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A TREATISE

ON THE

GEOMETRY OF SURFACES

GEOMETRY
Theory of Surfaces analytically treated
TEXT-BOOK on a special Topic
I.
APR 24 1929

A TREATISE

ON THE

GEOMETRY OF SURFACES

BY

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MATH. DEPT.

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PREFACE

THE last edition of Salmon's *Analytic Geometry of Three Dimensions*, which was published in 1884, has been out of print for some years; and although there are several excellent works on *Quadric Surfaces* and other special branches of the subject, such as those of Mr Blythe on *Cubic Surfaces* and of the late Mr Hudson on *Kummer's Quartic Surface*, yet there is no British treatise exclusively devoted to the theory of surfaces of higher degree than the second. I have therefore endeavoured to supply this want in the present work.

The Theory of Surfaces is an extensive one, and a thoroughly comprehensive treatise would necessarily be voluminous. I have therefore decided to limit this work to the more elementary portions of the subject, and have abstained from introducing investigations which require a knowledge of the Theory of Functions and of the higher branches of Modern Algebra. The ordinary methods of Analytical Geometry are quite sufficient to enable the properties of cubic and quartic surfaces and twisted curves, and also the point and plane singularities of surfaces, to be discussed with tolerable completeness, and to demonstrate a number of interesting and important theorems connected with them; but for the purpose of confining this treatise within a moderate compass, I have abstained from any general discussion of surfaces of higher degree than the fourth.

The properties of a point-singularity may usually be examined by means of a surface of low degree just as well as by one of the nth degree; but if the degree is less than a certain limit, which depends on the character of the singularity, the latter appears in an incomplete form on the surface. Thus the properties of a triple line cannot be fully investigated without employing a surface of the seventh degree, and this fact has rendered it necessary to partially discuss surfaces of higher degree than a quartic.

The resolution of a multiple point into its constituents has been discussed by Professor Segre of Turin, and other Italian mathematicians, in various papers published in the *Annali di Matematica*; and these researches have shown that an important analogy exists between the theories of plane curves and of surfaces. The class of an anautotomic plane curve of degree n, and also the reduction of class produced by a multiple point of order n, the tangents at which are distinct, are both equal to $n(n-1)$; whilst the constituents of the multiple point are $\frac{1}{2}n(n-1)$ nodes. The class of an anautotomic surface of degree n, and also the reduction of class produced by a multiple point of order n, the tangent cone at which is anautotomic, are both equal to $n(n-1)^2$; and from analogy I concluded that the constituents of the multiple point were $\frac{1}{2}n(n-1)^2$ conic nodes. In 1908 I succeeded in obtaining a formal proof of the last theorem, which enables a large number of singular points to be resolved into their constituent conic nodes and binodes.

In the present treatise I have incorporated a variety of results, originally due to Italian and German mathematicians, many of which have been published since the last edition of Salmon's work; and I have endeavoured to modernize the analysis and the terminology by discarding antiquated methods and inappropriate symbols and phrases. I have also to express my obligations to the late Professor Cayley's papers, references to which are denoted by the letters *C. M. P.*; as well as to the *Repertorio di Matematiche Superiori* by Professor E. Pascal, which contains a valuable epitome of the subject, together with an exhaustive collection of references to the original papers of British and foreign mathematicians, who have studied this subject.

FLEDBOROUGH HALL,
HOLYPORT, BERKS.
March, 1910.

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Page 2, last line, *add* or constant multiples of the distances.
„ 4, line 7, *read* points *for* point.
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„ 136, line 13 from bottom, *read* κ' *for* $2\kappa'$.
„ 164, line 8, *read* lmn *for* three.
„ 168, line 10 from bottom, *read* conic node *for* conic.
„ 175, Sections 239 and 240 require correction. Weddle's surface is the Jacobian of four quadrics. The surface (6) is a particular case of Weddle's surface.
„ 177, heading, *read* Octonodal Quartics.
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„ 233, line 7 from bottom, *read* $S(\overline{1, 1}, n)$ *for* $S(1, 1, n)$.

CHAPTER I

THEORY OF SURFACES

1. THE general equation of a surface of the nth degree, when expressed in Cartesian coordinates, is

$$u_0 + u_1 + u_2 + \dots u_n = 0 \quad \dots\dots\dots\dots\dots\dots(1),$$

where u_n is a ternary quantic of (x, y, z). The number of terms in (1) is equal to the sum of the series

$$\tfrac{1}{2}\{1 . 2 + 2 . 3 + \dots (n+1)(n+2)\},$$

that is to say $\quad \tfrac{1}{6}(n+1)(n+2)(n+3).$

The number of independent constants in (1) is one less than the preceding quantity and is therefore equal to

$$\tfrac{1}{6}n(n^2 + 6n + 11),$$

which determines the number of independent conditions that a surface of the nth degree can satisfy.

If in (1) we put $y = z = 0$, we obtain an equation of the nth degree for determining the points where the axis of x cuts the surface. Hence:—*every straight line intersects a surface of the nth degree in n points.* Also if we put $z = 0$ in (1), we obtain an equation of the nth degree in x and y, which determines the curve of intersection of the plane $z = 0$ and the surface. Hence:—*every plane intersects a surface of the nth degree in a curve of the same degree.*

Let $U_n = 0$, $V_m = 0$ be two surfaces of the nth and mth degrees respectively; then if z be eliminated we shall obtain an equation of degree mn in x and y, which represents a curve of degree mn which is the projection on the plane $z = 0$ of the curve of intersection of the two surfaces. Hence:—*two surfaces of degrees m and n intersect in a curve of degree mn; also every plane intersects the curve in mn points.*

Three surfaces of degrees l, m and n intersect one another in lmn points; for if $U_l = 0$, $V_m = 0$, $W_n = 0$ be the three surfaces, it is shown in treatises on Algebra that the result of eliminating y and z is an equation of degree lmn in x, which determines the values of x at the points of intersection of the three surfaces. And by parity of reasoning it follows that:—*every curve of degree n intersects a surface of degree m in mn points.*

The curve of intersection of two surfaces does not in general lie in a plane. There are consequently two kinds of curves in space, called *plane* and *twisted* according as they do or do not lie in a plane. It may also happen that a curve which is apparently a twisted one may degrade into two plane curves lying in different planes.

2. There are four distinct species of surfaces. First *ordinary* surfaces such as (1). Secondly *scrolls*, which are also called *skew* surfaces. These are generated by the motion of a straight line which moves in such a manner that two consecutive generators do not intersect. Thirdly, when each generator intersects the consecutive one the surface is called a *developable** surface, because it is capable of being unrolled into a plane. Fourthly *cones*, which are a special kind of developable surface, in which all the generators pass through a point. A cylinder is a special kind of cone which is obtained by projecting the vertex of the latter to infinity. It will hereafter be shown that all surfaces of a higher degree than the third are capable of assuming all four forms, provided they possess certain singularities. Thus a quartic surface which has a triple line is a scroll; if it has a cuspidal twisted cubic curve, it becomes a developable surface; whilst if it has a quadruple point, it becomes a cone.

Quadriplanar Coordinates.

3. In the quadriplanar system of coordinates, the position of a point P is determined by its distances $(\alpha, \beta, \gamma, \delta)$ from the four

* Some writers call a developable surface a *torse*. This is an inaccurate use of language, because *torse* is derived from *torsi* the perfect of *torquere*, to twist; and since a developable surface is formed by bending a plane, and therefore involves the idea of flexion alone, a word which connotes torsion is altogether inappropriate. The Italians call a developable surface *una sviluppabile*, and a scroll *una gobba*. Both scrolls and developables are included in the general term *superficie rigate* or ruled surfaces.

faces of a tetrahedron $ABCD$, called the tetrahedron of reference. The coordinate α is the length of the perpendicular drawn from P to the face BCD, and is positive or negative according as P lies on the same or the opposite side as A. If a, b, c, d be the areas of the four faces which are respectively opposite to A, B, C and D, and V the volume of the tetrahedron, then

$$a\alpha + b\beta + c\gamma + d\delta = 3V,$$

which we shall write $$I = 1 \quad\text{.............................(2)};$$

accordingly any algebraic function of the coordinates, which is not homogeneous, may be made so by multiplying each term by the proper power of I; hence we need only consider homogeneous functions of $(\alpha, \beta, \gamma, \delta)$, that is to say quaternary quantics of the coordinates. Any linear function of the coordinates represents a plane; $\alpha = 0$ is the plane BCD; whilst $I = 0$ is the plane at infinity. Any quaternary n-tic represents a surface of the nth degree; and any ternary n-tic of three coordinates represents a cone. Also since the equation of a sphere is $S + u = 0$, where S is a *given* sphere and u an arbitrary plane, the equation of any sphere may be expressed in the form $S + Iu = 0$, which shows that all spheres intersect the plane at infinity in the same circle. This circle is of course imaginary, and corresponds to the circular points in plane geometry.

4. We shall usually employ the symbols u_n, U_n to denote *ternary* quantics of (β, γ, δ); whilst such symbols as v_n, w_n, σ_n &c. will denote *binary* quantics of (γ, δ). With this notation the general equation of a surface of the nth degree is

$$\alpha^n u_0 + \alpha^{n-1} u_1 + \alpha^{n-2} u_2 + \dots u_n = 0 \quad\text{..............(3)},$$

where $$u_n = \beta^n \sigma_0 + \beta^{n-1}\sigma_1 + \dots \sigma_n.$$

To pass from quadriplanar to Cartesian coordinates, the plane BCD must be projected to infinity, and the lines AB, AC and AD projected so that they become the axes of x, y and z. Hence $\alpha =$ const., which may be taken as unity, and (3) becomes

$$u_0 + u_1 + u_2 + \dots = 0.$$

5. If a surface passes through the vertex A it follows from (3) that $u_0 = 0$, hence:—*if a surface passes through the vertices of the tetrahedron of reference, the terms involving the highest powers of the coordinates must be absent.*

Let the surface (3) pass through A, and let ABC be the plane u_1; then (3) becomes

$$\alpha^{n-1}\delta + \alpha^{n-2}u_2 + \alpha^{n-3}u_3 + \dots u_n = 0 \quad \text{...........(4).}$$

The equations of AB, which may be any line in the plane ABC, are $\gamma = \delta = 0$; whence if γ and δ are each put equal to zero in (4) it reduces to the binary quantic $\beta^2(\alpha, \beta)^{n-2} = 0$, which determines the planes passing through CD and the point where AB intersects the surface; and since β^2 appears as a factor, it follows that AB has bitactic* contact with the surface at A, whence ABC is the tangent plane at A. Accordingly:—*if a surface of the nth degree passes through the vertices of the tetrahedron of reference, the coefficients of the* $(n-1)$*th powers of the coordinates are the tangent planes at these points.*

The equation of the curve of section by the tangent plane at A is obtained by putting $\delta = 0$ in (4), which becomes

$$\alpha^{n-2}V_2 + \alpha^{n-3}V_3 + \dots V_n = 0 \quad \text{..............(5),}$$

where $V_n = (\beta, \gamma)^n$; accordingly the section has a node at A. Hence:—*the section of a surface by the tangent plane at any point is a plane curve having a node at the point of contact; also if the plane touches the surface at any other points, these will also be nodes on the section.*

Since the tangents at a node on a plane curve have tritactic contact with the curve, it follows that two lines can in general be drawn through the point of contact of the tangent plane to a surface which have tritactic contact with it at this point. If however these lines coincide, the condition for which is $V_2 = V_1^2$, the point of contact is a *cusp* on the section. Such points are called *parabolic* or *spinodal* points, and they lie on a certain curve called the *spinodal* curve which (as will hereafter be shown) is the intersection of the surface and its Hessian.

Since a straight line cannot intersect a quadric surface in more than two points, unless it lies in the surface, it follows that the tangent plane to every quadric surface intersects the latter in

* The phrase *sextactic points* was introduced by Cayley to designate the points where a conic touches a plane curve at six coincident points. I have therefore extended the use of this term, by defining a curve which has *n-tactic* contact with a given curve or surface at a point A, to be one which intersects the curve or surface in n coincident points at A. The introduction of this phrase avoids the ambiguity and confusion caused by defining the circle of curvature to be a circle which has a contact of the *second order* with a given curve.

a pair of straight lines. In the case of a hyperboloid of one sheet these lines are real; but in the case of an ellipsoid they are imaginary.

6. In (3) let $u_0 = u_1 = 0$; also let AB be one of the generators of the cone $u_2 = 0$, then (3) becomes

$$\alpha^{n-2}(\beta v_1 + v_2) + \alpha^{n-3}(\beta^3 w_0 + \beta^2 w_1 + \beta w_2 + w_3) + \dots \alpha^{n-4} u_4 + \dots u_n = 0 \qquad \dots\dots(6).$$

Putting $\gamma = \delta = 0$, it follows that β^3 is a factor of the resulting equation, which shows that the line AB has tritactic contact with the surface at A. Hence all the generators of the cone $u_2 = 0$ have tritactic contact with the surface at A, and the cone is consequently a tangent cone to the surface at this point. The point A is a singular point called a *conic node*, and possesses properties analogous to an ordinary node on a plane curve.

The cones u_2 and u_3 have six common generators, and if AB be one of them $w_0 = 0$. If therefore we put $\gamma = \delta = 0$ in (6) it follows that β^4 is a factor of the resulting equation and the line AB has quadritactic contact with the surface at A. No other generators can have a higher contact with the surface at A, and these six common generators are called the *lines of closest contact*.

This theory can be extended to multiple points of any order; for if the first term in (3) is $\alpha^{n-p} u_p$, then A is a multiple point of order p, at which there is a tangent cone of degree p. All the generators of this cone have $(p+1)$-tactic contact with the surface at A, except the $p(p+1)$ common generators of the cones u_p and u_{p+1}, which have $(p+2)$-tactic contact and are therefore the lines of closest contact at a multiple point of order p.

7. Returning to conic nodes, it is possible for the cone u_2 to degrade into a pair of planes ABC and ABD, and the first term of (6) becomes $\alpha^{n-2}\gamma\delta$. Such a singularity is called a *biplanar node* or shortly a *binode*; the two planes γ and δ are called the *biplanes*; and their line of intersection AB is called the *axis* of the binode. Putting $\gamma = \delta = 0$, it follows that the axis of the binode has tritactic contact with the surface at A. The section of the surface by either biplane is a curve having a triple point of the first kind at the binode, that is to say a triple point the three tangents at which are distinct; and the latter are the six lines of closest contact. If however a line of closest contact in each biplane coincides with the axis, we obtain a special kind of binode whose axis has quadritactic contact with the surface at A; also special kinds of binodes occur when any of the lines of closest contact coincide. We shall hereafter show that the ordinary conic node and binode are distinct singularities, the former of which reduces the class of the surface by 2 and the latter by 3, so that the binode is a singularity analogous to a cusp on a plane curve; but the binode whose axis has quadritactic contact with the surface is a *compound* singularity, whose point constituents will be shown to be two conic nodes and is therefore analogous to a tacnode on a plane curve.

There are two kinds of ordinary binodes, according as the biplanes are real or imaginary. In the latter case the singularity might be called a cuspidal point, since it is formed by the revolution of a cusp about its cuspidal tangent. It may also be regarded as the limiting form of a conic node whose nodal cone shrinks up into its axis. For example, the equation of the surface formed by the revolution of the cissoid $y^2 = x(x^2 + y^2)$ about its cuspidal tangent is $y^2 + z^2 = x(x^2 + y^2 + z^2)$, which shows that the singularity at the origin is a binode whose biplanes are the two imaginary planes $y \pm iz = 0$. The biplanes may also be regarded as the limit of the real cone $y^2 + z^2 = x^2 \tan^2 \alpha$, when $\alpha = 0$.

8. When the two biplanes coincide, the singularity is called a *uniplanar node*, or shortly a *unode*; and the pair of coincident planes the *uniplane*. The first term of (6) now becomes $\alpha^{n-2}\delta^2$, where δ or ABC is the uniplane. The section of the surface by the uniplane is a curve having a triple point of the first kind at A, the tangents at which are the lines of closest contact twice repeated. The unode will hereafter be shown to be a *compound* singularity, whose point constituents are three conic nodes which move up to coincidence in an arbitrary manner. Moreover there are three primary species of unodes according as the tangents at the triple point on the section by the uniplane are (i) all distinct; (ii) one distinct and two coincident; (iii) all three coincident; and it will be further shown in the last two cases that the point constituents are $C = 2, B = 1$; $C = 1, B = 2$. The unode therefore possesses features analogous to those of a triple point on a plane curve.

Singular Tangent Planes.

9. In plane geometry the equation of a straight line contains two independent constants, and since two conditions are necessary in order that a straight line should be a double or a stationary tangent to a plane curve, it follows that every such curve must have a determinate number of singular tangents. In the case of a curve it so happens that the two simple line singularities are the reciprocals of the two simple point singularities; but there is no such analogy between singular points on a surface and singular tangent planes to a surface. The connection between the point and plane singularities of surfaces, as we shall hereafter show, is complicated.

The equation of a plane contains three independent constants, which may be called its coordinates; and the condition that a plane should be an ordinary tangent plane to a surface is that the section of the surface by the plane should be a uninodal curve, whose node is the point of contact. Hence, if $F(\alpha, \beta, \gamma, \delta) = 0$ and $\xi\alpha + \eta\beta + \zeta\gamma + \omega\delta = 0$ be the equations of the surface and the plane respectively, the required condition of tangency is that the discriminant of the ternary quantic $F\{\alpha, \beta, \gamma, -(\xi\alpha + \eta\beta + \zeta\gamma)/\omega\}$ should vanish. This furnishes a single relation between the coordinates of the plane, which is called the *tangential* equation of the surface.

10. We shall now show that every surface has associated with it six important twisted curves and developable surfaces.

(i) *The Spinodal Curve and Developable.* Two equations of condition are necessary in order that a plane should touch the surface at a point which is a *cusp* on the section; hence the equation of such a plane contains only one independent constant, and its envelope is a developable surface called the *spinodal developable.* The point of contact of the tangent plane lies on a curve called the *spinodal curve,* and the tangent at the cusp on the section is a generator of the developable.

(ii) *The Flecnodal Curve and Developable.* Two equations of condition are necessary in order that a plane should touch the surface at a point which is a *flecnode* on the section. The envelope of such planes is called the *flecnodal developable*; and the points of contact of the tangent planes lie on a curve called the *flecnodal* *curve*; also the flecnodal tangent on the section is a generator of this developable.

(iii) *The Bitangential Curve and Developable.* Two equations of condition are also necessary in order that a plane should be a double tangent plane, in which case the section is a binodal curve whose nodes are the points of contact of the tangent plane. The envelope of these planes is called the *bitangential developable*; and their points of contact lie on a curve called the *bitangential curve.* The line joining the points of contact is a double tangent to the surface and is also a generator of the developable.

The three remaining curves are the edges of regression of these three developables; whilst the remaining three developables are those enveloped by the osculating planes to the spinodal, flecnodal and bitangential curves.

11. The six singular tangent planes arise from the fact that a tangent plane may be made to satisfy three equations of condition in six different ways. I shall denote these planes by ϖ_1, ϖ_2, ϖ_3, ϖ_4, ϖ_5 and ϖ_6.

ϖ_1 is a double tangent plane, one of whose points of contact is a *node* and the other a *cusp* on the section of the surface by the tangent plane.

ϖ_2 is a double tangent plane, one of whose points of contact is a *node* and the other a *flecnode* on the section.

ϖ_3 is a triple tangent plane, that is a plane which touches the surface at three points. Hence the section of the surface is a *trinodal* curve.

ϖ_4 is a tangent plane, whose point of contact is a *biflecnode* on the section.

ϖ_5 is a tangent plane, whose point of contact is a *tacnode* on the section.

ϖ_6 is a tangent plane, whose point of contact is a *hyperflecnode**, one of whose tangents has bitactic contact and the other quadritactic contact with their respective branches; and therefore the

* The Italians have introduced the phrase *a point of hyperinflexion* to designate a point, the tangent at which has a contact higher than tritactic. This very convenient term may be extended to nodes, when either tangent has a contact with its own branch which is higher than tritactic. The reader must recollect that when a nodal tangent has n-tactic contact with its own branch, its contact with the curve is $(n+1)$-tactic.

latter tangent has quinquetactic contact with the surface. Hence the point of contact is a point of undulation on its own branch, and no surface of lower degree than a quintic can possess this singularity in a complete form.

The number of tangent planes of each species is known when the surface is anautotomic, but the further consideration of them and their reciprocals must be postponed for the present.

Tangential Coordinates.

12. The Boothian system of tangential coordinates may be extended to surfaces; for in this system a surface is regarded as the envelope of the plane $x\xi + y\eta + z\zeta = 1$, subject to the condition $F(\xi, \eta, \zeta) = 0$, which is called the tangential equation of the surface. When the Cartesian equation is given, the tangential equation in these coordinates is obtained by writing

$$w = x\xi + y\eta + z\zeta$$

and making (1) homogeneous by multiplying each term by the proper power of w, and then equating the discriminant of the resulting ternary quantic to zero.

In the same way it can be shown that if the tangential equation of a surface is

$$u_0 + u_1 + u_2 + \dots u_n = 0 \quad\dots\dots\dots\dots\dots\dots\dots(7),$$

where u_n is a ternary quantic of (ξ, η, ζ), the Cartesian equation can be obtained in exactly the same manner. The following additional theorems can be proved by the same methods as those employed in the theory of plane curves.

(i) *If $f(x, y, z) = 0$ is the Cartesian equation of a surface whose tangential equation is $F(\xi, \eta, \zeta) = 0$; then $f(\xi, \eta, \zeta) = 0$ is the tangential equation of a surface whose Cartesian equation is*

$$F(x, y, z) = 0.$$

(ii) *If $F(\xi, \eta, \zeta) = 0$ is the tangential equation of a surface; then $F(x/k^2, y/k^2, z/k^2) = 0$, where k is a constant, is the Cartesian equation of its reciprocal polar. Hence, the degree of the tangential equation is the same as that of the reciprocal polar.*

When a surface is expressed by means of quadriplanar coordinates, the tangential equation may also be obtained by the usual methods for finding the envelope of a plane. Thus the tangential equation of a quadric may be obtained by finding the envelope of the plane

$$\xi\alpha + \eta\beta + \zeta\gamma + \omega\delta = 0$$

subject to the condition

$$(a, b, c, d, f, g, h, l, m, n \between \alpha, \beta, \gamma, \delta)^2 = 0,$$

and the result can be expressed by the determinantal equation

$$\begin{vmatrix} a, & h, & g, & l, & \xi \\ h, & b, & f, & m, & \eta \\ g, & f, & c, & n, & \zeta \\ l, & m, & n, & d, & \omega \\ \xi, & \eta, & \zeta, & \omega, & 0 \end{vmatrix} = 0,$$

and if $(\xi, \eta, \zeta, \omega)$ be now regarded as quadriplanar coordinates of a point, the last equation is the reciprocal polar of the quadric represented by the preceding one.

Polar Surfaces.

13. The theory of polar surfaces can be developed in exactly the same way as the theory of polar plane curves. Let

$$\left.\begin{aligned} \Delta &= f\frac{d}{d\alpha} + g\frac{d}{d\beta} + h\frac{d}{d\gamma} + k\frac{d}{d\delta} \\ \Delta' &= \alpha\frac{d}{df} + \beta\frac{d}{dg} + \gamma\frac{d}{dh} + \delta\frac{d}{dk} \end{aligned}\right\} \quad \text{..............(8)},$$

also let F be any quaternary quantic of $(\alpha, \beta, \gamma, \delta)$ of degree n, and F' the same function of (f, g, h, k). Then it can be shown that

$$\frac{1}{p!}\Delta^p F = \frac{1}{(n-p)!}\Delta'^{n-p} F' \quad \text{.................(9)}.$$

Either of the expressions (9) equated to zero is the pth polar of F with respect to the point (f, g, h, k); from which it follows that the pth polar with respect to A is

$$\frac{d^p F}{d\alpha^p} = 0 \quad \text{........................(10)}.$$

If (f, g, h, k) lie on the surface, the polar plane becomes the tangent plane at this point, and its equation is

$$\Delta' F' = 0 \text{........................(11)}.$$

The curve of intersection of the surface with its pth polar is called the pth polar curve, and its degree is $n(n-p)$.

The following theorems may be proved in the same manner as the corresponding ones in the theory of plane curves.

(i) *The tangent cone from any point A touches the surface along the first polar curve of A.*

(ii) *If the equation of any surface be written in the binary form* $(\alpha, 1)^n = 0$, *where the coefficients of* α *are ternary quantics of* (β, γ, δ) *of proper degrees, its discriminant equated to zero is the equation of the tangent cone from A.*

Let the equation of the surface be given by (3); then the first polar of A is

$$n\alpha^{n-1}u_0 + (n-1)\alpha^{n-2}u_1 + (n-2)\alpha^{n-3}u_2 + \dots u_{n-1} = 0 \qquad \text{......(12)},$$

and the equation of the tangent cone from A is obtained by eliminating α between (3) and (12); and this is the discriminant of the binary quantic $(\alpha, 1)^n = 0$.

(iii) *The locus of points whose polar planes pass through a fixed point is the first polar of that point.*

(iv) *The polar plane of every point on a surface is the tangent plane at that point.*

(v) *Every polar of a point on a surface touches the surface at that point.*

If A be the point, the equation of the surface is

$$F = \alpha^{n-1}u_1 + \alpha^{n-2}u_2 + \dots u_n = 0 \qquad \text{..............(13)},$$

and $u_1 = 0$ is the tangent plane at A to (13) and also to the surface $d^pF/d\alpha^p = 0$.

(vi) *Every multiple point of order p on a surface is a multiple point of order* $p - r$ *on the rth polar, where* $p > r$.

(vii) *The first polar of every point passes through each conic node; and the axis of every binode has tritactic contact with the first polar.*

The first part follows from (13) by putting $u_1 = 0$, and differentiating with respect to D, which may be any arbitrary point; in like manner the second part follows by putting $u_2 = L\gamma^2 + 2M\gamma\delta + N\delta^2$.

(viii) *If the first polar of B has a conic node at A, the polar quadric of A is a cone whose vertex is B.*

In order that the first polar of B should have a conic node at A, it is necessary that the surface should be of the form

$$\alpha^n u_0 + \alpha^{n-1} v_1 + \alpha^{n-2} v_2 + \alpha^{n-3} u_3 + \dots u_n = 0,$$

in which case the equation of the first polar of B is

$$d(\alpha^{n-3} u_3 + \dots u_n)/d\beta = 0,$$

showing that the surface has a conic node at A. The polar quadric of A is

$$n(n-1)\alpha^2 u_0 + 2(n-1)\alpha v_1 + 2v_2 = 0,$$

which represents a cone whose vertex is B.

(ix) *The second polar of a point A passes through the points where the generators of the tangent cone from A have tritactic contact with the surface; and the number of these generators is $n(n-1)(n-2)$.*

The surface and its first and second polars with respect to A obviously intersect in $n(n-1)(n-2)$ points; let B be one of them, and let the equation of the surface be given by (3); then if w, σ and τ denote binary quantics of (γ, δ)

$$u_n = \beta^{n-1}\tau_1 + \beta^{n-2}\tau_2 + \dots \tau_n.$$

The conditions that the first and second polars of A should pass through B require that

$$u_{n-1} = \beta^{n-2}\sigma_1 + \dots \sigma_n,$$

$$u_{n-2} = \beta^{n-3} w_1 + \dots w_n.$$

Substituting these values in (3), and then putting $\gamma = \delta = 0$, it follows that (3) reduces to the form $\alpha^3(\alpha, \beta)^{n-3} = 0$, which shows that the line AB has tritactic contact with the surface at B.

(x) *The condition that a surface should have a double point is that the discriminant of its equation should vanish.*

This may be proved in exactly the same manner as the corresponding theorem for a curve. Also since a cone is the only quadric surface which has a double point, the vanishing of the discriminant of a quaternary quadric is the condition that it should represent a quadric cone.

The preceding theorem shows that only one condition is necessary in order that a surface should have a conic node, and that every additional conic node involves one additional equation of condition. Two conditions are however necessary for a binode;

since the discriminant of the nodal cone at a conic node must also vanish in order that the cone should degrade into a pair of planes.

14. Surfaces are called *autotomic** or *anautotomic* according as they do or do not possess singular points or curves.

The *class* of a surface is equal to the number of tangent planes which can be drawn to the surface through an arbitrary fixed straight line.

From analogy to a curve it might be thought that the class of a surface ought to be equal to the degree of the tangent cone from an arbitrary point, which as will presently be shown is equal to $n(n-1)$; but in order to make the theory of surfaces and curves correspond, we ought to define the class in such a manner that it is equal to the *degree* of the reciprocal polar; and we shall now show that by virtue of the above definition :—

The class of a surface is equal to the degree of its reciprocal polar.

Let the given line be perpendicular to the plane of the paper and cut it in A. In the latter take any point O as the origin of reciprocation, and produce OA to A' so that $OA \,.\, OA' = k^2$, where k is a constant. Let OP be the perpendicular from O on to any tangent plane drawn to the surface through the given line; and produce OP to P' so that $OP \,.\, OP' = k^2$. Then P' is the point on the reciprocal surface which corresponds to the tangent plane AP. Since $OA \,.\, OA' = OP \,.\, OP'$, it follows that a circle can be drawn through the points A, A', P', P; and since the angle APP' is a right angle, the angle $AA'P$ is one also; hence all the points on the reciprocal surface, which correspond to the tangent planes drawn to the original surface through the given straight line, lie on the straight line $A'P'$ which is perpendicular to OA'; also the number of points in which the line $A'P'$ cuts the reciprocal surface is equal to the number of tangent planes which can be drawn to the original surface through the given line.

15. *The class of an anautotomic surface of degree n is equal to $n(n-1)^2$.*

Let A and B be any two points on the given straight line, P the point of contact of any tangent plane which passes through

* αὐτος = self, τέμνω = I cut.

AB. Then the first polars of A and B must pass through P; hence the number of points such as P must be equal to the number of points of intersection of the surface and the first polars of two arbitrary points, which is equal to $n(n-1)^2$.

The reduction of class produced by conic nodes and other singular points will be considered later on.

Tangent Cones.

16. *The degree of the tangent cone to any surface which does not possess singular lines and curves is equal to $n(n-1)$; and the class of the cone is equal to that of the surface. Accordingly when the surface is anautotomic, the class of the cone is equal to $n(n-1)^2$.*

Let O be the vertex of the tangent cone from an arbitrary point; then if the surface does not possess any singular lines or curves, it will be possible to draw a plane section of the surface through O which does not pass through any singular points on the surface. The section of the surface is therefore an anautotomic curve of degree n, and the number of tangents which can be drawn to it from O is $n(n-1)$; and since every tangent is a generator of the cone, its degree is equal to $n(n-1)$.

If however the surface possessed a nodal curve of degree b, the plane would cut the curve in b points which are nodes on the section, and the number of ordinary tangents would be $n(n-1)-2b$, and consequently the degree of the cone would be equal to $n(n-1)-2b$.

The class of the cone is equal to the number of tangent planes which can be drawn to it through any arbitrary line through the vertex, and since each of these planes is a tangent plane to the surface, the class of the cone is equal to that of the surface.

If OP be any generator of the cone, the tangent plane at P cuts the surface in a curve having a node at P, and every tangent which is drawn to the section through P is a double tangent to the surface; and for certain positions of P, one of these double tangents will pass through O. When the section is a uninodal curve, the number of double tangents to the surface which pass through P is $n(n-1)-6=(n+2)(n-3)$; but before we can ascertain the number of double tangents which can be drawn from an external point O, the following theorem must be proved.

17. *Every double tangent drawn from a point O to a surface is a nodal generator on the tangent cone from O; and every stationary tangent is a cuspidal generator. Also the tangent cone to an anautotomic surface possesses $n(n-1)(n-2)$ cuspidal generators.*

Let OAB be a double tangent which touches the surface at A and B; then the tangent planes to the surface at A and B are tangent planes to the tangent cone from O along the generator OAB; and since these tangent planes do not in general coincide, the cone has two tangent planes which touch it along the generator OAB, and therefore the latter is a nodal generator.

In the next place let A be the vertex of the tangent cone, and let the generator AB have tritactic contact with the surface at B. Then the equation of the surface must be of the form

$$\alpha^n u_0 + \alpha^{n-1} u_1 + \ldots \alpha^2 (\beta^{n-3} v_1 + \ldots v_{n-2}) + \alpha (\beta^{n-2} w_1 + \ldots w_{n-1}) \\ + \beta^{n-1} W_1 + \ldots W_n = 0 \ldots\ldots (14),$$

for when $\gamma = \delta = 0$, (14) reduces to $\alpha^3 (\alpha, \beta)^{n-3} = 0$. Draw any plane $\delta = k\gamma$ through AB, and let P_1, $P_2 \ldots$ be the points where this plane intersects the first polar curve of A; then the lines AP_1, $AP_2 \ldots$ will be generators of the tangent cone from A. Putting $\delta = k\gamma$ in (14) it reduces to

$$\beta^{n-1} \gamma W_1' + \beta^{n-2} \gamma (\alpha w_1' + \gamma W_2') + \ldots = 0 \ldots\ldots\ldots (15),$$

where the accents denote what the quantities become when $\gamma = 1$, $\delta = k$.

Writing down the first polar of A and making the same substitution, we obtain

$$\beta^{n-2} \gamma w_1' + \beta^{n-3} (3 V_0 \alpha^2 + 2\alpha\gamma v_1' + \gamma^2 w_2') + \ldots = 0,$$

which shows that every plane through AB cuts the surface and the first polar of A in two curves of degrees n and $n-1$ to which AB is the common tangent at B. Hence these curves intersect in $n(n-1) - 2$ other points, and therefore AB is a double generator of the tangent cone. Also since $W_1 = 0$ is the only tangent plane to the surface which passes through B, it follows that this plane is the only tangent plane to the cone along AB, and therefore AB is a cuspidal generator on the cone.

We have shown in § 13 (ix) that the points of contact of the stationary tangents are the intersections of the surface and its first and second polars with respect to the vertex of the cone; hence the number of cuspidal generators is $n(n-1)(n-2)$.

18. *Through any arbitrary point* $\frac{1}{2}n(n-1)(n-2)(n-3)$ *double tangents can be drawn to an anautotomic surface.*

Let ν and μ be the degree and class of the tangent cone; δ and κ the number of its nodal and cuspidal generators, then Plücker's first equation is

$$\mu = \nu(\nu-1) - 2\delta - 3\kappa \quad \text{..................(16).}$$

We have also shown in §§ 16 and 17 that

$$\nu = n(n-1), \quad \mu = n(n-1)^2, \quad \kappa = n(n-1)(n-2),$$

whence substituting in (16) we obtain

$$\delta = \tfrac{1}{2}n(n-1)(n-2)(n-3),$$

which determines δ, and therefore the number of generators of the tangent cone which are double tangents to the surface.

The following theorems can be proved in a similar manner.

(i) *The degree of the tangent cone whose vertex lies on an anautotomic surface is* $n(n-1)-2$, *and its class is* $n(n-1)^2$.

(ii) *The cone has* $\frac{1}{2}(n-3)(n-4)(n^2+n+2)$ *nodal and* $(n-3)(n^2+2)$ *cuspidal generators.*

19. *Every generator of the tangent cone from an arbitrary point which passes through a conic node is a nodal generator; and every generator which passes through a binode is a cuspidal generator. Hence if a surface possesses C conic nodes and B binodes, its class is determined by the equation*

$$m = n(n-1)^2 - 2C - 3B \quad \text{..............(17).}$$

Let A be a conic node, O the vertex of the cone; then the tangent planes through OA to the nodal cone at the conic node are tangent planes to the surface and therefore to the tangent cone from O. Hence OA is a nodal generator of the cone.

To prove the second part it will be sufficient to use a cubic surface, since the method and the result are the same when any surface of higher degree is employed. Let the surface be

$$\delta^3 + 3\delta^2(q\alpha + V_1) + 3\delta(p\alpha\gamma + V_2) + V_3 = 0,$$

where $V_n = (\beta, \gamma)^n$. The coefficient of α is $3\delta(p\gamma + q\delta)$, which shows that A is a binode and that AB is its axis. Forming the discriminant of this surface regarded as the binary cubic $(\delta, 1)^3 = 0$, it will be found that the tangent cone from D, which may be any arbitrary point, is of the sixth degree, and that the term involving the highest power of α is $3p^2q^2\alpha^4\gamma^2$, which shows that DA is a cuspidal generator of the cone.

It therefore follows from Plücker's first equation, that the class of the cone is given by (17); and since the class of the cone is the same as that of the surface, the theorem at once follows.

20. The following additional theorems can be proved without difficulty.

(i) *The number of ordinary tangent planes which can be drawn to a surface through a line passing through a conic node is* $m-4$.

(ii) *When the line passes through two conic nodes, the number is* $m-8$.

(iii) *When the line passes through a binode, the number is* $m-3$.

(iv) *When the line passes through two binodes, the number is* $m-6$.

(v) *When the line passes through a conic node and a binode, the number is* $m-7$.

To prove (i), let O be an arbitrary point, A a conic node; then by the theory of plane curves, the number of tangent planes which can be drawn through OA to the tangent cone from O, and therefore to the surface, is $m-4$.

To prove (ii), we observe that every one of the planes considered in (i) is a tangent plane to the tangent cone from a conic node to the surface; hence $m-4$ is the class of this cone. Also if B be any other conic node, AB is a nodal generator of this cone; hence the number of tangent planes which can be drawn through AB to this cone, and therefore to the surface, is $m-8$. The remaining three theorems can be proved in a similar manner.

21. We can now complete the theory of the tangent cone drawn to a surface from an arbitrary point.

Let the surface possess C conic nodes and B binodes; let ν, μ be the degree and class of the tangent cone; δ, κ the number of its nodal and cuspidal generators; τ, ι the number of its double and stationary tangent planes. Then

$$\nu = n(n-1), \quad \mu = m = n(n-1)^2 - 2C - 3B \quad \ldots(18).$$

$$\left.\begin{aligned} \delta &= \tfrac{1}{2}n(n-1)(n-2)(n-3) + C \\ \kappa &= n(n-1)(n-2) + B \end{aligned}\right\} \quad \ldots\ldots\ldots(19).$$

By Plücker's equations

$$\left.\begin{aligned} \iota &= 3\nu(\nu-2) - 6\delta - 8\kappa \\ 2\tau &= \mu(\mu-1) - \nu - 3\iota \\ \mu &= \nu(\nu-1) - 2\delta - 3\kappa \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(20).$$

Substituting the values of ν, δ, κ from (18) and (19) in the first of (20), we obtain

$$\iota = 4n(n-1)(n-2) - 6C - 8B \quad \ldots\ldots\ldots\ldots(21).$$

Eliminating δ and ι between (20) we obtain

$$\begin{aligned} 2\tau &= (\mu-5)^2 + 8\nu - 3\kappa - 25 \\ &= \{n(n-1)^2 - 2C - 3B - 5\}^2 - n(n-1)(3n-14) - 25 - 3B \\ &\qquad \ldots\ldots(22). \end{aligned}$$

Equation (21) determines the number of stationary tangent planes, and (22) the number of double tangent planes which the tangent cone possesses.

22. *The stationary tangent planes to the tangent cone touch the surface at the points where the curve of contact intersects the spinodal curve.*

To prove this it will be sufficient to employ the cubic surface

$$\alpha^3 u_0 + 3\alpha^2 u_1 + 3\alpha(\beta\delta + v_2) + \beta^2\delta + \beta\delta w_1 + w_3 = 0 \ldots(23).$$

The plane ABC or δ touches the surface at the point B, and if in (23) we put $\delta = 0$, it follows that B is a cusp on the section and BC is the cuspidal tangent. This shows that B is a point on the spinodal curve. Writing (23) in the form

$$\alpha^3 u_0 + 3\alpha^2 u_1 + 3\alpha u_2 + u_3 = 0$$

the tangent cone from A is

$$(u_0 u_3 - u_1 u_2)^2 = 4(u_0 u_2 - u_1^2)(u_1 u_3 - u_2^2),$$

and if in this we put $\delta = 0$, it will be found that γ^3 is a factor, which shows that ABC is a stationary tangent plane to the cone along AB.

23. *To find the maximum number of double points which a surface can have.*

It is obvious that a surface, like a curve, must have a maximum number of double points; also since a conic node reduces the class by 2 and a binode by 3, it follows that all the double points may be conic nodes, but only a limited number can be binodes. I shall now establish a formula for calculating this maximum number.

Putting $B=0$ in (22), we obtain

$$2\tau=\{n(n-1)^2-2C-5\}^2-n(n-1)(3n-14)-25 \quad \ldots(24).$$

Since τ is equal to the number of double tangent planes which can be drawn to the tangent cone from an arbitrary point, τ must be zero or a positive integer, but can never be negative; also

$$n(n-1)(3n-14)+25$$

is an odd integer. The conditions of the problem will therefore be satisfied by taking

$$n(n-1)^2-2C-5=\pm k,$$

where k is the least odd integer whose square is not less than

$$n(n-1)(3n-14)+25.$$

The sign of k must be determined from the conditions that m and ι must both be positive, and the value of m must not be less than a certain limit. Should the least value of k fail to satisfy these conditions, a greater one must be taken.

The maximum values of C and the corresponding values of m for the surfaces of the first twelve degrees are given in the following table:

n	3	4	5	6	7	8	9	10	11	12
C	4	16	34	66	114	181	270	383	524	696
m	4	4	12	18	24	30	36	44	52	60

The preceding results show that it is possible for a cubic surface to have three conic nodes and one binode; but if the equation of a cubic surface having four conic nodes be written down, and we attempt to make one of them a binode, it will be found that the surface degrades into an improper one. We shall hereafter show that the point constituents of a nodal line on a cubic surface are $C=3$, $B=1$; from which it appears that certain combinations of conic nodes and binodes cannot exist when the singularities are isolated, although they may exist in the form of a compound singularity.

Binodes and Unodes.

24. The theory of these singularities will be required when discussing cubic surfaces; and we shall commence with the following theorem due to Segre*.

When a surface has a multiple point of order p at A, the conditions that the singularity should possess an additional conic node indefinitely near to A in the direction AB are, (i) *that AB should be a nodal generator on the tangent cone at A;* (ii) *that AB should be a line of closest contact.*

Let the equation of the surface be

$$\alpha^{n-p}u_p + \alpha^{n-p-1}u_{p+1} + \dots u_n = 0 \quad \text{...........(25)},$$

where

$$u_p = \beta^p v_0 + \beta^{p-1}v_1 + \dots,$$

$$u_{p+1} = \beta^{p+1}w_0 + \beta^p w_1 + \dots.$$

When the surface has a conic node at a point B' on AB, the line AB must intersect the surface in p points at A and two points at B'; accordingly when B' moves up to coincidence with A the line AB must intersect the surface in $p+2$ coincident points at A. Putting $\gamma = \delta = 0$ in (25) this requires that $v_0 = w_0 = 0$. Also the first polar of any arbitrary point must pass through B' and have a multiple point of order $p-1$ at A; accordingly when coincidence takes place AB must intersect the first polar in p coincident points at A. This condition requires that $v_1 = 0$, which shows that AB is a nodal generator on the cone u_p and an ordinary one on the cone u_{p+1}.

Since every multiple point of order p gives rise to a multiple point of order $p-r$ on the rth polar, the theorem can easily be extended to the singularity formed by the union of two multiple points of orders p and p'.

25. In (25) let $p = 2$, and we obtain

$$\alpha^{n-2}v_2 + \alpha^{n-3}(\beta^2 w_1 + \beta w_2 + w_3) + \alpha^{n-4}u_4 + \dots u_n = 0 \text{...(26)},$$

which represents a surface having a binode at A, whose axis AB has quadritactic contact with the surface. This singularity is therefore equivalent to two conic nodes, and reduces the class by 4.

* *Annali di Matematica*, Serie II. vol. xxv. p. 28.

The theory of binodes whose axes have a contact with the surface which is higher than quadritactic, as well as that of binodes when one or more of the lines of closest contact coincide could easily be worked out; but it does not seem worth while to go into any further details*.

26. By means of the Theory of Birational Transformation which will be explained in Chapter V, it will be shown that the equation of a surface having at A the singularity formed by the union of a conic node and a binode is

$$\alpha^{n-4}(\alpha v_1 + W\beta^2)(\alpha w_1 + W\beta^2) + \alpha^{n-3}(\beta w_2 + w_3) + \alpha^{n-4}(\beta^3 W_1 + \ldots W_4) + \alpha^{n-5}u_5 + \ldots u_n = 0 \ldots (27),$$

where W is a constant, and all the suffixed letters except the u's denote binary quantics of (γ, δ). The section of (27) by the plane $v_1 = w_1$ is a curve having a rhamphoid† cusp at A; but when $n = 3$ the result fails, as might be expected, since the properties of the binode we are considering are different from those of the corresponding singularity on a cubic surface.

27. Let us now suppose that there is another conic node on AC indefinitely near A; then the form of (26) shows that v_2 must not contain γ and the coefficient of α^{n-3} must not contain γ^3. Hence (26) becomes

$$\alpha^{n-2}\delta^2 + \alpha^{n-3}(\beta^2 w_1 + \beta w_2 + \delta W_2) + \alpha^{n-4}u_4 + \ldots u_n = 0 \ldots (28),$$

which is the equation of a surface having a unode at A.

Since the plane δ intersects the cone $\beta^2 w_1 + \beta w_2 + w_3 = 0$ in AB, AC and a third line AE, it might be thought that there is a third conic node on AE, making altogether four conic nodes; but since it will be shown in the next section that a unode reduces the class by 6, it cannot be composed of more than three conic nodes. The explanation is that if A, B', C' are the nodes before coincidence, the line AE is the ultimate position of $B'C'$.

28. The equation of a surface having a unode at A and the plane ABC as the uniplane may also be written

$$\alpha^{n-2}\delta^2 + \alpha^{n-3}(\delta^3 V_0 + \delta^2 V_1 + \delta V_2 + V_3) + \alpha^{n-4}u_4 + \ldots u_n = 0 \ldots (29),$$

where $V_n = (\beta, \gamma)^n$; from which it follows that the section of the surface by the uniplane is a curve of the nth degree having

* See Rohn, *Math. Annalen*, vol. XXII. p. 124; *Sachsiche Bericht*, 1884.

† From ῥαμφος = a beak, εἶδον = appeared.

a triple point* at A, the tangents at which are given by the equation $V_3 = 0$. There are accordingly three primary species of unodes, according as the triple point is of the first, second or third kinds; and we shall show that their respective constituents are

$$C = 3,\ B = 0;\quad C = 2,\ B = 1;\quad C = 1,\ B = 2.$$

Also since the characteristics of a unode on a quartic surface are the same as on any other surface, we shall employ the surface

$$\delta^3 (\alpha V_0 + W_1) + 3\delta^2 (\alpha^2 + \alpha V_1 + W_2) + 3\delta (\alpha V_2 + W_3) + \alpha V_3 + W_4 = 0 \qquad \text{......(30)},$$

in which A is the unode and ABC the uniplane.

Writing down the discriminant of (30) regarded as the binary cubic $(\delta, 1)^3 = 0$, it follows that the tangent cone from D is of the 10th degree, and that the term containing the highest power of α is $\alpha^7 V_3$, which shows that DA is a triple generator of the first kind on the cone. Accordingly the number of tangent planes which can be drawn through DA to the surface is $m - 6$.

The tangent cone from A is

$$(\delta^3 V_0 + 3\delta^2 V_1 + 3\delta V_2 + V_3)^2 = 12\delta^2 (\delta^3 W_1 + 3\delta^2 W_2 + 3\delta W_3 + W_4) \qquad \text{......(31)},$$

which is of the 6th degree and has three nodal generators which are the lines of intersection of the planes $\delta = 0$, $V_3 = 0$; hence this cone is of the 24th class; and therefore 24 tangent planes can be drawn to it through DA. But each of these 24 tangent planes is a tangent plane to the surface; accordingly $m - 6 = 24$, giving $m = 30$. Also by § 15, the class of an anautotomic quartic surface is 36, and therefore the reduction of class produced by a unode is $36 - 30 = 6$.

When two of the tangents at the triple point on the section by the uniplane coincide, we may take $V_3 = \beta^2\gamma$. In this case AB is a nodal and AC a tacnodal generator of the cone (31), so that its class is still equal to 24. But the line DA now becomes a triple generator of the second kind on the tangent cone from D; hence the number of tangent planes which can be drawn†

* There are three kinds of triple points, which are of the first, second or third kinds, according as all the tangents are distinct, two coincident and one distinct, or all three coincident. I prefer this definition to the one given in *Cubic and Quartic Curves*, § 159.

† The number of tangents which can be drawn from a multiple point will be discussed in Chapter IV, but all the above results can be obtained directly by

through it is $m-5$. Accordingly $m-5=24$, giving $m=29$. The coincidence of two of the tangents therefore reduces the class by 7, and changes one of the three constituent conic nodes into a binode.

In the same way it can be shown that when all three tangents at the triple point coincide, the reduction of class is 8, and the constituents of the unode are $C=1$, $B=2$.

Tropes.

29. *A trope* is a tangent plane which touches a surface along a plane curve.*

When the curve of contact is a conic, the tangent plane is called a *conic trope*; and in like manner surfaces may possess cubic, quartic, &c., tropes.

The equation of a surface having a conic trope is

$$\alpha^n u_0 + \alpha^{n-1} u_1 + \dots \alpha^2 u_{n-2} + \alpha u_{n-1} + \Omega^2 u_{n-4} = 0,$$

where Ω is a quadric cone. The plane α, which is the conic trope, touches the surface along the conic (α, Ω) and intersects it in the residual curve (α, u_{n-4}). Also the cones Ω and u_{n-1} have $2(n-1)$ common generators; and if AB be one of them,

$$u_{n-1} = \beta^{n-2} w_1 + \dots w_{n-1}, \quad \Omega = \beta v_1 + v_2, \quad u_{n-4} = \beta^{n-4} W_0 + \dots W_{n-4},$$

which shows that the highest power of β is the $(n-2)$th, and that its coefficient is of the form $l\alpha^2 + \alpha w_1 + v_1^2 W_0$: hence, B is a conic node. Accordingly there are $2(n-1)$ conic nodes on the conic of contact, which are situated at the points of intersection of the cones Ω, u_{n-1} and the plane α. The reciprocal singularity is therefore a conic node of a very special character, since there are $2(n-1)$ tangent planes to the nodal cone which touch the reciprocal surface along a conic.

In like manner the reciprocal of a conic node is a conic trope of a special character, which consists of a plane touching the reciprocal surface along a conic which is the reciprocal of the nodal cone, and intersecting it in a residual curve which is the reciprocal of the tangent cone from the node. If the original

calculating the number of tangents which can be drawn from the point A to the plane quintic curve

$$\alpha^2 u_3 + \alpha u_4 + u_5 = 0.$$

* From τροπή = a turn, from which the word tropic is derived.

surface has only one conic node, the degree of the reciprocal surface is $n(n-1)^2-2$, and consequently the degree of the residual curve is $n(n-1)^2-6$. This shows that the class of the tangent cone from the node on the original surface is $n(n-1)^2-6$; also its degree is equal to the number of tangents which can be drawn from the node to any plane section of the original surface through the node, and is therefore equal to $n(n-1)-6=(n+2)(n-3)$. This is the class of the residual curve on the reciprocal surface; also, since the tangent planes to the nodal cone along the lines of closest contact are also tangent planes along the same lines to the tangent cone from the node, the conic of contact and the residual curve touch one another at six points which are the reciprocals of these tangent planes.

It thus appears that there is an important distinction between the theory of the singularities of plane curves and surfaces; for the reciprocal of an *ordinary* node on a plane curve is an *ordinary* double tangent and *vice versâ*; but the reciprocal of an ordinary conic node on a surface is a special kind of conic trope and *vice versâ*.

It is also possible for a conic trope to touch the surface at points lying on the residual curve; and such points must be nodes or cusps on the latter, although they are not necessarily singular points on the surface. Double points on the residual curve give rise to double and stationary tangent planes to the tangent cone from the node on the original surface.

Bitropes.

30. The reciprocal of a binode is called a bitrope; but it is a bitrope of a special kind. Since any arbitrary section through a binode is a nodal curve, the degree and class of the tangent cone from the node are $n(n-1)-6$ and $n(n-1)^2-6$ as in the case of a conic node, and these are respectively the class and degree of the residual curve on the reciprocal surface. The two biplanes are triple tangent planes to the tangent cone from the node, and respectively touch the cone along the lines of closest contact; also since the residual curve is the reciprocal of the tangent cone from the node, the former must have a pair of triple points P and Q which are the reciprocals of the biplanes. The axis of the binode reciprocates into the line PQ, and the three coincident points at A into three coincident tangent planes through PQ; hence the bitrope osculates the reciprocal surface at every point of PQ. Accordingly the section of the reciprocal surface by the bitrope consists of the line PQ repeated three times, and a residual curve of degree $n(n-1)^2 - 6$. The degree of the reciprocal surface is $n(n-1)^2 - 3$.

The reciprocal polar of a unode is called a *unitrope.*

31. *When a fixed plane touches a surface of the nth degree along a straight line, there are in general $n-1$ conic nodes on the line; but when the plane has a higher contact, the $n-1$ points are in general binodes.*

Let CD be the line, BCD the fixed plane; then the equation of the surface is

$$\alpha^n u_0 + \alpha^{n-1} u_1 + \dots \alpha^2 u_{n-2} + \alpha u_{n-1} + \beta^2 U_{n-2} = 0 \quad \text{......(32).}$$

The line CD intersects the cone u_{n-1} in $n-1$ points, and if C be one of them

$$u_{n-1} = \beta^{n-1} v_0 + \beta^{n-2} v_1 + \dots \beta v_{n-2} + \delta w_{n-2},$$

hence the highest power of γ in (32) is the $(n-2)$th, which shows that C is a conic node.

When the plane α has a higher contact, we must replace the last term of (32) by $\beta^s U_{n-s}$, and the coefficient of γ^{n-2} will be of the form $p\alpha^2 + \alpha(q\beta + r\delta)$, which shows that C is a binode.

Lines drawn on Surfaces.

32. A straight line cannot cut an irreducible plane curve of the nth degree in more then n points; for if it did, the curve would degrade into the straight line and a curve of lower degree. If however a straight line cuts a surface of the nth degree in $n+1$ points, the surface does not degrade, but the line lies altogether in the surface. Similar considerations apply to plane and twisted curves on surfaces. It is also possible for a surface to possess singular lines and curves; that is to say lines and curves such that if a plane be drawn through any point P on the line or curve, the section has a singular point at P, whose character determines that of the singular line or curve. We shall now consider the elementary theory of lines lying in a surface.

33. When a straight line lies altogether in a surface, the tangent plane at any point usually rotates about the line as the point of contact moves along it, and such a plane is therefore a *torsal** tangent plane, and the line is called a *torsal line.* It is also possible for the tangent plane to be *fixed* in space; and it has been shown in § 31 that such a line is a *singular* one and has in general $n-1$ conic nodes lying upon it. When the tangent plane osculates the line, so that every plane section has a point of inflexion where the line cuts the section, the tangent plane is called *oscular.* Similar considerations apply to multiple lines, the tangent planes along which may be torsal, fixed or oscular, or may even have a higher contact with their respective sheets.

34. *A surface of a higher degree than the third cannot in general possess straight lines lying in it.*

We have shown in § 11 that every surface possesses a determinate number of triple tangent planes, and that the section of the surface by such a tangent plane is a trinodal curve; and since a trinodal cubic curve consists of three straight lines forming a triangle, every cubic surface possesses a determinate number of straight lines lying in it. But when the surface is a quartic, a straight line cannot form part of a plane section unless the latter degrades into an anautotomic cubic curve and the straight line; and this involves the four conditions that the three nodes on the section should lie in the same straight line, which cannot in general be satisfied.

35. *Every plane containing a straight line lying in a surface touches the latter at $n-1$ points.*

The section of the surface by any plane through the straight line consists of the line and a curve of degree $n-1$; and since the $n-1$ points where the line and the curve cut one another are nodes on the section, the plane has $n-1$ points of contact.

Nodal Lines.

36. The tangent plane at any point on a nodal line may be fixed or torsal. We thus obtain three primary species of nodal lines.

* When a *fixed* plane touches a surface along a line, Cayley calls the latter a *torsal* line. This is a singularly inappropriate definition, since *torsal* is derived from *torsi* the perfect of *torquere*, to twist; whereas the essential feature of such a line is that the tangent plane is devoid of twisting.

I. Both tangent planes torsal.

II. One tangent plane fixed, the other torsal.

III. Both tangent planes fixed.

When both tangent planes coincide at every point on the line, the latter becomes a *cuspidal* line. There are consequently two species of cuspidal lines, according as the tangent plane is fixed or torsal.

Nodal lines of the first or second kinds possess a species of singular points called *pinch points*, which arise in the following manner. A point P which is constrained to move along a straight line possesses one degree of freedom, and consequently its position on the line is completely determined by a single parameter θ; hence the equation of any plane section through P contains this parameter, and therefore the value of θ may be chosen so that the node at P on the section changes its character and becomes a cusp.

37. *A nodal line of the first or second kinds on a surface of the nth degree possesses* $2n-4$ *pinch points; but when the line is of the second kind, the pinch points coincide in pairs so that there are only* $n-2$ *apparent pinch points.*

Let AB be the nodal line, then the equation of the surface must be of the form

$$(U, V, W \between \gamma, \delta)^2 = 0 \text{.....................} (33),$$

where U, V, W are quaternary quantics of all the coordinates of degree $n-2$. The discriminant of (33) equated to zero is

$$V^2 = UW \text{..........................} (34),$$

which is a surface of degree $2n-4$, and therefore the line AB intersects (34) in $2n-4$ points. Let A be one of them, then

$$U = A^2\alpha^{n-2} + \alpha^{n-3}U_1 + \dots,$$
$$V = AC\alpha^{n-2} + \alpha^{n-3}V_1 + \dots,$$
$$W = C^2\alpha^{n-2} + \alpha^{n-3}W_1 + \dots,$$

where $U_n \equiv V_n \equiv W_n \equiv (\beta, \gamma, \delta)^n$. Substituting these values in (33) it follows that the coefficient of α^{n-2} is $(A\gamma + C\delta)^2$, which shows that A is a pinch point. Hence these points are those in which the nodal line cuts the discriminantal surface (34), and there are consequently $2n-4$ of them.

When the nodal line is of the second kind, let γ be the fixed tangent plane; then since γ must have tritactic contact with the surface at every point of AB, it follows that

$$V = B\alpha^{n-2} + \alpha^{n-3} V_1 + \dots,$$
$$W = \delta\,(C\alpha^{n-3} + \alpha^{n-4} W_1 + \dots),$$

but if A be one of the points in which (34) is cut by AB, it follows that $B = 0$, and the coefficient of α^{n-2} in (33) is $A^2\gamma^2$, which shows that A is a pinch point. The term involving the highest power of α in (34) is now $A^2 C\alpha^{2n-5}\delta$, which shows that ABC is the tangent plane to (34) at A, and therefore the line AB touches the surface at A. Hence the pinch points coincide in pairs, and there are consequently only $n-2$ apparent pinch points.

37 A. *At a real pinch point, a nodal line changes from a crunodal to an acnodal one.*

Let the equation of the surface be

$$\alpha^{n-2}\delta^2 + \alpha^{n-3}\{\beta\,(A\gamma^2 + 2B\gamma\delta + C\delta^2) + v_3\} + \dots = 0,$$

on which A is a pinch point. Change the tetrahedron to $A'BCD$, where A' is a point on AB, $\beta = \lambda\alpha$ is the equation of the plane $A'CD$, and λ is a small quantity whose squares &c. are to be neglected. The condition that the nodal tangent planes at A' should be real is that

$$-A\lambda > 0.$$

Now λ is a small positive quantity when A' lies between A and B, and negative when A' lies on the side of A remote from B; if therefore the preceding inequality is satisfied when λ is positive, the planes will be real; but they will be imaginary when λ is negative.

38. *The tangent plane at a pinch point touches the first polar.*

Let A be a pinch point and ABC the tangent plane, then the equation of the surface is

$$\alpha^{n-2}\delta^2 + \alpha^{n-3}(\beta v_2 + v_3) + \dots = 0,$$

and the first polar of D, which is any arbitrary point, is

$$2\alpha^{n-2}\delta + \alpha^{n-3}(\beta v'_2 + v'_3) + \dots = 0,$$

where $v'_n = dv_n/d\delta$, which proves the theorem.

39. A pinch point, in common with all singular points of the same character, is an *incident* of a nodal line or curve, which cannot be created nor annihilated by assuming any relations between the constants of the surface, although such relations may apparently make some of them disappear by causing them to coincide. It is of course possible to make the nodal tangents at an arbitrary point coincide, but this does not introduce an additional pinch point but alters the character of the nodal line or of the surface. For example, the equation of a cubic surface on which AB is a nodal line and A and B pinch points is

$$p\alpha\gamma^2 + q\beta\delta^2 + v_3 = 0 \quad\dots\dots\dots\dots\dots\dots (34\,\text{A}).$$

Let the plane $\alpha' = \alpha - \lambda\beta = 0$ cut AB in B'; then referring the surface to the tetrahedron of reference $AB'CD$, (34 A) becomes

$$p\alpha'\gamma^2 + \beta(p\lambda\gamma^2 + q\delta^2) + v_3 = 0.$$

If the cubic has a third pinch point at B', $p = 0$ or $q = 0$, in either of which cases (34 A) becomes a cuspidal cubic cone.

40. *Every tangent plane to a nodal line of the first kind touches the surface at $n - 2$ points.*

The equation of the surface may be written in the form

$$Q_0\alpha^{n-2}\gamma\delta + \alpha^{n-3}\{\beta(P_1\gamma^2 + Q_1\gamma\delta + R_1\delta^2) + v_2\} + \dots$$
$$+ \beta^{n-2}(P_{n-2}\gamma^2 + Q_{n-2}\gamma\delta + R_{n-2}\delta^2) + \dots = 0.$$

Writing herein $\alpha' + \lambda\beta$ for α, the coefficient of β^{n-2} is

$$Q_0\lambda^{n-2}\gamma\delta + \lambda^{n-3}(P_1\gamma^2 + Q_1\gamma\delta + R_1\delta^2) + \dots P_{n-2}\gamma^2 + Q_{n-2}\gamma\delta + R_{n-2}\delta^2 = 0.$$

Hence, if λ be determined by the equation

$$\lambda^{n-3}R_1 + \lambda^{n-4}R_2 + \dots R_{n-2} = 0,$$

the plane γ will touch the surface at $n - 3$ other points, which together with A make $n - 2$ points. Similarly the plane δ touches the surface at the point A and the $n - 3$ points determined by the equation

$$\lambda^{n-3}P_1 + \lambda^{n-4}P_2 + \dots P_{n-2} = 0.$$

41. The equation of a surface having a nodal line of the third kind is of the form

$$(\alpha, \beta)^{n-2}v_2 + (P, Q, R, S \between \gamma, \delta)^3 = 0,$$

where P, Q, R, S are quaternary quantics of all the coordinates of degrees $n - 3$. The line has no pinch points, but it has $n - 2$ cubic nodes or triple points, which are determined by the equation $(\alpha, \beta)^{n-2} = 0$. If A be one of these points, the first term must be of the form $(\alpha, \beta)^{n-3}\beta v_2$, which shows that there is a tangent cone at A whose equation is of the form

$$p\beta v_2 + v_3 = 0,$$

and is therefore a cubic cone, on which AB is a nodal generator.

42. *If three surfaces of degrees l, m, n intersect in a straight line, which is a multiple line of orders p, q, r on each of them respectively; then the number of their ordinary points of intersection is*

$$lmn - lqr - mrp - npq + 2pqr.$$

In order to prove this theorem, we shall employ a method very successfully used by Salmon*, which depends upon the principle that the number of points of intersection of three surfaces is an invariable quantity; in other words a point of intersection cannot be created nor annihilated by means of any relations between the constants, or by making the surfaces degrade into improper ones. We may therefore replace the surface S_l by p planes U_p passing through the given straight line, and another surface S_{l-p} which does not contain the line. Treating each of the other surfaces in the same way, we have to find the number of points of intersection of the compound surfaces $S_{l-p}U_p$, $S_{m-q}U_q$, and $S_{n-r}U_r$ which do not lie in the line. Now S_{l-p}, S_{m-q} and S_{n-r} intersect in

$$(l-p)(m-q)(n-r)$$

points; also S_{l-p}, S_{m-q} and U_r intersect in $(l-p)(m-q)r$ points; hence the total number of points of intersection which do not lie in the line is

$$\begin{aligned}(l-p)(m-q)(n-r) + (l-p)(m-q)r& \\ + (m-q)(n-r)p + (n-r)(l-p)q& \\ = lmn - lqr - mrp - npq + 2pqr& \text{.............(35)},\end{aligned}$$

which shows that the number of points absorbed by the straight line is

$$lqr + mrp + npq - 2pqr.$$

If the three surfaces have a common straight line, $p = q = r = 1$, and the number of their ordinary points of intersection is

$$lmn - l - m - n + 2 \text{....................(36)}.$$

If however the line is a nodal line on the surface l, and an

* *Camb. and Dublin Math. Journ.* vol. II. p. 65.

ordinary line on the other two, $p = 2$, $q = r = 1$, and the number of points is

$$lmn - l - 2m - 2n + 4 \quad\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

43. *A nodal line of the first or second species on a surface of the nth degree reduces the class by* $7n - 12$.

Since a nodal line on a surface gives rise to an ordinary line on the first polar, we must put $l = m = n - 1$; $p = q = 1$, $r = 2$; and (35) becomes

$$n(n-1)^2 - 5n + 8.$$

This would give the reduction of class were it not for the pinch points; but we have shown in § 38 that the tangent plane at a pinch point touches the first polar, hence every pinch point absorbs an additional point of intersection. Accordingly the total number of ordinary points of intersection is

$$n(n-1)^2 - 5n + 8 - (2n - 4) = n(n-1)^2 - 7n + 12,$$

and therefore the reduction of class is $7n - 12$.

It follows from the preceding investigation that before the results of § 42 can be employed to determine the reduction of class produced by a singular line on a surface, it is necessary to examine the intersection of the surface and its first polar at the singular points on the line. Hence the above result is not true in the case of a nodal line of the third kind, since such lines possess cubic nodes but no pinch points.

44. *If a surface of the nth degree possesses a multiple line of order* $n - 2$, *it has* $2(3n - 4)$ *other lines lying in it.*

If AB be the multiple line, the equation of the surface is

$$\alpha^2 v_{n-2} + \alpha\beta w_{n-2} + \beta^2 \sigma_{n-2} + \alpha v_{n-1} + \beta w_{n-1} + v_n = 0,$$

and the section of the surface by the plane $\delta = k\gamma$ consists of AB repeated $n - 2$ times and the conic

$$(A, B, C, A', B', C' \between \alpha, \beta, \gamma)^2 = 0,$$

where A, B, C' are polynomials of k of degree $n - 2$; A', B' of degree $n - 1$; and C of degree n. If the plane is a tangent plane, the conic must degrade into a pair of straight lines, the condition for which is that its discriminant should vanish, which furnishes an equation of degree $3n - 4$ in k. Hence the surface contains twice this number of lines.

45. *If an anautotomic surface contains a straight line, the number of tangent planes which can be drawn through it is*

$$(n+2)(n-2)^2.$$

If AB be the line, the equation of the surface is

$$\alpha^{n-1}v_1 + \alpha^{n-2}(\beta w_1 + w_2) + \ldots = 0,$$

hence AB is a line lying in the first polar of every point on the line. Accordingly by (36) the number of points of intersection of the surface and its first polars with respect to A and B which are absorbed by the line is $3n-4$, and therefore the number of remaining points is

$$n(n-1)^2 - 3n + 4 = (n+2)(n-2)^2.$$

Either of these theorems show that 10 lines intersect every line lying in a cubic surface.

On the Intersections of Surfaces.

46. Three surfaces of degrees l, m, and n intersect in lmn points, but when three or more surfaces are given by a set of determinants, it frequently happens that they possess a common curve; and we shall now show how to determine the degree of this curve.

If u, u'; v, v'; w, w' be quaternary quantics of degrees l, m, n respectively, the degree of the common curve of intersection of the surfaces included in the set of determinants

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\| = 0$$

is $$mn + nl + lm.$$

The three surfaces are

$$vw' = wv', \quad wu' = uw', \quad uv' = vu' \quad \ldots\ldots\ldots\ldots (38),$$

and are of degrees $m+n$, $n+l$ and $l+m$ respectively; and the first and second surfaces intersect in a curve of degree

$$(m+n)(n+l).$$

But this curve is a compound curve consisting of the curve of intersection of the surfaces $w=0$, $w'=0$, which is of degree n^2, and a residual curve of degree

$$(m+n)(n+l) - n^2 = mn + nl + lm.$$

The coordinates of points on the curve $w = w' = 0$ obviously do not satisfy the third of (38); hence the residual curve alone is the one common to the three surfaces.

47. *The four surfaces included in the set of determinants*

$$\left\|\begin{matrix} u, & u', & u'', & u''' \\ v, & v', & v'', & v''' \\ w, & w', & w'', & w''' \end{matrix}\right\| = 0,$$

where the u's, v's and w's are of degrees l, m and n respectively, possess a common curve of intersection of degree

$$l^2 + m^2 + n^2 + mn + nl + lm.$$

Let $A = vw' - wv', \quad B = wu' - uw', \quad C = uv' - vu'$ ...(39),

then the surfaces formed by omitting the fourth and third columns respectively are

$$\left.\begin{matrix} Au'' + Bv'' + Cw'' = 0 \\ Au''' + Bv''' + Cw''' = 0 \end{matrix}\right\} \text{..................(40)}.$$

The two surfaces (40) are each of degree $l + m + n$, and their curve of intersection consists of the common curve of intersection of the three surfaces $A = 0$, $B = 0$, $C = 0$, which has been shown to be of degree $mn + nl + lm$, and a residual curve of degree

$$l^2 + m^2 + n^2 + mn + nl + lm.$$

Now if we write down the identity

$$Au' + Bv' + Cw' = 0 \text{...................(41)},$$

and eliminate A, B and C from (40) and (41), we shall obtain the determinant formed by omitting the first column, and the determinantal surface thereby formed will be satisfied by all values of the coordinates which satisfy (40) but which do not make A, B and C vanish; that is to say the coordinates of all points on the residual curve. Hence the determinantal surface formed by omitting the first column contains the residual curve. In like manner by writing down the identity

$$Au + Bv + Cw = 0 \text{ ....(42)},$$

and eliminating A, B and C between (40) and (42), it can be shown that the determinantal surface formed by omitting the second column also contains the residual curve. Hence the four surfaces included in the set of determinants intersect in a common curve of the degree above mentioned.

48. *The six surfaces included in the set of determinants*

$$\left\| \begin{matrix} u, & v, & w, & t \\ u', & v', & w', & t' \end{matrix} \right\| = 0,$$

where the u's, v's, w's and t's are quantics of degrees l, m, n and p, intersect in

$$lmn + mnp + npl + plm$$

common points.

The determinant is formed by eliminating the constant k between the four equations

$$\left. \begin{matrix} u = ku', & v = kv' \\ w = kw', & t = kt' \end{matrix} \right\} \quad \text{......................(43)},$$

and the condition that (43) should be satisfied by the same values of the coordinates is that the eliminant of (43) should vanish. Now it is known* that the highest power of k in the eliminant is equal to

$$lmn + mnp + npl + plm;$$

hence this is the number of sets of values of the coordinates which satisfy (43), and therefore the determinantal surfaces intersect in this number of common points.

When u, v, &c. are planes, the surfaces consist of six quadrics which possess four common points of intersection. This may be verified as follows. The three quadrics formed by omitting the last column intersect in a twisted cubic curve; and the quadrics $uv' = vu'$ and $ut' = tu'$ intersect in the line $u = 0$, $v = 0$ and a second twisted cubic, and the points of intersection of the two twisted cubics are those common to the system. Now if three quadric surfaces possess a common straight line, it appears from (36) that the latter absorbs four out of their eight points of intersection; hence the two twisted cubics intersect in four points, which are the ones in question.

The Hessian.

49. The Hessian of a surface is the locus of points whose polar quadrics with respect to the surface are cones.

The locus of the vertices of these cones is a second surface called the *Steinerian*.

* See Salmon's *Higher Algebra*, 4th edition, § 78. If P, Q, R, S be four quaternary quantics of degrees l, m, n, p respectively, the eliminant is a homogeneous function of degree mnp of the coefficients of P; of degree npl of those of Q; of degree plm of those of R; and of degree lmn of those of S.

Let us temporarily employ $(\xi, \eta, \zeta, \omega)$ to denote current coordinates; then the polar quadric of any point $(\alpha, \beta, \gamma, \delta)$ is

$$\left(\xi \frac{d}{d\alpha} + \eta \frac{d}{d\beta} + \zeta \frac{d}{d\gamma} + \omega \frac{d}{d\delta}\right)^2 F(\alpha, \beta, \gamma, \delta) = 0 \quad \ldots(44).$$

Let $$a = \frac{d^2F}{d\alpha^2}, \quad f = \frac{d^2F}{d\beta\, d\gamma}, \quad l = \frac{d^2F}{d\alpha\, d\delta}, \text{ \&c., \&c.} \quad \ldots(45),$$

then (44) may be written in the form

$$a\xi^2 + b\eta^2 + c\zeta^2 + d\omega^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta + 2l\xi\omega + 2m\eta\omega + 2n\zeta\omega = 0 \ldots(46),$$

or in the abbreviated one

$$(a, b, c, d, f, g, h, l, m, n \between \xi, \eta, \zeta, \omega)^2 = 0 \quad \ldots\ldots(47).$$

The discriminant of (46) may be expressed in the form of the symmetrical determinant*

$$H = \begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(48),$$

the vanishing of which expresses the condition that (46) should reduce to a cone.

If the original surface is of degree n, each of the constituents of H are of degrees $n-2$; *hence the degree of the Hessian is* $4(n-2)$.

The Hessian may be expressed in a variety of forms, one of the most convenient of which is the following. Let

$$\left.\begin{array}{lll} A = bc - f^2, & B = ca - g^2, & C = ab - h^2 \\ A' = gh - af, & B' = hf - bg, & C' = fg - ch \end{array}\right\} \ldots\ldots(49),$$

also let Δ be the determinant formed by erasing the last row and column in H; then

$$H = \Delta d - (A, B, C, A', B', C' \between l, m, n)^2 \quad \ldots\ldots(50).$$

The determinant obtained by putting $d = 0$ is a well known one, since it expresses the condition that the straight line

$$l\alpha + m\beta + n\gamma = 0$$

should touch the conic

$$(a, b, c, f, g, h \between \alpha, \beta, \gamma)^2 = 0;$$

* The letter n is used in two different senses in this and the following sections; but the reader will find no difficulty in avoiding confusing them.

hence the last term of (50) equated to zero is the tangential equation of the conic.

50. *The Hessian passes through every double point; also its curve of intersection with the surface is the spinodal curve.*

The equation of a surface passing through A is

$$\alpha^{n-1}u_1 + \alpha^{n-2}u_2 + \ldots u_n = 0 \quad \text{..............(51)},$$

and the polar quadric of A is

$$(n-1)\,\alpha u_1 + u_2 = 0 \quad \text{....................(52)}.$$

When A is a double point $u_1 = 0$ and the polar quadric reduces to the nodal cone, which shows that the Hessian passes through A. When A is a point on the spinodal curve, we may put $u_1 = \delta$, in which case

$$u_2 = \delta^2 v_0 + \delta\,(p\beta + q\gamma) + r\gamma^2,$$

and (52) becomes

$$\{(n-1)\,\alpha + \delta v_0 + p\beta + q\gamma\}\,\delta + r\gamma^2 = 0,$$

which is the equation of a quadric cone. This proves the second part and shows that the degree of the spinodal curve is $4n\,(n-2)$.

51. *Every conic node on a surface gives rise to a conic node on the Hessian, having the same nodal cone and the same lines of closest contact.*

We shall choose the tetrahedron of reference so that the equation of the surface is

$$\alpha^{n-2}\,(p\beta^2 + q\gamma\delta) + \alpha^{n-3}u_3 + \ldots u_n = 0 \quad \text{.........(53)},$$

the advantage of which is that when $p = 0$ the singularity at A becomes a binode, and when $q = 0$ it becomes a unode.

Retaining only the leading terms, we obtain

$$\left.\begin{aligned}
a &= (n-2)\,(n-3)\,(p\beta^2 + q\gamma\delta)\,\alpha^{n-4} \\
b &= 2p\alpha^{n-2} + \alpha^{n-3}d^2u_3/d\beta^2 \\
c &= \alpha^{n-3}d^2u_3/d\gamma^2 \\
d &= \alpha^{n-3}d^2u_3/d\delta^2 \\
f &= \alpha^{n-3}d^2u_3/d\beta\,d\gamma \\
g &= (n-2)\,q\alpha^{n-3}\delta \\
h &= 2\,(n-2)\,p\alpha^{n-3}\beta \\
l &= (n-2)\,q\alpha^{n-3}\gamma \\
m &= \alpha^{n-3}d^2u_3/d\beta\,d\delta \\
n &= q\alpha^{n-2} + \alpha^{n-3}d^2u_3/d\gamma\,d\delta
\end{aligned}\right\} \quad \text{.........(54)}.$$

Substituting these values in (50) it will be found that the only terms involving α^{4n-10}, which is the highest power of α, are

$$-(ab-h^2)n^2+2bgnl,$$

which is equal to

$$2pq^2(n-1)(n-2)(p\beta^2+q\gamma\delta)\alpha^{4n-10} \quad \text{.........(55)},$$

which proves the first part of the theorem.

To prove the second part of the theorem, we must calculate the coefficient of α^{4n-11}, which is rather a long expression and seems hardly worth while writing down. The result is as follows. Let AC and AD be two of the lines of closest contact, then

$$u_3=\beta^3+\beta^2(\lambda\gamma+\mu\delta)+\beta(L\gamma^2+2M\gamma\delta+N\delta^2)+\gamma\delta(F\gamma+G\delta),$$

and if U_3 be the coefficient of α^{4n-11} in the Hessian, it will be found that AC and AD are generators of the cone U_3.

52. *The spinodal curve has a sextuple point at a conic node.*

If m and n are the degrees of two surfaces, which intersect at a point A which is a multiple point of order p on one surface and of order q on the other and the nodal cones are not specially related to one another, any plane section through A will consist of two plane curves which have multiple points of orders p and q at A. These curves will therefore intersect in $mn-pq$ ordinary points, which shows that A is a multiple point of order pq on the curve of intersection of the two surfaces. If however $p=q$ and the nodal cones are identical, it can be shown as follows that the order of the multiple point on the curve is $p(p+1)$; for consider the surfaces

$$\alpha^{n-p}u_p+\alpha^{n-p-1}u_{p+1}+\dots u_n=0,$$

$$\alpha^{m-p}u_p+\alpha^{m-p-1}U_{p+1}+\dots U_m=0,$$

where $n>m$. Multiply the second equation by α^{n-m} and subtract from the first and we obtain

$$\alpha^{n-p-1}(u_{p+1}-U_{p+1})+\dots u_n=0,$$

which is the equation of a surface passing through the curve of intersection of the two surfaces and having a multiple point of order $p+1$ at A, whence A is a multiple point of order $p(p+1)$ on the curve.

In the case of a surface and its Hessian $p=2$, and $p(p+1)=6$.

53. *Every binode on a surface gives rise to a cubic node of the third kind on the Hessian, two of the tangent planes at which* *coincide with the biplanes; also the spinodal curve has an octuple point at the binode.*

When $p=0$, the double point becomes a binode, and (55) vanishes. In this case the highest power of α is the $(4n-11)$th, and the terms containing it are

$$2bgnl - abn^2 = (n-1)(n-2)\,\alpha^{4n-11}\gamma\delta d^2u_3/d\beta^2,$$

and consequently A is a cubic node on the Hessian whose nodal cone consists of the two biplanes and the plane $d^2u_3/d\beta^2 = 0$. Such a singularity is called a cubic node of the third kind.

To prove the last part consider the two surfaces

$$\alpha^{n-3}\beta\gamma\delta + \alpha^{n-4}u_4 + \dots\, u_n = 0,$$
$$\alpha^{m-2}\gamma\delta + \alpha^{m-3}U_3 + \dots\, U_m = 0,$$

where $n > m$, from which we deduce

$$\alpha^{n-4}(u_4 - \beta U_3) + \dots\, u_n = 0,$$

which shows that A is an octuple point on the curve of intersection of the first two surfaces.

54. *Every unode on a surface gives rise to a quartic node on the Hessian, whose nodal cone consists of the uniplane twice repeated and a quadric cone.*

When $q=0$, the singular point becomes a unode, and the term containing the highest power of α is

$$2p^2(n-1)(n-2)\,\beta^2\left\{\left(\frac{d^2u_3}{d\gamma d\delta}\right)^2 - \frac{d^2u_3}{d\gamma^2}\frac{d^2u_3}{d\delta^2}\right\}\alpha^{4n-12},$$

which is a quartic node of the species described.

55. *If a straight line lie in a surface, it will touch the Hessian and therefore the spinodal curve.*

Let AB be the line, let A be one of the points where AB cuts the Hessian, and let ABC be the tangent plane at A; then since the section of the surface by the plane δ must have a cusp at A, it follows that the equation of the surface must be

$$\gamma\{\alpha^{n-2}(p\gamma + q\delta) + \alpha^{n-2}u_2 + \dots\}$$
$$+ \delta(\alpha^{n-1}U_0 + \alpha^{n-2}U_1 + \dots) = 0 \quad \dots\dots\dots(56).$$

Now when $\gamma = \delta = 0$, the values of a, b, h are zero; and the Hessian reduces to $(fl - gm)^2 = 0$; and on calculating this expression it will be found that β^2 is a factor, which shows that the line AB touches the Hessian at A.

56. *The curve of contact of every trope on a surface forms part of the spinodal curve.*

Let α be the trope, (α, Ω_s) the curve of contact, B any point on it, then the equation of the surface is

$$\alpha^n u_0 + \alpha^{n-1} u_{n-1} + \dots \alpha^2 (\beta^{n-2}\sigma_0 + \beta^{n-3}\sigma_1 + \dots) \\ + \alpha (\beta^{n-1}\tau_0 + \beta^{n-2}\tau_1 + \dots) + \Omega_s^2 (\beta^{n-2s} w_0 + \dots) = 0,$$

where $\Omega_s = \beta^{s-1} v_1 + \beta^{s-2} v_2 + \dots.$

The polar quadric of B is

$$\alpha \{\alpha\sigma_0 + (n-1)\beta\tau_0 + \tau_1\} + v_1^2 w_0 = 0,$$

which represents a cone.

The reader is doubtless aware that Fresnel's wave surface possesses 16 tropes whose curves of contact are circles, and that 4 of these tropes are real whilst the remaining 12 are imaginary; hence the spinodal curve consists of these 16 circles, which make up an improper curve of the 32nd degree.

57. *If a fixed plane touches a surface along a straight line, it also touches the Hessian along the same line. Hence the line twice repeated forms part of the spinodal curve.*

The equation of the surface must be of the form

$$\alpha^{n-1}\delta + \alpha^{n-2} (\delta^2 v_0 + \delta v_1 + r\gamma^2) \\ + \alpha^{n-3} \{\delta^3 w_0 + \delta^2 w_1 + \delta w_2 + \gamma^2 V_1\} + \dots = 0,$$

where AB is the line δ the fixed tangent plane and A is *any* point on the line; also the suffixed letters are binary quantics of (β, γ). Forming the Hessian, the coefficient of α^{4n-9} will be found to be $A\delta$, where A is a constant.

58. *If a surface has a nodal curve of degree b and a cuspidal one of degree c, the degree of the spinodal curve is*

$$4n(n-2) - 8b - 11c.$$

This theorem is due to Cayley*, who proved it in a different manner; and it will be sufficient to consider the case of a surface having a nodal line, the equation of which is

$$\alpha^{n-2}\gamma\delta + \alpha^{n-3} u_3 + \dots = 0,$$

where $u_3 = \gamma^2 u + \gamma\delta v + \delta^2 w,$

u, v and w being arbitrary planes through A. Now if we proceed in the same way as in § 51, it will be found that the highest

* *C. M. P.* vol. VI. p. 342.

power of α in the Hessian is α^{4n-12}, and the term involving it is $(lf+mg)^2$. Omitting α, the coefficient is

$$(lf+mg)^2 = (n-2)^2 q^2 \left(\gamma \frac{d^2 u_3}{d\beta d\gamma} + \delta \frac{d^2 u_3}{d\beta d\delta}\right)^2$$
$$= (n-2)^2 q^2 (P\gamma^2 + Q\gamma\delta + R\delta^2)^2,$$

whence AB is a quadruple line on the Hessian having two pairs of coincident tangent planes; accordingly AB repeated 8 times forms part of the spinodal curve. In the same way it can be shown that if the surface has a twisted nodal curve of degree b, this curve is a quadruple curve on the Hessian, and that the former 8 times repeated forms part of the spinodal curve. Hence the degree of the residual intersection of the surface and its Hessian, which is the true spinodal curve is $4n(n-2)-8b$.

The equation of a quartic surface having a cuspidal line will hereafter be shown to be

$$(p\alpha\gamma + q\beta\delta)^2 + (P, Q, R, S \between \gamma, \delta)^3 = 0,$$

where P, Q, R, S are arbitrary planes, and by forming the Hessian by the same method it will be found that the line AB repeated 11 times is part of the spinodal curve, but the work is rather long.

59. In the same paper Cayley has also proved the corresponding theorem for the flecnodal curve which is:—

If a surface has a nodal curve of degree b and a cuspidal one of degree c, the degree of the flecnodal curve is

$$n(11n-24)-22b-27c.$$

A similar theorem undoubtedly exists for the bitangential curve, but so far as I am aware it has not been investigated.

60. *The Hessian of an anautotomic surface of degree n possesses* $10(n-2)^3$ *nodes**.

Let us for convenience suppose the original surface to be of degree $n+2$, so that each of the constituents of the Hessian is of degree n; let

$$\frac{\partial H}{\partial d} = \Delta, \quad \frac{\partial H}{\partial l} = -\Delta_1, \quad \frac{\partial H}{\partial m} = -\Delta_2, \quad \frac{\partial H}{\partial n} = -\Delta_3 \ldots\ldots(57),$$

* Cayley, *Proc. Lond. Math. Soc.* vol. III. p. 23. *C. M. P.* vol. VII. p. 133.

also let

$$D = \begin{vmatrix} b, & f, & m \\ f, & c, & n \\ m, & n, & d \end{vmatrix} \quad \text{.......................(58)},$$

then by means of (49) and (50) the following additional equations can easily be proved, viz.:

$$B'C' - AA' = \Delta f, \quad BC - A'^2 = \Delta a, \text{ \&c. ....(59)},$$

$$\left. \begin{aligned} \Delta &= Aa + B'g + C'h \\ \Delta_1 &= Al + C'm + B'n \\ \Delta_2 &= C'l + Bm + A'n \\ \Delta_3 &= B'l + A'm + Cn \end{aligned} \right\} \quad \text{..................(60)}.$$

Eliminating l between the second and third of (60) and taking account of the first, we obtain

$$\Delta_2 A - \Delta_1 C' = \Delta(mc - nf).$$

Hence if Q is any point at which two of the three determinants Δ, Δ_1, Δ_2 vanish, the third determinant will also vanish, provided none of the quantities A, C' and $mc - nf$ vanish at Q.

Now the equations $A = C' = mc - nf = 0$ are equivalent to

$$h/g = b/f = f/c = m/n \quad \text{.....................(61)},$$

so that the exceptional points, which we shall denote by P, are the common points of intersection of the six surfaces included in the set of determinants

$$\left\| \begin{matrix} h, & b, & f, & m \\ g, & f, & c, & n \end{matrix} \right\| = 0 \quad \text{..................(62)},$$

and since each constituent is of degree n, it follows from § 48 that there are $4n^3$ points such as P.

The system of $4n^3$ points included in (62) lies on the Hessian and also on the surface $D = 0$; for if we put each of the ratios (61) equal to k, and substitute in the determinants for H and D, they obviously vanish.

We have therefore proved that if Q be a point such that $\partial H/\partial d = 0$ and $\partial H/\partial l = 0$, then $\partial H/\partial m = 0$ and $\partial H/\partial n = 0$, provided Q does not coincide with any of the $4n^3$ points P; also the points at which these differential coefficients vanish lie on the Hessian.

In the next place write H in the form

$$H = \begin{vmatrix} d, & l, & m, & n \\ l, & a, & h, & g \\ m, & h, & b, & f \\ n, & g, & f, & c \end{vmatrix},$$

and let $\partial H/\partial c = 0$; then since $\partial H/\partial n$ has been shown to vanish it follows that $\partial H/\partial g = \partial H/\partial f = 0$.

Proceeding in the same way it can be shown that: *If any two of the differential coefficients of the Hessian with respect to the four letters a, b, c, d vanish; and any one of those with respect to the six letters f, g, h, l, m, n also vanishes, then all the differential coefficients will vanish, provided the points at which they vanish are not included amongst the points P.*

In explanation of the proviso, let the three equations be

$$\frac{\partial H}{\partial a} = 0, \quad \frac{\partial H}{\partial d} = 0, \quad \frac{\partial H}{\partial l} = 0 \quad\text{.................(63)},$$

then these equations will represent three surfaces of degrees $3n$ which intersect in a certain number of points, or which may possess a common curve; and if Q be any point common to the three surfaces all the differential coefficients of H will vanish at Q unless Q coincides with any of the $4n^3$ points P.

We shall now show that the points Q are nodes on the Hessian.

Let $$a = a_0\alpha^n + a_1\alpha^{n-1} + \ldots,$$

&c.; where $a_n = (\beta, \gamma, \delta)^n$; then

$$H = H_0\alpha^{4n} + \left(\frac{\partial H}{\partial a_0} a_1 + \ldots \frac{\partial H}{\partial l_0} l_1 + \ldots\right) \alpha^{4n-1} + \ldots.$$

Let the tetrahedron be chosen so that A coincides with one of the points Q, then H_0 and all its differential coefficients vanish, and the highest power of α is α^{4n-2}, which shows that A is a node on the Hessian. From these results it follows that the nodes are included amongst the intersections of the surfaces

$$D = 0,$$

and $$\left\| \begin{matrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \end{matrix} \right\| = 0 \quad\text{.................(64)}.$$

Now it follows from § 47 that the four determinantal surfaces (64) intersect in a common curve of degree $6n^2$; also the $4n^3$ points P lie on this curve and also on the surface D; and we have to show that the points P are not nodes, and that the number of the points Q is equal to $10n^3$.

Let A be one of the points P, which are the points common to the surfaces (61), then it follows that

$$h = \alpha^n + \alpha^{n-1}h_1 + \dots,$$

$$g = \alpha^n + \alpha^{n-1}g_1 + \dots,$$

&c., &c. Hence by (49)

$$A = (b_1 + c_1 - 2f_1)\,\alpha^{2n-1}, \quad B = (a_0 - 1)\,\alpha^{2n}, \quad C = (a_0 - 1)\,\alpha^{2n},$$

$$A' = -(a_0 - 1)\,\alpha^{2n}, \quad B' = (h_1 + f_1 - b_1 - g_1)\,\alpha^{2n-1},$$

$$C' = (f_1 + g_1 - c_1 - h_1)\,\alpha^{2n-1}.$$

Substituting in H and D we obtain

$$H = \{(a_0 - 1)\,d_0 - l_0^2 + 2l_0\}\,(b_1 + c_1 - 2f_1)\,\alpha^{4n-1} + \dots,$$

$$D = (d_0 - 1)\,(b_1 + c_1 - 2f_1)\,\alpha^{3n-1} + \dots.$$

The first equation shows that A is a point *but not a node* on the Hessian, and the second equation shows that the Hessian and D touch one another at A.

Now the common curve of intersection of (64) obviously lies in the Hessian; and since it is of degree $6n^2$ it intersects the surface D in $18n^3$ points and touches it at the $4n^3$ points P; hence the number of remaining points of intersection, which are the points Q, is equal to $18n^3 - 8n^3 = 10n^3$, which is therefore the number of nodes on the Hessian.

61. When the polar quadric degrades into a cone, the locus of the vertex of the cone is a surface called the *Steinerian.* This surface is the analogue of the plane curve of the same name, which is the locus of the points of intersection of the pair of straight lines constituting the polar quadric of points on the Hessian. The Hessian* and the Steinerian† are discussed in the papers referred to below.

* Hesse, *Crelle*, vols. XXVIII. and XLI.; Sylvester, *Phil. Trans.* CXLIII.; Del Pezzo, *Rendiconti, Napoli*, 1883; Brill, *Math. Annalen*, vol. XIII.; Segre, *Rendiconti, Lincei*, 1895.

† *Crelle*, vol. XLVII.

CHAPTER II

CUBIC SURFACES

62. THERE are altogether twenty-three different species of cubic surfaces, which depend upon the number and character of the point singularities which they possess. The class of an anautotomic cubic surface is $n(n-1)^2 = 12$; the degree of the tangent cone is $n(n-1) = 6$; also the cone has $n(n-1)(n-2) = 6$ cuspidal generators, but no nodal ones. From this it follows that a cubic surface cannot have more than four nodes; for if it had five nodes, the class of the surface would be two, and quadrics are the only surfaces of the second class.

63. *Every cubic surface can be expressed in either of the forms*

$$Su = S'u' \quad \text{.........................(I)},$$

or

$$\alpha\beta\gamma = uvw \quad \text{.........................(II)},$$

where S, S' are quadric surfaces, and α, β, γ, u, u', v, w are planes.

We have shown that every cubic surface possesses straight lines lying in it; hence if AB be one of these lines, the equation of the surface must be of the form $\gamma S = \delta S'$, which is of the form (I).

To prove (II) write the last equation in the form

$$\gamma(S + \delta w) = \delta(S' + \gamma w),$$

where w is an arbitrary linear function of $(\alpha, \beta, \gamma, \delta)$. Since two conditions are necessary in order that a quadric surface should degrade into a pair of planes, it is possible to determine the four constants in w so that the quadrics $S + \delta w$ and $S' + \gamma w$ should each become a pair of planes, which proves the second form.

64. *If a given quadric surface intersect a cubic surface in a quartic curve and a conic; then any other quadric surface drawn* *through the quartic curve intersects the cubic surface in a conic, which lies in a plane passing through a fixed straight line on the cubic surface.*

Equation (I) shows that the quadric S intersects the cubic surface in the quartic curve $S=0$, $S'=0$, and in the conic $S=0$, $u'=0$; also the plane $u'=0$, which contains the conic, intersects the cubic in the straight line $u=0$, $u'=0$. The equation $S'=\lambda S$, where λ is an arbitrary constant, represents any other quadric surface which intersects the cubic surface in the quartic curve; and if S and S' be eliminated we obtain the equation $u=\lambda u'$, which shows that the residual curve of intersection is a conic lying in the plane $u=\lambda u'$, which passes through the straight line $u=0$, $u'=0$.

The corresponding theorem for a plane cubic curve led to Sylvester's discovery of the Theory of Residuation of curves; and it will hereafter be shown that the preceding theorem is a particular case of a corresponding Theory of Residuation of surfaces.

65. *Through any straight line on a cubic surface,* 5 *planes can be drawn which intersect the surface in a pair of straight lines. Also* 27 *straight lines can be drawn on an anautotomic cubic surface.*

The first part of this theorem has already been proved in §§ 44 or 45; and the second part can be established as follows. Let any triple tangent plane intersect the cubic in the lines λ, μ, ν; then since four other planes can be drawn through λ, each of which intersects the cubic in two other straight lines, these four planes will furnish eight straight lines which together with λ make nine. Similarly, each of the other lines μ and ν will furnish nine more, making a total of 27.

66. *An anautotomic cubic surface has* 45 *triple tangent planes.*

Each of the 5 tangent planes drawn through a straight line on a cubic touches the latter in three points, which are the vertices of the triangle which is the section of the surface by the plane; and since 27 lines can be drawn on the surface, the number of such planes is $27 \times 5 = 135$. But every plane such as ABC contains each of the lines AB, AC and BC; hence the total number of distinct planes is $135 \div 3 = 45$.

67. *An anautotomic cubic surface has 54 tangent planes at which the point of contact is a tacnode on the section.*

The section of the surface by an arbitrary plane through the line AB consists of AB and a conic which in general cuts AB in two distinct points, and the plane is therefore a double tangent plane; but if the two points coincide, the point of contact is a tacnode on the section. Let the equation of the surface be

$$\{\alpha^2 U_0 + \alpha(P\beta + V_1) + \beta^2 W_0 + \beta W_1 + W_2\}\gamma$$
$$+ \{\alpha^2 u_0 + \alpha(p\beta + v_1) + \beta^2 w_0 + \beta w_1 + w_2\}\delta = 0 \quad \ldots\ldots(1).$$

In (1) write $\delta = k\gamma$, divide out by γ, and then put $\gamma = 0$, and (1) becomes

$$\alpha^2(U_0 - ku_0) + \alpha\beta(P - pk) + \beta^2(W_0 - kw_0) = 0 \quad \ldots\ldots(2).$$

Equation (2) determines the two points in which the line AB cuts the conic; but if AB touches the conic the two roots of (2) must be equal, which furnishes a quadratic equation for determining k. Hence on each of the 27 lines there are two points such that the section by the tangent plane consists of the straight line and a conic touching it at the point; accordingly there are 54 of such points.

68. The literature on the 27 lines of a cubic surface, and the different ways of arranging them is voluminous; and the reader is referred to the authorities cited below*. One arrangement is that of a *double-six*, which consists of two systems of lines

$$1, 2, 3, 4, 5, 6,$$
$$1', 2', 3', 4', 5', 6',$$

such that each line of one system intersects every line of the other system except the line represented by the figure above or below it, as the case may be. There are altogether 36 double-sixes.

The Hessian of a Cubic Surface.

69. We shall now give some theorems relating to a cubic surface and its Hessian, which is a quartic surface, since $4(n-2) = 4$ when $n = 3$.

* Schläfli, *Phil. Trans.* CLIII. (1863), p. 193; Cayley, *Ibid.* CLIX. (1869), p. 231, and *C. M. P.* vol. VI. p. 359, vol. VII. p. 316; Clebsch, *Math. Annalen*, vol. IV.; Cremona, *Crelle*, vol. LXVIII.; Dixon, *Quart. Journ.* vol. XL. p. 246; Burnside, *Ibid.* p. 381.

If the polar quadric of any point A with respect to a cubic be a cone whose vertex is B, the polar quadric of B is a cone whose vertex is A. Hence the Hessian and the Steinerian are identical.

This theorem has already been given in § 13 (viii); but for a cubic surface the proof is as follows. Let the equation of the surface be

$$\alpha^3 + \alpha^2 u_1 + \alpha u_2 + u_3 = 0 \quad \text{.....................(3)},$$

then the polar quadric of A is

$$3\alpha^2 + 2\alpha u_1 + u_2 = 0 \quad \text{.....................(4)},$$

and if (4) is a cone whose vertex is B, it cannot contain β; hence $u_1 = \mu\gamma + \nu\delta$, $u_2 = v_2$, and (3) becomes

$$\alpha^3 + \alpha^2(\mu\gamma + \nu\delta) + \alpha v_2 + u_3 = 0 \text{..................(5)}.$$

The polar quadric of B is now $du_3/d\beta = 0$, which is a cone whose vertex is A. The points A and B are points on the Hessian, and are called *conjugate poles*; and a theory exists with reference to them analogous to the corresponding theory for plane cubic curves.

70. *The tangent plane to the Hessian at A is the polar plane of B with respect to the cubic.*

The polar plane of B with respect to the cubic is $d^2u_3/d\beta^2 = 0$; and if we write down the Hessian of (5) it will be found that the only terms which contain α^3 are

$$b(acd + 2gln - an^2 - cl^2 - dg^2) \text{ ....(6)}.$$

Now $b = d^2u_3/d\beta^2$, whilst the term in brackets is equal to

$$A\alpha^3 + \alpha^2 U_1 + \ldots,$$

where A is a constant; hence the equation of the tangent plane to the Hessian at A is $d^2u_3/d\beta^2 = 0$, which proves the theorem.

Let
$$v_2 = p\gamma^2 + 2q\gamma\delta + r\delta^2 \quad \text{.....................(7)},$$

$$u_3 = \beta^3 w_0 + \beta^2(M\gamma + N\delta) + \beta(P\gamma^2 + 2Q\gamma\delta + R\delta^2) + F\gamma^3 + 3G\gamma^2\delta + 3H\gamma\delta^2 + K\delta^3 \quad \text{..................(8)},$$

then the tangent planes to the Hessian at A and B are

$$\left.\begin{aligned} 3\beta w_0 + M\gamma + N\delta &= 0 \\ 3\alpha + \mu\gamma + \nu\delta &= 0 \end{aligned}\right\} \text{.....................(9)}.$$

71. *If the line joining two conjugate poles A and B intersect the Hessian in two points A′ and B′, the tangent planes to the*

Hessian at A and B will intersect in the line joining the two poles which are conjugate to A′ and B′.

Let $\beta' = \beta - \lambda\alpha = 0$ be the plane $A'CD$; then since the points C and D are arbitrary, we may suppose that C is the pole conjugate to A'. Changing the tetrahedron to $A'BCD$, (5) becomes

$$\alpha^3 + \alpha^2(\mu\gamma + \nu\delta) + \alpha v_2 + (\lambda\alpha + \beta')^3 w_0 + (\lambda\alpha + \beta')^2 (M\gamma + N\delta)$$
$$+ (\lambda\alpha + \beta')(P\gamma^2 + 2Q\gamma\delta + \delta^2) + F\gamma^3 + \ldots = 0 \ldots\ldots(10).$$

The polar quadric of A' is a cone whose vertex is C, and must not therefore contain γ; hence

$$\mu = 0, \quad M = 0, \quad p + \lambda P = 0, \quad q + \lambda Q = 0 \quad \ldots\ldots(11).$$

Let D be the pole conjugate to B', and change the tetrahedron to $AB'CD$, where $\alpha' = \alpha - \lambda'\beta = 0$ is the equation of the plane $B'CD$, then the condition that the polar quadric of B' should be a cone whose vertex is D will be found to lead to a system of equations, which when combined with (11), give

$$\nu = 0, \quad N = 0, \quad q = Q = 0, \quad R = -\lambda' r \ldots\ldots\ldots\ldots(12).$$

Equations (9), which are the tangent planes to the Hessian at A and B, now become $\beta = 0$, $\alpha = 0$, which intersect in the line CD.

72. *The polar plane with respect to the cubic of any point on the line joining two conjugate poles, passes through the line of intersection of the tangent planes to the Hessian at these points.*

By the last section the equation of the cubic is

$$\alpha^3 + \alpha(p\gamma' + r\delta') + \beta^3 w_0 + \beta(P\gamma' + R\delta') + w_3 = 0 \ldots\ldots(13),$$

and the polar plane of any point (α', β') on AB is

$$\alpha\alpha'^2 + w_0\beta\beta'^2 = 0,$$

which passes through CD.

73. *The polar plane of the Hessian, with respect to any point on a cubic, intersects the tangent plane to the cubic at that point, in the line which passes through the three points of inflexion of the section of the cubic by the tangent plane.*

Consider the cubic

$$\alpha^2\beta + \alpha(\beta^2 v_0 + \beta v_1 + \gamma\delta) + \beta^3 w_0 + \beta^2 w_1 + \beta w_2 + \gamma^3 + \delta^3 = 0 \ldots(14).$$

This surface passes through the point A, and $\beta = 0$ is the tangent plane thereat; also the section of (14) by the plane β is

$$\alpha\gamma\delta + \gamma^3 + \delta^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Equation (15) is a cubic curve which has a node at A, and the line CD passes through the three points of inflexion on the section. Let

$$v_1 = p\gamma + q\delta, \quad w_2 = P\gamma^2 + Q\gamma\delta + R\delta^2,$$

then if we write down the Hessian of (14) it will be found that the only terms which involve α^4 and α^3 are

$$-abn^2 + h^2n^2 - 2ghmn - 2hfnl = 4\alpha^3(\alpha + 2Q\beta - 2pq\beta).$$

The polar plane of A with respect to the Hessian H is

$$\partial^3 H/\partial\alpha^3 = 48(2\alpha + Q\beta - pq\beta) = 0,$$

which passes through CD.

74. *If the polar quadric of a point A consists of a pair of planes, then A is a conic node on the Hessian.*

The polar quadric of A is in general a cone whose vertex is B; and since the line AB must possess *at least* one degree of freedom, the equation of the polar quadric of A must contain *at least* one variable parameter λ. If therefore λ is made to satisfy the condition that the discriminant of the polar quadric of A should vanish, the cone will degrade into a pair of planes whose line of intersection passes through B. From (5) and (7) the polar quadric of the cubic with respect to A is

$$3\alpha^2 + 2\alpha(\mu\gamma + \nu\delta) + p\gamma^2 + 2q\gamma\delta + r\delta^2 = 0 \quad \text{.........(16)},$$

and its discriminant equated to zero gives

$$3pr + 2q\mu\nu - 3q^2 - p\nu^2 - r\mu^2 = 0 \quad \text{..............(17)}.$$

Equation (6) gives the term involving α^3 in the Hessian, and if the values of a, c, &c., be substituted from (5) it will be found that (17) is the condition that this term should vanish. Hence A is a conic node on the Hessian.

75. Let BC be the line of intersection of the planes, then (16) must reduce to

$$3\alpha^2 + 2\alpha\nu\delta + r\delta^2 = 0,$$

which requires that $\mu = p = q = 0$, and (5) becomes

$$\alpha^3 + \nu\alpha^2\delta + r\alpha\delta^2 + u_3 = 0 \quad \text{.................(18)},$$

and since B may be any point on BC, it follows that the polar quadric of every point on BC consists of a cone whose vertex is A; hence BC is a line lying in the Hessian.

Let us now enquire whether the Hessian has any nodes lying on BC. Let C' be any point on BC, and let $\beta' = \beta - \lambda\gamma = 0$ be the equation of the plane $AC'D$; then in order to find the polar quadric of C', we must change the tetrahedron to $ABC'D$ and differentiate (18) with respect to γ. Accordingly the polar quadric of C' is

$$\frac{\partial u_3}{\partial \gamma} + \lambda \frac{\partial u_3}{\partial \beta} = 0,$$

where u_3 is given by (8). Writing this out at full length we obtain

$$(3\lambda w_0 + M)\beta^2 + (P\lambda + 3F)\gamma^2 + (R\lambda + 3H)\delta^2$$
$$+ 2(Q\lambda + 3G)\gamma\delta + 2(N\lambda + Q)\delta\beta + 2(M\lambda + P)\beta\gamma = 0 \ldots (19),$$

and its discriminant when equated to zero furnishes a cubic equation for λ, showing that there are three points on BC at which the polar quadric degrades into a pair of planes. Hence the Hessian has three conic nodes lying on BC.

76. *Every anautotomic cubic surface can be expressed in the canonical form*

$$a\alpha^3 + b\beta^3 + c\gamma^3 + d\delta^3 + eu^3 = 0,$$

where $$\alpha + \beta + \gamma + \delta + u = 0.$$

The preceding form of a quaternary cubic is due to Sylvester*, and we shall now show how it can be established by means of the foregoing results.

Let A be a node on the Hessian; then we have already shown that (i) the polar quadric of A consists of a pair of planes intersecting in a straight line LMN; (ii) the line LMN lies in the Hessian; (iii) it has three conic nodes upon it, which we shall suppose situated at the points L, M, N. Let $\alpha = 0$, $u = 0$ be the equations of LMN; then the polar quadric of A must be of the form

$$a\alpha^2 - eu^2 = 0.$$

Integrating with respect to α, it follows that the equation of the cubic surface must be of the form

$$U = a\alpha^3 + b\beta^3 + c\gamma^3 + d\delta^3 + eu^3 + \beta^2(\mu\gamma + \nu\delta)$$
$$+ \beta(P\gamma^2 + 2Q\gamma\delta + R\delta^2) + G\gamma^2\delta + H\gamma\delta^2 = 0 \ldots (20).$$

Let us construct a triangle by drawing three lines LBD, MCD and NCB in the plane α through L, M and N. Then each of

* A quaternary cubic can also be expressed as the sum of six cubes. See Dixon, *Proc. Lond. Math. Soc.* Series 2, vol. VII. p. 389.

these lines has one degree of freedom in the plane α; also since α may be any plane which passes through LMN, the plane α has also one degree of freedom. The coordinates of L may therefore be taken to be $\alpha=0$, $\gamma=0$, $\beta+\delta=0$; hence the polar quadric of L is

$$\frac{\partial U}{\partial \beta}-\frac{\partial U}{\partial \delta}=0,$$

which by virtue of (20) is

$$3b\beta^2+2\beta(\mu\gamma+\nu\delta)+P\gamma^2+2Q\gamma\delta+R\delta^2$$
$$-3d\delta^2-\gamma\beta^2-2\beta(Q\gamma+R\delta)-G\gamma^2-2H\gamma\delta=0 \quad \ldots(21).$$

This is the equation of a cone whose vertex is A; but since L is by hypothesis a node on the Hessian, it follows that (21) must degrade into a pair of planes intersecting in some line passing through A, which we may take to be AC, since AC may be any line passing through A. This requires that

$$\mu=P=Q=G=H=0,$$

which reduces (20) to

$$U=a\alpha^3+b\beta^3+c\gamma^3+d\delta^3+eu^3+\nu\beta^2\delta+R\beta\delta^2=0 \ldots(22).$$

The coordinates of M are $\alpha=0$, $\beta=0$, $\gamma+\delta=0$; and its polar quadric is

$$3c\gamma^2-3d\delta^2-\nu\beta^2-2R\beta\delta=0 \ldots\ldots\ldots\ldots\ldots(23),$$

which is a cone whose vertex is A. Now A and N are fixed points, and the line AC is a fixed line, hence ACN is a fixed plane, but AB may be any line through A in this plane. We shall therefore suppose it to coincide with one of the generators of the cone (23), in which case $\nu=0$. Also M is by hypothesis a conic node on the Hessian, hence the discriminant of (23) must vanish, which requires that $Rc=0$. Taking $R=0$, (22) reduces to

$$U=a\alpha^3+b\beta^3+c\gamma^3+d\delta^3+eu^3 \ldots\ldots\ldots\ldots\ldots(24),$$

the form of which shows that N is also a node on the Hessian.

77. *To find the equation of the Hessian.*

Let $(\alpha', \beta', \gamma', \delta')$ be any point on the Hessian, u' the corresponding value of u; then the polar quadric of this point with respect to the cubic is

$$\alpha'\left(\frac{\partial U}{\partial \alpha}-\frac{\partial U}{\partial u}\right)+\beta'\left(\frac{\partial U}{\partial \beta}-\frac{\partial U}{\partial u}\right)+\ldots=0,$$

which by virtue of (24) becomes

$$a\alpha'\alpha^2 + b\beta'\beta^2 + c\gamma'\gamma^2 + d\delta'\delta^2 + eu'u^2 = 0 \quad\text{.........(25).}$$

Since (25) is a cone, its discriminant must vanish; and the latter is obtained by eliminating $(\alpha, \beta, \gamma, \delta)$ between the equations

$$a\alpha'\alpha = b\beta'\beta = c\gamma'\gamma = d\delta'\delta = eu'u,$$

and

$$\alpha + \beta + \gamma + \delta + u = 0,$$

which gives

$$\frac{1}{a\alpha'} + \frac{1}{b\beta'} + \frac{1}{c\gamma'} + \frac{1}{d\delta'} + \frac{1}{eu'} = 0 \quad\text{..............(26),}$$

and since $(\alpha', \beta', \gamma', \delta')$ is by hypothesis a point on the Hessian, (26) is its equation.

From the way in which this result has been obtained, it follows that A is a node on the Hessian; and from symmetry it follows that B, C, D are also nodes. It has also been shown that the sides BC, CD and DB of the tetrahedron cut the plane $u = 0$ in three points which are also nodes; hence from symmetry the three points where AB, AC, AD cut the plane $u = 0$ are also nodes. Accordingly we obtain Sylvester's theorem that:—*The Hessian has ten nodes, which are the vertices of a pentahedron; also these nodes lie in triplets on the ten lines which form the edges of the pentahedron.* These ten lines lie in the Hessian, as is otherwise obvious; for since the Hessian is a quartic surface, the line joining three collinear nodes must necessarily lie in the surface.

Singularities of Cubic Surfaces.

78. Since a cubic surface may possess as many as four double points, it may have compound singularities formed by the union of two or more simple singularities; but one caution is necessary. The section of a surface by a plane through a conic node and the axis of a binode is a nodocuspidal curve, having an ordinary node at the conic node and a cusp at the binode; accordingly when the conic node and the binode coincide, the section will have a rhamphoid cusp at the point. But a quartic is the curve of lowest degree which can possess a proper rhamphoid cusp; and since the section of a cubic surface through two double points consists of a conic and the line joining them, the rhamphoid cusp must appear in a degraded form on the section. The consequence is that singularities composed of a determinate number of conic nodes and binodes possess special features, when the surface on which they lie is a cubic, which are quite different from those of the corresponding singularities on a surface of higher degree.

79. *The binode* $B_4 = 2C$. We have shown in § 25 that the equation of a surface having a binode formed by the union of two conic nodes is

$$\alpha^{n-2}\gamma\delta + \alpha^{n-3}(\beta^2 w_1 + \beta w_2 + w_3) + \alpha^{n-4}u_4 + \dots u_n = 0 \dots (27),$$

where AB is the axis of the binode. The axis has quadritactic contact with the surface at the binode, and the section of the latter by any plane through the axis has a tacnode thereat; also the axis is one of the lines of closest contact in each biplane, so that the axis is equivalent to two of such lines. Let $n = 3$, and let AC be one of the lines of closest contact in the biplane ABC, and AD one of them in ABD; then (27) becomes

$$\alpha\gamma\delta + \beta^2 w_1 + \beta w_2 + \gamma\delta v_1 = 0.$$

Change the tetrahedron to $A'BCD$, where $\beta' = \beta - \lambda\alpha = 0$ is the equation of the plane $A'CD$, then the equation becomes

$$\alpha\gamma\delta + (\beta' + \lambda\alpha)^2 w_1 + (\beta' + \lambda\alpha) w_2 + \gamma\delta v_1 = 0,$$

which shows that $w_1 = 0$ is the tangent plane along the axis, and is therefore fixed in space. The axis is therefore a singular line analogous to the curve of contact of a trope; hence a binode of this species on a cubic surface is a totally different singularity from one on a surface of higher degree.

80. *The binode* $B_5 = C + B$. Consider the equation

$$\alpha\gamma\delta + \mu\beta^2\gamma + \beta w_2 + w_3 = 0 \dots\dots\dots\dots\dots\dots (28).$$

This surface has a binode at A, and the biplane γ touches the surface along the axis AB; hence γ is a singular plane, and the axis a singular line analogous to the curve of contact of a trope. Writing down the first polars of C and D we obtain

$$\alpha\delta + \mu\beta^2 + \beta w_2' + w_3' = 0 \dots\dots\dots\dots\dots\dots (29),$$

$$\alpha\gamma + \beta w_2'' + w_3'' = 0 \dots\dots\dots\dots\dots\dots (30),$$

where $w' = dw/d\gamma$ and $w'' = dw/d\delta$. Eliminating α between (28) and (29), and between (28) and (30), we obtain

$$\left.\begin{aligned} \beta(w_2 - \gamma w_2') + w_3 - \gamma w_3' &= 0 \\ \mu\beta^2\gamma + \beta(w_2 - \delta w_2'') + w_3 - \delta w_3'' &= 0 \end{aligned}\right\} \dots\dots\dots\dots (31).$$

Equations (31) represent two cubic cones, and their number of common generators, exclusive of AB, gives the number of ordinary points of intersection of (28) and the first polars of C and D; and since AB is a nodal generator on the first of (31) and an ordinary one on the second, the number of common generators is equal to 7. Hence $m = 7$, and the constituents of this binode are $C = 1$, $B = 1$.

It follows from § 26, that when a surface of the nth degree possesses a binode formed by the union of a conic node and a binode, the singularity is of a totally different character. Also when a surface of the nth degree possesses a binode such that (i) the axis lies in the surface, (ii) one of the biplanes touches the surface along the axis, the constituents of the singularity are altogether different from those of the binode we are considering.

81. *The binode* $B_6 = 2B$. The equation of a cubic surface having this binode is

$$\alpha\gamma\delta + \mu\beta^2\gamma + \beta\gamma w_1 + w_3 = 0 \quad \text{..............(32)},$$

so that the axis is not only a singular line, but one of the biplanes *osculates* the surface along it.

We have shown in § 31 that when a fixed plane osculates a surface along a straight line, there are in general $n-1$ ordinary binodes lying in the line. Hence the equation of a cubic surface having a binode at A, and ABD as the osculating plane, is

$$\alpha\gamma(p\beta + q\gamma + r\delta) + \beta^2\gamma + \beta\gamma w_1 + w_3 = 0 \quad \text{.........(33)}.$$

Let the plane $\beta' = \beta - \lambda\alpha = 0$ cut AB at A'; then changing the tetrahedron to $A'BCD$, (33) becomes

$$\alpha\gamma\{p(\lambda\alpha + \beta') + q\gamma + r\delta\} + (\lambda\alpha + \beta')^2\gamma$$
$$+ (\lambda\alpha + \beta')\gamma w_1 + w_3 = 0 \quad \text{.........(34)};$$

accordingly if $\lambda + p = 0$, A' will be the other binode; and if $p = 0$ the latter will coincide with A. In this case (33) becomes

$$\alpha\gamma(q\gamma + r\delta) + \beta^2\gamma + \beta\gamma w_1 + w_3 = 0,$$

which is of the same form as (32). The constituents of this binode are therefore $B = 2$, and the reduction of class produced by it is 6.

82. *Unodes.* The theory of the unode has been discussed in §§ 27 and 28; and we have shown that there are three kinds of unodes, whose constituents are $C = 3$, $B = 0$; $C = 2$, $B = 1$; $C = 1$, $B = 2$; and respectively reduce the class by 6, 7 and 8. The equations of a cubic surface having the three species of unodes are

$$\left.\begin{aligned} &\alpha\delta^2 + u_3 = 0 \\ &\alpha\delta^2 + \mu\beta^2\delta + \beta w_2 \ + w_3 = 0 \\ &\alpha\delta^2 + \mu\beta^2\delta + \beta\delta w_1 + w_3 = 0 \end{aligned}\right\} \qquad (35),$$

in which δ is the uniplane. But a unode on a cubic surface is a singularity different from one lying on a surface of a higher degree, because the tangents at the triple point on the section of the surface by the uniplane lie in it; and a unode on a surface of the nth degree which possessed this property would be a special kind of unode. In the first species of unode on a cubic surface, the uniplane cuts the surface in three straight lines passing through the unode, which when twice repeated constitute the six lines of closest contact. In the second species, two of these lines coincide, and the uniplane δ touches the surface along the line AB which is a singular line analogous to the curve of contact of a trope. In the third species, all three lines coincide; and the uniplane osculates the cubic along AB, which is likewise a singular line, analogous to the curve of contact of a tangent plane which osculates a surface along a plane curve.

83. The first complete investigation of cubic surfaces was made by Schläfli*, who showed that there are twenty-three species; and his results were subsequently extended and discussed by Cayley†. The last author has also found the equations of the Hessian and the reciprocal surface, and the latter gives valuable information respecting the character of the reciprocal singularities. At the same time, since compound singularities on a cubic surface possess special features of their own, the character of the singularities on the reciprocal surface is not precisely the same as when the original surface is of higher degree.

The following table gives the twenty-three species, in which the letters m, l and t denote the class of the surface, and the number of lines and triple tangent planes.

* *Phil. Trans.* 1863, p. 193.

† *Ibid.* 1869, p. 231, and *C. M. P.* vol. VI. p. 359.

	Description of Surface	m	l	t
I	Anautotomic	12	27	45
II	One conic node	10	15	15
III	One binode	9	9	6
IV	Two conic nodes	8	7	3
V	One binode B_4	8	7	3
VI	One binode and one conic node	7	3	0
VII	One binode B_5	7	3	0
VIII	Three conic nodes	6	3	1
IX	Two binodes	6	0	0
X	One conic node and one binode B_4 ...	6	3	1
XI	One binode B_6	6	3	1
XII	One unode of the first kind	6	3	1
XIII	One binode and two conic nodes	5	1	0
XIV	One conic node and one binode B_5 ...	5	1	0
XV	One unode of the second kind	5	1	0
XVI	Four conic nodes	4	3	1
XVII	One conic node and two binodes	4	0	0
XVIII	Two conic nodes and one binode B_4 ...	4	3	1
XIX	One conic node and one binode B_6 ...	4	3	1
XX	One unode of the third kind	4	0	0
XXI	Three binodes	3	0	0
XXII	A nodal line of the first kind	3	...	...
XXIII	A nodal line of the second kind	3	...	...

Autotomic Cubics.

84. *One conic node.* The six lines of closest contact form 15 pairs, hence there are 15 planes each of which contains a pair of such lines and therefore intersects the cubic surface in a third straight line. From this it follows that when the cubic has a conic node there are 15 ordinary lines lying in the surface; but we shall also prove that each line of closest contact is equivalent to two ordinary lines*, hence the total number of lines is $15 + 2 \times 6 = 27$.

Let us first consider the surface

$$2\alpha^2 (f\beta + g\gamma + h\delta) + \alpha (P, Q, R, p, q, r)(\beta, \gamma, \delta)^2 + 2\beta\gamma\delta = 0 \qquad \text{.........(36)},$$

* Some of the results given in this and the following articles may be proved more shortly as follows. Equation (17) of § 19 shows that every plane through a line and a double point is equivalent to two or three tangent planes, according as the double point is a conic node or a binode. This shows that when a cubic surface has a conic node, only three proper tangent planes can be drawn to the surface through any ordinary line lying in it; and only two when the surface has a binode. This gives the 15 and 9 ordinary lines discussed in §§ 84 and 86.

in which α is one of the triple tangent planes. Putting $\delta = \lambda\alpha$, the section of the surface by an arbitrary plane through BC is the conic

$$2\alpha(f\beta + g\gamma + h\lambda\alpha) + P\beta^2 + Q\gamma^2 + R\lambda^2\alpha^2$$
$$+ 2(p\gamma + q\beta)\lambda\alpha + 2r\beta\gamma + 2\lambda\beta\gamma = 0 \quad \text{.........(37).}$$

Equating the discriminant to zero, we obtain

$$PQ(2h + R\lambda)\lambda + 2(r + \lambda)(g + p\lambda)(f + q\lambda)$$
$$- (2h + R\lambda)(r + \lambda)^2\lambda - P(g + p\lambda)^2 - Q(f + q\lambda)^2 = 0 \quad \text{...(38).}$$

Equation (38) is a quartic equation for determining λ, and shows that in addition to the plane α four other planes can be drawn through BC which cut the surface in a pair of straight lines. If however (36) has a conic node at A, $f = g = h = 0$, and λ^2 is a factor of (38); and since $\lambda = 0$ corresponds to the plane ABC, the fact that $\lambda = 0$ is a double root of (38) shows that the plane ABC is twice repeated, and therefore the two lines of closest contact in this plane are each equivalent to two ordinary lines. The quadratic factor of (38) which is

$$PQR + 2pq(r + \lambda) - R(r + \lambda)^2 - Pp^2 - Qq^2 = 0 \quad \text{......(39)}$$

furnishes two triple tangent planes through BC, which together with BCD make three; hence the 10 straight lines which cut BC consist of the two lines of closest contact twice repeated, which lie in the plane ABC, and six ordinary lines.

Again, three triple tangent planes can be drawn through each of the 15 ordinary lines making 45; but since this number includes every plane repeated three times, the number of distinct planes is 15. Every plane through a pair of lines of closest contact is equivalent to two triple tangent planes, since each line is counted twice; hence the total number of triple tangent planes ordinary and extraordinary is $15 + 2 \times 15 = 45$.

85. The following analytical investigation is due to Cayley; but it contains some errors which I have corrected. The equation of the surface may be taken to be

$$\alpha(1, 1, 1, l + l^{-1}, m + m^{-1}, n + n^{-1})(\beta, \gamma, \delta)^2 + pqrs\beta\gamma\delta/lmn = 0$$
$$\text{.........(40)}$$

where

$$p = mn - l, \quad q = nl - m, \quad r = lm - n, \quad s = lmn - 1 \quad \text{......(41).}$$

The six lines of closest contact lie in the planes β, γ and δ and their equations are

$$\left.\begin{array}{ll}\beta=0,\ \gamma+l\delta=0; & \beta=0,\ \gamma+\delta/l=0\\ \gamma=0,\ \delta+m\beta=0; & \gamma=0,\ \delta+\beta/m=0\\ \delta=0,\ \beta+n\gamma=0; & \delta=0,\ \beta+\gamma/n=0\end{array}\right\} \quad \text{.........(42).}$$

The 15 ordinary lines are first CD, DB and BC. Through CD two planes can be drawn one of which cuts the cubic in two lines 4, 5; whilst the other cuts it in two lines 6, 7. Similarly the lines DB and BC are respectively cut by the lines 8, 9, 10, 11 and 12, 13, 14, 15; and the equations of these lines are

4. $\alpha+ps\beta=0,\quad \beta+n\gamma+m\delta=0.$
5. $\alpha+ps\beta=0,\quad \beta+\gamma/n+\delta/m=0.$
6. $\alpha-qr\beta=0,\quad \beta+\gamma/n+m\delta=0.$
7. $\alpha-qr\beta=0,\quad \beta+n\gamma+\delta/m=0.$
8. $\alpha+qs\gamma=0,\quad \gamma+l\delta+n\beta=0.$
9. $\alpha+qs\gamma=0,\quad \gamma+\delta/l+\beta/n=0.$
10. $\alpha-rp\gamma=0,\quad \gamma+\delta/l+n\beta=0.$
11. $\alpha-rp\gamma=0,\quad \gamma+l\delta+\beta/n=0.$
12. $\alpha+rs\delta=0,\quad \delta+m\beta+l\gamma=0.$
13. $\alpha+rs\delta=0,\quad \delta+\beta/m+\gamma/l=0.$
14. $\alpha-pq\delta=0,\quad \delta+\beta/m+l\gamma=0.$
15. $\alpha-pq\delta=0,\quad \delta+m\beta+\gamma/l=0.$

To prove these results put $\alpha=-ps\beta$ in (40), and it becomes

$$\beta^2+\gamma^2+\delta^2+(l+l^{-1})\gamma\delta+(m+m^{-1})\delta\beta$$
$$+(n+n^{-1})\beta\gamma-\gamma\delta qr/lmn=0 \quad \text{......(43).}$$

Substituting the values of q and r from (41), the coefficient of $\gamma\delta$ becomes equal to $m/n+n/m$, which shows that (43) is equivalent to

$$(\beta+n\gamma+m\delta)(\beta+\gamma/n+\delta/m)=0,$$

and the remaining equations can be proved in a similar manner.

If three lines lie in the same plane, the points where they cut the plane BCD must obviously be collinear. Now the equation of the plane through 4 and 8 is

$$\alpha+ps\beta+(\beta+n\gamma+m\delta)\,ls=0,$$

which can be written in the form

$$\alpha+rs\delta+(\delta+m\beta+l\gamma)\,ns=0,$$

which shows that the plane passes through the line 12. The three lines 4, 8 and 12 therefore cut the plane α in three points which lie in the plane

4. 8. 12. $\beta/l + \gamma/m + \delta/n = 0$ ....(44),

which determines the line in which one triple tangent plane cuts the plane α. The lines in which seven other triple tangent planes cut the plane α are given by the following equations, and the figures on the left-hand side denote the lines they contain.

5. 9. 13. $l\beta + m\gamma + n\delta = 0.$

5. 11. 14. $\beta/l + m\gamma + n\delta = 0.$

6. 9. 15. $l\beta + \gamma/m + n\delta = 0.$

7. 10. 13. $l\beta + m\gamma + \delta/n = 0.$

6. 11. 12. $\beta/l + \gamma/m + n\delta = 0.$

4. 10. 15. $l\beta + \gamma/m + \delta/n = 0.$

7. 8. 14. $\beta/l + m\gamma + \delta/n = 0.$

We have thus accounted for eight out of the fifteen triple tangent planes; and the remainder are the planes which contain the lines CD, DB, BC; CD, 4, 5; CD, 6, 7; DB, 8, 9; DB, 10, 11; BC, 12, 13; BC, 14, 15.

86. *One binode.* When the cubic has a binode, let AB be its axis, ABC and ABD the biplanes; then three of the lines of closest contact AC, AC_1, AC_2 lie in the plane ABC, and three others AD, AD_1, AD_2 in the plane ABD. Nine planes can be drawn through any line lying in one plane and any line lying in the other, which intersect the cubic in a residual straight line; hence there are only 9 ordinary lines.

The condition for a binode at A is that the discriminant of the nodal cone should vanish, which shows that λ must be a factor of (39), and λ^3 a factor of (38). Accordingly each line of closest contact is equivalent to 3 ordinary lines, and therefore the total number of lines ordinary and extraordinary is equal to $9 + 3 \times 6 = 27$.

Again, through each ordinary line only two triple tangent planes can be drawn making $9 \times 2 = 18$; but since each plane occurs three times, the number of ordinary triple tangent planes is 6. The remaining extraordinary triple tangent planes are the following. Since each line of closest contact is equivalent to three ordinary lines, each plane through a pair of such lines is equivalent to three planes making $9 \times 3 = 27$; also each biplane is equivalent to 6 triple tangent planes, making the total number $6 + 27 + 12 = 45$.

87. Following Cayley, the equation of the cubic may be expressed in the form

$$\alpha(\beta+\gamma+\delta)(l\beta+m\gamma+n\delta)+(m-n)(n-l)(l-m)\beta\gamma\delta=0 \quad \text{.........(45)}.$$

The equations of the six lines of closest contact are

$$\begin{aligned} \beta &= 0, \ \gamma+\delta=0; \quad \beta=0, \ m\gamma+n\delta=0. \\ \gamma &= 0, \ \delta+\beta=0; \quad \gamma=0, \ n\delta+l\beta=0. \\ \delta &= 0, \ \beta+\gamma=0; \quad \delta=0, \ l\beta+m\gamma=0. \end{aligned}$$

The nine ordinary lines are BC, CD and DB, and six others whose equations are

$$\begin{aligned} &\alpha+l\,(m-n)\,\beta=0, \quad l\beta+m\gamma+l\delta=0. \\ &\alpha+l\,(m-n)\,\beta=0, \quad l\beta+l\gamma+n\delta=0. \\ &\alpha+m\,(n-l)\,\gamma=0, \quad l\beta+m\gamma+m\delta=0. \\ &\alpha+m\,(n-l)\,\gamma=0, \quad m\beta+m\gamma+n\delta=0. \\ &\alpha+n\,(l-m)\,\delta=0, \quad l\beta+n\gamma+n\delta=0. \\ &\alpha+n\,(l-m)\,\delta=0, \quad n\beta+m\gamma+n\delta=0. \end{aligned}$$

88. *One binode B_4.* Let AB be the axis of the binode, and let CD be one of the lines lying in the surface; then the equation of the latter may be written in the form

$$\alpha\gamma\delta+\beta^2(p\gamma+q\delta)+\beta(F\gamma^2+G\gamma\delta+H\delta^2)=0 \quad \text{......(46)}.$$

The fixed tangent plane which touches the surface along the axis intersects the surface in the axis twice repeated, and in the line

$$p\gamma+q\delta=0, \quad pq\alpha-(Fq^2-Gpq+Hp^2)\beta=0,$$

which is called the *transversal.* Hence the ordinary lines consist of the transversal and the four other lines which are the residual intersections of the planes containing the four pairs of lines of closest contact. Cayley considers that the axis ought to be regarded as equivalent to two ordinary lines, thus making a total of 7; but as this line is a singular one, I am inclined to think that it ought not to be included amongst the ordinary lines, but treated as one *sui generis.*

89. *One binode B_5.* When $q=0$, the tangent plane along the axis coincides with the biplane γ, and the singularity becomes the binode B_5. The transversal consequently disappears, since it coincides with the line of closest contact in the plane γ. There are therefore only two ordinary lines, but Cayley considers that the axis is equivalent to one ordinary line, thus making a total of 3.

90. It seems scarcely worth while to pursue the investigation of the lines which can be drawn on the remaining species of cubic surfaces; but for the purpose of reference, I shall give Cayley's form of the equations of the twenty-three different species, the proof of which may be left to the reader.

I.		$(\alpha, \beta, \gamma, \delta)^3 = 0.$
II.	$C.$	$\alpha(\beta, \gamma, \delta)^2 + 2k\beta\gamma\delta = 0.$
III.	$B.$	$\alpha(\beta+\gamma+\delta)(l\beta+m\gamma+n\delta) + k\beta\gamma\delta = 0.$
IV.	$2C.$	$\alpha\beta\delta + \gamma^2(p\alpha+s\delta) + (\beta, \gamma)^3 = 0.$
V.	$B_4.$	$\alpha\beta\delta + (\beta+\delta)(\gamma^2 - a\beta^2 - b\delta^2) = 0.$
VI.	$B+C.$	$\alpha\beta\delta + \gamma^2\delta + (\beta, \gamma)^3 = 0.$
VII.	$B_5.$	$\alpha\beta\delta + \gamma^2\delta + \gamma\beta^2 - \delta^3 = 0.$
VIII.	$3C.$	$\gamma^3 + \gamma^2(\alpha+\beta+\delta) + 4a\alpha\beta\delta = 0.$
IX.	$2B.$	$\alpha\beta\delta + (\beta, \gamma)^3 = 0.$
X.	$C+B_4.$	$\alpha\beta\delta + (\beta+\delta)(\gamma^2 - \beta^2) = 0.$
XI.	$B_6.$	$\alpha\beta\delta + \gamma^2\delta + \beta^3 - \delta^3 = 0.$
XII.	$U_1.$	$\alpha(\beta+\gamma+\delta)^2 + \beta\gamma\delta = 0.$
XIII.	$2C+B.$	$\alpha\beta\delta + \gamma^2(\beta+\gamma+\delta) = 0.$
XIV.	$C+B_5.$	$\alpha\beta\delta + \gamma^2\delta + \gamma\beta^2 = 0.$
XV.	$U_2.$	$\alpha\beta^2 + \beta\delta^2 + \gamma^2\delta = 0.$
XVI.	$4C.$	$\alpha(\beta\gamma+\gamma\delta+\delta\beta) + \beta\gamma\delta = 0.$
XVII.	$C+2B.$	$\alpha\beta\delta + \beta\gamma^2 + \gamma^3 = 0.$
XVIII.	$2C+B_4.$	$\alpha\beta\delta + (\beta+\delta)\gamma^2 = 0.$
XIX.	$C+B_6.$	$\alpha\beta\delta + \gamma^2\delta + \beta^3 = 0.$
XX.	$U_3.$	$\alpha\beta^2 + \beta\delta^2 + \gamma^3 = 0.$
XXI.	$3B.$	$\alpha\beta\delta + \gamma^3 = 0.$
XXII.	Nodal line of 1st kind	$\alpha\beta^2 + \gamma^2\delta = 0.$
XXIII.	do. 2nd kind	$\beta(\alpha\beta+\gamma\delta) + \gamma^3 = 0.$

91. *Nodal lines.* A cubic surface possessing a nodal line is evidently a scroll, since any plane through this line intersects the surface in the nodal line twice repeated and another line. We have already proved in §§ 37 and 43 that a nodal line on a surface of the nth degree reduces the class by $7n-12$, and that it has $2n-4$ pinch points. Putting $n=3$, these numbers become 9 and 2; hence cubic surfaces having a nodal line are of the third class and possess two pinch points. When both pinch points are distinct, both tangent planes are torsal and the line is one of the first kind; but when the pinch points coincide, one tangent plane is torsal and the other fixed, and the line is of the second kind.

92. *The point constituents of a nodal line on a cubic are* $C=3$, $B=1$.

The equation of a cubic having a unode at A is

$$\alpha(p\beta+q\gamma+r\delta)^2+\beta^3v_0+\beta^2v_1+\beta v_2+v_3=0 \quad \ldots\ldots(47),$$

and if (47) has a node of any kind at B, we must have $p=v_0=v_1=0$; and (47) becomes

$$\alpha(q\gamma+r\delta)^2+\beta v_2+v_3=0,$$

which represents a cubic surface on which AB is a nodal line of the first kind and A is one of the pinch points. Since the constituents of a unode are three double points, those of a nodal line are determined by the equations

$$2C+3B=9, \quad C+B=4,$$

giving $C=3$, $B=1$.

Since a nodal line of the second kind is produced by making the two pinch points coincide, its constituents are the same as those of a nodal line of the first kind.

A cubic surface cannot have a cuspidal line unless it is a cone, for if we attempt to make every point a pinch point, the surface reduces to the form

$$(p\alpha+q\beta)v_1^2+v_3=0.$$

93. *Four nodes.* We shall conclude by making a few remarks about species XVI, the equation of which is

$$l\beta\gamma\delta+m\gamma\delta\alpha+n\delta\alpha\beta+p\alpha\beta\gamma=0 \quad \ldots\ldots\ldots\ldots(48),$$

or

$$l/\alpha+m/\beta+n/\gamma+p/\delta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(49).$$

If we attempt to convert the conic node at A into a binode, the surface degrades into a quadric and a plane; hence a cubic surface cannot possess three conic nodes and one binode when the singularities are isolated. But we have shown that the constituents of a nodal line are $C = 3$, $B = 1$, from which it appears that certain combinations of conic nodes and binodes may exist in the form of a compound singularity, although they cannot exist when the double points are isolated.

Equation (49) is of the form

$$(l\alpha)^\nu + (m\beta)^\nu + (n\gamma)^\nu + (p\delta)^\nu = 0,$$

the reciprocal polar of which is obtained by changing ν into $\nu/(\nu - 1)$. Hence the reciprocal polar of (49) is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} + (p\delta)^{\frac{1}{2}} = 0 \quad \text{............(50)}.$$

Putting $l = m = n = p = 1$, and rationalizing, (50) becomes

$$(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta - 2\alpha\delta - 2\beta\delta - 2\gamma\delta)^2 = 64\alpha\beta\gamma\delta \quad \text{.........(51)}.$$

This surface is called Steiner's* quartic, and its properties will be discussed in the chapter on Quartic Surfaces.

A cubic having four conic nodes is also the envelope of the quadric $(A, B, C, F, G, H \between \lambda, \mu, \nu)^2 = 0$, where $A, B \dots$ are planes.

* *Crelle*, vols. LXIII. and LXIV.

CHAPTER III

TWISTED CURVES AND DEVELOPABLES

94. Every curve, plane or twisted, may be regarded as the limit of a polygon whose angles are denoted by the figures 1, 2, 3, &c. The lines 12, 23, &c. are tangents at successive points on the curve, and the plane 123 contains these lines. When the curve is twisted, the point 4 will in general lie in a different plane 234 which intersects the plane 123 in the line 23; also since these planes pass through three consecutive points on the curve they are *osculating* planes. A developable surface is a ruled surface each generator of which intersects the consecutive one; hence the envelope of the osculating planes to a twisted curve is a developable surface.

Since a point which is constrained to move along a twisted curve has only one degree of freedom, the osculating plane has likewise the same degree of freedom; hence the constants in its equation must be functions of a single parameter θ. Accordingly the equation of the osculating plane must be of the form

$$z = \theta x + y\phi(\theta) + \psi(\theta) \quad\quad (1),$$

where ϕ and ψ are arbitrary functions*.

95. When a developable surface is generated in the above manner, the curve whose osculating planes envelope the developable is called the *edge of regression*; and when the curve is an algebraic one, the edge of regression is a cuspidal† curve on the developable, that is to say the surface consists of two sheets which touch one another along a cusp. If a sheet of paper is

* The elimination of θ leads to the two well-known partial differential equations

$$q = \phi(p) \text{ and } rt = s^2.$$

† The Italians call a cusp *un regresso*; a rhamphoid cusp *un regresso di seconda specie* and so on.

bent along the lines 12, 23, &c., the continuations of these lines in the opposite directions will form a twisted curve on the developable; but if pieces of fine wire be gummed to the paper along the lines 12, 23, &c., whilst the continuations are left free, the latter portions will generate the other sheet of the developable. The surface therefore consists of two sheets, one of which is the bent paper, whilst the other is generated in the manner above described. The point where any plane section of the developable cuts the edge of regression is a cusp on the section; and it is obvious that if it were possible for a developable to consist of one sheet only, the point in question would be a *point d'arrêt*, and such a singularity cannot be possessed by an algebraic curve.

Any plane through a tangent, such as 12, cuts the developable in a curve having a point of inflexion at 1 and also in a generator which is a stationary tangent to the curve. If a point moves along the tangent from left to right keeping in the same sheet of the developable, the point will begin to move along the curve as soon as it has passed 1; for the continuation of the tangent is a generator on the other sheet of the developable.

96. *Every generator of a developable of degree ν is cut by $\nu-4$ other generators.*

Any plane through a generator 12 intersects the developable in this generator and in a curve of degree $\nu-1$; and since the generator 12 touches this curve at a point of inflexion, it cuts it in $\nu-4$ points which are the points where $\nu-4$ other generators cut 12.

97. *The reciprocal polar of a twisted curve is a developable surface and vice versâ.*

Let O and P, lying in the plane of the paper, be the origin of reciprocation and any point on the curve; and let the osculating plane at P be perpendicular to the plane of the paper and cut it in PA. Draw OA perpendicular to the osculating plane at P and produce it to A', so that $OA \cdot OA' = k^2$; also let P' be a point on OP such that $OP \cdot OP' = k^2$. Then the point A' is the reciprocal polar of the osculating plane at P, and its locus is the reciprocal of the developable; also since the osculating plane at P has only one degree of freedom, the point A' has the same degree of freedom, and therefore its locus is a twisted curve.

Since $OA . OA' = OP . OP' = k^2$, it follows that the four points A', A, P', P lie on a circle, and therefore the angle $A'P'P$ is a right angle. Hence the plane through $A'P'$ which is perpendicular to the plane of the paper is the reciprocal polar of P, and since it has only one degree of freedom its envelope is a developable surface.

98. Let E denote the original twisted curve, D the developable which is the envelope of its osculating planes, D' the developable which is the reciprocal polar of E; then it follows that the reciprocal polar of D is a twisted curve E' which lies in D'. Now the three consecutive points 2, 3, 4 on E lie in the osculating plane 234; and the three osculating planes 123, 234 and 345 each pass through the point 3; accordingly the tangent plane to D' which is the reciprocal polar of 3 contains the three points on E', which are the reciprocal polars of the three planes 123, 234, 345; hence this plane is an osculating plane to E', and therefore E' is the edge of regression of D'.

The curve and surface E and D' are the reciprocal polars of one another, and so also are the surface and curve D and E'; but the curves E and E' are not reciprocal polars of one another. At the same time they may be regarded as quasi-reciprocal curves, since the reciprocal polar of any point on E is an osculating plane to E' and *vice versâ*. Hence a complete theory of reciprocation exists between the curves E and E', in which the osculating plane takes the place of the tangent in the ordinary theory of the reciprocation of plane curves.

The degree ν of D is equal to the number of points in which an arbitrary straight line intersects it, and this is equal to the number of tangents to E which intersect an arbitrary straight line. Reciprocating, it follows that ν is equal to the number of tangent planes to E' which can be drawn through an arbitrary straight line, that is to the degree of D'. *Hence the degrees of D and D' are equal.*

99. Before considering the singularities of twisted curves, we must explain their generation and classification.

The degree of the complete curve of intersection of two surfaces is equal to the number of points in which it is cut by an arbitrary plane.

Two surfaces of degrees l and m intersect in a curve of degree

lm, and this is the number of points of intersection of the two surfaces and an arbitrary plane.

100. *The degree of a cone standing on a twisted curve of degree n is n; but if the vertex lie on the curve, the degree of the cone is $n-1$. Also a twisted curve cannot pass through more than $\frac{1}{2}n(n+1)$ arbitrary points.*

Draw any plane through the vertex O of the cone; then this will cut the twisted curve in n points; and since every line joining O with one of these points is a generator of the cone, its degree is n.

If however O lie on the curve, the plane will cut it in only $n-1$ other points, which is therefore the degree of the cone.

Let the curve pass through $r+1$ arbitrary points, and let the vertex O of the cone coincide with one of them; then the cone will pass through r arbitrary points. But a plane curve, and therefore a cone of degree $n-1$, cannot pass through more than $\frac{1}{2}(n-1)(n+2)$ arbitrary points; hence r cannot be greater than this quantity, and therefore $r+1$ cannot be greater than $\frac{1}{2}n(n+1)$.

101. When the degree of a twisted curve is a prime number it cannot be the complete intersection of two surfaces, for when n is prime its only factors are n and 1, and the curve would be a plane one. Consequently such a twisted curve must be the partial intersection of two surfaces of degrees l and m, whose complete curve of intersection is a compound curve consisting of one of degree n and of another curve of degree $lm-n$. For example, the complete curve of intersection of two quadric surfaces is a twisted quartic curve of the *first* species, but if the quadrics possess a common generator, the complete curve consists of the common generator and a twisted cubic curve. It also frequently happens that a curve whose degree is a prime number cannot be represented as the complete intersection of two surfaces; thus it is possible for a cubic and a quadric surface to intersect in two straight lines, lying in different planes, and in a residual quartic curve; but it can be shown that only one quadric surface can be drawn through such a quartic curve, and therefore the latter cannot be represented as the complete intersection of two quadric surfaces. Such a curve is called a quartic of the *second* species.

102. The first and most important step in the classification of twisted curves, which are the partial intersections of two surfaces, is to ascertain the two surfaces of *lowest degree* which can contain the curve. Thus the two surfaces of lowest degree which can contain a quintic curve are a quadric and a cubic surface which possess a common straight line; and such a curve is called a quintic of the *first* species. But two cubic surfaces can intersect in a quartic curve of the first species and a residual quintic curve, and we shall now show that the latter curve is a quintic of the first species.

The equations of the curve may be expressed in the form

$$\gamma S = \delta S', \quad \gamma u = \delta v \qquad \text{.......................(2)},$$

where S, S' are quadric surfaces and u, v are planes. Eliminating γ, δ we obtain

$$Sv = S'u \qquad \text{............................(3)}.$$

The two cubic surfaces (3) and the first of (2) intersect in a twisted quartic of the first species, which is the complete intersection of the quadrics S and S', and in a residual quintic curve which lies in both the surfaces (2); accordingly although (2) are the simplest equations for defining the curve, it may be expressed as the partial intersection of two surfaces of higher degrees. But the Theory of Residuation furnishes the simplest method of discovering the two surfaces in question.

103. We must now consider the characteristics of E and D.

The degree n of E is equal to the number of points in which the curve intersects an arbitrary plane. Now the *class* of a curve is defined to be that particular geometrical property which is the reciprocal of its degree; and, in the case of a twisted curve, this property is the number of tangent planes which can be drawn to D' through a point. But every tangent plane to D' is an osculating plane to E'; hence the class m of a twisted curve is equal to the number of osculating planes which can be drawn to it through a point.

A node δ can arise in three ways: (i) when the two surfaces containing the curve have ordinary contact at a point on the curve; (ii) when one surface passes through a double point on the other; (iii) when one of the surfaces possesses a nodal curve.

The reciprocal of a node is a doubly osculating plane ϖ; and such a plane is a double tangent plane to the developable D'. Since ϖ must intersect the curve in at least six points, it cannot occur on any curve of lower degree than a sextic; but, unlike a double tangent to a plane curve, it need not occur at all. Let the osculating plane at a point P cut the curve in points $Q, R, S, \ldots$; then since P has one degree of freedom, its position is determined by means of a single parameter θ; accordingly the distance QR can be expressed as some function F of θ. If, therefore, the parameter of P is a root of the equation $F(\theta)=0$, the points Q and R will coincide and the plane will touch the curve at Q; hence there is a determinate number of osculating planes which touch the curve elsewhere*; but a third point S cannot, in general, be made to coincide with Q without assuming some relation between the constants of the curve. The plane ϖ is equivalent to the two osculating planes at its points of contact.

A cusp κ occurs when the surfaces containing the curve have stationary contact at some point on it. A cusp may also, like a node, arise from the surfaces having singularities.

The reciprocal of a cusp is a stationary plane σ. Such a plane passes through four consecutive points on the curve, and at the point of contact two osculating planes coincide. The point is also one at which the tortuosity of the curve vanishes and changes sign; and since the tortuosity is some function f of θ, the plane σ occurs at the points whose parameters are the roots of the equation $f(\theta)=0$. This plane osculates the developable D' along a generator.

Since a cusp is the reciprocal of a stationary plane, the preceding argument shows that a cusp is a point through which four consecutive osculating planes pass. If, therefore, the developable is the envelope of the plane

$$a\alpha + b\beta + c\gamma + d\delta = 0,$$

where a, b, c, d are functions of a single parameter θ, the conditions for a cusp are obtained by differentiating the above equation three times with respect to θ and eliminating $(\alpha, \beta, \gamma, \delta)$.

A double tangent τ to E reciprocates into a double tangent τ' to E'. This singularity, unlike a double tangent to a plane curve, need not occur; for if a straight line touches a twisted curve, it need not intersect it elsewhere. The double tangent is a generator of D; but since the osculating planes to E at its points of contact are in general distinct, there are two tangent

* These planes will be discussed in § 111.

planes to D along a double tangent to E. Hence a double tangent to E is a nodal generator on D.

A stationary tangent ι to E reciprocates into a stationary tangent ι' to E'. Such a tangent touches the curve at a point of inflexion, which is a point where the curvature vanishes and changes sign; hence ι may always occur on a twisted curve *and must never be assumed to be zero.* A stationary tangent to E is a cuspidal generator on D.

It will hereafter appear that a considerable number of twisted curves do not possess stationary planes and tangents, the explanation of which is as follows. The tortuosity and curvature can always be expressed in the form $F(\theta)$; but if none of the roots of the equation $F(\theta)=0$ are the parameters of points on the curve, the tortuosity or curvature (as the case may be) can never vanish. A simple example is furnished by the expression for the curvature of an ellipse in terms of the excentric angle ϕ, which cannot be made to vanish for any *real* value of ϕ.

There are four other singularities which have to be taken account of.

If O be the vertex of any cone which stands on the curve, a generator which intersects the curve in two points P and Q is a nodal generator of the cone and gives rise to an *apparent node* on the curve*; since, to an eye situated at O, two branches of the curve *appear* to intersect one another. The number h of such nodal generators is equal to the number of apparent nodes on the curve.

The reciprocal singularity is called an *apparent double plane*, and consists of a pair of osculating planes whose line of intersection lies in a fixed plane. The number g of such pairs of planes is equal to the number of apparent double planes.

It is also possible for a pair of tangents to E to lie in a plane, which is consequently a double tangent plane to E; and the locus of the points of intersection of such pairs of tangents is a nodal curve on D. The degree x of this curve is equal to the number of

* Since a "double point" includes a cusp as well as a node, the phrase "apparent double point" is inappropriate; for a twisted curve cannot, in general, possess an *apparent cusp*. By properly choosing the vertex of the cone, it is quite possible for the curve to appear to have a cusp to an observer situated at the vertex; but if the vertex be shifted, the apparent cusp will be changed into an apparent node.

points in which it intersects an arbitrary plane; accordingly x is equal to the number of pairs of tangents to E, whose points of intersection lie in a fixed plane.

The reciprocal singularity y is equal to the number of pairs of tangents to E', which lie in planes passing through a fixed point; in other words, it is equal to the number of double tangent planes to the curve which pass through a fixed point.

The Plücker-Cayley Equations.

104. Thirteen quantities have therefore to be considered; and the reader will observe that I have assigned definite geometrical meanings to each of them, so that there is no need to employ such verbose and obscure phrases as "rank of the system," "planes through two lines," and the like. For the purpose of facilitating comparison, I subjoin my own notation and that of Cayley and Salmon*.

Basset—ν, n, m, δ, ϖ, κ, σ, τ, ι, h, g, x, y.

Cayley and Salmon—r, m, n, H, G, β, α, ω, v, h, g, x, y.

Cayley obtained equations connecting the thirteen characteristics of the curve in the following manner†.

Consider any plane section S of the developable D. Let $\mathfrak{N}$, $\mathfrak{M}$, $\mathfrak{D}$, $\mathfrak{K}$, $\mathfrak{T}$, $\mathfrak{I}$ denote the degree, class, and number of nodes, cusps, double and stationary tangents to S.

The degree $\mathfrak{N}$ of S is equal to the number of points in which it is cut by an arbitrary line lying in its plane; that is to say to the degree of D. Hence $\mathfrak{N} = \nu$.

The class $\mathfrak{M}$ of S is equal to the number of tangents which can be drawn to it through a fixed point O in its plane; and since each tangent lies in a tangent plane through O to D, this number is equal to the class of D, that is to the class of E. Hence $\mathfrak{M} = m$.

A node on S may arise in two ways. (i) If two tangents TP, TQ to E intersect the plane of S in a point T, the osculating planes to E at P and Q intersect the plane of S in two lines

* The symbols δ, κ, τ, ι have been used in England for many years to designate the number of nodes, cusps, double and stationary tangents which a *plane* curve possesses; and to employ different symbols for the corresponding singularities of a *twisted* curve introduces confusion and unnecessary complexity.

† *C. M. P.* vol. I. p. 207; *Liouville's Journ.* vol. x. p. 245.

TP', TQ', which are tangents to S at T. Hence two branches of S cross one another at T, and T is a node. The number of nodes arising from this cause is equal to the number x of pairs of tangents to E, whose points of intersection lie in the plane of S. (ii) In the next place the point where every nodal generator of D cuts the plane of S is a node on S; and since every such generator is a double tangent to E, there are τ nodes on S arising from this cause. Hence $\mathfrak{D} = x + \tau$.

A cusp on S also arises in two ways. (i) The points where the plane of S cuts E are obviously cusps on S, and their number is equal to the degree n of E. (ii) In the second place every stationary tangent to E gives rise to a cuspidal generator on D, and the points where these generators cut the plane of S are cusps on S; there are consequently ι cusps arising from this cause. Hence $\mathfrak{K} = n + \iota$.

A double tangent to S arises in two ways. (i) If P and Q are two points on E, the osculating planes at which intersect in a line pq lying in the plane of S, and Pp, Qq are the corresponding generators of D, the line pq is a double tangent to S whose points of contact are p and q. There are accordingly g double tangents arising from this cause. (ii) In the next place every doubly osculating plane ϖ to E intersects the plane of S in a line which is a double tangent to S. Hence $\mathfrak{T} = g + \varpi$.

A stationary tangent to S can only arise from the existence of stationary tangent planes to E; hence $\mathfrak{I} = \sigma$.

We must therefore write in Plücker's equations

$$\mathfrak{N} = \nu,\ \mathfrak{M} = m,\ \mathfrak{D} = x + \tau,\ \mathfrak{K} = n + \iota,\ \mathfrak{T} = g + \varpi,\ \mathfrak{I} = \sigma,$$

and we obtain

$$\left.\begin{aligned} m &= \nu(\nu - 1) - 2(x + \tau) - 3(n + \iota) \\ \sigma &= 3\nu(\nu - 2) - 6(x + \tau) - 8(n + \iota) \\ \nu &= m(m - 1) - 2(g + \varpi) - 3\sigma \\ n + \iota &= 3m(m - 2) - 6(g + \varpi) - 8\sigma \end{aligned}\right\} \quad \text{.........(4)},$$

of which only three are independent.

Four more equations can be obtained by considering the cone which stands on the curve; but although the use of the cone is instructive as a mathematical artifice, its employment is unnecessary, since the four remaining equations can be obtained from (4) by writing for each quantity its reciprocal. We thus obtain

$$\left.\begin{aligned} n &= \nu(\nu-1) - 2(y+\tau) - 3(m+\iota) \\ \kappa &= 3\nu(\nu-2) - 6(y+\tau) - 8(m+\iota) \\ \nu &= n(n-1) - 2(h+\delta) - 3\kappa \\ m+\iota &= 3n(n-2) - 6(h+\delta) - 8\kappa \end{aligned}\right\} \quad \text{.........(5)},$$

of which only three are independent.

105. It may be worth while to give a direct proof of (5).

The degree of the cone is equal to that of the curve, and its class is equal to the number of tangent planes which can be drawn through any arbitrary line passing through its vertex; and since the vertex may be any arbitrary point, this is equal to the number of tangents to E which intersect an arbitrary line, that is to the degree of D. Hence $\mathfrak{N} = n$, $\mathfrak{M} = \nu$.

Every generator of the cone which passes through an actual or an apparent node is a nodal generator. Hence $\mathfrak{D} = h + \delta$.

A cuspidal generator can only occur when the curve has a cusp. Hence $\mathfrak{K} = \kappa$.

A double tangent plane to the cone arises in two ways. (i) When the curve possesses a pair of tangents which lie in a plane passing through the vertex, and the number of such pairs of tangents is y. (ii) Every plane through the vertex and a double tangent to E is a double tangent plane to the cone. Hence $\mathfrak{T} = y + \tau$.

A stationary tangent plane to the cone also arises in two ways. (i) Every tangent plane to D, which passes through the vertex of the cone, is an osculating plane to E, and the number of such tangent planes is m. (ii) In the second place every tangent plane which passes through a stationary tangent to E is likewise a stationary tangent plane. Hence $\mathfrak{I} = m + \iota$.

Substituting in Plücker's equations for the cone we obtain (5).

106. There are certain other equations, called the *Salmon-Cremona* equations, the consideration of which will be postponed for the present, and we shall proceed to find the characteristics of the curve of intersection E of a pair of surfaces U and V whose degrees are M and N respectively, and which are arbitrarily situated with respect to one another.

The degree ν of the developable D is equal to

$$\nu = MN(M + N - 2).$$

The degree of D is equal to the number of tangents to E which intersect an arbitrary straight line. Let the equations of the latter be

$$\left.\begin{aligned} P\alpha + Q\beta + R\gamma + S\delta = 0 \\ p\alpha + q\beta + r\gamma + s\delta = 0 \end{aligned}\right\} \qquad \text{...........(6).}$$

The equations of the tangent to E are

$$\left.\begin{aligned} U_1\alpha + U_2\beta + U_3\gamma + U_4\delta = 0 \\ V_1\alpha + V_2\beta + V_3\gamma + V_4\delta = 0 \end{aligned}\right\} \qquad \text{...........(7),}$$

where $U_1 = dV/d\xi$, &c., $(\xi, \eta, \zeta, \omega)$ being the point of contact; and the condition that (6) and (7) should intersect is

$$\begin{vmatrix} P, & Q, & R, & S, \\ p, & q, & r, & s, \\ U_1, & U_2, & U_3, & U_4, \\ V_1, & V_2, & V_3, & V_4, \end{vmatrix} = 0 \qquad \text{...........(8).}$$

Equation (8) is a surface of degree $M + N - 2$, which passes through the points of contact of those tangents to the curve which intersect the line (6), and since the number of such points is equal to the number of points of intersection of U, V and (8), the former is equal to $MN(M + N - 2)$.

If $(\xi, \eta, \zeta, \omega)$ be the coordinates of any point O in space, (7) are the equations of the line of intersection of the polar planes of O with respect to the two surfaces; and (8) shows that if this line intersects a fixed straight line, the locus of O is a surface of degree $M + N - 2$.

107. The degree of the curve is obviously equal to MN; also if the surfaces do not touch one another $\delta = \kappa = 0$, whence, by the third of (5),

$$2h = MN(M-1)(N-1) \qquad \text{...........(9).}$$

Now the number of apparent nodes is obviously independent of the number of isolated singularities of the curve; hence the above value of h is true when the curve is autotomic. If, therefore, we substitute the value of h from (9) in (4) and (5), these equations may be reduced to the following six:—

$$\nu = MN(M + N - 2) - 2\delta - 3\kappa \qquad \text{...........(10),}$$

$$m = 3MN(M + N - 3) - 6\delta - 8\kappa - \iota \qquad \text{...........(11),}$$

$$\sigma = 2MN(3M + 3N - 10) - 12\delta - 15\kappa - 2\iota \qquad \text{.....(12),}$$

$$\begin{aligned}2(g+\varpi) = MN\{9MN(M+N-3)^2 - 22(M+N) + 71\} \\ - 6MN(M+N-3)(6\delta + 8\kappa + \iota) \\ + (6\delta + 8\kappa + \iota)^2 + 44\delta + 56\kappa + 7\iota \quad \ldots\ldots(13),\end{aligned}$$

$$\begin{aligned}2(x+\tau) = MN\{MN(M+N-2)^2 - 4(M+N) + 8\} \\ - 2MN(M+N-2)(2\delta + 3\kappa) \\ + (2\delta + 3\kappa)^2 + 8\delta + 11\kappa - 2\iota \quad \ldots\ldots(14),\end{aligned}$$

$$\begin{aligned}2(y+\tau) = MN\{MN(M+N-2)^2 - 10(M+N) + 28\} \\ - 2MN(M+N-2)(2\delta + 3\kappa) \\ + (2\delta + 3\kappa)^2 + 20\delta + 27\kappa \quad \ldots\ldots(15).\end{aligned}$$

We have already shown that, when the surfaces containing the curve are anautotomic and are arbitrarily situated with respect to one another, $\delta = \kappa = 0$; but Salmon has assumed, without proof, that $\iota = 0$, which is not permissible. That ι is zero may be proved as follows. In order that the curve may have a point of inflexion at P, the tangent at P must have tritactic contact with *both* surfaces. Let PT be the tangent at P, and let PM, PM' be the nodal tangents at P to the section of one of the surfaces by its tangent plane at P; and let PN, PN' be the corresponding tangents at the node on the section of the other surface by its tangent plane at P. Since P has only one degree of freedom, it is always possible to determine its parameter so that one or other of the tangents PM, PM' shall coincide with PT; but since a second equation of condition is necessary, in order that one or other of the tangents PN, PN' should coincide with PT, a stationary tangent cannot exist unless the surfaces are special ones, or are specially situated with reference to one another. It is also evident, from the discussion in § 103, that ϖ and τ cannot occur unless some special conditions are introduced.

108. *If the curve of intersection of two surfaces is an irreducible one, the surfaces cannot touch one another in more than*

$$\tfrac{1}{2}MN(M+N-4)+1$$

points.

Let δ be the maximum number of points of contact; then $\delta + h$ is the maximum number of nodal generators which any cone standing on the curve can possess; whence

$$2(\delta + h) = (MN-1)(MN-2).$$

Substituting the value of h from (9), we obtain the required result.

109. We shall now show that when the complete curve of intersection of two surfaces degrades into a pair of irreducible curves of degrees n_1 and n_2, the characteristics of one curve can be found when those of the other are known. We shall suppose that neither of the surfaces have stationary contact with one another, in which case neither curve can have any cusps; but if the two curves intersect, their points of intersection will be nodes on the compound curve and the two surfaces will touch at these points. Let the suffixes 1 and 2 refer to the two curves; and let H and δ' be the number of their apparent and actual intersections. Then

$$n_1 + n_2 = MN \quad\text{.......................(16)},$$

and, by (9),

$$2(h_1 + h_2 + H) = MN(M-1)(N-1) \quad\text{.........(17)}.$$

Applying the third of (5) to each of the curves 1 and 2, we obtain

$$\left.\begin{aligned} \nu_1 &= n_1(n_1 - 1) - 2h_1 \\ \nu_2 &= n_2(n_2 - 1) - 2h_2 \end{aligned}\right\} \quad\text{..................(18)},$$

whence, taking account of (16), we obtain

$$\nu_1 - \nu_2 = (n_1 - n_2)(MN - 1) - 2(h_1 - h_2) \quad\text{.........(19)}.$$

Applying the same equation to the compound curve, we obtain

$$\nu_1 + \nu_2 = MN(MN - 1) - 2(h_1 + h_2 + H + \delta').$$

Substituting the values of ν_1, ν_2 from (18), and taking account of (16), we obtain*

$$H + \delta' = n_1 n_2 \quad\text{.......................(20)}.$$

Equation (20) determines the number of actual intersections of the two curves, when the number of apparent intersections is known, and *vice versâ*.

The polar planes of U and V, with respect to a point O, intersect in a line L, and we have shown in § 106 that if L intersects a fixed straight line, then O will lie on a surface of degree $M + N - 2$. At a point where this surface intersects the curve n_1, L will be a generator of the developable enveloped by its osculating planes; but if O be a point of intersection of the curves n_1 and n_2, the two planes will coalesce with the common tangent plane to the two surfaces, and L will be the line joining O to the point where the tangent plane cuts the fixed line. Hence

* Otherwise thus. The common generators of the wo cones standing on the two curves must pass through the apparent and the actual intersections of the two curves.

$$\left.\begin{aligned} n_1(M+N-2) &= \nu_1 + \delta' \\ n_2(M+N-2) &= \nu_2 + \delta' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(21).$$

Substituting the values of ν_1, ν_2 from (18) and that of δ' from (20), we obtain

$$\left.\begin{aligned} 2h_1 + H &= n_1(M-1)(N-1) \\ 2h_2 + H &= n_2(M-1)(N-1) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(22),$$

accordingly $\quad 2(h_1 - h_2) = (n_1 - n_2)(M-1)(N-1) \quad \ldots\ldots\ldots(23),$

also from (21) $\quad \nu_1 - \nu_2 = (n_1 - n_2)(M+N-2) \ldots\ldots\ldots\ldots\ldots(24).$

Since the surfaces are supposed to be so situated that ι is zero, we obtain, in like manner, from the last of (5)

$$m_1 - m_2 = 3(n_1 - n_2)(M+N-3) \ldots\ldots\ldots\ldots(25),$$

and from the first two of (4)

$$\sigma_1 - \sigma_2 = 2(n_1 - n_2)(3M + 3N - 10) \quad \ldots\ldots\ldots(26).$$

The preceding equations contain the principal formulæ, and the reader can easily extend them to the case in which the two curves have actual nodes and cusps.

110. When the vertex of the cone standing on the curve has a special position, its characteristics are different from those of a cone whose vertex is arbitrary; and the following table, due to Cayley*, gives various results of importance.

	Vertex	Degree	Class	Nodal Generators	Cuspidal ditto	Double Tangent Planes	Stationary ditto
1.	On a tangent	n	$\nu-1$	$h-1+\delta$	$\kappa+1$	$y-\nu+4$ $+\tau$	$m-2$ $+\iota$
2.	On the curve	$n-1$	$\nu-2$	$h-n+2$ $+\delta$	κ	$y-2\nu+8$ $+\tau$	$m-3$ $+\iota$
3.	At a node	$n-2$	$\nu-4$	$h-2n+6$ $\delta-1$	κ	$y-2\nu+20$ $+\tau$	$m-6$ $+\iota$
4.	At a cusp	$n-2$	$\nu-3$	$h-2n+6$ $+\delta$	$\kappa-1$	$y-3\nu+13$ $+\tau$	$m-4$ $+\iota$
5.	On a stationary tangent	n	$\nu-2$	$h-2+\delta$	$\kappa+2$	$y-2\nu+9$ $+\tau$	$m-3$ $+\iota-1$
6.	At the point of contact of ditto	$n-1$	$\nu-3$	$h-n+1$ $+\delta$	$\kappa+1$	$y-3\nu+14$ $+\tau$	$m-4$ $+\iota-1$
7.	On a double tangent	n	$\nu-2$	$h-2+\delta$	$\kappa+2$	$y-2\nu+10$ $+\tau-1$	$m-4$ $+\iota$
8.	At a point of contact of ditto	$n-1$	$\nu-3$	$h-n+1$ $+\delta$	$\kappa+1$	$y-3\nu+15$ $+\tau-1$	$m-5$ $+\iota$

* *C. M. P.* vol. VIII. p. 72; *Quart. Journ.* vol. XI. p. 294.

The reciprocal of the cone is a plane section of the developable D', and the corresponding characteristics of the section are its *class, degree, double tangents, stationary tangents, nodes,* and *cusps.* Also in the eight special cases, the plane:—1 passes through a tangent; 2 is a tangent plane; 3 is a double tangent plane ϖ; 4 is a stationary plane σ; 5 passes through a stationary tangent ι; 6 is the tangent plane at contact of ditto; 7 passes through a double tangent τ; 8 is a tangent plane at one of the contacts of ditto. In each of these eight respective cases, the singularities of the section can be obtained from the table by writing for each quantity its reciprocal.

Denoting, as before, the characteristics of the section by old English letters, it follows from Plücker's equations that if the values of $\mathfrak{N}$, $\mathfrak{M}$ and $\mathfrak{I}$ can be found, the remaining three can be found from the equations

$$2\mathfrak{D} = \mathfrak{N}^2 - 10\mathfrak{N} + 8\mathfrak{M} - 3\mathfrak{I},$$
$$\mathfrak{K} = 3\mathfrak{N} - 3\mathfrak{M} + \mathfrak{I},$$
$$2\mathfrak{T} = \mathfrak{M}(\mathfrak{M} - 1) - \mathfrak{N} - 3\mathfrak{I}.$$

To prove 1, we observe that since the plane passes through a tangent to E, the section of D consists of the tangent and a residual curve of degree $\nu - 1$. Hence $\mathfrak{N} = \nu - 1$.

The class of the section is equal to m; hence $\mathfrak{M} = m$.

The tangent through which the plane passes is a stationary tangent to the section, hence $\mathfrak{I} = \sigma + 1$.

Substituting in the first equation, we obtain

$$2\mathfrak{D} = \nu^2 - 12\nu + 8 + 8m - 3\sigma,$$

and from the first two of (4), we obtain

$$8m - 3\sigma + \nu^2 = 10\nu + 2(x + \tau),$$

which gives $$\mathfrak{D} = x + \tau - \nu + 4,$$

and the rest may be proved in a similar manner.

The Salmon-Cremona Equations.

111. We have already shown that a determinate number of planes exists, which osculate the curve at one point and touch it at another. The reciprocal singularity, which we shall call γ, consists of a point the tangent at which intersects the curve. Let the tangent at P intersect the curve at Q; then Q is a point where two tangents intersect on the curve, and therefore the nodal curve x intersects E at Q.

Every double tangent plane to E contains a pair of tangents which intersect on the nodal curve x. The envelope of these planes is a developable called the *bitangential developable*; and its class is equal to the number of double tangent planes which can be drawn through a fixed point, that is to y. Its reciprocal polar is the nodal curve on the developable D'.

Since every double tangent plane has one degree of freedom, every curve possesses a determinate number t' of triple tangent planes; and the tangent lines at the three points of contact form a triangle whose vertices lie on the nodal curve on D. The reciprocal polar of a plane t' is a point on the nodal curve on D' at which there are three tangent planes; in other words it is a cubic node on D'. The point may also be regarded as one from which three tangent lines can be drawn to E'; or as a triple point on the nodal curve on D'.

Let k be the number of apparent nodes on the nodal curve on D; then the reciprocal singularity is the number of apparent double planes of the bitangential developable of E'.

Let q be the class of the nodal curve on D; then its reciprocal polar is the degree of the bitangential developable of E'.

We have therefore the following additional eight quantities, the last four of which have a reciprocal relation to the first four.

γ, the number of tangents which intersect E in one other point.

t, the number of triple points on the nodal curve x.

k, the number of apparent nodes on x.

q, the class of the nodal curve x.

γ', the number of tangent planes which osculate E at one point and touch it at another.

t', the number of triple tangent planes to E.

k', the number of apparent double planes of the bitangential developable of E.

q', the degree of the latter developable.

Cayley*, who was in correspondence with Cremona at the time, has given eight equations connecting these quantities with

* *C. M. P.* vol. VIII. p. 72.

the first thirteen characteristics of the curve; but instead of following Cayley's method I shall employ that of Zeuthen*, who made use of united points in the Theory of Correspondence. This will illustrate a totally different method of dealing with these and other problems.

112. The Theory of United Points, which is due to Chasles†, depends upon the following theorem.

On a given straight line let there be two sets of points, whose distances from a fixed point O on the line are x and y. Let λ points x correspond to a given point y; and let μ points y correspond to a given point x. Then if the distances between the two sets of points are connected by an algebraic equation, the number of points x which coincide with points y is $\lambda + \mu$.

These sets of coincident points are called *united points.*

By hypothesis, the distances between the two sets of points are expressed by means of an equation of the form

$$x^\lambda (Ay^\mu + By^{\mu-1} + \ldots) + x^{\lambda-1} (A'y^\mu + B'y^{\mu-1} + \ldots) + \ldots = 0 \ldots (1),$$

for if y has a determinate value b, equation (1) is of degree λ in x; whilst if x has a determinate value a, (1) is an equation of degree μ in y. When $x = y$, (1) becomes an equation of degree $\lambda + \mu$ in x, which proves the theorem.

Cayley‡ has extended this theorem to curves of deficiency p, and has shown that:—*If two points on a curve of deficiency p have a* (λ, μ) *correspondence, the number of united points is* $\lambda + \mu + 2kp$ *where k is a constant to be determined.* But it will not be necessary to consider this extension of Chasles' theorem.

Since the reciprocal of a point on a fixed line is a plane through another fixed line, Chasles' theorem is true in the case of two sets of planes through a fixed straight line, which have a (λ, μ) correspondence.

It frequently happens that p points x coincide with q points y, in which case the point is equivalent to pq united points; for if we select any single point x of the group, there are q points y which coincide with it; and since there are p coincident points x,

* *Annali di Matematica*, vol. III. Serie 2, p. 175.

† *Nouvelles Annales de Mathématiques*, 2e Serie, vol. V. p. 295.

‡ *C. M. P.* vol. VI. p. 9; *Proc. Lond. Math. Soc.* vol. I. April 16, 1866.

the total number must be pq. The numbers p and q are determined by the conditions of the problem under consideration.

A line which cuts a twisted curve three times, four times, &c., is called a *trisecant, quadrisecant,* &c.

We are now in a position to proceed with the proof of the Salmon-Cremona equations.

113. *The trisecants envelope a scroll whose degree T is*

$$T = (n-2)\{h - \tfrac{1}{6}n(n-1)\} \quad\text{.................(2).}$$

Let x be the distance of any point on a fixed line L from a fixed point on L; let the line xPQ cut the curve in P and Q; let RPy and RQy' be two lines through another point R on the curve, which cut L in y and y'; and take x and y, y' as corresponding points. Since the plane xPy cuts the curve in n points, there are $n-2$ points such as R; also since each of the lines RP and RQ gives rise to a y point, the plane xPy produces $2(n-2)$ points of type y corresponding to a single point of type x. But the number of lines such as xPQ, which can be drawn through a single point x, is equal to h; hence the total number of points y which correspond to a single point x is $2h(n-2)$. Accordingly

$$\lambda = \mu = 2h(n-2) \quad\text{....................(3).}$$

United points will occur:—

(i) When the point R coincides with Q. In this case Qy' is a tangent to the curve, and since there are $n-2$ points of type x in the plane through L and Q, this plane produces $n-2$ united points; but ν tangent planes can be drawn through any fixed line to the curve, hence the total number of united points is $\nu(n-2)$.

(ii) When the plane through L contains a double point. The degree ν of D, which is equal to the number of tangent planes that can be drawn through L to the curve, is given by the equation

$$\nu = n(n-1) - 2h - 2\delta - 3\kappa \quad\text{..............(4);}$$

accordingly when the curve is anautotomic, the value of ν is given by the first two terms; and (4) shows that every plane which passes through a node is equivalent to two tangent planes, and every plane which passes through a cusp is equivalent to three. From this it follows that when the curve possesses δ nodes and κ cusps, the number of united points is equal to $(n-2)(2\delta + 3\kappa)$.

(iii) When the line xPQ is a trisecant. Let R be the remaining point in which this line cuts the curve; then since the trisecant is a triple generator of the cone whose vertex is x, the number of its distinct apparent nodal generators is $h-3$, and the number of distinct points y produced by them is $2(n-2)(h-3)$. Also the number of points y arising from the plane through L and the trisecant is $3(n-3)$. But if we take any line yPS in this plane, we shall find that the number of distinct points x corresponding to y is

$$2(h-1)(n-2)+2(n-4)+2+1=2h(n-2)-1,$$

which shows that the point in which the trisecant cuts L is equivalent to two points x; and therefore each of the points y under consideration is equivalent to two points, making a total of $6(n-3)$. Accordingly the total number of united points produced by the trisecant is

$$2(n-2)h-2(n-2)(h-3)-6(n-3)=6,$$

and if T trisecants intersect L, the total number is $6T$.

Collecting our results we obtain

$$\lambda+\mu=(\nu+2\delta+3\kappa)(n-2)+6T.$$

Substituting the value of $\lambda+\mu$ from (3), and that of $2\delta+3\kappa$ from (4), we finally obtain

$$T=(n-2)\{h-\tfrac{1}{6}n(n-1)\} \quad \text{.................(5).}$$

114. *The degree of the bitangential developable is*

$$q'=\nu(n-3)-2\delta-3\kappa \quad \text{.................(6).}$$

Let P and Q be the points of contact of a double tangent plane to E; then PQ is a generator of the bitangential developable and the degree q' of the latter is equal to the number of lines such as PQ which intersect a fixed line L.

We must first find the number of points l on L, from which a pair of straight lines lying in a plane u through L can be drawn, each of which intersects the curve E in two points.

The plane u will cut E in n points $P, Q, \dots P_n$; the line PQ will cut L in a point x; the lines joining the $n-2$ remaining points will cut L in $\frac{1}{2}(n-2)(n-3)$ points y; and we shall take x and y as corresponding points. Since the cone, standing on the curve, whose vertex is x has h apparent nodal generators, there are h lines such as xPQ; hence the total number of points corresponding to a single point x is $\frac{1}{2}(n-2)(n-3)h$. We thus obtain

$$\lambda = \mu = \tfrac{1}{2}(n-2)(n-3)h \quad\text{.................(7).}$$

The points l are the only united points; but since each point l is equivalent to two x points, the former is equivalent to two united points. We thus obtain

$$\lambda + \mu = (n-2)(n-3)h = 2l \quad\text{..............(8).}$$

In the second place, $h-1$ apparent nodal generators can be drawn through the point x exclusive of xPQ. Let the planes through L and these $h-1$ generators be the planes v, and take u and v as corresponding planes. Since the plane u cuts the curve in n points, there will be $\frac{1}{2}n(n-1)$ points on L such as x, and consequently there will be $\frac{1}{2}n(n-1)(h-1)$ planes v corresponding to a single plane u; accordingly

$$\lambda' = \mu' = \tfrac{1}{2}n(n-1)(h-1) \quad\text{.................(9).}$$

United planes will occur:

(i) When the plane u contains a trisecant $x'PQR$. Since the trisecant is a triple generator of the cone whose vertex is x', the latter has only $h-3$ distinct apparent nodal generators, and consequently there are only $h-3$ distinct planes v; hence the trisecant gives rise to two planes v which coincide with u. But the number of distinct points x on L is now equal to

$$\tfrac{1}{2}(n-3)(n-4) + 3(n-3) = \tfrac{1}{2}n(n-1) - 3,$$

which shows that three points x coincide with x'; accordingly a trisecant gives rise to 6 united planes, and since T of them cut L, the total number is $6T$.

(ii) When the plane u contains a generator of the bitangential developable. Let $x'PQ$ be the generator; then since the tangents to the curve at P and Q lie in the tangent plane to the cone along $x'PQ$, this generator is an *apparent tacnodal* generator which is formed by the union of two apparent nodal generators. Hence there are only $h-2$ remaining apparent nodal generators, which give rise to one united plane; accordingly the total number is q'.

(iii) The ν ordinary tangent planes which can be drawn through L to the curve do not give rise to any united planes; but it is otherwise when a plane u passes through a double point. Let N be a node, P any other point on the curve in the plane u, and let NP cut L in x; then the line xPN is a triple generator of the cone whose vertex is x, because the node N gives rise to two tangent planes and the point P to a third, all of which touch the cone along the generator xPN. Let h' be the number of apparent nodal generators exclusive of xPN, δ the number of actual nodes on the curve, then $h'+3+\delta-1=h+\delta$, giving $h'=h-2$; hence one v plane coincides with a u plane. But since the plane in question is equivalent to two u planes, the node and the point P give rise to two united planes; also since there are $n-2$ points P and δ nodes, the latter produce $2(n-2)\delta$ united planes.

(iv) Since a plane through L and a cusp is equivalent to three u planes, it can be shown in the same manner that the κ cusps produce $3(n-2)\kappa$ united planes.

(v) The l points mentioned above obviously give rise to $2l$ united planes. We thus obtain

$$\lambda'+\mu'=6T+q'+(n-2)(2\delta+3\kappa)+2l.$$

Substituting the value of $\lambda'+\mu'$ from (9) and those of T and l from (2) and (8), we finally obtain

$$q'=(n-3)\{n(n-1)-2h\}-(n-2)(2\delta+3\kappa),$$

which by virtue of (4) may be expressed in the form (6) or by the alternative equation

$$q'=2h+\nu(n-2)-n(n-1) \quad \text{..............(10).}$$

115. *The number of tangents which cut the curve in one other point is*

$$\gamma=\nu(n-4)+4h-2n(n-3)-2\iota-4\tau \quad \text{......(11).}$$

Let a plane u pass through a line L and cut the curve in n points P. Through one of these points draw a line PQR cutting the curve in Q and R; also through the line L and the two points Q and R draw two planes v, v'; and take u and v as corresponding planes. Since by the table to § 110 the cone whose vertex is P has $h-n+2$ apparent nodal generators, the number of planes v arising from a point P is $2(h-n+2)$, and since there are n points P, the total number of planes v corresponding to a single plane u is $2n(h-n+2)$. Hence

$$\lambda=\mu=2n(h-n+2) \quad \text{..................(12).}$$

United planes will occur:

(i) When a point Q coincides with P, in which case PQR is one of the tangents γ.

(ii) When one of the points P is a node. The cone whose vertex is P now possesses $h-2n+6$ apparent nodal generators, hence the effect of a node is to produce

$$2(h-n+2)-2(h-2n+6)=2(n-4)$$

united planes. Accordingly δ nodes produce $2(n-4)\delta$ of such planes.

(iii) When one of the points P is a cusp. If one of the planes u is a tangent plane to the curve no united planes are produced, but there are $2(h-n+2)$ pairs of coincident planes v none of which coincides with a u plane. But a plane which passes through L and a node is equivalent to two tangent planes, and the fact that a node produces $2(n-4)$ united planes indicates that $n-4$ planes v coincide with each of the two u planes. A plane through L and a cusp is equivalent to three tangent planes, from which we conclude that a cusp produces $3(n-4)$ united planes making a total of $3(n-4)\kappa$.

(iv) Let one of the points P be the point of contact of a stationary tangent; then by § 110 the number of united planes is

$$2(h-n+2)-2(h-n+1)=2,$$

which makes a total of 2ι united planes.

(v) Let one of the points P be the point of contact of a double tangent; then the number of united planes is in like manner 2, but since there are two points of contact this number must be doubled making a total of 4τ.

(vi) Let the plane u contain a trisecant PQR; then since this line is an apparent nodal generator on each of the cones whose vertices are P, Q and R, the number of distinct planes v is

$$2(n-3)(h-n+2)+6(h-n+1)=2n(h-n+2)-6,$$

so that the number of united planes is 6, making a total of $6T$.

We thus obtain

$$\lambda+\mu=\gamma+(n-4)(2\delta+3\kappa)+2\iota+4\tau+6T,$$

which by (12), (4) and (2) may be expressed in the form (11), or by the alternative equation

$$\gamma=n(\nu+4)-6(\nu+\kappa)-4(\delta+\tau)-2\iota\ldots\ldots\ldots(13).$$

116. *The number of triple tangent planes is given by the equation*

$$4y(\nu-5)=6t'+3\gamma'+\gamma+12\varpi+4\delta+3\iota(\nu-6)+2\iota$$
$$+2\tau(\nu-6)+\nu(\nu-4)(\nu-5)\quad\ldots\ldots(14).$$

Let L be a fixed line, O a fixed point on it, u an arbitrary point on L. Through u draw a double tangent plane U to the curve, touching it at P and Q; let TP, TQ be the tangents at P and Q; through TP and TQ respectively draw a series of double tangent planes V, V' intersecting L in v, v'; and take u and v as corresponding points.

The line TP is a generator of the developable D, and it is intersected by $\nu-4$ other generators, hence $\nu-4$ tangent planes can be drawn through TP to the curve; but since one of these is the plane U, the number of remaining planes which determine the points v is $\nu-5$. Accordingly the number of v points which correspond to a single point u is $2(\nu-5)$; but y double tangent planes can be drawn through the point u to the curve, hence $2y(\nu-5)$ points v correspond to a single point u; and therefore

$$\lambda=\mu=2y(\nu-5)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

United points will occur:

(i) When U is a triple tangent plane t'. Let R be the third point of contact; then U cuts D in three straight lines and in a residual curve of degree $\nu-3$, but since TP touches this curve at a point of inflexion at P, it intersects it in $\nu-6$ remaining points and consequently $\nu-6$ generators of D pass through TP. But TQ and the tangent at R are two of these, therefore the number of remaining generators is $\nu-8$; accordingly the number of distinct double tangent planes, which can be drawn through TP, is $\nu-8$. It therefore follows that there are $2(\nu-8)=2(\nu-5)-6$ planes V corresponding to the plane U, hence the number of united points due to the t' planes is $6t'$.

(ii) When U is a plane γ'. Let this plane osculate the curve at P and touch it at Q; then the residual curve on D is of degree $\nu-3$ and consequently $\nu-7$ double tangent planes, exclusive of U, can be drawn through TP. Through TQ, $\nu-4$ double tangent planes can be drawn which include U twice repeated by reason of the fact that U osculates the curve at P;

hence the number of distinct planes V due to TQ is $\nu-6$, making the total number of points v equal to $\nu-7+\nu-6=2(\nu-5)-3$. Accordingly each plane γ' gives rise to three united points.

(iii) When U contains a tangent γ. Let P be the point of contact, Q the point where the tangent intersects the curve; then the number of planes V is $\nu-6+\nu-5=2(\nu-5)-1$, so that one united point is produced making a total equal to γ.

(iv) When U is a doubly osculating plane to E. The degree of the section of D is $\nu-4$; also if TP, TQ be the lines of contact of U, each of them osculates the section at P and Q, hence the number of generators which cut TP, exclusive of TQ, is $\nu-4-6-1=\nu-11$. Accordingly the total number of V planes is $2(\nu-11)=2(\nu-5)-12$; and therefore the number of united points is 12ϖ.

(v) When U is an osculating plane at a node P. Let PT, PT' be the nodal tangents; then the degree of the section is $\nu-2$, and the line PT intersects the section in $3+1=4$ coincident points at P; hence the number of generators which cut PT, exclusive of PT', is $\nu-2-4-1=\nu-7$; and as there are two osculating planes at a node, the number of planes V is $2(\nu-7)=2(\nu-5)-4$; hence δ nodes give rise to 4δ united points.

(vi) A stationary tangent TP to E is a cuspidal generator on D, and the tangent plane to D along it is the cuspidal tangent plane. The degree of the section is therefore $\nu-3$, and the number of generators which intersect TP is $\nu-3-3=\nu-6$. Let one of the planes through TP and a generator TQ be a plane U, then the number of points v due to TP is $\nu-7$, and to TQ is $\nu-1-3-2=\nu-6$, making a total of $2\nu-13=2(\nu-5)-3$, giving 3 united points. And since the total number of points u is $\nu-6$ and there are ι stationary tangents, the number of united points is $3\iota(\nu-6)$.

(vii) We have still to consider the case in which U is the cuspidal tangent plane to D. Since a stationary tangent is equivalent to two ordinary tangents, the point in which U intersects D is equivalent to two u points; also the only generator lying in the plane U is the coincident tangent PT, and there is consequently only one v point, so that the total number due to this cause is 2ι.

(viii) Let a double tangent touch the curve at P and Q; then since PQ is a nodal generator on D, the degree of any section through D is $\nu-2$, and therefore $\nu-2-3-3=\nu-8$ other generators intersect PQ; and if R is the point of contact of one of them with E, the plane PQR is an improper triple tangent plane to E, which is equivalent to one actual triple tangent plane. Zeuthen has omitted the terms due to τ, and his method does not apparently enable them to be determined; I shall therefore assume that each plane such as PQR produces r united points, and that the two osculating planes at P and Q produce s more, making the total number equal to $r(\nu-8)+s$, where r and s are positive integers which will hereafter be determined.

(ix) Let u be one of the points in which L cuts D. Then if l be the generator through u, $\nu-4$ double tangent planes can be drawn to E through l, and since U is one of them the number of V planes is $\nu-5$. But the number of U planes corresponding to the point u is $\nu-4$, which shows that $(\nu-4)(\nu-5)$ points v coincide with u, and since there are ν points u, the total number of united points is $\nu(\nu-4)(\nu-5)$.

Collecting our results we obtain

$$4y(\nu-5)=6t'+3\gamma'+\gamma+12\varpi+4\delta+3\iota(\nu-6)+2\iota$$
$$+r\tau(\nu-8)+s\tau+\nu(\nu-4)(\nu-5)\ldots(16).$$

In this equation substitute the value of γ from (13) and the value of γ', which is obtained from (13) by writing for each quantity its reciprocal; and we shall obtain

$$6t'=4y(\nu-5)-(3m+n)(\nu+4)+18\sigma-3\iota(\nu-8)$$
$$+6\kappa-\nu(\nu^2-9\nu-4)+\{16-r(\nu-8)-s\}\tau\ldots(17).$$

From the Plücker-Cayley equations we easily obtain

$$3m+n=\sigma+3\nu-\iota,$$
$$2y=\nu(\nu-10)+8n-3\kappa-2\tau,$$

and if these values of $3m+n$ and $2y$ be substituted in (17), it follows that all the quantities involved are independent of τ except t'. The term involving τ is $\{36-4\nu-r(\nu-8)-s\}\tau$, and since we have already shown that the effect of a double tangent is to reduce the number of *proper* triple tangent planes by $\nu-8$, it follows that

$$36-4\nu-r(\nu-8)-s=-6(\nu-8),$$

or
$$r(\nu-8)+s=2(\nu-6),$$
which gives $r=2, s=4$. Substituting in (16), we obtain (14).

Equation (17) may now be reduced to
$$3t'=-\nu(\nu-6)(\nu-7)+8n+y(3\nu-26)-2\tau \quad \text{......(18)}$$
by substituting the values of γ, γ' from (13) and reducing by means of the Plücker-Cayley equations.

117. *The number of apparent double planes of the bitangential developable is given by the equation*
$$2k'=y(y-1)-q'-6t'-3\gamma'-3\iota(\nu-6)-2\tau(\nu-8)-2\delta-18\varpi \quad \text{......(19)}.$$

Let D_b be the bitangential developable, E_b its edge of regression; and let us denote the characteristics of E_b by suffixed letters. Then
$$\nu_1=q', \quad m_1=y, \quad g_1=k',$$
so that the third of (4) of §104 becomes
$$2k'=y(y-1)-q'-2\varpi_1-3\sigma_1 \quad \text{..............(20)}.$$

Every double tangent plane to E touches D_b along the chord of contact of the two tangents to E, and is an osculating plane to E_b; hence if PQ, QR, RP be the tangents to E at the points of contact of a triple tangent plane, the latter will osculate E_b at three points and will therefore be equivalent to three doubly osculating planes. Accordingly the t' triple tangent planes to E give rise to $3t'$ doubly osculating planes to E_b.

Let TPQ be a double tangent to E; P and Q its points of contact; then $\nu-8$ double tangent planes can be drawn to E through TPQ. Hence if R be the point of contact of one of them, this plane will touch D_b along the generators RP, RQ, and will therefore be a doubly osculating plane to E_b. Accordingly τ double tangents to E give rise to $\tau(\nu-8)$ doubly osculating planes to E_b.

Let PT, PT' be the tangents at a node on E; then the degree of the section of D by the tangent plane along PT is $\nu-2-4=\nu-6$, which shows that PT is intersected by $\nu-6$ generators of D, exclusive of PT'. Accordingly the plane TPT' is equivalent to two double tangent planes to E and to two osculating planes to E_b. Hence this plane is a doubly osculating plane to E_b and the lines TP, $T'P$ are the tangents at its points of contact; and therefore δ nodes on E produce δ doubly osculating planes to E_b.

A doubly osculating plane to E produces on E_b a compound plane singularity, whose constituents are a certain number of doubly osculating and stationary planes to E_b. I have not been able to determine the constituents of this singularity directly, and I shall therefore assume that a doubly osculating plane ϖ to E produces $\lambda\varpi$ tangent planes ϖ_1 and $\mu\varpi$ planes σ_1 to E_b, and we thus obtain

$$\varpi_1 = 3t' + \tau(\nu - 8) + \delta + \lambda\varpi \;\ldots\ldots\ldots\ldots\ldots(21).$$

Let a plane γ' osculate the curve at P and touch it at Q; then since every double tangent plane to E is an osculating plane to E_b, and the plane γ' is equivalent to two tangent planes at P, the former is equivalent to two coincident osculating planes to E_b, that is to a plane σ_1.

If PT is a stationary tangent to E, $\nu - 6$ double tangent planes can be drawn through it to E, and since each of them osculates the curve at P, each plane is a stationary plane to E_b, the tangent at which is the line joining P to its other point of contact with E. Hence there are $\iota(\nu - 6)$ stationary planes due to this cause; accordingly

$$\sigma_1 = \gamma' + \iota(\nu - 6) + \mu\varpi \;\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Substituting these values of ϖ_1 and σ_1 from (21) and (22) in (20) we obtain

$$2k' = y(y-1) - q' - 6t' - 2\tau(\nu - 8) - 2\delta$$
$$- 3\gamma' - 3\iota(\nu - 6) - (2\lambda + 3\mu)\varpi \;\ldots\ldots(23),$$

adding this to (14) we obtain

$$2k' + 4y(\nu - 5) = y(y-1) - q' + 2\delta + \gamma + 4\tau$$
$$+ \nu(\nu - 4)(\nu - 5) + (12 - 2\lambda - 3\mu)\varpi \;\ldots\ldots(24).$$

The singularities δ, κ, τ are independent of one another and need not exist; and if we examine the Plücker-Cayley equations it will be found that ν depends on n, δ and κ; whilst y depends on n, ν, κ and τ, since $m + \iota$ can be eliminated from the first two of (5) of § 104. Also from (6), q' depends on ν, n, δ, κ; whilst by (11), γ depends on ν, n, h, ι and τ. Hence in (24), k' is the only quantity whose value is affected by changes in the value of ϖ; and we can therefore obtain the value of $2\lambda + 3\mu$ by ascertaining how many planes k' are equivalent to a plane ϖ.

Let TP, TQ be two ordinary generators of D lying in the same plane, which are indefinitely close to the lines of contact of a double tangent plane ϖ to D. Let the generator $T'P'$ intersect TQ at a point T' near T, and let the generator $T''Q'$ intersect TP at another point T'' near T. When the generator $T'P'$ coincides with TP, and $T''Q'$ coincides with TQ the plane TPQ becomes a double tangent plane ϖ to D, which is equivalent to the three coincident osculating planes TPQ, $T'P'Q$ and $T''PQ'$ to E_b. These three coincident planes are equivalent to $\frac{1}{2}(3 \times 2) = 3$ planes k'; and a doubly osculating plane ϖ therefore reduces the number of planes k' by 3 and $2k'$ by 6. Whence $12 - 2\lambda - 3\mu = -6$, giving $2\lambda + 3\mu = 18$, and substituting in (23) we obtain (19).

Equation (19) gives the value of k' in the form obtained by Cayley, *C. M. P.*, Vol. VIII. p. 76; and by means of the Plücker-Cayley equations it can be reduced to the form

$$2k' = \nu^3 - 9\nu^2 + 17\nu + y(y - 4\nu + 19) + 4n - 3\kappa - 6\varpi \quad \ldots(25).$$

Cayley has given a direct proof of the value of k, and his result may be obtained from (19) by writing for each quantity its reciprocal, so that the last term is 18δ. According to Cayley, $\lambda = 6$, $\mu = 2$, but for the reasons I have stated in the *Quart. Jour.* Vol. XL. note to p. 217, I am not satisfied with this result and am inclined to think that $\lambda = 9$, $\mu = 0$. At the same time it is satisfactory to have established, by an independent investigation, that the coefficient of ϖ is 18.

118. Equations (10), (11), (18) and (25) give the four Salmon-Cremona equations for q', γ, t' and k', and the remaining four can be obtained by writing for each quantity its reciprocal. The eight equations are thus :—

$$\gamma = \nu(n-4) + 4h - 2n(n-3) - 2\iota - 4\tau,$$
$$3t = -\nu(\nu-6)(\nu-7) + 8m + x(3\nu - 26) - 2\tau,$$
$$q = 2g + \nu(m-2) - m(m-1),$$
$$2k = \nu^3 - 9\nu^2 + 17\nu + x(x - 4\nu + 19) + 4m - 3\sigma - 6\delta,$$
$$\gamma' = \nu(m-4) + 4g - 2m(m-3) - 2\iota - 4\tau,$$
$$3t' = -\nu(\nu-6)(\nu-7) + 8n + y(3\nu - 26) - 2\tau,$$
$$q' = 2h + \nu(n-2) - n(n-1),$$
$$2k' = \nu^3 - 9\nu^2 + 17\nu + y(y - 4\nu + 19) + 4n - 3\kappa - 6\varpi.$$

We have also the alternative equations for q, q', viz.

$$q = \nu(m-3) - 3\sigma - 2\varpi,$$
$$q' = \nu(n-3) - 3\kappa - 2\delta.$$

Unicursal Twisted Curves.

119. If the coordinates of any point on a twisted curve can be expressed as rational integral functions of a parameter θ, the curve is called a *unicursal* or *rational* curve.

120. *A unicursal curve is the edge of regression of the developable enveloped by the plane*

$$u = (a, b, c, d, e, f \dots a_m \between \theta, 1)^m = 0 \quad \dots\dots\dots\dots (1),$$

where $a, b, c \dots$ *are arbitrary planes, and* θ *is a variable parameter.*

Let D be the developable enveloped by (1), E its edge of regression; E' and D' the reciprocal polars of D and E. If the coordinates of any point O be substituted in (1), the roots of the resulting equation will determine the parameters of the points of contact $P_1, P_2 \dots P_m$ of the osculating planes to E which pass through O; hence the class of E and therefore of D is equal to m. These planes intersect in the lines $OQ_{12} \dots$, where OQ_{12} is the line of intersection of the osculating planes at P_1 and P_2; but if O is so situated that (1) has a pair of equal roots, two of the points P_1 and P_2 will coincide, and OQ_{12} becomes the tangent to E at P_1. Let Δ_m, Δ_{m-r} be the discriminants of (1) and of its rth differential coefficient with respect to θ; then the condition for a pair of equal roots is

$$\Delta_m = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (2),$$

and this furnishes a relation between the coordinates of O, which gives the equation of the surface on which O lies. Hence (2) is the equation of the developable enveloped by (1).

If three roots of (1) are equal, O must be a point through which three osculating planes to E can be drawn; hence O must lie on E. Accordingly the equations of E are

$$\Delta_m = 0, \quad \Delta_{m-1} = 0 \quad \dots\dots\dots\dots\dots (3),$$

but (3) may usually be simplified by means of the Theory of Invariants. Thus if $m = 4$, the simplest equations, which are equivalent to (3), are $I = 0$, $J = 0$, where I and J are the invariants of the quartic.

121. Equations (3) are equivalent to the result of eliminating θ between the three equations

$$u = 0, \quad \partial u/\partial\theta = 0, \quad \partial^2 u/\partial\theta^2 = 0 \quad \dots\dots\dots\dots (4),$$

which give two surfaces intersecting in the curve E. Equations (4) are also those of three planes intersecting at a point on E, in which the coefficients of the coordinates are rational integral functions of θ; and their solution by the ordinary methods leads to a system of equations of the form

$$\alpha/A = \beta/B = \gamma/C = \delta/D,$$

where A, B, C, D are rational integral functions of the coordinates. But since we are concerned with the ratios of any three of the quantities $\alpha, \beta, \gamma, \delta$ to the fourth, and not with their actual values, we may without loss of generality take

$$\alpha = A, \quad \beta = B, \quad \gamma = C, \quad \delta = D \quad\quad (5),$$

which determine the coordinates of any point on E in terms of θ.

The point which is determined by (5) is the reciprocal polar of the plane

$$A\alpha + B\beta + C\gamma + D\delta = 0,$$

and the envelope of this plane is the developable D'. The coordinates of any point on E' in terms of θ can be determined as in the preceding section, but more simply as follows. Let (1) be written in the form

$$\mathfrak{A}\alpha + \mathfrak{B}\beta + \mathfrak{C}\gamma + \mathfrak{D}\delta = 0,$$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ are rational functions of θ; then the coordinates of any point on E' are determined by the equations

$$\alpha = \mathfrak{A}, \quad \beta = \mathfrak{B}, \quad \gamma = \mathfrak{C}, \quad \delta = \mathfrak{D} \quad\quad (6).$$

These reciprocal relations are of considerable importance.

If four roots of (1) are equal, O must be a point on E through which four osculating planes can be drawn; hence O is a cusp. Accordingly the conditions for a cusp are

$$\Delta_m = 0, \quad \Delta_{m-1} = 0, \quad \Delta_{m-2} = 0 \quad\quad (7),$$

which can in like manner be simplified.

122. *Every unicursal twisted curve must in general possess a determinate number of cusps.*

Equations (7) are equivalent to (4) combined with the additional equation $\partial^3 u/\partial\theta^3 = 0$, and if $\alpha, \beta, \gamma, \delta$ be eliminated from this equation and (4), the result will be a determinantal equation for θ, which determines the parameters of the cusps.

123. If (1) has two pairs of equal roots, then (1) is a double tangent plane to E, and O is the point of intersection of the tangents to E at the points of contact of the plane. Hence the locus of O is the nodal curve x on D.

If (1) has three roots equal to u and two equal to v, then O must be a point on the curve through which another tangent can be drawn. The point O is the singular point which is denoted by γ; and no curve of lower degree than a quintic can possess this point.

If (1) has three pairs of equal roots, then O is a point through which three tangents can be drawn to E; hence O is a triple point on the nodal curve x.

When $\theta = 0$, equation (1) reduces to $a_m = 0$; hence a_m is the osculating plane at the point $\theta = 0$. In like manner the equations $a_m = 0$, $a_{m-1} = 0$ are those of the tangent at this point, whilst the point itself is the intersection of the three planes $a_m = 0$, $a_{m-1} = 0$, $a_{m-2} = 0$.

Putting $\theta = \phi^{-1}$, in (1) it can be shown in a similar manner that $a = 0$ is the osculating plane at the point $\theta = \infty$; $a = 0$, $b = 0$ are the equations of the tangent at this point; whilst the point itself is the intersection of the planes $a = 0$, $b = 0$, $c = 0$.

124. *When all the planes a, b, c ... are arbitrary, the singularities σ, ϖ, τ, ι and δ are zero.*

(i) Let $\theta = 0$ or P be the point of contact of a stationary plane; then since two osculating planes must coincide at P, it follows that the two equations $a_m = 0$, $a_m + a_{m-1}\,\partial\theta = 0$ must differ by a factor. This requires that $a_{m-1} = \lambda a_m$, where λ is a constant. In this case the tangent at P is determined by the equation $a_m = 0$, $a_{m-2} = 0$, and the point P by these equations combined with $a_{m-3} = 0$.

If $\theta = \infty$ had been the point of contact of a stationary plane, we should have obtained in like manner the condition $b = \lambda a$, and if the equation is transformed by writing $\theta = \phi - \lambda$, (1) becomes

$$(a,\ 0,\ c',\ d' \ldots \between \phi,\ 1)^m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

in which the coefficient of ϕ^{m-1} is zero. This result is important, since it shows that canonical and semi-canonical forms of binary quantics cannot be employed in this subject, since they lead to twisted curves and developables which possess special singularities.

(ii) Let $\theta = 0$, $\theta = \infty$ be the points of contact of a doubly osculating plane; then the two planes a and a_m must be identical, which requires that $a_m = \mu a$.

(iii) Let $\theta = 0$, $\theta = \infty$ be the points of contact of a double tangent; then since the planes $a_m = 0$ and $a_{m-1} = 0$ must intersect in the line (a, b) it follows that

$$a_m = pa + qb, \quad a_{m-1} = ra + sb \quad \ldots\ldots\ldots\ldots\ldots(9).$$

(iv) Let $\theta = \infty$ be a point of inflexion, then three consecutive osculating planes must pass through the same straight line, the condition for which is that $c = \lambda a + \mu b$. Transform (1) by writing $\theta = \phi - \frac{1}{2}\mu$ and it will be found that the coefficient of ϕ^{m-2} is of the form λa; whence (1) becomes

$$(a, b, \lambda a, d \ldots a_m \between \phi, 1)^m = 0 \quad \ldots\ldots\ldots\ldots\ldots(10).$$

(v) A node possesses two parameters* which we shall take to be $\theta = 0$ and $\theta = \infty$. Let $\theta = \infty$ be the parameter of A, then a, b, and c must be three planes which pass through A, which without loss of generality may be taken to be β, γ, δ; but if A is a node, it follows that a_{m-2}, a_{m-1}, a_m must also be three planes passing through A, and therefore cannot contain α.

We have thus shown that whenever any of the five singularities exist, the coefficients of θ in (1) cannot be arbitrary planes.

125. The *deficiency* of a twisted curve, like that of a plane curve, is the difference between its maximum number of double points, and those which the curve possesses. But in calculating the deficiency, it is necessary to take into account apparent as well as actual double points. Let n be the degree of the curve; h and δ the number of apparent and actual nodes, κ the number of cusps; then the deficiency p of any plane section of the cone standing on the curve is

$$p = \tfrac{1}{2}(n-1)(n-2) - h - \delta - \kappa \quad \ldots\ldots\ldots\ldots(11),$$

and this is the deficiency of the twisted curve. The maximum number of double points is $\frac{1}{2}(n-1)(n-2)$, for if the curve had

* Let s be the length of the arc AN measured from some fixed point A, so that $AN = s$. Let l be the length of the loop of the node; then as we proceed round the loop to N, the distance AN becomes $s + l$, showing that there are two values of the parameter which correspond to N. When $l = 0$, the node becomes a cusp; hence a cusp has two parameters both of which are equal. The same argument applies when the parameter θ is some function of s.

any greater number, a plane section of the cone would become an improper curve consisting of two or more curves of lower degrees.

126. The eight Plücker-Cayley equations are equivalent to the six independent equations

$$\left.\begin{aligned} 2(h+\delta) &= n^2 - 10n - 3m + 8\nu - 3\iota \\ 2(g+\varpi) &= m^2 - 10m - 3n + 8\nu - 3\iota \\ 2(x+\tau) &= \nu(\nu-1) - m - 3n - 3\iota \\ 2(y+\tau) &= \nu(\nu-1) - n - 3m - 3\iota \\ \sigma &= n - 3\nu + 3m + \iota \\ \kappa &= m - 3\nu + 3n + \iota \end{aligned}\right\}\quad\ldots\ldots\ldots(12),$$

by means of which it can be easily shown that

$$p = \tfrac{1}{2}(m-1)(m-2) - g - \varpi - \sigma \ldots\ldots\ldots\ldots(13),$$

from which it follows that the deficiency of the reciprocal curve E' is the same as that of E. When all the planes which are the coefficients of θ in (1) are independent, the discriminant Δ_m is of degree $2(m-1)$, which gives the value of ν; accordingly if we put $\sigma = \varpi = \tau = \iota = \delta = 0$ in (12) and use the above value of ν, we shall obtain

$$\left.\begin{aligned} &\nu = 2(m-1), \quad m = m, \quad n = 3(m-2), \quad \kappa = 4(m-3) \\ &x = 2(m-2)(m-3), \quad y = 2(m-1)(m-3) \\ &h = \tfrac{1}{2}(9m^2 - 53m + 80), \quad g = \tfrac{1}{2}(m-1)(m-2) \end{aligned}\right\}\ldots(14),$$

from which it follows that

$$h + \kappa = \tfrac{1}{2}(3m-7)(3m-8) = \tfrac{1}{2}(n-1)(n-2),$$

showing that the deficiency of E is zero. Hence E and therefore E' are unicursal curves.

127. When $p = 0$, (11) becomes

$$2(h + \delta + \kappa) = (n-1)(n-2)\ldots\ldots\ldots\ldots\ldots(15),$$

by means of which and (12), we obtain

$$\left.\begin{aligned} n &= 3(m-2) - 2\sigma - \iota \\ \nu &= 2(m-1) - \sigma \\ \kappa &= 4(m-3) - 3\sigma - 2\iota \\ g + \varpi &= \tfrac{1}{2}(m-1)(m-2) - \sigma \\ x + \tau &= 2(m-2)(m-3) - \tfrac{1}{2}(4m-11)\sigma + \tfrac{1}{2}\sigma^2 \\ y + \tau &= 2(m-1)(m-3) - \tfrac{1}{2}(4m-7)\sigma + \tfrac{1}{2}\sigma^2 - \iota \end{aligned}\right\}\ldots(16),$$

which give all the quantities in terms of m, σ and ι, and agree with those given by Salmon when $\sigma = \iota = 0$.

Twisted Cubic Curves.

128. *Every twisted cubic curve is the partial intersection of two quadric surfaces.*

Every quadric surface contains 9 arbitrary constants, and therefore an infinite number of quadric surfaces can be drawn through 7 points on a cubic curve; but since a quadric cannot intersect the curve in more than 6 points, it follows that every quadric drawn through 7 points must contain the curve.

129. *A cubic curve, which is the intersection of two quadric surfaces having a common generator, cuts all the generators of the same system as the common one in two points, and those of the opposite system in one point.*

Every generator of a quadric cuts any other quadric in two points which lie on their curve of intersection; but when the quadrics have a common generator, any generator of the same system does not intersect the common generator and must therefore cut the cubic twice; but any generator of the opposite system cuts the common generator once, and must therefore cut the cubic once.

130. The most convenient way of representing a twisted curve is by means of the equations of three surfaces which contain the curve; and by § 46 a twisted cubic can be represented by the system of determinants

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\| = 0 \quad \text{.....................(1)},$$

where u, u', &c. are planes. The determinant is equivalent to the system of equations

$$u/u' = v/v' = w/w' \quad \text{.......................(2)},$$

but in practice a simpler method is preferable. Let A and D be any points on the curve, then the cones having these points as vertices which contain the curve are quadric cones; and by properly choosing the tetrahedron of reference, their equations may be taken to be

$$\beta\delta - \gamma^2 = 0, \quad \alpha\gamma - \beta^2 = 0,$$

from which we deduce $\alpha\delta = \beta\gamma$, which also contains the cubic. Hence (1) may be replaced by the system of determinants

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma \\ \beta, & \gamma, & \delta \end{matrix} \right\| = 0 \quad \text{.....................(3).}$$

131. *Every twisted cubic is the edge of regression of the developable enveloped by the plane*

$$\alpha\theta^3 + 3\beta\theta^2 + 3\gamma\theta + \delta = 0 \quad \text{..................(4).}$$

The theory of these developables has already been considered; and the equation of the envelope of (4) is its discriminant, and is

$$(\alpha\delta - \beta\gamma)^2 = 4(\alpha\gamma - \beta^2)(\beta\delta - \gamma^2) \quad \text{..............(5),}$$

which is a quartic surface. The equations of the edge of regression are obtained from the conditions that (4) should have three equal roots and are

$$\alpha/\beta = \beta/\gamma = \gamma/\delta \quad \text{.......................(6),}$$

which are equivalent to (3). The solution of (4) of § 121 leads to the three equations

$$\alpha = -\beta/\theta = \gamma/\theta^2 = -\delta/\theta^3,$$

hence the coordinates of any point on a twisted cubic curve may be expressed in terms of a parameter by means of the equations

$$\alpha = 1, \quad \beta = -\theta, \quad \gamma = \theta^2, \quad \delta = -\theta^3 \quad \text{...........(7),}$$

but when we are dealing with more than one point on the curve, each equation must be multiplied by a quantity ϕ, where ϕ is the value of α at each point in question. *Accordingly all twisted cubic curves are unicursal.*

That the cubic curve is a cuspidal curve on (5) may be put in evidence in the following manner. Let A be any point on the cubic and write $\alpha + u_1$, $\alpha + v_1$, $\alpha + w_1$, $\alpha + t_1$ for $(\alpha, \beta, \gamma, \delta)$, where the suffixed letters are linear functions of (β, γ, δ); then the highest power of α is α^2, and its coefficient is $(u_1 - 3v_1 + 3w_1 - t_1)^2$, which gives the cuspidal tangent plane at A.

132. *If a family of quadrics have a common curve, the locus of the poles of any fixed plane is a twisted cubic.*

Let U, V be two given quadrics, and let the fixed plane be

$$\alpha + \beta + \gamma + \delta = 0 \quad \text{.......................(8),}$$

then the equation of any other quadric passing through their curve of intersection is

$$U + \lambda V = 0 \quad \text{.........................(9).}$$

Let (f, g, h, k) be the pole of (8) with respect to (9), and let $U_1 = dU/df$, &c.; then the equation of the polar plane is

$$\alpha(U_1 + \lambda V_1) + \beta(U_2 + \lambda V_2) + \gamma(U_3 + \lambda V_3) + \delta(U_4 + \lambda V_4) = 0 \quad \ldots\ldots(10),$$

and if (8) and (10) represent the same plane

$$U_1 + \lambda V_1 = U_2 + \lambda V_2 = U_3 + \lambda V_3 = U_4 + \lambda V_4.$$

Eliminating λ, we obtain

$$\frac{U_1 - U_2}{V_1 - V_2} = \frac{U_2 - U_3}{V_2 - V_3} = \frac{U_3 - U_4}{V_3 - V_4},$$

which are equivalent to (1).

133. *All twisted cubics are anautotomic curves.*

The equations of two quadric surfaces which intersect along the line AB and touch one another at D are

$$(P\alpha + Q\beta + R\gamma + S\delta)\gamma + (P'\alpha + Q'\beta)\delta = 0,$$
$$(p\alpha + q\beta + r\gamma + S\delta)\gamma + (P'\alpha + Q'\beta)\delta = 0,$$

which shows that they intersect in the line AB and also in the line $\gamma = 0$, $P'\alpha + Q'\beta = 0$. Hence the residual curve is a conic.

134. The characteristics of the cubic can now be obtained from (14) of § 126 by putting $m = 3$, $\kappa = 0$, and are $\nu = 4$, $m = 3$, $n = 3$, $h = 1$, $g = 1$, and all the other characteristics are zero.

Since $n = m = 3$, it follows that all cubic curves are their own reciprocals in the extended sense of the word, since any point on a cubic corresponds to an osculating plane to another cubic. Also since every twisted cubic possesses one apparent node, it follows that every cone standing on the curve is a nodal cubic cone. Hence every property of a nodal *plane* cubic curve furnishes a property of a twisted cubic curve; and this property is capable of furnishing a reciprocal theorem for such curves.

135. Under these circumstances, it seems unnecessary to enter into any detailed discussion of twisted cubic curves; and the following examples will illustrate the method.

Let C be the twisted cubic, and S the plane nodal cubic which is the section of any cone whose vertex is O, which stands upon C. Let P, Q, R be the three points of inflexion of S; then these points lie in a straight line, and consequently in a plane passing through O. Let the generators OP, OQ, OR cut the twisted cubic in p, q, r; then the tangent planes to the cone along OP, OQ, OR osculate the twisted cubic at p, q, r, and these points lie in the plane $OPQR$. Hence: (i) *If from any point O three osculating planes be drawn to a twisted cubic curve, their points of contact lie in a plane passing through O.*

For a plane nodal cubic curve, the theorem of § 108, *Cubic and Quartic Curves*, becomes:

If AP, AQ be the two tangents drawn from a point A on the curve, and R be the third point where the chord of contact PQ cuts the curve, the tangent at R intersects the tangent at A at a point on the curve.

Hence: (ii) *Through a point A on a twisted cubic curve and any point O draw two tangent planes OAP, OAQ; and let the plane OPQ cut the cubic in a third point R. Then the tangent planes at R and A which pass through O intersect in a straight line which intersects the curve.*

A plane nodal cubic has 3 sextactic points. Hence: (iii) *With any point O as a vertex three quadric cones can be described which have sextactic contact with a twisted cubic at three distinct points**.

Twisted Quartic Curves.

136. There are two distinct species of quartic curves, the first of which is the complete intersection of a pair of quadric surfaces. These consist of three subsidiary divisions according as the quadrics

* Sextactic points on plane curves have been discussed by Cayley, *C. M. P.* vol. v. pp. 221, 545 and 618, vol. VI. p. 217. He shows that when a plane curve of degree n possesses δ nodes and κ cusps, the number of sextactic points is

$$3n(4n-9)-24\delta-27\kappa;$$

from which it can be easily shown by means of Plücker's equations that this number is also equal to $3m(4m-9)-24\tau-27\iota$. On p. 618, some remarks are made with regard to the connection between these points and the reciprocant called the *Mongean*, see Sylvester's *Lectures on Reciprocants*. Some further details with respect to plane quartic curves have been given by myself, *Quart. Jour.* 1903, p. 1.

The following is a list of some of the principal memoirs on twisted cubic curves: Möbius, *Barycentric Calculus*, 1827, *Crelle*, vol. X.; Chasles, *Aperçu Historique*, Note XXXIII.; Schröter, *Crelle*, vol. LVI.; Cremona, *Ibid.* vols. LVIII., LX.; Sturm, *Ibid.* vols. LXXIX., LXXX., LXXXVI.; Müller, *Math. Ann.* vol. I.

The following papers relate to the connection between these curves and the theory of invariants of binary quantics: Beltrami, *Ist. Lomb.* 1868; Voss, *Math. Ann.* vol. XIII.; D'Ovidio, *Acc. Torino*, vol. XXXII.; Pittarelli, *Giorn. di Batt.* vol. XVII.

(i) do not touch one another, (ii) have ordinary contact, (iii) have stationary contact; in which three respective cases the curve is *anautotomic, nodal* or *cuspidal.* But $\iota = 0$, otherwise a tangent would have tritactic contact with both quadrics, and would therefore lie in both of them, in which case the curve would degrade into a straight line and a twisted cubic. Similarly $\tau = 0$, otherwise a tangent would be a double tangent to both quadrics, which is impossible. Lastly $\varpi = 0$, since no curve of a lower degree than a sextic can possess this singularity.

Quartics of the second species are the partial intersection of a quadric and a cubic which possess two common straight lines lying in different planes. They cannot possess any *actual* double points, since as will hereafter be shown, a quadric and a cubic so situated cannot touch one another; but they may possess one or two points of inflexion, which will occur whenever a generator of the quadric through a point on the curve has tritactic contact with the cubic. This shows that the second species constitutes a totally different kind of curve; also that there are three subsidiary divisions, according as the quartic possesses none, one or two points of inflexion.

137. We shall give for reference a table of the values of the singularities of the two kinds of quartic curves

	n	ν	m	δ	ϖ	κ	σ	τ	ι	h	g	x	y	γ	γ'	t	t'	q	q'	k	k'
1st Species.	4	8	12	0	0	0	16	0	0	2	38	16	8	0	0	16	0	24	0	60	126
	4	6	6	1	0	0	4	0	0	2	6	6	4	0	0	0	0	6	4	3	3
	4	5	4	0	0	1	1	0	0	2	2	2	2	0	0	0	0	6	6	0	0
2nd Species.	4	6	6	0	0	0	4	0	0	3	6	6	4	4	0	0	0	6	6	6	3
	4	6	5	0	0	0	2	0	1	3	4	5	4	2	0	0	0	6	6	4	3
	4	6	4	0	0	0	0	0	2	3	3	4	4	0	0	0	0	6	6	3	3

and the deficiency p is given by (11) of § 125. It therefore follows that nodal and cuspidal quartics of the first species and all quartics of the second species are unicursal curves, and are therefore the envelopes of the planes (1) of § 120, where in the five respective cases m is equal to 6, 4, 6, 5 or 4.

When the quartic is of the first species, the first thirteen characteristics can be found from equations (10) to (15) of § 107 by putting $M = N = 2$. For the second species, let the suffix 1 refer to the quartic; then $n_1 = 4$, $n_2 = 2$, $M = 3$, $N = 2$; also the two straight lines form an improper conic having one apparent node; hence $h_2 = 1$. Substituting in (23) of § 109 we obtain $h_1 = 3$. The remainder of the thirteen characteristics can be obtained from (4) and (5) of § 104 by putting $n = 4$, $h = 3$, $\delta = \kappa = \tau = \varpi = 0$, and $\iota = 0$, 1 or 2. The last eight can be obtained in both cases from the Salmon-Cremona equations.

Quartics of the First Species.

138. *Through every quartic of the first species four, three or two quadric cones can be drawn according as the curve is anautotomic, nodal or cuspidal.*

Let S, S' be the two quadrics containing the curve, then the equation of any other quadric passing through it is $S + \lambda S' = 0$, and the condition that this should be a cone is that its discriminant should vanish, which furnishes a quartic equation for determining λ.

When the quartic is nodal, let A be the vertex of one of the cones, B the node and ABD the tangent planes to both quadrics at B; then their equations may be written

$$\left.\begin{aligned} S &= a\alpha^2 + c\gamma^2 + d\delta^2 + 2f\beta\gamma + 2g\gamma\alpha + 2l\alpha\delta + 2n\gamma\delta = 0 \\ S' &= \delta^2 + 2\beta\gamma = 0 \end{aligned}\right\} \quad \ldots(1),$$

and the discriminant of $S + \lambda S'$ is

$$(f + \lambda)^2 (l^2 - ad - a\lambda) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which shows that two of the cones coincide.

To find the condition that the quadrics should have stationary contact at B, eliminate γ from (1) and we obtain

$$a\alpha^2 + c\delta^4/4\beta^2 + (d - f)\,\delta^2 + 2l\alpha\delta - (g\alpha + n\delta)\,\delta^2/\beta = 0.$$

This is the equation of a quartic cone, whose vertex is C, which stands on the curve, and the condition that the coefficient of β^2 should be a perfect square is $l^2 = a\,(d - f)$, which reduces (2) to $a\,(f + \lambda)^3 = 0$, and shows that three of the cones coincide.

139. *If a plane passing through two fixed points on the quartic intersects the curve in two other points P and Q, the line PQ envelopes a quadric which contains the quartic; also four planes of the system touch the curve.*

Let B and C be the two fixed points; A and D the vertices of two of the quadric cones which contain the quartic; then its equations may be expressed in the form

$$\left.\begin{array}{r}\delta^2+2\beta\gamma=0\\ p\alpha^2+2p'\beta\gamma+2q'\gamma\alpha+2r'\alpha\beta=0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(3).$$

The equation of any plane through BC is $\delta=\lambda\alpha$, whence the chord PQ is the intersection of this plane and the plane

$$(p-p'\lambda^2)\,\alpha+2q'\gamma+2r'\beta=0 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

and the envelope of this line is obtained by eliminating λ and is

$$p\alpha^2-p'\delta^2+2q'\gamma\alpha+2r'\alpha\beta=0,$$

which is the result of eliminating $\beta\gamma$ between (3). The condition that the plane $\delta=\lambda\alpha$ should touch the quartic is that the cone

$$\lambda^2\alpha^2+2\beta\gamma=0,$$

and the second of (3) should touch; which by eliminating β can be shown to be

$$(p-p'\lambda^2)^2=8q'r'\lambda^2,$$

and furnishes a quartic equation for determining λ.

140. When the vertex O of the cone standing on the quartic lies on the curve, the cone will be a cubic cone which is anautotomic, nodal or cuspidal according as the quartic belongs to one or other of these species; for since a straight line cannot cut a quadric surface in more than two points, the cone cannot have any apparent nodal generators. For the same reason the quartic cannot have any trisecants. Also any stationary tangent plane to the cone is an osculating plane to the quartic; and since anautotomic cubic curves possess 9 points of inflexion, it follows that 9 osculating planes can be drawn to the quartic through any point on the curve. Again let P, Q, R be any three collinear points of inflexion on a plane section, and let the generators OP, OQ, OR cut the quartic in p, q and r; then p, q and r form a triplet of points which possess the property of lying in the same plane, the osculating planes at which pass through a point O on the curve. Moreover since a real straight line can be drawn through the three real points of inflexion, and also through each real and two conjugate imaginary points of inflexion, there are altogether four triplets corresponding to a point O on the curve.

141. These results can be generalized. Let O be any point in space; then since an anautotomic twisted quartic curve possesses two apparent nodes, the projection of the curve on a plane is a plane binodal quartic curve; and since such a curve possesses 12 points of inflexion, it follows that through any point O, 12 osculating planes can be drawn to the curve; in other words, the curve is of the 12th class as we have already shown by means of the Plücker-Cayley equations. By a known theorem*, the 12 points of inflexion of a plane binodal quartic will lie on a cubic curve, provided the four points in which the nodal tangents intersect the curve are collinear; if therefore the point O be chosen so that the four points in which the apparent nodal tangent planes intersect the curve lie in a plane passing through O, the points of contact of the 12 osculating planes passing through O will lie on a cubic cone.

When the twisted quartic possesses an actual node, the projection of the curve will be a trinodal quartic, in which case only 6 osculating planes can be drawn through O, and their points of contact lie on a quadric cone which passes through two generators OS, OS' of the quartic cone, which correspond to the S points of a plane trinodal quartic curve. Also the theorems of plane trinodal quartics relating to the conics which pass through (i) the points where the nodal tangents intersect the curve and (ii) the points where the tangents from the nodes touch the curve can be adapted in like manner to nodal twisted quartics†.

142. When the excentricity of an ellipse is equal to $(\sqrt{5}-1)/2\sqrt{2}$, the circles of curvature at the extremities of the minor axis intersect in two points E, E', which respectively lie on the circles of curvature at the extremities of the major axis; and the inverse of the ellipse with respect to one of these points is a trinodal quartic having 3 points of undulation. Now the four σ planes of a nodal twisted quartic form a tetrahedron, and the cone standing on the curve whose vertex is any one of the vertices of the tetrahedron is a trinodal quartic cone of this character.

The developables enveloped by the osculating planes to the three kinds of quartics of the first species have been discussed by Cayley‡.

* Richmond, *Proc. Lond. Math. Soc.* vol. XXXIII. p. 218; Basset, *Quart. Jour.* vol. XXXVI. p. 44.

† Basset, *American Journal*, vol. XXVI. p. 169. See also Appendix I.

‡ *C. M. P.* vol. I. p. 486; *Camb. and Dublin Math. Jour.* vol. V. p. 46. The following papers relate to these curves. Hesse, *Crelle*, vol. XXVI.; Reye, *Ibid.*

143. *A quartic of the first species is the partial intersection of a quadric and a cubic which possess a common conic.*

Let $\delta = 0$, $S = 0$ be the equations of the conic; then the equations of any cubic and quadric surface which contain this conic are

$$\delta\Sigma = Su, \quad \delta v = S \quad \text{.......................(5)},$$

where Σ is another quadric, and u, v are planes. Eliminating S and δ we obtain

$$\Sigma = uv \quad \text{............................(6)},$$

which shows that the quartic is the complete intersection of (6) and the second of (5). This theorem is true when the conic degrades into a pair of intersecting straight lines.

Quartics of the Second Species.

144. *A quartic of the second species is the partial intersection of a quadric and a cubic surface possessing a line in common, which is an ordinary line on the quadric and a nodal line on the cubic.*

Let a quadric and a cubic intersect in the lines CD and (u, v); then their equations may be taken to be

$$\alpha u = \beta v, \quad \alpha(uU + vV) = \beta(uU' + vV') \quad \text{...........(1)},$$

whence eliminating (u, v) we obtain an equation of the form

$$P\alpha^2 + 2Q\alpha\beta + R\beta^2 = 0 \quad \text{....................(2)},$$

where P, Q, R are planes. The quartic is therefore the partial intersection of (2) and the first of (1), which proves the theorem.

145. *A quartic of the second species cannot have any actual double points.*

Since CD is an ordinary line on the quadric and a nodal line on the cubic, we may suppose that the two surfaces touch at A, in which case A will be a node on the quartic. Hence if ABC be the common tangent plane to the two surfaces, we must have

$$u = p\beta + r\delta, \quad v = p\alpha + P'\beta + Q'\gamma + R'\delta.$$

Also if
$$P = G\beta + H\gamma + K\delta,$$
$$Q = f\alpha + g\beta + h\gamma + k\delta,$$

vol. C.; and *Annali di Mat.* vol. II.; Chasles, *Comptes Rend.* vols. LII., LIV.; Zeuthen, *Acta Math.* vol. XII.; Schrötter's treatise on the *Theorie der Raumcurven 4ter Ordnung*, Leipzig, 1890.

the conditions that δ should be the tangent plane to (2) at A are

$$G + 2f = 0, \quad H = 0,$$

which reduce (2) and the first of (1) to the forms

$$k\alpha^2\delta + 2\alpha\beta(g\beta + h\gamma + k\delta) + R\beta^2 = 0,$$
$$r\alpha\delta = \beta(P'\beta + Q'\gamma + R'\delta),$$

and show that the line (β, δ) or AC lies in the quadric and cubic. Hence the two surfaces intersect in three straight lines and a residual twisted cubic curve.

146. The developable D which is the envelope of the plane

$$(a, b, c, d, e \between \theta, 1)^4 = 0 \quad \text{..................(3)}$$

has been discussed by Cayley* and various other writers; and we shall show that it is the reciprocal polar of a curve which includes all nodal and cuspidal quartics of the first species and all quartics of the second species. Its characteristics are

$$\nu = 6, \quad n = 6, \quad m = 4, \quad \kappa = 4, \quad x = 4, \quad y = 6, \quad h = 6, \quad g = 3,$$

and therefore those of the reciprocal polar E' of the developable are

$$\nu = 6, \quad m = 6, \quad n = 4, \quad \sigma = 4, \quad x = 6, \quad y = 4, \quad h = 3, \quad g = 6,$$

which are those of a quartic curve of the second species and first kind.

Writing as usual

$$\left.\begin{aligned} I &= ae - 4bd + 3c^2 \\ J &= ace + 2bcd - ad^2 - b^2e - c^3 \end{aligned}\right\} \quad \text{..............(4)},$$

it follows that the equation of D, which is the discriminant of (3), is

$$I^3 = 27J^2 \quad \text{.........................(5)},$$

* "On the developable derived from an equation of the fifth order," *C. M. P.* vol. I. p. 500; *Camb. and Dublin Math. Jour.* vol. v. p. 152. In this paper the discriminant of a binary quintic is given in a form which would repay a geometrical examination. "On certain developable surfaces," *C. M. P.* vol. v. p. 267; *Quart. Jour.* vol. VI. p. 108. "On the reciprocation of a certain quartic developable," *C. M. P.* vol. v. p. 505; *Quart. Jour.* vol. VII. p. 87. "On a special sextic developable," *C. M. P.* vol. v. p. 511; *Quart. Jour.* vol. VII. p. 105.

The conditions for equalities amongst the roots of a quintic equation have been discussed by Sylvester, *Phil. Trans.* 1864, *Collected Papers*, vol. II. p. 452; and these results have important applications with reference to the developables and curves derived from the binary quintic $(a, b, c, d, e, f \between \theta, 1)^5 = 0$.

and those of E, which are the conditions that (3) should have three equal roots, are

$$I = 0, \quad J = 0 \quad \text{.........................(6)},$$

and the four cusps, which are the conditions that (3) should have four equal roots, are determined by the equations

$$a/b = b/c = c/d = d/e \quad \text{.......................(7)}.$$

The nodal curve on D is found from the conditions that (3) should have a pair of equal roots, and its equations are

$$ad^2 = b^2e, \quad 2b^3 + a^2d = 3abc \quad \text{..................(8)}.$$

147. *Every cusp on E is a cubic node of the sixth species on D and an ordinary point on the nodal curve; also the latter is a quartic.*

Let A be the cusp, ABC the osculating plane at A, AC the cuspidal tangent; then we may take

$$a = \delta, \quad b = \beta, \quad c = \gamma, \quad e = \alpha \quad \text{.................(9)}.$$

From (7) and (9) we obtain the equation $\alpha\gamma = d^2$; and since this is the equation of a cone which has to pass through A, it follows that $d = p\delta + q\beta + r\gamma = p\delta + u$, say. Hence (6) reduces to

$$\alpha^3\delta^3 + 3\alpha^2 \{(4\beta d - 3\gamma^2)\,\delta^2 + 9\,(\gamma\delta - \beta^2)^2\} + \ldots = 0,$$

which proves the first part. Equations (8) become

$$\delta\,(p\delta + u)^2 = \alpha\beta^2,$$
$$\delta^2\,(p\delta + u) = 3\beta\gamma\delta - 2\beta^3,$$

whence eliminating δ we obtain

$$(\alpha + 2p\beta)^3\,\beta + (\alpha + 2p\beta)\,(2u^2 - 3\alpha\gamma + 12p\beta\gamma)\,u$$
$$+ 9p\,(2u^2 - 3\alpha\gamma)\,\gamma^2 - 4p\beta u^3 = 0 \quad \text{...(10)}.$$

Equation (10) represents the cone standing on the nodal curve (8) whose vertex is an arbitrary point D; also DA is an ordinary generator of this cone, and the latter is a quartic cone.

148. Equation (3) may be written in the form

$$A\alpha + B\beta + C\gamma + D\delta = 0 \quad \text{.................(11)},$$

where A, B, C, D are polynomials of θ of degree four; whence it follows from § 121 that the coordinates of any point on the quartic E' can be expressed by the equations

$$\alpha = A, \quad \beta = B, \quad \gamma = C, \quad \delta = D,$$

but since the quartic is of the sixth class, it will be instructive to find the parametric equation of its osculating plane. To do this, we shall make use of the property that every curve of a higher degree than the fourth possesses a determinate number of planes which osculate it at one point and touch it at another.

Since the curve E, of which (6) are the equations, is of the sixth degree, let the plane δ osculate it at A and touch it at B; let AC be the tangent at A, and BC that at B. Then

$$a=\delta, \quad b=\beta, \quad c=\gamma, \quad d=p\alpha+s\delta, \quad e=\alpha \quad \text{......(12)},$$

so that (3) becomes

$$u=\delta\theta^4+4\beta\theta^3+6\gamma\theta^2+4(p\alpha+s\delta)\theta+\alpha=0.$$

The coordinates of any point on E are determined by (4) of § 121, which are equivalent to

$$\begin{aligned} \delta\theta^2+2\beta\theta+\gamma&=0,\\ \beta\theta^2+2\gamma\theta+p\alpha+s\delta&=0,\\ \gamma\theta^2+2(p\alpha+s\delta)\theta+\alpha&=0, \end{aligned}$$

the solution of which is

$$\begin{aligned} \alpha&=-\theta^6-4s\theta^3,\\ \beta&=-3p\theta^4-2\theta^3+s,\\ \gamma&=2p\theta^5+\theta^4-2s\theta,\\ \delta&=4p\theta^4+3\theta^2, \end{aligned}$$

multiplied by a constant, which without loss of generality may be taken to be unity.

Hence the reciprocal developable, of which the quartic E' is the edge of regression, is the envelope of the plane

$$\alpha\theta^6-2p\gamma\theta^5+(3p\beta-\gamma)\theta^4+(4s\alpha+2\beta-4p\delta)\theta^3 \\ -3\delta\theta^2+2s\gamma\theta-s\beta=0 \quad \text{......(13)},$$

and the discriminant of this binary sextic will give the developable, whose edge of regression is a quartic of the second species.

149. Since nodal and cuspidal quartics of the first species are unicursal curves, and are therefore included amongst the curves which are the reciprocal polars of the developable (5), the theories of the two species of quartics overlap; and we shall now proceed to consider the first species further.

Nodal quartics and their reciprocals. The reciprocal developable D must have a doubly osculating plane ϖ, and by § 124

$e = \lambda a$, where λ is a constant. Let ABC be such a plane; A and B its points of contact; AC and BC the tangents at these points. Then we may take

$$a = \delta, \quad b = \beta, \quad c = \gamma, \quad d = \alpha, \quad e = \lambda\delta \quad \text{.........(14)},$$

and the equations of E are

$$\left.\begin{aligned} \lambda\delta^2 - 4\alpha\beta + 3\gamma^2 = 0 \\ \lambda\gamma\delta^2 + 2\alpha\beta\gamma - \alpha^2\delta - \lambda\beta^2\delta - \gamma^3 = 0 \end{aligned}\right\} \quad \text{......(15)},$$

which are those of a sextic curve which is the complete intersection of a quadric and a cubic surface; and the reciprocal curve E' is a nodal quartic of the first species.

By § 121, the parametric expressions for the coordinates of any point on E are

$$\left.\begin{aligned} \alpha &= -\theta^6 - 3\lambda\theta^2 \\ \beta &= -3\theta^4 - \lambda \\ \gamma &= 2\theta^5 + 2\lambda\theta \\ \delta &= 4\theta^3 \end{aligned}\right\} \quad \text{.....................(16)},$$

and the nodal quartic E' is the edge of regression of the developable which is the envelope of the plane

$$\alpha\theta^6 - 2\gamma\theta^5 + 3\beta\theta^4 - 4\delta\theta^3 + 3\lambda\alpha\theta^2 - 2\lambda\gamma\theta + \lambda\beta = 0 \text{...(17)},$$

and the node is at the point D.

150. *The tangents at the points of contact of a doubly osculating plane intersect at a point, which is a node on the nodal curve.*

By (8) and (14) the equations of the nodal curve are

$$\alpha^2 = \lambda\beta^2, \quad 2\beta^3 + \alpha\delta^2 = 3\beta\gamma\delta,$$

which represent a pair of conics whose planes intersect in the line CD, which does not form part of the nodal curve. Also since both conics intersect in the point C, and nowhere else, C is a node on the nodal curve; and this is the point where the tangents AC and BC, at the points of contact of the doubly osculating plane, intersect. This theorem is a general one.

151. *Cuspidal quartics and their reciprocals.* We have shown in § 124 that if in (3) we put $b = 0$, the curve E will possess a stationary tangent plane. Let it be ABC; A its point of contact; then we may take

$$a = \delta, \quad b = 0, \quad c = \beta, \quad d = \gamma, \quad e = \alpha \quad \text{........(18)},$$

and the equation for D becomes

$$\alpha^3\delta^2 + 9\alpha^2\beta\delta + 27\alpha\beta^4 = 27(\alpha\beta - \gamma^2)\{(\alpha\beta - \gamma^2)\delta - 2\beta^3\} \quad \ldots(19),$$

which shows that any plane section of (19) through A has a tacnode thereat, hence: *The points of contact of the stationary planes are tacnodal points on E.* This theorem is a general one.

152. Equations (6), which determine the edge of regression, now become

$$\alpha\delta + 3\beta^2 = 0, \quad \alpha\beta\delta - \delta\gamma^2 - \beta^3 = 0 \quad \ldots\ldots\ldots\ldots(20),$$

from which we deduce

$$4\alpha\beta - 3\gamma^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

which shows that the curve is the complete intersection of (21) and the first of (20). Accordingly the curve is a cuspidal quartic of the first species, which possesses one cusp and one stationary plane, and is therefore its own reciprocal. Hence:

A cuspidal quartic is the edge of regression of the developable enveloped by the plane

$$\delta\theta^4 + 6\beta\theta^2 + 4\gamma\theta + \alpha = 0,$$

and the parametric equations for the coordinates are

$$\alpha = 3\theta^4, \quad \beta = \theta^2, \quad \gamma = -3\theta^3, \quad \delta = -1.$$

153. *Quartics of the second species having points of inflexion.* We have shown in § 124 (iv) that the condition for such a point is $c = \lambda a$; and if we put $m = 4$, $\iota = 1$, $\sigma = \varpi = \tau = 0$ in (16) of § 127, it will be found that the characteristics of E are the reciprocals of those of a quartic curve of the second species which has one point of inflexion. Hence such curves are the reciprocal polars of the developables enveloped by the plane

$$(a, b, \lambda a, d, e \between \theta, 1)^4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

154. *A quartic curve which has two points of inflexion is the reciprocal polar of the developable*

$$(\alpha\delta - 4\beta\gamma)^3 = 27(\alpha\gamma^2 + \beta^2\delta)^2 \quad \ldots\ldots\ldots\ldots\ldots(23).$$

Equation (23) is the discriminant of (22), when $\lambda = 0$ and

$$a = \delta, \quad b = \gamma, \quad d = \beta, \quad e = \alpha,$$

and it may be established as follows. Let A be one point of inflexion, D the other; also let AB be the tangent at A, DC that at D. Then we may take

$$a = \delta, \quad b = p\gamma + q\delta, \quad c = 0, \quad d = r\alpha + s\beta, \quad e = \alpha,$$

and the discriminant of (22) becomes

$$\{\alpha\delta - 4(p\gamma + q\delta)(r\alpha + s\beta)\}^3 = 27\{\delta(r\alpha + s\beta)^2 + \alpha(p\gamma + q\delta)^2\}^2 \quad (24).$$

The form of (24) shows that AB and CD are double lines on the developable; also the term involving the highest power of β is $27s^4\beta^4\delta^2$, and since B may be any point on AB, it follows that the line is cuspidal. In like manner CD is a cuspidal line.

The plane $r\alpha + s\beta$ may be any plane through CD, let us therefore choose it for the plane β; then $r = 0$ and we may take $s = 1$; hence the term involving the highest power of α is $\alpha^3\delta^3$. The point A is now one where the stationary tangent touches the curve, and is therefore a cubic node of the sixth kind on the developable D. In like manner if D is the point of contact of the other stationary tangent, $p = 1$, $q = 0$, and (24) becomes

$$(\alpha\delta - 4\beta\gamma)^3 = 27(\alpha\gamma^2 + \beta^2\delta)^2 \quad \text{..............}(25).$$

155. It thus appears that anautotomic quartic curves of the first species constitute a class of curves *sui generis*; but that nodal and cuspidal quartics, and also all quartics of the second species, constitute a class of curves which possess many features in common. In particular they are all unicursal curves, and are also included amongst those which are the reciprocal polars of the developables enveloped by (3).

No quartic of the second species can possess a double tangent, since the latter would be a line lying in the cubic and quadric surfaces of which the quartic is the partial intersection; in which case the quartic would degrade into the double tangent and a cubic.

A historical account of unicursal quartic curves, together with a list of memoirs, has been given by Mr Richmond, in *Trans. Camb. Phil. Soc.* vol. XIX. p. 132.

Quintic Curves.

156. There are four primary species of twisted quintic curves*.

I. Quintics which are the partial intersection of a quadric and a cubic surface, the residual intersection being a common straight line. These possess four apparent nodes and may also have two actual double points, which may be nodes or cusps.

* Cayley, *C. M. P.* vol. v. p. 15.

II. Quintics which are the partial intersection of two cubic surfaces, the residual intersection being a quartic curve of the second species. These have five apparent nodes, and may also possess an actual double point.

III. Quintics which are the partial intersection of two cubic surfaces, the residual intersection being a twisted cubic curve and a straight line. These have six apparent nodes.

IV. Quintics which are the partial intersection of a quadric and a quartic surface, the residual intersection being three generators of the quadric belonging to the same system.

The number of apparent nodes is obtained from the equation

$$2(h-h')=(n-n')(M-1)(N-1),$$

where the unaccented and accented letters refer to the quintic and the residual curve. In the four respective cases $h'=0, 3, 4, 3$; $n'=1, 4, 4, 3$; which gives $h=4, 5, 6, 6$.

Since the cone standing on a twisted quintic curve is a quintic cone having at least four double generators, a great many properties of such curves may be derived from those of plane quintic curves, which have been discussed by myself*; I shall therefore briefly consider the four species.

157. *First Species.* If U, V are quadric surfaces, the simplest form of the equations of curves of this species is

$$\left\| \begin{matrix} U, & \alpha, & \beta \\ V, & \gamma, & \delta \end{matrix} \right\| = 0 \qquad \text{.....................(1)},$$

from which it can be shown as in § 102 that the quintic is also the partial intersection of two cubic surfaces, whose residual intersection is a quartic of the first species. We shall now show that:

158. *A quintic of the first species is the partial intersection of a quadric and a quartic surface, the residual intersection being a twisted cubic.*

Let U, V, W be quadric surfaces; p, q, r constants; u, u', &c. planes; also let

$$\lambda = vw' - v'w, \quad \mu = wu' - w'u, \quad \nu = uv' - u'v,$$

* *Quart. Jour.* vol. XXXVII. pp. 106 and 199. See also, "On plane quintic curves with four cusps," *Rend. Palermo*, vol. XXVI. p. 1.

and consider the equations

$$\left.\begin{aligned} U\lambda + V\mu + W\nu &= 0 \\ p\lambda + q\mu + r\nu &= 0 \\ u\lambda + v\mu + w\nu &= 0 \end{aligned}\right\} \quad \text{.....................(2)};$$

the first two equations represent a quartic and a quadric surface which intersect in the twisted cubic (λ, μ, ν) and in a residual quintic curve, whilst the last one is an identity. Eliminating (λ, μ, ν) we obtain

$$U(rv - qw) + V(pw - ru) + W(qu - pv) = 0 \quad \text{......(3)},$$

whilst the second of (2) may be written

$$u'(rv - qw) + v'(pw - ru) + w'(qu - pv) = 0 \quad \text{......(4)}.$$

Equations (3) and (4) represent a cubic and a quadric which both contain the residual quintic, and consequently the latter is of the first species.

159. *Second Species.* Let

$$U = \alpha\beta - \gamma\delta, \quad V = (a\beta + b\gamma)\alpha + (c\beta + d\gamma)\delta \text{ ......(5)},$$

where (a, b, c, d) are arbitrary planes; then $U = 0$, $V = 0$ represent a quadric and a cubic surface which intersect in the lines BC and AD; hence the residual intersection is a quartic of the second species. From (5) eliminate successively (β, γ) and (α, δ) and we obtain

$$\left.\begin{aligned} b\alpha^2 + (a + d)\alpha\delta + c\delta^2 &= 0 \\ b\gamma^2 + (a + d)\beta\gamma + c\beta^2 &= 0 \end{aligned}\right\} \quad \text{..................(6)},$$

which represent a pair of cubic surfaces on which BC and AD are nodal lines respectively. These intersect in a quartic of the second species and a residual quintic curve of the same species.

160. *Third Species.* The equations of these quintics may be expressed by means of the system of determinants

$$\left\| \begin{matrix} p, & s, & P, & S \\ q, & t, & Q, & T \\ r, & u, & R, & U \end{matrix} \right\| = 0 \text{..................(7)},$$

where the small letters represent arbitrary planes; whilst the capital letters represent six planes passing through the same straight line but otherwise arbitrary. For if

$$\lambda = qu - rt, \quad \mu = rs - pu, \quad \nu = pt - qs,$$

the determinants are equivalent to

$$P\lambda + Q\mu + R\nu = 0,$$
$$S\lambda + T\mu + U\nu = 0,$$

and these are the equations of two cubic surfaces each passing through the twisted cubic (λ, μ, ν) and the common line of intersection of the six planes. The residual curve is therefore a quintic.

161. *Fourth Species.* The lines CD and AB are generators of the quadric $\alpha\gamma = \beta\delta$, and the equations of any other generator of the same system are $\alpha = \lambda\delta$, $\beta = \lambda\gamma$; and the equation of any quartic containing the curve may be taken to be

$$P(\alpha - \lambda\delta) + Q(\lambda\gamma - \beta) = 0 \quad \ldots\ldots\ldots\ldots\ldots(8),$$

where P and Q are quaternary cubics, which have to be determined so that (8) vanishes when $\alpha = 0$, $\beta = 0$; or when $\gamma = 0$, $\delta = 0$.

Let

$$\Omega = a\alpha^2 + b\alpha\beta + c\beta^2, \quad \Omega' = A\gamma^2 + B\gamma\delta + C\delta^2,$$

where a, A ... are constants; then the values of P and Q may be written

$$P = \beta\Omega + \gamma\Omega' + \alpha(\alpha v_1 + \beta\sigma_1 + v_2) + \beta(\alpha w_1 + \beta\tau_1 + w_2),$$
$$Q = \alpha\Omega + \delta\Omega' + \alpha(\alpha v_1' + \beta\sigma_1' + v_2') + \beta(\alpha w_1' + \beta\tau_1' + w_2'),$$

where the suffixed letters denote quantics of (γ, δ). Denoting the last two terms by U, U', (8) becomes

$$(\alpha\gamma - \beta\delta)(\lambda\Omega + \Omega') + (\alpha - \lambda\delta)U + (\lambda\gamma - \beta)U' = 0,$$

which shows that the curve is the intersection of the quadric $\alpha\gamma = \beta\delta$ and the quartic

$$(\alpha - \lambda\delta)U + (\lambda\gamma - \beta)U' = 0 \quad \ldots\ldots\ldots\ldots\ldots(9).$$

By means of the equation of the quadric, (9) may be reduced to

$$\mathfrak{A}_3\alpha + \mathfrak{B}_3\beta + \mathfrak{C}_4 = 0,$$

where the old English letters denote binary quantics of (γ, δ), hence: *The curve is the partial intersection of a quartic which has a triple line, and a quadric which passes through the line.*

The following papers* relate to quintic curves; and the consideration of sextic curves will be postponed until we discuss the Theory of Residuation.

* Bertini, *Collect. Math.* 1881; Berzolari, *Lincei*, 1893; Weyl, *Wiener Berichte*, 1884–5–6; Montesano, *Acc. Napoli*, 1888.

CHAPTER IV

COMPOUND SINGULARITIES OF PLANE CURVES

162. ALTHOUGH the geometry of surfaces is the object of this treatise, yet the theory of their singularities cannot be properly understood without a more detailed account of the corresponding portion of the theory of plane curves, than is contained in my treatise on *Cubic and Quartic Curves.* I shall therefore devote the present chapter to the consideration of the compound singularities of plane curves*.

163. Plücker's equations show that the simple singularities of a curve are four in number, viz. the node, the cusp, and their reciprocals the double and the stationary tangent; and also that every algebraic curve possesses a determinate number of these singularities which can be calculated from the formulae he gave. From this it follows that every other singularity, which an algebraic curve can possess, is a compound singularity formed by the union of two or more simple singularities.

Compound singularities may be divided into three primary species. First, *point singularities,* which are exclusively composed of nodes and cusps. Secondly, *line singularities,* which are exclusively composed of double and stationary tangents. Thirdly, *mixed singularities,* which are composed of a combination of point and line singularities.

164. The point constituents of a singularity can be determined in the following manner. Plücker's first equation is

$$2\delta + 3\kappa = n(n-1) - m \quad \text{.................(1)},$$

where δ and κ are the number of constituent nodes and cusps, and $2\delta + 3\kappa$ is the *reduction of class* produced by the singularity; and since the degree n of the surface is given, it follows that as soon

* Basset, *Quart. Jour.* vols. XXXVI. p. 359, XXXVII. p. 313.

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as its class m has been ascertained, (1) furnishes one relation between the unknown quantities δ and κ.

Another equation exists of the form

$$\delta + \kappa = \lambda \quad \text{.........................(2)},$$

where λ is the number of constituent double points; and as soon as λ has been found, (1) and (2) furnish two equations for determining δ and κ.

The line constituents can usually be found by forming the reciprocal singularity, and ascertaining the number of its constituent nodes and cusps.

The only *point* singularities which exist are multiple points of order p, the tangents at which have $(p+1)$-tactic contact with the curve at the point. If any tangent has a higher contact, the singularity is a *mixed* one.

165. *If r tangents at a multiple point of order p coincide, its constituents are*

$$\delta = \tfrac{1}{2}p(p-1) - r + 1, \quad \kappa = r - 1.$$

Since the properties of a multiple point of this kind are the same on a curve of degree $p+1$ as on one of higher degree, we may employ the curve

$$\alpha\gamma^r u_{p-r} + u_{p+1} = 0 \quad \text{.....................(3)},$$

the triangle of reference being chosen so that A is the multiple point, and AB the line which coincides with the r coincident tangents. The first polar of C, which may be any arbitrary point, is

$$\alpha\gamma^{r-1}(ru_{p-r} + \gamma u'_{p-r}) + u'_{p+1} = 0 \text{..............(4)},$$

where the accents denote differentiation with respect to γ. Eliminating α between (3) and (4) we obtain

$$\gamma^{r-1}\{\gamma u_{p-r} u'_{p+1} - (ru_{p-r} + \gamma u'_{p-r})\, u_{p+1}\} = 0,$$

which shows that the first polar of C intersects the curve in $2p - r + 1$ ordinary points; hence

$$m = 2p - r + 1,$$

and since the degree of the curve is $p+1$, we obtain from (1)

$$2\delta + 3\kappa = p(p-1) + r - 1 \text{..................(5)}.$$

Since the point C is arbitrary, it follows that if the curve has another double point we may suppose it situated at C, in which case the terms in γ^{p+1} and γ^p must be absent, and (3) reduces to the improper curve

$$\beta(\alpha\gamma^r v_{p-r-1} + \beta w_{p-1}) = 0,$$

showing that the deficiency of (3) is zero; whence

$$\delta + \kappa = \tfrac{1}{2}p(p-1) \text{ ....(6).}$$

Solving (5) and (6) we obtain the required result.

When all the tangents are distinct, $r = 1$, and the constituents of the point are $\delta = \frac{1}{2}p(p-1)$, $\kappa = 0$. It can also be shown that if r tangents coincide with a particular line AP, and s tangents with another line AQ, the constituents of the point are

$$\delta = \tfrac{1}{2}p(p-1) - r - s + 2, \quad \kappa = r + s - 2.$$

It is impossible for a multiple point to be composed exclusively of cusps, for if all the tangents coincide $r = p$, and the constituents are

$$\delta = \tfrac{1}{2}(p-1)(p-2), \quad \kappa = p - 1.$$

166. Reciprocating the theorem of § 165, we obtain: *If a multiple tangent of order p has $(r+1)$-tactic contact at one point, and bitactic contact at $p - r$ points, its constituents are*

$$\tau = \tfrac{1}{2}p(p-1) - r + 1, \quad \iota = r - 1 \text{ ....(7).}$$

167. *Let r tangents at a multiple point of order p coincide; then if t be the number of tangents which can be drawn from the point, and m the class of the curve*

$$t = m - 2p + r - 1 \text{ ....(8).}$$

The reciprocal polar of the multiple point is a multiple tangent to the reciprocal curve, whose degree is m. The tangent has $(r+1)$-tactic contact at one point, bitactic contact at $p - r$ points, and intersects the curve at t ordinary points; hence

$$t + 2(p - r) + r + 1 = m,$$

giving $$t = m - 2p + r - 1,$$

and the number of ordinary points of intersection are the reciprocal polars of the tangents drawn from the multiple point on the original curve.

When all the tangents are distinct $r = 1$ and

$$t = m - 2p \text{ ....(9).}$$

In the same way it can be shown that if r tangents coincide with a line AP and s with a line AQ, the value of t is

$$t = m - 2p + (r-1) + (s-1) \quad \text{...........(9 A)}.$$

168. When the number of constituent double points in a singularity is unequal to $\frac{1}{2}p(p-1)$, the latter cannot be a multiple point but must be a mixed singularity. It is also possible for a singularity to possess this number of double points without being a multiple point. Thus the point constituents of an oscnode are $\delta = 3$; and the distinction between a triple point of the first kind and an oscnode is that (i) the three nodes move up to coincidence in an arbitrary manner, whereas in an oscnode they move up to coincidence along a continuous curve; (ii) the triple point has no line constituents, whereas those of an oscnode are $\tau = 3$.

169. If an arbitrary straight line through a point P, which is not a multiple point of order p, intersects the curve in p coincident points at P, then P is called *a singular point of order p*. The rhamphoid cusp and the oscnode are examples of singular points of order 2. Also if from an arbitrary point on a tangent, which is not a multiple tangent of order p, $m-p$ tangents can be drawn to a curve of class m, the tangent is called *a singular tangent of order p*. The distinction between multiple points and singular points is of importance in the theory of compound singularities.

170. The theorem of § 24 is applicable to plane curves, and affords a ready means of determining the number of constituent point singularities. It is:

If a node moves up to coincidence with a multiple point of order p along the line AB, the equation of the curve is

$$\alpha^{n-p}\gamma^2 u_{p-2} + \alpha^{n-p-1}\gamma u_p + \alpha^{n-p-2}u_{p+2} + \dots u_n = 0 \quad \text{...(10)}.$$

The equation of a curve having an ordinary multiple point of order p at A is

$$\alpha^{n-p}v_p + \alpha^{n-p-1}v_{p+1} + \dots u_n = 0 \quad \text{...........(11)}.$$

If the curve has a node at a point P on AB, the line AB must have p-tactic contact with the curve at A and bitactic contact at P; hence when P coincides with A, the line AB must have $(p+2)$-tactic contact at A. Similarly the first polar of C, which is any arbitrary point, must have p-tactic contact at A. These conditions reduce (11) to (10), and the point constituents of the singularity are $\delta = \frac{1}{2}p(p-1) + 1$.

171. To find the line constituents, we must consider the reciprocal singularity, and for this purpose we may employ the curve

$$\alpha^2\gamma^2 u_{p-2} + \alpha\gamma u_p + u_{p+2} = 0 \quad \text{..............(12).}$$

From § 170 it follows that $m = 4p$, also $2p$ tangents can be drawn from A to the curve; hence the reciprocal singularity is a tangent to a curve of degree $4p$, which touches it at $p-2$ distinct points, corresponding to the distinct nodal tangents $u_{p-2} = 0$; also the tangent intersects the curve in $2p$ points, corresponding to the $2p$ tangents drawn from A; and it touches it at

$$4p - 2p - 2(p-2) = 4$$

coincident points at a point A', which is the reciprocal of the tangent AB to the original curve.

If we write down the first polar of (12) with respect to B, which may be any arbitrary point on AB, and eliminate $\alpha\gamma$, the result is a binary quantic of (β, γ) of degree $4p-2$, which shows that $4p-2$ tangents can be drawn to (12) from an arbitrary point on AB. Hence an arbitrary line through A' cuts the reciprocal curve in $4p-2$ ordinary points, and therefore A' is a singular point of the second order.

The reciprocal singularity is therefore a tacnodal tangent, which has bitactic contact with the reciprocal curve at $p-2$ points, and its constituents are

$$\delta = 2, \quad \tau = \tfrac{1}{2}p(p-1) + 1,$$

whilst the original singularity is a multiple point having one pair of tacnodal and $p-2$ ordinary branches, and its constituents are

$$\delta = \tfrac{1}{2}p(p-1) + 1, \quad \tau = 2.$$

172. The above results are true when there are any number of tacnodal branches, and may be generalized as follows:

(i) *If a multiple point of order p has s pairs of tacnodal branches and $p-2s$ distinct ordinary branches, its constituents are*

$$\delta = \tfrac{1}{2}p(p-1) + s, \quad \tau = 2s.$$

Putting $p = 2s$, it follows that

(ii) *If a multiple point of order $2s$ has s pairs of tacnodal and no ordinary branches, its constituents are*

$$\delta = 2s^2, \quad \tau = 2s.$$

The reciprocals of these singularities are:—

(iii) *A multiple tangent which touches a curve at s tacnodes and has bitactic contact with the curve at $p-2s$ points; and its constituents are*

$$\delta = 2s, \quad \tau = \tfrac{1}{2}p(p-1)+s.$$

(iv) *A multiple tangent which touches the curve at s tacnodes and nowhere else; and its constituents are*

$$\delta = 2s, \quad \tau = 2s^2.$$

173. We must now consider how these results are modified when some of the branches coincide; and we shall show that every ordinary branch which coincides with a tacnodal branch changes a node into a cusp, whilst every pair of tacnodal branches which coincides with another pair of tacnodal branches changes two nodes into two cusps. The first theorem is as follows:—

If a multiple point of order p consists (i) *of one pair of tacnodal branches,* (ii) *of r ordinary branches which coincide with the pair of tacnodal branches,* (iii) *of $p-r-2$ distinct ordinary branches; its constituents are*

$$\delta = \tfrac{1}{2}p(p-1)-r+1, \quad \kappa = r, \quad \tau = 2.$$

The curve

$$\alpha^2(\lambda\beta+\mu\gamma)^{r+2}u_{p-r-2}+\alpha u_{p+1}+u_{p+2}=0 \quad \text{........(13)}$$

has a multiple point at A consisting of $p-r-2$ distinct and $r+2$ coincident branches; and if an additional double point moves up to coincidence with A along AB, it can be shown as in § 170 that (13) becomes

$$\alpha^2\gamma^{r+2}u_{p-r-2}+\alpha\gamma u_p+u_{p+2}=0 \quad \text{..............(14).}$$

Write down the first polar of C, which may be any arbitrary point, and eliminate α, and the result will be a binary quantic of (β, γ) of degree $4p-r$. Whence

$$\begin{aligned} 2\delta+3\kappa &= (p+2)(p+1)-4p+r \\ &= p(p-1)+r+2. \end{aligned}$$

Also $\qquad \delta+\kappa = \tfrac{1}{2}p(p-1)+1,$

whence $\qquad \delta = \tfrac{1}{2}p(p-1)-r+1, \quad \kappa = r$ ....(15),

which give the point constituents of the singularity.

The reciprocal singularity consists of a multiple tangent which has bitactic contact with the reciprocal curve at $p-r-2$ points and touches it at q points at a tacnode, and also cuts the curve at $2p$ points, which correspond to the $2p$ tangents which can be drawn to (14) from A. Hence

$$4p - r = q + 2(p - r - 2) + 2p,$$

which gives $$q = r + 4.$$

Accordingly the line constituents of the original singularity are $\tau = 2$; and the reciprocal singularity is:—

(i) *A multiple tangent which touches the curve at $r+4$ points at a tacnode and has bitactic contact with it at $p-r-2$ distinct points; and its constituents are*

$$\delta = 2, \quad \tau = \tfrac{1}{2}p(p-1) - r + 1, \quad \iota = r.$$

Also the coincidence of each successive ordinary point increases the contact by 1, *and converts a double tangent into a stationary one.*

Let $p - r - 2 = 0$, then:—

(ii) *The constituents of a tacnodal tangent which touches the curve at $r+4$ points at a tacnode and nowhere else, are*

$$\delta = 2, \quad \tau = \tfrac{1}{2}r(r+1) + 2, \quad \iota = r.$$

Also each additional point of contact after the $(r+4)$th adds one stationary and $r+1$ double tangents to the constituents of the singularity.

174. The theory of coincident tacnodal branches is contained in the following theorem:—

*If a multiple point of order p has s pairs of tacnodal branches of which r pairs are coincident, $r > 1$, and $p - 2s$ ordinary branches; its constituents are**

$$\delta = \tfrac{1}{2}p(p-1) + s - 2r + 2, \quad \kappa = 2r - 2, \quad \tau = 2s - r + 1.$$

This theorem, so far as its point constituents are concerned, may be proved by the previous methods; but the portion relating to the line singularities will be proved in the next section. We notice the following special cases.

If there are no ordinary branches $p = 2s$, whence

(i) *If a multiple point of order $2s$ consists of r pairs of coincident and $s-r$ pairs of distinct tacnodal branches, its constituents are*

$$\delta = 2s^2 - 2r + 2, \quad \kappa = 2r - 2, \quad \tau = 2s - r + 1.$$

* When $r=1$, all the tacnodal branches are distinct.

If all the tacnodal branches coincide, $s = r$, whence

(ii) *If a multiple point of order* $2s$ *consists of* s *pairs of coincident tacnodal and no ordinary branches; its constituents are*

$$\delta = 2s^2 - 2s + 2, \quad \kappa = 2s - 2, \quad \tau = s + 1.$$

(iii) *If all the tacnodal branches coincide, and there are* $p - 2s$ *ordinary branches, the constituents of the multiple point are*

$$\delta = \tfrac{1}{2}p\,(p-1) - s + 2, \quad \kappa = 2s - 2, \quad \tau = s + 1.$$

175. We must now examine the reciprocal singularity. Consider the two curves

$$\alpha^2\gamma^2 u_1^2 + \alpha\gamma u_1 u_3 + u_6 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

$$\alpha^2\gamma^4 + \alpha\gamma^2 u_3 + u_6 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

The first curve is a sextic having a quadruple point at A, which consists of two pairs of distinct tacnodal and no ordinary branches; consequently, putting $p = 4$, $s = 2$, $r = 1$ in the theorem of § 174, it follows that the point constituents of the singularity are $\delta = 8$, $\kappa = 0$; which gives $m = 14$. The reciprocal of the quadruple point is a tangent which touches the reciprocal curve at two tacnodes and intersects it at six ordinary points, and consequently its point constituents are $\delta = 4$, and therefore the line constituents of the original singularity are $\tau = 4$.

In the second curve the two pairs of tacnodal branches coincide, so that $r = 2$, which gives $\delta = 6$, $\kappa = 2$; hence $m = 12$.

Also since six tangents can be drawn from A to (17), the reciprocal singularity consists of a tangent which intersects the reciprocal curve in six ordinary points and in six coincident points formed by the union of the two tacnodes. Hence a node is lost by the union of the two tacnodes on the reciprocal curve, and a double tangent is lost by the coincidence of the two pairs of tacnodal branches on the original curve. Accordingly the constituents of the original singularity are $\delta = 6$, $\kappa = 2$, $\tau = 3$, and those of the reciprocal are $\delta = 3$, $\tau = 6$, $\iota = 2$. Generalizing this result we obtain $\tau = 2s - r + 1$ for the line constituents of the singularity discussed in § 174.

We thus obtain the following reciprocal theorems:—

(α) *A tangent touches a curve* (i) *at* $s - r$ *distinct tacnodes;* (ii) *at a point composed of the union of* $r + 1$ *collinear nodes;* (iii) *at* $p - 2s$ *ordinary points; its constituents are*

$$\delta = 2s - r + 1, \quad \tau = \tfrac{1}{2}p\,(p-1) + s - 2r + 2, \quad \iota = 2r - 2.$$

(β) *A tangent touches a curve* (i) *at* $s-r$ *distinct tacnodes;* (ii) *at a point composed of the union of* $r+1$ *collinear nodes, and at no ordinary points; its constituents are*

$$\delta = 2s - r + 1, \quad \tau = 2s^2 - 2r + 2, \quad \iota = 2r - 2.$$

(γ) *A tangent touches a curve at a point composed of* $s+1$ *coincident collinear nodes; its constituents are*

$$\delta = s + 1, \quad \tau = 2s^2 - 2s + 2, \quad \iota = 2s - 2.$$

(δ) *A tangent touches a curve at a point composed of* $s+1$ *coincident collinear nodes and at* $p-2s$ *ordinary points; its constituents are*

$$\delta = s + 1, \quad \tau = \tfrac{1}{2}p(p-1) - s + 2, \quad \iota = 2s - 2.$$

Birational Transformation.

176. We shall now explain the theory of birational transformation, and shall show how it may be employed to investigate the constituents of the compound singularities of curves.

The conic $\alpha^2 = \beta\gamma$ touches the sides AB, AC of the triangle of reference at B and C. Let P be any point (ξ, η, ζ); and let AP cut the polar of P with respect to this conic in a point P', whose coordinates are (ξ', η', ζ'). The polar of P is

$$2\alpha\xi - \beta\zeta - \gamma\eta = 0,$$

and since this passes through P', we have

$$2\xi\xi' - \zeta\eta' - \eta\zeta' = 0 \quad \text{.....................(1)}.$$

But the equation of AP' is

$$\beta/\eta' = \gamma/\zeta' = k \text{ (say)},$$

whence

$$\eta/\eta' = \zeta/\zeta' = k \quad \text{..........................(2)}.$$

Substituting in (1) we obtain

$$\xi\xi' = k\eta'\zeta'.$$

Accordingly from (2), we have

$$\xi\xi' = \eta\zeta' = \zeta\eta',$$

or

$$\frac{\xi}{\eta'\zeta'} = \frac{\eta}{\xi'\eta'} = \frac{\zeta}{\xi'\zeta'}, \quad \text{.......................(3)},$$

which is the equation connecting the coordinates of P and P'.

It follows from the above construction, that any point on BC except B and C corresponds to A; any point on AB except A

corresponds to B; and any point on AC except A corresponds to C.

177. *A node which does not lie on the sides of the triangle ABC transforms into a node.*

The curve
$$u^2U + uvV + v^2W = 0,$$
where (u, v) are straight lines, and U, V, W are ternary quantics of degree $n - 2$, has a node at the point of intersection of the lines $u = 0$, $v = 0$; and if this curve be transformed by means of (3), u and v will become conics circumscribing the triangle ABC and intersecting in a fourth point P' which corresponds to the node (u, v). The point P' is obviously a node; and the theorem can be extended to multiple points of any order.

178. Let a curve cut BC in two ordinary points P and Q, which can always be effected by making B and C multiple points; then the transformed curve will have a node at A. And generally, if the curve cut BC in s ordinary points, the transformed curve will have a multiple point of order s at A; also since any pair of ordinary points gives rise to a node at A, it follows that the number of constituent nodes of a multiple point of order s is equal to the number of combinations of s things taken two at a time, that is, to $\frac{1}{2}s(s-1)$.

The directions of the nodal tangents at A are determined as follows. Let there be two ordinary points P and Q on BC; and let p, q be two points on the curve in the neighbourhood of P and Q. Then if p', q' be the corresponding points, Ap' and Aq', and ultimately AP and AQ, will be the directions of the nodal tangents at A. Hence if P and Q coincide, AP and AQ will also coincide; accordingly, if the curve touches BC at P and does not intersect it at any ordinary points, the transformed curve will have a cusp at A.

If BC touches the curve at P and intersects it at one ordinary point Q, the transformed curve will have a triple point of the second kind at A, consisting of a cusp and a branch through it; and its constituents are $\delta = 2$, $\kappa = 1$. And generally if the curve touches BC in r coincident points and intersects it in $s - r$ points, the transformed curve will have a multiple point of order s at A at which r tangents coincide; and the constituents of such a point are $\delta = \frac{1}{2}s(s-1) - r + 1$, $\kappa = r - 1$.

If BC cuts a curve at a node and in no ordinary points, the transformed curve has a tacnode at A; hence each of the two coincident points of which the node is composed transforms into a node, whilst the two branches which pass through the node transform into the two branches which touch one another at the tacnode. For example the equation

$$\alpha^2 u_4 + \alpha\beta\gamma v_1 u_2 + \beta^2\gamma^2 v_1^2 = 0$$

represents a sextic having nodes at B and C, and a third node at the point $\alpha = 0$, $v_1 = 0$; and this transforms into the curve

$$\alpha^2 v_1^2 + \alpha v_1 u_2 + u_4 = 0,$$

which is a quartic having a tacnode at A.

If BC cuts the curve at a node and $p-2$ ordinary points, it can be shown in the same way that the transformed curve has a multiple point of order p at A consisting of one pair of tacnodal and $p-2$ distinct ordinary branches. And since its point constituents have been shown to be $\delta = \frac{1}{2}p(p-1)+1$, it follows that each of the two points which coincide at the node gives rise to a node, whilst every ordinary point in combination with either of the nodal points or with another ordinary point gives rise to a node. Also the theorem of § 172 (i) shows that this is true for any number of nodes and ordinary points on BC; and it follows from § 173 that if r ordinary points on BC coincide with a node, the effect is to convert r of the constituent nodes of the transformed singularity into cusps.

179. Before considering the case of a cusp, it will be useful to state that the equations of a quartic curve which has a tacnode, a rhamphoid cusp, an oscnode and a tacnode cusp at A may be written in the forms

$$(\alpha u_1 + u_2)^2 + u_4 = 0 \quad\ldots\ldots(4),$$

$$(\alpha u_1 + u_2)^2 + u_1 u_3 = 0 \quad\ldots\ldots(5),$$

$$(\alpha u_1 + u_2)^2 + u_1^3(l\alpha + m\beta + n\gamma) = 0 \quad\ldots\ldots(6),$$

$$(\alpha u_1 + u_2)^2 + u_1^3(m\beta + n\gamma) = 0 \quad\ldots\ldots(7).$$

180. Transform (5) birationally and it becomes

$$(\alpha u_2 + \beta\gamma u_1)^2 + \alpha^2 u_1 u_3 = 0 \quad\ldots\ldots(8),$$

which represents a sextic curve having nodes at B and C and a double point on BC at the point C' where $u_1 = 0$ cuts it; and if ABC' be taken as a new triangle of reference, the double point will be found to be a cusp. *Hence a cusp on BC transforms into a rhamphoid cusp at A.*

In the same way it can be shown that a tacnode and a rhamphoid cusp at C' respectively transform into an oscnode and a tacnode cusp at A, provided the tangent at the singularity is an arbitrary line through C. These results may however be proved differently by means of the theorem of § 177; for a tacnode is formed by the union of two nodes situated at C' and at an arbitrary point P near C', such that ultimately $C'P$ is the tacnodal tangent. Now the node at C' transforms into a tacnode at A and AC' is the tacnodal tangent; the line $C'P$ transforms into a conic which touches AC' at A; whilst the node at P transforms into a node at a point P' lying on the conic. Hence when P coincides with C', P' coincides with A, and the singularity is an oscnode which is formed by the union of three nodes which move up to coincidence along a conic. The case of a tacnode cusp can be dealt with in a similar manner.

181. By means of the theory of birational transformation, all the preceding theorems relating to multiple points with tacnodal branches can be proved (see *Quart. Jour.* vol. XXXVI. p. 313); and by the same method the following theorems relating to rhamphoid cuspidal branches can be established.

(i) *If the line BC intersects a curve at a cusp and at $p-2$ ordinary points, the transformed singularity consists of a multiple point of order p composed of a rhamphoid cusp and $p-2$ ordinary branches through it; and its constituents are*

$$\delta = \tfrac{1}{2}p(p-1), \quad \kappa = 1, \quad \tau = 1, \quad \iota = 1.$$

(ii) *If a multiple point of order p consists of a rhamphoid cusp and one coincident and $p-3$ distinct ordinary branches; its constituents are*

$$\delta = \tfrac{1}{2}p(p-1)-1, \quad \kappa = 2, \quad \tau = 1, \quad \iota = 1.$$

The reciprocal singularities are

(iii) *A rhamphoid cusp, whose cuspidal tangent touches the curve at $p-2$ ordinary points; and its constituents are*

$$\delta = 1, \quad \kappa = 1, \quad \tau = \tfrac{1}{2}p(p-1), \quad \iota = 1.$$

(iv) *A rhamphoid cusp, whose cuspidal tangent has quinquetactic contact with the curve at the cusp, and touches the curve at $p-3$ ordinary points; and its constituents are*

$$\delta = 1, \quad \kappa = 1, \quad \tau = \tfrac{1}{2}p(p-1)-1, \quad \iota = 2.$$

Putting $p = 3$, it follows that the constituents of a rhamphoid cusp whose tangent has quinquetactic contact with the curve are $\delta = 1$, $\kappa = 1$, $\tau = 2$, $\iota = 2$. It can also be shown by birational transformation that the equation of a curve having a rhamphoid cusp at A, and AB as the cuspidal tangent, is

$$\alpha^{n-4}(M\beta^2 + L\alpha\gamma)^2 + \alpha^{n-3}\gamma^2 u_1 + \alpha^{n-4}\gamma u_3 + \alpha^{n-5}u_5 + \dots u_n = 0 \quad \dots(9),$$

and if the tangent has quinquetactic contact at A, $M = 0$ and (9) becomes

$$L\alpha^{n-2}\gamma^2 + \alpha^{n-3}\gamma^2 u_1 + \alpha^{n-4}\gamma u_3 + \alpha^{n-5}u_5 + \dots u_n = 0 \quad \dots(10).$$

182. *If the line BC intersects a curve at an n-tuple point of the first kind and at no ordinary points, the transformed singularity consists of a multiple point formed by the union of two n-tuple points. Its constituents are*

$$\delta = n(n-1), \quad \tau = n(n-1),$$

and the equation of the curve is

$$\alpha^n\gamma^n + \alpha^{n-1}\gamma^{n-1}u_2 + \dots \alpha\gamma u_{2n-2} + u_{2n} = 0.$$

It will be sufficient to prove this theorem for a sextic curve, since the method of proof is the same for any other curve.

The sextic curve

$$\alpha^3 u_3 + \alpha^2 u_4 + \alpha u_5 + u_6 = 0 \quad \dots\dots\dots\dots(11)$$

has a triple point of the first kind at A, and if it has another triple point at a point P on AB, it follows that A and P are nodes on the first polar and ordinary points on the second polar of C, which may be any arbitrary point. Hence when A and P coincide, AB must have sextactic contact with the curve, quadritactic contact with the first polar, and bitactic contact with the second polar of C. This will be found to reduce (11) to the form

$$\alpha^3\gamma^3 + \alpha^2\gamma^2 u_2 + \alpha\gamma u_4 + u_6 = 0 \quad \dots\dots\dots(12).$$

Hence the singularity is a singular point of the third order, the tangent at which has sextactic contact with the curve. Writing (12) in the form

$$\alpha^3 v_1^3 + \alpha^2 v_1^2 u_2 + \alpha v_1 u_4 + u_6 = 0 \quad \dots\dots\dots(13),$$

and transforming birationally, we obtain

$$\alpha^3 u_6 + \alpha^2\beta\gamma v_1 u_4 + \alpha\beta^2\gamma^2 v_1^2 u_2 + \beta^3\gamma^3 v_1^3 = 0 \quad \dots\dots(14),$$

which is the equation of a curve of the 9th degree, having triple points at B and C and also at the point $\alpha = 0$, $v_1 = 0$ on BC.

The portion relating to the line constituents may be proved as follows.

The reciprocal curve is of degree 18; also since the discriminant of (12) is of degree 12, it follows that 12 tangents can be drawn from A to (12); hence if A' be the point on the reciprocal curve corresponding to AB, the tangent at A' has sextactic contact with the reciprocal curve at A'. The first polar of B, which may be any arbitrary point on AB, is

$$\alpha^2\gamma^2 u_2' + \alpha\gamma u_4' + u_6' = 0 \text{ ....(15).}$$

Eliminating $\alpha\gamma$ between (12) and (15) by Sylvester's dialytic method*, it will be found that the eliminant contains the term $u_2 u_4' u_6'^2$, and is therefore of the 15th degree; accordingly 15 ordinary tangents can be drawn from B; and therefore an arbitrary line through A' intersects the reciprocal curve in 15 ordinary points. Hence A' is a singular point of order three, the tangent at which has sextactic contact with the curve; accordingly the singularity at A is its own reciprocal.

183. The singularity consists of n branches touching one another at the same point. This is evident by considering a sextic curve having such a singularity at the origin O and the axis of y as the tangent. The Cartesian equation of the curve is

$$ax^3 + x^2 U_2 + xU_4 + U_6 = 0 \text{ ....(16),}$$

where $U_n = (x, y)^n$. If ρ be the radius of curvature at the origin, $y^2 = 2\rho x$; whence substituting in (16), dividing out by x^3 and then putting $x = 0$, (16) becomes

$$a + b\rho + c\rho^2 + d\rho^3 = 0,$$

where a, b, c, d are constants. This proves the theorem and determines the radii of curvature of the three branches. It also follows that n complete branches touching one another at the same

* The easiest way of employing this method is to use polynomials instead of binary quantics. Let $u = (a, b, c, d \between x, 1)^3$ and $v = (A, B, C \between x, 1)^2$; and write down the five equations $xu = 0$, $x^2 v = 0$, $u = 0$, $xv = 0$, $v = 0$. Then eliminating x^4, x^3, x^2 and x, we obtain the determinant

$$\begin{vmatrix} a, & b, & c, & d, & 0 \\ A, & B, & C, & 0, & 0 \\ 0, & a, & b, & c, & d \\ 0, & A, & B, & C, & 0 \\ 0, & 0, & A, & B, & C \end{vmatrix} = 0.$$

point cannot be possessed by any curve of lower degree than the $2n$th; but a curve of lower degree may have n branches, some of which are partial branches which commence at a point and proceed in only one direction. Thus a quintic can have a cusp and a complete branch through it which touches the cuspidal tangent.

The foregoing theory of the transformation of multiple points may be extended to cases in which a p-tuple, instead of a double point, is the elementary singularity.

The Reciprocal Theorem.

184. There appears to be a remarkable reciprocal theorem, which may be enunciated as follows:—

Let there be two curves such that BC intersects them at a certain singularity at a point C', and at no other points except B and C. In the first curve let the singularity be related in a given manner to the line BC; and in the second curve let the singularity be related in the same manner to the line AC'. Then when the curves are transformed, the two singularities at A are the reciprocals of one another.

I shall not attempt any general proof of this theorem, nor venture to assert that it is true in every conceivable case that may arise. I shall content myself with verifying its truth in the case of one of the theorems previously discussed.

Let AC' and BC have r-tactic contact with their respective curves at C', then the transformed singularities are their own reciprocals.

The equation

$$\alpha^3 u_0 + \alpha^2 v_1 + \alpha v_1 u_1 + \beta\gamma v_1 = 0$$

represents a cubic which passes through B and C, and has a point of inflexion at C', and v_1 or AC' is the stationary tangent. The transformed equation is

$$\alpha^3 v_1 + \alpha^2 v_1 u_1 + \alpha\beta\gamma v_1 + \beta^2\gamma^2 u_0 = 0,$$

which represents a quartic having nodes at B and C, and a point of undulation at A, the tangent at which is AC'. In the same way it can be shown that if AC' has r-tactic contact with a curve at C', and BC does not intersect the curve at any other points except B, C and C', the line AC' has $(r+1)$-tactic contact with the transformed curve at A.

If BC has r-tactic contact with another curve at C' and does not intersect it at any other points except B and C, the transformed curve has a multiple point of order r at A, the r tangents at which are coincident; and this is the reciprocal of a multiple tangent which has $(r+1)$-tactic contact and does not touch the curve elsewhere.

185. The theory of the birational transformation of curves and surfaces has been discussed by Cayley* at considerable length: and the transformation which we have employed is a particular case of the quadric transformation†. The theory has also formed the subject of numerous memoirs by Italian mathematicians‡ which are to a large extent connected with the theory of the singularities of surfaces, to which as will be shown in the next chapter a similar theory applies.

* *Proc. Lond. Math. Soc.* vol. III. p. 127; *C. M. P.* vol. VII. p. 189.

† *Ibid.* p. 170.

‡ Segre, "Sulla scomposizione dei punti singolari delle superficie algebriche," *Ann. di Mat.* Serie II. vol. XXV. p. 219; and various other papers published in this journal.

CHAPTER V

SINGULARITIES OF SURFACES

186. BEFORE proceeding to consider the theory of quartic surfaces, it will be desirable to discuss the general theory of the compound singularities of surfaces; for just as most compound singularities, which a cubic surface can possess, appear on the latter in a special form, so various singularities which a quartic surface can possess are deficient in certain peculiarities which appear when the singularity occurs on a surface of higher degree.

From analogy to the theory of plane curves, it is to be anticipated that the conic node and the binode are the only *simple* point singularities which an algebraic surface can possess; and that every other singularity is a compound one which, so far as its point constituents are concerned, is formed by the union of two or more conic nodes or binodes. I have found this to be the case in every compound singularity which I have examined; and I shall commence with the theory of the multiple point.

187. *If three cones of degrees l, m, n have a common vertex and no common generators, the vertex absorbs lmn of their points of intersection.*

Since the cones have no common generators, the common vertex is their only point of intersection; hence the theorem at once follows; but if the cones had a common generator, every point on it would be common to the three cones, and the theorem would no longer be true.

188. *If three surfaces have multiple points of orders p, q, r at A, and the nodal cones are not specially related to one another, then A absorbs pqr of their points of intersection*.*

* Berzolari, *Ann. di Matem.* Serie II. vol. xxiv. p. 165.

Let l, m, n be the degrees of the three surfaces; then in order to prove the theorem, we shall employ the method of Salmon explained in § 42. Let the first surface S_l be replaced by a proper cone S_p, whose vertex is A, and a surface S_{l-p} which does not pass through A. Treating each of the other surfaces in the same way, it follows that the number of ordinary points of intersection consists of (i) the intersections of any cone such as S_p with the two surfaces S_{m-q} and S_{n-r}; (ii) the intersections of any surface such as S_{l-p} with the cones S_q and S_r; (iii) the intersections of the three surfaces S_{l-p}, S_{m-q}, S_{n-r}. Hence the total number of ordinary points of intersection is

$$(m-q)(n-r)p+(n-r)(l-p)q+(l-p)(m-q)r+(l-p)qr$$
$$+(m-q)rp+(n-r)pq+(l-p)(m-q)(n-r)=lmn-pqr$$

which shows that pqr points of intersection are absorbed at A.

189. *When three surfaces intersect at a point A which is a multiple point of order p on the first two, and whose nodal cones are identical, whilst A is a multiple point of order q on the third, the number of points of intersection absorbed at A is $pq(p+1)$.*

We may without loss of generality suppose the first two surfaces to be of degrees $p+1$, in which case the equations of their curves of intersection by any arbitrary plane δ through A are

$$\alpha u_p + u_{p+1} = 0, \quad \alpha u_p + U_{p+1} = 0.$$

These curves obviously intersect in $p+1$ ordinary points; hence the number of points absorbed at A is $p(p+1)$. This shows that A is a multiple point of order $p(p+1)$ on the curve of intersection of the two surfaces.

If the nodal cone at the multiple point on the third surface is not specially related to the two first surfaces, each branch of their curve of intersection will cut the third surface in q coincident points at A; hence the total number of points absorbed at A is $pq(p+1)$.

This result is also true when the first two surfaces have the same lines of closest contact; for since they are of degree $p+1$, their curve of intersection consists of the $p(p+1)$ lines of closest contact and a residual curve of degree $p+1$. Let the equation of the third surface be

$$\alpha^2 U_q + \alpha U_{q+1} + U_{q+2} = 0 \quad \text{.................(1)},$$

then the residual curve intersects (1) in $(p+1)(q+2)$ ordinary points, whilst the lines of closest contact intersect (1) in $2p(p+1)$ of such points; hence the number of points absorbed at A is

$$(p+1)^2(q+2)-(p+1)(q+2)-2p(p+1)=pq(p+1).$$

(i) *If the nodal cone of the third surface contains all the lines of closest contact of the first two surfaces, and $q>p$, the number of points absorbed at A is $p(p+1)(q+1)$.*

(ii) *But, if these are also lines of closest contact on the third surface, the number absorbed at A is $p(p+1)(q+2)$.*

If U_q contain all the lines of closest contact, each will intersect (1) in only one ordinary point; hence the number of points absorbed at A is

$$(p+1)^2(q+2)-(p+1)(q+2)-p(p+1)=p(p+1)(q+1).$$

And if these are lines of closest contact on (1), they will not intersect (1) in any ordinary points; hence the number of points absorbed is

$$(p+1)^2(q+2)-(p+1)(q+2)=p(p+1)(q+2).$$

The last two theorems are true* when $p=q$.

190. *A multiple point of order p, the tangent cone at which is anautotomic, reduces the class† by $p(p-1)^2$.*

When a surface has a multiple point of order p at A, the first polars of any two points have multiple points of orders $p-1$ at A; also if the nodal cone is anautotomic, this cone and the nodal cones at A to the first polars have no common generators; hence A absorbs $p(p-1)^2$ of the points of intersection of the surface and the first polars of any two points. Accordingly

$$m=n(n-1)^2-p(p-1)^2.$$

191. Before finding the constituents of a multiple point, a few additional remarks on the compound singularities of plane curves will be necessary.

If δ nodes on a plane curve move up to coincidence in any manner whatever, the point constituents of the resulting compound

* If a surface of degree $p+2$ has a multiple point of order p at A, the tangent cone from A obviously cannot possess any generators which have tritactic contact with the surface at some other point P; hence the surface and its first and second polars with respect to A cannot intersect at any ordinary points, and therefore the number of points absorbed at A is $p(p+1)(p+2)$. A similar argument frequently gives a short cut to theorems of this character.

† Segre, *Ann. di Matem.* Serie II. vol. xxv. p. 28.

singularity are δ nodes; but it is otherwise in the case of cusps. For if κ cusps move up to coincidence, it frequently happens that $2p$ of them are changed into $3p$ nodes, and this is especially the case when the cusps move up to coincidence along a continuous curve; also since the reductions of class produced by a node and a cusp are respectively equal to 2 and 3, the class of the curve remains unaltered. The simplest example of the conversion of cusps into nodes is furnished by the oscnode. For the equation of a bicuspidal quartic curve can be expressed in the form

$$S^2 = uv^3 \quad \text{.............................(2)},$$

where u is the double tangent, v the line joining the cusps and S is a conic, which passes through the points of contact of the double tangent and has tritactic contact with the curve at each cusp; but when the line v touches the conic S, the two cusps coincide and the resulting singularity becomes an oscnode. If however the point constituents of an oscnode were two cusps, it would be possible for the quartic to have a third double point; but if one be introduced, it can be shown in the following manner that the quartic will degrade into a pair of conics which osculate one another.

Let ABC be the triangle of reference, A the oscnode, AB the oscnodal tangent; then (2) becomes

$$(\alpha\gamma + P\beta^2 + Q\beta\gamma + R\gamma^2) = (l\alpha + m\beta + n\gamma)\gamma^3 \quad \text{......(3)}.$$

Since C is an arbitrary point we may suppose it to be an additional node, the conditions for which are

$$l = 2R, \quad m = 2QR, \quad n = R^2,$$

and (3) becomes

$$(P\beta^2 + Q\beta\gamma + \alpha\gamma)^2 + 2PR\beta^2\gamma^2 = 0,$$

which represents a pair of conics. This shows that the union of the two cusps produces a compound singularity whose point constituents are three nodes; and many other similar examples might be given.

192. *The constituents of a multiple point of order p, the tangent cone at which is anautotomic, are**

$$C = \tfrac{1}{2}p(p-1)^2, \quad B = 0,$$

* Basset, *Rend. del Circolo Mat. di Palermo*, vol. XXVI. p. 329.

where C and B are the number of constituent conic nodes and binodes.

Let A be the multiple point, D any point in space; then since A is a multiple point of orders $p-1$ and $p-2$ respectively on the first and second polars of the surface with respect to D, it follows that A absorbs $p(p-1)(p-2)$ of the points of intersection of the surface and its first and second polars with respect to D. Hence the number of *distinct* generators of the tangent cone from D, which have tritactic contact with the surface, and which are therefore cuspidal generators of the cone, is

$$\kappa = n(n-1)(n-2) - p(p-1)(p-2) \quad \text{.........(4).}$$

Let ν and μ be the degree and class of the tangent cone from D, then

$$\nu = n(n-1), \quad \mu = n(n-1)^2 - p(p-1)^2 \quad \text{.........(5);}$$

also let δ be the number of *distinct* generators which are double tangents to the surface, and which are therefore nodal generators of the cone.

Since the tangent cone at A is anautotomic, its class is $p(p-1)$, and therefore DA is a multiple generator of the tangent cone from D of order $p(p-1)$, the tangent planes at which are distinct; hence A is a multiple point of the same character on the section of the cone by the plane ABC, and its point constituents are $\frac{1}{2}p(p-1)(p^2-p-1)$ nodes.

Applying Plücker's equations to the section of the tangent cone from D, we obtain

$$\mu = \nu(\nu-1) - p(p-1)(p^2-p-1) - 2\delta - 3\kappa \quad \text{......(6).}$$

Substituting the values of κ, μ and ν from (4) and (5) in (6), we obtain

$$\delta = \tfrac{1}{2}n(n-1)(n-2)(n-3) - \tfrac{1}{2}p(p-1)(p-2)(p-3) \quad \text{...(7),}$$

hence if δ' and κ' are the number of nodal and cuspidal generators which are absorbed by the multiple point

$$\delta' = \tfrac{1}{2}p(p-1)(p-2)(p-3), \quad \kappa' = p(p-1)(p-2) \quad \text{...(8).}$$

We shall now suppose that the multiple point at A is formed by the union of C conic nodes and B binodes. These double points are originally supposed to be isolated and to be arranged in any manner on the surface; hence the tangent cone from D will possess two species of nodal and cuspidal generators, the first of which arises from the double points on the surface, whilst the second arises from generators which are double and stationary tangents to the surface. When the C conic nodes and B binodes coincide at A, all the generators of the first species and $\delta' + \kappa'$ of the second species will coincide with the line DA; and we have to find the number of those of the first species.

The multiple point at A on the section of the tangent cone from D is of order $p(p-1)$, and is composed of double points of both species; hence

$$C + B + \delta' + \kappa' = \tfrac{1}{2}p^2(p-1)^2 - \tfrac{1}{2}p(p-1) \quad \text{.........(9)},$$

where δ' is given by the first of (8), but nothing at present is supposed to be known about κ' except that it represents the effect of the coincidence of the $p(p-1)(p-2)$ cuspidal generators of the second species. Also since the reduction of the class of the surface is $p(p-1)^2$, it follows that

$$2C + 3B = p(p-1)^2 \quad \text{....................(10)}.$$

Substituting the value of δ' from the first of (8) we obtain from (9) and (10)

$$\left.\begin{aligned} C &= \tfrac{1}{2}p(p-1)(10p-19) - 3\kappa' \\ 2\kappa' - B &= 3p(p-1)(p-2) \end{aligned}\right\} \quad \text{.........(11)}.$$

Now if we supposed that the κ' *distinct* cuspidal generators of the second species were equivalent *after coincidence* to $p(p-1)(p-2)$ cusps, we should obtain from the last of (11)

$$B = -p(p-1)(p-2),$$

which is impossible, since B cannot be a negative quantity. This shows that the effect of coincidence is to convert the $2\kappa'$ cusps into $\tfrac{3}{2}\kappa'$ nodes, which produces no alteration of the class of the tangent cone or of the surface, but makes

$$C = \tfrac{1}{2}p(p-1)^2, \quad B = 0.$$

193. When the tangent cone is autotomic, the investigation of the point constituents of any multiple point involves the solution of two distinct problems. In the first place the class m of the surface is determined by the equation

$$m = n(n-1)^2 - 2C - 3B \quad \text{.................(12)},$$

and in the second place an equation exists of the form

$$C + B = \lambda \quad \text{.........................(13)}.$$

When the tangent cone is anautotomic the value of λ by the preceding theorem is $\tfrac{1}{2}p(p-1)^2$, and the theorem of § 24 usually enables us to ascertain without much difficulty whether any change in the character of the multiple point is produced by the conversion of conic nodes into binodes, or by the union of additional double points with the multiple point. The principal difficulty is to determine the value of m, and we shall proceed to explain the methods by which this can be effected*.

194. *When the nodal cone at a multiple point of order p has δ nodal and κ cuspidal generators, all of which are distinct, the reduction of class is*

$$p(p-1)^2+\delta+2\kappa,$$

and the point constituents of the singularity are

$$C=\tfrac{1}{2}p(p-1)^2-\delta-2\kappa, \quad B=\delta+2\kappa.$$

Let the equation of the surface be

$$\alpha u_p+u_{p+1}=0 \quad\text{.......................(14)},$$

then the first polars of C and D are

$$\alpha u'_p+u'_{p+1}=0 \quad\text{.......................(15)},$$

$$\alpha u''_p+u''_{p+1}=0 \quad\text{.......................(16)},$$

where the single and double accents denote differentiation with respect to γ and δ respectively; whence eliminating α between (14), (15) and (16) we obtain

$$u_p u'_{p+1}=u_{p+1}u'_p \quad\text{....................(17)},$$

$$u_p u''_{p+1}=u_{p+1}u''_p \quad\text{....................(18)}.$$

Equations (17) and (18) represent two cones of degree $2p$, and their $4p^2$ common generators intersect the surface (14) at the points where it is intersected by (15) and (16); but these generators include the $p(p+1)$ lines of closest contact, which do not give rise to ordinary points of intersection; hence the number of the latter is reduced by $p(p+1)$.

Again, if we temporarily regard the cones u_p, u'_p and u''_p as curves lying in the plane BCD, the last two will be the first polars of u_p with respect to C and D; accordingly if AB is a nodal generator, it must be repeated *once* on the cones u'_p and u''_p, and *twice* if it is a cuspidal generator, but the three cones u_p, u'_p and u''_p will not in general have any other common generator except AB. Hence every nodal generator on u_p produces a further reduction

* Basset, "Multiple points on Surfaces," *Quart. Jour.* vol. xxxix. p. 1.

in the number of common generators equal to 1, and every cuspidal generator reduces it by 2. Accordingly the number of ordinary points of intersection of (14), (15) and (16) is $4p^2-p(p+1)-\delta-2\kappa$, giving

$$m = 4p^2 - p(p+1) - \delta - 2\kappa$$
$$= (p+1)p^2 - p(p-1)^2 - \delta - 2\kappa,$$

which shows that the reduction of class is given by the last three terms. We thus obtain

$$2C + 3B = p(p-1)^2 + \delta + 2\kappa \quad \text{..............(19).}$$

From the theorem of § 24, it is easily seen that the reduction of class is not produced by the union of any additional double points with the multiple point; hence by § 193

$$C + B = \tfrac{1}{2}p(p-1)^2 \quad \text{....................(20).}$$

Solving (19) and (20) we obtain the required result.

195. *If the nodal cone at a multiple point of order p possesses a multiple generator of order q, such that r of the tangent planes are coincident, the constituents of the singularity are*

$$C = \tfrac{1}{2}p(p-1)^2 - (q-1)^2 - r + 1, \quad B = (q-1)^2 + r - 1.$$

(i) Let all the tangent planes along the multiple generator AB be distinct; then since a multiple point of order q on a curve gives rise to a multiple point of order $q-1$ on the first polar, it follows that the first polars of two arbitrary points intersect in $(q-1)^2$ coincident points at B; hence if s be the additional reduction produced by the generator, $s = (q-1)^2$.

(ii) Let r of the tangent planes along AB coincide; then AB is a multiple point of order $q-1$ on the first polar, having $r-1$ coincident tangent planes; accordingly AB will be repeated $r-1$ additional times on the first polars of two arbitrary points, so that $s=(q-1)^2+r-1$. This gives

$$2C + 3B = p(p-1)^2 + (q-1)^2 + r - 1,$$

also

$$C + B = \tfrac{1}{2}p(p-1)^2,$$

which proves the theorem.

It does not appear to make any difference whether the preceding compound singularities occur on a proper or an improper cone. Putting $q=3$, $r=1$, it follows that the additional reduction of class produced by a triple generator on the nodal cone is 4; and it can be shown, by an independent investigation, that the additional reduction of class produced by a triple generator when the nodal cone is a quartic cone is the same, whether the cone is (i) a proper one, (ii) a nodal cubic cone and a plane through the nodal generator, (iii) two planes and a quadric cone passing through their line of intersection.

Multiple Points in which the Cone consists of Planes intersecting in the same Straight Line.

196. We shall now discuss multiple points in which the cone degrades into p planes intersecting in the same straight line, and shall commence with the following theorem.

When a multiple point of order p consists of p distinct planes intersecting in a point, the reduction of class is

$$\tfrac{1}{2}p(p-1)(2p-1),$$

and the point constituents of the singularity are

$$C=\tfrac{1}{2}p(p-1)(p-2), \quad B=\tfrac{1}{2}p(p-1).$$

But when the planes intersect in the same straight line, the reduction of class is $(p+1)(p-1)^2$, *and the point constituents of the singularity are*

$$C=\tfrac{1}{2}(p-1)^2(p-2), \quad B=(p-1)^2.$$

By means of the theorem of § 24, it can be shown in both cases that the reduction of class does not arise from the union of any additional conic nodes or binodes with the multiple point; hence the reduction is caused by the conversion of conic nodes into binodes.

In the first case, when the planes intersect in the same point, the number of their lines of intersection is $\tfrac{1}{2}p(p-1)$; hence

$$2C+3B=p(p-1)^2+\tfrac{1}{2}p(p-1),$$

also

$$C+B=\tfrac{1}{2}p(p-1)^2,$$

whence

$$C=\tfrac{1}{2}p(p-1)(p-2), \quad B=\tfrac{1}{2}p(p-1) \text{ ....(1).}$$

To prove the second case, we may employ a surface of degree $p+1$, which is

$$\alpha v_p+u_{p+1}=0 \text{ ....(2),}$$

where

$$u_{p+1}=\beta^{p+1}w_0+\beta^p w_1+\dots w_{p+1}.$$

The first polars of C and D are

$$\alpha v'_p+u'_{p+1}=0, \quad \alpha v''_p+u''_{p+1}=0 \text{ ....(3),}$$

Multiplying the first of (3) by γ and the second by δ and adding, we obtain

$$p\alpha v_p + \beta^p w_1 + 2\beta^{p-1} w_2 + \ldots (p+1)\, w_{p+1} = 0 \quad \ldots\ldots(4).$$

Eliminating α between (2) and (4), we obtain

$$p\beta^{p+1} w_0 + (p-1)\,\beta^p w_1 + \ldots - w_{p+1} = 0 \quad \ldots\ldots\ldots(5).$$

Eliminating α between (3), we obtain

$$(\beta^p w_1' + \beta^{p-1} w_2' + \ldots)\, v_p'' = (\beta^p w_1'' + \beta^{p-1} w_2'' + \ldots)\, v_p' \ldots(6).$$

Equations (5) and (6) represent two cones of degrees $p+1$ and $2p-1$, which possess $(p+1)(2p-1)$ common generators; and this number is equal to the number of ordinary points of intersection of (2) and the first polars of two arbitrary points. Hence

$$m = (p+1)(2p-1) = (p+1)\, p^2 - 2C - 3B,$$

accordingly $\quad 2C + 3B = (p+1)(p-1)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$

also $\quad C + B = \tfrac{1}{2} p\,(p-1)^2,$

whence $\quad C = \tfrac{1}{2}(p-1)^2 (p-2), \quad B = (p-1)^2 \ldots\ldots\ldots\ldots(8).$

197. *When s tangent planes coincide, the reduction of class is*

$$(p+1)\{(p-1)^2 + s - 1\}.$$

In this case $v_p = \delta^s v_{p-s}$; hence

$$v_p' = \delta^s v'_{p-s}, \quad v_p'' = \delta^{s-1}(\delta v''_{p-s} + s v_{p-s});$$

accordingly (6) contains δ^{s-1} as a factor which must be rejected, and the resulting cone is of degree $2p - s$. Whence

$$m = (p+1)(2p-s) = (p+1)\, p^2 - 2C - 3B,$$

giving $\quad 2C + 3B = (p+1)\{(p-1)^2 + s - 1\} \ldots\ldots\ldots\ldots(9),$

from which it follows that each successive coincident plane produces an additional reduction of class equal to $p+1$. When all the planes coincide, $p = s$, and the reduction becomes $p\,(p^2-1)$.

198. We shall now explain a method for determining the number of constituent conic nodes and binodes, when some of the planes coincide, which depends upon the theorem of § 24.

Two cones of degree n which have a common vertex possess n^2 common generators; and a pair of such cones may be regarded as an improper cone of degree $2n$ which has n^2 nodal generators. If the two cones have an additional common generator they must coincide; hence a cone of degree n twice repeated may be regarded as an improper cone of degree $2n$ which has n^2+1 nodal generators. From this it follows that a pair of coincident planes may be regarded as a *binodal* quadric cone, the positions of whose nodal generators are indeterminate. Similarly three coincident planes may be regarded as an improper cubic cone having $2+2+2=6$ common generators; and generally if t_s be the number of nodal generators when there are s coincident planes,

$$t_s = t_{s-1} + 2(s-1),$$

the solution of which is $t_s = s(s-1)$. Accordingly s coincident planes may be regarded as a cone of degree s which has $s(s-1)$ nodal generators, the positions of which are indeterminate.

199. *When s coincident planes coincide, the constituents of the singularity are*

$$C = \tfrac{1}{2}(p-1)^2(p-2) - (p+1)(s-4),$$
$$B = (p-1)^2 + (p+1)(s-3); \text{ when } p+1 \geqq s(s-1),$$

and

$$C = \tfrac{1}{2}(p-1)^2(p-2) + (s-1)(3s-p-1),$$
$$B = (p-1)^2 - (s-1)(2s-p-1); \text{ when } p+1 \leqq s(s-1).$$

The equation of the surface is

$$\alpha^{n-p}\delta^s v_{p-s} + \alpha^{n-p-1}u_{p+1} + \dots u_n = 0 \text{ ....(10).}$$

In the first case, each of the $p+1$ lines of intersection of the cone u_{p+1} with the plane δ may be regarded as nodal generators of the cone δ^s; hence the number of additional nodes is $p+1$, and

$$C + B = \tfrac{1}{2}p(p-1)^2 + p + 1 \text{ ....(11).}$$

Combining this with (9) we obtain the first result. But in the second case there are only $s(s-1)$ additional nodes, whence

$$C + B = \tfrac{1}{2}p(p-1)^2 + s(s-1) \text{ ....(12),}$$

which by virtue of (9) gives the second result.

When $p=s$, $p+1 < p(p-1)$ except when $p=1$ and $p=2$; and in the latter case the singularity is an ordinary unode, and the second result gives $C=3$, $B=0$, which is right. But when $p=s>2$, the proper formulae are the first ones, and we obtain

$$\left.\begin{aligned} C &= \tfrac{1}{2}(p-1)^2(p-2) - (p+1)(p-4) \\ B &= (p-1)^2 + (p+1)(p-3) \end{aligned}\right\} \text{ ......(13),}$$

where $p>2$.

200. I shall now explain another method of finding the reduction of class produced by a multiple point.

Let A be a multiple point of order p, the tangent cone at which is anautotomic, then the class of the cone is $p(p-1)$; hence if D be any point of space, $p(p-1)$ tangent planes can be drawn to the cone through DA, and therefore DA is a multiple generator of the tangent cone from D of order $p(p-1)$. Now the class m of the tangent cone from D is the same as that of the surface; hence by § 167, $m-2p(p-1)$ tangent planes can be drawn to the surface through the line DA. This number is obviously equal to the class μ of the tangent cone from A to the surface; hence

$$m-2p(p-1)=\mu \quad \text{.....................(1)},$$

which reduces the problem to the determination of the class of the tangent cone from A.

Let the surface on which the multiple point exists be of degree $p+2$; then since none of the generators of the tangent cone from A can be double or stationary tangents to the surface, it follows that the tangent cone is anautotomic; and since the equation of the surface is

$$a^2u_p+2au_{p+1}+u_{p+2}=0,$$

that of the tangent cone from A is

$$u^2_{p+1}=u_pu_{p+2} \quad \text{.......................(2)},$$

whence its degree is $2p+2$, and its class $\mu=(2p+2)(2p+1)$.

Substituting in (1), we obtain

$$m-2p(p-1)=(2p+2)(2p+1) \quad \text{...........(3)},$$

giving $$m=(p+2)(p+1)^2-p(p-1)^2,$$

and the last term is the reduction of class produced by the multiple point.

Equation (2) shows that the lines of closest contact are generators; hence a singular generator of u_p which is not a line of closest contact will not affect the value of μ, but the left-hand side (1) will be altered. Let the nodal cone at A possess δ nodal and κ cuspidal generators, then the section of this cone by the plane BCD will be a curve of degree p having δ nodes and κ cusps, and the number of tangent lines which can be drawn through D to the section is $p(p-1)-2\delta-3\kappa$, and this is consequently the number of *distinct* tangent planes which can be drawn through DA to the nodal cone. But every plane through DA and a nodal generator is equivalent to two coincident tangent planes, and every such plane through a cuspidal generator is equivalent to three coincident tangent planes. Hence DA is a singular generator of the tangent cone from D of order $p(p-1)$ having δ pairs of coincident tangent planes, corresponding to each nodal generator, and κ planes consisting of three coincident tangent planes which correspond to each cuspidal generator. Putting $r=2$ and $s=3$ in (9 A) of § 167, it follows that the number of tangent planes which can be drawn through DA to the tangent cone from D is

$$m-2p(p-1)+(r-1)\delta+(s-1)\kappa=m-2p(p-1)+\delta+2\kappa;$$

and since this is equal to the number of tangent planes which can be drawn through DA to the surface, (3) must be replaced by

$$m-2p(p-1)+\delta+2\kappa=(2p+2)(2p+1),$$

giving $$m=(p+2)(p+1)^2-p(p-1)^2-\delta-2\kappa,$$

which furnishes another proof of the theorem of § 194.

201. In § 169 we have called attention to the distinction which exists between *multiple* points and *singular* points on plane curves; and we shall now prove that:

If the nodal cone at a multiple point of order p possesses a singular generator of order 2, *whose constituents are δ nodal and κ cuspidal generators, which move up to coincidence along a continuous curve, the total reduction of class is*

$$p(p-1)^2+2\delta+3\kappa-1$$

and the point constituents are

$$C=\tfrac{1}{2}p(p-1)^2-2\delta-3\kappa+1, \quad B=2\delta+3\kappa-1.$$

In this case, the number of distinct tangent planes which can be drawn through DA to the nodal cone at A is $p(p-1)-2\delta-3\kappa$ as before; but the number of coincident tangent planes is $2\delta+3\kappa$. Hence the number of tangent planes which can be drawn through DA to the surface is $m-2p(p-1)+2\delta+3\kappa-1$. Substituting this quantity for left-hand side of (3) we obtain

$$m=(p+2)(p+1)^2-p(p-1)^2-2\delta-3\kappa+1.$$

Also since the value of $C+B$ is given by (20) of § 194, the theorem at once follows.

Let us write $s=2\delta+3\kappa-1$, then the following special cases may be noted when the nodal cone has the following singular generators:

(i) *Tacnodal generator.* Here $\delta = 2$, $\kappa = 0$; whence $s = 3$.

(ii) *Rhamphoid cuspidal generator.* Here $\delta = 1$, $\kappa = 1$; whence $s = 4$.

(iii) *Oscnodal generator.* Here $\delta = 3$, $\kappa = 0$; whence $s = 5$.

(iv) *Tacnode cuspidal generator.* Here $\delta = 2$, $\kappa = 1$; whence $s = 6$.

202. The preceding theorem requires modification when the singular generator is a line of closest contact; and we shall show that:

If AB is a singular generator of order 2 *on the nodal cone, which produces an additional reduction of class equal to s; then when AB is a line of closest contact, the additional reduction is* $s+1$, *and the point constituents of the singularity are*

$$C = \tfrac{1}{2}p(p-1)^2 - s + 2, \quad B = s - 1.$$

The equation of the surface must be of the form

$$\alpha^{n-p}(\beta^{p-2}v_2 + \beta^{p-3}v_3 + \dots) + \alpha^{n-p-1}(\beta^{p+1}w_0 + \beta^p w_2 + \dots) + \alpha^{n-p-2}u_{p+2} + \dots u_n = 0,$$

where the w's are arbitrary binary quantics of (γ, δ), but the v's are connected in a manner which depends on the character of the singular generator AB. The latter has $(p+1)$-tactic contact with the surface at A, and p-tactic contact with the first polar at A; but if $w_0 = 0$, so that AB becomes a line of closest contact, then AB has $(p+2)$-tactic contact with the surface at A and $(p+1)$-tactic contact with the first polar at A. This shows that the surface and the first polars of any two points intersect in an additional point at A; hence the total reduction of class is

$$2C + 3B = p(p-1)^2 + s + 1.$$

Also by § 24, this additional reduction of class is produced by an additional double point which moves up to coincidence with A; accordingly

$$C + B = \tfrac{1}{2}p(p-1)^2 + 1,$$

which gives the required result.

The reader will be assisted in understanding the process which takes place, by considering the case of an ordinary conic node. When the nodal cone has a nodal generator (that is becomes two planes) the conic node is converted into a binode; but when this generator becomes a line of closest contact, the binode is reconverted into a conic node, and an additional conic node added, so that the singularity becomes the special binode whose axis has quadritactic contact with the surface.

203. The following theorem is an extension of this result.

If the nodal cone at a multiple point of order p possesses δ nodal generators, each of which is a line of closest contact; then when all coincide, the constituents of the singularity are

$$C = \tfrac{1}{2}p(p-1)^2 - \delta + 2, \quad B = 2\delta - 2.$$

To prove this theorem it will be sufficient to consider the case of a tacnodal generator on a quartic cone.

When two generators AB, AB' are lines of closest contact the effect is to add two conic nodes to the constituents of the singularity; so that in the case of a quartic node, the total reduction of class is $2C + 3B = 36 + 2 + 2 = 40$; and we shall now show that when AB and AB' coincide, the effect is to produce a further reduction of class equal to 2. Consider the surface

$$\alpha(\beta^2 v_1^2 + 2\beta v_1 v_2 + 2v_4) + k\beta^4 v_1 + \beta^3 w_2 + \dots w_5 = 0 \ldots\ldots(4),$$

where $v_1 = \gamma + \delta$ and k is a constant. Equation (17) of § 194 now becomes

$$(\beta^2 v_1^2 + 2\beta v_1 v_2 + 2v_4)(k\beta^4 + \beta^3 w_2' + \dots w_5')$$
$$= 2(k\beta^4 v_1 + \beta^3 w_2 + \dots w_5)\{\beta^2 v_1 + \beta(v_2 + v_1 v_2') + v_4'\} \; (5),$$

or
$$k\beta^6 v_1^2 + \beta^5 v_1(2kv_1 v_2' + 2w_2 - kv_1 w_2') + \dots = 0 \quad \ldots\ldots(6).$$

Now write down the equation corresponding to (18) of § 194 and subtract, and it will be found that we shall obtain an equation of the form

$$\beta^5 v_1^2 \Omega_1 + \beta^4 v_1 \Omega_3 + \beta^3 \Omega_5 + \dots \Omega_8 = 0 \quad \ldots\ldots\ldots\ldots(7),$$

from which it is easily shown* that (6) and (7) intersect in 54 ordinary generators and in 10 coincident generators along AB.

* Regarding (6) and (7) as plane curves, we have to find the number of coincident points in which they intersect at B; and we may replace them by two equations of the form

$$\beta^2\gamma^2 + \beta\gamma v_2 + v_4 = 0,$$
$$\beta^2\gamma^2 w_1 + \beta\gamma w_3 + w_5 = 0.$$

Eliminating $\beta\gamma$, we obtain a binary decimic of (γ, δ) which shows that the two curves intersect in 10 ordinary points, and therefore the number of points absorbed at B is $20 - 10 = 10$.

But since AB is four times repeated amongst the lines of closest contact of (4), the total number of ordinary lines of intersection of (6) and (7) is

$$64 - 10 - (20 - 4) = 38,$$

whence $$m = 38 = 80 - 2C - 3B,$$

giving $$2C + 3B = 42 = 36 + 6.$$

From this it follows that the coincidence of the two generators AB, AB' produces a further reduction of class equal to 2; and by taking a third generator AB'', which is a nodal generator on the cone and is also a line of closest contact, it can be shown that an additional reduction of class $2 + 2 = 4$ is produced. Generalizing it follows that when there are δ coincident nodal generators, all of which before coincidence are lines of closest contact,

$$2C + 3B = p(p-1)^2 + 4\delta - 2,$$
$$C + B = \tfrac{1}{2}p(p-1)^2 + \delta,$$

which proves the theorem.

Cubic Nodes.

204. There are six primary species of cubic nodes.

I. In the first species the nodal cone is an irreducible cubic cone. Of these there are three subsidiary species, which occur when the cone is (i) anautotomic, $C = 6$, $B = 0$; (ii) nodal, $C = 5$, $B = 1$; (iii) cuspidal, $C = 4$, $B = 2$.

II. In this species the cone consists of a quadric cone and a plane; and there are two subsidiary species according as the plane (i) intersects the cone in two distinct generators, $C = 4$, $B = 2$; or (ii) touches the cone, $C = 3$, $B = 3$. In the latter case the cone is a reducible or improper cubic cone having a *tacnodal* generator, and the values of the constituents follow from § 201.

III. Three planes intersecting in a point, $C = 3$, $B = 3$.

IV. Three planes intersecting in the same straight line, $C = 2$, $B = 4$.

V. One distinct and two coincident planes, $C = 4$, $B = 4$.

VI. Three coincident planes, $C = 6$, $B = 4$.

All these results follow from the preceding theorems. With regard to V, it follows from § 199 that $p = 3$, $s = 2$, so that the second formulae are the proper ones; but in the case of VI, $p=s=3$ and the first formulae must be used.

205. It will be noticed that I (iii) and II (i) have the same point constituents, and a similar remark applies to II (ii) and III. Exactly the same thing occurs in the theory of plane curves, for the point constituents of a triple point of the second kind and of a tacnode cusp are both equal to $\delta=2$, $\kappa=1$; but the line constituents are different, for in the former case they are $\tau=0$, $\iota=0$, and in the latter $\tau=2$, $\iota=1$. And since surfaces possess *plane* as well as *point* singularities, it is practically certain that the *plane* constituents in the above respective cases are different; although the theory has not yet been worked out.

Quartic Nodes.

206. The theory of quartic nodes is coextensive with that of quartic curves, since a plane section of the nodal cone may be any quartic curve proper or improper. The theorems of § 201 give the reduction of class when the nodal cone has a tacnodal, a rhamphoid cuspidal, an oscnodal and a tacnode cuspidal generator which is not a line of closest contact; the theorem of § 202 solves it when the generator is a line of closest contact; whilst that of § 203 solves it when the cone u_{p+1} touches the nodal cone along a tacnodal generator, or osculates it along an oscnodal one. Triple generators are discussed in § 195, and the various cases in which the nodal cone degrades into four planes are dealt with in §§ 196—9. I shall therefore only discuss two additional cases for the purpose of illustrating the method employed.

207. *When the nodal cone consists of a quadric cone and two coincident planes, the point constituents of the quartic node are*

$$C=12, \quad B=8.$$

We shall employ the sextic surface

$$\alpha^2(\delta^2+\delta v_1+\beta\gamma)\delta^2+\alpha(\delta^4 V_1+4\delta^3 V_2+6\delta^2 V_3+4\delta V_4+V_5) \\ +\delta^4 W_2+\dots W_6=0 \quad \dots(1),$$

where the suffixed letters denote binary quantics of (β, γ). Write (1) in the binary form

$$(a, b, c, d, e \between \delta, 1)^4=0,$$

then the equation of the tangent cone from D is

$$I^3 - 27J^2 = 0,$$

where
$$I = ae - 4bd + 3c^2,$$
$$J = ace + 2bcd - ad^2 - b^2e - c^3,$$

and the values of a, b, c, d, e are

$$a = \alpha^2 + \alpha V_1 + W_2,$$
$$b = \tfrac{1}{4}\alpha^2 v_1 + \alpha V_2 + W_3,$$
$$c = \tfrac{1}{6}\alpha^2 \beta\gamma + \alpha V_3 + W_4,$$
$$d = \alpha V_4 + W_5,$$
$$e = \alpha V_5 + W_6.$$

Writing down the discriminantal equation of the tangent cone from D, it will be found that the highest power of α is α^{11}, and the term involving it is $27c^3e(3ac - 2b^2)$; and that its coefficient (rejecting constant factors) is $\beta^3\gamma^3 V_5(\beta\gamma - \frac{1}{4}v_1^2)$; also the coefficient of α^{10} does not involve β or γ as a factor. From this it follows that if μ be the class of the cone, the number of tangent planes which can be drawn through DA to the surface is $\mu - 22 = m - 26$, since $\mu = m - 4$.

The tangent cone *from* A has five nodal generators, which are the lines of intersection of the plane δ and the cone V_5; also AD is another nodal generator of the tangent cone; hence its class is $90 - 10 - 2 = 78$. The number of tangent planes which can be drawn to this cone, and therefore to the surface, through AD is accordingly $78 - 4 = 74$; and we thus obtain $m - 26 = 74$, giving $m = 100$.

Let x be the reduction of class produced by each of the lines AB and AD, then

$$m = 100 = 150 - 36 - 2x - 2 - 2 - 2,$$

giving
$$x = 4.$$

Since the surface possesses an isolated conic node at D, it follows that if this were absent we should have $m = 102$, whence

$$2C + 3B = 48.$$

Also from § 198 the union of the two planes produces two additional conic nodes, whence

$$C + B = 20,$$

from which we obtain $C = 12, \quad B = 8.$

208. *When the nodal cone at a quartic node consists of a quadric cone twice repeated, the constituents of the singularity are*

$$C = 15, \quad B = 8.$$

Consider the surface

$$\alpha u_2^2 + u_5 = 0,$$

then, proceeding according to the first method, the two cones whose common generators determine the class are the sextic cones

$$u_2 u_5' = u_2' u_5,$$
$$u_2 u_5'' = u_2'' u_5,$$

which possess 36 common generators; but, since 10 of these common generators are the 10 lines of closest contact, which are the common generators of the cones u_2 and u_5, the total number of ordinary generators is 26. Hence

$$m = 26 = 80 - 2C - 3B,$$

whence $$2C + 3B = 54.$$

Now, from § 198, it appears that a quadric cone twice repeated may be regarded as a quinquenodal quartic cone whose nodal generators lie on the quadric cone u_2, but are otherwise indeterminate; also these five nodal generators may be regarded as coinciding with five of the lines of closest contact. Hence

$$C + B = 18 + 5 = 23,$$

giving $$C = 15, \quad B = 8.$$

The foregoing result is capable of extension to multiple points whose nodal cones are of the form $u_p^s u_q$.

Singular Lines and Curves.*

209. A surface may possess any line or curve lying in it, such that an arbitrary plane section through any point P on the line or curve has a singular point at P of any species which an algebraic plane curve can possess. Moreover all singular lines and curves possess singular points, analogous to pinch points, at which the singularity changes its character. Thus a cuspidal line possesses certain points at which the cusp changes into a tacnode, and a triple line of the first kind points at which the triple point changes into one of the second kind and so on. We also saw in § 41 that

* Basset, *Quart. Jour.* vol. xxxix. p. 334.

nodal lines of the third kind possess cubic nodes but no pinch points; and in like manner it will be found that singular lines and curves possess multiple points lying in them of a higher order of singularity than that of the line or curve.

The tangent planes at any point on a multiple line are in general *torsal* tangent planes; it is however possible for any tangent plane to be fixed in space, and such lines usually possess distinct features of their own. There are consequently two species of cuspidal, tacnodal &c. lines, in the first of which the tangent plane is torsal and in the second it is fixed in space.

Cuspidal Lines.

210. The general equation of a surface having a cuspidal line of the first species is

$$(L\alpha\gamma + M\beta\delta)^2 (\alpha, \beta)^{n-4} + (P, Q, R, S \between \gamma, \delta)^3 = 0 \text{......(1)},$$

where P, Q, R, S are quaternary quantics of $(\alpha, \beta, \gamma, \delta)$ of degree $n-3$. This equation when written out at full length is

$$(L\alpha\gamma + M\beta\delta)^2 (p_0\alpha^{n-4} + p_1\alpha^{n-5}\beta + \dots p_{n-4}\beta^{n-4}) + \alpha^{n-3}v_3$$
$$+ \alpha^{n-4}(\beta V_3 + V_4) + \dots + \beta^{n-3}w_3 + \beta^{n-4}w_4 + \dots w_n = 0 \text{...(2)}.$$

Cuspidal lines possess two kinds of singular points which occur (i) when the cusp changes into a tacnode, (ii) when there are cubic nodes on the line. Let the plane $\alpha = \lambda\beta$ cut AB in B', then the equation of the section is

$$\beta^{n-2}(L\lambda\gamma + M\delta)^2 (p_0\lambda^{n-4} + p_1\lambda^{n-5} + \dots p_{n-4})$$
$$+ \beta^{n-3}(\lambda^{n-3}v_3 + \lambda^{n-4}V_3 + \dots w_3) + \dots = 0 \text{ ...(3)},$$

which, for brevity, we shall write in the form

$$A\beta^{n-2}\Omega_1^2 + B\beta^{n-3} + \dots = 0 \text{.................(4)}.$$

211. *The cuspidal line possesses n tacnodal points, and $n-4$ cubic nodes, at which there is a cuspidal cubic cone.*

The condition for a tacnodal point is that Ω_1 should be a factor of B, which requires that the eliminant of Ω_1 and B should vanish. This furnishes an equation of the nth degree in λ.

The points where the cuspidal line cuts the planes $(\alpha, \beta)^{n-4} = 0$ are cubic nodes on the line, and there are $n-4$ of them. If A be one of these points $p_0 = 0$, and the coefficient of α^{n-3} equated to zero gives

$$L^2 p_1 \beta\gamma^2 + v_3 = 0,$$

which shows that AB is a cuspidal generator of the cone. A cuspidal line on a quartic surface has 4 tacnodal points, but no cubic nodes; hence on such a surface the line appears in an incomplete form.

212. *A cuspidal line of the second species possesses $n-3$ tacnodal points and $n-2$ cubic nodes.*

If γ be the fixed tangent plane, the equation of the surface must be of the form

$$\gamma^2(\alpha, \beta)^{n-2} + (P, Q, R, S \between \gamma, \delta)^3 = 0 \quad \text{...........(5).}$$

Proceeding as in § 210, the first term of the equation corresponding to (3) must be of the form

$$\beta^{n-2}\gamma^2\{p_0\lambda^{n-2} + p_1\lambda^{n-3} + \dots p_{n-2}\},$$

and the condition for a cubic node is that this should vanish, which furnishes an equation of degree $n-2$ in λ. The condition for a tacnodal point is that the coefficient of δ^3 in the expression

$$\lambda^{n-3}v_3 + \lambda^{n-4}V_3 + \dots w_3$$

should vanish, which furnishes an equation of degree $n-3$ in λ. A quartic surface having a cuspidal line of this character possesses both species of singular points.

213. The discussion of other species of singular lines is very similar, and I shall therefore merely give the results, referring the reader to my paper on Singular Lines and Curves on Surfaces.

Tacnodal Lines. The equation of a surface having a tacnodal line of the first species is

$$(L\alpha\gamma + M\beta\delta)^2(\alpha, \beta)^{n-4} + 2(L\alpha\gamma + M\beta\delta)(F, G, \dots \between \alpha, \beta)^{n-4}$$
$$+ (P, Q, R, S, T \between \gamma, \delta)^4 = 0 \text{...(6)},$$

where $F, G, \dots$ are binary quantics of (γ, δ); and P, Q, R, S, T are quaternary quantics of all the coordinates. The singular points consist of (i) $2n-4$ points where the tacnode changes into a rhamphoid cusp; (ii) $n-4$ points which are cubic nodes, the nodal cone at which consists of a quadric cone and a plane touching the latter along the tacnodal line.

The equation of a tacnodal line of the second kind is

$$\gamma^2(\alpha, \beta)^{n-2} + 2\gamma(F, G \dots \between \alpha, \beta)^{n-3}$$
$$+ (P, Q, R, S, T \between \gamma, \delta)^4 = 0 \text{...(7)},$$

and it possesses $2n-6$ rhamphoid cuspidal points and $n-2$ cubic nodes.

Rhamphoid Cuspidal Lines. The equation of the surface when the line is of the first kind is

$$(L\alpha\gamma + M\beta\delta + p\gamma^2 + q\gamma\delta + r\delta^2)^2 (\alpha, \beta)^{n-4}$$
$$+ (L\alpha\gamma + M\beta\delta)(F, G, \ldots \between \alpha, \beta)^{n-5} + (P, Q, R \ldots \between \gamma, \delta)^5 = 0 \ldots (8),$$

where $F, G, \ldots$ are binary cubics of (γ, δ), and $P, Q, \ldots$ are quaternary quantics of all the coordinates. These lines possess $n-4$ cubic nodes, and n points where the rhamphoid cusp changes into an oscnode.

When the line is of the second kind, its equation is

$$(\alpha\gamma + p\gamma^2 + q\gamma\delta + r\delta^2)^2 (\alpha, \beta)^{n-4} + \gamma^2 (\alpha^{n-3} v_1 + \alpha^{n-4}\beta w_1 + \ldots)$$
$$+ \gamma (\alpha^{n-4} v_3 + \alpha^{n-5}\beta w_3 + \ldots)$$
$$+ \alpha^{n-5} w_5 + \alpha^{n-6}(\beta W_5 + W_6) + \ldots W_n = 0 \ldots (9),$$

and the line possesses $n-4$ cubic nodes and $n-3$ oscnodal points.

214. The highest singular line of the second order and first species which a quartic surface can possess is a *tacnodal* line; but when the line is of the second species, such a surface may possess a *rhamphoid cuspidal* and an *oscnodal line.* The equations of the surface in the two respective cases may be reduced to the forms

$$(\alpha\gamma + \delta^2)^2 + \gamma(\alpha\gamma v_1 + \beta\gamma w_1 + w_3) = 0 \ \ldots\ldots\ldots (10),$$

and

$$(\alpha\gamma + \delta^2)^2 + \gamma^3 (P\alpha + Q\beta + R\gamma + S\delta) = 0 \ \ldots\ldots\ldots (11).$$

The section of (11) by the plane $\alpha = \lambda\beta$ is

$$(\lambda\beta\gamma + \delta^2)^2 + \gamma^3 \{(P\lambda + Q)\beta + R\gamma + S\delta\} = 0,$$

and the condition that B' should be a tacnode cusp is that $\lambda = -Q/P$. An oscnodal line of the second kind on a quartic has therefore one tacnode cuspidal point on it. It is not possible for a quartic to have a tacnode cuspidal line, since the conditions are that $P\lambda + Q = 0$ for all values of λ, which require that $P = Q = 0$, in which case (11) becomes a cone.

Triple Lines.

215. There are ten *primary* species of triple lines.

I. Three distinct tangent planes; all of which are torsal.

II. Three distinct tangent planes; one fixed and two torsal.

III. Three distinct tangent planes; two fixed and one torsal.

IV. Three distinct tangent planes; all three fixed.

V. Two coincident fixed tangent planes; one distinct torsal plane.

VI. Two coincident fixed tangent planes; one distinct fixed plane.

VII. Three coincident fixed tangent planes.

VIII. Two coincident torsal tangent planes; one distinct torsal plane.

IX. Two coincident torsal tangent planes; one distinct fixed plane.

X. Three coincident torsal tangent planes.

Triple lines possess a variety of species of singular points, which we shall proceed to consider. Thus when the line is of the first kind, points exist at which a pair of tangent planes coincide, so that the section of the surface through the point has a triple point of the second species thereat; also in certain cases points exist which are quartic nodes on the triple line.

I. *A triple line of the first kind on a surface of the nth degree has* $4n-12$ *points at which two of the tangent planes coincide.*

The equation of the surface is of the form

$$(P, Q, R, S \between \gamma, \delta)^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where P, Q, R, S are quaternary quantics of degree $n-3$. Equation (12), when written out at full length, becomes

$$\alpha^{n-3} v_3 + \alpha^{n-4}(\beta w_3 + w_4) + \ldots \beta^{n-3} W_3 + \beta^{n-4} W_4 + \ldots W_n = 0 \ldots (13),$$

and the equation of the section by the plane $\alpha = \lambda\beta$ is

$$\beta^{n-3}(\lambda^{n-3} v_3 + \lambda^{n-4} w_3 + \ldots W_3)$$
$$+ \beta^{n-4}(\lambda^{n-4} w_4 + \lambda^{n-5} V_4 + \ldots W_4) + \ldots = 0 \ldots (14),$$

or

$$A\beta^{n-3} + B\beta^{n-4} + \ldots = 0 \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The points at which a pair of tangent planes coincide will be called pinch points, and the condition for their existence is that the discriminant of A should vanish; and since A is a binary cubic of (γ, δ) whose coefficients are polynomials of λ of degree $n-3$, the discriminant is of degree $4n-12$ in λ.

Every tangent plane touches the surface at $n-3$ *distinct points.*

The condition that γ should be a tangent plane at the point B' where the plane $\alpha = \lambda\beta$ cuts AB, is that the coefficient of δ^3 in A should vanish. This furnishes an equation of degree $n-3$ in λ, which shows that there are $n-3$ of such points.

The theory of coincident pinch points will be considered in § 216, but in the meantime I shall enunciate the theorems concerning them.

II. *When $2n-6$ pinch points coincide in pairs, one of the tangent planes is fixed in space, and the line becomes one of the second species.*

III. *When all the pinch points coincide in pairs, two tangent planes are fixed in space.*

IV. *When all the tangent planes are fixed in space, the triple line possesses $n-3$ quartic nodes.*

In this case

$$A = \gamma\delta v_1 (\lambda^{n-3} v_0 + \dots W_0) = \gamma\delta v_1 A_0,$$

and a quartic node will occur whenever A_0 vanishes. It will hereafter be shown that the pinch points coincide in quartettes at each quartic node.

The equation of the surface may be written in the form

$$\gamma\delta v_1 (\alpha, \beta)^{n-3} + (P, Q, R, S, T \between \gamma, \delta)^4 = 0,$$

and the points where quartic nodes occur are given by the equation $(\alpha, \beta)^{n-3} = 0$; hence if A is a quartic node, the nodal cone is of the form

$$\beta\gamma\delta v_1 + v_4 = 0,$$

which is the equation of a quartic cone having a triple generator of the first kind.

V. *When all the pinch points coincide in quartettes, two coincident tangent planes are fixed in space, and the third one is torsal; also the line possesses $n-4$ singular points, at which the triple point of the second kind changes to one possessing a pair of tacnodal branches and one distinct ordinary branch.*

The value of A in equation (15) is of the form

$$A = \gamma^2 B_1,$$

and the condition for a pinch point is that the coefficient of δ in B_1 should vanish, which shows that there are $n-3$ apparent pinch points.

Equation (15) now becomes

$$B_1\beta^{n-3}\gamma^2 + B\beta^{n-4} + \ldots = 0,$$

and if the coefficient of δ in B vanishes, γ will be a factor of B, and the point consists of a pair of tacnodal branches and one ordinary branch through it. Since B is of degree $n-4$ in λ, there are $n-4$ of such points, and, like pinch points, they affect the class of the surface.

VI. *When two coincident tangent planes are fixed in space, and the distinct plane is also fixed, the triple line possesses* $n-3$ *quartic nodes, and* $n-4$ *of the points considered in* V.

For the equation of the section by the plane $\alpha = \lambda\beta$ is

$$(\lambda^{n-3}v_0 + \ldots W_0)\,\beta^{n-3}\gamma^2\delta + B\beta^{n-4} + \ldots = 0.$$

VII. *When three coincident tangent planes are fixed in space, the line possesses* $n-3$ *quartic nodes, and* $n-4$ *points consisting of a pair of tacnodal branches and a coincident ordinary branch.*

The equation of the section is

$$A_0\beta^{n-3}\gamma^3 + B\beta^{n-4} + \ldots = 0.$$

The first kind of points occur when $A_0 = 0$, and the second when γ is a factor of B. The constituents of the latter point (on a plane curve) are three nodes and one cusp; and both kinds of points affect the class of the surface.

216. The theory of coincident pinch points is best investigated by the following method. Let Δ be the discriminant of (12), so that

$$\Delta = P^2S^2 - 6PQRS + 4PR^3 + 4Q^3S - 3Q^2R^2 \quad \ldots\ldots(16),$$

then $\Delta = 0$ is a surface of degree $4n-12$, and we shall first show that the pinch points occur where AB intersects the surface $\Delta = 0$. Let

$$P = P_0\alpha^{n-3} + P_1\alpha^{n-4} + \ldots P_{n-3},$$

with similar expressions for Q, R, S, where $P_{n-3} = (\beta,\ \gamma,\ \delta)^{n-3}$; then if A be a pinch point and $\gamma^2\delta = 0$ the equation of the tangent planes thereat, it follows from (12) that

$$P_0 = R_0 = S_0 = 0.$$

The term $4Q^3S$ in (16) contains the highest power of α which is the $(4n-13)$th, and shows that the surface $\Delta = 0$ passes through A.

In the next plane consider the line II in which $\gamma = 0$ is the fixed tangent plane. The values of P, Q, and R remain unaltered, but $S = \Sigma\gamma + T\delta$, where $\Sigma, T \equiv (\alpha, \beta, \gamma, \delta)^{n-4}$. Let

$$\Sigma = \sigma_0\alpha^{n-4} + \sigma_1\alpha^{n-5} + \dots \sigma_{n-4},$$

$$T = t_0\alpha^{n-4} + t_1\alpha^{n-5} + \dots t_{n-4},$$

where $\sigma_n, t_n \equiv (\beta, \gamma, \delta)^n$; then the two terms in (16) which contain the highest powers of α are

$$4Q_0^3(\gamma\sigma_0 + \delta t_0)\,\alpha^{4n-13} - 3Q_0^2R_1^2\alpha^{4n-14},$$

putting $\gamma = \delta = 0$, this reduces to $k\beta^2\alpha^{4n-14}$, where k is a constant, which shows that the line AB touches the surface $\Delta = 0$ at A, and therefore two pinch points coincide. If however we had supposed that the tangent planes at A were $\gamma\delta^2$, we should find that $\Delta = 0$ intersects but does not touch AB at A, so that A is an ordinary pinch point. Accordingly the discriminantal surface cuts AB in $2n-6$ points and touches it at $n-3$ points, which shows that there are $2n-6$ distinct pinch points and $2n-6$ which coincide in pairs.

In the same way any other case may be treated.

We shall now inquire what becomes of the pinch points in the case of line IV. Let M be a binary quantic of (α, β) of degree $n-3$; also let P, P', &c. be quaternary quantics of degree $n-4$. Then the equation of the surface may be written in the form

$$\gamma^3(P\gamma + Q\delta) + 3\gamma^2\delta(Mf + P'\gamma + Q'\delta)$$
$$+ 3\gamma\delta^2(Mg + R'\gamma + S'\delta) + \delta^3(R\gamma + S\delta) = 0,$$

where $\gamma\delta(f\gamma + g\delta) = 0$ are the three fixed tangent planes. Writing down the equation of the discriminantal surface, and then putting $\gamma = \delta = 0$, it will be found to reduce to $-3f^2g^2M^4 = 0$, which shows that the surface Δ has quadritactic contact with the line AB at the quartic nodes. This shows that the pinch points coincide in quartettes at the quartic nodes.

217. VIII. The remaining three species present many features in common with cuspidal lines of the first kind; and the equation of a surface having a line of the eighth species is

$$(L\alpha\gamma + M\beta\delta)^2(F, G, \dots \between \alpha, \beta)^{n-5}$$
$$+ (P, Q, R, S, T \between \gamma, \delta)^4 = 0 \quad \dots(17),$$

where $F, G, \dots$ are linear functions of (γ, δ), and $P, Q, \dots$ are quaternary quantics of $(\alpha, \beta, \gamma, \delta)$. Equation (17) when written out at full length is of the form

$$(L\alpha\gamma + M\beta\delta)^2 (F\alpha^{n-5} + G\alpha^{n-6}\beta + \dots K\beta^{n-5})$$
$$+ \alpha^{n-4}v_4 + \alpha^{n-5}(\beta V_4 + V_5) + \dots + \beta^{n-4}w_4 + \dots w_n = 0 \dots (18).$$

Equation (18) shows that no surface of a lower degree than a sextic can possess a line of this species; for if $n = 5$, (18) reduces to the form

$$(L\alpha\gamma + M\beta\delta)^2 v_1 + \alpha v_4 + \beta w_4 + w_5 = 0 \dots\dots\dots\dots (19),$$

in which the distinct tangent plane is fixed in space, and the line therefore belongs to species IX.

The section of (18) by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^2 (F\lambda^{n-5} + G\lambda^{n-6} + \dots) \beta^{n-3}$$
$$+ (\lambda^{n-4}v_4 + \lambda^{n-5}V_4 + \dots) \beta^{n-4} + \dots = 0,$$

which we shall write

$$Av_1^2\beta^{n-3} + B\beta^{n-4} + \dots = 0 \quad \dots\dots\dots\dots\dots (20).$$

(*a*) *The first kind of singular point occurs when all the tangent planes coincide, and there are $n - 4$ of them.*

The condition for these points is that $A = kv_1$, where k is a constant, which furnishes an equation of degree $n - 4$ in λ.

(*b*) *There are n points at which the triple point of the second kind changes into one consisting of one pair of tacnodal branches and an ordinary branch passing through it.*

The condition for these points is that v_1 should be a factor of B, which furnishes an equation of degree n in λ.

IX. When the distinct tangent plane is fixed in space, the equation of the surface is

$$(L\alpha\gamma + M\beta\delta)^2 (p\gamma + q\delta)(\alpha, \beta)^{n-5}$$
$$+ (P, Q, R, S, T \between \gamma, \delta)^4 = 0 \dots (21),$$

and the section by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^2 (p\gamma + q\delta)(F\lambda^{n-5} + \dots) \beta^{n-3}$$
$$+ (\lambda^{n-4}v_4 + \lambda^{n-5}V_4 + \dots) \beta^{n-4} + \dots = 0,$$

or

$$Av_1^2 w_1 \beta^{n-3} + B\beta^{n-4} + \dots = 0 \dots\dots\dots\dots\dots (22).$$

(*a*) *There are $n - 5$ quartic nodes which are the intersections of the triple line and the planes $(\alpha, \beta)^{n-5} = 0$.*

A quintic surface cannot possess these quartic nodes, and therefore the singularity occurs on such a surface in an incomplete form. When $n > 5$ the equation of the nodal cone is of the form

$$\gamma^2 (p\gamma + q\delta) \beta + v_4 = 0.$$

(*b*) *There is one pinch point, which occurs when* $Lq\lambda = Mp$.

(*c*) *There are n points where the triple point of the second kind changes into one consisting of a pair of tacnodal branches and one distinct ordinary branch.*

X. The equation of the surface is

$$(L\alpha\gamma + M\beta\delta)^3 (\alpha, \beta)^{n-6} + (P, Q, R, S, T \between \gamma, \delta)^4 = 0 \ldots (23),$$

and the section by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^3 (F\lambda^{n-6} + \ldots) \beta^{n-3} + (\lambda^{n-4} v_4 + \ldots) \beta^{n-4} + \ldots = 0,$$

or

$$A\beta^{n-3} v_1^3 + B\beta^{n-4} + \ldots = 0.$$

(*a*) *There are* $n - 6$ *quartic nodes, and the equation of the nodal cone is of the form*

$$\gamma^3\beta + v_4 = 0.$$

A sextic is the surface of lowest degree which can possess this line, and since there are no quartic nodes the singularity occurs in an incomplete form.

(*b*) *There are n points at which the triple point of the third kind changes into one consisting of a pair of tacnodal branches and one coincident ordinary branch.*

Both these singular points affect the class of the surface.

Nodal Curves.

218. The equation of a surface of the *n*th degree which has a plane nodal curve of degree *s* is

$$\alpha^2 V_{n-2} + 2\alpha\Omega_s u_{n-s-1} + \Omega_s^2 u_{n-2s} = 0 \ldots\ldots\ldots\ldots (24),$$

where V is a quaternary quantic of $(\alpha, \beta, \gamma, \delta)$, and Ω, u are ternary quantics of (β, γ, δ) of the degrees indicated by the suffixes. We shall usually omit the suffix s in Ω.

219. *The nodal curve possesses* $2n(n - s - 1)$ *pinch points, which are the points of intersection of the curve and surface*

$$u^2_{n-s-1} = V_{n-2} u_{n-2s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (25).$$

Let B be one of the points of intersection of (25) and the nodal curve, and let u_{n-2} be the portion of V_{n-2} which is independent of α; then (25) may be replaced by

$$u^2_{n-s-1} = u_{n-2}u_{n-2s} \qquad \text{.....................(26).}$$

Since $\delta = 0$ is any arbitrary section of the surface through B, we have to show that the section has a cusp at B; also since (26) has to pass through B which is a point on Ω, it follows that when $\delta = 0$

$$\left.\begin{aligned} u_{n-2} &= p^2\beta^{n-2} + p_1\beta^{n-3}\gamma + \dots p_{n-2}\gamma^{n-2} \\ u_{n-s-1} &= pq\beta^{n-s-1} + q_1\beta^{n-s-2}\gamma + \dots \\ u_{n-2s} &= q^2\beta^{n-2s} + r_1\beta^{n-2s-1}\gamma + \dots \\ \Omega_s &= k_1\beta^{s-1}\gamma + k_2\beta^{s-2}\gamma^2 + \dots \end{aligned}\right\} \qquad \text{.........(27).}$$

From these results it follows that the highest power of β on the section of (29) by δ is the $(n-2)$th and that its coefficient is $(p\alpha + qk_1\gamma)^2$, which shows that B is a pinch point.

220. The plane α intersects the surface in the nodal curve twice repeated, and in the curve $\alpha = 0$, $u_{n-2s} = 0$, which is called the *residual curve*; and the latter curve intersects the nodal curve in $s(n-2s)$ points. When the curve is of a higher order of singularity, these points are as a general rule multiple points on the singular curve, but when the latter is nodal the plane α is a tangent plane at these points; in other words, α is one of the two nodal tangent planes. To prove this, let B be one of the points in question; then

$$\begin{aligned} V_{n-2} &= \beta^{n-2}u_0' + \beta^{n-3}u_1' + \dots, \\ u_{n-s-1} &= \beta^{n-s-1}v_0 + \dots, \\ u_{n-2s} &= \beta^{n-2s-1}w_1 + \dots, \\ \Omega &= \beta^{s-1}\Omega_1 + \dots, \end{aligned}$$

where $u_n' = (\alpha, \gamma, \delta)^n$, and $v_n \equiv w_n \equiv \Omega_n \equiv (\gamma, \delta)^n$, from which it follows that the coefficient of β^{n-2} in (24) is $\alpha(\alpha u_0' + 2v_0\Omega_1)$.

Cuspidal Curves.

221. When the singular curve is cuspidal, every point on it must be a pinch point, which requires that equation (26) should contain Ω as a factor. Accordingly

$$u^2_{n-s-1} - u_{n-2}u_{n-2s} = \Omega\Omega' \qquad \text{.................(28),}$$

where $\Omega' = (\beta, \gamma, \delta)^{2n-3s-2}$. The right-hand side of (28) vanishes at every point on the cuspidal curve, hence the left-hand side of (28) must do so also. Now u_{n-2s} vanishes at the points where the residual curve intersects the cuspidal curve, hence u_{n-s-1} must also vanish at these points; accordingly

$$u_{n-s-1} = u_{n-2s}\sigma_{s-1} + \Omega w_{n-2s-1} \quad\text{..............(29)},$$

where σ and w are undetermined ternary quantics of (β, γ, δ). Substituting from (29) in (28), we obtain

$$u^2_{n-2s}\sigma^2_{s-1} + 2\Omega u_{n-2s}\sigma_{s-1}w_{n-2s-1} + \Omega^2 w^2_{n-2s-1} - u_{n-2}u_{n-2s} = \Omega\Omega',$$

and since Ω has to be a factor of the left-hand side, we must have

$$u_{n-2} = u_{n-2s}\sigma^2_{s-1} + \Omega\phi_{n-s-2} \quad\text{..............(30)},$$

where ϕ is another ternary quantic of (β, γ, δ).

Substituting from (29) and (30) in (24) and writing

$$S = \alpha\sigma_{s-1} + \Omega \quad\text{......................(31)},$$

we obtain

$$S^2(u_{n-2s} + 2\alpha w_{n-2s-1}) + S\alpha^2(\phi_{n-s-2} - 4\sigma_{s-1}w_{n-2s-1}) + \alpha^3 U'_{n-3} = 0,$$

which is the equation of a surface having a plane cuspidal curve of degree s; but it can be easily verified that such a surface may be represented by the more general equation

$$S^2 U_{n-2s} + 2\alpha^2 S V_{n-s-2} + \alpha^3 W_{n-3} = 0 \quad\text{...........(32)},$$

where U, V, W are quaternary quantics of the degrees indicated by the suffixes.

Some writers seem to have thought it possible that a surface possessing a cuspidal curve, plane or twisted, might be represented by an equation of the form $S^2U + \Sigma^3V = 0$, where S and Σ are two surfaces whose intersections determine the cuspidal curve; but although the above equation undoubtedly does represent such a surface, equation (32) indicates that it is deficient in generality, since it does not appear possible to transform (32) so as to get rid of the second term. Omitting suffixes, (32) may be written in the form

$$(SU + \alpha^2 V)^2 + \alpha^3(UW - V^2\alpha) = 0 \quad\text{...........(33)},$$

which shows that the surface $UW = V^2\alpha$ touches (32) along its curve of intersection with the surface $SU + \alpha^2 V = 0$.

222. *The surface possesses $s(n-3)$ tacnodal points, which are the intersections of the surface $W_{n-3} = 0$ and the cuspidal curve.*

Let B be one of the points of intersection; then

$$S = \beta^{s-1}(\Omega_1 + \alpha\sigma_0) + \ldots \quad \ldots\ldots\ldots\ldots(34),$$

$$W_{n-3} = \beta^{n-4}w_1 + \ldots,$$

and the coefficient of β^{n-2} in (32) is $(\Omega_1 + \alpha\sigma_0)^2$, and that of β^{n-3} contains $\Omega_1 + \alpha\sigma_0$ as a factor, which shows that B is a tacnodal point.

223. *The surface possesses $s(n-2s)$ cubic nodes, which are the points of intersection of the cuspidal and residual curves; also the nodal cone possesses a cuspidal generator, which is the tangent to the cuspidal curve at the point.*

Let B be one of the points of intersection of the two curves; then S is given by (34), whilst the value of U_{n-2s}, whose intersection with the plane α is the residual curve, is

$$U_{n-2s} = \beta^{n-2s-1}u_1 + \ldots,$$

where $u_n = (\alpha, \gamma, \delta)^n$. The highest power of β in (32) is the $(n-3)$th, and its coefficient equated to zero is the cuspidal cubic cone

$$(\alpha\sigma_0 + \Omega_1)^2 u_1 + 2\alpha^2(\alpha\sigma_0 + \Omega_1)v_0 + \alpha^3 w_0 = 0,$$

which is the cone in question.

224. In the paper from which these investigations have been taken, I have discussed the cases of a tacnodal and a rhamphoid cuspidal curve. The results are as follows:

(i) *A tacnodal curve possesses $2s(n-s-2)$ points where the tacnode changes into a rhamphoid cusp.*

(ii) *The $s(n-2s)$ points where the tacnodal curve intersects the residual curve are cubic nodes, whose nodal cone consists of a quadric cone and a plane.*

(iii) *A rhamphoid cuspidal curve possesses $s(n-5)$ oscnodal points.*

(iv) *The $s(n-2s)$ points where the residual curve intersects the rhamphoid cuspidal curve are cubic nodes of the fifth species.*

Nodal Twisted Cubic Curves.

225. The theory of nodal curves, which are the partial intersections of two surfaces, has been discussed by Cayley*; but he has not given the number of pinch points nor considered curves of a higher singularity. Salmon has also, by a very obscure method, arrived at the conclusion that when the surface is a quartic, the number of pinch points is 4. Let u, u'; v, v'; w and w' be quaternary quantics of degrees l, m, and n respectively; then we have shown in § 46 that the system of determinants

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(35)$$

represents three surfaces having a common curve of intersection of degree $mn + nl + lm$. Hence if

$$\lambda = vw' - wv', \quad \mu = wu' - uw', \quad \nu = uv' - vu' \ldots\ldots(36),$$

the equation

$$(U, V, W, U', V', W' \between \lambda, \mu, \nu)^2 = 0 \quad \ldots\ldots\ldots(37),$$

where U, V, ... are quaternary quantics of proper degrees, represents a surface having a nodal twisted curve, which is the common curve of intersection of (35).

226. To apply this to a cubic curve, all the quantities u, u', &c. must be planes, and U, V, &c. must be quaternary quantics of degrees $n-4$. Also if A be a point on the cubic, the quadrics λ, μ, ν must pass through A, the condition for which may be satisfied by taking

$$u = \alpha u_0 + u_1, \; u' = k\alpha u_0 + u_1' \quad \ldots\ldots\ldots\ldots\ldots(38),$$

with similar expressions for v, v'; w, w'. Hence

$$\lambda = \alpha \{(kv_1 - v_1')\, w_0 - (kw_1 - w_1')\, v_0\} + v_1 w_1' - w_1 v_1'$$
$$= \alpha\lambda_1 + \lambda_2 \quad \text{(say)},$$

accordingly $$\lambda_1 u_0 + \mu_1 v_0 + \nu_1 w_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(39).$$

Let $$U = U_0 \alpha^{n-4} + U_1 \alpha^{n-5} + \ldots$$

with similar expressions for V, W, &c.; then if these values be substituted in (37), the highest power of α will be α^{n-2}, and its coefficient equated to zero gives

$$(U_0, V_0, W_0, U_0', V_0', W_0' \between \lambda_1, \mu_1, \nu_1)^2 = 0 \quad \ldots\ldots(40),$$

* "On a Singularity of Surfaces," *C. M. P.* vol. VI. p. 123; and *Quart. Jour.* vol. IX. p. 332.

which by virtue of (39) is resolvable into two linear factors. Hence (40) combined with (39) give the nodal tangent planes at A.

To find the number of pinch points, consider the determinantal surface

$$\Delta = \begin{vmatrix} U, & W', & V', & u \\ W', & V, & U', & v \\ V', & U', & W, & w \\ u, & v, & w, & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (41),$$

which is of degree $2n-6$, and the first term of which is $\Delta_0 \alpha^{2n-6}$, where Δ_0 is the value of Δ, when U_0, V_0, ... are substituted for U, V, ...

If A be a pinch point (40) must become a perfect square, the condition for which is the same as the condition that the line (39) should touch the conic (40), where λ_1, μ_1, ν_1 are regarded as ordinary trilinear coordinates of a point; that is to say $\Delta_0 = 0$. Hence the determinantal surface passes through all the pinch points, and there are apparently $6(n-3)$ of them. But the point of intersection of the three planes u, v, and w is a conic node on Δ, and this point can be shown to be an ordinary point on the cubic curve. For if the tetrahedron of reference be changed by writing (β, γ, δ) for (u, v, w), the coefficient of α^{n-2} in (37) becomes

$$(U_0, V_0, \ldots \between \gamma w_0' - \delta v_0', \delta u_0' - \beta w_0', \beta v_0' - \gamma u_0')^2,$$

which is not a perfect square. Hence the cubic curve intersects the surface Δ at a node and at $6n-20$ ordinary points, which is the number of pinch points on the nodal curve. When the surface is a quartic, $6n-20=4$, which agrees with Salmon's result.

227. These results are capable of generalization. For if the degrees of the coefficients in (37) are as follows:

$$\begin{aligned} U &= s - 2m - 2n; & U' &= s - 2l - m - n; \\ V &= s - 2n - 2l; & V' &= s - l - 2m - n; \\ W &= s - 2l - 2m; & W' &= s - l - m - 2n; \end{aligned}$$

equation (37) will represent a surface of degree s having a nodal twisted curve of degree $mn + nl + lm$. In the place of (38), we must write

$$\begin{aligned} u &= \alpha^l u_0 + \alpha^{l-1} u_1 + \ldots, \\ u' &= k u_0 \alpha^l + \alpha^{l-1} u_1' + \ldots, \end{aligned}$$

&c., and equation (39) will still hold good. Also, if

$$A = VW - U'^2, \quad A' = V'W' - UU', \text{ \&c.,}$$

the determinantal equation (41) becomes

$$(A, B, C, A', B', C' \between u, v, w)^2 = 0 \text{ ....(42),}$$

which is of degree $2(s - l - m - n)$; and the degrees of A, A' are respectively equal to $2(s - 2l - m - n)$ and $2s - 2l - 3m - 3n$, so that there are apparently $2(mn + nl + lm)(s - l - m - n)$ pinch points. But the three points of intersection of the surfaces u, v, and w are nodes on (42) and are ordinary points on the nodal curve; hence the total number of pinch points is

$$2\{s(mn + nl + lm) - l^2(m + n) - m^2(n + l) - n^2(l + m) - 4lmn\}.$$

These results hold good whenever it is possible to represent a surface having a nodal twisted curve by means of an equation such as (37).

*Birational Transformation**.

228. In Chapter IV we employed the theory of birational transformation to investigate the constituents of the point, line, and mixed singularities of plane curves; and we shall now show that the same transformation may be used to analyse the point constituents of various kinds of multiple points on surfaces. One result of the investigation is to show that there is an important analogy between the theories of curves and surfaces, and that a fairly complete theory exists with respect to the changes produced in the constituents of a multiple point on a surface, when the tangent cone possesses singular generators.

229. Let P be any point whose coordinates are $(\xi, \eta, \zeta, \omega)$; let $\Omega = (\beta, \gamma, \delta)^2$; and let AP cut the polar plane of P with respect to the quadric $\alpha^2 = \Omega$ in P'. Then the equation of the polar plane of P is

$$2\alpha\xi - \beta d\Omega_1/d\eta - \gamma d\Omega_1/d\zeta - \delta d\Omega_1/d\omega = 0 \text{ ....(1),}$$

where $\Omega_1 = (\eta, \zeta, \omega)^2$; and those of AP are

$$\beta/\eta = \gamma/\zeta = \delta/\omega = h \text{ (say)..................(2);}$$

whence, if $(\xi', \eta', \zeta', \omega')$ be the coordinates of P', we obtain from (1) and (2)

$$2\xi\xi' = h(\eta d\Omega_1/d\eta + \zeta d\Omega_1/d\zeta + \omega d\Omega_1/d\omega) = 2h\Omega_1 \text{......(3),}$$

* Basset, *Quart. Jour.* vol. XXXIX. p. 250.

accordingly, by (2) and (3),

$$\frac{\xi\xi'}{\Omega_1} = \frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = \frac{\omega'}{\omega}.$$

From this it follows that if $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ be the coordinates of a pair of corresponding points P and P', these quantities are connected by the equations

$$\frac{\alpha'}{\Omega} = \frac{\beta'}{\alpha\beta} = \frac{\gamma'}{\alpha\gamma} = \frac{\delta'}{\alpha\delta} = k \text{ (say)} \quad \text{..............(4).}$$

Substituting the values of β', γ', δ' from the last three of (4), we obtain

$$\Omega' = k^2\alpha^2\Omega = \alpha^2\alpha'^2/\Omega,$$

whence

$$\frac{\alpha}{\Omega'} = \frac{\beta}{\alpha'\beta'} = \frac{\gamma}{\alpha'\gamma'} = \frac{\delta}{\alpha'\delta'} \quad \text{....................(5).}$$

The value of Ω will usually be written in the form

$$\Omega = \beta^2 + \beta\Omega_1 + \Omega_2 \quad \text{....................(6),}$$

where $\Omega_n = (\gamma, \delta)^n$.

230. *A conic node, whose position is arbitrary, gives rise to a conic node on the transformed surface.*

To prove this it will be sufficient to consider the quartic surface

$$\alpha^2(\beta^2 + \beta v_1 + v_2) + \alpha(2\lambda\beta^3 + \beta^2 w_1 + \beta w_2 + w_3)$$
$$+ \lambda^2\beta^4 + (\lambda w_1 - \lambda^2 v_1)\beta^3 + \beta^2 V_2 + \beta V_3 + V_4 = 0,$$

which has a conic node at A and also at the point B', where AB is cut by the plane $\alpha + \lambda\beta = 0$. The transformed equation may be put into the form

$$(\Omega + \lambda\alpha\beta)^2\beta^2 + (\Omega^2 - \lambda^2\alpha^2\beta^2)\beta v_1 + (\Omega + \lambda\alpha\beta)\alpha\beta^2 w_1$$
$$+ \alpha^2(\beta^2 V_2 + \beta V_3 + V_4) + \alpha\Omega(\beta w_2 + w_3) + \Omega^2 v_2 = 0.$$

This is the equation of a sextic surface which has a quadruple point at A, and a conic node at the point B', where AB is cut by the plane $\lambda\alpha + \beta = 0$. The quadric $\Omega + \lambda\alpha\beta = 0$, into which the plane $\alpha + \lambda\beta = 0$ is transformed, also passes through B'.

If the conic node on the original surface is situated at B, $\lambda = 0$; whence the conic node on the transformed surface is situated at A. Hence: *Every conic node situated at a point B in the plane α becomes a conic node at A on the transformed surface, which moves up to coincidence with A along the line AB.*

231. We are now in a position to apply the theory of birational transformation to multiple points on surfaces.

(i) *A multiple point of order p on a surface corresponds to a curve of degree p on the transformed surface, which is identical with the section of the tangent cone by the plane α.*

The equation of the original surface is of the form

$$\alpha^{n-p}u_p + \alpha^{n-p-1}u_{p+1} + \alpha^{n-p-2}u_{p+2} + \dots u_n = 0 \quad\dots\dots\dots(7),$$

and that of the transformed surface is

$$\alpha^{n-p}u_n + \alpha^{n-p-1}\Omega u_{n-1} + \dots \alpha^2\Omega^{n-p-2}u_{p+2} + \alpha\Omega^{n-p-1}u_{p+1} + \Omega^{n-p}u_p = 0 \quad\dots\dots(8).$$

The latter surface is therefore of degree $2n-p$, and has a multiple point of order n at A; also the section of (8) by the plane α is a multiple conic of order $n-p$ and a curve of degree p.

(ii) *Every nodal generator of the tangent cone at A, which is not a line of closest contact, corresponds to a node on the section of the transformed surface by the plane α.*

Let

$$\left.\begin{aligned} u_p &= \beta^{p-2}v_2 + \beta^{p-3}v_3 + \dots v_p \\ u_{p+1} &= \beta^{p+1}w_0 + \beta^{p}w_1 + \dots w_{p+1} \\ u_{p+2} &= \beta^{p+2}W^2 + \beta^{p+1}W_1 + \dots w_{p+2} \end{aligned}\right\} \quad\dots\dots\dots\dots(9),$$

then AB is a nodal generator of the cone u_p. Also the highest power of β in (8) is the $(2n-p-1)$th, and its coefficient is αw_0, which shows that the plane α touches the transformed surface at B; hence B is a node on the section.

It follows from § 194 that every nodal generator of the tangent cone at A converts one of the constituent conic nodes into a binode; and the preceding result shows that, if the plane α touches a surface at δ points, the transformed surface has a multiple point, the tangent cone at which has δ nodal generators.

(iii) *Every cuspidal generator of the tangent cone at A, which is not a line of closest contact, corresponds to a cusp on the section of the transformed surface by the plane α.*

This may be proved in a similar manner by putting $v_2 = v_1^2$. In the same way it can be shown that if the tangent cone at A has a singular generator of any species whatever, the section of the transformed surface by the plane α possesses a singular point of the same character.

We shall now suppose that the singular generator of the tangent cone at A is a line of closest contact, in which case $w_0 = 0$.

(iv) *Every nodal generator of the tangent cone at A, which is a line of closest contact, corresponds to a conic node on the section of the transformed surface by the plane α; and such a generator adds an additional conic node to the constituents of the multiple point.*

When $w_0 = 0$, the highest power of β is the $(2n - p - 2)$th, and its coefficient is

$$\alpha^2 W^2 + \alpha w_1 + v_2 \qquad\text{.......................(10)},$$

which shows that B is a conic node on the transformed surface. Also we have shown in § 230 that this conic node becomes a conic node on the original surface, which moves up to coincidence with A along the line AB. This furnishes another proof of the theorem § 24.

(v) *Every cuspidal generator of the tangent cone at A, which is a line of closest contact, corresponds to a conic node on the section of the transformed surface such that the tangent cone thereat touches the plane α; and such a generator adds an additional binode to the constituents of the multiple point.*

In this case $v_2 = v_1^2$, and the expression (10) equated to zero gives the tangent cone at the point B on the transformed surface. It follows from § 230, or from § 24, that an additional double point is added to the multiple point at A on the original surface, and § 202 shows that this double point is a binode.

The results (iii) and (v) fail when $p = 2$; for in this case the cone u_p becomes a pair of coincident planes and the singularity at A on the original curve is a unode.

(vi) *Every binode on the section of the transformed surface by the plane α adds a binode to the constituents of the multiple point on the original surface.*

Let $w_1 = W(v_1 + \tau_1)$, $v_2 = v_1\tau_1$, then the expression (10) becomes

$$(\alpha W + v_1)(\alpha W + \tau_1),$$

which shows that the singularity at B on the transformed surface is a binode whose axis is arbitrary. The original surface may now be written in the form

$$\alpha^{n-p-2}\beta^{p-2}(\alpha v_1 + W\beta^2)(\alpha\tau_1 + W\beta^2) + \alpha^{n-p}(\beta^{p-3}v_3 + \dots v_p)$$
$$+ \alpha^{n-p-1}(\beta^{p-1}w_2 + \dots w_{p+1}) + \alpha^{n-p-2}(\beta^{p+1}W_1 + \dots W_{p+2})$$
$$+ \alpha^{n-p-3}u_{p+3} + \dots u_n = 0 \quad \text{.......(11).}$$

By means of the methods explained in Chapter IV, it can be shown that the section of (11) by the plane $v_1 = \tau_1$ is a curve having a rhamphoid cusp at A and $p-2$ ordinary branches through it. To find the reduction of class, it will be sufficient to consider a quartic surface, since the method employed is applicable to any surface. Putting $n = 4$, $p = 2$, $v_1 = \delta$, $\tau_1 = p\gamma + q\delta$, the surface (11) becomes

$$(\alpha\delta + W\beta^2)\{\alpha(p\gamma + q\delta) + W\beta^2\} + \alpha(\beta w_2 + w_3)$$
$$+ \beta^3 W_1 + \beta^2 W_2 + \beta W_3 + W_4 = 0 \text{...(12),}$$

and the first polar of D, which may be any arbitrary point, is

$$\alpha\{\alpha(p\gamma + q\delta) + W\beta^2\} + q\alpha(\alpha\delta + W\beta^2) + \alpha(\beta w_2'' + w_3'')$$
$$+ \beta^3 W_1'' + \dots W_4'' = 0 \text{...(13).}$$

The sections of (12) and (13) by the plane $\delta = p\gamma + q\delta$ may be written in the form

$$(\alpha\delta + W\beta^2)^2 + \alpha\delta^2 A_1 + \delta A_3 = 0 \quad \text{.......(14),}$$

and

$$\alpha(\alpha\delta + W\beta^2) + \alpha\delta B_1 + B_3 = 0 \quad \text{.......(15),}$$

where $A_n \equiv B_n \equiv (\beta, \delta)^2$.

Eliminating α between (14) and (15), it will be found that the eliminant is a binary septimic of (β, δ), which shows that (14) and (15) have quinquetactic contact with one another at A. Accordingly the surface (12) and its first polar with respect to any arbitrary point intersect one another in five coincident points at A; hence the reduction of class is 5, which shows that the singularity at A is formed by the union of a conic and a binode.

The case of a cubic surface is peculiar. Let us consider the equation

$$\alpha u_2 + u_3 = 0 \quad \text{.......(16),}$$

which represents a cubic surface having a conic node at A. The transformed surface, when written at full length, becomes

$$\alpha(\beta^3 w_0 + \beta^2 w_1 + \beta w_2 + w_3) + \Omega(\beta^2 v_0 + \beta v_1 + v_2) = 0 \text{ ...(17).}$$

In order that (17) should have a conic node at B, it is necessary that $w_0 = v_0 = v_1 = 0$, in which case the equation of the nodal cone is

$$\alpha w_1 + v_2 = 0,$$

and, if B is a binode, $v_2 = w_1\tau_1$ and (16) becomes

$$\alpha w_1\tau_1 + \beta^2 w_1 + \beta w_2 + w_3 = 0 \quad\quad (18),$$

and the singularity at A is Salmon's binode B_5, which, as pointed out in § 80, is a singularity of a different character to the singular point $C = 1$, $B = 1$ on a surface of higher degree than the third.

232. The preceding results enable us to develop an important analogy between the theory of the birational transformation of curves and surfaces.

Let ABC be the triangle of reference of the curve, and $ABCD$ the tetrahedron of reference of the surface; then the following correspondence exists between the different elements of a curve and a surface, which is shown in the table on page 170.

233. The first theorem has already been proved, and the others may be established as follows. For brevity we shall write $\lambda = \frac{1}{2}p(p-1)$, $\mu = \frac{1}{2}p(p-1)^2$.

(ii) The first two portions are proved in § 165, where it is shown that when $2r$ tangents coincide in pairs, each pair being distinct, the constituents of the multiple points are

$$\delta = \lambda - r, \quad \kappa = r.$$

In § 194 it is shown that when the tangent cone has δ distinct nodal generators, the constituents of the multiple point on the surface are

$$C = \mu - \delta, \quad B = \delta.$$

(iii) From § 165 and § 194 it follows that if a multiple point on a curve has r tangents, each of which consists of three coincident tangents, the constituents of the multiple point are

$$\delta = \lambda - 2r, \quad \kappa = 2r,$$

whilst, if the tangent cone at the multiple point on the surface has κ cuspidal generators, its constituents are

$$C = \mu - 2\kappa, \quad B = 2\kappa.$$

(iv) The first two portions follow from § 165; and the constituents of the multiple point on the curve are

$$\delta = \lambda - 2r + 1, \quad \kappa = 2r - 1,$$

whilst the latter part follows from § 201, which shows that the constituents of the multiple point on the surface are

$$C = \mu - 2\delta + 1, \quad B = 2\delta - 1.$$

Original Curve.	Transformed Curve.	Original Surface.	Transformed Surface.
(i) A multiple point of order p at A. Tangents distinct.	p distinct points on BC.	A multiple point of order p at A. Tangent cone anautotomic.	An anautotomic curve of degree p in the plane BCD.
(ii) Do. having a pair of coincident tangents.	2 coincident and $p-2$ distinct points on BC.	Do. A nodal generator on the tangent cone.	A node on the section by the plane BCD.
(iii) Do. having 3 coincident tangents.	3 coincident and $p-3$ distinct points on BC.	Do. A cuspidal generator on the tangent cone.	A cusp on the section by the plane BCD.
(iv) Do. having $2r$ coincident tangents, $r > 0$.	$2r$ coincident and $p-2r$ distinct points on BC.	Do. δ nodal generators on the tangent cone, which move up to coincidence along a continuous curve.	δ nodes on the section, which move up to coincidence along a continuous curve.
(v) Do. having a pair of tacnodal and $p-2$ ordinary branches.	A node and $p-2$ distinct points on BC.	Do. a nodal generator on the tangent cone, which is a line of closest contact.	A conic node on the section by the plane BCD.
(vi) Do. having one pair of tacnodal and $2r$ ordinary branches which coincide with the tacnodal branches, and $p-2r-2$ ordinary branches.	A node and $2r$ ordinary points, which coincide with the node; and $p-2r-2$ distinct points on BC.	Do. A nodal generator on the tangent cone, which is a line of closest contact; and δ ordinary nodal generators which move up to coincidence along a continuous curve.	A conic node and δ ordinary nodes on the section by the plane BCD, which move up to coincidence along a continuous curve.
(vii) Do. having s pairs of tacnodal branches, r of which are coincident, and $p-2s$ ordinary branches.	s nodes on BC, r of which are coincident; and $p-2s$ distinct points.	Do. δ nodal generators on the tangent cone, all of which are lines of closest contact, and of which r are coincident.	δ conic nodes on the section by the plane BCD, r of which are coincident.
(viii) Do. consisting of a pair of tacnodal branches and one coincident ordinary branch, and $p-3$ distinct ordinary branches.	A node, with BC as one of the nodal tangents; and $p-3$ distinct points on BC.	A cuspidal generator on the tangent cone, which is a line of closest contact.	A conic node on the section, whose tangent cone touches the plane BCD.
(ix) Do. consisting of a rhamphoid cusp and $p-2$ distinct ordinary branches passing through it.	A cusp and $p-2$ distinct points on BC.	A singular point formed by the union of an ordinary multiple point and a binode.	A binode on the section by the plane BCD.

(v) By § 172 (i) it follows that if a multiple point on a curve has s pairs of tacnodal branches and $p-2s$ ordinary branches, all of which are distinct, its constituents are

$$\delta=\lambda+s, \quad \kappa=0,$$

and it follows from the theorem, § 24, that if the tangent cone at a multiple point on a surface has δ nodal generators, all of which are lines of closest contact, the constituents are

$$C=\mu+\delta, \quad B=0.$$

(vi) Putting $2r$ for r in § 173, it follows that the constituents of the multiple point on the curve are

$$\delta=\lambda-2r+1, \quad \kappa=2r.$$

Let the tangent cone at the multiple point on the surface have $\delta+1$ distinct nodal generators, and let one of them AB be a line of closest contact. Then it is shown in § 201 that if the cone possesses $\delta+1$ nodal generators, which move up to coincidence along a continuous curve, the total reduction of class is $2\mu+2\delta+1$; whence $s=2\delta+1$. Also it follows from § 202 that if one of the generators before coincidence is a line of closest contact the constituents of the singularity are

$$C=\mu-2\delta+1, \quad B=2\delta,$$

since we have shown that $s=2\delta+1$.

(vii) The first two portions follow from § 174, which shows that the constituents of the multiple point on the curve are

$$\delta=\lambda+s-2r+2, \quad \kappa=2r-2.$$

To prove the third part, it follows from § 203 that if the tangent cone at the multiple point has r coincident nodal generators, all of which before coincidence are lines of closest contact,

$$2C+3B=2\mu+4r-2,$$
$$C+B=\mu+r.$$

But if the tangent cone possesses $\delta-r$ additional distinct nodal generators, all of which are lines of closest contact, their effect is to produce an additional reduction of class equal to $2\delta-2r$, and to add $\delta-r$ double points to the constituents of the multiple point; whence

$$2C+3B=2\mu+2\delta+2r-2,$$
$$C+B=\mu+\delta,$$

accordingly $\quad C=\mu+\delta-2r+2, \quad B=2r-2.$

(viii) If in § 173 we put $r=1$, we obtain the singularity in question, and its point constituents are

$$\delta=\lambda, \quad \kappa=1,$$

and it follows from § 202 that if the tangent cone at a multiple point possesses a cuspidal generator, which is a line of closest contact, the constituents are

$$C=\mu, \quad B=1.$$

(ix) The first two portions are proved in § 181; and the last two portions in § 231 (vi). The point constituents of the singularities are $\delta=\lambda$, $\kappa=1$ for a curve; and $C=\mu$, $B=1$ for a surface.

234. It may, at first sight, appear strange that two such different singularities as the ones discussed in (v) and (vi) of § 231 should have the same point constituents; but the theory of curves supplies the explanation. The singularity corresponding to (v) in plane geometry is a multiple point consisting of one pair of tacnodal branches, one coincident ordinary branch, and $p-3$ distinct ordinary branches; and its constituents are given by the equations $\delta=\lambda$, $\kappa=1$, $\tau=2$, $\iota=0$, whilst the one corresponding to (vi) is a multiple point consisting of a rhamphoid cusp and $p-2$ distinct ordinary branches passing through it, and its constituents are given by the equation $\delta=\lambda$, $\kappa=1$, $\tau=1$, $\iota=1$.

Both singularities are therefore *mixed* ones, whose point constituents are the same, but whose line constituents are different; and from analogy we should anticipate that the singularities (v) and (vi) of § 231 in solid geometry are *mixed* ones, whose constituents consist partly of point and partly of plane singularities; but that the plane constituents of (v) and (vi) are different.

CHAPTER VI

QUARTIC SURFACES

235. THE class of a quartic surface may be any number lying between 3 and 36. In the latter case the surface is anautotomic and its equation contains 34 independent constants; whilst in the former it is the reciprocal polar of a cubic surface. A quartic surface may also possess as many as 16 double points, which may be isolated or may coalesce so as to form a variety of compound point singularities as well as singular lines and curves. A complete investigation of quartic surfaces would require a separate treatise, and all that can be attempted in the present chapter is to give an account of some of the principal results, with references to the authorities where further information may be obtained.

Nodal Quartics.

236. The theory of these surfaces has been worked out by Cayley* at considerable length. When the surface has not more than four nodes, these may be taken as the vertices of the tetrahedron of reference, and the highest power of the coordinate α corresponding to any node A must be α^2.

The existence of each node involves one equation of condition; but if the node is situated at a *given* point, three more equations are required to determine the point. Hence a *given* node involves four equations of condition; accordingly if the surface has k given nodes, it cannot contain more than $34-4k$ constants. When $k>8$, this expression becomes negative, the explanation of which is that a quartic surface cannot possess as many as 9 nodes which are arbitrarily situated; but the nodes must lie on one or more given surfaces called *dianodal* surfaces. We shall hereafter show

* *Proc. Lond. Math. Soc.* vol. III. pp. 19, 198, 234; and *C. M. P.* vol. VII. pp. 133, 256, 264.

that 7 is the greatest number of arbitrarily situated nodes which a quartic surface can possess.

237. *Five given nodes.* Let A, B, C, D be four of the nodes, and let the fifth be at the point (f, g, h, k). Let P, Q, R, S, T be five quadric surfaces, each of which passes through the five nodes; then the equation of the quartic is

$$(P, Q, R, S, T)^2 = 0 \quad \text{.....................(1)},$$

for it obviously possesses nodes at the five given points, and since it contains 14 independent constants, it is the most general form of the required surface. The five quadrics may be taken to be

$$P = \beta(k\gamma - h\delta), \quad Q = \beta(f\gamma - h\alpha), \quad R = \gamma(k\beta - g\delta),$$
$$S = \gamma(f\beta - g\alpha), \quad T = fk\beta\gamma - gh\alpha\delta,$$

from which if we eliminate $(\alpha, \beta, \gamma, \delta)$ it can be shown that there is one relation between the five quadrics, which is a *cubic* and not a quadric function.

238. *Six given nodes.* In the last article, the analysis may be simplified by writing $\alpha' = \alpha/f$, in which case the coordinates of the fifth node are $(1, 1, 1, 1)$, and we shall take the coordinates of the sixth node to be (f, g, h, k). Let P, Q, R, S be four quadrics passing through the six points, then the equation of a quartic having these points as nodes is

$$(P, Q, R, S)^2 = 0 \quad \text{........................(2)},$$

but since this contains nine instead of ten constants, it is not the most general form of a sexnodal quartic. Let J be the Jacobian of the four quadrics, then it can be shown that the latter is a surface having the six points as nodes which is not included in (2). Hence the required equation is

$$(P, Q, R, S)^2 + \lambda J = 0 \quad \text{....................(3)},$$

where λ is a constant.

The four pairs of planes

$$\left.\begin{aligned} \beta\{\alpha(h-k) + \gamma(k-f) + \delta(f-h)\} &= 0 \\ \gamma\{\alpha(k-g) + \beta(f-k) + \delta(g-f)\} &= 0 \\ \delta\{\alpha(g-h) + \beta(h-f) + \gamma(f-g)\} &= 0 \\ \alpha\{\beta(k-h) + \gamma(g-k) + \delta(h-g)\} &= 0 \end{aligned}\right\} \quad \text{.........(4)}$$

pass through each of the six points; but if we add together the equations of the second planes in each pair, the result vanishes, which shows that the planes are not independent. We shall therefore take the first three pairs as the surfaces P, Q, R; and the surface S to be the cone

$$S = g(k-h)\gamma\delta + h(g-k)\delta\beta + k(h-g)\beta\gamma = 0 \text{......(5)}$$

whose vertex is A, and which passes through the remaining five points.

The Jacobian of P, Q, R, S will be found to be

$$\begin{aligned} J &= (h-k)(g\alpha^2 - f\beta^2)\gamma\delta + (g-f)(h\delta^2 - k\gamma^2)\alpha\beta \\ &+ (k-g)(h\alpha^2 - f\gamma^2)\delta\beta + (h-f)(k\beta^2 - g\delta^2)\alpha\gamma \\ &+ (g-h)(k\alpha^2 - f\delta^2)\beta\gamma + (k-f)(g\gamma^2 - h\beta^2)\alpha\delta \quad \text{...(6)}, \end{aligned}$$

from which it can be shown that J cannot be expressed as a quadric function of P, Q, R, S; also J has nodes at each of the points A, B, C, D and it can easily be shown that it has nodes at the two remaining points.

Weddle's Surface.

239. Weddle showed* that the locus of the vertex of a quadric cone which passes through six given points is a quartic surface; and we shall now show that this is the surface (6).

The surface

$$lP + mQ + nT + pS = 0 \text{....................(7)},$$

where (l, m, n, p) are arbitrary constants, represents any quadric surface passing through the six points. If this surface has a node, the coordinates of the latter are obtained by differentiating (7) with respect to $(\alpha, \beta, \gamma, \delta)$; but since only three equations are necessary to determine a point, the elimination of $(\alpha, \beta, \gamma, \delta)$ furnishes a relation between (l, m, n, p), viz. the discriminant of the quadric, which is the condition that (7) should be a cone. If on the other hand we eliminate (l, m, n, p) we shall obtain the equation of the locus of the vertex of the cone, which is the Jacobian of (P, Q, R, S). Hence (6) is the equation of Weddle's surface.

240. Weddle's surface possesses several remarkable properties, among which the following may be noticed.

(i) *The six given points are nodes on the surface.*

* *Camb. and Dublin Math. Jour.* vol. v. p. 69; Bateman, *Proc. Lond. Math. Soc.* vol. III. (2nd Series), p. 225.

For the highest power of α in (6) is α^2, and its coefficient is the cone S.

(ii) *The surface contains* 25 *straight lines, which consist of two sets of* 15 *and* 10.

When $\gamma=\delta=0$, $J=0$; hence AB lies in the surface. This shows that J contains each of the 15 lines joining the 15 pairs of nodes. Let E and F be the two remaining nodes; then there are 10 lines which are the intersections of a pair of planes such as ABC and DEF; and the equations of this line are

$$\delta=0, \quad (g-h)\alpha+(h-f)\beta+(f-g)\gamma=L=0.$$

Putting $\delta=0$ in (6) it reduces to $kL\alpha\beta\gamma=0$, which shows that J contains the line in question.

241. *Seven given nodes.* Let P, Q, R be three quadric surfaces passing through the 7 points; then the equation

$$(P, Q, R)^2=0 \quad \text{.........................(8)}$$

represents a quartic surface having the 7 points as nodes, but since three quadrics intersect in 8 points, the eighth point of intersection is also a node. The required quartic must contain six constants, whilst (8) contains only five; but if Δ is some particular quartic which has a node at 7 of the points of intersection, the equation

$$(P, Q, R)^2+\lambda\Delta=0 \quad \text{.......................(9)}$$

represents a quartic having only seven nodes and containing six constants.

Let $$\Omega=a\beta\gamma\delta+b\gamma\delta\alpha+c\delta\alpha\beta+d\alpha\beta\gamma \quad \text{...........(10)},$$

then $\Omega=0$ is the equation of a quadrinodal cubic surface, whose nodes are A, B, C, D. Since the cubic contains three independent constants, these may be determined so that Ω passes through the remaining three nodes; and if u be the plane passing through the latter, the equation

$$(P, Q, R)^2+\lambda\Omega u=0 \quad \text{....................(11)}$$

represents the required quartic.

242. *Eight nodes.* Let θ, ϕ, ψ be the differential coefficients of (8) with respect to P, Q, R respectively; then if (9) has an eighth node at any point $(\alpha, \beta, \gamma, \delta)$, the equations determining the node are

$$\theta dP/d\alpha+\phi dQ/d\alpha+\psi dR/d\alpha+\lambda d\Delta/d\alpha=0 \quad \text{......(12)}$$

with three similar ones. The elimination of $(\alpha, \beta, \gamma, \delta)$ from (12) gives rise to a relation between the constants of (9) which reduces them to five; but if $(\theta, \phi, \psi, \lambda)$ be eliminated, we shall obtain the equation

$$J(P, Q, R, \Delta) = 0 \quad \text{.....................(13)},$$

where J is the Jacobian of P, Q, R, Δ. This is a surface of the sixth degree, and (12) shows that the eighth node may be anywhere on this surface; moreover the latter passes through the remaining seven nodes, and is therefore the dianodal surface.

Since any given point on (13) requires two equations of condition for its determination, the equation of a quartic which possesses seven arbitrary nodes and an eighth one, which lies on the dianodal surface (13), contains three arbitrary constants.

We have therefore shown that a quartic surface cannot possess more than seven conic nodes which are arbitrarily situated. If a quartic possesses more than this number, the nodes must lie on a certain surface (which need not be a proper one) called the *dianodal surface.*

243. The octonodal quartic (8) has been discussed by Cayley* and is one of considerable importance. It will hereafter be shown that all quartic surfaces having a singular conic can be reduced to this form; the equation also includes the reciprocals of the following surfaces, viz. parabolic ring $n=6$; elliptic ring $n=8$; parallel surface of a paraboloid, and first negative pedal of an ellipsoid $n=10$; centro-surface of an ellipsoid and parallel surface of an ellipsoid $n=12$. Also the general torus, or surface generated by the revolution of a conic about any axis whatever.

The proof of these theorems belongs to the theory of quadric surfaces rather than to that of quartics; I shall therefore merely give the following investigation due to Cayley† in order to illustrate the method to be employed.

244. (i) The centro-surface of an ellipsoid is the locus of the centres of principal curvature. Let P be any point on the surface, (x, y, z) the coordinates of either of the centres of principal

* *C. M. P.* vol. VII. p. 304; *Quart. Jour.* vol. X. p. 24; *C. M. P.* vol. VIII. pp. 2 and 25.

† "On the Centro-Surface of an Ellipsoid," *C. M. P.* vol. VIII. p. 363; *Trans. Camb. Phil. Soc.* vol. XII. pp. 319—365.

curvature corresponding to P, ρ either of the radii of curvature, p the perpendicular from the centre of the ellipsoid on to the tangent plane at P, then it is shown in Treatises on Quadric Surfaces* that (x, y, z) satisfy the equation

$$\frac{a^2x^2}{(a^2+\xi)^2}+\frac{b^2y^2}{(b^2+\xi)^2}+\frac{c^2z^2}{(c^2+\xi)^2}=1 \text{ ....(13 A)},$$

where $\xi = p\rho$. Since the quantity ξ is a function of the position of P, the equation of the centro-surface is the envelope of (13 A) where ξ is a variable parameter, and its equation is therefore the discriminant of (13 A) regarded as a binary sextic $(\xi, 1)^6 = 0$. But since the surface is the envelope of the ellipsoid (13 A), the reciprocal surface must be the envelope of the reciprocal ellipsoid

$$(a^2+\xi)^2x^2/a^2+(b^2+\xi)^2y^2/b^2+(c^2+\xi)^2z^2/c^2=k^4,$$

and since this is a quadratic equation in ξ, its discriminant is

$$(a^2x^2+b^2y^2+c^2z^2-k^4)(x^2/a^2+y^2/b^2+z^2/c^2)=(x^2+y^2+z^2)^2,$$

which is of the form (8).

(ii) The *rings* in question are the envelopes of a given sphere of constant radius c, whose centre moves on a conic section. Let $z=0$, $y^2=4ax$ be the equations of a parabola; then the coordinates of any point on the curve are $x=a\theta^2$, $y=2a\theta$, $z=0$. Hence the equation of the sphere is

$$(x-a\theta^2)^2+(y-2a\theta)^2+z^2=c^2,$$

and the discriminant of this equation regarded as the binary quartic $(\theta, 1)^4=0$ gives that of the ring, which will be found to be of the sixth degree.

The reciprocal polar is the envelope of the reciprocal of the sphere, whose equation can easily be shown to be the quadric

$$k^2=a\theta^2x+2a\theta y+cr,$$

where k is the constant of reciprocation; whence the equation of the reciprocal surface is

$$(ay^2+k^2x)^2=c^2x^2(x^2+y^2+z^2),$$

which is a quartic. Therefore the original surface is of the fourth class.

245. I shall not attempt to discuss the remaining cases in detail; but there are a few points which require consideration.

* Frost's *Solid Geometry*, vol. I. (1875), § 618.

The equation

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix} = 0 \quad\text{.................... (14),}$$

where the letters represent arbitrary planes, is a quartic surface called the *Symmetroid.* It possesses 10 nodes, which lie on the cubic surfaces obtained by equating the minors of this determinant to zero. The Hessian of a cubic surface is a particular case of the symmetroid, and the existence of the nodes on the latter has been proved in § 60.

It can also be shown that the vanishing of any of the four quantities a, b, c and d produces an additional node. When all four vanish, (14) assumes the form

$$(lf)^{\frac{1}{2}} + (mg)^{\frac{1}{2}} + (nh)^{\frac{1}{2}} = 0 \quad\text{.................. (15),}$$

which is a special case of a quartic with 14 nodes.

The equation of a quartic surface having a conic node at A is

$$\alpha^2 u_2 + 2\alpha u_3 + u_4 = 0 \quad\text{.................... (16),}$$

and the tangent cone from A is the sextic cone

$$u_3^2 = u_2 u_4 \quad\text{.......................... (17).}$$

Now a proper sextic cone cannot possess more than 10 nodal generators; if therefore a quartic surface possesses more than 11 conic nodes, the tangent cone (17) will degrade into an improper cone, and this fact has been made use of by Cayley* for finding the equations of quartic surfaces with more than 11 nodes.

Kummer's Surface.

246. This surface has been so fully discussed in Mr Hudson's recent treatise† that only a slight sketch will be given. The equations of a quartic having a node at A and of the tangent cone from A, are given by (16) and (17) and the latter obviously touches the nodal cone u_2 along the lines of closest contact. The line joining A to any other node on (16) must be a nodal generator of (17); and since Kummer's surface possesses 16 nodes, the sextic cone (17) must possess 15 nodal generators and must therefore

* *Proc. Lond. Math. Soc.* vol. III. p. 234.

† *Kummer's Quartic Surface,* Cambridge University Press.

degrade into six planes. Each of these planes is intersected by the five other planes, and their five lines of intersection connect A with five nodes; hence each of the six planes contains six nodes. But since each of the six planes forms a part of the improper tangent cone from A, each plane must touch the quartic along its curve of intersection and therefore the latter must be a conic twice repeated; in other words, each of the six planes is a *conic trope* on which lie six nodes. Also since the surface is of the fourth class, it is its own reciprocal; moreover the reciprocal polar of a conic node is a conic trope on the reciprocal surface, and since the original surface possesses 16 conic nodes, it must also possess the same number of conic tropes.

247. Kummer starts with the irrational equation

$$(\alpha - u)^{\frac{1}{2}} \beta^{\frac{1}{2}} + (\alpha - v)^{\frac{1}{2}} \gamma^{\frac{1}{2}} + (\alpha - w)^{\frac{1}{2}} \delta^{\frac{1}{2}} = 0 \ldots\ldots\ldots (18)$$

or

$$(\alpha - u)^2 \beta^2 + (\alpha - v)^2 \gamma^2 + (\alpha - w)^2 \delta^2 - 2(\alpha - v)(\alpha - w)\gamma\delta$$
$$- 2(\alpha - w)(\alpha - u)\delta\beta - 2(\alpha - u)(\alpha - v)\beta\gamma = 0 \ldots\ldots (19),$$

where u, v, w are any three planes passing through A. Equation (19) may be written in the form

$$A\alpha^2 - 2B\alpha + C = 0,$$

where

$$A = \beta^2 + \gamma^2 + \delta^2 - 2\gamma\delta - 2\delta\beta - 2\beta\gamma,$$
$$B = \beta^2 u + \gamma^2 v + \delta^2 w - (v + w)\gamma\delta - (w + u)\delta\beta - (u + v)\beta\gamma,$$
$$C = \beta^2 u^2 + \gamma^2 v^2 + \delta^2 w^2 - 2vw\gamma\delta - 2wu\delta\beta - 2uv\beta\gamma;$$

hence A is a node. Writing (18) in the form

$$(X\beta)^{\frac{1}{2}} + (Y\gamma)^{\frac{1}{2}} + (Z\delta)^{\frac{1}{2}} = 0$$

it is obvious that the points of intersection of the eight triplets of planes viz. β, γ, δ; β, γ, Z; β, Y, δ; β, Y, Z; X, γ, δ; X, γ, Z; X, Y, δ; X, Y, Z are nodes; but we shall show that (18) possesses altogether 13 nodes.

Let $\sigma = \beta + \gamma + \delta, \quad \Sigma = \beta u + \gamma v + \delta w,$

then

$$A = 2(\beta^2 + \gamma^2 + \delta^2) - \sigma^2,$$
$$B = 2(\beta^2 u + \gamma^2 v + \delta^2 w) - \Sigma\sigma,$$
$$C = 2(\beta^2 u^2 + \gamma^2 v^2 + \delta^2 w^2) - \Sigma^2,$$

and if the above values of A, B and C be substituted in the equation $AC = B^2$ it reduces to

$$2\beta\gamma\delta\{\beta(u - v)(w - u) + \gamma(u - v)(v - w) + \delta(v - w)(w - u)\} = 0;$$

hence the tangent cone from A consists of three planes and a proper cubic cone. The complete cone has accordingly 12 nodal generators, viz. the lines AB, AC and AD, and the three lines in which each of the planes ABC, ACD and ADB intersect the cubic cone

$$\beta/(v-w)+\gamma/(w-u)+\delta/(u-v)=0 \quad \text{.........(20).}$$

The quartic therefore possesses 13 nodes; but it will have 14 when (20) has a nodal generator; 15 when (20) consists of a quadric cone and a plane; and 16 when (20) consists of three planes.

The condition that the plane $\beta/l+\gamma/m+\delta/n=0$ should touch the cone $A=0$ is that $l+m+n=0$, and Kummer takes

$$u=l\,(n_1n_2\gamma-m_1m_2\delta), \quad v=m\,(l_1l_2\delta-n_1n_2\beta),$$
$$w=n\,(m_1m_2\beta-l_1l_2\gamma),$$

where $l+m+n=l_1+m_1+n_1=l_2+m_2+n_2=0$; from which it can be easily shown that the equation of the tangent cone becomes

$$K\beta\gamma\delta\,(\beta/l+\gamma/m+\delta/n)\,(\beta/l_1+\gamma/m_1+\delta/n_1)\,(\beta/l_2+\gamma/m_2+\delta/n_2)=0,$$

where $K=4ll_1l_2mm_1m_2nn_1n_2$.

248. A particular case of Kummer's 16 nodal quartic surface is the *Tetrahedroid*, which can be projected into Fresnel's wave surface

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0.$$

The sections of the surface by each of the coordinate planes consist of a circle and an ellipse, and if $a>b>c$ the four points of intersection in the plane $y=0$ are real and give rise to four real conic nodes, which produce external conical refraction; the eight points of intersection in the planes $x=0$, $z=0$ are imaginary, and give rise to eight imaginary conic nodes; and the section by the plane at infinity consists of the factors

$$x^2+y^2+z^2 \text{ and } a^2x^2+b^2y^2+c^2z^2,$$

showing that there are four nodes on the imaginary circle at infinity*.

* In 1871 Lord Rayleigh proposed a theory of double refraction, which is discussed in Chapter XV of my *Physical Optics*, in which the velocity of propagation is determined by the equation

$$\frac{l^2a^2}{v^2-a^2}+\frac{m^2b^2}{v^2-b^2}+\frac{n^2c^2}{v^2-c^2}=0.$$

From this it follows that the pedal of Lord Rayleigh's surface is Fresnel's, and that

The surface has also 16 conic tropes, four of which are real and the remaining 12 imaginary. The Hessian intersects the surface in the 16 circles of contact and the latter constitutes the spinodal curve. The flecnodal and bitangential curves do not appear to have been investigated.

Quartics with Singular Lines.

249. The theory of surfaces with singular lines has already been given, and we shall now enquire what lines of this character a quartic surface can possess.

Nodal line of the first kind. We have shown in §§ 43 and 37 that when the surface is of the nth degree, the reduction of class $R = 7n - 12$, and that the number of pinch points is $2n - 4$. Hence when $n = 4$, $R = 16$, $m = 20$ and the number of pinch points is four; accordingly if A and B are two of them, the equation of the quartic is

$$p\alpha^2\delta^2 + 2\alpha\beta v_2 + q\beta^2\gamma^2 + 2\alpha v_3 + 2\beta w_3 + w_4 = 0 \text{ ......(1).}$$

250. *The surface has* 16 *lines lying in it, all of which intersect the nodal line.*

The section of the surface by the plane $\delta = k\gamma$ consists of AB twice repeated and a conic; and if the discriminant of this conic be equated to zero, it will furnish an equation of the 8th degree in k, which shows that there are eight planes in which the conic degrades into a pair of straight lines. There are thus 16 lines, which lie in pairs in eight planes passing through AB. See also § 44.

its reciprocal polar is the inverse of Fresnel's surface, and is of the 6th degree. The surface is therefore of the 6th class, and if a certain inequality existed between the optical constants, biaxal crystals would be capable of producing *triple refraction.* A principal section of the reciprocal surface consists of a circle and the inverse of an ellipse with respect to its centre, and since the last curve is a trinodal quartic, and therefore of the 6th class, the principal sections of Lord Rayleigh's surface consist of a circle and a sextic curve of the 4th class. The surface is therefore of the 8th degree. For a uniaxal crystal, this wave surface degrades into a sphere, and the reciprocal polar of the inverse of a spheroid with respect to its centre. The inverse of a spheroid can possess a pair of real tropes having real circles of contact, which reciprocate into a pair of real conic nodes having real nodal cones; hence Lord Rayleigh's theory leads to the result that uniaxal crystals might not only possess *two* extraordinary rays as well as one ordinary ray, but might also produce *external* conical refraction.

251. An arbitrary plane cuts the surface in a uninodal quartic, but a triple tangent plane cuts the surface in a pair of conics which pass through the point where AB intersects the plane, and intersect one another in three other points which are the points of contact of the plane. Let BCD be a triple tangent plane, then the equation of the surface must be of the form

$$(\alpha U + PP')\gamma^2 + (\alpha V + PQ' + P'Q)\gamma\delta + (\alpha W + QQ')\delta^2 = 0 \ldots (2),$$

where U, V, W are arbitrary planes, and P, P', &c. are planes passing through A; for when $\alpha = 0$, (2) becomes

$$(P\gamma + Q\delta)(P'\gamma + Q'\delta) = 0 = SS' \text{ (say) } \ldots\ldots\ldots (3).$$

Let AC be one of the 16 lines, ABD any one of the eight planes, D the remaining point where BD cuts S; then

$$U = f\beta + h\delta, \quad P = \lambda\beta + \nu\delta, \quad Q = G\beta + H\gamma,$$
$$P' = \lambda'\beta + \mu'\gamma + \nu'\delta, \quad Q' = G'\beta + H'\gamma + K'\delta,$$
$$W = F(F'\alpha + G'\beta + K'\delta) + F'G\beta + E\gamma.$$

Putting $\delta = 0$, (2) becomes

$$\beta\{f\alpha + \lambda(\lambda'\beta + \mu'\gamma)\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which shows that AC is one of the lines lying in the plane ABC, whilst the other line is given by the remaining factor of (4). Let C' be the point where the last line cuts BC; then it follows from (3) and (4) that C lies on the conic S, and C' on the conic S'.

Let D' be the remaining point where BD cuts the conic S'; then putting $\gamma = 0$ in (2), it follows that the equation of the lines lying in the plane ABD is

$$(F\alpha + G\beta)(F'\alpha + G'\beta + K'\delta) = 0,$$

the first of which passes through the point D which lies on the conic S, whilst the second passes through the point D' which lies on the conic S'. Hence:—*If BCD be any triple tangent plane, the section of the surface by it consists of two conics S and S'; also one of the lines in each of the eight planes intersects the conic S, whilst the other intersects the conic S'.*

Since the nodal tangent planes at B are $\lambda\gamma + G\delta = 0$ and $\lambda'\gamma + G'\delta = 0$, it follows that:—*The nodal tangent planes at B touch the two conics respectively.*

252. The theorems of § 250 or § 44 show that eight tangent planes can be drawn to the quartic through the nodal line AB. Now a node diminishes the class of the surface, and therefore the number of tangent planes which can be drawn through an arbitrary straight line, by 2; hence the plane through the line and the node is an *improper* tangent plane, which is equivalent to two *ordinary* tangent planes. If therefore the surface possesses an isolated conic node, only six ordinary tangent planes can be drawn through AB, giving 12 ordinary lines, whilst the two remaining lines consist of a pair passing through AB and the node, each of which is equivalent to two ordinary lines. Since a binode reduces the class by 3, it follows that if the surface possessed an isolated binode, there would be only five planes and 10 ordinary lines, and each of the lines through AB and the binode would be equivalent to three ordinary lines. When the surface possesses four conic nodes, there are only eight lines, which consist of four pairs such that the lines belonging to each pair intersect at a conic node; and in this case the equation of the surface may be expressed in the form

$$(U, V, W)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where U, V, W are three quadric surfaces, which possess a common straight line. The latter is the nodal line on the quartic, and the four distinct points in which the quadrics intersect are the four nodes on the quartic.

253. Since not more than eight tangent planes can be drawn through the nodal line AB, it follows that if the quartic has a fifth node it must lie in a plane through AB and one of the four other nodes; for if not, *five* improper tangent planes, which are equivalent to ten ordinary tangent planes, could be drawn through AB to the surface, which is impossible. Now it follows from § 31 that if a surface of the nth degree possesses $n-1$ conic nodes lying in the same straight line, the latter not only lies in the surface, but the tangent plane along it is a *fixed* instead of a *torsal* tangent plane; accordingly if two conic nodes P and Q lie in the plane $ABPQ$, the point where the line PQ cuts AB is a third node on the section and therefore the plane must touch the quartic at every point on PQ.

This result may be proved more simply for a quartic as follows. If there is a conic node at P, the section by the plane ABP must consist of AB twice repeated and two straight lines Pp, Pq; but if there is another node at Q, Pp and Pq must pass through it and must therefore coincide. It therefore follows that:—*When* *a quartic surface possesses a nodal line AB and eight conic nodes, the latter lie in pairs in four planes passing through AB; also each of these planes touches the quartic along the line joining the pair of nodes lying in it.*

This surface is called *Plücker's Complex Surface*, by whom it was studied in connection with the theory of the Line Complex *.

254. *The equation of Plücker's surface may be expressed in the form*

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & 1 \\ \alpha, & \beta, & 1, & 0 \end{vmatrix} = 0,$$

where $a \equiv b \equiv h \equiv (\gamma, \delta)^2$; $f \equiv g \equiv (\gamma, \delta)^1$; *and* c *is a constant.*

By § 49 the determinant when expanded becomes

$$(bc - f^2)\,\alpha^2 + (ca - g^2)\,\beta^2 + ab - h^2$$
$$+ 2\,(gh - af)\,\beta + 2\,(hf - bg)\,\alpha + 2\,(fg - ch)\,\alpha\beta = 0 \ldots (6).$$

Let $\delta = k\gamma$ be any arbitrary plane through AB, and let a', b', &c. denote what a, b, &c. become when $\gamma = 1$, $\delta = k$; then (6) reduces to γ^2 multiplied by the conic

$$(b'c - f'^2, \ldots \between \alpha, \beta, \gamma)^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

and if the plane $\delta = k\gamma$ touch the quartic, the discriminant of (7) must vanish; but this is equal to the square of the discriminant of

$$(a', b', c, f', g', h' \between \alpha, \beta, \gamma)^2 = 0,$$

hence the discriminantal equation for k is one of the eighth degree having four pairs of equal roots, and the quartic has a node in each of the planes corresponding to the four values of k.

Let us now suppose that $\delta = 0$ is any one of the four planes through a node; and let $a = a_0\gamma^2 + a_0'\gamma\delta + a_0''\delta^2$. Then the equation of the section is obtained by writing a_0 for a &c. in (6); but if C is a node, the section must reduce to a pair of straight lines intersecting at C, whence

$$g_0/f_0 = a_0/h_0 = h_0/b_0 = q \text{ (say)},$$

so that the section becomes

$$(b_0 c - f_0^2)\,(\alpha - q\beta)^2 = 0,$$

* *Neue Geometrie des Raumes.* Jessop, *Treatise on the Line Complex*, § 86.

and therefore consists of a straight line through B twice repeated, so that the plane δ is a fixed tangent plane along this line.

255. *Nodal line of the second kind.* This line has only two actual pinch points, owing to the fact that the four pinch points coincide in pairs; and the equation of the quartic is

$$p\alpha^2\gamma^2 + 2\alpha\beta\gamma v_1 + q\beta^2\gamma^2 + 2\alpha v_3 + 2\beta w_3 + w_4 = 0 \ldots\ldots(8).$$

The section of the surface by the plane $\gamma = 0$ consists of AB three times repeated and another line; also proceeding as before, the discriminantal equation will be of the seventh degree in k. Hence the 16 lines consist of AB and 15 other lines.

256. *Nodal line of the third kind.* The equation of the quartic is

$$\alpha\beta v_2 + \alpha v_3 + \beta w_3 + w_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

and it has two cubic nodes at A and B. The lines of closest contact at each cubic node lie in the surface, and they consist of the line AB six times repeated and six other lines, making altogether 12; also the discriminantal equation furnishes 12 more, making a total of 24 lines. The class of the surface is obtained by differentiating (9) with respect to γ and δ and eliminating (α, β); whence $m = 14$.

257. *Cuspidal line of the first kind.* The general theory of these lines has been discussed in §§ 210, 211. When the surface is a quartic the line has four tacnodal points but no cubic nodes; and if A and B be two of these, the equation of the surface may be written

$$(\alpha\gamma + \beta\delta)^2 + \alpha\gamma v_2 + \beta\delta w_2 + w_4 = 0 \ldots\ldots\ldots\ldots(10).$$

258. *The surface is of the twelfth class.*

Let $$\Omega = \alpha\gamma + \beta\delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Differentiating (10) with respect to γ and δ, we obtain

$$2\Omega\alpha + \alpha v_2 + \alpha\gamma v_2' + \beta\delta w_2' + w_4' = 0 \ldots\ldots\ldots(12),$$

$$2\Omega\beta + \alpha\gamma v_2'' + \beta w_2 + \beta\delta w_2'' + w_4'' = 0 \ldots\ldots\ldots(13),$$

from which we deduce

$$\alpha\gamma v_2 + \beta\delta w_2 + 2w_4 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

$$\Omega^2 = w_4 \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Eliminate (α, β) between (11), (12) and (14) and we shall obtain an equation of the form

$$\Omega^2 V_2 + \Omega V_4 + V_6 = 0 \quad \text{..................(16)},$$

where $V_n = (\gamma, \delta)^n$. Eliminating Ω between (15) and (16) the result is a binary quantic of the 12th degree, which shows that $m = 12$.

259. Putting $\delta = 0$ in (10) it follows that the section of the surface by the tangent plane at a tacnodal point is a pair of straight lines which intersect at the point. Thus there are four pairs of lines lying in the surface which intersect at these points. That there are no other lines can be shown by putting $\delta = k\gamma$ in (10) and equating the discriminant of the resulting conic to zero, which will be found to be of the form $k^2(w_2' - v_2')^2 = 0$, where v_2', w_2' are the values of v_2 and w_2 when $\gamma = 1$, $\delta = k$. The double root $k^2 = 0$, corresponds to the point A, whilst the other two double roots correspond to the two other tacnodal points exclusive of B. There are consequently no proper tangent planes to the quartic through AB.

260. It is possible for a quartic having a cuspidal line to possess as many as four conic nodes. Changing the planes γ and δ to any arbitrary planes through AB, the equation when there is a node at C is

$$(\alpha v_1 + \beta w_1)^2 + \delta(\alpha v_1 \rho_1 + \beta w_1 \sigma_1 + \delta w_2) = 0,$$

and the section of the surface by the plane ABC consists of AB, and a line CE through the node both twice repeated. Hence the plane δ touches the quartic along this line, which is therefore a singular line of the nature of the curve of contact of a *trope*. When the quartic has four nodes, its equation is

$$(\alpha v_1 + \beta w_1)^2 + \gamma\delta(L\gamma^2 + M\gamma\delta + N\delta^2) = 0 \quad \text{......(17)}.$$

261. *Cuspidal line of the second kind.* These lines possess three tacnodal points and two cubic nodes, and the equation of the quartic is

$$\alpha\beta\gamma^2 + \alpha v_3 + \beta w_3 + w_4 = 0 \quad \text{..............(18)}.$$

262. *Tacnodal line of the first kind.* The equation of the quartic is

$$(\alpha\gamma + \beta\delta)^2 + 2(\alpha\gamma + \beta\delta) v_2 + v_4 = 0 \text{...........(19)},$$

and there are four points where the tacnode changes into a rhamphoid cusp, see § 213. The surface is of the fourth class and is a scroll, for if we put $\delta = k\gamma$, the section consists of the line AB twice repeated, and the curve

$$(\alpha + k\beta)^2 + 2P(\alpha + k\beta) + Q\gamma^2 = 0,$$

which represents two straight lines.

263. *Tacnodal line of the second kind.* The equation of the surface is

$$(L\alpha^2 + M\alpha\beta + N\beta^2)\gamma^2 + 2\gamma(\alpha v_2 + \beta w_2) + w_4 = 0 \quad \ldots(20),$$

and it possesses two cubic nodes at the points where AB intersects the planes $L\alpha^2 + M\alpha\beta + N\beta^2 = 0$, and two rhamphoid cuspidal points.

264. The highest singular line of the second order and first kind, which a quartic surface can possess, is a tacnodal line; but when the line is of the second species, a quartic surface as shown in § 214 may possess a rhamphoid cuspidal and an oscnodal line. Quartic surfaces may also possess triple lines, the discussion of which will be postponed for the present*.

When a quartic surface possesses a singular conic, the latter can degrade into a pair of straight lines. These surfaces will be considered under the head of quartics having nodal conics.

Nodal Conics.

265. Professor Segre† has enumerated as many as 76 different species of quartic surfaces which possess a singular conic, the principal of which are the following :—(i) a nodal conic, $m = 12$;

* The following papers relate to the subjects discussed in §§ 249 to 264. Clebsch, *Crelle*, vol. LXIX. p. 355; *Math. Ann.* vol. I. p. 260; *Atti del R. Ist. Lomb.* 12th Nov. 1868. Cayley, "Quartic and Quintic Surfaces," *Proc. Lond. Math. Soc.* vol. III. p. 186 and the authorities there cited. Basset, *Quart. Jour.* vol. XXXVIII. p. 160; vol. XXXIX. p. 334.

† *Math. Ann.* vol. XXIV. p. 313. The following are some of the principal memoirs on the subject. Berzolari, *Annali di Matematica*, Serie II. vol. XIII. p. 81; Zeuthen, *Ibid.* vol. XIV. p. 31; Kummer, *Borchardt*, vol. LXIV. p. 66; Clebsch, *Ibid.* vol. LXIX. p. 142; Geiser, *Ibid.* vol. LXX. p. 249; Cremona, *Rendiconti del R. Istituto Lombardo*, 1871; Sturm, *Math. Annalen*, vol. IV. p. 265; Moutard, *Nouvelles Annales*, 1864, pp. 306 and 536; Loria, *Mem. dell' Accad. delle Scienze di Torino*, Serie II. vol. XXXVI.; Veronese, *Atti di R. Istituto Veneto*, Serie VI. vol. II.

(ii) a cuspidal conic, $m = 6$; (iii) two nodal lines intersecting in a point, $m = 12$; (iv) a nodal and a cuspidal line intersecting in a point; (v) two cuspidal lines, $m = 6$; (vi) two coincident nodal lines. All these six species may be divided into various subsidiary ones, owing to the fact that the surface may possess isolated conic nodes and binodes, as well as a variety of compound singular points.

266. *The point constituents of a nodal conic on a quartic are 12 conic nodes, and it has four pinch points; also* $m = 12$.

Consider the quartic

$$\alpha^2 (\alpha^2 u_0 + \alpha u_1 + u_2) + (\alpha U_1 + U_2)(\alpha U_1' + U_2') = 0 \quad \ldots(1).$$

The section of the surface by the plane BCD consists of two conics, and if B, C, D be three of their four points of intersection we may take

$$\left.\begin{aligned} U_1 &= P\beta + Q\gamma + R\delta, \quad U_1' = P'\beta + Q'\gamma + R'\delta \\ U_2 &= L\gamma\delta + M\delta\beta + N\beta\gamma, \quad U_2' = L'\gamma\delta + M'\delta\beta + N'\beta\gamma \end{aligned}\right\} \ldots(2).$$

$$u_2 = F\beta^2 + G\gamma^2 + H\delta^2 + f\gamma\delta + g\delta\beta + h\beta\gamma.$$

The coefficient of β^2 in (1) is

$$F\alpha^2 + (P\alpha + M\delta + N\gamma)(P'\alpha + M'\delta + N'\gamma),$$

which shows that B is a conic node; hence the four points in which the two conics intersect one another are conic nodes on the quartic. Let $F = 0$, $P'/P = M'/M = N'/N$, then B is a unode; from which it follows that if the quadrics $\alpha U_1 + U_2$ and $\alpha U_1' + U_2'$ become identical, the quartic will have unodes at each of the four points of intersection of the plane α and the cones u_2 and U_2. But when this happens, these two conics coalesce into a nodal conic, whose pinch points are the four points of intersection of α, u_2 and U_2; also since a unode is equivalent to three conic nodes, the nodal conic is equivalent to 12 conic nodes, whence $m = 36 - 24 = 12$.

Equation (1) now becomes

$$\alpha^2 (\alpha^2 u_0 + \alpha u_1 + u_2) + (\alpha U_1 + U_2)^2 = 0,$$

which is of the form

$$\alpha^2 W + V^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and is a special case of the quartic $(U, V, W)^2 = 0$, where U, V, W are quadric surfaces. The equation may also be written in the form

$$\alpha^2 U + 2\alpha\Omega U_1 + \Omega^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(4),$$

where U is an arbitrary quadric, U_1 any plane through A, and Ω a cone whose vertex is A.

In accordance with the theory explained in § 219, the pinch points are the intersections of the nodal conic and the surface

$$U_1^2 = U,$$

for if B is one of their points of intersection

$$U = p^2\beta^2 + \beta V_1 + V_2, \quad U_1 = p\beta + q\gamma + r\delta,$$
$$\Omega = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma,$$

where $V_n = (\alpha, \gamma, \delta)^n$ and the coefficient of β^2 in (4) is

$$(p\alpha + \mu\delta + \nu\gamma)^2.$$

267. *The surface possesses five bitangential quadric cones.*

Equation (3) shows that the quadric V intersects the quartic in the nodal conic twice repeated and in a twisted quartic curve, and that the quadric W touches the quartic along this curve. Now (3) may be written in the form

$$\alpha^2 (W - 2\lambda V - \lambda^2\alpha^2) + (V + \lambda\alpha^2)^2 = 0 \qquad (5),$$

where λ is an arbitrary constant; and if the discriminant of the quadric

$$W - 2\lambda V - \lambda^2\alpha^2 = 0$$

be equated to zero, it will become a cone. Since the discriminant furnishes a quintic equation for λ, it follows that there are five of such cones, which are called after the name of their discoverer *Kummer's cones**; and the quartic may be represented by an equation of the form (3), where W is one of Kummer's cones.

268. *The surface possesses* 16 *straight lines lying in it, each of which intersects the nodal conic; also each line is intersected by* 5 *others none of which lie in the same plane.*

Let p and q be the points of contact of any double tangent plane; then since the latter intersects the nodal conic in two points P and Q, the section of the surface by the plane in general consists of two conics which intersect in p, q, P, Q. But since every double tangent plane possesses one degree of freedom, and therefore contains a single variable parameter, it is possible to determine the latter so that three of the points P, p and q should lie in the same straight line, in which case the section consists of the straight line Ppq and a nodal cubic curve whose node is at Q.

* *Crelle*, vol. LXIV. p. 66.

Let AB be one of the lines, then the section of the quartic by any plane $\delta = k\gamma$ through AB consists of this line and a cubic curve which intersects AB in the point B and in two other points, whilst the cubic has a node at the point C' in which the plane cuts the nodal conic. If therefore the value of k be determined so that the cubic curve degrades into a conic and a straight line, the latter must intersect AB, and we shall now show that there are 5 of such lines.

The equation of the quartic may be written in the form

$$\alpha^2 (\alpha v_1 + \beta w_1 + w_2) + \alpha\Omega (P\beta + Q\gamma + R\delta) + \Omega^2 = 0 \ldots (6),$$

where $$\Omega = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma,$$

and if the tetrahedron be changed to $ABC'D$, we must write

$$\delta = \delta' + k\gamma, \quad \beta = (\beta' - \lambda k\gamma)/\rho,$$

where $\rho = \mu k + \nu$. Let V_1, W_1, W_2 be what v_1, w_1, w_2 become when $\gamma = 1$, $\delta = k$; then making these substitutions the section of (6) by the plane δ' will be the line AB and the cubic curve

$$\alpha (\alpha^2 V_1\rho + \alpha\beta' W_1 + P\beta'^2) + \gamma [(W_2\rho - \lambda k W_1) \alpha^2 \\ + \{(Q + Rk) \rho - \lambda kP\} \alpha\beta' + \rho\beta'^2] = 0 \ldots (7).$$

In order that (7) should degrade into a conic and a line through C', it is necessary that the coefficients of α and γ in brackets should have a common factor; the condition for which is that the eliminant of

$$x^2 V_1\rho + xW_1 + P = 0,$$
$$x^2 (W_2\rho - \lambda k W_1) + x \{(Q + Rk) \rho - \lambda kP\} + \rho = 0,$$

should vanish; but if the eliminant be written down it will be found that the term which does not explicitly involve ρ vanishes, hence ρ is a factor of the eliminant and the remaining one furnishes an equation of the fifth degree in k. The root $\rho = 0$ corresponds to the plane $\delta = 0$, that is to the line AB; and the remaining quintic factor gives five other lines, and since none of the roots are equal, all these lines lie in different planes.

Let 2, 3, 4, 5 and 6 denote the five lines which cut the line 1; then we have to find a certain number of other lines such that each of the first five lines are intersected by five others, the arrangement being such that only two lines lie in the same plane. The 16 lines are shown in the following table, in which the top row of figures denotes the line we are considering, and the column underneath denotes the five lines which intersect it.

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.
2	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
3	7	7	8	9	10	3	4	5	6	4	5	6	5	6	6
4	8	11	11	12	13	14	12	11	11	9	8	8	7	7	7
5	9	12	14	14	15	15	13	13	12	10	10	9	10	9	8
6	10	13	15	16	16	16	16	15	14	16	15	14	13	12	11

The first five columns furnish 16 lines; also the line 7 is cut by 2 and 3, hence no other line which cuts 2 and 3 can cut 7, otherwise more than two lines would lie in the same plane. Accordingly the only other lines which can cut 7 are 14, 15 and 16; whence continuing this process we obtain the remainder of the table.

269. *Through any point on the nodal conic* 10 *planes can be drawn which intersect the quartic in a pair of conics. These* 10 *planes consist of five pairs, such that each pair touches one of Kummer's cones.*

Let B be the point; then if the suffixed letters denote ternary quantics of (α, γ, δ) the equation of the surface is of the form

$$\alpha^2 (\beta^2 U_0 + \beta U_1 + U_2) + (\beta u_1 + u_2)^2 = 0,$$

and the equation of the tangent cone from B is

$$(\alpha^2 U_1 + 2u_1 u_2)^2 = 4 (\alpha^2 U_0 + u_1^2)(\alpha^2 U_2 + u_2^2) \quad \text{.........(8).}$$

Multiplying out and dividing out by α^2, (8) reduces to a quartic cone which may be put into the form

$$(\alpha^2 U_0 + u_1^2)(U_1^2 - 4U_0 U_2) = (2U_0 u_2 - U_1 u_1)^2 \quad \text{......(9).}$$

Equation (9) shows that the tangent cone from any point on the nodal conic is a quartic cone; a result which may be obtained otherwise as follows. The section of the surface by any plane through B is a binodal quartic curve, one of whose nodes is at B whilst the other is situated at the remaining point E in which the plane cuts the nodal conic. From either node of the quartic curve four tangents can be drawn to the latter, and since each of these tangents is a generator of the tangent cone to the surface from B, it follows that any plane through B must cut the cone in four straight lines and therefore the cone must be a quartic one. Also since every generator of the cone intersects the quartic in two coincident points at B, it follows that no double nor stationary tangents can be drawn to the surface from this point; hence the cone must be anautotomic. Accordingly it has 28 double tangent planes; but the form of (9) shows that two of them are $\alpha^2 U_0 + u_1^2 = 0$, which are the nodal tangent planes at B; hence the cone has 26 remaining double tangent planes.

Each of these 26 double tangent planes touches the surface at two points P and Q, and intersects the nodal conic at B and a fourth point E; hence the section of the quartic by such a plane is a pair of conics unless three of the points lie in the same straight line, in which case the section will consist of a straight line and a nodal cubic. Now the tangent plane at any point on one of the sixteen lines is a torsal plane; hence there must be a certain position of the point of contact for which the tangent plane passes through B. From this it follows that 16 of the 26 double tangent planes to the cone contain one of the 16 lines, whilst the remaining 10 cut the quartic in a pair of conics.

To prove the remaining part of the theorem, let A be the vertex of one of Kummer's cones, B any point on the nodal conic, and let the tangent planes to the cone through AB touch it along AC and AD. Then the equation of the surface may be written in the form

$$\alpha^2 (\beta^2 + k\gamma\delta) = U^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where $\beta^2 + k\gamma\delta = 0$ is the Kummer's cone whose vertex is A. The section of the surface by the tangent plane γ to the cone is

$$\alpha\beta \pm U' = 0,$$

where U' is what U becomes when $\gamma = 0$, which represents a pair of conics.

270. *Each of the* 16 *lines touches each of Kummer's cones.*

The tetrahedron can always be chosen so that A is the vertex of one of Kummer's cones, C the point where one of the lines intersects the nodal conic, whilst ABC contains this line. Also D may be any point on the conic. Hence the equation of the surface may be taken to be

$$\alpha^2 (\delta^2 V_0 + \delta V_1 + F^2\beta^2 + G\beta\gamma + H^2\gamma^2)$$
$$- \{\alpha^2 + \alpha (P\beta + Q\gamma + R\delta) + \Omega\}^2 = 0 \ldots (11).$$

Let $\delta = 0$, $\alpha = k\beta$ be the equations of the line through C, then the conditions are that

$$k^2 (F^2\beta^2 + G\beta\gamma + H^2\gamma^2) - \{k^2\beta + k (P\beta + Q\gamma) + \nu\gamma^2\}^2 = 0,$$

which requires that

$$G = 2FH, \quad F = P + k, \quad kH = Qk + \nu.$$

The first condition reduces the equation of the cone to $\delta^2 V_0 + \delta V_1 + (F\beta + H\gamma)^2 = 0$, which shows that the plane δ, and therefore every line lying in it, touches the cone.

Pinch Points.

271. We shall now give a few theorems concerning these points.

The tangent planes at the four pinch points intersect at a point.

Let the equation of the quartic be

$$\alpha^2 (\alpha^2 u_0 + \alpha u_1 + u_2) + (\alpha U_1 + U_2)^2 = 0 \quad \text{.........(12)},$$

where

$$\left.\begin{aligned} u_1 &= p\beta + q\gamma + r\delta, & U_1 &= P\beta + Q\gamma + R\delta \\ u_2 &= f\gamma\delta + g\delta\beta + h\beta\gamma, & U_2 &= F\gamma\delta + G\delta\beta + H\beta\gamma \end{aligned}\right\} \ldots(13).$$

The four pinch points, being the points of intersection of u_2 and U_2, are B, C, D and a fourth point E; also the equations of the tangent planes at B, C, D are

$$\left.\begin{aligned} P\alpha + H\gamma + G\delta = 0 \\ Q\alpha + H\beta + F\delta = 0 \\ R\alpha + G\beta + F\gamma = 0 \end{aligned}\right\} \quad \text{..................(14)},$$

and since these planes cannot pass through the same straight line, they must intersect at a point; and if this be taken as the vertex A, $P = Q = R = 0$.

The coordinates of E are

$$(Hg - Gh)\,\beta = (Fh - Hf)\,\gamma = (Gf - Fg)\,\delta$$

or

$$B\beta = C\gamma = D\delta \ \text{(say)},$$

whence changing the tetrahedron to $ABDE$ by writing

$$\beta' = B\beta - C\gamma, \quad \delta' = C\gamma - D\delta,$$

it will be found that the tangent plane at E is

$$\beta' (GC + HD) - \delta' (FB + GC) = 0$$

or

$$B^2 F\beta + C^2 G\gamma + D^2 H\delta = 0 \quad \text{..............(15)},$$

which passes through A.

272. *The section of the surface by the tangent plane at a pinch point has a triple point of the second kind thereat.*

Recollecting that $P=Q=R=0$, the section of (12) by the plane $H\gamma+G\delta=0$, which is the tangent plane at B, is a quartic curve in which the term involving β is

$$\alpha^2\{p\alpha+(Gh-Hg)\gamma/G\}\beta,$$

which shows (i) that B is a triple point of the second kind on the section; (ii) the cuspidal tangent at the triple point is the tangent to the nodal conic; (iii) the line $Gp\alpha+(Gh-Hg)\gamma=0$ is the tangent at the ordinary branch.

273. *Each of Kummer's five cones passes through the four pinch points; also the tangent planes to each cone along the generators, which pass through a pinch point, contain the ordinary tangent at the pinch point.*

If A be the vertex of one of Kummer's cones, the equation of the quartic must be of the form

$$\alpha^2u_2+(\alpha^2+\alpha U_1+U_2)^2=0\quad\text{.................(16)},$$

where the U's and u_2 are given by (13). The form of (16) shows that the cone u_2 passes through the pinch points.

The tangent plane along the generator of u_2 which passes through B is

$$h\gamma+g\delta=0\quad\text{.......................(17)},$$

and the tangent plane at B is

$$P\alpha+H\gamma+G\delta=0\quad\text{....................(18)},$$

and if δ be eliminated between (16) and (18), the ordinary tangent at B is given by the intersection of (18) and the plane

$$Gh\gamma-g(P\alpha+H\gamma)=0\quad\text{.................(19)}.$$

Eliminating α between (18) and (19) it follows that this line lies in the plane (17).

274. *There are* 40 *triple tangent planes*.*

Let TP, TQ be a pair of lines lying in the same plane, P and Q the points where they cut the nodal conic; then the section consists of the two lines and a conic passing through P and Q and intersecting TP, TQ in p and q. Hence the plane is a triple tangent plane which touches the surface at T, p and q. Now the first six columns of the table of § 268 furnish 25 planes which contain a pair of lines, columns 7 to 11 furnish 13 more and

* Berzolari, *Ann. di Mat.* Serie II. vol. XIV. p. 31.

columns 12 to 16 furnish 2; accordingly the total number of planes is 40.

275. *There are 52 planes whose point of contact is a tacnode on the section.*

The equation of the surface may be written $\alpha^2 W = V^2$, where W is one of Kummer's cones. Let O be the vertex of the cone W; P and Q the points where any generator touches the quartic; then these will be the points where the generator intersects the quadric V, also the tangent plane to W along OPQ will be a double tangent plane to the quartic surface. Now P and Q will coincide when OPQ is a generator of the tangent cone from O to the quadric V; and since this cone and the cone W have four common generators, there will be four planes ϖ_5 corresponding to each of Kummer's five cones. Hence the total number due to this cause is 20.

Let AB be one of the 16 lines, then the equation of the quartic is given by (6). Let $\delta = k\gamma$ be any plane through AB, then the section consists of this line and the cubic curve

$$\alpha^2(\alpha v_1' + \beta w_1' + \gamma w_2') + \nu\alpha\beta(P\beta + Q\gamma + Rk\gamma) + \nu^2\beta^2\gamma = 0,$$

where v' &c. denote what v &c. become when $\gamma = 1$, $\delta = k$. The line AB intersects this cubic curve at B, and in two points p and q which are determined by the equation

$$\alpha^2 v_1' + \alpha\beta w_1' + P\nu\beta^2 = 0 \quad\text{.................}(20),$$

and the condition that p and q should coincide is $w_1'^2 = 4P\nu v_1'$, which is a quadratic equation for determining k. This shows that each line gives rise to two planes ϖ_5, so that the 16 lines produce 32 planes, making a total of $20 + 32 = 52$.

276. Equation (20) shows that every tangent plane to the quartic along one of the 16 lines is a double tangent plane; hence these lines form part of the bitangential curve. Also the five curves of contact of each of Kummer's cones, which are quartic curves, form part of the bitangential curve, which together with the 16 lines make up a curve of degree 36. It will be shown in Chapter IX that the degree of the complete bitangential curve on a quartic is 320, and therefore the degree of the residual curve is $320 - 36 = 284$. Moreover if P and Q are the points of contact of a double tangent plane, it follows from what has preceded that PQ is a generator of one of Kummer's cones; hence the residual curve consists of the nodal conic repeated 142 times.

277. Cremona* has shown that it is possible to transform a quartic having a nodal conic into an anautotomic cubic surface and *vice versâ* by means of the equations

$$\frac{\alpha}{\alpha'^2} = \frac{\beta}{\alpha'\beta'} = \frac{\gamma}{\alpha'\gamma'} = \frac{\delta}{\beta'\delta' - \gamma'^2} \quad \text{..............(21)},$$

which are equivalent to

$$\frac{\alpha'}{\alpha\beta} = \frac{\beta'}{\beta^2} = \frac{\gamma'}{\beta\gamma} = \frac{\delta'}{\alpha\delta + \gamma^2} \quad \text{..................(22)}.$$

Let AD be one of the 27 lines lying in the cubic; draw any plane through AD, then the section of the cubic will consist of the line AD and a conic, which intersects AD in A and D; let CA, CD be the tangents to the conic at A and D; and let B be any arbitrary point. Then the equation of the cubic will be

$$\beta(\delta^2 + \delta u_1 + u_2) + (\alpha\delta + \gamma^2)\gamma = 0 \quad \text{...........(23)},$$

where $u_n = (\alpha, \beta, \gamma)^n$. Transforming (23) by means of (21), it becomes

$$\alpha'^2(u_2' + \gamma'\delta') + \alpha' u_1' \Omega' + \Omega'^2 = 0 \quad \text{...........(24)},$$

where $\Omega' = \beta'\delta' - \gamma'^2$; which is the equation of a quartic having a nodal conic $\alpha' = 0$, $\Omega' = 0$.

(i) Since the plane ABD is any plane through AD, let us choose it so that it contains one of the pairs of lines which cut AD; then we must have

$$u_1 = f\alpha + g\beta + h\gamma, \quad u_2 = pu_1^2 + \gamma(F\alpha + G\beta + H\gamma).$$

Substituting these values in (23), and putting $\gamma = 0$, it becomes

$$\beta(\delta^2 + \delta v + pv^2) = 0 \quad \text{....................(25)},$$

where $v = f\alpha + g\beta$; and the second factor gives the equation of one pair of lines which intersect AD. Transforming the factor by means of (21), it becomes

$$p\alpha'^2 v'^2 + \alpha'\beta'\delta' v' + \beta'^2\delta'^2 = 0 \quad \text{..............(26)},$$

which represents a pair of conics.

The form of (24) shows that the point D' lies on the nodal conic, and that the section of the quartic by the plane γ' consists of the pair of conics (26). Hence the 10 straight lines on the cubic which intersect AD transform into 10 conics on the quartic, which lie in five planes passing through D'.

(ii) The remaining 16 lines which lie in the cubic must pass through the conic $\beta = 0$, $\alpha\delta + \gamma^2 = 0$. Let us therefore take B as

* *Rend. Ist. Lombardo*, 1871; Geiser, *Crelle*, LXX.

the point where two of them intersect; then $u_2 = \beta v_1 + v_2$, where $v_n = (\alpha, \gamma)^n$; also these lines must be generators of the cone and the quadric

$$\alpha\delta + \gamma^2 = 0,$$

$$\delta^2 + \delta u_1 + \beta v_1 + v_2 = 0,$$

whence eliminating δ, we obtain

$$\gamma^4 - \alpha\gamma^2 u_1 + \alpha^2(\beta v_1 + v_2) = 0 \quad \text{..............(27).}$$

Equation (27) is that of a quartic cone whose vertex is D, on which DB is a triple generator; hence the constants must be determined so that (27) degrades into a pair of planes and a quadric cone which pass through DB; but it will not be necessary to work out the necessary conditions, because (21) transforms (27) into itself, and therefore a pair of intersecting straight lines on the cubic which pass through the conic, transforms into a pair of intersecting straight lines on the quartic which pass through the nodal conic. This pair of lines on the quartic lie in the plane δ', for if we put $\delta' = 0$ in (24) it reduces to (27). We thus obtain the theorem:

If a cubic surface pass through the conic $\beta = 0$, $\alpha\delta + \gamma^2 = 0$, *and is not touched at D by the plane* α; *equations* (21) *transform the cubic into a quartic having a nodal conic whose equations are* $\alpha' = 0$, $\beta'\delta' - \gamma'^2 = 0$. *The* 16 *lines which intersect the conic through which the cubic passes transform into the* 16 *lines on the quartic; the* 10 *lines which intersect the line AD on the cubic transform into* 10 *conics, which form five pairs lying in five planes passing through the point D′ on the nodal conic; and the line AD on the cubic transforms into the point D′ on the quartic.*

278. The theory of quartics furnishes the following theorem with respect to cubics:

Let a plane cut a cubic surface in any line AD and a conic S. Then (i) 16 *lines pass through S;* (ii) *each of these* 16 *lines is intersected by five others which pass through S and five which pass through AD;* (iii) *of the first set no two lie in the same plane, and the same is true of the second set, but a plane can be drawn through any line of the first set and one line of the second set;* (iv) *when two lines passing through S intersect, the residual intersection of the plane and the cubic is a line passing through AD.*

Cuspidal Conics.

279. The equation of a quartic surface having a nodal conic is given by (4) of § 266, where $U = \alpha^2 u_0 + \alpha u_1 + u_2$; and the pinch points are the four points of intersection of the nodal conic with the quadric cone

$$U_1^2 = u_2.$$

But if the conic is cuspidal every point must be a pinch point, which requires that $U_1^2 - u_2 = k\Omega$, and (4) is reducible to the form

$$\alpha^3 u + U^2 = 0 \qquad \text{.........................(1)},$$

where u is a plane and U a quadric surface.

The quadric U has tritactic contact with the surface along the cuspidal conic, and intersects it in a conic along which the quartic is touched by the plane u; hence u is a *conic trope*. Let C and D be the points where the trope intersects the cuspidal conic; then

$$\left.\begin{array}{l} u = l\alpha + m\beta, \quad U_1 = P\beta + Q\gamma + R\delta \\ \qquad\qquad U_2 = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma \end{array}\right\} \qquad \text{...........(2)},$$

and (1) becomes

$$\alpha^3 (l\alpha + m\beta) + (\alpha U_1 + U_2)^2 = 0 \qquad \text{..............(3)}.$$

Since (3) may be written in the form

$$\beta^2 u^2 + \beta u \Omega_1 + \Omega_2 = 0,$$

it follows that C and D are tacnodal points. These are the only singular points on the cuspidal conic; also the form of (3) shows that the cuspidal tangent planes envelope a quadric cone, whose vertex is the pole of α with respect to the quadric $\alpha U_1 + U_2 = 0$.

280. *Every plane passing through the tangent to the cuspidal conic at the tacnodal points cuts the surface in a quartic curve having a tacnode cusp* thereat, the tangent at which is the tangent to the cuspidal conic.*

The equation of any plane through the tangent at C to the conic is

$$\lambda\delta + \nu\beta = k\alpha \qquad \text{.........................(4)},$$

whence eliminating δ between (3) and (4) we obtain

$$\alpha^3 (l\alpha + m\beta) + [\alpha \{P\beta + Q\gamma + R(k\alpha - \nu\beta)/\lambda\} + k\alpha\gamma + \mu\beta(k\alpha - \nu\beta)/\lambda]^2 = 0 \quad \text{...(5)},$$

* The equations of a quartic curve having tacnodes &c. are given in § 179.

which is the equation of a quartic curve having a tacnode cusp at C, and $\alpha = 0$ is the cuspidal tangent.

281. *Every plane through the tangent to the conic of contact at the tacnodal points cuts the surface in a quartic curve having a rhamphoid cusp at this point, whose cuspidal tangent is the tangent to the conic.*

Let the conic trope cut AB in B', and let $\alpha' = l\alpha + m\beta$; then changing the tetrahedron to $AB'CD$, the equation of the surface becomes

$$\alpha'^3 (\alpha' - m\beta)^3 + l\{(\alpha' - m\beta) U_1 + l U_2\}^2 = 0 \quad \text{......(6)}.$$

The equation of any plane through the tangent at C to the conic of contact is

$$l(\lambda\delta + \nu\beta) - mQ\beta = k\alpha' \quad \text{..................(7)},$$

whence proceeding as before the equation of the section will be found to be

$$\alpha' (\alpha' - m\beta)^3 + l\{(Q + k)\alpha'\gamma + L\alpha'^2 + M\alpha'\beta + N\beta^2\}^2 = 0,$$

which is the equation of a quartic having a rhamphoid cusp at C.

282. *The tangent planes at the tacnodal points cut the surface in two quartettes of straight lines. These eight straight lines may be divided into four pairs, such that each pair lies in a plane passing through both the tacnodal points.*

The tangent plane to the surface at C is

$$Q\alpha + \nu\beta + \lambda\delta = 0 \quad \text{........................(8)},$$

and the section of (3) by it consists of the four straight lines

$$\alpha^3 (l\alpha + m\beta) + \{P\alpha\beta - (R\alpha + \mu\beta)(Q\alpha + \nu\beta)/\lambda\}^2 = 0 \quad \text{...(9)},$$

and the section of the surface by the tangent plane at D, which is

$$R\alpha + \mu\beta + \lambda\gamma = 0 \quad \text{.....................(10)},$$

consists of four straight lines which are the intersections of (9) and (10).

283. *There are three Kummer's cones, whose vertices lie on the line of intersection of the tangent planes to the quartic at the tacnodal points; also each cone touches these planes and also passes through the cuspidal conic.*

Let C and D be the tacnodal points, A the vertex of one of Kummer's cones, then the equation of the quartic may be written in the form

$$l\alpha^3 (l\alpha + 2m\beta) + (\alpha U_1 + \Omega)^2 = 0 \quad \text{..............(11)}$$

or

$$\alpha^2(l^2\alpha^2 + 2lm\alpha\beta - 2\lambda\alpha U_1 - 2\lambda\Omega - \lambda^2\alpha^2) + (\lambda\alpha^2 + \alpha U_1 + \Omega)^2 = 0 \quad \ldots(12).$$

Since A is the vertex of one of Kummer's cones, it follows from § 267 that $l = \lambda$, $U_1 = m\beta$, which reduces (12) to

$$2l\alpha^2\Omega - (l\alpha^2 + m\alpha\beta + \Omega)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(13),$$

and shows that the cone Ω, which contains the cuspidal conic, is one of Kummer's cones. Also if

$$\Omega = L\beta^2 + \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma,$$

the tangent planes at C and D are

$$\lambda\delta + \nu\beta = 0, \quad \lambda\gamma + \mu\beta = 0,$$

which pass through A. These planes obviously touch the cone Ω. The other two cones are obtained by equating to zero the discriminant of the quadric

$$(l^2 - \lambda^2)\alpha^2 + 2m(l - \lambda)\alpha\beta - 2\lambda\Omega = 0,$$

which will be found to furnish a cubic equation for λ, one of whose roots is $\lambda = l$, which corresponds to the cone Ω.

Since Kummer's cones pass through the cuspidal conic, they are not strictly speaking double tangent cones; for each generator is an ordinary tangent at one point, but the contact at the other point is of the same character as that of a line passing through a double point.

284. It is known from the theory of plane quartic curves, that every bicuspidal quartic is of the form (1), where α is the line passing through the cusps, u the double tangent, and U a conic which has tritactic contact with the curve at each cusp and also passes through the points of contact of the double tangent. Now every quartic curve of this species can be projected into an oval of Descartes in which the cusps are at the circular points; and in like manner every quartic having a cuspidal conic can be projected into the surface formed by the revolution of an oval of Descartes about its axis, in which case the cuspidal conic is the imaginary circle at infinity.

The equation of the surface is

$$(x^2 + y^2 + z^2 + ax + c^2)^2 + Ax + B = 0 \quad \ldots\ldots\ldots\ldots(14),$$

and if this be transformed to quadriplanar coordinates, it becomes

$$(\beta^2 + \gamma^2 + \delta^2 + \alpha\beta + \alpha^2)^2 + (A\beta + B\alpha)\alpha^3 = 0 \quad \ldots\ldots(15).$$

The form of (14) shows that the triangle BCD is self-conjugate to the cuspidal conic; whence if C' and D' are the points where CD cuts the conic, and we change the tetrahedron to $ABC'D'$ by writing $\gamma' = \gamma + \iota\delta$, $\delta' = \gamma - \iota\delta$, (15) becomes

$$(\beta^2 + \gamma'\delta' + \alpha\beta + \alpha^2)^2 + (A\beta + B\alpha)\,\alpha^3 = 0 \quad\text{.........(16).}$$

The points C', D' are the tacnodal points, and the tangent planes thereat intersect in the line AB; hence the axis of x in (14) is the line of intersection of the tangent planes at the tacnodal points.

285. Since an oval of Descartes possesses eight stationary tangents which intersect in four pairs on the axis of x, it follows that:

Every quartic which possesses a cuspidal conic has four stationary tangent quadric cones, that is to say cones whose generators have tritactic contact with the surface; also the vertices of these cones lie on the line of intersection of the tangent planes at the tacnodal points.

This result affords a verification of Cayley's theorem § 58 for the degree of the spinodal curve; for the latter consists of the cuspidal conic 11 times repeated, the conic of contact of the four cones and the conic of contact of the trope, which together make up a curve of the 32nd degree. It also follows from § 55 that each of the eight lines lying in the surface must touch these conics of contact.

286. Again the eight points of inflexion of an oval of Descartes lie on a circular cubic, whence: *A cubic surface can be described which intersects a quartic with a cuspidal conic in the curves of contact of these four cones and also in the cuspidal conic.*

287. In the same way the existence of the three quasi-Kummer's cones can be proved. An oval of Descartes is a curve of the sixth class, consequently three tangents can be drawn from each cusp, which intersect in pairs on the axis of x, hence:

The quartic surface possesses three tangent quadric cones which also pass through the cuspidal conic; also the vertices of these cones lie on the line of intersection of the tangent planes at the tacnodal points.

288. Quartics with a cuspidal conic may have one other double point, which may either be a conic node or a binode. In the former case the quartic is of the fourth class and may be projected into the surface formed by the revolution of a limaçon about its axis of symmetry. In the latter case the quartic is of the third class, and may be projected into the surface formed by the revolution of a cardioid or of a three-cusped hypocycloid about such an axis.

Two intersecting Double Lines.

289. *Two intersecting Nodal Lines.* Let the nodal conic degrade into the two lines BC and BD; then the equation of the surface is

$$\alpha^2 U + 2\alpha\gamma\delta U_1 + \gamma^2\delta^2 = 0 \quad \text{.................(1)},$$

where

$$\left.\begin{aligned} U &= \alpha^2 u_0 + \alpha u_1 + u_2 \\ u_1 &= p\beta + q\gamma + r\delta \\ U_1 &= P\beta + Q\gamma + R\delta \\ u_2 &= F\beta^2 + G\gamma^2 + H\delta^2 + f\gamma\delta + g\delta\beta + h\beta\gamma \end{aligned}\right\} \text{.........(2)}.$$

The pinch points are the intersections of the nodal lines with the quadric

$$U_1^2 = U \text{.............................(3)},$$

and there are four of them, one pair lying in each line.

Equation (1) may also be written in the form

$$F\beta^2\alpha^2 + \beta\alpha\Omega_2 + \Omega_4 = 0,$$

where $\Omega_n = (\alpha, \gamma, \delta)^n$; which shows that B is a peculiar kind of pinch point, the tangent plane at which is a factor of the coefficient of β. The line AB may be any arbitrary line through B, and therefore the plane $\delta = k\gamma$ is any arbitrary plane through B, hence: *The section of the quartic by any plane through the point of intersection of the nodal lines has a tacnode thereat.*

290. *The surface possesses* 16 *lines lying in it, of which eight intersect one of the nodal lines and the remaining eight intersect the other. Also every line of one system is intersected by four lines of the other.*

Let AD be one of the lines; then $u_0 = r = H$; and by proceeding as in § 268 it can be shown that there are seven additional lines which intersect BD.

Writing $\beta = \lambda\gamma$ in (1), it becomes

$$\alpha^2 \{(p\lambda + q)\alpha + (f + g\lambda)\delta\} + \gamma \{\alpha^2 (F\lambda^2 + h\lambda + G) + 2\alpha\delta (P\lambda + Q) + \delta^2\} = 0,$$

showing that the section consists of AD and a cubic, whose node is the point C' where the section intersects BC. The condition that the cubic should degrade into a conic and a line through C' is obtained by eliminating α and δ between the coefficients of α^2 and γ in this equation, which furnishes a quartic equation for λ.

The surface possesses only four Kummer's cones.

291. *A Nodal and a Cuspidal Line.* When BC is a cuspidal line, every point on it must be a pinch point; hence BC must be a generator of the quadric (3), the conditions for which are

$$F = P^2, \quad G = Q^2, \quad h = 2PQ \quad \ldots\ldots\ldots\ldots\ldots(4),$$

and (1) becomes

$$\alpha^2 \{\alpha^2 u_0 + \alpha u_1 + (H\delta + f\gamma + g\beta)\delta\} + 2R\alpha\gamma\delta^2 + (P\alpha\beta + Q\alpha\gamma + \gamma\delta)^2 = 0 \quad \ldots(5).$$

Putting $\alpha = \gamma = 0$, (3) now becomes

$$\{(g - 2PR)\beta + (H - R^2)\delta\} = 0,$$

which shows that one of the pinch points coincides with B, so that there is only one distinct pinch point on BD.

292. *The cuspidal line possesses two tacnodal points, through each of which a pair of straight lines can be drawn which lie in the surface.*

Let C be one of these points, ABC the tangent plane; then $Q = q = 0$, and (5) becomes

$$\alpha^2 \{\alpha^2 u_0 + \alpha (p\beta + r\delta) + (H\delta + f\gamma + g\beta)\delta\} + 2R\alpha\gamma\delta^2 + (P\alpha\beta + \gamma\delta)^2 = 0 \quad \ldots(6).$$

To find the other point, let $\beta = k\gamma$, and change the tetrahedron to $ABC'D$; then the required condition is

$$k(p - Pf + 2RP^2k) = 0,$$

and since the root $k = 0$ corresponds to C', there is one other tacnodal point.

To prove the second part, put $\delta = 0$ in (6) and it reduces to

$$\alpha^2 (\alpha^2 u_0 + p\alpha\beta + P^2\beta^2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

which shows that two straight lines can be drawn through C, and similarly for the other tacnodal point.

293. *There are four straight lines lying in the surface which intersect the nodal line; and each of these lines intersects one of the lines which pass through the cuspidal line.*

Let AD be one of these lines; then $u_0 = r = H = 0$, and (6) and (7) become

$$\alpha^2 (p\alpha\beta + f\gamma\delta + g\beta\delta) + 2R\alpha\gamma\delta^2 + (P\alpha\beta + \gamma\delta)^2 = 0 \quad \ldots(8),$$

and $$\alpha^2\beta\,(p\alpha + P^2\beta) = 0,$$

which shows that the line AC through the point C intersects AD. Putting $\gamma = 0$ in (8), it becomes

$$\alpha^2\beta\,(p\alpha + g\delta + P\beta) = 0,$$

which gives the other line lying in this plane; and if in (8) we put $\gamma = k\alpha$, the discriminantal equation will furnish one other value of k, showing that there are two other lines.

294. *The section of the surface by any plane through the point of intersection of the two lines, has a rhamphoid cusp thereat.*

The section by the plane $\delta = k\gamma$ is

$$(P\alpha\beta + k\gamma^2)^2 + \alpha\,(\alpha,\ \beta,\ \gamma)^3 = 0,$$

which shows that B is a rhamphoid cusp.

295. *Two Cuspidal Lines.* The line BD must also be a generator of (3), which involves the additional equations $H = R^2$ $g = 2PR$; whence the equation of the surface becomes

$$\alpha^2 (\alpha^2 u_0 + \alpha u_1 + f\gamma\delta) + (\alpha U_1 + \gamma\delta)^2 = 0 \quad \ldots\ldots\ldots\ldots(9).$$

Each line has one tacnodal point lying in it; and if C and D be these points (9) becomes

$$\alpha^2 \{\alpha^2 u_0 + p\alpha\beta + f(Q\gamma + R\delta)\,\alpha + f\gamma\delta\} + (\alpha U_1 + \gamma\delta)^2 = 0 \ldots(10),$$

and the section of the surface by β, which may be any plane through C and D, consists of a pair of conics which touch one another at these points. Also the tangent planes at C and D are $Q\alpha + \delta = 0$ and $R\alpha + \gamma = 0$.

296. *The tangent plane at each tacnodal point intersects the surface in a pair of straight lines; each line of one system intersects one line of the other; also the two points of intersection of the lines lie in a line passing through the point of intersection of the cuspidal lines.*

The section of the surface by the tangent planes at C and D are both represented by the equation

$$(P\beta - QR\alpha)^2 + (u_0 - QRf)\,\alpha^2 + p\alpha\beta = 0 \quad\text{.........(11)},$$

which gives the projection on the plane ABC of the two lines; and since they are identical, each line of one system intersects each line of the other system in a line passing through B.

297. *The section of the surface by an arbitrary plane through the point of intersection of the cuspidal lines has an oscnode thereat.*

Let $V = (Q + Rk)\,\alpha + k\gamma$; then the section of the surface by the plane $\delta = k\gamma$ may be put into the form

$$(P\alpha\beta + V\gamma)^2 + \alpha^3\left\{(u_0 - \tfrac{1}{4}f^2)\,\alpha + (p - Pf)\,\beta\right\} = 0,$$

which represents a quartic having an oscnode at B, and the line of intersection of the plane, with that containing the cuspidal lines, is the oscnodal tangent.

298. It is possible for the singular lines to coincide, in which three respective cases the surface will possess a tacnodal, a rhamphoid cuspidal, and an oscnodal line. These have been considered in § 214; and in § 191 we have explained why it is that an oscnode can be formed by the union of two cusps, although its point constituents are three nodes.

Cyclides.

299. When the nodal conic is the imaginary circle at infinity, the quartic is called a *cyclide*. The cyclide with four additional conic nodes was first studied by Dupin*, and is usually called Dupin's cyclide; but the subject was afterwards taken up by Casey† in a paper, which contains an extension to quartic surfaces of his previous investigations on bicircular quartic curves‡. The reader may easily adapt the investigations of Chapter IX of my treatise on *Cubic and Quartic Curves* to cyclides, and I shall therefore deal briefly with the subject.

* *Applications de Géometrie*, 1822.

† *Phil. Trans.* CLXI. (1871), p. 585. See also Clerk-Maxwell, *Scientific Papers*, vol. II. p. 144; and *Quart. Jour.* 1867.

‡ *Trans. Roy. Irish Acad.* vol. XXIV. p. 457.

300. *If OT be the perpendicular from any fixed point O on to the tangent plane at any point Q of a fixed quadric surface; and if two points P, P' be taken on OT such that $TP = TP'$, and*

$$OT^2 - PT^2 = \delta^2,$$

where δ is a constant, the locus of P and P' is a cyclide.

When the fixed quadric is an ellipsoid or a hyperboloid, the imaginary circle at infinity is nodal; when the quadric is a sphere, the circle is cuspidal; and when the quadric is a paraboloid, the surface degrades into a cubic which contains the imaginary circle at infinity.

Let EY be the perpendicular from the centre E of the quadric on to the tangent plane at any point Q. Let (f, g, h) be the coordinates of O referred to E; (x, y, z) those of P referred to O. Let $OP = r$, $EY = p$, and let (λ, μ, ν) be the direction cosines of EY.

Then
$$OT = p - f\lambda - g\mu - h\nu,$$
$$PT = r - OT,$$
$$\delta^2 = 2rOT - r^2,$$
whence
$$r^2 + 2r(f\lambda + g\mu + h\nu) + \delta^2 = 2rp \quad \text{...........(1).}$$

(i) When the quadric is central

$$p^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2,$$

and (1) becomes

$$(r^2 + 2fx + 2gy + 2hz + \delta^2)^2 = 4(a^2x^2 + b^2y^2 + c^2z^2) \quad \text{...(2).}$$

(ii) When the quadric is a sphere, $a = b = c$; and (2) may be put into the form

$$(r^2 + 2fx + 2gy + 2hz + \delta^2 - 2a^2)^2$$
$$= 4a^2(a^2 - 2fx - 2gy - 2hz - \delta^2) \quad \text{...(3),}$$

which is the equation of a quartic having a cuspidal conic.

(iii) When the quadric is the paraboloid

$$y^2/l + z^2/m = 2x,$$

then
$$2p\lambda + l\mu^2 + n\nu^2 = 0,$$

and (1) becomes

$$x(r^2 + 2fx + 2gy + 2hz + \delta^2) + ly^2 + mz^2 = 0 \quad \text{......(4),}$$

which represents a cubic surface passing through the imaginary circle.

301. *A cyclide is the envelope of a variable sphere, whose centre moves on a fixed quadric called the focal quadric, and which cuts a fixed sphere orthogonally.*

The moving sphere is called the generating sphere, and the fixed sphere is the sphere of inversion; and the theorem may be proved in the same manner as the corresponding one for bicircular quartic curves. See *Cubic and Quartic Curves*, § 205.

302. *There are five centres of inversion, which are the vertices of Kummer's five cones.*

In (2) the origin is the centre O of the fixed sphere, and if (2) be inverted with respect to O, the inverse surface is another cyclide; hence O is one centre of inversion. Also the form of (2) shows that the surface $a^2x^2 + b^2y^2 + c^2z^2 = 0$ is one of Kummer's cones, and that its vertex is O; and since there are five of such cones, the surface has five centres of inversion.

A surface which is its own inverse with respect to a point is called by Darboux an *anallagmatic* surface.*

303. *When the sphere of inversion touches its corresponding focal quadric, the point of contact is a node on the cyclide.*

Let R be the point of contact; (ξ, η, ζ) the coordinates of R referred to E. Then

$$\xi = f + \delta\lambda, \quad \eta = g + \delta\mu, \quad \zeta = h + \delta\nu,$$
$$a^2\lambda = p\xi, \quad b^2\mu = p\eta, \quad c^2\nu = p\zeta,$$
$$p = f\lambda + g\mu + h\nu + \delta.$$

Substituting in (2) it becomes

$$(r^2 + 2x\xi + 2y\eta + 2z\zeta + 2p\delta)^2 = 4\{a^2x^2 + b^2y^2 + c^2z^2 + 2p\delta(x\xi + y\eta + z\zeta) + p^2\delta^2\}.$$

The terms of lowest dimensions are

$$x^2(\xi^2 + p\delta - a^2) + y^2(\eta^2 + p\delta - b^2) + z^2(\zeta^2 + p\delta - c^2) + 2yz\eta\zeta + 2zx\zeta\xi + 2xy\xi\eta = 0,$$

which shows that R is a conic node on the surface. The condition

* α privat. ἄλλαγμα = that which is given or taken in exchange. Inversion usually changes a surface into a different one, but this does not occur when these surfaces are inverted with respect to the five centres of inversion.

that R should be a binode is that the discriminant of the nodal cone should vanish, which furnishes the equation

$$\frac{\xi^2}{p\delta - a^2} + \frac{\eta^2}{p\delta - b^2} + \frac{\zeta^2}{p\delta - c^2} = 0.$$

304. The following theorem furnishes a method of generating nodal cyclides.

The inverse of a quadric surface with respect to an arbitrary point O is a uninodal cyclide. That of a quadric cone is a binodal cyclide, whose second node is the inverse point of the vertex of the cone. That of a quadric of revolution is a trinodal cyclide. That of a circular cone or cylinder is a quadrinodal cyclide.

Uninodal Cyclides. If $u_n = (x, y, z)^n$, the equation of any quadric surface referred to an arbitrary origin O is

$$(k^2 + u_1)^2 = u_2 \quad \text{......................(5)},$$

whence inverting with respect to O, (5) becomes

$$(r^2 + u_1)^2 = u_2 \quad \text{......................(6)},$$

which is the equation of a uninodal cyclide, whose node is at O.

These cyclides are also the pedals of central quadrics with respect to an arbitrary point O, whose coordinates referred to its centre are (f, g, h). For if r and p be the perpendiculars from O and the centre on to the tangent plane at any point P,

$$r + \lambda f + \mu g + \nu h = p,$$

whence $$(r^2 + fx + gy + hz)^2 = a^2x^2 + b^2y^2 + c^2z^2,$$

which is of the form (6).

Since ordinary inversion is a special case of the quadric transformation in the Theory of Birational Transformation, it follows that any quartic having a node and a nodal conic can be transformed into a quadric surface and *vice versâ*. Let the quadric be

$$(\alpha + u_1)^2 = u_2 \quad \text{......................(7)},$$

and employ the equations

$$\alpha\alpha'/\Omega' = \beta/\beta' = \gamma/\gamma' = \delta/\delta',$$

and (7) becomes $$(\alpha' u_1' + \Omega')^2 = \alpha'^2 u_2',$$

which is the equation of the quartic in question.

305. *Transformation of a Surface into a Plane.* The preceding theorem is a very simple example of a general theory, which has been studied by Cremona and various other geometers, and consists in ascertaining—*what surfaces are capable of being transformed into a plane, in such a manner that each point on the surface corresponds to a single point on the plane and vice versâ?*

A quartic having a conic node at an arbitrary point and a nodal conic in an arbitrary plane can be birationally transformed into a central quadric, and if the centre be taken as the origin, the Cartesian equation of the latter is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. In this write $x/a = x'$, &c., and it becomes a sphere; and if the latter be inverted with respect to a point on its surface, it becomes a plane. Hence the quartic in question can be transformed into a plane, by means of an algebraic system of equations between the coordinates of any point on the surface and the corresponding point on the plane; also every curve on the surface can be similarly transformed into a plane curve. Surfaces of this character are called *unicursal* by Cayley and *homaloidal** by Cremona; the algebraic relation between the two sets of coordinates is called a *birational* or *Cremonian*† transformation; and the condition that such a relation should be possible is, that the coordinates of any point on the surface to be transformed should be expressible as rational functions of two parameters θ and ϕ.

When a surface is not homaloidal, its coordinates are usually expressible in terms of two parameters by means of elliptic or other transcendental functions; but this branch of the subject cannot be studied without a knowledge of the Theory of Functions.

306. *Binodal Cyclides.* Since the inverse of a conic node is another conic node situated at the inverse point, it follows that when the cyclide is binodal the quadric must be a cone. Let the vertex of the cone be the point $x = f$, then its equation is

$$(x-f)^2 + Gy^2 + Hz^2 + 2F'yz + 2(x-f)(gy+hz) = 0 \quad \text{...(8)},$$

the inverse of which with respect to the origin is of the form

$$(r^2 - fx - gy - hz)^2 = Ay^2 + Byz + Cz^2 \quad \text{.........(9)}.$$

The planes $Ay^2 + Byz + Cz^2 = 0$ are conic tropes, which touch the cyclide along a pair of circles intersecting at the two nodes.

* ὁμαλός = level; εἶδον = appeared.

† Cayley, *Proc. Lond. Math. Soc.* vol. III. p. 171; *C. M. P.* vol. VII. p. 189; Cremona, *Gött. Nach.* 1871; *Math. Ann.* vol. IV.; *Rend. Ist. Lombardo*, 1871; *Ann. di Mat.* vol. V.; *Acc. Bologna*, 1871–2; Nöther, *Math. Ann.* vol. III.

By § 29, it follows that when a surface possesses a conic trope there are in general $2(n-1)$ conic nodes lying in the conic of contact, which in the case of a quartic equals 6. The curves of contact of the two tropes intersect at the two conic nodes on the surface; and to find the position of the other four, the simplest course is to transform (9) to quadriplanar coordinates, and it becomes

$$(\alpha u_1 + p\beta^2 + \beta v_1 + s\gamma\delta)^2 = \alpha^2\gamma\delta \quad \text{..............(10)},$$

where γ and δ are the two tropes, C is one of the points where the plane γ cuts the nodal conic and D is one of the points where δ cuts this conic. The point C is a pinch point on the nodal conic, and the other point C' where γ intersects the conic is another pinch point; hence each of these two pinch points are equivalent to two conic nodes on the conic of contact, which makes 6.

307. A special case arises when the tropes intersect in a line lying in the plane of the nodal conic; for the equation of the surface must be of the form

$$(r^2 + 2fx + 2gy + 2hz)^2 + (x-a)(x-b) = 0,$$

whence transferring the origin to the centre of the sphere, this becomes

$$(r^2 + c^2)^2 + (x-A)(x-B) = 0 \quad \text{...........(11)},$$

which is a surface of revolution, whose meridian curve is a hemi-symmetrical bicircular quartic curve. Since these curves possess six pairs of stationary tangents which intersect on the axis of x, the surface possesses six stationary quadric tangent cones whose vertices lie in a straight line. The curve also possesses a pair of double tangents perpendicular to the axis of x, and three pairs which intersect on that axis; hence Kummer's cones consist of the two tropes each repeated twice, and three cones whose vertices lie on the line passing through the vertices of the stationary tangent cones.

When (11) is transformed to quadriplanar coordinates, we must recollect that the point A, the line AB and the plane ACD are determinate, but the planes γ and δ may be any planes through AB. We shall therefore choose them so as to pass through the two points where the tropes intersect the nodal conic. Hence (11) becomes

$$(\beta^2 + \gamma\delta + \alpha^2)^2 + (A\alpha - \beta)(B\alpha - \beta)\alpha^2 = 0,$$

which shows that the points C and D are *tacnodal* points on the nodal conic.

308. *Trinodal cyclides.* When a cyclide is trinodal, it must possess a pair of tropes, and two of the nodes are situated at the points of intersection of the circles of contact, whilst the third node is isolated. If therefore the cyclide be inverted with respect to the third node, the surface becomes a quadric, and the two tropes become a pair of spheres passing through the third node, which touch the quadric along a pair of circles. Hence the quadric must be one of revolution.

Let $u = fx + gy + hz$, then the equation of the quadric is

$$ax^2 + b(y^2 + z^2) + u + 1 = 0,$$

the inverse of which is

$$r^4 + k^2r^2(u + bk^2) + k^4(a - b)x^2 = 0,$$

which may be written in the form

$$\{2r^2 + k^2(u + bk^2)\}^2 + k^4\{4(a - b)x^2 - (u + bk^2)^2\} = 0 \ldots (12),$$

and the last term equated to zero gives the two tropes.

309. Equation (12) includes several well-known surfaces. When $u = fx$, the third node is situated on the axis of revolution, and the meridian curve is a hemisymmetrical trinodal bicircular quartic; and the surfaces include those formed by the revolution of the lemniscate, the limaçon, the cardioid and the three-cusped hypocycloid about an axis of symmetry. In the case of the lemniscate the imaginary circle is a biflecnodal one, and the origin is a biflecnodal point. The quadriplanar equation is of the form $\alpha^2 u_2 + u_4 = 0$, which shows that the eight common generators of the cones u_2 and u_4 lie in the surface; whilst in the case of the three last surfaces the imaginary circle is cuspidal.

310. *Quadrinodal Cyclides.* In this case the quadric is a circular cone, and if (f, g, h) be the coordinates of its vertex, its equation is

$$A(x - f)^2 + B(y - g)^2 + B(z - h)^2 = 0 \ldots\ldots\ldots (13),$$

the inverse of which is

$$\{Af^2 + B(g^2 + h^2)\}r^4 - 2r^2(Afx + Bgy + Bhz) + Ax^2 + By^2 + Bz^2 = 0 \quad \ldots (14).$$

One of the nodes is at the origin, the second is the vertex of the cone, and the two others are the points of intersection of the circles of contact of the two tropes

$$(Af^2 + Bg^2 + Bh^2)(A - B)x^2 - (Afx + Bgy + Bhz - \tfrac{1}{2}B)^2 = 0 \quad \ldots(15).$$

311. This cyclide is usually known as *Dupin's cyclide*, by whom it was first studied. It may also be defined as:—(i) *The envelope of a sphere, whose centre moves in a plane and which touches two given spheres.* (ii) *The envelope of a sphere whose centre moves on a conic and touches a given sphere.* (iii) *The envelope of a sphere whose centre moves on a conic and cuts orthogonally another sphere. Also the lines of curvature are circles.*

312. When the centre of inversion lies on the axis of the cone, the cyclide becomes an anchor ring, and is sometimes called the *ring-cyclide.* When the line about which the circle revolves lies outside it, all the double points are imaginary; but when it cuts the circle two of them are real. The remaining two imaginary nodes lie on the imaginary circle.

313. There are two other forms of quadrinodal cyclides called the *horned-cyclide,* consisting of two sheets external to one another, which are connected at the two real double points; and the *spindle-cyclide,* consisting of two sheets internal to one another similarly connected. When the cyclide is generated as the envelope of a sphere whose centre lies on a parabola, the surface becomes a quadrinodal cubic, which passes through the imaginary circle, and there are two forms called the *parabolic horned-cyclide* in which two of the double points are real, and the *parabolic ring-cyclide* in which all these points are imaginary. Figures of these surfaces have been drawn by Maxwell*.

314. Since a nodal conic on a quartic reduces the class by 24, the classes of the four species of nodal cyclides are respectively equal to 10, 8, 6 and 4. When the imaginary circle is cuspidal the class of the cyclide is 6, and there are only two subsidiary species in which the surface has one isolated double point, which may be a conic node or a binode.

* *Quart. Jour.* 1867; *Scientific Papers,* vol. II. p. 144.

Steiner's Quartic.

315. The reciprocal of a quadrinodal cubic surface is called *Steiner's quartic*, and in § 93 its equation is shown to be

$$(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta - 2\alpha\delta - 2\beta\delta - 2\gamma\delta)^2 = 64\alpha\beta\gamma\delta \quad \ldots(1).$$

316. *The surface has four conic tropes which form a tetrahedron ABCD. The conic of contact of the trope α touches the lines CD, DB and BC at the points C', F' and H'; those situated in the planes β, γ, δ touch these lines at the same points respectively; and the conics in the planes γ and δ; δ and β; β and γ respectively touch AB, AC and AD in three points B', E' and G'. Also the three lines $B'C'$, $E'F'$, $G'H'$ intersect at a point O, whose coordinates are $\alpha = \beta = \gamma = \delta$.*

In (1) the planes α, β, γ, δ are conic tropes, and the conic of contact in the plane α touches CD at a point C' whose coordinates are $\alpha = 0$, $\beta = 0$, $\gamma = \delta$; and this is the point where the conic of contact in the plane β touches CD. And similarly for the other five edges of the tetrahedron.

The equations of $B'C'$ are

$$\alpha = \beta, \quad \gamma = \delta,$$

hence $B'C'$ passes through the point O; and similarly $E'F'$, $G'H'$ pass through the same point.

317. *The lines $B'C'$, $E'F'$, $G'H'$ are nodal lines on the surface, and their point of intersection O is a cubic node of the third kind. Also each nodal line intersects two of the conics of contact in points which are pinch points on the nodal line.*

Change the tetrahedron to $AB'C'D$; then we must write

$$\alpha' = \alpha - \beta, \quad \delta' = \delta - \gamma,$$

and (1) becomes

$$\{(\alpha' - \delta')^2 - 4\alpha'\gamma - 4\beta\delta'\}^2 = 16\beta\gamma(\alpha' + \delta')^2 \quad \ldots\ldots\ldots(2),$$

which shows that the line $\alpha' = 0$, $\delta' = 0$ or $B'C'$ is a nodal line on the surface. Similarly $E'F'$ and $G'H'$ are nodal lines which intersect in the point O.

Again the coefficients of β^2 and γ^2 in (2) are $4\delta'^2$ and $4\alpha'^2$ respectively, which shows that B' and C' are pinch points on $B'C'$; accordingly there are only two pinch points on each nodal line, the remaining two being absorbed at the point O.

In (2) transfer the tetrahedron to $AOC'D$ by writing $\gamma = \beta - \gamma'$ and it becomes

$$\{(\alpha' - \delta')^2 + 4\alpha'\gamma'\}^2 = 8\beta\,(\alpha'^2 - \delta'^2)(\alpha' - \delta' + 2\gamma'),$$

which shows that O is a cubic node of the third species.

318. *Every quartic which has three nodal lines intersecting at a point is a Steiner's quartic.*

Let AB, AC, AD be the nodal lines, then the equation of the surface is

$$\gamma^2\delta^2 + \delta^2\beta^2 + \beta^2\gamma^2 + \beta\gamma\delta\,(l\beta + m\gamma + n\delta) + k\alpha\beta\gamma\delta = 0 \ldots (3).$$

The section of (3) by the arbitrary plane $\alpha + p\beta + q\gamma + r\delta = 0$ is

$$\gamma^2\delta^2 + \delta^2\beta^2 + \beta^2\gamma^2 + \beta\gamma\delta\,\{(l - pk)\,\beta + (m - qk)\,\gamma + (n - rk)\,\delta\} = 0 \ldots (4).$$

There are four sets of values of p, q, r, such that (4) becomes a perfect square, one of which is $l - pk = m - qk = n - rk = 2$; hence the surface has four conic tropes, and by changing the tetrahedron so that its faces are the tropes, (4) can be reduced to (1).

319. An arbitrary plane section of the surface is a trinodal quartic, whose nodes are the points where the plane is intersected by the three nodal lines; and every tangent plane cuts the surface in a pair of conics, which intersect in these three points and also in the point of contact.

We have shown in § 56 that the Hessian of a surface passes through the curve of contact of every conic trope; and in § 58 that every nodal line on a surface gives rise to a quadruple* line on the Hessian; and since the Hessian is a surface of the 8th degree, the spinodal curve consists of the three nodal lines each repeated eight times and the four conics of contact, which together make up a curve of the 32nd degree.

320. Properties of Steiner's quartic may also be obtained by reciprocating known properties of a quadrinodal cubic. Thus an arbitrary section of the cubic is an anautotomic cubic curve,

* The equation

$$\alpha v_n + \beta w_n + v_{n+1} = 0$$

represents a surface of the $(n+1)$th degree on which AB is a multiple line of order n; from which it can easily be shown that AB is a multiple line of order $4n - 4$ on the Hessian. Putting $n = 2$, we obtain the theorem in question.

hence:—*The tangent cone from an arbitrary point to Steiner's quartic is a sextic cone having nine cuspidal generators.* Also since the tangent plane to the cubic cuts the surface in a nodal cubic curve, *the tangent cone from a point on the quartic is a tricuspidal quartic cone**.

Nodal Twisted Cubic Curve.

321. The theory of surfaces having a nodal twisted cubic curve is given in §§ 225—6, where it is shown that the number of pinch points is $6n-20$. Hence a quartic surface possesses four pinch points.

Every quartic having a nodal twisted cubic is a scroll; also the two lines joining any point on the curve with the points in which the nodal tangent planes respectively intersect the cubic curve, are generators of the scroll.

Let A be any point on the cubic; then since the nodal tangent planes at A have bitactic contact with the curve at A, each plane can cut the cubic in only one other point. Let these points be P and Q. Then since every line through A in a nodal tangent plane has tritactic contact with the surface at A, it follows that the line AP has tritactic contact with the surface at A and bitactic contact at P, and therefore intersects the surface in five points. Hence AP, and therefore AQ, both lie in the surface; and therefore the latter is a scroll of which AP and AQ are generators.

322. *Only one generator passes through each pinch point, and the former is a singular line on the quartic.*

When A is a pinch point, the two nodal tangent planes, and also the lines AP and AQ coincide; and the tangent plane at the pinch point touches the quartic along the line AP and intersects the surface in a residual conic. Hence the line AP is a singular line analogous to the curve of contact of a trope.

323. The preceding theorems can be proved analytically as follows. Let

$$\lambda = \alpha\gamma - \delta^2, \quad \mu = \gamma\delta - \alpha\beta, \quad \nu = \beta\delta - \gamma^2 \quad \text{.........(1)},$$

* For further information, see Cremona, *Crelle*, vol. LXIII. p. 315; Kummer, *Ibid.* vol. LXIV. p. 66; Shrötter, *Ibid.* p. 79; Cayley, *C. M. P.* vol. IX. p. 1; *Proc. Lond. Math. Soc.* vol. V. p. 14.

then the equation of the quartic is

$$(a^2, b^2, c^2, f, g, h)(\lambda, \mu, \nu)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and that of the nodal tangent planes at A is

$$a^2\gamma^2 + b^2\beta^2 - 2h\beta\gamma = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Accordingly the latter intersect in the line AD, which is the tangent to the cubic at A. The two generators of the quartic, which pass through A, must obviously be generators of the cone $\nu = 0$, and therefore lie on the quartic

$$a^2\lambda^2 + b^2\mu^2 + 2h\lambda\mu = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Eliminating δ between $\nu = 0$ and (4), we obtain

$$(\alpha\beta^2 - \gamma^3)(a^2\gamma^2 + b^2\beta^2 - 2h\beta\gamma) = 0 \quad \ldots\ldots\ldots\ldots(5),$$

which shows that the cone ν intersects the quartic in the nodal cubic twice repeated, and in two straight lines AP and AQ which lie in the nodal tangent planes (3). Writing $\beta\delta$ for γ^2 in (3), it follows that the equation of the plane PAQ is

$$a^2\delta + b^2\beta - 2h\gamma = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

When A is a pinch point, $h = ab$, and (3) and (6) become

$$a\gamma - b\beta = 0, \quad a^2\delta + b^2\beta - 2ab\gamma = 0 \quad \ldots\ldots\ldots\ldots(7),$$

the first of which is the tangent plane at the pinch point, and the second is the tangent plane along the singular line. Equation (2) now becomes

$$(a\lambda + b\mu)^2 + \nu(c^2\nu + 2f\mu + 2g\lambda) = 0,$$

the section of which by the second of (7) can be put into the form

$$(a\delta - b\gamma)^2 \{(a\alpha - b\delta)^2 - c^2\nu - 2f\mu - 2g\lambda\} = 0,$$

which shows that the section consists of the singular line twice repeated and a conic.

324. *When a quartic possesses a cuspidal twisted cubic curve, it is the developable enveloped by the plane*

$$u\theta^3 + 3v\theta^2 + 3w\theta + t = 0,$$

and its equation is

$$(wv - ut)^2 = 4(uw - v^2)(vt - w^2) \ldots\ldots\ldots\ldots\ldots(8).$$

Let $\lambda = vt - w^2, \quad \mu = wv - ut, \quad \nu = uw - v^2,$

and let the equation of a quartic having a *nodal* twisted cubic be

$$b\mu^2 = 2g\lambda\nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

By § 226, the equation of the quadric passing through the pinch points is obtained from the condition that the line

$$\lambda u + \mu v + \nu w = 0$$

should touch (9), and is

$$2buw = gv^2 \text{..........................} (10),$$

and when the cubic is cuspidal every point must be a pinch point, in which case (9) must contain the cubic and therefore $2b = g$, which reduces (9) to (8).

Quartic Scrolls.

325. The theory of quartic scrolls has been discussed by Cayley* and Cremona†, who divided them into 12 species. The general theory of scrolls will be considered in the next chapter, and we shall proceed to examine the properties of quartic scrolls.

326. Every quartic which has a triple line must necessarily be a scroll, since any section through the triple line consists of the line three times repeated and another line. By § 215, a triple line of the first kind has four pinch points, and if A and B be two of them and (γ, δ) the tangent planes thereat, the equation of the surface must be of the form

$$\alpha\gamma^2(p\gamma + q\delta) + \beta\delta^2(r\gamma + s\delta) = (F, P, Q, R, G \between \gamma, \delta)^4 \text{...} (1)$$

or

$$\alpha\gamma^2 v_1 + \beta\delta^2 w_1 = v_4.$$

The section of the surface by the plane δ is the line AB^3 and the line $p\alpha = F\gamma$, and if the latter be taken as the side BC, $F = 0$. Similarly the section by the plane γ consists of AB^3 and the line $s\beta = G\delta$, and if this be taken as the side AD, $G = 0$. We can therefore reduce (1) to the form

$$\alpha\gamma^2 v_1 + \beta\delta^2 w_1 = \gamma\delta(P\gamma^2 + Q\gamma\delta + R\delta^2) \text{...........} (2).$$

The tangent plane at C, which may be any point on BC, is $p\alpha = P\delta$, and the section of (2) by this plane consists of BC^2 and a conic passing through B. Hence BC is a singular line the tangent plane along which is fixed, and if this plane be taken as the plane α, $P = 0$. In like manner the tangent plane along AD is fixed in

* *C. M. P.* vol. v. pp. 168, 201 and vol. VI. p. 312; *Phil. Trans.* 1864 and 1869.

† *Mem. di Bologna*, vol. VIII. (1868). See also, Chasles, *C. R.* 1861; Rohn, *Math. Ann.* vols. XXIV. and XXVIII.; Segen, *Crelle*, CXII.

space, and if the plane be taken as the plane β, $R=0$. Accordingly (2) can be reduced to

$$\alpha\gamma^2 v_1 + \beta\delta^2 w_1 = Q\gamma^2\delta^2 \qquad (3).$$

The preceding argument shows that:—*Through each of the four pinch points a singular line passes, the tangent plane along which is fixed in space.*

327. The surface (3) gives rise to four species of scrolls.

(i) *9th species of Cayley; 8th of Cremona.* Since every plane through AB intersects the surface in AB^3 and a line which passes through AB, all the generators pass through this line. Also writing the equation of the surface in the form

$$\alpha\gamma\delta(p\gamma + q\delta) + \beta v_3 = v_4$$

it follows that each of the tangent planes at the point A intersect the surface in a line passing through A, hence:—*Through every point on the triple line three generators can be drawn, which lie in different planes.*

328. (ii) *3rd species of Cayley; 9th of Cremona.* When $Q=0$, the line CD lies in the surface; hence there is one generator which does not cut the triple line. The equation of the surface is now

$$\alpha\gamma^2 v_1 + \beta\delta^2 w_1 = 0 \qquad (4),$$

and the section of the surface by the plane $\beta = \lambda\alpha$ is

$$\gamma^2 v_1 + \lambda\delta^2 w_1 = 0,$$

and therefore consists of three straight lines which intersect on the triple line. Hence:—*Through every point on the triple line three generators can be drawn, which lie in a plane passing through the generator which does not intersect the triple line.*

329. (iii) *12th species of Cayley; 3rd of Cremona.* When $ps=qr$, the triple line becomes one of the second kind, since one of the tangent planes is fixed in space; and (3) may be written

$$(\alpha\gamma^2 + \beta\delta^2)(p\gamma + q\delta) = Q\gamma^2\delta^2 \qquad (5).$$

There are only three distinct pinch points, since by § 216 one pair coincide; and by forming the equation of the discriminantal surface as in § 216, it follows that the points A and B are the distinct pinch points, and that the other two coincide at a point

A', such that $q^2\alpha + p^2\beta = 0$ is the equation of the plane $A'CD$. Writing $\beta' = q^2\alpha + p^2\beta$, (5) becomes

$$\alpha(p\gamma - q\delta)(p\gamma + q\delta)^2 + \beta'\delta^2(p\gamma + q\delta) = Qp^2\gamma^2\delta^2 \quad \text{......(6)},$$

which shows that A' is a pinch point.

Writing (6) in the form

$$\alpha\gamma\delta(p\gamma + q\delta) + \beta\gamma v_2 = v_4,$$

it follows that the planes δ and $p\gamma + q\delta$ intersect the surface in AB^3 and two lines passing through A; but that the fixed plane γ intersects the surface in AB^4, hence:—*Through every point on the triple line only two distinct generators can be drawn, since the third one coincides with AB.*

330. (iv) *6th species of Cayley; 10th of Cremona.* In this case the triple line is of the third species, one of the tangent planes being torsal and the other two fixed in space. There are only two distinct pinch points, by reason of the fact that they coincide in pairs. The equation of the surface is

$$\alpha\gamma^2\delta + \beta\gamma\delta^2 = v_4 \quad \text{......................(7)},$$

where A and B are the pinch points.

331. (v) *10th species of Cayley; 1st of Cremona.* The quartic has a proper nodal twisted cubic, and its equation is

$$(a, b, c, f, g, h \between \lambda, \mu, \nu)^2 = 0,$$

where (λ, μ, ν) have the same meanings as in § 323, where this surface has been discussed.

332. (vi) *8th species of Cayley; 7th of Cremona.* The scroll* $S(1, 3^2)$ is a special case of the last species, since every generator intersects the cubic twice and also intersects a fixed straight line, whose equations may be taken to be

$$l\alpha + m\beta + n\gamma + p\delta = 0,$$
$$l'\alpha + m'\beta + n'\gamma + p'\delta = 0.$$

Also if

$$a = mn' - m'n, \quad f = lp' - l'p,$$
$$b = nl' - n'l, \quad g = mp' - m'p,$$
$$c = lm' - l'm, \quad h = np' - n'p,$$

where a, b, c, f, g, h are the six coordinates of the line, the identical relation

$$af + bg + ch = 0 \quad \text{......................(8)}$$

exists between the six coordinates.

* The notation for scrolls will be explained in the next Chapter.

Let $\lambda = \mu = \nu = 0$ be the equations of the cubic, where

$$\lambda = \beta\delta - \gamma^2, \quad \mu = \beta\gamma - \alpha\delta, \quad \nu = \alpha\gamma - \beta^2 \text{.........} (8\text{A}),$$

then the parametric values of the coordinates of any point on the cubic are ρ, $\rho\theta$, $\rho\theta^2$, $\rho\theta^3$, and therefore the equations of the line through any two points θ and ϕ on the cubic are

$$\frac{\alpha - \rho}{\rho - \sigma} = \frac{\beta - \rho\theta}{\rho\theta - \sigma\phi} = \frac{\gamma - \rho\theta^2}{\rho\theta^2 - \sigma\phi^2} = \frac{\delta - \rho\theta^3}{\rho\theta^3 - \sigma\phi^3} = r,$$

which shows that the coordinates of any point on the line are given by equations of the form

$$\alpha = 1 + \omega, \quad \beta = \theta + \omega\phi, \quad \gamma = \theta^2 + \omega\phi^2, \quad \delta = \theta^3 + \omega\phi^3,$$

where ω is a variable parameter which depends upon the position of the point $(\alpha, \beta, \gamma, \delta)$ on the line. Substituting these values in (8A) and rejecting the common factor $\omega^2(\theta - \phi)^2$, we obtain

$$\lambda : \mu : \nu = \theta\phi : -\theta - \phi : 1.$$

The conditions that the variable line should intersect the directrix line are

$$l + m\theta + n\theta^2 + p\theta^3 + \omega(l + m\phi + n\phi^2 + p\phi^3) = 0,$$
$$l' + m'\theta + n'\theta^2 + p'\theta^3 + \omega(l' + m'\phi + n'\phi^2 + p'\phi^3) = 0,$$

eliminating ω and dividing out by the common factor $\theta - \phi$, we obtain

$$c - b(\phi + \theta) + f\{(\phi + \theta)^2 - \phi\theta\} + a\phi\theta + g\phi\theta(\phi + \theta) + h\phi^2\theta^2 = 0.$$

Substituting the values of $\phi\theta$ and $\phi + \theta$ in terms of λ, μ, ν this becomes

$$c\nu^2 + b\mu\nu + f(\mu^2 - \lambda\nu) + a\lambda\nu - g\lambda\mu + h\lambda^2 = 0,$$

or $$(A, B, C, F, G, H \between \lambda, \mu, \nu)^2 = 0 \text{..............} (9),$$

where $$A, B, C, 2F, 2G, 2H = h, f, c, b, a - f, -g,$$

which by virtue of (8) becomes

$$AC + B^2 + 2BG - 4FH = 0 \text{ ....} (10).$$

Hence :—*In order that the scroll* (9) *should belong to the species* $S(1, n^2)$, *which is generated by a straight line which intersects the cubic twice and the given straight line once, it is necessary that the coefficients should be connected by the relation* (10).

Since a twisted cubic cannot possess a trisecant, it is impossible for scrolls of the species $S(3^3)$ to exist.

333. (vii) *7th species of Cayley; 2nd of Cremona.* The surface possesses a nodal conic and a nodal line which cuts the conic.

Let AB be the nodal line, (α, Ω) the nodal conic where

$$\Omega = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma;$$

then the equation of the surface is

$$\alpha^2 (L\gamma^2 + M\gamma\delta + N\delta^2) + \alpha (p\gamma + q\delta) \Omega + \Omega^2 = 0 \text{ ...(11).}$$

Since the points C and D are any points on the nodal conic, let them be those in which the nodal tangent planes at A cut the conic; then $L = N = 0$. The section of the surface by the plane ACD now becomes

$$\gamma\delta \{M\alpha^2 + \lambda\alpha (p\gamma + q\delta) + \lambda^2\gamma\delta\} = 0 \text{............(12),}$$

which consists of a conic cutting the nodal conic at C and D, and of the straight lines AC and AD. These lines are obviously generators of the scroll.

Returning to the more general equation (11), in which C is any point on the nodal conic, transfer the vertex A of the tetrahedron to A', where $\beta = k\alpha$ is the equation of the plane $A'CD$. Then (11) becomes

$$\alpha^2 v_2 + \alpha v_1 \{\lambda\gamma\delta + (\mu\delta + \nu\gamma)(k\alpha + \beta')\} + \{\lambda\gamma\delta + \dots\}^2 = 0,$$

and the condition that the line $A'C$ should lie in the surface is

$$L + p\nu k + k^2\nu^2 = 0,$$

which shows that there are two points A', corresponding to any point C on the nodal conic, such that two lines CA', CA'' lie in the quartic surface. Hence the latter is a scroll. Also if A be one of these points, $L = 0$ and (11) becomes

$$\alpha^2\delta (M\gamma + N\delta) + \alpha v_1 \Omega + \Omega^2 = 0 \text{ ....(13),}$$

the section of which by the plane δ is

$$\gamma^2\beta (p\alpha + \nu\beta) = 0.$$

Hence :—*Through every point on the nodal conic two straight lines can be drawn lying in the surface, both of which intersect the nodal line; also the two points of intersection have one common tangent plane in which both the two lines lie.*

The section of (13) by an arbitrary plane through AC consists of this line and a cubic curve; but from the first portion of this article, it is obvious that there is a certain position of this plane such that the section consists of the line AC, another line AD' intersecting the nodal conic in D', and a conic cutting the nodal conic in C and D'. Hence :—*The scroll may be generated by a line which intersects* (i) *the line* AB; (ii) *a conic passing through* B *and lying in the plane* BCD; (iii) *a conic lying in the plane* ACD, *which intersects the first conic in* C *and* D, *but which does not pass through* A.

334. (viii) 11*th species of Cayley; 4th of Cremona.* Equation (11) shows that both nodal tangent planes are torsal, hence the nodal line is of the first kind; but if $N=0$ the tangent plane γ is fixed and the nodal line becomes one of the second kind. The section of the surface by the plane γ consists of AB^3 and a line through the point B, where the nodal line intersects the nodal conic. In both cases an arbitrary section of the surface by a plane through the point B has a tacnode thereat; and therefore the section belongs to the same species of curves as the conchoid of Nicomedes, which possesses a node and a tacnode.

335. (ix) 2*nd species of Cayley; 5th of Cremona.* In this case the nodal conic degrades into two straight lines; hence the nodal curve consists of three straight lines, one of which cuts the remaining two which lie in different planes. Hence if AB, BD and DC be the three lines, the equation of the surface is

$$\alpha^2\delta (M\gamma + N\delta) + \alpha (p\gamma + q\delta) \beta\gamma + \beta^2\gamma^2 = 0 \quad\text{......(14)}.$$

336. (x) 4*th species of Cayley; 12th of Cremona.* The surface possesses a tacnodal line of the *first* kind, and its equation may always be reduced to the form

$$(\alpha\delta - \beta\gamma + v_2)^2 + v_4 = 0 \quad\text{..................(15)},$$

and the section of the surface by an arbitrary plane through the tacnodal line consists of two straight lines.

It is a remarkable fact that quartic surfaces which possess tacnodal lines of the second kind are not scrolls; for the equation of the quartic is

$$\alpha\beta\gamma^2 + 2\gamma (\alpha v_2 + \beta w_2 + w_3) + v_4 = 0,$$

and the section by an arbitrary plane through AB consists of AB^2 and a conic, and it is only for certain positions of the plane that the conic degrades into two straight lines. Similar observations apply to quartic surfaces possessing rhamphoid cuspidal and oscnodal lines.

337. (xi) *5th species of Cayley; 6th of Cremona.* It is possible for a quartic to have a tacnodal line AB of the first kind, and a nodal line BD intersecting it, in which case its equation is

$$(\alpha\delta - \beta\gamma + \gamma v_1)^2 + \gamma^2 v_2 = 0 \quad \text{..............(16)}.$$

338. (xii) *1st species of Cayley; 11th of Cremona.* The surface possesses two nodal lines AB, CD which lie in different planes, and the equation of the surface is

$$\alpha^2 v_2 + \alpha\beta w_2 + \beta^2 \sigma_2 = 0 \quad \text{..................(17)},$$

from which it can be shown that if A' be any point on AB, there are two positions of A' such that the line CA' lies in the surface.

CHAPTER VII

SCROLLS

339. THE envelope of a plane which possesses one degree of freedom is a developable surface; but that of a line is a scroll, since it is not necessary that each line should intersect the next consecutive one. The theory of scrolls has been discussed in three memoirs by Cayley*, of which some account will be given.

340. *When a straight line moves in such a manner that it intersects three twisted curves, its envelope is a scroll.*

Let O be any point on the first curve; and with O as a centre describe the cones standing on the second and third curves. Then the common generators of the two cones intersect the three curves; and if OPQ be one of them, the direction cosines of this line will be functions of the parameter θ of O and of the constants of the other two curves. Accordingly the coordinates of the line will be functions of a single parameter θ, and therefore its envelope is a scroll.

The three curves are called *directing curves.*

341. We shall now explain the notation employed.

The symbols $S(l, m, n)$, $S(m, n^2)$, $S(n^3)$ will be used respectively to denote the scrolls generated by a straight line which intersects (i) three curves of degrees l, m and n; (ii) the curve m once and n twice; (iii) the curve n three times. Hence the last scroll is the one generated by the trisecants of the curve n. In like manner, $S(1, m, n)$, $S(1, 1, n)$ &c. denote the scrolls generated

* Cayley, *C. M. P.* vol. v. pp. 168 and 201, vol. VI. p. 312; Salmon, *Camb. and Dublin Math. Jour.* vol. VIII. p. 45; Cremona, *Mem. di Bologna*, vol. VIII. (1868); Schwarz, "On quintic scrolls," *Crelle*, vol. LXVII. (1868).

by a line which intersects (i) a straight line and the curves m and n; (ii) two straight lines and the curve n. These symbols will also be frequently used to denote the degrees of the scrolls.

When the curve m consists of a curve l indefinitely close to the first curve l, the symbol $S(\overline{l, l}, n)$ is used to denote the scroll. In this case the curve l is called a *doubly directing* curve, and we shall hereafter show how to find the equation of the scroll when l is a straight line.

342. Since every straight line which intersects three curves possesses one degree of freedom, a determinate number of lines exists which intersect four curves of degrees l, m, n and p; and the symbol $G(l, m, n, p)$ will be used to denote the number of such lines; also such symbols as $G(1, 1, n, p)$ &c. denote the number of lines when l and m are straight lines. The symbols $G(l, m, n^2)$, $G(m^2, n^2)$, $G(m, n^3)$ respectively denote the number of lines which intersect (i) the curves l and m once and n twice; (ii) the curves m and n twice; (iii) the curves m once and n three times.

343. *To prove that*

$$S(l, m, n) = lS(1, m, n) = lmS(1, 1, n) = lmnS(1, 1, 1) \ldots (1),$$

where S denotes the degrees of the respective scrolls.

The degree of a scroll is equal to the number of points in which an arbitrary straight line intersects it, and this is equal to the number of generators which intersect the line. Now the curve l intersects the scroll $S(1, m, n)$ in $lS(1, m, n)$ points, and since the generators through each of these points intersect the line 1, this must be the number of generators of the scroll $S(l, m, n)$ which pass through 1, and is therefore equal to the degree of this scroll. Hence

$$S(l, m, n) = lS(1, m, n).$$

Proceeding in the same manner we obtain the remaining formulæ (1).

344. *The degree of the scroll whose directing curves are l, m, n is $2lmn$.*

From the last article it follows that the degree of the scroll is $lmnS(1, 1, 1)$, and we shall now show that the second scroll is a hyperboloid.

The hyperboloid $\mu\alpha\gamma = \lambda\beta\delta$ ....(2)
obviously passes through the lines AB and CD, and also through the line

$$\alpha + \lambda\delta = 0, \quad \beta + \mu\gamma = 0 \quad \text{.................(3).}$$

Now the line $\mu\gamma + \theta\delta = 0, \quad \alpha + \lambda\beta/\theta = 0$ ....(4),
where θ is a variable parameter, is a generator of (2); also the three planes (3) and the first of (4), intersect in the point

$$\alpha = -\lambda\beta/\theta = \lambda\mu\gamma/\theta = -\lambda\delta,$$

which lies in the second of (4); hence the hyperboloid (2) is the envelope of the line (4), which intersects AB, CD and (3). Accordingly $S(1, 1, 1) = 2$, which gives

$$S(l, m, n) = 2lmn.$$

345. The three directing curves l, m and n are multiple curves of orders mn, nl and lm respectively on the scroll; for if P be any point on l, the cones whose common vertex is P and which stand on the curves m and n have mn common generators, hence mn generators of the scroll pass through P.

346. *Every scroll of degree ν has in general a nodal curve, which is intersected by every generator in $\nu - 2$ points. Also the tangent plane along a generator is a torsal plane, which touches the generator at only one point.*

The section of the scroll by any plane through a generator G consists of the latter and a curve of degree $\nu - 1$, which cuts G in the same number of points $P, P_1 \ldots P_{\nu-2}$. Through every ordinary point on this curve only one generator of the system in general passes; hence the generator G will belong to one of these points P, whilst a different generator will pass through the remaining $\nu - 2$ points. The plane will therefore be a proper tangent plane at P; but at each of the other points $P_1, P_2 \ldots$ there will be two tangent planes corresponding to the two generators passing through it; hence these are fixed points lying in a nodal curve; also since the point of contact P moves along the generator as the plane rotates round it, the tangent plane to a scroll is a torsal plane.

347. *Let the directing curves m, n; n, l; l, m intersect one another in p, q and r points respectively; then the degree of the scroll is*

$$2lmn - pl - qm - rn,$$

whilst l, m and n are multiple curves of orders $mn-p$, $nl-q$, $lm-r$ respectively.

Let P be one of the points of intersection of m and n; then the complete scroll will be an improper one, consisting of the cone whose vertex is P which stands upon the curve l, and a residual scroll of degree $2lmn-l$. Accordingly if the curves m and n intersect in p points, the degree of the proper scroll is $2lmn-pl$. This proves the first part.

Since the cone l forms one sheet of the complete scroll, only $mn-1$ sheets of the residual scroll pass through the curve l; hence if the curves m and n intersect in p points, the multiplicity of the curve l is $mn-p$.

This theorem requires modification when the directing curves intersect at a multiple point*.

348. *Every generator intersects every other generator in*

$$2lmn-mn-nl-lm+1$$

points not on the directing curves.

The point where any generator G intersects the curve l is a multiple point of order mn, hence $mn-1$ other generators pass through this point. Accordingly the total number of ordinary generators which intersect G is

$$2lmn-2-(mn-1)-(nl-1)-(lm-1)$$
$$=2lmn-mn-nl-lm+1.$$

349. *The degree of the scroll generated by a line which cuts a curve l once and a curve n twice is*

$$l\{h+\tfrac{1}{2}n(n-1)\},$$

where h is the number of apparent nodes possessed by n.

Equations (1) apply to all three species of scrolls, hence

$$S(l, n^2)=lS(1, n^2).$$

Let O be any point on the line 1, h the number of apparent nodes possessed by n; then since h generators of $S(1, n^2)$ pass through O, the line 1 is a multiple generator of order h on the scroll. The section of the scroll by any plane through 1 consists of 1 repeated h times and a residual curve of degree $S-h$; moreover the plane intersects the curve n in n points, and from the mode of generation

* See Picquet, *Compt. Rend.* vol. LXXVII. (1873); Guccia, *Rend. Palermo*, vol. I.

it follows that the residual curve must consist of the $\frac{1}{2}n(n-1)$ lines which connect these n points; hence $S-h=\frac{1}{2}n(n-1)$, giving

$$S(1, n^2)=h+\tfrac{1}{2}n(n-1).$$

350. *The degree of the scroll* $S(n^3)$ *is*

$$(n-2)\{h-\tfrac{1}{6}n(n-1)\}.$$

This result has been proved in § 113 by the Theory of Correspondence.

351. *The class of the tangent cone is equal to the degree of the scroll.*

Let O be any point, OL a fixed straight line through O. Then since each generator which passes through OL possesses one torsal tangent plane, there must be a certain position of the point of contact, such that this plane passes through OL. Hence the class of the cone is equal to the number of generators of the scroll which pass through OL, that is to its degree.

The G formulæ.

352. The number of lines which cut four curves of degrees l, m, n and p is obviously equal to the number of points in which the curve p cuts the scroll $S(l, m, n)$; hence

$$G(l, m, n, p)=pS(l, m, n)=2lmnp \quad \text{............(5).}$$

In (5) put $p=1$, and we obtain

$$G(1, l, m, n)=S(l, m, n)=lS(1, m, n)$$

and

$$G(l, m, n, p)=pG(1, l, m, n)=lpG(1, 1, m, n)$$
$$=lmpG(1, 1, 1, n)=lmnpG(1, 1, 1, 1),$$

so that $G(1, 1, 1, 1)=2$, as is otherwise obvious since a straight line can intersect a hyperboloid in only two points. The last result also shows that two straight lines can be drawn which intersect four given straight lines.

353. *The number of straight lines which intersect a given curve* n *twice and two curves* l *and* m *once is*

$$lm\{h_3+\tfrac{1}{2}n(n-1)\},$$

where h_3 *is the number of apparent nodes possessed by* n.

By means of the last article, it can be shown that

$$G(l, m, n^2) = lmS(1, n^2) = lm\{h_3 + \tfrac{1}{2}n(n-1)\} \quad\text{......(6)}$$

by § 349.

354. *The number of trisecants of a curve n, which intersect a curve l once is*

$$l(n-2)\{h_3 - \tfrac{1}{6}n(n-1)\}.$$

It can be shown as in § 352 that

$$G(l, n^3) = lS(n^3) = l(n-2)\{h_3 - \tfrac{1}{6}n(n-1)\} \quad\text{........(7)}$$

by § 350.

355. *To prove that*

$$G(l^2, m^2) = h_1h_2 + \tfrac{1}{4}lm(l-1)(m-1) \quad\text{........(8)},$$

where h_1, h_2 *are the number of apparent nodes possessed by l and m.*

To establish this result, I shall adopt an indirect method.

Let $\qquad L = \tfrac{1}{2}l(l-1), \quad M = \tfrac{1}{2}m(m-1),$

then we have shown in § 353 that

$$G(1, 1, l^2) = h_1 + L; \quad G(1, 1, m^2) = h_2 + M.$$

Now $G(1, 1, l^2)$ may be regarded in two lights; first, as the number of lines which intersect the curve l twice and two given straight lines, lying in different planes, once; or secondly, the two straight lines may be regarded as an improper conic which the line intersects twice. Hence under these circumstances,

$$G(1, 1, l^2) = G(2^2, l^2),$$

and the required formula must therefore reduce to $G(1, 1, l^2)$, when the curve m consists of two straight lines. We shall therefore assume

$$G(l^2, m^2) = ALM + BLh_2 + CMh_1 + Dh_1h_2 \quad\text{........(9)},$$

where A, B, C, D are constants. This formula satisfies the condition of being symmetrical with respect to the two curves l and m.

When the curve m becomes two straight lines, $m = 2$, $h_2 = 1$, $M = 1$; and (9) becomes

$$h_1 + L = (A + B)L + (C + D)h_1,$$

whence $\qquad A + B = 1; \quad C + D = 1 \quad\text{................(10)}.$

When the curve l becomes two straight lines, we shall obtain in like manner

$$A + C = 1, \quad B + D = 1 \quad \dots\dots\dots\dots\dots\dots (11),$$

but (10) and (11) are only equivalent to the three independent equations

$$B = 1 - A, \quad C = 1 - A, \quad D = A \quad \dots\dots\dots\dots (12).$$

To obtain a fourth equation, let l and m be a pair of conics lying in different planes, then the only line which intersects both conics twice is the line of intersection of their planes; hence $G(2^2, 2^2) = 1$; also $l = m = 2$, $h_1 = h_2 = 0$. We thus obtain

$$A = D = 1; \quad B = C = 0,$$

which gives the required result.

356. This result may be verified as follows. Let $G(m + m')^2$ denote the number of lines which intersect the curve l twice, and a compound curve $m + m'$ twice, consisting of two simple curves of degrees m and m'. Then this number is obviously equal to the number of lines which intersect l twice, and also (i) m twice; (ii) m' twice; (iii) m and m' each once. The latter quantity is given by § 353; hence

$$G(m + m') = G(m^2) + G(m'^2) + mm' \{h_1 + \tfrac{1}{2} l (l - 1)\} \dots (13),$$

where $G(m^2)$ is written for $G(l^2, m^2)$. And if the value of the G's be substituted from (8), it will be found that (13) is satisfied.

357. *The number of quadrisecants of a twisted curve of degree n is*

$$\tfrac{1}{2} h (h - 4n + 11) - \tfrac{1}{24} n (n - 2)(n - 3)(n - 13) \dots\dots (14),$$

where h is the number of apparent nodes.

This result was first obtained by Cayley*; and other proofs have been given by Zeuthen† by means of the Theory of Correspondence, and also by Berzolari‡. According to the functional notation of § 352, the number in question is represented by $G(n^4)$; and as in the last article, I shall proceed to form the functional equation for a compound curve consisting of two simple curves of degrees n and n'.

* *C. M. P.* vol. v. p. 179.

† *Annali di Matematica*, Serie II. vol. III. p. 189.

‡ *Rend. Palermo*, vol. IX. (1895).

The value of $G(n+n')^4$ for the compound curve is equal to (i) the number of quadrisecants of the two simple curves n and n', that is to $G(n^4)+G(n'^4)$; (ii) the number of lines which cut n three times and n' once, that is to $G(n^3, n')$; (iii) the number which cut n once and n' three times, that is to $G(n, n'^3)$; (iv) the number which cut n and n' twice, that is to $G(n^2, n'^2)$. We thus obtain

$$G(n+n')^4 = G(n^4) + G(n'^4) + G(n^3, n') + G(n, n'^3) + G(n^2, n'^2) \ldots (15).$$

The value of $G(n^3, n')$ is obtained from (7) by writing $h_3 = h$, $l = n'$, where h is the number of apparent nodes of n; whilst that of $G(n^2, n'^2)$ is obtained from (8) by writing $h_1 = h'$, $h = h_2$, $l = n'$, $m = n$; accordingly the last three terms of (15) become

$$n'(n-2)\{h - \tfrac{1}{6}n(n-1)\} + n(n'-2)\{h' - \tfrac{1}{6}n'(n'-1)\} + hh' + \tfrac{1}{4}nn'(n-1)(n'-1).$$

The solution of the functional equation (15) thus consists of a particular solution and of the complementary function, and Cayley has deduced the value of $G(n^4)$ by solving this equation. But if its value be substituted in (15) from (14) the functional equation will be satisfied.

The reader will find some further information on the general theory of scrolls, together with references to the original authorities, in Pascal's *Repertorio di Matematiche Superiori,* vol. II. pp. 515—525.

358. *To find the equation of the scroll* $S(1, 1, n)$.

Let AB be the directing line l; and let the line m be any line CD' lying in the plane ACD. Then the equations of the directing lines are

$$\gamma = 0, \quad \delta = 0; \quad \text{and} \quad \alpha + \delta = 0, \quad \beta = 0 \ \ldots\ldots\ldots (1),$$

and the directing curve may be taken to be any curve in the plane BCD, and therefore its equations are

$$\alpha = 0, \quad \beta^n v_0 + \beta^{n-1} v_1 + \ldots\ldots v_n = 0 \ \ldots\ldots\ldots\ldots (2).$$

Let A' be any point on AB, $\theta\alpha = \beta$ the plane $A'CD$; then the coordinates of A' are $(\xi, \theta\xi, 0, 0)$. Also if $(0, g, h, k)$ be the coordinates of any point P on (2), the equations of the line $A'P$ are

$$\frac{\alpha - \xi}{-\xi} = \frac{\beta - \theta\xi}{g - \theta\xi} = \frac{\gamma}{h} = \frac{\delta}{k} = \lambda \ \text{(say)} \ \ldots\ldots\ldots\ldots (3),$$

whence

$$\left.\begin{aligned} \alpha &= \xi(1-\lambda) \\ \beta &= \lambda g + \theta\xi(1-\lambda) \\ \gamma &= \lambda h \\ \delta &= \lambda k \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

from the first two of which we obtain

$$\beta = \lambda g + \theta\alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The conditions that (3) should intersect the directing line CD' are

$$\xi(1-\lambda) + \lambda k = 0,$$

$$\lambda g + \theta\xi(1-\lambda) = 0,$$

from which we obtain $\qquad g - \theta k = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$

which by the last of (4) becomes

$$\lambda g = \theta\delta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Eliminating θ between (5) and (7) we obtain

$$\lambda g = \frac{\beta\delta}{\alpha+\delta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Let v_n' be what v_n becomes when $\gamma = h$, $\delta = k$; then since P lies on the curve (2)

$$g^n v_0 + g^{n-1}v_1' + g^{n-2}v_2' + \ldots\ldots v_n' = 0,$$

which by the last two of (4) becomes

$$(\lambda g)^n v_0 + (\lambda g)^{n-1} v_1 + \ldots\ldots v_n = 0 \ldots\ldots\ldots\ldots(9).$$

Eliminating λg between (8) and (9) we obtain

$$\beta^n\delta^n v_0 + (\alpha+\delta)\beta^{n-1}\delta^{n-1}v_1 + \ldots\ldots (\alpha+\delta)^n v_n = 0 \ldots (10),$$

which is the required equation of the scroll.

If the tetrahedron be changed to $ABCD'$, so that the directing line CD' lies in the plane BCD', (10) becomes

$$\beta^n\delta^n v_0 + \alpha'\beta^{n-1}\delta^{n-1}v_1 + \ldots\ldots \alpha'^n v_n = 0 \quad \ldots\ldots(10\text{ A}).$$

359. *To find the equation of the scroll* $S(1, 1, n)$.

Since a hyperboloid can be described through any two straight lines, the doubly directing line may be supposed to consist of two generators belonging to the same system of a hyperboloid, which are indefinitely close together. Let AB be the doubly directing line, A' any point on AB; (2) the directing curve n; also let the tangent plane to the hyperboloid at A' intersect (2) in P; then

$A'P$ is the line which generates the scroll. The equation of the hyperboloid may be taken to be

$$\alpha\gamma = \beta\delta \quad \text{.........................(11)},$$

and if $\theta\alpha = \beta$ be the plane $A'CD$, the tangent plane to (11) at A' is

$$\gamma = \theta\delta \quad \text{.........................(12)},$$

and the coordinates of P are $(0, g, \theta k, k)$. Hence the equations of $A'P$ are

$$\frac{\alpha - \xi}{-\xi} = \frac{\beta - \theta\xi}{g - \theta\xi} = \frac{\gamma}{\theta k} = \frac{\delta}{k} = \lambda \quad \text{..............(13)}.$$

From the first two of (13) we deduce (5), whence eliminating θ by (12) we obtain

$$\lambda g = (\beta\delta - \alpha\gamma)/\delta.$$

Accordingly from (9) we get

$$(\beta\delta - \alpha\gamma)^n v_0 + (\beta\delta - \alpha\gamma)^{n-1} \delta v_1 + \ldots\ldots \delta^n v_n = 0 \text{......(14)},$$

which is the required equation of the scroll.

360. Equations (10) and (14) furnish a method of classifying the scrolls $S(1, 1, n)$ and $S(\overline{1, 1}, n)$, which depends on the character of the curve n and not on the degree of the scroll. Let this curve have a multiple point of order p at B and of order q at C. Let p have any value from 0 to $n-1$, and q any value from 0 to r, where r is a number whose limiting value is obtained from the condition that the curve n is always a proper curve. Then by considering all possible curves of given degree subject to these conditions, we obtain the equations of all possible scrolls generated by them.

361. *Cubic Scrolls.* (i) Let the nodal line AB be the curve l, and the line $\beta = 0$, $\alpha + \delta = 0$ or CD' be the line m; and let the curve n be a plane cubic whose node is at B. Then in the formulæ of § 347, we must put $l = m = 1$, $n = 3$, $q = 2$, $r = 0$, in which case the lines AB, CD' and the plane cubic will be multiple lines of orders $3 - p$, 1, 1; but since AB is a nodal line $p = 1$, and consequently the line CD' must intersect the cubic curve in one point. Let C be this point, then the equation of the cubic curve is

$$\beta v_2 + \delta w_2 = 0,$$

and by (10) that of the cubic scroll is

$$\beta v_2 + (\alpha + \delta)\, w_2 = 0,$$

which is the equation of a cubic surface having a nodal line of the *first* kind, and is of the form $S(1, 1, 3)$.

(ii) In the case of the cubic scroll $S(\overline{1, 1}, 3)$ the line CD' becomes one indefinitely close to AB, and therefore BC must be the tangent at B to the section by the plane α, the equation of which is therefore

$$\beta\delta v_1 + v_3 = 0.$$

Accordingly by (14) the equation of the scroll is

$$(\beta\delta - \alpha\gamma) v_1 + v_3 = 0,$$

which is the equation of a cubic scroll having a nodal line of the *second* kind.

Quartic Scrolls.

362. We have already considered the different species of quartic scrolls, and we shall now explain Cayley's method of generating them. There are three species of the form $S(1, 1, 4)$ and three of the form $S(\overline{1, 1}, 4)$.

1*st species.* This scroll is of the species $S(1, 1, 4)$; AB and CD' are nodal generators, and therefore the section by the plane α is

$$\beta^2 v_2 + \beta\delta w_2 + \delta^2\sigma_2 = 0,$$

and therefore by (10 A) the equation of the scroll is

$$\alpha'^2\sigma_2 + \alpha'\beta w_2 + \beta^2 v_2 = 0.$$

2*nd species.* Let the generating curve have a tacnode at B and a node at C; let BD be the tacnodal tangent and CD one of the nodal tangents at C. Then the equation of the section is

$$\beta^2\gamma^2 + \beta\gamma\delta v_1 + \delta^3 w_1 = 0,$$

and that of the scroll is

$$\beta^2\gamma^2 + \alpha'\beta\gamma v_1 + \alpha'^2\delta w_1 = 0,$$

which is of the same form as (14) of § 335.

3*rd species.* Let the generating curve have a triple point at B and pass through C. Then its equation is

$$\beta v_3 + \delta w_3 = 0,$$

and that of the scroll is

$$\beta v_3 + \alpha' w_3 = 0,$$

which can be reduced to (4) of § 328 by taking A and B as two of the pinch points.

363. The next three species are of the form $S(\overline{1, 1}, 4)$.

4th species. Let the generating curve have a tacnode at B and let BC be the tacnodal tangent. Then the equation of the curve is

$$\beta^2\delta^2 + \beta\delta v_2 + v_4 = 0,$$

and by (14) that of the scroll is

$$(\beta\delta - \alpha\gamma)^2 + (\beta\delta - \alpha\gamma)\, v_2 + v_4 = 0,$$

which can be reduced to (15) of § 336.

5th species. Let the generating curve possess a node at D in addition to the tacnode at B. Then its equation is

$$\beta^2\delta^2 + \beta\delta\gamma v_1 + \gamma^2 v_2 = 0,$$

and by (14) that of the scroll is

$$(\beta\delta - \alpha\gamma)^2 + (\beta\delta - \alpha\gamma)\, \gamma v_1 + \gamma^2 v_2 = 0,$$

which can be reduced to (16) of § 337.

6th species. Let the generating curve have a triple point at B, and let BC be one of the tangents. Then its equation is

$$\beta\delta v_2 + v_4 = 0,$$

and that of the scroll is

$$(\beta\delta - \alpha\gamma)\, v_2 + v_4 = 0,$$

and the latter possesses a triple line having one torsal and two fixed tangent planes, and can be reduced to (7) of § 330.

364. *7th species.* This species is of the form $S(1, 2, 2)$. The line AB is the directing line and there are two directing conics; one of which passes through B, C and D, whilst the other lies in the plane ACD and intersects the first conic in C and D, but does not pass through A. Hence in § 347

$$l = 1,\ m = 2,\ n = 2,\ p = 2,\ q = 1,\ r = 0,$$

and the scroll is of the fourth degree. The line AB and the conic BCD are nodal curves on the scroll, whilst the conic in the plane ACD is an ordinary conic; the section of the scroll by the plane ACD must therefore consist of the last conic and the lines AC, AD. This scroll has been discussed in § 333.

8th species. This is the scroll $S(1, 3^2)$, which has been discussed in § 332.

Cayley's 9th and 12th species are quartics with triple lines and have been fully discussed in §§ 327 and 329. The 11th species has been considered in § 334 (viii), where it appears as a particular case of (vii). The 10th species is a quartic having a nodal twisted cubic and has been dealt with in §§ 321 and 331. It is a special case of the scroll $S(4, 3^2)$, where 4 is a plane trinodal quartic whose nodes lie on the twisted cubic 3; and the scroll may be generated by a straight line which intersects the cubic twice and the quartic once.

CHAPTER VIII

THEORY OF RESIDUATION

365. THE theory of residuation of plane curves was discovered by the late Prof. Sylvester, and has formed the subject of numerous investigations by German and Italian* mathematicians. We shall commence by explaining this theory so far as it relates to plane curves, and shall afterwards show how it can be extended to surfaces.

366. We shall denote a given curve of the nth degree by C_n; and one whose coefficients are wholly or partially arbitrary by S_n. The curve C_n whose properties we are considering will be called the *primitive* curve.

If a group of points on a curve contain p points, p is called the degree of the group.

Let two curves C_n and C_l intersect in ln ordinary points, and divide them into two groups p and q, so that $ln = p + q$. Then the group p is called a *residual* of the group q and *vice versâ*. Hence two point groups p and q on the primitive curve are said to be residual to one another, whenever it is possible to draw another curve through them which does not intersect the primitive curve elsewhere. Also the group $p + q$ is said to have a *zero residual*, which we shall express by means of the symbolic equation

$$[p + q] = 0 \qquad (1).$$

If two point groups p and q have a common residual r, they are called *coresidual* point groups, which we shall express by means of the symbolic equation

$$[p - q] = 0 \qquad (2).$$

* F. S. Macaulay, *Proc. Lond. Math. Soc.* vol. XXVI. p. 495; *Ibid.* vol. XXIX. p. 673 and the authorities there cited.

367. The theory of residuation depends upon three subsidiary theorems, which may be respectively called the *addition theorem*, the *multiplication theorem*, and the *subtraction theorem*.

I. The Addition Theorem. *If p and q be two point groups on a curve C_n, each of which has a zero residual, then the group $p+q$ has a zero residual.*

Since $[p]=0$, this group must be the complete intersection of a curve C_l with C_n; and for the same reason the group q must be the complete intersection of another curve C_m with C_n. Let $l+m \geqq n$; then the curve

$$C_n S_{l+m-n} + C_l C_m = 0$$

obviously passes through the group $p+q$ and intersects C_n nowhere else; hence $[p+q]=0$.

Let $l+m<n$, and $l \geqq m$; then the group p consists of ln points which are common to the curves C_l and C_n; and since a proper curve of degree $l+m$ cannot intersect a curve of degree l in more than $l(l+m)$ points, it follows that if $nl > l(l+m)$ or $n > l+m$, a proper curve of degree $l+m$ cannot be drawn through the group $p+q$. Hence the only curve of degree $l+m$ which can be drawn through the group $p+q$ is the improper curve $C_l C_m = 0$; accordingly in this case also $[p+q]=0$.

II. The Multiplication Theorem. *If the group p has a zero residual, then np where n is any positive integer has also a zero residual.*

This at once follows as a corollary of the addition theorem; but there is no *division* theorem, that is to say if np has a zero residual it does not follow that sp, where s is any factor of n, also has a zero residual. This may be proved as follows. Let a proper conic touch a proper cubic at A, B and C; then $[2A+2B+2C]=0$; but $A+B+C$ cannot have a zero residual unless the three points lie in the same straight line, which is contrary to the hypothesis that the conic is a proper one.

III. The Subtraction Theorem. *If $p+q$ and p be two point groups on a curve C_n, each of which has a zero residual, then q has a zero residual.*

The groups $p+q$ and p are the complete intersections of C_n with two curves C_{l+m} and C_l respectively; hence if $l+m \geqq n$, the equation

$$C_{l+m} S_0 + C_n S_{l+m-n} + C_l S_m = 0$$

represents some curve which passes through the group p. But by hypothesis it must be possible to determine the arbitrary constants so that this curve passes through the group $p+q$; hence it must be possible to determine a curve S_m which intersects C_n in the group q and nowhere else.

Let $l+m<n$, and $l \geqq m$. The curve C_{l+m} intersects C_n in $n(l+m)$ points of which, by hypothesis, nl points are the complete intersections of a curve C_l with C_n; but since a proper curve of degree $l+m$ cannot be drawn through more than $l(l+m)$ points on a curve of degree l, it follows that if $nl > l(l+m)$ or $n > l+m$, the curve C_{l+m} must be an improper curve consisting of two curves C_l and C_m, of which the latter intersects C_n in the group q and nowhere else. Hence in both cases q has a zero residual.

IV. The Theorem of Residuation. *If two point groups p and q have a common residual, then any residual of p is a residual of q.*

Let r be the common residual of p and q; and let s be a residual of p. Then by hypothesis

$$[p+r]=0, \qquad [q+r]=0, \qquad [p+s]=0;$$

adding the second and third we obtain

$$[p+q+r+s]=0,$$

and subtracting the first, we get

$$[q+s]=0,$$

which shows that s is a residual of q.

In the preceding theory we have expressly assumed that none of the curves pass through a multiple point on any other curve, so that all the points are *ordinary* points. The case of a node will be discussed later on. The theory is also subject to certain exceptions*, when the points composing any group such as p are so situated that a curve of lower degree than l can be described through them. For example, a cubic is the curve of lowest degree which can be described through six arbitrary points on a given curve; but if the six points were so situated that a conic could be described through them, an exceptional case would arise.

* Bacharach, *Math. Annalen*, vol. XXVI. p. 275; Cayley, *C. M. P.* vol. XII. p. 500. An exceptional case occurs in a theorem proved by myself, *Quart. Jour.* vol. XXXVI. pp. 50 and 51.

368. We shall now give some examples.

(i) *If a straight line intersects a curve of the nth degree in n ordinary points, the tangents at these points intersect the curve in* $n(n-2)$ *points which lie on a curve of degree* $n-2$, *which is called the satellite curve.*

Let the straight line intersect the curve in the group P, and let the tangents at P intersect it in a group Q. Then

$$[P]=0 \text{ and therefore } [2P]=0.$$

The tangents form an improper curve of degree n, which intersects the primitive curve in the groups $2P$ and Q; hence

$$[2P+Q]=0.$$

Accordingly by the subtraction theorem

$$[Q]=0.$$

Since the group Q contains $n^2-2n=n(n-2)$ points, a curve of degree $n-2$ can be drawn through them. Also every curve of degree n can be expressed in the form

$$\alpha^2 S_{n-2}+t_1t_2\ldots t_n=0,$$

where α is the line, $t_1 \ldots t_n$ the n tangents at the points where it cuts the curve, and S_{n-2} the satellite curve.

(ii) *If from any point* O, $n(n-1)$ *tangents be drawn to an anautotomic curve, the points where the tangents intersect the curve lie on one of degree* $(n-1)(n-2)$.

The $n(n-1)$ points of contact form a group P, which is the complete intersection of the curve and its first polar with respect to O; hence

$$[P]=0 \text{ and therefore } [2P]=0.$$

The tangents form an improper curve of degree $n(n-1)$, which touches the curve at the group $2P$ and intersects it at a group Q consisting of $n(n-1)(n-2)$ points; hence

$$[2P+Q]=0,$$

whence by the subtraction theorem

$$[Q]=0;$$

hence the group Q is the complete intersection of the primitive curve with one of degree $(n-1)(n-2)$.

(iii) *If six of the points of intersection of a cubic and a quartic lie on a conic, the remaining six points of intersection lie on another conic; also the four remaining points where the two conics intersect the quartic are collinear.*

Let 6 and 6′ be the two groups of six points in which the cubic intersects the quartic, and let the group 6 lie on a conic. Then the conic will intersect the quartic in two points 2, and the straight line through 2 will intersect the quartic in two other points 2′. Hence

$$[6+6']=0, \quad [6+2]=0, \quad [2+2']=0.$$

Adding the first and third and subtracting the second, we obtain

$$[6'+2']=0,$$

which shows that the eight points $6'+2'$ lie on a conic.

Let a straight line intersect a quartic in the four points S, S', T, T'; then since the straight line repeated three times forms an improper cubic, it follows that if a conic can be described osculating the quartic at S and S', another conic can be described which osculates the quartic at T and T'.

(iv) *A cubic can be drawn through the six points, where the stationary tangents of a trinodal quartic intersect the curve, which osculates the quartic at the T points*.*

Let I denote the six points of inflexion, and J the points where the tangents at the former points intersect the quartic. Then since the six stationary tangents form an improper sextic,

$$[3I+J]=0.$$

It is a known theorem that if S and T denote the S and T points, the eight points I and S lie on a conic; hence

$$[I+S]=0\text{; also } [S+T]=0\text{;}$$

accordingly $\quad [3I+3S]=0, \quad [3S+3T]=0.$

Subtracting the first and fourth and adding the fifth, we obtain

$$[J+3T]=0,$$

* See Appendix I in which the S, T and Q points of plane quartic curves are explained. The S points are those denoted by P and Q in *Cubic and Quartic Curves*, § 193 (iv); and the two remaining points in which the line joining the S points cuts the quartic are called the T points. If the tangents *at* a node intersect the quartic in D, D', the line DD' cuts it in two other points called the Q points. See also Basset, *Amer. Jour.* vol. xxvi. p. 169.

which shows that the six points J and the two T points three times repeated lie on a cubic.

(v) *A conic can be described through the six Q points and the two T points of a trinodal quartic.*

Let D denote the six points where the nodal tangents intersect the quartic; then since the three straight lines passing through each pair of D points and the corresponding pair of Q points form an improper cubic

$$[D+Q]=0.$$

It is a known theorem that a conic can be described through the six D points and the two S points, whence

$$[D+S]=0, \text{ also } [S+T]=0;$$

accordingly $$[Q+T]=0.$$

All the preceding theorems can be proved by trilinear coordinates; and (iv) and (v) by the parametric methods applicable to trinodal quartics*.

369. We shall now consider how these results are modified when a node† forms part of the group; and we shall confine our attention to the case in which two curves have the same node and the same nodal tangents, and such curves will be called *nodotangential* curves. We shall define a *cluster* of points to be any *special arrangement* of points indefinitely close together.

If a nodal curve be cut by two nodotangential curves in two groups of ordinary points p and $p+r$, then r has a zero residual.

Let the curve C_n have a node at A, and let the nodotangential curve C_l cut C_n in a group of ordinary points of degree p. Then

$$\left.\begin{aligned} C_n &= \alpha^{n-2}u_2 + \alpha^{n-3}u_3 + \dots u_n \\ C_l &= \alpha^{l-2}u_2 + \alpha^{l-3}v_3 + \dots v_n \end{aligned}\right\} \qquad (3),$$

also let $$S_r = \Sigma_r{}^0 w_k \alpha^{r-k}, \quad S_r' = \Sigma_r{}^0 w_k' \alpha^{r-k} \qquad (4),$$

and consider the curve

$$\Sigma_{l+m} = C_n S'_{l+m-n} + C_l S_m = 0 \qquad (5),$$

where $l+m \geqq n$. Multiplying out, we obtain

$$\Sigma_{l+m} = \alpha^{l+m-2}(w_0' + w_0)\,u_2 + \dots\dots \qquad (6),$$

* See R. A. Roberts, *Proc. Lond. Math. Soc.* vol. XVI. p. 44.

† Basset, *Quart. Jour.* vol. XXXVI. p. 43.

which shows that Σ_{l+m} is a nodotangential curve; hence C_n, C_l and Σ_{l+m} each pass through the same cluster A of six points at the node. Let $p+r$ be the number of ordinary points in which Σ_{l+m} intersects C_n; then

$$p = ln - A,$$

$$p + r = (l+m)\,n - A,$$

whence $$r = mn.$$

Now by hypothesis

$$[p+A]=0 \text{ and } [p+r+A]=0,$$

and equation (5) shows that it is possible to determine a curve S_m which intersects C_n in a group of mn ordinary points and nowhere else; hence r has a zero residual. When $l+m<n$, Σ_{l+m} is of the form C_lS_m, and the same result follows.

370. *Let two nodotangential curves intersect a nodal curve in two groups of ordinary points p and q; then p and q are coresidual.*

By hypothesis we have the following equations

$$[A+p]=0, \quad [A+q]=0 \text{..................(7)}.$$

Draw any other nodotangential curve cutting the primitive curve in the groups p and q and in a further group of ordinary points r; then

$$[A+p+q+r]=0 \text{ ....(8)}.$$

The theorem of the last article shows that we may subtract (7) from (8) in the same way as if they were groups of ordinary points; we thus obtain the two equations

$$[p+r]=0, \quad [q+r]=0,$$

which show that p and q are coresidual. We may therefore apply the theory of residuation to nodotangential curves in the same way as to groups composed of ordinary points; also the theory applies to nodotangential curves having any number of nodes, and is also true when the double points are biflecnodes.

371. Since a curve, which has none but ordinary nodes, its Hessian and its nodal tangents form a nodotangential system, it follows that:—

(i) *On a curve which has none but ordinary nodes, the points of inflexion and the points where the nodal tangents cut the curve form a pair of coresidual point groups**.

* Richmond, *Proc. Lond. Math. Soc.* vol. XXXIII. p. 218.

Also since the first polar with respect to the node of a uninodal curve (including the case of a biflecnode) is a nodotangential curve, it follows that:—

(ii) *The points of contact of the tangents drawn from the node of any uninodal curve, the points of inflexion and the points where the nodal tangents intersect the curve form a coresidual system.*

Let I, D and E denote the number of points of inflexion, the number of points where the nodal tangents intersect the curve, and the number of points of contact of the tangents drawn from the node. Then for a uninodal quartic $I = 18$, $D = 2$, $E = 6$; also the two Q points are a residual of D. Hence:—

(iii) *The* 18 *points of inflexion of a uninodal quartic lie on a quintic, which passes through the Q points.*

It will hereafter be shown that every quintic which passes through the points of inflexion passes through the Q points; hence if C_4 be the quartic and C_5 the quintic, there is a triply infinite system of such quintics which are determined by the equation

$$C_5 + (l\alpha + m\beta + n\gamma) C_4 = 0.$$

When the node becomes a biflecnode the D points coincide with the node, hence:—

(iv) *The* 16 *points of inflexion of a unibiflecnodal quartic lie on a quartic.*

For a binodal quartic, $I = 12$, $D = 4$; also the D points lie on a conic passing through the nodes; but if the conic degrades into a straight line passing through the D points and one passing through the nodes, $[D] = 0$, hence $[I] = 0$. Accordingly:—

(v) *If the four points, where the nodal tangents of a binodal quartic intersect the curve are collinear, the* 12 *points of inflexion lie on a cubic.*

When the nodes are biflecnodes, this becomes:—

(vi) *The eight points of inflexion of a quartic with two biflecnodes lie on a conic*.*

Assuming the theorem of *Cubic and Quartic Curves*, § 194, we obtain:—

* The expression for the radius of curvature of a Cassinian, see *Cubic and Quartic Curves*, § 251, combined with the theory of projection, furnishes a direct proof of this theorem.

(vii) *The six points of inflexion of a trinodal quartic lie on a conic which passes through the S points.*

For a uninodal quartic $E=6$; hence:—

(viii) *The six points, where the tangents drawn from the node of a uninodal quartic touch the curve, lie on a conic passing through the Q points.*

372. It is a well known theorem that every cubic which passes through eight of the nine points of intersection of two given cubics passes through the remaining one; and we shall now prove a more general theorem.

If l be any integer not less than $n-2$, any curve of degree l which passes through

$$ln - \tfrac{1}{2}(n-1)(n-2)$$

of the points of intersection of two given curves C_l and C_n passes through all the rest.

Let the points of intersection of the curves C_l and C_n be divided into two groups p and $ln-p$; then since the coordinates of the points of the group $ln-p$ satisfy the equation $C_n=0$, and consequently satisfy $ln-p$ equations of condition, the number of available constants, which any curve of degree l passing through this group contains, is

$$\tfrac{1}{2}l(l+3) - ln + p \quad \text{.....................(9)}.$$

The equation of any curve of degree l which passes through the points of intersection of C_l and C_n is

$$C_l + C_n S_{l-n} = 0,$$

provided $l \geqq n$, and it therefore contains

$$\tfrac{1}{2}(l-n+1)(l-n+2) \quad \text{..................(10)}$$

available constants; and if the curve through the group $ln-p$ passes through p, the expressions (9) and (10) must be equal; whence

$$\tfrac{1}{2}l(l+3) - ln + p = \tfrac{1}{2}(l-n+1)(l-n+2),$$

giving $$p = \tfrac{1}{2}(n-1)(n-2).$$

When $l=n-1$ or $n-2$ the theorem is also true; since in this case $ln-p=\tfrac{1}{2}l(l+3)$.

373. Let $l=5$, $n=4$; then $p=3$; hence every quintic which passes through 17 of the points of intersection of a quartic and a given quintic passes through all the rest. Accordingly every quintic which passes through the 18 points of inflexion of a uninodal quartic passes through the Q points.

374. The theorem of § 372 is due to Cayley*; but Bacharach† has pointed out an important exception to it. Let $n > 3$; then the value of p may be written in the form

$$p = \tfrac{1}{2}n(n-3)+1.$$

Now it is not in general possible to describe a curve of degree $n-3$ through the group of points p; but whenever this can be done, Cayley's theorem is not true. This may be proved as follows.

Through the group p describe a curve C_{n-3}, which cuts the curve C_n in a group of s ordinary points, where $s = \tfrac{1}{2}n(n-3)-1$; and through the group s describe another curve C'_{n-3}, which cuts C_n in a group of q points, where $q=p$. Then $s+q=n(n-3)$. Now the curve

$$C_l C'_{n-3} + C_n S_{l-3} = 0$$

is one of degree $l+n-3$ which passes through the group ln and also through the group $q+s$; hence

$$[ln+q+s]=0.$$

But since the group $p+s$ is the complete intersection of C_n and C_{n-3}, it follows that

$$[p+s]=0,$$

whence $$[ln-p+q]=0,$$

which shows that q is a residual of the group $ln-p$.

375. *Every curve of degree m, which passes through*

$$ln-\tfrac{1}{2}(l+n-m-1)(l+n-m-2)$$

of the points of intersection of two curves C_l and C_n passes through all the rest, provided $m \geqq l$ and $m \geqq l+n-2$.

Any curve of degree m which passes through $ln-p$ of the points of intersection of C_l and C_n contains

$$\tfrac{1}{2}m(m+3)-ln+p \quad\text{.....................(11)}$$

* *C. M. P.* vol. I. p. 25.

† *Math. Ann.* vol. XXVI. p. 275.

constants; but the equation of a curve of degree m which passes through the complete intersection of C_l and C_n is

$$S_{m-l}C_l + C_nS_{m-n} = 0,$$

and the number of available constants which it contains is

$$\tfrac{1}{2}(m-l+1)(m-l+2)+\tfrac{1}{2}(m-n+1)(m-n+2)-1 \ldots (12),$$

and if the curve which passes through the group $ln-p$ also passes through p, the expressions (11) and (12) must be equal, which gives

$$p=\tfrac{1}{2}(l+n-m-1)(l+n-m-2).$$

In this theorem various exceptional cases arise, which have been discussed by Bacharach in the paper referred to.

A corresponding theory exists with respect to the intersections of surfaces, a brief account of which together with references to the original authorities will be found in Pascal's *Repertorio di Matematiche Superiori*, vol. II. pp. 297—303.

Theory of Residuation of Surfaces.

376. When we attempt to apply this theory to surfaces, we are at once confronted with a difficulty. Let the primitive surface C_n be intersected by another surface C_l in a multipartite curve of degree ln, which does not pass through any singular points or curves on either surface. This curve may be divided into two groups of curves p and r; but if the curve of intersection of the two surfaces is a proper curve, it will be impossible to describe an *algebraic* surface through the group r which does not pass through the group p. It is of course possible to perform the mechanical operation of describing a surface, such as a cone, whose vertex is any arbitrary point and whose generators pass through the group of curves r; but if such a surface could be represented by an equation, the latter would be a transcendental and not an algebraic one, and the ordinary theory of algebraic surfaces would not apply. It is therefore necessary to suppose that the curve of intersection of the two surfaces is a *compound* one, consisting of two complete curves of degrees p and r, in which case it will be possible to describe an algebraic surface through r which does not pass through p.

We may therefore extend the theory of the residuation of plane curves to surfaces in the following manner. Let the curve of intersection of two algebraic surfaces be a compound curve consisting of two complete curves of degrees p and r; then the curve p will be called a residual of r and *vice versâ*. Hence two curves p and r on the primitive surface are said to be *residual* to one another, whenever it is possible to draw another surface through them which does not intersect the primitive surface elsewhere. Also the compound curve $p+r$ is said to have a *zero residual*, which is expressed by the symbolic equation

$$[p+r]=0 \qquad\qquad (1).$$

Through the curve r draw a surface C_m which intersects the primitive surface in another curve of degree q, where $mn = q + r$; then the curves p and q have a common residual r, and are called *coresidual* curves, and this is expressed by the symbolic equation

$$[p-q]=0 \qquad\qquad (2).$$

377. The theory of residuation of surfaces, like that of plane curves, depends on three subsidiary theorems, which may be respectively called the *addition theorem*, the *multiplication theorem* and the *subtraction theorem*.

The Addition Theorem. *If p and q be two curves on a surface, each of which has a zero residual, then the compound curve $p+q$ has also a zero residual.*

Since $[p]=0$, this curve must be the complete intersection of a surface C_l with the primitive surface C_n; and for the same reason the curve q must be the complete intersection of a surface C_m with C_n. Let $l + m \geqq n$; then the surface

$$C_n S_{l+m-n} + C_l C_m = 0 \qquad\qquad (3)$$

obviously passes through the compound curve $p+q$; also since the degree of the surface is $l+m$, whilst that of the curve $p+q$ is $n(l+m)$, the surface (3) cannot intersect C_n elsewhere.

When $l+m<n$, the only surface of degree $l+m$ which can be drawn through the two curves is the improper surface $C_l C_m$.

The Multiplication Theorem. *If p has a zero residual, then np, where n is any positive integer, has also a zero residual.*

This follows at once as a corollary of the addition theorem.

The Subtraction Theorem. *If $p+q$ and p be two curves on a surface, each of which has a zero residual, then q has a zero residual.*

Let the curves $p+q$ and p be the complete intersections of C_n with two surfaces C_{l+m} and C_l; and let $l+m \geqq n$. Then the surface C_{l+m} must be of the form

$$C_{l+m}S_0 + C_n S_{l+m-n} + C_l S_m = 0,$$

for this surface is of degree $l+m$ and passes through the curve p, whose degree is ln, which lies in the three surfaces C_{l+m}, C_n and C_l. Now the curve q lies in the surfaces C_{l+m} and C_n, and therefore it must be possible to determine a surface S_m which intersects C_n in the curve q and nowhere else; hence q has a zero residual.

When $l+m<n$, the only surface of degree $l+m$ which can be drawn through the curve $p+q$ is the improper surface $C_l S_m$; hence in this case also, q has a zero residual.

The Theorem of Residuation. *If two curves p and q on a surface C_n have a common residual r, then any residual of p is a residual of q.*

Let s be some other curve which is a residual of p; then by hypothesis

$$[p+r]=0, \quad [q+r]=0, \quad [p+s]=0 \quad \text{.........(4).}$$

By means of the addition theorem, we obtain from the last two of (4)

$$[p+q+r+s]=0 \quad \text{.....................(5),}$$

whence by the subtraction theorem, (5) and the first of (4) give

$$[q+s]=0,$$

which shows that s is a residual of q.

This theory is subject to certain exceptions, similar to those discussed by Bacharach in the case of plane curves.

378. We shall now illustrate this theory by some examples.

The tangent cone to an anautotomic surface intersects it in a curve of degree $n(n-1)(n-2)$, which is the complete intersection of the primitive surface and another surface of degree

$$(n-1)(n-2).$$

Let P be the curve of contact of the tangent cone, and Q the curve in which the latter intersects the surface; then $[2P+Q]=0$. But since P is the complete intersection of the primitive surface with its first polar with respect to the vertex of the cone, $[P]$ and therefore $[2P]=0$; hence by the subtraction theorem $[Q]=0$, which shows that another surface can be drawn through Q which intersects the primitive surface nowhere else.

Since the degree of the tangent cone is $n(n-1)$, it follows that the degree of its complete curve of intersection is $n^2(n-1)$; also the degree of the curve P is $n(n-1)$, hence that of Q is

$$n^2(n-1) - 2n(n-1) = n(n-1)(n-2).$$

379. In § 102 we discussed the classification of curves; and we shall now apply the theory of residuation to show that:—*If two cubic surfaces have a common quartic curve of the first species, their residual intersection is a quintic curve of the first species.*

Take one of the cubic surfaces as the primitive one, and denote the quartic and quintic curves by 4 and 5. Through 4 draw a quadric surface, which *must* cut the cubic surface in a conic 2; and let the plane of the conic intersect the cubic surface in the straight line 1. Then since 4 is a residual of 2 and 5, the latter are coresidual curves; and since 1 is a residual of 2, 5 and 1 have a zero residual and therefore lie on a quadric surface.

In the same way it can be shown that:—*If a quadric and a quartic intersect in a twisted cubic and a quintic, the latter is of the first species.*

380. There is another branch of the theory of considerable importance. Cremona showed that a twisted quartic curve of the second species is the partial intersection of a cubic surface, which possesses a nodal line, and a quadric surface which contains the line; and Cayley showed that a quintic curve of the fourth species is the partial intersection of a quartic surface, which has a triple line, and a quadric surface which contains the line. These results have already been obtained by writing down the equations of the curve from the usual definitions; but what is required is a general method which will give the result without going through the labour of proving it in each particular case. Moreover the equations furnished by this method are usually the simplest ones for expressing the curve, since the residual intersection is concentrated in a single straight line. Similar observations apply to twisted curves which can be expressed as the partial intersections of two surfaces, the residual intersection being a multiple curve on one or both surfaces.

The extension of the theory, which I shall proceed to explain, furnishes an answer to the following two questions. Let a given line L be a multiple line on a surface Σ and an ordinary line on two other surfaces S, S'; let the residual intersection of Σ and S; of Σ and S'; and of S and S' be three curves P, Q and R respectively. Required the conditions that (i) when Σ is the primitive surface, P and Q should be ordinary coresidual curves, (ii) when S is the primitive surface, P and R should be ordinary coresidual curves.

381. We shall first prove the following theorem.

Let two surfaces C_l, C_n intersect in a line AB, which is an ordinary line on C_l and a multiple line of order p on C_n; and let the residual curve of intersection be P. Draw any other surface which has p-tactic contact with C_l at every point on AB, and which intersects C_l in a residual curve $P+Q$. Then Q has a zero residual.

Let

$$C_l = \alpha^{l-1} v_1 + \alpha^{l-2}(\beta w_1 + w_2) + \alpha^{l-3}(\beta^2 \sigma_1 + \beta\sigma_2 + \sigma_3) + \ldots\ldots \quad (6),$$

$$C_n = \alpha^{n-p} v_p + \alpha^{n-p-1}(\beta w_p + w_{p+1}) + \ldots\ldots \quad \ldots\ldots(7),$$

then AB is an ordinary line on C_l and a multiple line of order p on C_n. Change the tetrahedron of reference to $A'BCD$ by writing $\lambda\alpha - \beta$ for β, where $\lambda\alpha = \beta$ is the equation of the plane $A'CD$ referred to $ABCD$; then the equation of the tangent plane to C_l at A' is

$$v_1 + \lambda w_1 + \lambda^2 \sigma_1 + \ldots\ldots = 0 \quad \ldots\ldots(8).$$

Let

$$\left.\begin{aligned} S_r &= \alpha^r u_0 + \alpha^{r-1} u_1 + \ldots\ldots u_r \\ S_r' &= \alpha^r u_0' + \alpha^{r-1} u_1' + \ldots\ldots u_r' \end{aligned}\right\} \ldots\ldots(9),$$

and consider the surface

$$\Sigma_{l+m} = C_n S'_{l+m-n} + C_l S_m = 0 \quad \ldots\ldots(10),$$

where $l+m \geqq n$.

The highest power of α occurs in the last term of (10) and is $\alpha^{l+m-1} v_1 u_0$, which shows that C_l and Σ have the same tangent plane at A; and if the tetrahedron be changed to $A'BCD$, it will be found that the coefficient of α^{l+m-1} becomes the expression (8) multiplied by a constant. Hence the two surfaces have a common tangent plane at every point on AB.

Take C_l as the primitive surface. Then from (10) it follows that the surfaces Σ and C_l intersect in a curve of degree

$$l(l+m-n) = Q,$$

which is the complete curve of intersection of C_l and S'_{l+m-n}; and in another curve which is the complete curve of intersection of C_l and C_n. The latter consists of the line AB repeated p times and a residual curve of degree $ln-p=P$; and since AB is an ordinary line on the two surfaces Σ and C_l, and both surfaces have been shown to have a common tangent plane at every point on AB, it follows that Σ and C_l must have p-tactic contact with one another at every point on AB. Denoting therefore by A the cluster of p lines AB which are common to the three surfaces Σ_{l+m}, C_n and C_l, it follows that the intersection of C_n and C_l gives

$$[A+P]=0 \quad \text{.......................(11).}$$

The intersection of Σ_{l+m} and C_l gives

$$[A+P+Q]=0 \quad \text{....................(12),}$$

and the intersection of S'_{l+m-n} and C_l gives

$$[Q]=0 \quad \text{..........................(13),}$$

which proves the theorem.

382. *Let AB be an ordinary line on the primitive surface S, and a multiple line of order p on another surface S'; and let S and S' intersect in a residual curve P. Draw a second surface S'' which has p-tactic contact with S at every point on AB, and intersects S in a residual curve Q. Then P and Q are coresidual curves on S.*

The three surfaces intersect one another along AB in a cluster A of lines, which consist of AB repeated p times. Hence

$$[A+P]=0, \quad [A+Q]=0 \quad \text{..............(14).}$$

Through the curves P and Q draw any other surface which has p-tactic contact with S at every point on AB, and intersects S in a residual curve R; then

$$[A+P+Q+R]=0,$$

whence by the last article

$$[P+R]=0, \quad [Q+R]=0,$$

which shows that P and Q have an ordinary residual R, and are therefore coresidual curves.

383. The theory is of a similar character when C_n is taken as the primitive surface. The form of (10) shows that the surfaces Σ and C_n intersect in a curve of degree $nm=Q'$, which is the complete curve of intersection of C_n and S_m; and in another curve which is the complete curve of intersection of C_l and C_n. Hence as before we obtain

$$[A+P]=0.$$

Also the intersection of Σ_{l+m} and C_n gives

$$[A+P+Q']=0,$$

and that of C_n and S_m gives

$$[Q']=0.$$

By means of these equations we can prove, as in § 382, the theorem:—

Let AB be a multiple line of order p on the primitive surface S, and an ordinary line on another surface S′; and let S and S′ intersect in a residual curve P. Draw a second surface S″ which has p-tactic contact with S′ at every point on AB, and intersects S in a residual curve Q. Then P and Q are coresidual curves on S.

384. *If a twisted curve is the partial intersection of two surfaces C_l and C_n, where $l \geqq n$, which are such that AB is a multiple line of order p on C_l and an ordinary line on C_n; then the curve is the partial intersection C_n with another surface S_l, which has p-tactic contact with C_n at every point on AB.*

Consider the surface

$$C_l S_0 + C_n S_{l-n} = S_l = 0.$$

Since S_{l-n} is a general quaternary quantic of degree $l-n$, the highest powers of α and β in S_l are the $(l-1)$th powers; hence AB is an ordinary line on S_l; also the form of S_l shows that it intersects C_n in the complete curve of intersection of C_l and C_n and nowhere else. The curve of intersection of S_l and C_n must therefore consist of the above mentioned curve and the line AD repeated p times; and since AB is an ordinary line on both surfaces, they must have p-tactic contact at every point on AB.

This result is of importance in the classification of twisted curves. Also it can be proved in the same manner that:—

If a twisted curve is the partial intersection of two surfaces C_l and C_n, where $l \geqq n$, which are such that (i) *AB is a multiple line of order $p-1$ on C_l, and* (ii) *the tangent plane to C_n along AB is one of the tangent planes to C_l along the same line; then the curve is the partial intersection of C_n with another surface S_l which has p-tactic contact with C_n at every point on AB.*

385. *A twisted curve which is the partial intersection of a quadric surface and a surface C_n, where the residual intersection consists of p distinct lines lying in different planes, is the partial intersection of the quadric with another surface S_n, which intersect in a common line, which is an ordinary line on the quadric and a multiple line of order p on S_n.*

Let the surfaces be

$$(P\alpha + Q\beta)\gamma = (R\alpha + S\beta)\delta \quad \text{..............(15)},$$

$$\alpha\gamma = \beta\delta \quad \text{..........................(16)},$$

where P, Q, R, S are quaternary quantics of degree $n-2$; whence eliminating (γ, δ) it follows that (15) may be replaced by

$$R\alpha^2 + (S-P)\alpha\beta - Q\beta^2 = 0 \quad \text{.............(17)},$$

on which CD is a nodal line. Let another generator (u, v) of the same system be common to (15) and (16); then (16) must be expressible in the form

$$\alpha u = \beta v \quad \text{.........................(18)},$$

and P, Q, R, S must be linear functions of (u, v); whence eliminating (u, v) between (15) and (18) we obtain an equation of the form

$$(A, B, C, D \between \alpha, \beta)^3 = 0,$$

on which CD is a triple line. Proceeding in this way, we obtain the theorem.

Twisted Sextic Curves.*

386. There are five primary species of twisted sextic curves.

I. The *complete* intersection of a quadric and a cubic surface.

II. The *partial* intersection of two cubic surfaces, when the residual intersection consists of a twisted cubic curve. Their equations may be expressed by means of the system of determinants

$$\left\| \begin{matrix} A, & A', & u, & u' \\ B, & B', & v, & v' \\ C, & C', & w, & w' \end{matrix} \right\| = 0 \quad \text{.................(1)},$$

where all the quantities represent planes.

* Clebsch, *Crelle*, vol. LXIII.; Nöther, *Crelle*, vol. XCIII.; Pascal, *Lincei*, 1893, p. 120. Septimic curves have been discussed by Weyr, *Wiener Berichte*, vol. LXIX.

III. The partial intersection of two cubic surfaces, when the residual curve consists of a conic and a straight line lying in a different plane. Let AB be the straight line; and let the conic be the intersection of the plane α and the quadric S; then the equations of the sextic are

$$(p\gamma + q\delta) S = (u\gamma + v\delta)\alpha \quad \text{.................(2)},$$

$$(P\gamma + Q\delta) S = (u'\gamma + v'\delta)\alpha \quad \text{.................(3)},$$

where P, p, Q, q are constants, and u, v, u', v' are planes. Eliminating S and α, we obtain

$$(p\gamma + q\delta)(u'\gamma + v'\delta) = (u\gamma + v\delta)(P\gamma + Q\delta) \quad \text{.........(4)}.$$

Equation (4) represents a cubic surface on which AB is a nodal line, and it also contains another line EF which is the residual intersection of the plane $p\gamma + q\delta = 0$ and the quadric $u\gamma + v\delta = 0$. Also since AB and EF lie in the same plane, they are generators of opposite systems on the quadric. The sextic may therefore be regarded as the partial intersection of the cubics (2) and (4), which contain two straight lines lying in the same plane, one of which is a nodal line on the second cubic.

IV. The partial intersection of two cubics, when the residual intersection consists of three straight lines lying in different planes.

Let AB and CD be two of the lines, then the equations of the third line may be taken to be

$$\lambda\alpha + \delta = 0, \quad \mu\beta + \gamma = 0,$$

and the equation of the quadric having these three straight lines for generators is

$$\lambda\alpha\gamma = \mu\beta\delta,$$

and the equations of the two cubics which contain the sextic may be written

$$(\lambda\alpha\gamma - \mu\beta\delta)\, u = (\alpha v_1 + \beta w_1)\{P\,(\lambda\alpha + \delta) + Q\,(\mu\beta + \gamma)\} \ldots (5),$$

$$(\lambda\alpha\gamma - \mu\beta\delta)\, u' = (\alpha v_1' + \beta w_1')\{P'\,(\lambda\alpha + \delta) + Q'\,(\mu\beta + \gamma)\} \ldots (6),$$

where u, u' are arbitrary planes; v_1, w_1 ... are linear functions of (γ, δ), and P, Q, P', Q' arbitrary constants.

V. The partial intersection of a quadric and a quartic surface, when the residual intersection consists of two straight lines lying in different planes.

The equations of the surfaces containing the curve may be written

$$\alpha\gamma + \beta\delta = 0 \quad \text{.........................(7)},$$

$$(P\alpha + Q\beta)\gamma + (R\alpha + S\beta)\delta = 0 \quad \text{...............(8)},$$

where P, Q, R, S are quadric surfaces.

387. In considering the possible intersections of two cubic surfaces, we have the following additional cases to consider.

(i) When the two cubics osculate one another along a line *AB*. By virtue of § 384, this curve is the same as the partial intersection of two cubic surfaces which possess a common straight line, which is a triple line on one of them; but since the only cubic of this species consists of three planes intersecting in the line, the sextic is an improper one consisting of three conics lying in different planes which intersect in a line.

(ii) When one of the cubics has a nodal line, and the other cubic contains the line and is touched along it by one of the nodal tangent planes to the first cubic. By the corollary to § 384, this is of the same species as (i).

(iii) When one of the cubics touches the other along a line, and intersects it along a third line lying in a different plane.

(iv) When the two cubics intersect in two straight lines lying in different planes, one of which is a nodal line on one of the cubics.

The equations of the two cubics in (iii) are

$$\alpha^2\gamma + 2\alpha\beta v_1 + \beta^2\delta + \alpha v_2 + \beta w_2 = 0 \quad \text{...........(9)},$$

$$\alpha^2\gamma + 2\alpha\beta v_1 + \beta^2\delta + \alpha v_2' + \beta w_2' = 0 \quad \text{...........(10)},$$

whence by subtraction

$$\alpha(v_2 - v_2') + \beta(w_2 - w_2') = 0 \quad \text{..............(11)},$$

which shows that the curves (iii) and (iv) are identical. Write (11) in the form

$$\alpha\omega_2 + \beta\omega_2' = 0 \quad \text{......................(12)},$$

by virtue of which (9) may be written

$$(\alpha\gamma + \beta v_1 + v_2)\,\omega_2' = (\beta\delta + \alpha v_1 + w_2)\,\omega_2 \quad \text{.........(13)}.$$

Let the capital letters denote what these quantities become when $\delta = k$, $\gamma = 1$; then the sections of (12) and (13) by the plane $\delta = k\gamma$ are

$$\left.\begin{aligned} &\alpha\Omega_2 + \beta\Omega_2' = 0 \\ &(\alpha + \beta V_1 + \gamma V_2)\,\Omega_2' = (k\beta + \alpha V_1 + \gamma W_2)\,\Omega_2 \end{aligned}\right\} \quad \text{...(14)},$$

which are the equations of two planes. The equation $\delta = k\gamma$ combined with (14) determines the six points in which the plane intersects the sextic curve; and since only one of them lies outside AB, it follows that five of them lie on this line, which is therefore a *quinquesecant*. To determine these points put $\gamma = 0$ in the second of (14), and eliminate (α, β) and we obtain

$$\Omega_2'^2 - 2V_1\Omega_2\Omega_2' + k\Omega_2^2 = 0,$$

which is a quintic equation for determining k, and shows that the five points are distinct. Also since a curve cannot in general have a quinquesecant, this species is a special kind of a more general one.

By means of § 384 or directly, it can be shown that when (i) a quadric surface passes through a nodal line on a quartic surface, or (ii) a quadric and a quartic surface touch one another along a line, the residual sextic belongs to species V.

CHAPTER IX

SINGULAR TANGENT PLANES TO SURFACES

388. THE theory of the singularities of plane curves is comparatively easy, owing to the fact (i) that such curves possess only four simple singularities, viz. the node and the cusp which are point singularities, and the double and the stationary tangent which are line singularities; (ii) that the two simple point singularities are the reciprocal polars of the two simple line singularities. But the theory of the singularities of surfaces is much more difficult, (i) because surfaces possess two simple point singularities, viz. the conic node and the binode, and six simple plane singularities, the nature of which has been explained in § 11; (ii) because the reciprocal polar of a conic node or a binode is a compound plane singularity of a special kind, and no theory of reciprocation exists between the simple point and plane singularities of surfaces analogous to the corresponding one for plane curves. When the surface is anautotomic, the values of ϖ_1, ϖ_3 and ϖ_5 were first obtained by Salmon*; those of ϖ_4 and ϖ_6 by Schubert†; but the value of ϖ_2 appears to have been first given by myself‡ in 1908. In a subsequent paper§ I obtained the values of the six singular planes, when a surface possesses C conic nodes and B binodes which are isolated; but certain portions of this investigation are subject to the limitation, that the double points must not be so numerous as to cause the tangent cone from any one of them to degrade into an improper cone. These portions do not therefore apply to quartic surfaces possessing more than 11 conic nodes, since the tangent cone from a conic node being a sextic one would degrade. Cayley in his paper on reciprocal surfaces|| has attempted

* *Trans. Roy. Irish Acad.* vol. XXIII. p. 461.

† *Math. Annalen*, vol. X. p. 102; vol. XI. p. 348.

‡ *Quart. Jour.* vol. XL. p. 210.

§ *Ibid.* vol. XLII. p. 21.

|| *Phil. Trans.* vol. CLIX. p. 210; *C. M. P.* vol. VI. p. 329, see p. 347.

to find the value of ϖ_5, which he calls β', for a surface possessing a nodal and a cuspidal curve of degrees b and c respectively and also C conic nodes and B binodes; but the investigation is not very intelligible. Amongst special results, we may notice that Berzolari* found that a quartic surface having a nodal conic possesses 40 triple tangent planes, see § 274; while Pascal† states that for such a surface $\varpi_5 = 52$, see § 275. The same author also states that when a quartic surface possesses 12 conic nodes, $\varpi_3 = 0$, $\varpi_5 = 32$; and the last result agrees with that given by my own formula.

In §§ 10 and 11 the various curves and developables connected with this branch of the subject, as well as the notation employed, have been defined and explained; and we shall commence with a discussion of the spinodal, the flecnodal and the bitangential curves and the surfaces associated with them. I shall denote the spinodal, the flecnodal and the bitangential developables by the symbols D_s, D_f, D_b, and their edges of regression by E_s, E_f, E_b.

The Spinodal Curve.

389. The surface

$$\alpha^{n-1}\delta + \alpha^{n-2}\{\delta^2 v_0 + \delta(p\beta + q\gamma) + r\gamma^2\} + \alpha^{n-3}u_3 + \dots u_n = 0 \quad \dots(1)$$

is one on which A is a point on the spinodal curve, ABC is the tangent plane at A, and AB is the cuspidal tangent to the section of the surface by the plane ABC.

The first step is to examine the intersection of (1) and its Hessian at A. The Hessian will be found to be of the form

$$-8r(n-1)\left\{(n-1)\frac{d^2u_3}{d\beta^2} - 3(n-2)p^2\delta\right\}\alpha^{4n-9} + \dots = 0 \quad \dots(2).$$

Let $$u_3 = P\beta^3 + 3(Q\gamma + R\delta)\beta^2 + \dots,$$

then the equation of the tangent plane to the Hessian at A is

$$2(n-1)(P\beta + Q\gamma + R\delta) - (n-2)p^2\delta = 0 \quad \dots\dots\dots(3),$$

and the tangent line AE to the spinodal curve is the intersection of (3) with the plane δ, and therefore does not coincide with AB.

When $P = 0$, the section of (2) by the plane δ is the curve

$$r\alpha^{n-2}\gamma^2 + (\beta, \gamma)^2\alpha^{n-3}\gamma + (\beta, \gamma)^4\alpha^{n-4} + \dots = 0 \quad \dots\dots(4),$$

* *Annali di Matematica*, Serie II. vol. XIV. p. 31.

† *Repertorio di Matematiche Superiori*, vol. II. p. 424.

so that the point of contact is a *tacnode* on the section. In this case the tangent at the tacnode is the tangent to the spinodal curve.

The spinodal curve does not possess any stationary tangents, for such a tangent must have tritactic contact with (1) and also with the Hessian at the point of contact. Now the tangent cannot have tritactic contact with (1) except at the tacnodal points, where the contact is quadritactic; but at such points it appears, from (2) and (3), that the contact with the Hessian is bitactic. Hence $\iota = 0$.

We have shown in § 55 that when a straight line lies in the surface it touches but does not intersect the spinodal curve; from which it follows that τ is, in general, zero.

The spinodal curve cannot have any double points; for at such points it is necessary that the Hessian should touch the surface, which requires that $P = Q = 0$. The section of the surface by the tangent plane is now of the form

$$r\alpha^{n-2}\gamma^2 + (\beta, \gamma)\,\alpha^{n-3}\gamma^2 + (\beta, \gamma)^4\,\alpha^{n-4} + \ldots = 0 \quad \ldots\ldots(5),$$

and consequently the singularity at A on the section is the particular kind of tacnode formed by making the two tangents at a *biflecnode* coincide. Now four conditions must be satisfied in order that the point of contact of the tangent plane should be a singularity of this character, which is in general impossible since the equation of a plane contains only three constants. Hence

$$\delta = \kappa = 0.$$

Let us now denote the degree of the original surface by N; then the characteristics of the spinodal curve and the developable enveloped by its osculating planes are obtained from equations (10) to (15) of § 107 by writing

$$M = 4(N-2), \quad \delta = \kappa = \tau = \iota = 0 \quad \ldots\ldots\ldots\ldots(6),$$

accordingly

$$\left.\begin{aligned} n &= 4N(N-2) \\ h &= 2N(N-1)(N-2)(4N-9) \\ \nu &= 20N(N-2)^2 \\ m &= 3N(N-2)(5N-11) \\ \sigma &= 8N(N-2)(15N-34) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(7).$$

These formulæ are the most important. The values of x and y can be obtained from (14) and (15) and that of $g + \varpi$ from (13) of

§107. Further investigation is required before the values of g and ϖ can be determined. With this exception we have obtained the characteristics of the spinodal curve and the developable enveloped by its osculating planes.

The Spinodal Developable D_s and its Edge of Regression E_s.

390. This developable may be regarded indifferently as the envelope of the tangent planes to the surface at points on the spinodal curve; or as the developable generated by the cuspidal tangents to the section of the surface by the tangent planes at these points.

391. *The degree ν of the spinodal developable is given by the equation*

$$\nu = 2N(N-2)(3N-4) \quad \text{.................}(8).$$

Let L be any fixed line, O any point on it; then $N(N-1)(N-2)$ stationary tangents can be drawn from O to the surface; hence as O moves along L these tangents will generate a scroll on which L is a multiple generator of order $N(N-1)(N-2)$. Let OP be any generator of this scroll, (f, g, h, k) the coordinates of the point P where it touches the surface; then since OP lies in the tangent plane and the polar quadric of P, it follows that if we eliminate $(\alpha, \beta, \gamma, \delta)$ between the equations of the two planes which determine L and also those of the tangent plane and polar quadric of P, we shall obtain a relation between (f, g, h, k) which is the equation of a surface Σ intersecting the original surface in the locus of P. Let U be the original surface; let $U(f, g, h, k) = U'$; and let the equations of L be

$$\left.\begin{aligned} P\alpha + Q\beta + R\gamma + S\delta = 0 \\ p\alpha + q\beta + r\gamma + s\delta = 0 \end{aligned}\right\} \quad \text{..................}(9);$$

also let Δ' denote the operator

$$\Delta' = \alpha d/df + \beta d/dg + \gamma d/dh + \delta d/dk \quad \text{.........}(10),$$

then the equations of the tangent plane and polar quadric at P are

$$\Delta' U' = 0, \quad \Delta'^2 U' = 0 \quad \text{.................}(11),$$

and since the result of eliminating $(\alpha, \beta, \gamma, \delta)$ between (9) and (11) furnishes an equation of degree $2(N-1) + N - 2 = 3N - 4$ in (f, g, h, k) this is the degree of the surface Σ.

Now if P be one of the points where Σ cuts the spinodal curve, the generator of the spinodal developable which passes through P is also a generator of the scroll and therefore passes through the line L; hence the number of such generators is apparently equal to $4N(N-2)(3N-4)$; but since the tangent plane to the surface at P touches the polar quadric of P (which is a cone) along a generator, this number must be halved, which gives (8).

392. *The class m of the spinodal developable is given by the equation*

$$m = 4N(N-1)(N-2) \quad\text{.................(12).}$$

Let O be any arbitrary point; then every tangent plane to the surface through O, which touches it at a point P on the spinodal curve, is a tangent plane to the developable. Hence its class is equal to the number of points in which the first polar of O intersects the spinodal curve.

393. Equation (12) determines the class of the edge of regression E_s of D_s, and we must now consider this curve. If E_s had any double or stationary tangents, these would give rise to nodal and cuspidal generators on D_s, and therefore to nodes and cusps on the spinodal curve; and since we have shown that this curve has no double points when the surface is anautotomic, it follows that $\tau = \iota = 0$. If, however, the surface were autotomic, τ and ι need not be zero.

At a tacnodal point, the tacnodal tangent on the section is the tangent to the spinodal curve, and is therefore equivalent to the cuspidal tangent at two points P and P' which ultimately coincide. Hence at such a point two osculating planes to E_s coincide, and therefore the tangent plane to the surface at a tacnodal point osculates D_s along the tacnodal tangent, and is therefore a stationary plane σ to E_s. Now a tacnode is a compound singularity which has several penultimate forms. In particular, it may be regarded as a cusp whose cuspidal tangent has quadritactic contact at the cusp, or as a flecnode whose two tangents coincide. Hence the tacnodal points are points where the cuspidal and flecnodal curves intersect. We shall hereafter prove that the flecnodal curve is the complete intersection of the surface and one of degree $11N-24$; hence the spinodal and flecnodal curves apparently intersect in $4N(N-2)(11N-24)$ points. We shall also show that these two curves touch one another, but do not intersect; accordingly the number of tacnodal points is half this number, and a direct proof may be given by means of the theory of united points, which has been explained in Chapter III.

394. *The number of singular tangent planes whose point of contact is a tacnode on the section is*

$$\varpi_5 = 2N(N-2)(11N-24) \quad\ldots\ldots\ldots\ldots\ldots(13),$$

also each of these planes is a stationary plane σ *to the curve* E_s.

Let L be any fixed line; through L draw a plane x cutting the spinodal curve in a series of points P; then the tangent to the cusp at P on the section of the surface by the tangent plane at P will cut the surface in $N-3$ points Q; through L and Q draw a series of planes y; and take x and y as corresponding planes.

Since there are n points P lying in the plane x, it follows that there are $n(N-3)$ planes y corresponding to a single plane x; hence

$$\mu = n(N-3).$$

The spinodal developable intersects the surface in the spinodal curve three times repeated and in a residual curve of degree n', where

$$n' + 3n = N\nu,$$

and since n' planes x correspond to each plane y, it follows that

$$\lambda = n' = N\nu - 3n.$$

United planes will occur:—

(i) When P is the point of contact of one of the planes ϖ_5.

(ii) When the line PQ passes through the line L; but since each line PQ contains $N-3$ points Q, and ν is the degree of the spinodal developable, the number of united planes due to this cause is $(N-3)\nu$. We thus obtain

$$\lambda + \mu = N\nu + n(N-6) = \varpi_5 + (N-3)\nu.$$

Substituting the value of ν from (8) and recollecting that $n = 4N(N-2)$, we obtain

$$\varpi_5 = 2N(N-2)(11N-24).$$

395. Equations (8), (12), and (13) accordingly furnish the following formulæ for the characteristics of D_s and E_s, viz.

$$\left.\begin{aligned} \nu &= 2N(N-2)(3N-4) \\ m &= 4N(N-1)(N-2) \\ \sigma &= 2N(N-2)(11N-24) \\ \tau &= 0,\ \iota = 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(14),$$

and by means of (4) and (5) of § 104 the following additional formulæ can be obtained, viz.

$$\left.\begin{aligned} n &= 4N(N-2)(7N-15) \\ \kappa &= 10N(N-2)(7N-16) \\ x &= N(N-2)(18N^4-84N^3+160N^2-111N+96) \\ g+\varpi &= 2N(N-2)(4N^4-16N^3+20N^2-27N+39) \\ h+\delta &= 2N(N-2)(196N^4-1232N^3+2580N^2-1861N+135) \end{aligned}\right\}$$
$$\ldots\ldots\ldots(15).$$

The formulæ (14) and (15) agree with those obtained by Salmon* by a different method with this exception. The coefficient of the last term in the last of equations (15) is, according to Salmon, 274 instead of 270; and he has also assumed, without proof, that ϖ and δ are zero.

The Flecnodal Curve, its Developable D_f and the Edge of Regression E_f of the Latter.

396. The flecnodal curve has been defined in § 10; and there are three species of singular points lying on it. In the first place the points, where the curve touches the spinodal curve, are the tacnodal points which have already been considered. In the second place the *biflecnodal* points, where the planes ϖ_4 touch the surface, are nodes on the flecnodal curve, for at such points two generators of the flecnodal developable intersect on the curve. The latter cannot, however, have any cusps, for such a singularity could only occur if the point of contact of the section by the tangent plane were the particular kind of tacnode which is formed by the coincidence of the two tangents at a biflecnode; and we have shown that such points cannot in general exist. In the third place the points, where the planes ϖ_6 touch the surface, lie on this curve.

* *Geometry of Three Dimensions*, p. 580.

397. *The degree of the flecnodal developable is determined by the equation*

$$\nu = 2N(N-3)(3N-2) \quad\text{.................}(16).$$

Let A be a point on the flecnodal curve; ABC the tangent plane at A; AB, AC the tangents at A to the section by the plane ABC, of which AB is the flecnodal tangent. Then the equation of the surface U is

$$\alpha^{n-1}\delta + \alpha^{n-2}(\delta^2 v_0 + \delta v_1 + p\beta\gamma) + \alpha^{n-3}(\delta^3 w_0 + \delta^2 w_1 + \delta w_2 + \gamma W_2)$$
$$+ \alpha^{n-4}u_4 + \ldots u_n = 0 \quad\text{........}(17).$$

Writing down the polar quadric and cubic of A, it follows that both the tangents at A to the section lie in the polar quadric, and that the flecnodal tangent AB lies in the polar cubic.

Let P be any point (f, g, h, k) on the flecnodal curve; PO the flecnodal tangent to the section by the tangent plane at P. Then the equations of the tangent plane, the polar quadric and cubic of P, are

$$\Delta' U' = 0, \quad \Delta'^2 U' = 0, \quad \Delta'^3 U' = 0 \quad\text{...........}(18),$$

where Δ' is given by (10); also, since (f, g, h, k) lies on the surface,

$$U' = 0 \quad\text{.........................}(19).$$

The point P lies on the four surfaces (18) and (19), whilst $(\alpha, \beta, \gamma, \delta)$ are the coordinates of *any* point on the line PO, which is common to the three surfaces (18); if, therefore, we eliminate (f, g, h, k) between (18) and (19), we shall obtain a relation between $(\alpha, \beta, \gamma, \delta)$ which connects the coordinates of any point on the flecnodal tangent PO, and is therefore the equation of the flecnodal developable. By the usual rule, the degree of the eliminant in $(\alpha, \beta, \gamma, \delta)$ is apparently equal to

$$n(n-2)(n-3) + 2n(n-1)(n-3) + 3n(n-1)(n-2)$$
$$= 6n^3 - 22n^2 + 18n \quad\text{........}(20),$$

but we shall now show that this result must be reduced by $6n$.

Equations (18) and (19) may be regarded in another light; for if $(\alpha, \beta, \gamma, \delta)$ were a fixed point O on the flecnodal tangent at a point P on the surface, and (f, g, h, k) a variable point, equations (18) would be the first, second, and third polars of the surface with respect to O. Hence the result of eliminating (f, g, h, k) between (18) and (19) gives the locus of points, such as O, whose first, second, and third polars intersect on the surface U, and the degree of this locus is given by (20). But if we write down the first, second, and third polars of (17) with respect to A, it can easily be shown that they intersect in six coincident points at A; hence the original surface U six times repeated forms part of the locus. Accordingly the degree of the residual surface, which is the developable in question, is

$$6n^3 - 22n^2 + 18n - 6n = 2n(n-3)(3n-2).$$

Changing n into N, we obtain (16).

398. *The flecnodal curve is the complete intersection of the surface U and one of degree $11N - 24$, and the degree of the flecnodal curve is**

$$n = N(11N - 24) \quad \text{.....................(21).}$$

Let $(\alpha, \beta, \gamma, \delta)$ be any point O on D_f. Then the result of eliminating these quantities between (18) and the equation $D_f(\alpha, \beta, \gamma, \delta) = 0$ gives a relation between (f, g, h, k) of degree

$$6\nu(N-1) + 3\nu(N-2) + 2\nu(N-3) = \nu(11N - 18),$$

hence $11N - 18$ is the degree of a surface which contains the flecnodal curve. But if O be regarded as a fixed point on D_f, and (f, g, h, k) or P a variable point, (18) may, as in the last section, be regarded as the first, second, and third polars of U with respect to O; and since these surfaces intersect U in six coincident points, which lie on the flecnodal curve, the eliminant will furnish a locus which includes D_f six times repeated. Hence if F be the residual surface, we must have

$$\nu(11N - 18) = \nu F + 6\nu,$$

giving $$F = 11N - 24,$$

so that the degree of the flecnodal curve is given by (21).

399. *The class of the flecnodal developable is*

$$m = N(N-1)(11N-24) \quad \text{..............(22),}$$

for this is equal to the number of points in which the first polar

* Otherwise thus. The point $(\alpha, \beta, \gamma, \delta)$ is common to the four surfaces $U(\alpha, \beta, \gamma, \delta) = 0$ and (18); and if we eliminate $(\alpha, \beta, \gamma, \delta)$, we obtain a quantic of (f, g, h, k) of degree $11n - 18$, which, when equated to zero, gives a surface which contains the flecnodal curve. But the point $(\alpha, \beta, \gamma, \delta)$ six times repeated is common to these four surfaces, hence U^6 forms part of the locus; accordingly the degree of the residual surface is $11n - 24$. The form of this result shows that the locus consists of the original and residual surfaces, and the intersection of these two surfaces determines the flecnodal curve.

of U, with respect to any arbitrary point, intersects the flecnodal curve.

400. *The flecnodal and spinodal curves touch one another but do not intersect.*

Since a tacnode may be regarded either as a particular kind of flecnode or cusp, and therefore partakes of the character of both singularities, the points of contact of the tangent planes ϖ_5 must be the points where the spinodal and flecnodal curves intersect one another. The total number of these points is

$$4N(N-2)(11N-24);$$

but since the number of planes ϖ_5 is half this number, the two curves must touch one another at the points of contact of ϖ_5.

401. The 27 lines lying in an anautotomic cubic surface constitute the flecnodal curve; also any line lying in a surface of higher degree forms part of this curve, and the theorem of § 55 is a particular case of the preceding one. If the flecnodal curve consists entirely of straight lines lying in the surface, their number is $N(11N-24)$, hence :—*A surface of the Nth degree cannot possess more than* $N(11N-24)$ *straight lines lying in it.*

402. Before explaining Schubert's method for finding the number of planes ϖ_4 and ϖ_6, some preliminary theorems will be necessary.

The flecnodal developable intersects the surface in a residual curve of degree

$$n_f = 2N(N-4)(3N^2+N-12) \qquad \text{.............(23).}$$

Let n, ν_f denote the degrees of the flecnodal curve and developable respectively; then since the developable intersects the surface in the flecnodal curve four times repeated, the degree n_f of the residual curve is given by the equation

$$N\nu_f = 4n + n_f \qquad \text{........................(24).}$$

Substituting the values of ν_f and n from (16) and (21) we obtain (23).

403. *If P be any point on the flecnodal curve, the ordinary tangent at P to the section of the surface by the tangent plane at P, generates a developable whose degree ν_0 is*

$$\nu_0 = N(27N^2-94N+84) \qquad \text{...............(25).}$$

Let P be any point on the flecnodal curve, then the flecnodal and ordinary tangents at P will generate two developable surfaces ν_f and ν_0; but if P be one of the points where the flecnodal curve intersects the surface Σ, which has been discussed in § 391, one of these two tangents must intersect the fixed line L. Accordingly the degree of the compound surface generated by both tangents is

$$\nu_f + \nu_0 = N(11N - 24)(3N - 4).$$

Substituting the value of ν_f from (16) we obtain (25).

404. *The surface ν_0 intersects the original surface in a residual curve of degree n_0, where*

$$n_0 = N\nu_0 - 3n \quad \text{.......................(26).}$$

For every generator of ν_0 intersects the surface in the flecnodal curve three times repeated and in a residual curve n_0.

405. *The number of singular tangent planes, whose point of contact is a biflecnode on the section, is*

$$\varpi_4 = 5N(7N^2 - 28N + 30) \quad \text{..............(27).}$$

A plane x through a fixed line L intersects the flecnodal curve in n points, where n is given by (21). Let P be one of them, then the ordinary tangent to the surface at P intersects it in $N-3$ points Q, all of which lie on the curve n_0. Let the planes through L and the points Q be the planes y, and take x and y as corresponding planes.

To every point P correspond $N-3$ planes y; and since there are n points P, there are $(N-3)n$ planes y corresponding to a single plane x; hence

$$\mu = (N-3)n.$$

A plane y intersects the curve n_0 in n_0 points, to each of which corresponds a plane x; hence

$$\lambda = n_0.$$

United planes will occur:—

(i) When one of the points Q coincides with P, in which case P is a point of contact of a tangent plane ϖ_4. But since both the tangents at P are flecnodal ones, and Q may be supposed to coincide with either of them, this plane must be counted twice; hence the number of united planes due to this cause is $2\varpi_4$.

(ii) When P is a tacnodal point, one of the points Q will coincide with P; hence the number of united planes due to this cause is ϖ_5.

(iii) Let P be a point where the ordinary tangent intersects the line L; then since $N-3$ points Q lie on this tangent, the number of united planes due to this cause is $(N-3)\nu_0$.

We thus obtain

$$\lambda + \mu = n_0 + (N-3)n = 2\varpi_4 + \varpi_5 + (N-3)\nu_0.$$

Substituting the values of n_0, ν_0, n and ϖ_5 from (27), (26), (21) and (13) we obtain (27).

406. *The number of tangent planes, whose point of contact is a hyperflecnode on the surface, one of whose tangents has ordinary contact and the other quadritactic contact with their respective branches, is*

$$\varpi_6 = 5N(N-4)(7N-12) \quad \text{..............(28).}$$

The planes x are the same as before; but the points Q are those where the flecnodal tangent at P intersects the surface; hence

$$\mu = (N-4)n.$$

A plane y intersects the curve n_f in n_f points, to each of which corresponds one plane x; hence

$$\lambda = n_f.$$

United planes will occur:—

(i) When a point Q coincides with P, in which case P is the point of contact of a plane ϖ_6; hence the number of united planes due to this cause is ϖ_6.

(ii) Let P be a point where the flecnodal tangent intersects the line L; then since $N-4$ points Q lie on this tangent, the number of united planes due to this cause is $(N-4)\nu_f$.

We thus obtain

$$\lambda + \mu = n_f + (N-4)n = \varpi_6 + (N-4)\nu_f.$$

Substituting the values of n_f and ν_f from (24) and (23) we obtain (28).

407. *The Flecnodal Curve.* The characteristics of this curve and the developable enveloped by its osculating planes can now be partially found by means of equations (10) to (15) of § 107; for we have shown in § 398 that

$$M = 11N - 24 \quad \text{........................(29),}$$

whence, by (9) of § 107,

$$2h = N(N-1)(11N-24)(11N-25) \quad\ldots\ldots\ldots(30).$$

The points of contact of the planes ϖ_4 are nodes on the curve, whence, by (27), we obtain

$$\delta = 5N(7N^2 - 28N + 30) \quad\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

Substituting these values and recollecting that $\kappa = 0$, we obtain

$$\left.\begin{aligned} \nu &= 2N(N-3)(31N-54) \\ m &= 3N(62N^2 - 305N + 348) - \iota \\ \sigma &= 4N(93N^2 - 463N + 534) - 2\iota \end{aligned}\right\} \quad\ldots\ldots\ldots(32),$$

and the value of $y + \tau$ can be found from (15) of § 107. The first of (32) gives the degree of the developable enveloped by the osculating planes to the flecnodal curve; but whether or not the curve possesses any points of inflexion cannot be ascertained without further investigation. It appears to me possible that the points of contact of ϖ_6 might be points of this character.

408. *The Flecnodal Developable and its Edge of Regression.* Our knowledge of this surface and curve is confined to the equations

$$\left.\begin{aligned} \nu &= 2N(N-3)(3N-2) \\ m &= N(N-1)(11N-24) \\ \varpi &= 5N(7N^2 - 28N + 30) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(33),$$

$$\tau = \iota = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

The third of (33) arises from the fact that the planes ϖ_4 are double tangent planes to D_f, and therefore doubly osculating planes to E_f. The planes ϖ_6 also, in all probability, give rise to some singularity.

The Bitangential Curve, its Developable D_b and the Edge of Regression E_b of the Latter.

409. *The class of the bitangential developable is*

$$m = \tfrac{1}{2}N(N-1)(N-2)(N^3 - N^2 + N - 12) \quad\ldots\ldots(35).$$

Let O be the vertex of the tangent cone to the surface from an arbitrary point; then every double tangent plane to the cone is a tangent plane to D_b; hence m is equal to the number of double tangent planes to the cone. Let ν, μ be the degree and class of the cone; δ, κ the number of its nodal and cuspidal generators; then, by Chapter I,

$$\nu = n(n-1), \quad \mu = n(n-1)^2,$$

$$\delta = \tfrac{1}{2}n(n-1)(n-2)(n-3), \quad \kappa = n(n-1)(n-2),$$

whence, by Plücker's equations, we obtain

$$2\tau = n(n-1)(n-2)(n^3 - n^2 + n - 12).$$

Changing τ into m and n into N, we obtain (35).

410. *The degree of the bitangential curve is*

$$n = N(N-2)(N^3 - N^2 + N - 12) \quad \text{..........(36).}$$

Let T be the degree of the bitangential surface, that is the surface which intersects the original one in the bitangential curve; let OPQ be a double tangent plane to the tangent cone from O, which touches the cone along the generators OP, OQ; and let P and Q be the points where these generators touch the surface U. Then the number of points such as P and Q is obviously equal to $2m$; but these points are the intersections of the bitangential surface, the original surface and its first polar with respect to O; hence their number is equal to $TN(N-1)$. Accordingly

$$TN(N-1) = 2m.$$

Substituting the value of m from (35), we obtain

$$T = (N-2)(N^3 - N^2 + N - 12) \quad \text{..........(37).}$$

Equation (37) gives the degree of the bitangential surface, and the degree of the bitangential curve is this quantity multiplied by N.

411. *The spinodal and bitangential curves touch one another at the tacnodal points. They intersect one another at the points which are the cuspidal points on the tangent planes ϖ_1; and the number of such planes is*

$$\varpi_1 = 4N(N-2)(N-3)(N^3 + 3N - 16) \quad \text{......(38),}$$

also the planes ϖ_1 are stationary planes to the edge of regression of the bitangential developable.

Let P and Q be the points of contact of any double tangent plane to the surface; then P and Q are nodes on the section by the plane. But a tacnode may be formed by the union of two nodes, hence if P and Q coincide, P becomes a tacnodal point on the surface, and the tacnodal tangent PQ becomes a tangent to the bitangential curve. Hence the bitangential curve touches the spinodal and also the flecnodal curve at the tacnodal points.

The tangent plane ϖ_1 touches the surface at a point p, which is a cusp on the section, and at another point q which is a node. Hence p must be a point where the spinodal and bitangential curves intersect one another. Accordingly the number of such points plus twice the number of tacnodal points is equal to the number of points in which the spinodal and bitangential curves intersect; whence

$$\varpi_1 + 4N(N-2)(11N-24) = 4N(N-2)^2(N^3 - N^2 + N - 12),$$

giving $$\varpi_1 = 4N(N-2)(N-3)(N^3+3N-16).$$

To prove the last part of the theorem, let ABC be the plane ϖ_1; B the node, A the cusp, and AC the cuspidal tangent in the section. Then if we write down the equation of the surface and its first polar with respect to any point T on AB, and then put $\delta = 0$, we shall obtain exactly the same equations as if we had *first* put $\delta = 0$. Hence these equations represent the section by the plane δ of the surface and of its first polar with respect to T; and we know from the theory of plane curves that these two sections have tritactic contact with one another at A, and that AC is the common tangent. Hence AC is the tangent to the bitangential curve at A, and the generator AB of D_b is equivalent to three coincident generators through three coincident points at A. From this it follows that the plane δ or ϖ_1 osculates D_b along AB, and is therefore a stationary plane to E_b. We thus obtain the equation

$$\sigma = \varpi_1 \quad \text{..........................(39)}.$$

We have also proved that:—*The tangent to the bitangential curve at a point, where it intersects (but does not touch) the spinodal curve, is the tangent at the cusp on the section of the surface by the tangent plane* ϖ_1.

412. *The points, where the bitangential and flecnodal curves intersect one another, are the flecnodal points on the tangent planes* ϖ_2; *and the number of such planes is*

$$\varpi_2 = N(N-2)(11N-24)(N^3 - N^2 + N - 16) \text{...(40)}.$$

The tangent plane ϖ_2 touches the surface at a point P, which is a flecnode on the section, and at another point Q, which is a node. Hence P is a point where the bitangential and flecnodal curves intersect one another. Accordingly

$$\varpi_2 + 4N(N-2)(11N-24)$$
$$= N(N-2)(11N-24)(N^3 - N^2 + N - 12),$$

giving $\varpi_2 = N(N-2)(11N-24)(N^3 - N^2 + N - 16)$.

Reciprocal Surfaces.

413. Let S be an anautotomic surface of degree n, S' the reciprocal surface; and let the unaccented and accented letters refer to the original and the reciprocal surface respectively. Let T be any plane section of S; then, since the characteristics of an anautotomic plane curve are

$$\left.\begin{array}{lll} m = n(n-1), & \tau = \tfrac{1}{2}n(n-2)(n^2-9) & \\ \iota = 3n(n-2), & \delta = 0, & \kappa = 0 \end{array}\right\} \text{.........(41)},$$

the reciprocal of a plane section of S is a tangent cone to S', whose characteristics are

$$\left.\begin{array}{lll} n' = n(n-1), & m' = n, & \delta' = \tfrac{1}{2}n(n-2)(n^2-9) \\ \kappa' = 3n(n-2), & \tau' = 0, & \iota' = 0 \end{array}\right\} \text{...(42)}.$$

(i) Let the plane T have ordinary contact with S at a point O. Then O is a node on T, and therefore $\delta = 1$, $\tau' = 1$; also the vertex O' of the cone lies on S', and the double tangent plane to the cone is the tangent plane to S' at O'. The two generators along which this plane touches the cone are the nodal tangents at O' to the section of S' by the tangent plane, and they are the reciprocals of the nodal tangents to T at O.

(ii) Let O be a cusp on T. Then $\kappa = 1$, and $\iota' = 1$; hence the tangent plane at O' to S' osculates the cone along a generator. Through O' draw an arbitrary plane P', then the reciprocal of P' is a point P lying in the plane T; and the reciprocal of the section of S' by P' is the tangent cone to S from P. Now the plane T can easily be shown to osculate this cone along a generator PO*;

* For the purpose of proving this result, it is sufficient to employ the cubic surface

$$\alpha^2\delta + \alpha(\delta^2 v_0 + \delta v_1 + p\gamma^2) + u_3 = 0.$$

The tangent plane δ touches the surface at A, which is a cusp on the section; also C is any point in this plane. Writing the cubic in the binary form $(\gamma, 1)^3 = 0$, and equating its discriminant to zero, we obtain the equation of the tangent cone from C, which shows that the plane ABC osculates the cone along the generator AC.

hence O' is a cusp on the section of S' by P', and consequently the locus of O' is a cuspidal curve on S', which is the reciprocal of the spinodal developable of S. The characteristics of the latter are given by equations (14) and (15); hence, reciprocating, we obtain the following formulæ for the cuspidal curve on S', viz.

$$\left.\begin{aligned} \nu &= 2N(N-2)(3N-4) \\ n &= 4N(N-1)(N-2) \\ \kappa &= 2N(N-2)(11N-24) \\ m &= 4N(N-2)(7N-15) \\ \sigma &= 10N(N-2)(7N-16) \\ \tau &= \iota = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(43).$$

The remaining characteristics can be obtained from the last three of (15) by writing y, h, δ, g and ϖ for x, g, ϖ, h and δ respectively. These formulæ show that the tacnodal points on S correspond to cusps on the cuspidal curve on S'.

The reciprocal of the spinodal curve is the developable enveloped by the tangent planes to S' at points on the cuspidal curve. The characteristics of this developable and of its edge of regression are obtained by reciprocating (7) and the last four of (6).

(iii) Let T be a double tangent plane, and let P and Q be its points of contact. Then $\delta = 2$, and $\tau' = 2$; hence the cone has a pair of double tangent planes, both of which are tangent planes to S' at O'. Accordingly the locus of O' is a nodal curve on S', which is the reciprocal of the bitangential developable. The characteristics of the latter have only been partially obtained; but by reciprocating (35) and (38), and recollecting that ϖ_1 is a stationary plane to E_b, and therefore gives rise to a cusp on the nodal curve, we obtain the following formulæ for the nodal curve on S'.

$$\left.\begin{aligned} n &= \tfrac{1}{2}N(N-1)(N-2)(N^3-N^2+N-12) \\ \kappa &= 4N(N-2)(N-3)(N^3+3N-16) \end{aligned}\right\} \ldots\ldots(44).$$

The reciprocal of the bitangential curve on S is the developable enveloped by the tangent planes to S' at points on the nodal curve.

(iv) Let O be a flecnode on T. Then $\delta = 1$, $\iota = 1$; so that $\tau' = 1$, $\kappa' = 1$. Hence the cone has a cuspidal generator, whose cuspidal tangent plane touches the cone along another generator, and is therefore a double tangent plane. This plane is the tangent plane to S' at O', and the locus of O' is a curve on S', which is the reciprocal of the flecnodal developable. Its character may be investigated by means of the quartic surface

$$\alpha^3\delta + 3\alpha^2(\delta^2 v_0 + \delta v_1 + p\beta\gamma) + 3\alpha(\delta^3 w_0 + \delta^2 w_1 + \delta w_2 + \gamma W_2) + u_4 = 0 \quad \ldots(45)$$

or

$$\alpha^3\delta + 3\alpha^2 u_2 + 3\alpha u_3 + u_4 = 0,$$

in which the plane δ touches the surface at a point A, which is a flecnode on the section, and AB is the flecnodal tangent. The tangent cone at A is

$$(\delta u_4 - u_2 u_3)^2 = 4(\delta u_3 - u_2^2)(u_2 u_4 - u_3^2),$$

from which it follows that the plane δ is a double tangent plane, which has ordinary contact along the generator AC, but AB is a cuspidal generator whose cuspidal tangent plane is δ. This shows that O' is a point on the flecnodal curve on S', accordingly:—*The reciprocals of the flecnodal curve and developable on S are the flecnodal developable and curve on S'.* Reciprocating (33), the degrees of the flecnodal curve and developable on S' are

$$\left.\begin{aligned} n &= N(N-1)(11N-24) \\ \nu &= 2N(N-3)(3N-2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(46).$$

To avoid circumlocution, I shall denote the degrees of the nodal, cuspidal, and flecnodal curves on S' by the letters b, c, and f, and shall frequently refer to them as the curves b, c, and f; whilst the degree of the bitangential curve on S will be denoted by ρ. By (44), (43), (46), and (36) their values are

$$\left.\begin{aligned} b &= \tfrac{1}{2}N(N-1)(N-2)(N^3-N^2+N-12) \\ c &= 4N(N-1)(N-2) \\ f &= N(N-1)(11N-24) \\ \rho &= N(N-2)(N^3-N^2+N-12) \end{aligned}\right\} \quad \ldots\ldots(47).$$

Moreover, it is possible for the nodal curve, considered as a curved line drawn on the surface, to possess nodes, cusps, and other singularities; and these must be carefully distinguished from singular points, such as pinch points, which are singular points on the surface, but not necessarily such on the *curved line*, which constitutes the nodal curve.

(v) Let O be a point where the bitangential and spinodal curves intersect; then the plane T has ordinary contact with S at some point Q on the former curve and is the double tangent plane ϖ_1, one of whose points of contact is a cusp on the section, and its cuspidal tangent is some line OP, whilst Q is a node. Hence $\delta = 1$, $\kappa = 1$; and therefore $\tau' = 1$, $\iota' = 1$. Accordingly the cone has a double and a stationary tangent plane, both of which touch S' at O'. The latter plane is the cuspidal tangent plane at O' to S' along the curve c, and is one of the nodal tangent planes to the curve b; whilst the former plane is the other nodal tangent plane at O' to the curve b. From this it follows that O' is a cubic node of the *fifth* species on S'.

We have also shown in (iii) that O' is a cusp on the curve b; but since the generators OP and OQ of the spinodal and bitangential developables to S at O do not, in general, coincide, the curves b and c on S' intersect, but do not touch at O'.

Furthermore, if P', Q' be two points on the curve b near O', the curve c cuts the plane $O'P'Q'$ at a finite angle; but if $O'R'$ is the cuspidal tangent at O' to the curve b, the three tangent planes to the surface at O' all pass through $O'R'$.

(vi) Let O be a tacnode; then O is a point where the spinodal and bitangential curves touch, and T is the singular tangent plane ϖ_5. Also $\delta = 2$, $\tau = 2$; so that $\delta' = 2$, $\tau' = 2$; accordingly the cone has a tacnodal tangent plane, which is the tangent plane to S' at O'. In this case the curves b and c touch one another at O', and O' is a pinch point on the former. Moreover, from (ii), O' is a cusp on the curve c, and the two coincident nodal tangent planes to b coincide with the cuspidal tangent plane to c at O'. These three coincident planes pass through the cuspidal tangent to c at O'; and O' is a cubic node of the *sixth* species on S'.

Any plane section through a cubic node of the sixth species has a triple point of the third kind thereat; and we can verify this by the method explained in (i) by means of the quartic surface

$$\gamma^4 + 4\gamma^3(\alpha v_0 + v_1) + 6\gamma^2(\alpha^2 w_0 + \alpha w_1 + w_2) + 4\gamma(\alpha^2\delta V_0 + \alpha V_2 + V_3) + \alpha^3\delta W_0 + \alpha^2\delta W_1 + \alpha\delta W_2 + W_4 = 0 \quad \ldots\ldots\ldots(48),$$

where the suffixed letters denote binary quantics of (β, δ). The plane ABC is the tangent plane at A; also this point is a tacnode on the section, and AB is the tacnodal tangent. The equation of the tangent cone from C, which is any arbitrary point on the section, is obtained in the usual manner by equating to zero the discriminant of $(\gamma, 1)^4 = 0$, viz. $I^3 = 27J^2$, from which it will be found that the plane δ has quadritactic contact with the cone along AC. This shows that A is a point of undulation on the section of the cone by the plane ABD, and that AB is the tangent at this point; and since the reciprocal of the tangent at a point of undulation on a plane curve is a triple point of the third kind, O' is such a point on the plane section of S'.

(vii) Let O be a point where the bitangential and flecnodal curves intersect; then the plane T has ordinary contact with S at some other point Q on the bitangential curve, and T is the double tangent plane ϖ_2, one of whose points of contact is a flecnode on T. Hence $\delta = 2$, $\iota = 1$; and therefore $\tau' = 2$, $\kappa' = 1$. Accordingly the cone has one ordinary double tangent plane corresponding to Q, and a singular tangent plane corresponding to O, which has ordinary contact with the cone along one generator and is the cuspidal tangent plane to the cone along another generator. These two planes are the nodal tangent planes at O', but the one corresponding to O touches the flecnodal curve and the latter intersects the nodal curve at O'. The value of ϖ_2 is given by (40).

(viii) Let T be a triple tangent plane to S; and let P, Q, R be its points of contact. Then these points are nodes on the section, and are also points on the bitangential curve. Hence $\delta = 3$ and $\tau' = 3$. The tangent cone from O' has therefore three double tangent planes which are tangent planes to S', at O'; hence O' is a cubic node of the third kind on S', and a triple point of the first kind on the nodal curve. The number of these points will be considered later on.

We have now completed the discussion of the spinodal and bitangential curves, but the flecnodal curve remains to be considered.

(ix) Let O' be a biflecnode on T. Then $\delta = 1$, $\iota = 2$; so that $\tau' = 1$, $\kappa' = 2$; and by employing a similar method to that of (iii) it can be shown that the cone possesses two cuspidal generators having a common cuspidal tangent plane, and that O' is a biflecnode on the section of the tangent plane at O' to S'.

(x) Let O be a hyperflecnode on T, one of whose tangents has quadritactic contact with its own branch, and consequently quinquetactic contact with S at O. Then $\delta=1$, $\tau=1$, $\iota=2$; hence $\tau'=1$, $\delta'=1$, $\kappa'=2$; and the tangent plane at O' touches the cone along two generators, one of which is a triple generator of the third kind, whilst the contact along the other is ordinary bitactic contact. To ascertain the character of the singularity at O', let us consider the quintic surface

$$\alpha^4\delta + 4\alpha^3(\delta^2 v_0 + \delta v_1 + p\beta\gamma) + 6\alpha^2(\delta u_2 + \gamma t_2) + 4\alpha(\delta u_3 + \gamma T_3) + u_5 = 0,$$

where the u's are ternary quantics of (β, γ, δ), and the other letters are binary quantics of (β, δ). The section of this surface by the tangent plane δ is the singularity in question, AB being the tangent which has quinquetactic contact with the surface; and if the equation to the tangent cone from A be written down, it will be found that ABC is a double tangent plane to the cone along the generators AB and AC, and that AB is a triple generator of the third kind, whilst the contact is bitactic along AC. This shows that the singularity at O' on the reciprocal surface is of the same character as that on the original one.

414. Equations (46) and (47) furnish a verification of Cayley's theorem of § 59; for the degree n of the flecnodal curve on the reciprocal surface is given by the equation

$$n = M(11M - 24) - 22b - 27c,$$

where $M = N(N-1)^2$. Substituting the values of b and c from (47), it will be found that this equation reduces to the first of (46).

The corresponding equation, which gives the degree of the spinodal curve on S', is by § 58

$$n = 4M(M-2) - 8b - 11c \quad \text{...........(48 A).}$$

Now the spinodal curve on S' gives rise to a spinodal developable, the reciprocal polar of which is a cuspidal curve on S. But since S is anautotomic, it possesses no cuspidal curve and therefore S' possesses no proper spinodal curve, and the degree of the latter is therefore zero; hence the curve of intersection of S' and its Hessian must consist of the nodal and cuspidal curves on S' repeated a certain number of times. And if the values of M, b and c be substituted in (48 A), it will be found that $n=0$ as ought to be the case.

I have not succeeded in ascertaining the reduction in the degree of the bitangential curve which is produced by a nodal and a cuspidal curve; but if the reduction is denoted by $xb + yc$, the method of the preceding paragraph indicates that x and y are functions of the degree N of the surface. This is confirmed by the fact that a double point on the original surface gives rise to a multiple point of order $N(N-1)^2 - 6$ on the bitangential surface*; and we should therefore anticipate that a nodal or a cuspidal curve on the original surface gives rise to a multiple curve on the bitangential surface, whose multiplicity is a function of the degree of the original surface.

415. *To find the number of triple tangent planes to an anautotomic surface.*

We shall prove the formula

$$b(n'-2) = \rho + 2\varpi_5 + 3\varpi_1 + 3\varpi_3 \text{ ....(49)},$$

where $n' = N(N-1)^2$ is the degree of the reciprocal surface; ϖ_3 is the number of triple tangent planes to S, and ρ is the degree of the bitangential curve. The values of b and ρ are given by the first and last of (47).

Let A be any point in space; let a surface of degree n' possess a nodal curve of degree b, and a cuspidal one of degree c; also let a be the number of ordinary tangents which can be drawn from A to any plane section of S' through A. Then, by Plücker's equations,

$$a = n'(n'-1) - 2b - 3c \text{ ....(50)}.$$

The *complete* tangent cone from A to S' is of degree $n'(n'-1)$; and (50) shows that it consists of the cone twice repeated, which stands on the nodal curve b, the cone three times repeated, which stands on the cuspidal curve c, and a proper cone whose degree a is given by (50).

Equation (49) is proved by examining the character of the points of intersection of the second polar of A with the nodal curve b. These points are *ordinary and singular*.

At every ordinary point B, in which the curve of contact of the cone a intersects the curve b, one of the nodal tangent planes must pass through A, and we shall first show that these points lie on the second polar of A.

* "Singular tangent planes to autotomic surfaces," *Quart. Jour.* vol. XLII. p. 37.

Consider the surface*

$$\alpha^n u_0 + \alpha^{n-1} u_1 + \dots \alpha^2 u_{n-2} + \alpha \Omega_s u_{n-s-1} + \Omega_s^2 u_{n-2s} = 0 \dots (51),$$

where

$$\left.\begin{aligned} \Omega_s &= \beta^{s-1} w_1 + \beta^{s-2} w_2 + \dots w_s \\ u_{n-2} &= \beta^{n-2} v_0 + \beta^{n-3} v_1 + \dots v_{n-2} \end{aligned}\right\} \dots\dots\dots\dots(52).$$

Equation (51) represents a surface having a plane nodal curve (α, Ω_s), which passes through B. The nodal tangent planes at B are obtained by equating the coefficient of β^{n-2} to zero, and are of the form

$$\alpha^2 v_0 + V_0 \alpha w_1 + W_0 w_1^2 = 0,$$

and if one of them passes through A, $v_0 = 0$. The second polar of A is obtained by differentiating (51) twice with respect to α, and $v_0 = 0$ is the condition that it should pass through B.

Let us now reciprocate this result. The point A becomes an arbitrary plane P; the tangent cone a becomes the section of S by this plane; the points, where the curve of contact of a intersects the nodal curve b, become the tangent planes at the points where the section of S by P intersects the bitangential curve, and the number of these points is equal to ρ.

It follows from (vi) that the points on S' corresponding to ϖ_5 are cubic nodes of the sixth kind, and such points are ordinary points on the second polar†, and the latter has ordinary contact with the surface at such points. Also the points in question are ordinary points on b, hence the second polar and this curve have bitactic contact with one another at these points. Accordingly the number of points of intersection arising from this cause is $2\varpi_5$.

It follows from (v) that points on S' corresponding to ϖ_1 are cubic nodes of the fifth species on the surface; and such points are ordinary points on the second polar, but the latter does not touch the surface. The tangent plane, however, passes through

* Although the method of proof only applies to surfaces having a plane nodal curve, there can be no doubt that the theorem is true when the nodal curve is twisted.

† The equations of a surface having a cubic node of the sixth species at A, and of its second polar with respect to D are

$$\alpha^{n-3}\delta^3 + \alpha^{n-4} u_4 + \dots u_n = 0,$$

and

$$6\alpha^{n-3}\delta + \alpha^{n-4} u_4'' + \dots u_n'' = 0,$$

where $u_n'' = du_n/d\delta$.

the line of intersection of the tangent planes at the cubic node* on S'; and this line is the line $O'R'$ considered at the end of (v), which is the cuspidal tangent to the curve b at O'. Hence the second polar intersects the nodal curve b at O' in three coincident points; accordingly the number of points of intersection arising from this cause is $3\varpi_1$.

Since every multiple point of order k on a surface gives rise to a multiple point of order $k-2$ on the second polar, it follows that the second polar passes through every cubic node on the surface. Now we have shown in (viii) that every triple tangent plane gives rise to a cubic node of the third kind on S', and to a triple point of the first kind on the curve b. Accordingly the number of points of intersection arising from this cause is $3\varpi_3$.

We have therefore proved the formula (49), and we have to substitute the values of b, ρ, ϖ_5 and ϖ_1 from (47), (13) and (38); also $n' = N(N-1)^2$; we thus obtain

$$\varpi_3 = \tfrac{1}{6}N(N-2)(N^7 - 4N^6 + 7N^5 - 45N^4 + 114N^3 - 111N^2 + 548N - 960) \ldots (53),$$

which determines the number of triple tangent planes.

416. *The degree of the bitangential developable is*

$$\nu = N(N-2)(N-3)(N^2+2N-4) \quad \ldots\ldots\ldots (54).$$

By (13) and the fourth of (47) equation (54) is equivalent to

$$\nu = \rho - \tfrac{1}{2}\varpi_5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55),$$

and we shall prove the last equation by the Theory of Correspondence.

Through any fixed line L draw a plane x cutting the bitangential curve in ρ points P; and let the generator of the bitangential developable through P intersect the curve in Q; through L and Q draw a plane y, and take x and y as corresponding planes. Then to every point P one point Q corresponds and vice versa; hence

$$\lambda = \mu = \rho.$$

* This may be proved by considering such a surface as

$$\alpha^{n-3}(p\gamma + q\delta)\,\delta^2 + \alpha^{n-4}u_4 + \ldots u_n = 0,$$

for its second polar with respect to D is

$$2\alpha^{n-3}(p\gamma + 3q\delta) + \alpha^{n-4}u_4'' + \ldots u_n'' = 0.$$

United planes will occur:—

(i) When P and Q coincide, in which case P will be the point of contact of a tangent plane ϖ_5; hence the number of united planes due to this cause is ϖ_5.

(ii) When the line PQ intersects L. There are obviously ν of such lines, but since the plane LPQ may be regarded as a plane x or y, this plane is equivalent to two united planes; hence the number due to this cause is 2ν.

We thus obtain

$$\lambda + \mu = 2\rho = \varpi_5 + 2\nu,$$

which is the required result.

417. We have therefore proved the following formulæ for the bitangential developable and its edge of regression, viz.

$$\left.\begin{aligned} m &= \tfrac{1}{2}N(N-1)(N-2)(N^3-N^2+N-12) \\ \sigma &= 4N(N-2)(N-3)(N^3+3N-16) \\ \nu &= N(N-2)(N-3)(N^2+2N-4) \end{aligned}\right\}\text{......(56)},$$

and from (4) and (5) of § 104 we easily obtain

$$\left.\begin{aligned} n &= \tfrac{1}{2}N(N-2)(5N^4-11N^3+12N^2-221N+420) \\ \kappa &= \tfrac{1}{2}N(N-2)(16N^4-41N^3+44N^2-686N+1224) \end{aligned}\right\}\text{...(57)},$$

which determine the degree of the edge of regression, and also the number of its cusps.

The arguments that we have already used show that τ and ι are zero; also, since the curve b does not possess any isolated nodes, there are no isolated planes ϖ, for these are included in the triple tangent planes ϖ_3, each of which osculates E_b at three distinct points. The remaining quantities x, y, and g can be obtained from (4) and (5) of § 104.

418. *To find the number k' of apparent double planes of the bitangential developable.*

The value of k' is equal to the number of apparent nodes of the nodal curve on the reciprocal surface. Now we have already shown that the tangent planes ϖ_1 and ϖ_3 to S respectively give rise to cusps and triple points of the first kind on the nodal curve b on S'. Also every triple point is equivalent to three actual nodes, but the curve has no other actual nodes except those included in the triple point; we must therefore write in (5) of § 104

$$n = b, \quad \delta = 3\varpi_3, \quad \kappa = \varpi_1, \quad h = k',$$

and we obtain

$$\nu = b(b-1) - 2k' - 6\varpi_3 - 3\varpi_1,$$

in which all the quantities except k' are known. We thus obtain

$$8k' = N(N-2)(N^{10} - 6N^9 + 16N^8 - 54N^7 + 164N^6 - 288N^5 + 547N^4 - 1058N^3 + 1068N^2 - 1214N + 1464) \ldots (58).$$

419. The following equations give the numbers of the six singular tangent planes to an anautotomic surface of degree N; and the number attached to each equation indicates where it is to be found in the text:

$$\varpi_1 = 4N(N-2)(N-3)(N^3 + 3N - 16) \ldots\ldots (38),$$

$$\varpi_2 = N(N-2)(11N-24)(N^3 - N^2 + N - 16) \ldots\ldots (40),$$

$$\varpi_3 = \tfrac{1}{6}N(N-2)(N^7 - 4N^6 + 7N^5 - 45N^4 + 114N^3 - 111N^2 + 548N - 960) \ldots (53),$$

$$\varpi_4 = 5N(7N^2 - 28N + 30) \ldots\ldots (27)$$

$$\varpi_5 = 2N(N-2)(11N-24) \ldots\ldots (13),$$

$$\varpi_6 = 5N(N-4)(7N-12) \ldots\ldots (28).$$

The preceding analysis gives a fairly complete investigation of the six curves and developables mentioned in §§ 10 and 11, with the exception of the developable and curve D_f and E_f, with respect to which further investigation is required to complete the theory.

Autotomic Surfaces.

420. I shall not give any detailed account of the theory of singular tangent planes to autotomic surfaces, which possess C conic and B binodes, since the investigation is lengthy, and for the reasons stated in my paper* the results must be regarded as provisional until verified by some independent method, such as the Theory of Correspondence.

Let A be a conic node; draw the tangent cone from A, and let the curve of contact cut the spinodal curve at P. Then since the tangent plane along the generator AP intersects the surface

* "Singular tangent planes to autotomic surfaces," *Quart. Jour.* vol. XLII. p. 21.

in a curve which has a node at A and a cusp at P, the plane is a singular tangent plane of the species ϖ_1; but it is an *improper* plane, because the contact at A is not ordinary contact, but is of a special character due to the fact that A is a conic node. The true tangent planes ϖ_1 are those which touch the surface at two points P and Q, one of which is an *ordinary* point of intersection of the spinodal and bitangential curves, whilst the other is an *ordinary* point on the latter curve. In like manner, when a surface possesses C conic nodes, the improper triple tangent planes are (i) every double tangent plane to the tangent cone from a conic node, (ii) every tangent plane to the surface through a pair of conic nodes. Similar observations apply to surfaces which possess binodes as well as conic nodes; from which it follows that every double point on a surface must produce a diminution in the number of singular tangent planes to the surface, similar to that produced by a double point on a plane curve in the number of double and stationary tangents. Accordingly a set of formulæ exists for surfaces similar to Plücker's equations for plane curves.

421. One of Plücker's equations for a plane curve is

$$3n(n-2) = \iota + 6\delta + 8\kappa,$$

in which the left hand side is equal to the number of stationary tangents possessed by an anautotomic plane curve, whilst the right hand side shows that, when the curve is autotomic, each nodal tangent is equivalent to three and each cuspidal tangent to eight stationary tangents. And by considering the surface formed, by the revolution about the axis of x, of a plane curve symmetrical about this axis which has a node upon the latter, it follows that the tangent cone from the node *three times* repeated forms part of the spinodal developable. Hence every conic node reduces the degree of the spinodal developable by 6; and in a similar manner it can be shown that a binode reduces it by 8; accordingly the degree ν of the true spinodal developable is

$$\nu = 2N(N-2)(3N-4) - 6C - 8B.$$

422. In the paper referred to I have worked out the degrees and classes of most of these developables; and the method employed in calculating the singular tangent planes is to find the number of improper tangent planes of each species, and to subtract their number from the value of ϖ for an anautotomic surface, which is denoted by ϖ'. The value of ϖ_6 is unaltered by ordinary double points, and the change produced when the singularity is a compound one has not been considered. The final results are as follows:—

$$\varpi_1 = \varpi_1' - 2\{N(N-1)(7N-11) - 6C - 54\}C$$
$$- 4\{N(N-1)(5N-8) - 6B - 36\}B + 32BC,$$
$$\varpi_2 = \varpi_2' - 2\{N(N-1)(17N-30) - 12C - 84\}C$$
$$- 3\{N(N-1)(17N-30) - 18B - 96\}B + 66BC,$$
$$\varpi_3 = \varpi_3' - 2Ct_1 - 3Bt_2 - 12B - 2(M-8)C(C-1)$$
$$- \tfrac{9}{2}(M-6)B(B-1) - 6(M-7)BC,$$
$$\varpi_4 = \varpi_4' - 30C - 45B,$$
$$\varpi_5 = \varpi_5' - 24C - 36B,$$
$$\varpi_6 = \varpi_6'.$$

The value of M is

$$M = N(N-1)^2 - 2C - 3B,$$

t_1 is the number of double tangent planes to the tangent cone from a conic node, and t_2 the number when the vertex is a binode. Their values are

$$2t_1 = (M-9)^2 - N(N-1)(3N-14) - 3B - 1,$$
$$2t_2 = (M-8)^2 - N(N-1)(3N-14) - 3B - 10.$$

APPENDIX

I. *On Plane Trinodal Quartics.*

Let the tangents at the node of a uninodal quartic cut the curve at B and C; and let the line BC cut the curve in Q, Q'. Then these points, which are called the Q points, possess various important properties which have been discussed by Roberts*. He employs the parametric method, but all his results can be obtained much more simply by the ordinary methods of trilinear coordinates. A trinodal quartic possesses three pairs of Q points, of which one pair corresponds to each node; hence Roberts' results are capable of extension to these curves.

The three conics mentioned in § 194 of my treatise on *Cubic and Quartic Curves* pass through two points, which I call the S points; and the line SS' intersects the quartic in two other points, which I call the T points; and both pairs of points possess various important properties. Let the equation of the quartic be

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma\,(l\alpha + m\beta + n\gamma) = 0 \;\ldots\ldots(1);$$

also let

$$\sigma = l\beta\gamma + m\gamma\alpha + n\alpha\beta,$$

$$\tau = \beta\gamma/l + \gamma\alpha/m + \alpha\beta/n,$$

$$u = k_1\alpha + k_2\beta + k_3\gamma,$$

$$k_1 = m/n + n/m - 2l, \;\; \&c.,$$

then (1) can be written in the form

$$\sigma\tau - \alpha\beta\gamma u = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The conic σ passes through the nodes and intersects (1) in the S points. The conic τ intersects it in the T points; and the S and T points lie on the line u.

Writing $2l$, $2m$, $2n$ for l, m, n in § 194 of *Cubic and Quartic Curves*, the equation of the conic which passes through the six

* *Proc. Lond. Math. Soc.* vol. xxv. p. 151.

points in which the nodal tangents intersect the curve can be put into the form

$$\Delta\sigma + u(\alpha/l + \beta/m + \gamma/n) = 0 \quad \ldots\ldots\ldots\ldots\ldots(3),$$

where $$\Delta = -4 - \frac{1}{l^2} - \frac{1}{m^2} - \frac{1}{n^2} + \frac{4(l^2+m^2+n^2)-1}{2lmn}.$$

The equation of the conic which passes through the six points of contact of the tangents drawn from the nodes is

$$\Delta_1\sigma + u\{(l^{-1}-l)\alpha + (m^{-1}-m)\beta + (n^{-1}-n)\gamma\} = 0 \ldots(4),$$

where $$\Delta_1 = \frac{1+l^2+m^2+n^2}{lmn} - 1 - \frac{1}{l^2} - \frac{1}{m^2} - \frac{1}{n^2}.$$

And the equation of the conic which passes through the six points of inflexion is

$$\Delta_2\sigma + 2lmn\{(l^{-1}-l)\alpha + (m^{-1}-m)\beta + (n^{-1}-n)\gamma\}u = 0 \ldots(5),$$

where $\Delta_2 = 3 + l^2 + m^2 + n^2 - 2(m^2n^2 + n^2l^2 + l^2m^2)/lmn.$

The forms of (3), (4) and (5) show that the three conics pass through the S points.

The equation of the conic which passes through the six Q points and the two T points is

$$\Delta_3\tau - u(\alpha/l + \beta/m + \gamma/n) = 0 \quad \ldots\ldots\ldots\ldots\ldots(6),$$

where

$$\Delta_3 = 1 + 4(l^2 + m^2 + n^2 - 2lmn) - 2(l^2m^2 + n^2l^2 + l^2m^2)/lmn.$$

All these theorems, together with several others of a similar character, have been proved by myself in a paper published in the *American Journal of Mathematics*, vol. XXVI. p. 169.

II. *If three surfaces of degrees L, M, N have a common anautotomic curve of degree n, which is a multiple curve of orders p, q, r on these surfaces respectively; and if ν is the degree of the developable enveloped by the osculating planes to the curve, the number of points of intersection which are absorbed by the curve is*

$$n(Lqr + Mrp + Npq - 2pqr) - \nu pqr \ldots\ldots\ldots\ldots(1).$$

Let us first suppose that the curve is a multiple one of order p on the surface L and an ordinary one on the two surfaces M and N. Then the surfaces M and N intersect in the curve n and in a residual curve of degree $n' = MN - n$; and the latter curve intersects the surface L in Ln' points. But the curves n and n' intersect in δ' points, where δ' is given by writing $n = n_1$, $n' = n_2$, $\nu = \nu_1$ in the first of (21) of § 109; hence since n is a multiple curve of order p on L, the total number of points of intersection of the three surfaces, which do not lie on the curve, is

$$Ln' - p\delta' = L(MN - n) - p\delta'$$
$$= LMN - n(L + pM + pN - 2p) + p\nu,$$

which shows that the number of points absorbed is

$$n(L + pM + pN - 2p) - p\nu \quad \text{.................(2).}$$

The formula (1) may now be established by induction; for if we successively put $q = r = 1$, $r = p = 1$, $p = q = 1$, it will be found in each case to reduce to one which is equivalent to (2).

When a surface of degree N has a nodal curve of degree n, the reduction of class is obtained by putting $L = N$, $M = N = N - 1$, $p = 2$, which gives

$$n(5N - 8) - 2\nu + j \quad \text{....................(3),}$$

where j is the number of pinch points. In the case of a quartic scroll, which possesses a nodal twisted cubic, $n = 3$, $N = 4$, $\nu = 4$, $j = 4$, and therefore the reduction of class is equal to 32.

Equation (3) enables us to prove the following theorem due to Salmon*.

When the nodal curve is the complete intersection of two surfaces of degrees k and l, the reduction of class is

$$kl\{7N - 4(k + l + 1)\},$$

and the number of pinch points is

$$2kl(N - k - l).$$

The equation of the surface is

$$P_k^2 U + 2P_k Q_l V + Q_l^2 W = 0 \quad \text{..................(4),}$$

and the pinch points are the intersections of the nodal curve (P, Q) and the surface $V^2 = UW$, which is of degree $2(N - k - l)$; hence their number is $j = 2kl(N - k - l)$. By § 107, equation (10),

$$\nu = kl(k + l - 2).$$

Substituting these values of j and ν in (3) and recollecting that $n = kl$, we obtain the required result.

The formula (1) presupposes that if P be any arbitrary point on the curve n, the sections of all three surfaces by any arbitrary plane through P has a multiple point thereat, the tangents at

* *Camb. and Dublin Math. Jour.* vol. II. p. 65.

which are distinct. The result would therefore require modification if the multiple point on the section were of a different character; or if the multiple curve possessed any singular points such as pinch points, multiple points of a higher order and the like.

The corresponding results for curves of higher singularity await investigation; but when the curve is cuspidal, every point must be a pinch point, which requires that

$$V^2 - UW = P\Phi + Q\Psi \qquad (5).$$

Let N be even and equal to $2n$; then (5) is satisfied by

$$U = \Omega^2_{n-k} + P\theta + Q\phi,$$

$$V = \Omega_{n-k}\Omega'_{n-l} + P\chi + Q\psi,$$

$$W = \Omega'^2_{n-l} + P\rho + Q\sigma,$$

whence substituting in (4) and omitting suffixes, we obtain

$$(P\Omega + Q\Omega')^2 + (A, B, C, D \between P, Q)^3 = 0 \qquad (6),$$

which is the equation of a surface having a cuspidal curve, which is the *complete* intersection of the surfaces (P, Q).

The surface $P\Omega + Q\Omega' = 0$ intersects the surface (6) in the cuspidal curve three times repeated and in a residual curve; and it can be shown that the tacnodal points on the former one are the points of intersection of the two curves. The case in which N is an odd integer may be left to the reader.

III. *Every multiple point of order p on a surface in general gives rise to a multiple point of order $4p - 6$ on the Hessian. Also the nodal cone at the latter is a compound one, which consists of the nodal cone at the multiple point on the surface and a second cone of degree $3p - 6$.*

Let the equation of the surface be

$$\alpha^{n-p}u_p + \alpha^{n-p-1}u_{p+1} + \ldots\ldots u_n = 0,$$

then if the values of a, b, &c. be calculated in the same manner as was done in § 51, it can easily be shown that the highest power of α in the Hessian is the $(4n - 4p - 2)$th; and since the degree of the Hessian is $4n - 8$, it follows that the latter surface has a multiple point of order $4p - 6$ at A.

Let AB be any generator of the cone u_p; then

$$u_p = (\mu\gamma + \nu\delta)\,\beta^{p-1} + (L\gamma^2 + 2M\gamma\delta + N\delta^2)\,\beta^{p-2} + \ldots\ldots,$$

and to find the number of coincident points in which AB cuts the nodal cone to the Hessian at A, we must put $\gamma = \delta = 0$ in the values of a, b, &c. We thus obtain $a = b = h = 0$, which reduces the first term of the Hessian to $(fl - gm)^2 = 0$; and if the values of f, g, l, m when $\gamma = \delta = 0$ be substituted, the left-hand side will vanish. This shows that the line AB, and consequently every generator of the nodal cone at the multiple point on the surface, is a generator of the nodal cone at the multiple point on the Hessian, and therefore the former cone forms part of the latter one.

The corresponding theorem for a plane curve is given in §§ 45 and 46 of my *Cubic and Quartic Curves*. It can also be shown that, when all the tangents at the multiple point on the curve coincide, the order of the multiple point on the Hessian is $3p - 3$; of which I have given a proof in the case of a cusp. We should therefore anticipate that the preceding theorem is subject to various exceptions, when the nodal cone at the multiple point on the surface is an autotomic or an improper one. See also §§ 53 and 54.

IV. *Quartic Scrolls*, 13*th species*. In the enumeration of these scrolls I have followed Cayley and Cremona; but whilst this treatise has been going through the press I have discovered a 13th species, which occurs when the triple line is of the 5th species. By § 215, these lines possess two coincident fixed tangent planes and one distinct torsal tangent plane; and the equation of a quartic surface possessing such a line is

$$(\beta\delta - \alpha\gamma)\,v_1^2 + v_4 = 0,$$

and by § 359, the generating curve is

$$\beta\delta v_1^2 + v_4 = 0,$$

which has a triple point of the second kind where the line AB intersects the curve. The surface therefore belongs to the species $S\,(\overline{1, 1}, 4)$.

THE END.

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