are given in terms of the old points, e_0, e_1, e_2 may be found by elimination. Thus, if $e_0' = \Sigma le$, $e_1' = \Sigma me$, $e_2' = \Sigma ne$, we have at once

$$\begin{vmatrix} l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \\ n_0 & n_1 & n_2 \end{vmatrix} e_0 = \begin{vmatrix} e_0' & l_1 & l_2 \\ e_1' & m_1 & m_2 \\ e_2' & n_1 & n_2 \end{vmatrix},$$

with similar values for e_1 and e_2.

As a numerical example let the new reference triangle be formed by joining the middle points of the sides of the old one. Then $e_0' = \frac{1}{2}(e_1 + e_2)$, $e_1' = \frac{1}{2}(e_2 + e_0)$, $e_2' = \frac{1}{2}(e_0 + e_1)$; whence $e_0 = - e_0' + e_1' + e_2'$, $e_1 = e_0' - e_1' + e_2'$, $e_2 = e_0' + e_1' - e_2'$. Thus $p = x_0e_0 + x_1e_1 + x_2e_2$

$$= (- x_0 + x_1 + x_2)e_0' + (x_0 - x_1 + x_2)e_1' + (x_0 + x_1 - x_2)e_2'.$$

Exercise 6.—Three points being given in terms of the reference points e_0, e_1, e_2, find the condition that must hold between their weights when they are collinear.

Let $p_0 = \sum_0^2 le$, $p_1 = \sum_0^2 me$, $p_2 = \sum_0^2 ne$; then, k_0, k_1, k_2 being scalars, we must have for collinearity, by (18),

$$k_0p_0 + k_1p_1 + k_2p_2 = 0,$$

that is, $$k_0 \Sigma le + k \Sigma_1 me + k \Sigma ne = 0,$$

whence $$(k_0 l_0 + k_1 m_0 + k_2 n_0)e_0 + (k_0 l_1 + k_1 m_1 + k_2 n_1)e_1 + (k_0 l_2 + k_1 m_2 + k_2 n_2)e_2 = 0,$$

and, as e_0, e_1, e_2 are not collinear, the coefficients must be zero, by Art. 4; hence

$$k_0 l_0 + k_1 m_0 + k_2 n_0 = k_0 l_1 + k_1 m_1 + k_2 n_1 = k_0 l_2 + k_1 m_2 + k_2 n_2 = 0,$$

and, by elimination of the k's,

$$\begin{vmatrix} l_0 & m_0 & n_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0, \qquad (31)$$

which is the required condition of collinearity.

Prob. 7. If $p = 3e_0 - e_1 - e_2$, $4e_0' = 3e_1 + e_2$, $4e_1' = 3e_2 + e_0$, $4e_2' = 3e_0 + e_1$, show that $7p = -19e_0' - 3e_1' + 29e_2'$.

Prob. 8. Find the condition that four points $\sum_0^3 ke$, $\sum_0^3 le$, $\sum_0^3 me$, $\sum_0^3 ne$ shall be coplanar. Ans. $[k_0, l_1, m_2, n_3] = 0$.

Prob. 9. If $p = we_0 + xe_1 + ye_2$, and there exist between the scalars w, x, y a linear relation such as $Aw + Bx + Cy = 0$, A, B, C being scalar constants, show that p will always lie on a straight line which cuts the reference lines in $Ae_1 - Be_0$, $Ae_2 - Ce_0$, and $Ce_1 - Be_2$. Consider the special cases when $A = B$, $B = C$, $C = A$, $A = B = C$, $A = 0$, $B = 0$, and $C = 0$.

Prob. 10. If $p = we_0 + xe_1 + ye_2 + ze_3$, and there exist also an equation $Aw + Bx + Cy + Dz = 0$, show that p will lie on a plane which cuts the edges of the reference tetrahedron in $\frac{e_1}{B} - \frac{e_0}{A}$, $\frac{e_2}{C} - \frac{e_0}{A}$, etc. Also, if a second relation between the variables, such as $A'w + B'x + C'y + D'z = 0$, be given, then p lies on a line which pierces the faces of the reference tetrahedron in

$$\begin{vmatrix} e_0 & e_1 & e_2 \\ A & B & C \\ A' & B' & C' \end{vmatrix}, \quad \begin{vmatrix} e_3 & e_0 & e_1 \\ D & A & B \\ D' & A' & B' \end{vmatrix}, \quad \text{etc.}$$

ART. 6. NATURE OF GEOMETRIC MULTIPLICATION.*

The fundamental idea of geometric multiplication is, that a product of two or more factors is that which is determined by those factors.

Thus, two points determine a line passing through them, and also a length, viz., the shortest distance between them; hence $p_1p_2 = L$ is the sect † drawn from p_1 to p_2, or generated by a point moving rectilinearly from p_1 to p_2.

The student should note carefully the difference between p_1p_2 and $p_2 - p_1$; they have the same length and direction, but the sect p_1p_2 is confined to the line through these two points, while the vector $p_2 - p_1$ is not. The sect has position in addition to the direction and length possessed by the vector.

Again, in plane space, two sects determine a point, the intersection of the lines in which they lie, and also an area, as will appear later, so that $L_1L_2 = p$, in which p is not in general a unit point. In solid space, however, two lines do not, in general, meet, and hence cannot fix a point; but two sects, in this case, determine a tetrahedron of which they are opposite edges.

It appears, therefore, that a product may have different interpretations in spaces of different dimensions. Hence we will consider separately products in plane space, or planimetric products, and those in solid space, or stereometric products.

Products of the kind here considered are termed "combinatory," because two or more factors combine to form a new quantity different from any one of them. This is the fundamental difference between this algebra and the linear associative algebras of Peirce, of which quaternions are a special case.

Before discussing in detail the various products that may arise, we will give a table which will serve as a sort of bird's-eye view of the subject.

* Grassmann (1844), Chap. 2; (1862), Chap. 2.

† See Art. 1.

In this table and generally throughout the chapter we shall use p, p_1, p_2, etc., for points; $\epsilon, \epsilon_1, \epsilon_2$, etc., for vectors; L, L_1, etc., for sects, or lines; η, η_1, etc,, for plane-vectors; and P, P_1, etc., for plane-sects, or planes. Also p, p_1, etc., as used in this table will not generally be unit points.

The products are arranged in two columns, so as to bring out the geometric principle of duality.

PLANIMETRIC PRODUCTS.

$p_1p_2 = L.$	$L_1L_2 = p.$
$p_1p_2p_3 =$ area (scalar).	$L_1L_2L_3 =$ (area)^2(scalar).
$pL =$ area (scalar).	$Lp =$ area (scalar).
$p_1 . L_1L_2 = L.$	$L_1 . p_1p_2 = p.$
$p_1p_2 . p_3p_4 = p.$	$L_1L_2 . L_3L_4 = L.$
$p_1p_2 . p_3p_4 . p_5p_6 =$(area)^2(scalar).	$L_1L_2.L_3L_4.L_5L_6 =$(area)^4(scalar)

$\epsilon_1\epsilon_2 =$ area (scalar).

STEREOMETRIC PRODUCTS.

$p_1p_2 = L.$	$P_1P_2 = L.$
$p_1p_2p_3 = P.$	$P_1P_2P_3 = p.$
$p_1p_2p_3p_4 =$ volume (scalar).	$P_1P_2P_3P_4 =$ (volume)^3 (scalar).
$pP =$ volume (scalar).	$Pp =$ volume (scalar).
$L_1L_2 =$ volume (scalar).	$L_1L_2 =$ volume (scalar).
$pL = Lp = P.$	$PL = LP = p.$
$p . P_1P_2 = P.$	$P . p_1p_2 = p.$
$p . P_1P_2P_3 = L.$	$P . p_1p_2p_3 = L.$
$L . p_1p_2p_3 = p.$	$L . P_1P_2P_3 = P.$
$\epsilon_1\epsilon_2 = \eta.$	$\eta_1\eta_2 = \epsilon.$
$\epsilon_1\epsilon_2\epsilon_3 =$ volume (scalar).	$\eta_1\eta_2\eta_3 =$ (volume)^2 (scalar).
$\epsilon_1\epsilon_2 . \epsilon_3\epsilon_4 = \epsilon.$	$\eta_1\eta_2 . \eta_3\eta_4 = \eta.$

Laws of Combinatory Multiplication.—All combinatory products are assumed to be subject to the distributive law expressed by the equation

$$A(B+C)=AB+AC.$$

The planimetric product of three points or of three lines, and the stereometric product of three points or planes, or of four points or planes, are subject to the associative law. That is,

In Plane Space:

$$p_1p_2p_3=p_1p_2\cdot p_3=p_1\cdot p_2p_3;\quad L_1L_2L_3=L_1L_2\,.\,L_3=L_1\,.\,L_2L_3.$$

In Solid Space:

$$p_1p_2p_3=p_1\cdot p_2p_3=p_1p_2\ p_3;\quad P_1P_2P_3=P_1\,.\,P_2P_3=P_1P_2\,.\,P_3.$$

$$p_1p_2p_3p_4=p_1\cdot p_2p_3p_4=p_1p_2\cdot p_3p_4;$$
$$P_1P_2P_3P_4=P_1\,.\,P_2P_3P_4=P_1P_2\,.\,P_3P_4.$$

The commutative law of scalar algebra does not, in general, hold. Instead of this, in the products just given as being associative, a law prevails which may be expressed by the equation

$$AB=-BA,$$

from which it follows that the interchange of any two single factors of those products changes the sign of the product.*

Since vectors are equivalent to points at ∞, the associative law holds for $\epsilon_1\epsilon_2\epsilon_3$ and $\eta_1\eta_2\eta_3$.

ART. 7. PLANIMETRIC PRODUCTS.

Product of Two Points.†—This has been fully defined in Art. 6, and it is evident from its nature as there given that

$$p_1p_2=-p_2p_1. \tag{32}$$

If $p_2=p_1$, this becomes $p_1p_1=0$, which must evidently be true, since the sect is now of no length.

Also, $$p_1(p_2-p_1)=p_1p_2-p_1p_1=p_1p_2. \tag{33}$$

* Grassmann (1862), Chap. 3. † Grassmann (1862), Arts. 245, 246, 247.

But $p_2 - p_1$ is a vector, say, ϵ; hence

$$p_1\epsilon = p_1p_2; \tag{34}$$

or the product of a point and a vector is a sect having the direction and magnitude of the vector; or, again, multiplying a vector by a point fixes its position by making it pass through the point.

To find under what conditions pp' will be equal to p_1p_2. Take any other point p_3 in the plane space under consideration, and write $p = x_1p_1 + x_2p_2 + x_3p_3$, $p' = y_1p_1 + y_2p_2 + y_3p_3$, with the conditions for unit points $\Sigma x = \Sigma y = 0$.

Then $$pp' = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} p_1p_2 + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} p_2p_3 + \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix} p_3p_1.$$

If this is to reduce to p_1p_2, we must have the third condition $x_2y_3 - x_3y_2 = x_3y_1 - x_1y_3 = 0$, which requires that $x_3 = y_3 = 0$, unless the coefficient of p_1p_2 is to vanish also. Thus pp' must be in the same straight line with p_1p_2. If, moreover, in addition $x_1y_2 - x_2y_1 = 1$, we shall have $pp' = p_1p_2$. Hence pp' is equal to p_1p_2 when, and only when, the four points are collinear, and p' is distant from p by the same amount and in the same direction that p_2 is from p_1.

Product of Three Points.—By Art. 6 the product is what is determined by the three points. In solid space they would fix a plane, but, as we are now confined to plane space, this is not the case. The points evidently fix either a triangle or a parallelogram of twice its area, and the product $p_1p_2p_3$ will be taken as the area of this, or an equivalent, parallelogram.

This area is taken rather than that of the triangle, because it is what is generated by p_1p_2 as it is moved parallel to its initial position till it passes through p_3.

We have $p_1p_2p_3 = p_1 \cdot p_2p_3 = -p_1 \cdot p_3p_2 = -p_1p_3p_2$, so that if we go around the triangle in the opposite sense the sign is changed. As this product possesses only the properties of magnitude and sign it is scalar.

Write $p = \sum_1^3 xp$, $p' = \sum_1^3 yp$, $p'' = \sum_1^3 zp$; then

$$pp'p'' = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} p_1p_2p_3; \qquad (35)$$

that is, any triple point product in plane space differs from any other only by a scalar factor.*

$$\text{Finally,} \quad p_1p_2p_3 = p_1(p_2 - p_1)(p_3 - p_1) = p_1\epsilon\epsilon', \qquad (36)$$

if $\epsilon = p_2 - p_1$ and $\epsilon' = p_3 - p_1$.

Product of Two Vectors.—Using the values of ϵ and ϵ' just given, we see that ϵ and ϵ' determine the same parallelogram that p_1, p_2, and p_3 do; hence the meaning of the product is the same in all respects in two-dimensional space.

We shall have $\epsilon\epsilon' = -\epsilon'\epsilon$, for

$$\epsilon\epsilon' = (p_2 - p_1)(p_3 - p_1) = -(p_3 - p_1)(p_2 - p_1) = -\epsilon'\epsilon;$$

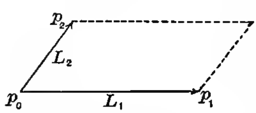
since we have shown that inverting the order changes the sign in a product of points. The result may be obtained also by regarding ϵ and ϵ' as points at infinity, or by consideration of a figure.

As we have seen that $\epsilon\epsilon'$ has, in plane space, precisely the same meaning as $p_1p_2p_3$ we may write

$$\begin{aligned} p_1p_2p_3 &= p_1\epsilon\epsilon' = \epsilon\epsilon' \\ &= (p_2 - p_1)(p_3 - p_1) = p_2p_3 + p_3p_1 + p_1p_2. \qquad (37) \end{aligned}$$

Thus the sum of three sects which form the sides of a triangle, all taken in the same sense as looked at from outside the triangle, is equal to the area of the triangle.

Product of Two Sects.—Any two sects in plane space, L_1, L_2, determine a point, the intersection of the lines in which they lie, and an area, that of a parallelogram as in the figure. Let p_0 be the intersection,



and take p_1 and p_2 so that $L_1 = p_0p_1$ and $L_2 = p_0p_2$. The area

* Grassmann (1862), Art. 255.

determined by L_1 and L_2 is then the same that we have given as the value of $p_0p_1p_2$. We write therefore

$$L_1L_2 = p_0p_1 \cdot p_0p_2 = p_0p_1p_2 \cdot p_0. \tag{38}$$

The third member of (38) is not to be regarded as derived from the second by ordinary transposition and reassociation of the points, for the associative law does not hold for the four points taken together, since $p_0p_1p_0 \cdot p_2 = 0$. The third member simply results from the definition of L_1L_2.* It may be taken as a model form which will be found to apply to several other cases, for instance to (38) when points and lines are interchanged throughout. Thus, if $p_1 = L_0L_1$ and $p_2 = L_0L_2$ we have

$$p_1p_2 = L_0L_1 \,.\, L_0L_2 = L_0L_1L_2 \,.\, L_0. \tag{39}$$

For take p_1' and p_2' so that $p_1p_1' = L_1$ and $p_2p_2' = L_2$; p_1p_2 is evidently some multiple of L_0, say nL_0; hence

$$p_1p_2 = nL_0 = \frac{1}{n^2}(p_1p_2 \cdot p_1p_1') \cdot (p_1p_2 \cdot p_2p_2')$$

$$= \frac{1}{n^2}(p_1p_2p_1' \cdot p_1) \cdot (p_1p_2p_2' \cdot p_2), \text{ by (38)},$$

$$= \frac{1}{n^2} \cdot p_1p_2p_1' \cdot p_1p_2p_2' \cdot p_1p_2, \text{ because } p_1p_2p_1' \text{ and } p_1p_2p_1' \text{ are scalar},$$

$$= \frac{1}{n} \cdot (p_1p_2 \cdot p_1p_1' \cdot p_2p_2') \,.\, L_0, \text{ by (38)},$$

$$= L_0L_1L_2 \,.\, L_0, \text{ which was to be proved.}$$

Product of Three Sects.—The method has just been indicated, but we may also proceed thus: Let the lines be L_0, L_1, L_2, and let p_0, p_1, p_2 be their common points. Take scalars n_0, n_1 n_2 so that $L_0 = n_0p_1p_2$, etc., then

$$L_0L_1L_2 = n_0n_1n_2 \cdot p_1p_2 \cdot p_2p_0 \cdot p_0p_1 = - n_0n_1n_2 \cdot p_2p_1p_2p_0 \cdot p_0p_1$$
$$= - n_0n_1n_2 \cdot p_2p_1p_0 \cdot p_2p_0p_1 = n_0n_1n_2(p_0p_1p_2)^2. \tag{40}$$

* Grassmann applies the terms "eingewandt" and "regressiv" to a product of this kind, the first term being used in the Ausdehnungslehre of 1844, and the second in that of 1862. See Chapter 3 of the first, and Chapter 3, Art. 94, of the second.

Product of a Point and Two Sects.—Let p be any point and let L_1 and L_2 be as in (38); then

$$pL_1L_2 = p \cdot p_0p_1 \cdot p_0p_2 = p \cdot p_0p_1p_2 \cdot p_0 = p_0p_1p_2 \cdot pp_0. \quad (41)$$

It has been here assumed that $pL_1L_2 = p \,.\, L_1L_2$. The product is not associative, for $pL_1 \,.\, L_2$ is the line L_2 times the scalar pL_1, a different meaning from that assigned in (41). As a rule, to avoid ambiguity, the grouping of such products will be indicated by dots.

Product of Two Parallel Sects.—Let them be $p_1\epsilon$ and $np_2\epsilon$; then, as in (38),

$$p_1\epsilon \,.\, np_2\epsilon = n \,.\, p_1\epsilon \,.\, p_2\epsilon = n \,.\, \epsilon p_1 \,.\, \epsilon p_2 = n \,.\, \epsilon p_1p_2 \,.\, \epsilon, \quad (42)$$

that is, a scalar times the common point at ∞.

Addition and Subtraction of Sects.—Let L_1 and L_2 be two sects, p_0 their common point, and p_1 and p_2 so taken that $L_1 = p_0p_1$, $L_2 = p_0p_2$; then

$$L_1 + L_2 = p_0p_1 + p_0p_2 = p_0(p_1 + p_2) = 2p_0\bar{p}, \quad (43)$$

$\bar{p}$ being the mean of p_1 and p_2; hence the sum is that diagonal of the parallelogram which passes through p_0. Also

$$L_1 - L_2 = p_0(p_1 - p_2), \quad (44)$$

so that the difference of the two passes also through p_0 and is parallel to the other diagonal of the parallelogram determined by L_1 and L_2.

If the two sects are parallel let them be $n_1p_1\epsilon$ and $n_2p_2\epsilon$; then

$$n_1p_1\epsilon + n_2p_2\epsilon = (n_1p_1 + n_2p_2)\epsilon = (n_1 + n_2)\bar{p}\epsilon_1, \quad (45)$$

so that the sum is a sect parallel to each of them, having a length equal to the sum of their lengths, and at distances from them inversely proportional to their lengths.

If $n_2 = -n_1$ the two sects are oppositely directed and of equal length, and the sum is

$$n_1(p_1\epsilon - p_2\epsilon) = n_1(p_1 - p_2)\epsilon, \quad (46)$$

which, being the product of two vectors, is a scalar area.

Consider next n sects $p_1\epsilon_1, p_2\epsilon_2, \ldots p_n\epsilon_n$, and let e_0 be some arbitrarily chosen point; then

$$\sum_1^n p\epsilon \equiv e_0\sum_1^n \epsilon - e_0\sum_1^n \epsilon + \sum_1^n p\epsilon \equiv e_0\sum_1^n \epsilon + \sum_1^n (p - e_0)\epsilon. \qquad (47)$$

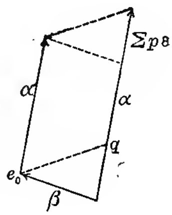
The second term of the third member of this equation, being a sum of double vector products, that is, a sum of areas, is itself an area, and is equal to the product of any two non-parallel vectors of suitable lengths. Therefore, α and β being such vectors, write $\Sigma\epsilon = \alpha$ and $\Sigma(p - e_0) = \alpha\beta$. Hence (47) become

$$\Sigma p\epsilon = e_0\alpha + \alpha\beta = (e_0 - \beta)\alpha. \qquad (48)$$

Let q be some point on the line $\Sigma p\epsilon$; then

$$q\Sigma p\epsilon = 0 = qe_0\alpha + q\alpha\beta = qe_0\alpha + \alpha\beta,$$

by (37), hence $\quad qe_0\alpha = -\alpha\beta = \beta\alpha.$



The figure presents the geometrical meaning of the equation, and hence it appears that $q\alpha(=\Sigma p\epsilon)$ is at a perpendicular distance from e_0 of

$$\frac{\alpha\beta}{T\alpha} = \frac{\Sigma(p - e_0)\epsilon}{T\Sigma\epsilon}. \qquad (49)$$

It is easily seen that a sect possesses the exact geometrical properties of a force, namely, magnitude, direction, and position, and the discussion of the summation of sects which has just been given corresponds completely to the discussion of the resultant of a system of forces in a plane. In this algebra, then, the resultant of any system of forces is simply their sum, and this will be found hereafter to be equally true in three-dimensional space. The expression in (46) corresponds to a couple, as does also the $\Sigma(p - e_0)\epsilon$ of (47); and this equation proves the proposition that any system of forces in a plane is equivalent to a single force acting at an arbitrary point, e_0, and a couple. Equation (49) gives the distance of the resultant from this arbitrary point.

Exercise 7.—To find x, y, z from the scalar equations

$$a_1x + b_1y + c_1z = d_1, \quad a_2x + b_2y + c_2z = d_2, \quad a_3x + b_3y + c_3z = d_3.$$

Multiply the equations by p_1, p_2, and p_3 respectively, and add; hence

$$x\sum_1^3 ap + y\sum_1^3 bp + z\sum_1^3 cp = \sum_1^3 dp.$$

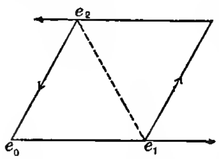
Now Σap, Σbp, etc., are points: multiply the equation just written by $\Sigma ap . \Sigma bp$; thus

$$z\Sigma ap . \Sigma bp\Sigma cp = \Sigma ap . \Sigma bp . \Sigma dp,$$

because $\Sigma ap . \Sigma ap = 0$, etc.; therefore

$$z = \Sigma ap . \Sigma bp . \Sigma dp \div \Sigma ap . \Sigma bp\Sigma cp = [a_1, b_2, d_3] \div [a_1, b_2, c_3],$$

a very simple proof of the determinant solution. Of course x and y will be found by multiplying by the other pairs of points.



Exercise 8.—Forces are represented by given multiples of the sides of a parallelogram; determine their resultant.

Let the parallelogram be double the triangle $e_0e_1e_2$, and the forces

$$k_0e_0e_1 + k_1e_1(e_2 - e_0) + k_2e_2(e_0 - e_1) + k_3e_2e_0 = \Sigma pe$$
$$= (k_0 + k_1)e_0e_1 + (k_1 + k_2)e_1e_2 + (k_2 + k_3)e_2e_0.$$

Multiply by e_0e_1 to find where the resultant cuts this line; then

$$(k_1+k_2)e_0e_1 . e_1e_2 + (k_2+k_3)e_0e_1 . e_2e_0 = e_0e_1e_2 . [(k_1+k_2)e_1 - (k_2+k_3)e_0],$$

or e_0e_1 cuts the resultant at the point

$$[(k_1 + k_2)e_1 - (k_2 + k_3)e_0] \div (k_1 - k_3).$$

Similarly the resultant cuts the other sides of the reference triangle at $[(k_2 + k_3)e_2 - (k_0 + k_1)e_1] \div (k_2 + k_3 - k_0 - k_1)$ and at $[(k_0 + k_1)e_0 - (k_1 + k_2)e_2] \div (k_0 - k_2)$.

Suppose $k_0 = k_1 = k_2 = k_3$; then each of the three points just found recedes to infinity; but in this case Σpe reduces to $2k_0(e_0e_1 + e_1e_2 + e_2e_0) = 2k_0(e_1 - e_0)(e_2 - e_0)$, and the system is equivalent to a couple.

Prob. 11. Construct the resultant of Exercise 8 when $k_0 = 1$, $k_1 = 2$, $k_2 = 3$, $k_3 = 4$; when $k_0 = 1$, $k_1 = -2$, $k_2 = 3$, $k_3 = -4$; when $k_0 = 3$, $k_1 = k_3 = 2$, $k_2 = 1$; and when $k_1 = k_2 = 1$, $k_0 = k_3 = -2$.

Prob. 12. There are given n points $p_1 \ldots p_n$; to find a point e such that forces represented by the sects ep_1, ep_2, etc., shall be in equilibrium. (The equation of equilibrium is $\Sigma ep \equiv e\Sigma p \equiv \frac{1}{n}e\bar{p} = 0$. Hence e coincides with the mean point of the p's.)

Prob. 13. If a harmonic range e_1, p, e_2, p' be given, together with some point e_0 not collinear with these points, show that

$$e_0e_1p \cdot e_0e_2p' = -e_0pe_2 \cdot e_0p'e_1.$$

(Let $p = m_1e_1 + m_2e_2$ and $p' = m_1e_1 - m_2e_2$, as in Exercise 2 of Art. 3.)

Prob. 14. Show that the relation of Prob. 13 holds for any four points whatever taken respectively on the four lines e_0e_1, e_0p, e_0e_2, e_0p'. If the four points are all at the same distance from e_0, show that the areas e_0e_1p, etc., become proportional to the sines of the angles between e_0e_1 and e_0p, etc.

ART. 8. THE COMPLEMENT.*

Taking point reference systems, or unit normal vector reference systems, as in Art. 5, the product of the reference units taken in order being in any case unity, the complement of any reference unit is the product of all the others so taken that the unit times its complement is unity.

To find the complements of quantities other than reference units the following properties are assumed:

(a) The complement of a product is equal to the product of the complements of its factors.

(b) The complement of a sum is equal to the sum of the complements of the terms added together.

(c) The complement of a scalar quantity is the scalar itself.

Considering now the point system in plane space e_0, e_1, e_2 with the constant condition $e_0e_1e_2 = 1$, the sides of the reference triangle taken in order are the complements of the opposite vertices, and vice versâ.

The complement of a quantity is indicated by a vertical line, as $|p$, read, complement of p.

* See Ausdehnungslehre of 1862, Art. 89.

Thus $\quad |e_0 = e_1e_2, \qquad |e_1e_2 = |(|e_0) = e_0,$

$\qquad |e_1 = e_2e_0, \qquad |e_2e_0 = |(|e_1) = e_1,$

$\qquad |e_2 = e_0e_1, \qquad |e_0e_1 = |(|e_2) = e_2.$

For $e_0|e_0 = e_0e_1e_2 = 1$, which agrees with the definition; $|e_1e_2 = |e_1 . |e_2 = e_2e_0 . e_0e_1 = - e_0e_2 . e_0e_1 = - e_0e_2e_1 . e_0 = e_0$, by ($a$) and (38); $|e_0e_1e_2 = |e_0 . |e_1 . |e_2 = e_1e_2 . e_2e_0 . e_0e_1 = (e_0e_1e_2)^2 = 1 = e_0e_1e_2$, which agrees with ($c$); $e_0|e_1 = e_0e_2e_0 = 0 = e_0|e_2 = e_1|e_2$.

Next take any point $p_1 = \sum_0^2 le$, and we have, by (b),

$$|p_1 = \sum_0^2 l|e = l_0e_1e_2 + l_1e_2e_0 + l_2e_0e_1 = l_0l_1l_2\left(\frac{e_1}{l_1} - \frac{e_0}{l_0}\right)\left(\frac{e_2}{l_2} - \frac{e_0}{l_0}\right) = L_1. \quad (50)$$

Thus the complement of a point is a line,* which may be easily constructed by the fourth member of (50); which expresses this line as the product of the points in which it cuts the sides e_0e_1 and e_0e_2 of the reference triangle. Comparing this equation with Ex. 3 in Art. 4, it appears that $|p_1$ above is related to the point $\sum_0^2 \frac{e}{l}$ as the line p_0p_2 of Ex. 3 is to the point Σne. Hence $|p_1$ may be found by constructing this line corresponding to $\sum_0^2 \frac{e}{l}$ as shown in the figure of Ex. 3, Art. 4.

Again, the line $|p_1$ may be shown to be the anti-polar of p with respect to an ellipse of such dimensions, and so placed upon $e_0e_1e_2$ that, with reference to it, each side of the reference triangle is the anti-polar of the opposite vertex.* From this it appears that complementary relations are polar reciprocal relations. Take any point $p_2 = \sum_0^2 me$, and we have

$$p_1|p_2 = (l_0e_0 + l_1e_1 + l_2e_2)(m_0e_1e_2 + m_1e_2e_0 + m_2e_0e_1)$$
$$= \sum_0^2 lm = \Sigma me . \Sigma l|e = p_2|p_1, \quad (51)$$

* See Hyde's Directional Calculus, Arts. 41–43 and 121–123.

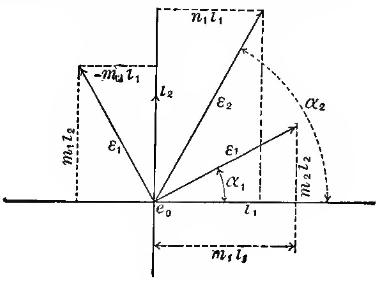
so that this product is commutative about the complement sign, and scalar. This is true of all such products when the quantities on each side of the complement sign are of the same order in the reference units. Take for instance the product $p_1p_2|p_3p_4$. This is scalar because $|p_3p_4$ is a point, so that the whole quantity is equivalent to a triple-point product; and we have $p_1p_2|p_3p_4 = |p_3p_4 \cdot p_1p_2 = |(p_3p_4|p_1p_2) = p_3p_4|p_1p_2$, by ($a$) and ($c$). If, however, such a quantity be taken as $p_1p_2 \cdot |p_3$ it is neither scalar nor commutative about the sign $|$; for, $|p_3$ being a line, the product is that of two lines, that is, a point, and

$$p_1p_2 \cdot |p_3 = -|p_3 \cdot p_1p_2 = -|(p_3 \cdot |p_1p_2). \qquad (52)$$

Such products as we have just been considering are called by Grassmann "inner products,"* and he regards the sign $|$ as a multiplication sign for this sort of product. Inasmuch, however, as these products do not differ in nature from those heretofore considered, it appears to the author to conduce to simplicity not to introduce a nomenclature which implies a new species of multiplication. For instance, $p|q$ will be treated as the combinatory product of p into the complement of q, and not as a different kind of product of p into q.

The term co-product may be applied to such expressions, regarded as an abbreviation merely, after the analogy of cosine for complement of the sine.

Consider next a unit normal vector system. By the definition we have



$$|\iota_1 = \iota_2, \ |\iota_2 = |(|\iota_1) = -\iota_1,$$

because $\quad \iota_1|\iota_1 = \iota_1\iota_2 = 1,$

$$\iota_2|\iota_2 = \iota_2(-\iota_1) = -\iota_2\iota_1 = \iota_1\iota_2 = 1.$$

Also, $\quad \iota_1|\iota_2 = -\iota_1\iota_1 = 0 = \iota_2|\iota_1.$

Next let

$$\epsilon_1 = m_1\iota_1 + m_2\iota_2 \quad \text{and} \quad \epsilon_2 = n_1\iota_1 + n_2\iota_2;$$

* Grassmann (1862), Chapter 4.

then, by (*b*) and (*c*),

$$|\epsilon_1 = m_1|\iota_1 + m_2|\iota_2 = m_1\iota_2 - m_2\iota_1. \tag{53}$$

By the figure it is evident that $|\epsilon_1$ is a vector of the same length as ϵ_1 and perpendicular to it, or, in other words, taking the complement of a vector in plane space rotates it positively through 90°.

The co-product $\epsilon_1|\epsilon_2$ is the area of the parallelogram, two of whose sides are ϵ_1 and $|\epsilon_2$ drawn outwards from a point; if ϵ_1 is parallel to $|\epsilon_2$, this area vanishes, or $\epsilon_1|\epsilon_2 = 0$; but, since $|\epsilon_2$ is perpendicular to ϵ_2, ϵ_1 must in this case be perpendicular to ϵ_2; hence the equation

$$\epsilon_1|\epsilon_2 = 0 \tag{54}$$

is the condition that two vectors ϵ_1 and ϵ_2 shall be perpendicular to each other.

The co-product $\epsilon_1|\epsilon_1$, which will usually be written $\epsilon_1{}^{\underline{2}}$, and called the co-square of ϵ_1, is the area of a square each of whose sides has the length $T\epsilon_1$; hence

$$T\epsilon_1 = \sqrt{\epsilon_1|\epsilon_1} = \sqrt{\epsilon_1{}^{\underline{2}}}. \tag{55}$$

Let α_1 and α_2 be the angles between ι_1 and ϵ_1 and between ι_1 and ϵ_2 respectively, as in the figure. Then

$$\epsilon_1\epsilon_2 = m_1n_2 - m_2n_1 = T\epsilon_1 T\epsilon_2 \sin(\alpha_2 - \alpha_1), \tag{56}$$

the third member being the ordinary expression for the area of the parallelogram $\epsilon_1\epsilon_2$. Also

$$\epsilon_1|\epsilon_2 = (m_1\iota_1 + m_2\iota_2)(n_1\iota_2 - n_2\iota_1)$$

$$= m_1n_1 + m_2n_2 = T\epsilon_1 T\epsilon_2 \cos(\alpha_2 - \alpha_1), \tag{57}$$

the last member being found as before, remembering that $\sin(90° + \alpha_2 - \alpha_1) = \cos(\alpha_2 - \alpha_1)$.

If in (57) we let $\epsilon_2 = \epsilon_1$, whence $n_1 = m_1$ and $n_2 = m_2$, we have

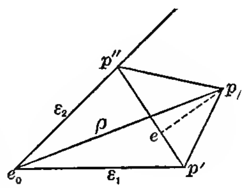
$$T\epsilon_1 = \epsilon_1{}^{\underline{2}} = \sqrt{m_1{}^2 + m_2{}^2}. \tag{58}$$

If $T\epsilon_1 = T\epsilon_2 = 1$, then $m_1 = \cos\alpha_1$, $m_2 = \sin\alpha_1$, $n_1 = \cos\alpha_2$, $n_2 = \sin\alpha_2$, and equations (56) and (57) give the ordinary trigonometrical formulas $\sin(\alpha_2 - \alpha_1) = \sin\alpha_2\cos\alpha_1 - \cos\alpha_2\sin\alpha_1$, and $\cos(\alpha_2 - \alpha_1) = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2$. Squaring and adding (56) and (57), there results

$$T^2\epsilon_1 \,.\, T^2\epsilon_2 = \epsilon_1{}^2\epsilon_2{}^2 = (\epsilon_1\epsilon_2)^2 + (\epsilon_1 | \epsilon_2)^2. \tag{59}$$

Attention is called to the fact, which the student may have already noticed, that such an equation as $AB = AC$, in which AB and AC are combinatory products, does not, in general, imply that $B = C$, for the reason that the equation $A(B - C) = 0$ can usually be satisfied without either factor being itself zero. Thus $pL_1 = pL_2$ means simply that the two quantities which are equated have the same magnitude and sign, which permits L_2 to have an infinity of lengths and positions, when p and L_1 are given. The equation $p_1p_2 = p_1p_3$, or $p_1(p_2 - p_3) = 0$, p_2 and p_3 being unit points, implies, however, that $p_2 = p_3$, unless p_1 is at ∞, that is, a vector.

Exercise 9.—A triangle whose sides are of constant length moves so that two of its vertices remain on two fixed lines: find the locus of the other vertex.



Let $e_0\epsilon_1$ and $e_0\epsilon_2$ be the two fixed lines, and $pp'p''$ the triangle. Let pe be perpendicular to $p'p''$, $p' - e_0 = x\epsilon_1$ and $p'' - e_0 = y\epsilon_2$; then $p'' - p' = y\epsilon_2 - x\epsilon_1$, $T(y\epsilon_2 - x\epsilon_1) = c =$ constant, by the conditions. Also, $Tp'e =$ constant $= mc$, say, and $Tep =$ constant $= nc$, say. Hence

$$e - p' = Tp'e \,.\, U(e - p') = mc \,.\, \frac{y\epsilon_2 - x\epsilon_1}{T(y\epsilon_2 - x\epsilon_1)} = m(y\epsilon_2 - x\epsilon_1),$$

and similarly $p - e = n|(y\epsilon_2 - x\epsilon_1)$. Therefore

$$p - e_0 = \rho = x\epsilon_1 + m(y\epsilon_2 - x\epsilon_1) + n|(y\epsilon_2 - x\epsilon_1),$$

an equation which, with the condition $T(y\epsilon_2 - x\epsilon_1) = c$, or

$$y^2\epsilon_2{}^2 - 2xy\epsilon_1|\epsilon_2 + x^2\epsilon_1{}^2 = c^2,$$

determines the locus to be a second-degree curve, which must in fact be an ellipse, since it can have no points at infinity. Let us rearrange the equation in ρ thus:

$$\rho = x[(1 - m)\epsilon_1 - n|\epsilon_1] + y[m\epsilon_2 + n|\epsilon_2] = x\epsilon + y\epsilon', \text{ say,}$$

so that $\epsilon = (1 - m)\epsilon_1 - n|\epsilon_1$ and $\epsilon' = m\epsilon_2 + n|\epsilon_2$; then multiply successively into ϵ and ϵ'; therefore $\rho\epsilon = y\epsilon'\epsilon$ and $\rho\epsilon' = x\epsilon\epsilon'$. Substituting these values of x and y in the equation of condition, we have

$$\epsilon_2^2 . (\rho\epsilon)^2 + 2\epsilon_1|\epsilon_2 . \rho\epsilon . \rho\epsilon' + \epsilon_1^2(\rho\epsilon')^2 = c^2(\epsilon\epsilon')^2,$$

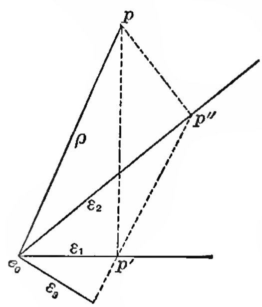
a scalar equation of the second degree in ρ.

Exercise 10.—There is given an irregular polygon of n sides: show that if forces act at the middle points of these sides, proportional to them in magnitude, and directed all outward or else all inward, these forces will be in equilibrium.

Let e_0 be a vertex of the polygon, and let $2\epsilon_1, 2\epsilon_2, \ldots 2\epsilon_n$ represent its sides in magnitude and direction. Then the middle points will be $e_0 + \epsilon_1$, $e_0 + 2\epsilon_1 + \epsilon_2$, etc., and, using the complement in a vector system, we have

$$\Sigma p\epsilon = (e_0+\epsilon_1)|\epsilon_1+(e_0+2\epsilon_1+\epsilon_2)|\epsilon_2+(e_0+2\epsilon_1+2\epsilon_2+\epsilon_3)|\epsilon_3+\ldots$$
$$+ (e_0 + 2\epsilon_1 + \ldots + 2\epsilon_{n-1} + \epsilon_n)|\epsilon_n$$
$$= e_0\Big|\sum_1^n\epsilon + \sum_1^n\epsilon^2 + 2\epsilon_1\Big|\sum_2^n\epsilon + 2\epsilon_2\Big|\sum_3^n\epsilon + \ldots + 2\epsilon_{n-1}|\epsilon_n$$
$$= e_0\Big|\sum_1^n\epsilon + \left(\sum_1^n\epsilon\right)^2 = 0,$$ which was to be proved.

Exercise 11.—A line passes through a fixed point and cuts two fixed lines; at the points of intersection perpendiculars to the fixed lines are erected; find the locus of the intersection of these perpendiculars.



Let the fixed lines be $e_0\epsilon_1$ and $e_0\epsilon_2$, and the fixed point $e_0 + \epsilon_3$; the moving line cuts the fixed lines in p' and p'', at which points perpendiculars are erected meeting in p.

Let $p - e_0 = \rho$, $p' - e_0 = x\epsilon_1$, $p'' - e_0 = y\epsilon_2$, $T\epsilon_1 = T\epsilon_2 = 1$;

then $\rho = x\epsilon_1 + x'|\epsilon_1 = y\epsilon_2 + y'|\epsilon_2$, whence $\rho|\epsilon_1 = x$ and $\rho|\epsilon_2 = y$.

Also, since $e_0 + \epsilon_3$, p', p'' are collinear points,

$$(x\epsilon_1 - \epsilon_3)(y\epsilon_2 - \epsilon_2) = 0 = xy\epsilon_1\epsilon_2 + y\epsilon_2\epsilon_3 + x\epsilon_3\epsilon_1;$$

or, substituting values of x and y,

$$\rho|\epsilon_1 . \rho|\epsilon_2 . \epsilon_1\epsilon_2 + \rho|\epsilon_2 . \epsilon_2\epsilon_3 + \rho|c_1 . \epsilon_3\epsilon_1 = 0,$$

an equation of the second degree in ρ, and hence representing a conic.

Prob. 15. If a, b, c are the lengths of the sides of a triangle, prove the formula $a^2 = b^2 + c^2 - 2bc \cos A$, by taking vectors ϵ_1, ϵ_2, and $\epsilon_2 - \epsilon_1$ equal to the respective sides.

Prob. 16. If $e_0\epsilon_1$ and $e_0\epsilon_2$ are two unit lines, show that the vector perpendicular from e_0 on the line $(e_0 + a\epsilon_1)(e_0 + be_2)$ is $\dfrac{ab\epsilon_1\epsilon_2}{(b\epsilon_2 - a\epsilon_1)^2} . |(b\epsilon_2 - a\epsilon_1)$, of which the length is $\dfrac{ab\epsilon_1\epsilon_2}{T(b\epsilon_2 - a\epsilon_1)}$. From this derive the Cartesian expression for the perpendicular from the origin upon a straight line in oblique coordinates,

$ab \sin \omega \div (a^2 + b^2 - 2ab \cos \omega)^{1/2}$, ω being angle between the axes.

Prob. 17. If three points, $me_0 + ne_1$, $me_1 + ne_2$, $me_2 + ne_0$, be taken on the sides of the reference triangle, then the sides of the complementary triangle, $|(me_0 + ne_1)$, etc., will be respectively parallel to the corresponding sides of the triangle formed by the assumed points $(me_1 + ne_2)$, $(me_2 + ne_0)$, etc.

ART. 9. EQUATIONS OF CONDITION, AND FORMULAS.

Several equations of condition are placed here together for convenient reference: some have been already given; others follow from the results of Arts. 7 and 8. When we have

$p_1p_2 = 0,$ or $n_1p_1 + n_2p_2 = 0,$ the two points coincide;	$L_1L_2 = 0,$ or $n_1L_1 + n_2L_2 = 0,$ the two lines coincide;	(60)
$p_1p_2p_3 = 0,$ or $\sum_1^3 np = 0,$ the three points are collinear;	$L_1L_2L_3 = 0,$ or $\sum_1^3 nL = 0,$ the three lines are confluent.	(61)

$$\epsilon_1\epsilon_2 = 0, \quad \text{or} \quad n_1\epsilon_1 + n_2\epsilon_2 = 0, \tag{62}$$

the two vectors are parallel (points at infinity coincide);

$$\epsilon_1|\epsilon_2 = 0, \tag{63}$$

the two vectors are perpendicular;

$p_1 \| p_2 = 0$,	$L_1 \| L_2 = 0$, (64)
either point lies on the complementary line of the other.	either line passes through the complementary point of the other.

If we write the equation

$$\rho = x_1\epsilon_1 + x_2\epsilon_2,$$

$x_1\epsilon_1$ is the projection of ρ on ϵ_1 parallel to ϵ_2, and $x_2\epsilon_2$ is the projection of ρ on ϵ_2 parallel to ϵ_1. Multiply both sides of the equation into ϵ_2; therefore $\rho\epsilon_2 = x_1\epsilon_1\epsilon_2$, or $x_1 = \rho\epsilon_2 \div \epsilon_1\epsilon_2$. Similarly, multiplying into ϵ_1, we have $\rho\epsilon_1 = x_2\epsilon_2\epsilon_1$, or $x_2 = \rho\epsilon_1 \div \epsilon_2\epsilon_1$, whence

$$\rho = \frac{\epsilon_1 \, . \, \rho\epsilon_2}{\epsilon_1\epsilon_2} + \frac{\epsilon_2 \, . \, \rho\epsilon_1}{\epsilon_2\epsilon_1}. \tag{65}$$

The two terms of the second member of (65) are therefore the projections of ρ on ϵ_1 parallel to ϵ_2, and on ϵ_2 parallel to ϵ_1, respectively.*

Let ϵ_1 and ϵ_2 be unit normal vectors, say, ι and $|\iota$; then (65) becomes

$$\rho = \iota \, . \, \rho|\iota - |\iota \, . \, \rho\iota = \iota \, . \, \rho|\iota + \iota\rho \, . \, |\iota\,; \tag{66}$$

or, if ι_1 and ι_2 be used instead of ι and $|\iota$,

$$\rho = \iota_1 \, . \, \rho|\iota_1 + \iota_2 \, . \, \rho|\iota_2. \tag{67}$$

Again, in (65) let $\rho = \epsilon_3$, clear of fractions, and transpose; therefore

$$\epsilon_1\epsilon_2 \, . \, \epsilon_3 + \epsilon_2\epsilon_3 \, . \, \epsilon_1 + \epsilon_3\epsilon_1 \, . \, \epsilon_2 = 0, \tag{68}$$

a symmetrical relation between any three directions in plane space. Let $T\epsilon_1 = T\epsilon_2 = T\epsilon_3 = 1$, and multiply (68) into $|\epsilon_3$, thus

$$\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 \, . \, \epsilon_1|\epsilon_3 + \epsilon_3\epsilon_1 \, . \, \epsilon_2|\epsilon_3 = 0, \tag{69}$$

which is equivalent to

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

the upper or lower sign corresponding to the case when ϵ_3 is

* Grassmann (1844), Chapter 5 (1862), Art. 129. Hyde's Directional Calculus, Arts. 46 and 47.

between ϵ_1 and ϵ_2, or outside, respectively. Writing in (69) $|\epsilon_2$ instead of ϵ_2, we have

$$\epsilon_1|\epsilon_2 - \epsilon_2|\epsilon_3 \,.\, \epsilon_3|\epsilon_1 + \epsilon_3\epsilon_1 \,.\, \epsilon_2\epsilon_3 = 0, \tag{70}$$

which gives the cos $(\alpha \pm \beta)$. These formulas being for any three directions in plane space, are independent of the magnitude of the angles involved.

There is given below a set of formulas for points and lines, arranged in complementary pairs, and all placed together for convenient reference, the derivation of them following after.

$$\left.\begin{aligned} p&=(p_0p_1p_2)^{-1}[p_0 \cdot pp_1p_2 + p_1 \cdot pp_2p_0 + p_2 \cdot pp_0p_1], \\ L&=(L_0L_1L_2)^{-1}[L_0 \,.\, LL_1L_2 + L_1 \,.\, LL_2L_0 + L_2 \,.\, LL_0L_1] \end{aligned}\right\}, \tag{71}$$

$$\left.\begin{aligned} p&=(p_0p_1p_2)^{-1}[|p_1p_2 \cdot p|p_0 + |p_2p_0 \cdot p|p_1 + |p_0p_1 \cdot p|p_2], \\ L&=(L_0L_1L_2)^{-1}[|L_1L_2 \,.\, L|L_0 + |L_2L_0 \,.\, L|L_1 + |L_0L_1 \,.\, L|L_2] \end{aligned}\right\}, \tag{72}$$

$$\left.\begin{aligned} p_1p_2 \cdot p_3p_4 &= -\, p_1 \cdot p_2p_3p_4 + p_2 \cdot p_3p_4p_1 \\ &= \quad p_3 \cdot p_4p_1p_2 - p_4 \cdot p_1p_2p_3, \\ L_1L_2 \,.\, L_3L_4 &= -\, L_1 \,.\, L_2L_3L_4 + L_2 \,.\, L_3L_4L_1 \\ &= \quad L_3 \,.\, L_4L_1L_2 - L_4 \,.\, L_1L_2L_3 \end{aligned}\right\}, \tag{73}$$

$$p_1p_2 \cdot |q_1 = -\begin{vmatrix} p_1 & p_1|q_1 \\ p_2 & p_2|q_1 \end{vmatrix}, \quad L_1L_2|M_1 = -\begin{vmatrix} L_1 & L_1|M_1 \\ L_2 & L_2|M_1 \end{vmatrix}, \tag{74}$$

$$p_2|q_1q_2 = \begin{vmatrix} |q_1 & p_2|q_1 \\ |q_2 & p_2|q_2 \end{vmatrix}, \quad L_2|M_1M_2 = \begin{vmatrix} |M_1 & L_2|M_1 \\ |M_2 & L_2|M_2 \end{vmatrix}, \tag{75}$$

$$p_1p_2|q_1q_2 = \begin{vmatrix} p_1|q_1 & p_1|q_2 \\ p_2|q_1 & p_2|q_2 \end{vmatrix}, \quad L_1L_2|M_1M_2 = \begin{vmatrix} L_1|M_1 & L_1|M_2 \\ L_2|M_1 & L_2|M_2 \end{vmatrix}, \tag{76}$$

$$p_0p_1p_2 \cdot q_0q_1q_2 = \begin{vmatrix} p_0|q_0 & p_0|q_1 & p_0|q_2 \\ p_1|q_0 & p_1|q_1 & p_1|q_2 \\ p_2|q_0 & p_2|q_1 & p_2|q_2 \end{vmatrix} \tag{77}$$

The complementary formula to (77) is not given, but may be obtained by putting L's and M's for p's and q's.

Derivation of Equations (71)–(77).—Equation (71). Write $p = x_0p_0 + x_1p_1 + x_2p_2$, and multiply this equation by p_1p_2; then $\quad p_1p_2p = x_0p_1p_2p_0, \quad$ or $\quad x_0 = pp_1p_2 \div p_0p_1p_2$.

Multiplying similarly by $p\,p_2$ and by p_0p_1, we find $x_1 = pp_2p_0 \div p_0p_1p$ and $x_2 = pp_0p_1 \div p_0p_1p_2$. The substitution of these values gives the first of (71), and the second is similarly obtained or may be found by simply putting L's for p's in the first.

Equation (72). Write $p = x_0|p_1p_2 + x_1|p_2p_0 + x_2|p_0p_1$, and multiply into $|p_0$; thus $p|p_0 = x_0p_0p_1p_2$. Find in the same way values of x_1 and x_2, and substitute.

Equation (73). Write $p_1p_2 \cdot p_3p_4 = xp_1 + yp_2$, and multiply by pp_2; therefore $pp_2 \cdot p_1p_2 \cdot p_3p_4 = xpp_2p_1$, or, by Art. 23, $p_2pp_1 \cdot p_2p_3p_4 = xpp_2p_1 = -xp_2pp_1$; or, $x = -p_2p_3p_4$. Multiplying by pp_1 we find $y = p_3p_4p_1$, and on substituting obtain the first of (73). For the second put $p_1p_2 \cdot p_3p_4 = xp_3 + yp_4$, and proceed in a similar way.

Equation (74). In the first of (73) put $p_3p_4 = |q_1$.

Equation (75). In the fourth of (73) put

$$L_1L_2 = p_2,\ L_3 = |q_1,\ L_4 = |q_2.$$

Equation (76). Multiply (75) by p_1.

Equation (77). In the first of (72) put q_2 for p, and multiply by $p_0p_1p_2 \cdot q_0q_1$; then

$$p_0p_1p_2 \cdot q_0q_1q_2 = q_0q_1|p_1p_2 \cdot q_2|p_0 + q_0q_1|p_2p_0 \cdot q_2|p_1 + q_0q_1|p_0p_1 \cdot q_2|p_2$$
$$= p_0|q_2 \cdot \begin{vmatrix} p_1|q_0 & p_1|q_1 \\ p_2|q_0 & p_2|q_1 \end{vmatrix} + p_1|q_2 \cdot \begin{vmatrix} p_2|q_0 & p_2|q_1 \\ p_0|q_0 & p_0|q_1 \end{vmatrix} + p_2|q_2 \cdot \begin{vmatrix} p_0|q_0 & p_0|q_1 \\ p_1|q_0 & p_1|q_1 \end{vmatrix},$$

by (76), which is equivalent to the third order determinant of equation (77).*

Exercise 12.—To show the product of two determinants as a determinant of the same order.

Let $p_0 = \sum_0^2 le,\ p_1 = \Sigma me,\ p_2 = \Sigma ne,\ q_0 = \Sigma\lambda e,\ q_1 = \Sigma\mu e,\ q_2 = \Sigma\nu e$; then $p_0p_1p_2 = [l_0,\ m_1,\ n_2]$, $q_0q_1q_2 = [\lambda_0,\ \mu_1,\ \nu_2]$; also $p_0|q_0 = l_0\lambda_0 + l_1\lambda_1 + l_2\lambda_2$, $p_1|q_0 = m_0\lambda_0 + m_1\lambda_1 + m_2\lambda_2$, etc. Substituting these values in (77), we have the required result. A solution may also be obtained directly without the use of (77). Let the q's be as above, but write $p_0 = \sum_0^2 lq,\ p_1 = \Sigma mq,\ p_2 = \Sigma nq$. Then

$$p_0p_1p_2 = \Sigma lq \cdot \Sigma mq \cdot \Sigma nq = [l_0,\ m_1,\ n_2]q_0q_1q_2 = [l_0,\ m_1,\ n_2][\lambda_0,\ \mu_1,\ \nu_2].$$

* Grassmann (1862), Art. 173.

Also $p_0 = l_0\Sigma\lambda e + l_1\Sigma\mu e + l_2\Sigma\nu e$

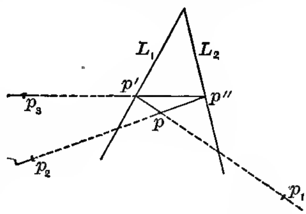
$$= (l_0\lambda_0 + l_1\mu_0 + l_2\nu_0)e_0 + (l_0\lambda_1 + l_1\mu_1 + l_2\nu_1)e_1 + (l_0\lambda_2 + l_1\mu_2 + l_2\nu_2)e_2,$$

with similar values for p_1 and p_2, which on being substituted in $p_0p_1p_2$ give the result. Equation (77), however, exhibits the product in a very compact, symmetrical, and easily remembered form.*

Exercise 13.—Show that the sides p_1p_2, p_2p_3, p_3p_1 of the triangle $p_1p_2p_3$ cut the corresponding sides $|p_3$, $|p_1$, $|p_2$ of the complementary triangle in three collinear points.

The three points of intersection are, using (74), $p_1p_2 \cdot |p_3 = -p_1 \cdot p_2|p_3 + p_2 \cdot p_1|p_3$, $p_2p_3 \cdot |p_1 = -p_2 \cdot p_3|p_1 + p_3 \cdot p_2|p_1$, $p_3p_1 \cdot |p_2 = -p_3 \cdot p_1|p_2 + p_1 \cdot p_3|p_2$, of which the sum is zero, showing that the points are collinear. It may be shown in the same way that the lines joining corresponding vertices are confluent.

Exercise 14.—If the sides of a triangle pass through three fixed points, and two of the vertices slide on fixed lines, find the locus of the other vertex.



Let the fixed points and lines be p_1, p_2, p_3, L_1, L_2, and p, p', p'' the vertices of the triangle, as in the figure. Then $p'p_3p'' = 0$; p' coincides with $pp_1 . L_1$ and p'' with $pp_2 . L_2$; hence substituting $(pp_1 . L_1)p_3(L_2 . p_2p) = 0$, the equation of the locus, which, being of the second degree in p, is that of a conic.

Prob. 18. Show that if the three fixed points of the last exercise are collinear, then the locus of p breaks up into two straight lines. Use equation (73).

Prob. 19. If the vertices of a triangle slide on three fixed lines, and two of the sides pass through fixed points, find the envelope of the other side. (This statement is reciprocally related to that of Exercise 14, that is, lines and points are replaced by points and

* These methods may be applied to determinants of any order by using a space of corresponding order.

lines respectively, and the resulting equation will be an equation of the second order in L, a variable line.)

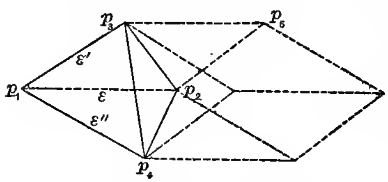
Prob. 20. Show that if the three fixed lines of Exercise 5 are confluent, then the envelope of L reduces to two points and the line joining them.

ART. 10. STEREOMETRIC PRODUCTS.

The product of two points in solid space is the same as in plane space. See Art. 7.

Product of Three Points.—Any three points determine a plane, and also, as in Art. 7, an area; hence $p_1p_2p_3$ is a plane-sect or a portion of the plane fixed by the three points whose area is double that of the triangle $p_1p_2p_3$. It may be shown, in the manner used in Art. 7 for the sect, that no plane-sect, not in this plane, can be equal to $p_1p_2p_3$, and that any plane-sect in this plane having the same area and sign will be equal to $p_1p_2p_3$.* Of course $p_1p_2p_3$ is not now scalar.

Product of Four Points.—Any four non-coplanar points determine a tetrahedron, say $p_1p_2p_3p_4$, and six times the volume of this tetrahedron is taken for the value of the product, because this is the volume of the parallelepiped generated by the product $p_1p_2p_3$,—i.e. the parallelogram p_1, p_6,—when it moves parallel to its initial position from p_1 to p_4. Let



$p_2 - p_1 = \epsilon$, $p_3 - p_1 = \epsilon'$, $p_4 - p_1 = \epsilon''$, then

$$p_1p_2p_3p_4 = p_1p_2p_3\epsilon'' = p_1p_2\epsilon'\epsilon'' = p_1\epsilon\epsilon'\epsilon''. \qquad (78)$$

If $p_1 = \sum_0^3 ke$, $p_2 = \sum_0^3 le$, $p_3 = \sum_0^3 me$, $p_4 = \sum_0^3 ne$, then

$$p_1p_2p_3p_4 = \Sigma ke\Sigma le\Sigma me\Sigma ne = [k_0, l_1, m_2, n_3] \,.\, e_0e_1e_2e_3; \quad (79)$$

from which it appears that any two quadruple products of points differ from each other only by a scalar factor, that is, they differ only in magnitude, or sign, or both; hence such products are themselves scalar.† If $p_1p_2p_3p_4 = 0$, the volume of the tetrahedron vanishes, so that the four points are coplanar.

* Grassmann (1862), Art. 255. † Grassmann (1862), Art. 263.

Product of Two Vectors.—The two vectors determine an area as in Art. 7, but they also determine now a plane direction, so that the product $\epsilon_1\epsilon_2$ is a plane-vector, and is not scalar as in plane space. Also, $\epsilon_1\epsilon_2$ differs from $p_1\epsilon_1\epsilon_2$ now just as ϵ differs from $p\epsilon$; namely, $\epsilon_1\epsilon_2$ has a definite area and plane direction, that is, toward a certain line at infinity, while $p_1\epsilon_1\epsilon_2$ is fixed in position by passing through p_1. Equation (37) therefore does not hold in solid space.

Product of Three Vectors.—Three vectors determine a parallelepiped as in the figure above, and $\epsilon\epsilon'\epsilon''$ is therefore the volume of this parallelepiped. Any other triple vector product can differ from this only in magnitude and sign. For let $\epsilon_1\epsilon_2\epsilon_3$ be such a product, and write

$$\epsilon = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 = \sum_1^3 x\epsilon,\ \epsilon' = \sum_1^3 y\epsilon,\ \epsilon'' = \sum_1^3 z\epsilon;\ \text{then}$$

$$\epsilon\epsilon'\epsilon'' = \Sigma x\epsilon \Sigma y\epsilon \Sigma z\epsilon = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \epsilon_1\epsilon_2\epsilon_3, \tag{80}$$

so that the two products only differ by the scalar determinant factor. Hence the product of three vectors must be itself a scalar, by Art. 1. Since, then, the product of four points has precisely the same signification as that of three vectors, we may write

$$\begin{aligned} p_1p_2p_3p_4 &= p_1\epsilon\epsilon'\epsilon'' = \epsilon\epsilon'\epsilon'' = (p_2 - p_1)(p_3 - p_1)(p_4 - p_1) \\ &= p_2p_3p_4 - p_3p_4p_1 + p_4p_1p_2 - p_1p_2p_3. \end{aligned} \tag{81}$$

Thus the sum of the plane-sects forming the doubles of the faces of a tetrahedron, all taken positively in the same sense as looked at from outside the tetrahedron, is equal to the volume of the tetrahedron. Compare equation (37).

If $\epsilon\epsilon'\epsilon'' = 0$, the volume of the parallelepiped vanishes, and the three vectors must be parallel to one plane.

Product of Two Sects.—In solid space two sects determine a tetrahedron of which they are opposite edges. Thus

$$p_1p_2p_3p_4 = p_1p_2 \cdot p_3p_4 = L_1L_2 = p_3p_4 \cdot p_1p_2 = L_2L_1, \tag{82}$$

so that the stereometric product of two sects is commutative, and has the same meaning as that of four points.

Product of a Sect and a Plane-Sect.—Let them be L and P, and let p_0 be their common point; take p_1, p_2, p_3 so that $L = p_0 p_1$ and $P = p_0 p_2 p_3$. L and P evidently determine the point p_0, and also the parallelepiped of which one edge is L and one face is P, so that the product should be made up of these two factors. Hence we write

$$\left.\begin{aligned} LP &= p_0 p_1 \cdot p_0 p_2 p_3 = p_0 p_1 p_2 p_3 \cdot p_0; \\ PL &= p_0 p_2 p_3 \cdot p_0 p_1 = p_0 p_2 p_3 p_1 \cdot p_0 = LP. \end{aligned}\right\} \qquad (83)$$

If L is parallel to P, p_0 is at infinity, and, replacing it by ϵ, (83) becomes

$$PL = LP = \epsilon p_1 \cdot \epsilon p_2 p_3 = \epsilon p_1 p_2 p_3 \cdot \epsilon. \qquad (84)$$

Product of Two Plane-Sects.—Let them be P_1 and P_2, and let L be their intersection, while p_1 and p_2 are such points that $P_1 = Lp_1$ and $P_2 = Lp_2$; then P_1 and P_2 determine the line L and also a parallelepiped of which they are two adjacent faces, and

$$P_1 P_2 = Lp_1 \cdot Lp_2 = Lp_1 p_2 \cdot L = - P_2 P_1. \qquad (85)$$

If P_1 and P_2 are parallel, L is at infinity, and is equivalent to a plane-vector, say to η; hence, substituting in (84),

$$P_1 P_2 = \eta p_1 \cdot \eta p_2 = \eta p_1 p_2 \cdot \eta = - P_2 P_1. \qquad (86)$$

Product of Three Plane-Sects.—By (85) and (83) this must be the square of a volume times the common point of the three planes; or, if p_0, p_1, p_2, p_3 be taken in such manner that $P_1 = p_0 p_2 p_3$, $P_2 = p_0 p_3 p_1$, $P_3 = p_0 p_1 p_2$, then

$$P_1 P_2 P_3 = 023 \,.\, 031 \,.\, 012 = 023 \,.\, 0123 \,.\, 01 = (p_0 p_1 p_2 p_3)^2 \cdot p_0; \qquad (87)$$

the suffixes being used instead of the corresponding points. If p_0 be at infinity, the three planes are parallel to a single line, and may be written $P_1 = n_1 \epsilon p_2 p_3$, etc., and then treated as above.

Product of Four Plane-Sects.*—Let the planes be $P_0 \ldots P_3$, and let $p_0 \ldots p_3$ be the four common points of the planes taken three by three. $n_0 \ldots n_3$ may be so taken that $P_0 = n_0 p_1 p_2 p_3$, etc.; then

$$\begin{aligned} P_0 P_1 P_2 P_3 &= n_0 n_1 n_2 n_3 \,.\, 123 \,.\, 230 \,.\, 301 \,.\, 012 \\ &= n_0 n_1 n_2 n_3 (p_0 p_1 p_2 p_3)^3. \end{aligned} \qquad (88)$$

* Grassmann (1862), Art. 300.

Product of Two Plane-Vectors.—Let η_1 and η_2 be two plane-vectors or lines at infinity; let ϵ be parallel to each of them, and ϵ_1 and ϵ_2 so taken that $\eta_1 = \epsilon\epsilon_1$, $\eta_2 = \epsilon\epsilon_2$, then

$$\eta_1\eta_2 = \epsilon\epsilon_1 \cdot \epsilon\epsilon_2 = \epsilon\epsilon_1\epsilon_2 \cdot \epsilon = -\eta_2\eta_1, \tag{89}$$

because η_1 and η_2 determine a common direction ϵ, and a parallelepiped of which three conterminous edges are equal to ϵ, ϵ_1, ϵ_2, respectively.

Product of Three Plane-Vectors.—Take ϵ_1, ϵ_2, ϵ_3 so that

$$\eta_1\eta_2\eta_3 = n \cdot \epsilon_2\epsilon_3 \cdot \epsilon_3\epsilon_1 \cdot \epsilon_1\epsilon_2 = n(\epsilon_1\epsilon_2\epsilon_3)^2. \tag{90}$$

The directions $\epsilon_1 \ldots \epsilon_3$ are common to the plane-vectors $\eta_1 \ldots \eta_3$ taken two by two.

Several conditions are given here together which follow from the results of this article.

$p_1p_2 = 0$, Two points coincide.	$P_1P_2 = 0$, Two planes coincide.	(91)
$p_1p_2p_3 = 0$, Three points collinear.	$P_1P_2P_3 = 0$, Three planes collinear.	(92)
$p_1p_2p_3p_4 = p_1p_2 \cdot p_3p_4$ $= L_1L_2 = 0$, Four points coplanar; two lines intersect.	$P_1P_2P_3P_4 = P_1P_2 \cdot P_3P_4$ $= L_1L_2 = 0$, Four planes confluent; two lines intersect.	(93)
$\epsilon_1\epsilon_2 = 0$, Vectors parallel.	$\eta_1\eta_2 = 0$, Plane-vectors parallel.	(94)
$\epsilon_1\epsilon_2\epsilon_3 = 0$, Three vectors parallel to one plane.	$\eta_1\eta_2\eta_3 = 0$, Three plane-vectors parallel to one line.	(95)

Sum of Two Planes.—Let them be P_1 and P_2, let L be a sect in their common line, and take p_1 and p_2 so that $P_1 = Lp_1$, $P_2 = Lp_2$; then

$$P_1 + P_2 = L(p_1 + p_2) = 2L\bar{p}, \tag{96}$$

$\bar{p}$ being the mean of p_1 and p_2. Also

$$P_1 - P_2 = L(p_1 - p_2); \tag{97}$$

whence the sum and difference are the diagonal plane through L, and a plane through L parallel to the diagonal plane which is itself parallel to L, of the parallelepiped determined by P_1

and P_2. If $TP_1 = TP_2$, $P_1 \pm P_2$ will evidently be the two bisecting planes of the angle between them. The bisecting planes may also be written

$$\frac{P_1}{TP_1} \pm \frac{P_2}{TP_2} \quad \text{or} \quad P_1 TP_2 \pm P_2 TP_1. \tag{98}$$

If the two planes are parallel, let η be a plane-vector parallel to each of them, that is, their common line at infinity, and let p_1 and p_2 be points in the respective planes; then we may write $P_1 = n_1 p_1 \eta$, $P_2 = n_2 p_2 \eta$, whence

$$P_1 + P_2 = (n_1 p_1 + n_2 p_2)\eta = (n_1 + n_2)\bar{p}\eta. \tag{99}$$

If $n_1 + n_2 = 0$, this becomes

$$P_1 + P_2 = n_2(p_2 - p_1)\eta, \tag{100}$$

the product of a vector into a plane-vector and therefore a scalar, by (80).

Two plane-vectors may be added similarly, since they will have a common direction, namely, that of the vector parallel to both of them.

Exercise 15.—If two tetrahedra $e_0 e_1 e_2 e_3$ and $e_0' e_1' e_2' e_3'$ are so situated that the right lines through the pairs of corresponding vertices all meet in one point, then will the corresponding faces cut each other in four coplanar lines.

The given conditions are equivalent to $e_0 e_0' \,.\, e_1 e_1' = 0$ $= e_0 e_0' \,.\, e_2 e_2' = e_0 e_0' \,.\, e_3 e_3' = e_1 e_1' \,.\, e_2 e_2' = e_2 e_2' \,.\, e_3 e_3' = e_3 e_3' \,.\, e_1 e_1'$. Two of the intersecting lines of faces are $e_0 e_1 e_2 \,.\, e_0' e_1' e_2'$ and $e_1 e_2 e_3 \,.\, e_1' e_2' e_3'$, and, if these intersect, we must accordingly have, by (92), $012 \,.\, 0'1'2' \,.\, 123 \,.\, 1'2'3' = 0 = 012 \,.\, 123 \,.\, 0'1'2' \,.\, 1'2'3$ $= 0123 \,.\, 0'1'2'3' \,.\, 121'2'$, the last factor of which is equivalent to the fourth condition above, since quadruple-point products in solid space are associative. Similarly all the other pairs of intersections may be treated.

Exercise 16.—The twelve bisecting planes of the diedral angles of a tetrahedron fix eight points, the centers of the inscribed and escribed spheres, through which they pass six by six.

The sum and difference of two unit planes are their two bisecting planes, by (97). Let the tetrahedron be $e_0e_1e_2e_3$, and let the double areas of its faces be $A_0 = Te_1e_2e_3$, etc.; then a pair of bisecting planes will be $\frac{e_0e_1e_2}{A_3} \pm \frac{e_0e_1e_3}{A_2}$ or $e_0e_1(A_2e_2 \pm A_3e_3)$. The pair through the opposite edge will be $e_2e_3(A_0e_0 \pm A_1e_1)$. If there be a point through which the six internal bisecting planes pass, it must be on the intersection of these two planes taken with the upper signs, and we infer by symmetry that it must be the point $\sum_0^3 Ae$. Another internal bisecting plane is $e_3e_0(A_1e_1 + A_2e_2)$, which gives zero when multiplied into ΣAe, as do also the other three.

To obtain all the points we have only to use the double signs, so that they are $\pm A_0e_0 \pm A_1e_1 \pm A_2e_2 \pm A_3e_3$. This gives eight cases, namely,

+ + + +	− + + +
+ + + −	+ + − −
+ + − +	+ − − +
+ − + +	+ − + −

The eight apparent cases that would arise by changing all the signs are included in these because the points must be essentially positive. Moreover, no positive point could have three negative signs, because the sum of any three faces of the tetrahedron must be greater than the fourth face. It will be found on trial that six of the bisecting planes will pass through $\Sigma(\pm Ae)$ with any one of the above arrangements of sign.

Prob. 21. The twelve points in which the edges of a tetrahedron are cut by the bisecting planes of the opposite diedral angles fix eight planes, each of which passes through six of them.

Prob. 22. The centroid of the faces of a tetrahedron coincides with the center of the sphere inscribed within the tetrahedron whose vertices are the centroids of the respective faces of the first tetrahedron.

Prob. 23. If any plane be passed through the middle points of two opposite edges of a tetrahedron, it will divide the volume of the tetrahedron into two equal parts.

ART. 11. THE COMPLEMENT IN SOLID SPACE.

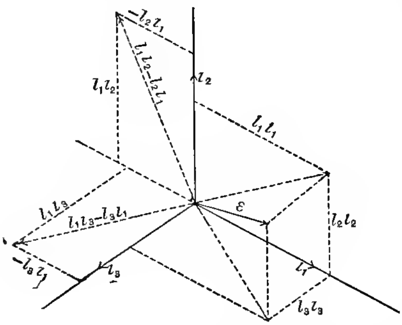
According to the definitions of Art. 8 the complementary relations in a unit normal vector system are as follows:

$$\left.\begin{aligned} |\iota_1 &= \iota_2\iota_3, & |\iota_2\iota_3 &= |(|\iota_1) = \iota_1 \\ |\iota_2 &= \iota_3\iota_1, & |\iota_3\iota_1 &= |(|\iota_2) = \iota_2 \\ |\iota_3 &= \iota_1\iota_2, & |\iota_1\iota_2 &= |(|\iota_3) = \iota_3 \end{aligned}\right\}. \qquad (101)$$

Let $\epsilon = \sum_1^3 l\iota$; then

$$|\epsilon = l_1\iota_2\iota_3 + l_2\iota_3\iota_1 + l_3\iota_1\iota_2 = \frac{1}{l_1}(l_1\iota_2 - l_2\iota_1)(l_1\iota_3 - l_3\iota_1), \qquad (102)$$

so that $|\epsilon$ is a plane-vector. The figure, which is drawn in isometric projection, shows that the two vectors $l_1\iota_2 - l_2\iota_1$ and $l_1\iota_3 - l_3\iota_1$, whose product is $l_1 . |\epsilon$, are both perpendicular to ϵ; for the first is perpendicular to $l_1\iota_1 + l_2\iota_2$, which is the orthogonal projection of ϵ upon $\iota_1\iota_2$, and to ι_3, and therefore is also perpendicular to ϵ, while the second is perpendicular to $l_1\iota_1 + l_3\iota_3$ and to ι_2, and therefore to ϵ. Hence $|\epsilon$ is a plane-vector perpendicular to ϵ; and, since $|(|\epsilon) = \epsilon$, the converse is also true, i.e. the complement of a plane-vector is a line-vector normal to it.



The figure shows that ϵ is equal to the vector diagonal of the rectangular parallelepiped whose edges have the lengths l_1, l_2, l_3, hence

$$T\epsilon = \sqrt{l_1^2 + l_2^2 + l_3^2}. \qquad (103)$$

Multiply equation (102) by ϵ; therefore

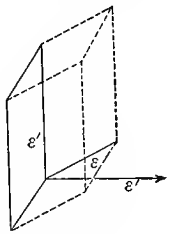
$$\begin{aligned} \epsilon|\epsilon &= (l_1\iota_1 + l_2\iota_2 + l_3\iota_3)(l_1\iota_2\iota_3 + l_2\iota_3\iota_1 + l_3\iota_1\iota_2) \\ &= l_1^2 + l_2^2 + l_3^2 = T^2\epsilon = \epsilon_-^2, \end{aligned} \qquad (104)$$

so that the co-square of a vector is equal to the square of its tensor. The product $\epsilon|\epsilon$ is that of a vector ϵ into a plane-vector perpendicular to it, as has just been shown; it is therefore a volume which is equivalent to $T\epsilon\,.\,T|\epsilon$; hence, by (104), $\epsilon|\epsilon = T\epsilon\,.\,T|\epsilon = T^2\epsilon$, or $T\epsilon = T|\epsilon$. Hence, the complement of a vector in solid space is a plane-vector perpendicular to it and having the same tensor, or numerical measure of magnitude.*

Let a second vector be $\epsilon' = \sum_1^3 m\iota$; then

$$\epsilon|\epsilon' = l_1m_1 + l_2m_2 + l_3m_3 = \epsilon'|\epsilon. \qquad (105)$$

Now $\epsilon|\epsilon'$, being the product of ϵ into the plane-vector $|\epsilon'$, is the volume of the parallelepiped in the figure, that is, $T\epsilon T\epsilon'$ sin (angle between ϵ and $|\epsilon'$) $= T\epsilon T\epsilon' \cos {}^{\epsilon'}_{\epsilon}$. Hence



$$\epsilon|\epsilon' = \epsilon'|\epsilon = l_1m_1 + l_2m_2 + l_3m_3 = T\epsilon T\epsilon' \cos {}^{\epsilon'}_{\epsilon}. \qquad (106)$$

If $T\epsilon = T\epsilon' = 1$, $l_1 \ldots l_3$, $m_1 \ldots m_3$ are direction cosines, and (105) gives a proof of the formula for the cosine of the angle between two lines in terms of the direction cosines of the lines. We have also in this case

$\epsilon\epsilon' = (l_1m_2 - l_2m_1)|\iota_3 + (l_2m_3 - l_3m_2)|\iota_1 + (l_3m_1 - l_1m_3)|\iota_2$, and, taking the co-square,

$$(\epsilon\epsilon')^2 = (\sin {}^{\epsilon'}_{\epsilon})^2 = (l_1m_2 - l_2m_1)^2 + (l_2m_3 - l_3m_2)^2 + (l_3m_1 - l_1m_3)^2. \qquad (107)$$

If
$$\epsilon|\epsilon' = 0, \qquad (108)$$

ϵ is parallel to the plane-vector perpendicular to ϵ', that is, ϵ is perpendicular to ϵ', as is also shown by (106).

Let $\eta = |\epsilon$, $\eta' = |\epsilon'$; then

$$\eta|\eta' = |\epsilon\,.\,\epsilon' = \epsilon'|\epsilon = \epsilon|\epsilon' = T\epsilon T\epsilon' \cos {}^{\epsilon'}_{\epsilon} = T\eta T\eta' \cos {}^{\eta'}_{\eta}, \qquad (109)$$

and
$$\eta|\eta' = 0 \qquad (110)$$

is the condition of perpendicularity of two plane-vectors. Also either

$$\epsilon|\eta' = 0, \quad \text{or} \quad \eta'|\epsilon = 0, \qquad (111)$$

is the condition that a vector shall be perpendicular to a plane-vector, for the first means that ϵ is parallel to a vector which is

* Grassmann (1862), Art. 335.

perpendicular to η', and the second that η' is parallel to a plane-vector which is perpendicular to ϵ.

Equations (71)–(77) of Art. 9 become stereometric vector formulæ if ϵ_1, ϵ_2, etc., be substituted for p_1, p_2, etc., and η_1, η_2, etc., for L_1, L_2, etc. For instance, (76) gives the vector formulas

$$\epsilon_1\epsilon_2|\epsilon_1'\epsilon_2' = \begin{vmatrix} \epsilon_1|\epsilon_1' & \epsilon_1|\epsilon_2' \\ \epsilon_2|\epsilon_1' & \epsilon_2|\epsilon_2' \end{vmatrix}, \quad \eta_1\eta_2|\eta_1'\eta_2' = \begin{vmatrix} \eta_1|\eta_1' & \eta_1|\eta_2' \\ \eta_2|\eta_1' & \eta_2|\eta_2' \end{vmatrix}. \qquad (112)$$

For lack of space no treatment of the complement in a point system in solid space is given.

Exercise 17.—To prove the formulas of spherical trigonometry $\cos a = \cos b \cos c + \sin b \sin c \cos A$, and

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Take three unit vectors ϵ_1, ϵ_2, ϵ_3 parallel to the radii to the vertices of the spherical triangle, then a=(angle bet. ϵ_2 and ϵ_3), A=(angle bet. $\epsilon_1\epsilon_2$ and $\epsilon_1\epsilon_3$), etc. In eq. (112) put $\epsilon_1\epsilon_3$ for $\epsilon_1'\epsilon_2'$; hence

$$\epsilon_1\epsilon_2|\epsilon_1\epsilon_3 = \sin b \sin c \cos A = \epsilon^2.\ \epsilon_2|\epsilon_3 - \epsilon_1|\epsilon_2 . \epsilon_1|\epsilon_3$$
$$= \cos a - \cos b \cos c.$$

Again,

$$T(\epsilon_1\epsilon_2 . \epsilon_1\epsilon_3) = T(\epsilon_1\epsilon_2\epsilon_3 . \epsilon_1) = T\epsilon_1\epsilon_2\epsilon_3 = T(\epsilon_2\epsilon_3 . \epsilon_2\epsilon_1) = T(\epsilon_3\epsilon_1 . \epsilon_3\epsilon_2);$$

or $\sin b \sin c \sin A = \sin a \sin c \sin B = \sin a \sin b \sin C$,

whence we have the second result by dividing by $\sin a \sin b \sin c$.

Exercise 18.—Show that in a spherical triangle taken as in Exercise 17, $\cos \dfrac{A}{2} = \dfrac{U\epsilon_1\epsilon_2|(U\epsilon_1\epsilon_2 + U\epsilon_1\epsilon_3)}{T(U\epsilon_1\epsilon_2 + U\epsilon_1\epsilon_3)}$, whence derive the ordinary value $\sqrt{\dfrac{\sin s \sin (s-a)}{\sin b \sin c}}$.

Expanding, the numerator becomes $1 + U\epsilon_1\epsilon_2|U\epsilon_1\epsilon_3$, and the denominator $\sqrt{2(1 + U\epsilon_1\epsilon_2|U\epsilon_1\epsilon_3)}$. Also there is obtained $U\epsilon_1\epsilon_2|U\epsilon_1\epsilon_3 = \dfrac{\epsilon_1\epsilon_2|\epsilon_1\epsilon_3}{T\epsilon_1\epsilon_2 T\epsilon_1\epsilon_3}$. The remainder is left to the student.

Prob. 24. If ϵ_1, ϵ_2, ϵ_3, drawn outward from a point, are taken as three edges of a tetrahedron, show that the six planes perpendicular to the edges at their middle points all pass through the end of the vector $\rho = \frac{1}{2\epsilon_1\epsilon_2\epsilon_3}(|\epsilon_2\epsilon_3 . \epsilon_1^2 + |\epsilon_3\epsilon_1 . \epsilon_2^2 + |\epsilon_1\epsilon_2 . \epsilon_3^2)$. (Suggestion. We must have $(\rho - \frac{1}{2}\epsilon_1)|\epsilon_1 = 0$, with two other similar expressions.)

Prob. 25. Show that ϵ, $|\epsilon\epsilon'$ and $\epsilon\epsilon'.|\epsilon$ are three mutually perpendicular vectors, no matter what the directions of ϵ and ϵ' may be.

Prob. 26. Let ϵ_1, ϵ_2, ϵ_3 be taken as in Prob. 24; let A_0 be the area of the face of the tetrahedron formed by joining the ends of these vectors, and $2A_1 = T\epsilon_2\epsilon_3$, etc.; also θ_1 = Angle between $\epsilon_1\epsilon_2$ and $\epsilon_1\epsilon_3$, etc.: then show that we have the relation, analogous to that of Prob. 15, Art. 8,

$$A_0^2 = A_1^2 + A_2^2 + A_3^2 - 2A_2A_3 \cos\theta_1 - 2A_3A_1 \cos\theta_2 - 2A_1A_2 \cos\theta_3.$$

If $\theta_1 \ldots \theta_3$ are right angles, this becomes the space-analog of the proposition regarding the hypotenuse and sides of a right-angled triangle. (Suggestion. $2A_0 = T(\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_1)$.)

Prob. 27. There are given three non-coplanar lines $e_0\epsilon_1$, $e_0\epsilon_2$, $e_0\epsilon_3$; planes cut these lines at right angles, the sum of the squares of their distances from e_0 being constant. Show that the locus of the common point of these three planes is $(\rho|\epsilon_1)^2 + (\rho|\epsilon_2)^2 + (\rho|\epsilon_3)^2 = c^2$, if $T\epsilon_1 = T\epsilon_2 = T\epsilon_3 = 1$.

ART. 12. ADDITION OF SECTS IN SOLID SPACE.

Two lines in solid space will not in general intersect, so that their sum will not be, as in eq. (43), a definite line. For let $p_1\epsilon_1$ and $p_2\epsilon_2$ be any two sects: then

$$\begin{aligned} p_1\epsilon_1 + p_2\epsilon_2 &= p_1\epsilon_1 + p_2\epsilon_2 + e_0(\epsilon_1 + \epsilon_2) - e_0(\epsilon_1 + \epsilon_2) \\ &= e_0(\epsilon_1 + \epsilon_2) + (p_1 - e_0)\epsilon_1 + (p_2 - e_0)\epsilon_2; \end{aligned}$$

that is, the sum is a sect passing through an arbitrary point e_0, and a plane-vector, the sum of the two in the equation. The sum cannot be a single sect unless the two are coplanar; for let $p_2 = p_1 + x\epsilon_1 + y\epsilon_2 + z\epsilon_3$, ϵ_3 being a vector not parallel to $\epsilon_1\epsilon_2$; hence

$$\begin{aligned} p_1\epsilon_1 + p_2\epsilon_2 &= p_1\epsilon_1 + (p_1 + x\epsilon_1 + y\epsilon_2 + z\epsilon_3)\epsilon_2 \\ &= p_1(\epsilon_1 + \epsilon_2) + x\epsilon_1(\epsilon_1 + \epsilon_2) + z\epsilon_3\epsilon_2 \\ &= (p_1 + x\epsilon_1)(\epsilon_1 + \epsilon_2) + z\epsilon_3\epsilon_2; \end{aligned}$$

and this cannot reduce to a single sect unless $z = 0$, that is, unless $p_1\epsilon_1$ and $p_2\epsilon_2$ are coplanar. Since a plane-vector is a line at

∞, the sum of two lines may always be presented as the sum of a finite line and a line at ∞.

If the sum of any two sects is equal to the sum of any other two, their products will also be equal, that is, the two pairs will determine tetrahedra of equal volumes. For let $L_1 + L_2 = L_3 + L_4$; then squaring we have $L_1L_2 = L_3L_4$, since $L_1L_1 = 0$, etc.

An infinite number of pairs of sects can be found such that the sum of each pair is equal to the sum of any given pair; for let a given pair be $p_1\epsilon_1 + p_2\epsilon_2$, and take a new pair

$$(x_1p_1 + x_2p_2)(u_1\epsilon_1 + u_2\epsilon_2) + (y_1r_1 + y_2p_2)(v_1\epsilon_1 + v_2\epsilon_2)$$
$$= (x_1u_1 + y_1v_1)p_1\epsilon_1 + (x_2u_2 + y_2v_2)p_2\epsilon_2 + (x_1u_2 + y_1v_2)p_1\epsilon_2 + (x_2u_1 + y_2v_1)p_2\epsilon_1.$$

This will be equal to the given pair if we have
$x_1u_1 + y_1v_1 = x_2u_2 + y_2v_2 = 1$, and $x_1u_2 + y_1v_2 = x_2u_1 + y_2v_1 = 0$).

Since there are eight arbitrary quantities with only four equations of condition, the desired result can evidently be accomplished in an infinite number of ways.

Let $p_1\epsilon_1, p_2\epsilon_2 \dots . p_n\epsilon_n$ be n sects, and let S be their sum, and e_0 any point, then

$$S = \sum_1^n p\epsilon \equiv e_0\Sigma\epsilon - e_0\Sigma\epsilon + \Sigma p\epsilon = e_0\Sigma\epsilon + \Sigma(p - e_0)\epsilon, \dots . \quad (113)$$

the sum of a sect and a plane-vector as before.

If $\Sigma(p - e_0)\epsilon$ is parallel to $\Sigma\epsilon$ it may be written as the product of some vector ϵ' into $\Sigma\epsilon$, that is, $\epsilon'\Sigma\epsilon$, when the sum becomes $S = e_0\Sigma\epsilon + \epsilon'\Sigma\epsilon = (e' + \epsilon')\Sigma\epsilon$, a sect, because $e_0 + \epsilon'$ is a point. In no other case does S reduce to a single sect. If $\Sigma\epsilon = 0$. S becomes a plane-vector. Of the two parts composing S, the sect will be unchanged in magnitude and direction if e_0 be moved to a new position, while the plane-vector will in general be altered. It is proposed to show that a point q may be substituted for e_0 such that the plane-vector will be perpendicular to $\Sigma\epsilon$. Writing

$$S \equiv q\Sigma\epsilon - (q - e_0)\Sigma\epsilon + \Sigma(p - e_0)\epsilon,$$

and, for brevity, putting $q - e_0 = \rho$, $\Sigma\epsilon = \alpha$, $\Sigma(p - e_0)\epsilon = |\beta$, so that

$$S \equiv q\alpha - \rho\alpha + |\beta, \quad (114)$$

we must have for perpendicularity, by (111),

$$(|\beta - \rho\alpha)|\alpha = 0 = |\beta\alpha - \rho\alpha.|\alpha,$$

or $$\rho\alpha.|\alpha \equiv \alpha.\rho|\alpha - \rho.\alpha^2 = |\beta\alpha. \qquad (115)$$

The second member is obtained from the first by substituting in eq. (74) ρ for p_1 and α for p_2 and q_1, in accordance with the statement at the end of Art. 11. If in (115) we make $\rho|\alpha = 0$, ρ will be the vector from e_0 to q taken perpendicularly to α, say

$$\rho_1 = |\alpha\beta \div \alpha^2 = q_1 - e_0. \qquad (116)$$

Since α and β are known, the required point has been found. Multiply (115) by α; then, using (75),

$$-\alpha\rho.\alpha^2 \equiv \rho\alpha.\alpha^2 = \alpha|\beta\alpha = |\beta.\alpha^2 - |\alpha.\alpha|\beta,$$

whence, substituting in (114),

$$S = q\alpha + \frac{\alpha|\beta}{\alpha^2}.|\alpha = q\Sigma\epsilon + \frac{\Sigma\epsilon\Sigma(p - e_0)\epsilon}{(\Sigma\epsilon)^2}.\Sigma\epsilon. \qquad (117)$$

This may be called the normal form of S.*

The sects of this article represent completely the geometric properties of forces, hence all that has been shown applies immediately to a system of forces in solid space. We have only to substitute the words force and couple for sect and plane-vector. The resultant action of any system of forces is S, called by Ball in his Theory of Screws "a wrench." The condition for equilibrium is $S = 0$, which gives at once

$$\Sigma\epsilon = 0 \quad \text{and} \quad \Sigma(p - e_0)\epsilon = 0; \qquad (118)$$

since otherwise we must have $e_0\Sigma\epsilon = -\Sigma(p - e_0)\epsilon$, which is an impossibility. The line $q\Sigma\epsilon$ is the central axis of the system of forces S.

Lack of space forbids a further development of the subject, but what has been given in this article will indicate the perfect adaptability of this method to the requirements of mechanics.

Exercise 19.—Reduce $p_1\epsilon_1 + p_2\epsilon_2 = S$ to its normal form. $S \equiv e_0(\epsilon_1 + \epsilon_2) + (p_1 - e_0)\epsilon_1 + (p_2 - e_0)\epsilon_2$. For convenience suppose p_1 and p_2 to be taken at the ends of the common per-

* Grassmann (1862), Art. 346.

pendicular on $p_1\epsilon_1$ and $p_2\epsilon_2$, and moreover let $e_0 = \frac{1}{2}(p_1 + p_2)$, $p_1 - e_0 = \iota = -(p_2 - e_0)$; then $\iota|\epsilon_1 = \iota|\epsilon_2 = 0$. Accordingly

$$S \equiv e_0(\epsilon_1 + \epsilon_2) + \iota(\epsilon_1 - \epsilon_2) = q(\epsilon_1+\epsilon_2) + \frac{(\epsilon_1+\epsilon_2)\iota(\epsilon_1-\epsilon_2)}{(\epsilon_1+\epsilon_2)^2}\,.\,|\epsilon_1+\epsilon_2)$$

$$= q(\epsilon_1 + \epsilon_2) + \frac{\iota\epsilon_1\epsilon_2}{(\epsilon_1+\epsilon_2)^2}\,.\,|(\epsilon_1 + \epsilon_2).$$

$$\text{By (116)}, q - e_0 = -\frac{|\beta\,.\,|\alpha}{\alpha^2} = -\frac{\iota(\epsilon_1 - \epsilon_2\,.\,|(\epsilon_1 + \epsilon_2)}{(\epsilon_1+\epsilon_2)^2}$$

$$= \frac{\iota\,.(\epsilon_1 - \epsilon_2)|(\epsilon_1+\epsilon_2)}{(\epsilon_1+\epsilon_2)^2}, \text{ by (74)}, = \frac{\epsilon_1^2 - \epsilon_2^2}{(\epsilon_1+\epsilon_2)^2}\,.\,\iota.$$

Hence the normal form of S is

$$S = \left(e_0 + \frac{\epsilon_1^2 - \epsilon_2^2}{(\epsilon_1+\epsilon_2)^2}\,.\,\iota\right)(\epsilon_1 + \epsilon_2) + \frac{\iota\epsilon_1\epsilon_2}{(\epsilon_1+\epsilon_2)^2}\,.\,|(\epsilon_1 + \epsilon_2).$$

Exercise 20.—Forces are represented by the six edges of a tetrahedron e_0e_1, e_0e_2, e_0e_3, e_2e_3, e_3e_1, e_1e_2; find the S, reduce to normal form, and consider the special case when three diedral angles are right angles. $S \equiv e_0(e_1 + e_2 + e_3) + e_2e_3 + e_3e_1 + e_1e_2$ $\equiv e_0(\epsilon_1+\epsilon_2+\epsilon_3)+(e_2-e_1)(e_3-e_1) \equiv e_0(\epsilon_1+\epsilon_2+\epsilon_3)+(\epsilon_2-\epsilon_1)(\epsilon_3-\epsilon_1)$ $\equiv e_0(\epsilon_1 + \epsilon_2 + \epsilon_3) + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2$, in which $\epsilon_1 = e_1 - e_0$, etc. Hence

$$S \equiv \left(e_0 + \frac{(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2)|(\epsilon_1 + \epsilon_2 + \epsilon_3)}{(\epsilon_1 + \epsilon_2 + \epsilon_3)^2}\right)(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

$$+ \frac{3\epsilon_1\epsilon_2\epsilon_3}{(\epsilon_1 + \epsilon_2 + \epsilon_3)^2}\,.\,|(\epsilon_1 + \epsilon_2 + \epsilon_3).$$

For the rectangular tetrahedron let $\epsilon_1 = a\iota_1$, $\epsilon_2 = b\iota_2$, $\epsilon_3 = c\iota_3$, ι_1, ι_2, ι_3 being unit normal vectors. Then we find

$$S \equiv \left(e_0 + \frac{a(c^2 - b^2)\iota_1 + b(a^2 - c^2)\iota_2 + c(b^2 - a^2)\iota_3}{a^2 + b^2 + c^2}\right)(a\iota_1 + b\iota_2 + c\iota_3)$$

$$+ \frac{3abc}{a^2 + b^2 + c^2}\,.\,|(a\iota_1 + b\iota_2 + c\iota_3).$$

Exercise 21.—A pole 50 feet high stands on the ground and is held erect by three guy-ropes symmetrically arranged about it, attached to its top and to pegs in the ground 50 feet from the pole. The wind blows against the pole with a pressure of 50 pounds in the direction $e_0 - p$, when e_0 is at the bottom of the pole, and p divides the distance between two of the pegs in the ratio $\frac{m}{n}$: find the tension on the guys and the pressure on the ground.

Evidently only two of the guys will be in tension; let their pegs be at e_1 and e_3, and let e_2 be at the top of the pole, and w the weight of the pole. Then $p = \frac{me_1 + ne_3}{m+n}$, and the equation of equilibrium is

$$50 . \frac{(e_0+e_2)(e_0-p)}{2T(e_0-p)} + \frac{25e_0(p-e_0)}{T(e_0-p)} + \frac{(x+w)e_0e_2}{Te_0e_2} + \frac{ye_2e_1}{Te_2e_1} + \frac{ze_2e_3}{Te_2e_3} = 0.$$

$$Te_0e_2 = 50,\ Te_2e_1 = Te_2e_3 = 50\sqrt{2},\ T(p - e_0) = T\left(\frac{me_1+ne_2}{m+n} - e_0\right)$$

$$= T\left(\frac{m(e_1-e_0)+n(e_3-e_0)}{m+n}\right) = \frac{50}{m+n}T(m\epsilon_1+n\epsilon_3),\text{ if } \epsilon_1 = U(e_1-e_0)$$

and $\epsilon_2 = U(e_2 - e_0)$; then $T(p - e_0) = \frac{50}{m+n}\sqrt{m^2 + n^2 - mn}$,

because $\epsilon_1^2 = \epsilon_3^2 = 1$, and $\epsilon_1 | \epsilon_3 = \cos 120^\circ = -\frac{1}{2}$. Hence the equation of equilibrium becomes

$$\frac{25e_2((m+n)e_0 - me_1 - ne_3)}{\sqrt{m^2+n^2-mn}} + (x+w)e_0e_2 + \frac{y}{\sqrt{2}}e_2e_1 + \frac{z}{\sqrt{2}}e_2e_3 = 0.$$

Multiply successively by e_2e_1, e_0e_3, and e_0e_1, and we obtain

$$\frac{x+w}{m+n} = \frac{y}{m\sqrt{2}} = \frac{z}{n\sqrt{2}} = \frac{25}{\sqrt{m^2+n^2-mn}},$$

y and z being the tensions, and $x + w$ the upward pressure.

Prob. 28. Three equal poles are set up so as to form a tripod, and are mutually perpendicular; a weight w hangs upon a rope which passes over a pulley at the top of the tripod, and thence down under a pulley at the ground at a point $p = \sum_1^3 le$, in which $e_1 \ldots e_3$ are at the feet of the poles, and $\sum_1^3 l = 1$; if the rope is pulled so as to raise w, show that the pressures on the poles, supposing the pulleys frictionless, are

$$w\left(\frac{l_1}{\sqrt{\Sigma l^2}} + \frac{1}{\sqrt{3}}\right), \quad w\left(\frac{l_2}{\sqrt{\Sigma l^2}} + \frac{1}{\sqrt{3}}\right), \quad w\left(\frac{l_3}{\sqrt{\Sigma l^2}} + \frac{1}{\sqrt{3}}\right).$$

Prob. 29. Six equal forces act along six successive edges of a cube which do not meet a given diagonal; show that if the edges of the cube be parallel to ι_1, ι_2, ι_3, and F be the magnitude of each force, then $S = -2F|(\iota_1 + \iota_2 + \iota_3)$, if the diagonal taken be parallel to $\iota_1 + \iota_2 + \iota_3$.

Prob. 30. Three forces whose magnitudes are 1, 2, and 3 act along three successive non-coplanar edges of a cube; show that the normal form of S is

$$S = (e_0 + \tfrac{13}{14}\iota_1 + \tfrac{1}{2}\iota_2 - \tfrac{9}{14}\iota_3)(\iota_1 + 2\iota_2 + 3\iota_3) + \tfrac{3}{14}|(\iota_1 + 2\iota_2 + 3\iota_3).$$

Prob. 31. Forces act at the centroids of the faces of a tetrahedron, perpendicular and proportional to the faces on which they act, and all directed inwards, or else all outwards; show that they are in equilibrium.