0

ELEMENTS

OF

GEOMETRY,

CONIC SECTIONS

AND

PLANE TRIGONOMETRY.

By ELIAS LOOMIS, LL.D.,

PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN YALE COLLEGE,
AND AUTHOR OF "A COURSE OF MATHEMATICS."

REVISED EDITION, WITH APPENDIX.

NEW YORK:
HARPER & BROTHERS, PUBLISHERS,
FRANKLIN SQUARE.
1880.

PREFACE.

THE stereotype plates of my Elements of Geometry and Conic Sections having become so much worn by long-continued use that it was found necessary to recast them, the opportunity has been improved to give the entire book a thorough revision. As the general plan of the original work has met with very extensive approval, it has not been thought best to modify it materially; nevertheless, the minor changes which have been made are numerous and of considerable importance.

The volume commences with a brief sketch of the history of Elementary Geometry, which, it is hoped, may increase the student's interest in a subject which has occupied the attention of so many gifted minds. The definitions of Book I. have been somewhat amplified, for the purpose of giving clearer ideas of the philosophy of the subject; and several notes have been added to the first pages of the book which, it is hoped, may be found useful and suggestive, although they are generally such as any competent teacher might easily have supplied.

In Book II. the subject of Ratio has been expanded, especially for the purpose of meeting the difficulty of incommensurable quantities; and in this I have followed substantially the method of Vincent in his Cours de Géométrie. A few new propositions have been added to Books III., IV., and V.; and at the close of Book VI. is given a considerable collection of new theorems and problems, with some numerical exercises on the preceding books. These theorems and problems are so simple that it is hoped many students may be encouraged to labor upon them; for no one can be considered as master of the subject of Geometry who has not acquired the ability to discover the demonstration of new theorems and the solution of new problems. Those who find these exercises too difficult may be benefited by practice upon the numerical examples here given, and such similar ones as any competent teacher can readily furnish.

Occasional alterations of some importance will be noticed in Books VII., VIII., and IX., and at the close of Book X. will be found more numerical exercises, designed to impress the preceding principles upon the mind of the student.

In the Treatise on the Conic Sections the alterations will be found more numerous. Several new propositions have been added, and the mode of demonstration has in several instances been materially changed. At the close of each chapter is given a small collection of new theorems and several numerical exercises. If the former should be found too difficult, the latter will not be beyond the power of any student who thoroughly understands the preceding principles; and the teacher can easily supply a greater number of similar exercises if it should be thought expedient.

As the demand has frequently been made for a brief treatise on Trigonometry to accompany the volume on Geometry, a concise outline of Plane Trigonometry has been added to this volume, together with a Table of Logarithms, and of Sines and Tangents sufficiently extensive for the solution of all the problems contained in this Treatise.

Throughout the entire volume I have aimed to remove difficulties to such an extent as not to discourage any faithful student, and yet have designed to leave sufficient difficulty to call out his best exertions; since that does not deserve the name of education which does not summon the student to grapple with difficulties; and each one is conscious that every difficulty which is overcome by his own efforts imparts increased power to surmount fresh difficulties.

I have again to acknowledge my obligations to Professor H. A Newton, who has carefully read all the proofs of that part of this volume embracing Conic Sections and Plane Trigonometry.

CONTENTS.

PLANE TRIGONOMETRY.

N.B.—When reference is made to a Proposition in the same Book, only the number of the Proposition is usually given; but when the Proposition is found in a different Book, the number of the Book is also specified.

SKETCH OF THE HISTORY

OF

ELEMENTARY GEOMETRY.

THE term Geometry is derived from γεωμετρία, a Greek word, signifying the *science of land-measuring*. Ancient writers have generally supposed that this science was first cultivated in Egypt, and Herodotus ascribed the origin of Geometry to the time when Sesostris divided the country among the inhabitants. Aristotle attributed the invention to the Egyptian priests, who, living secluded from the world, had abundant leisure for study.

Thales of Miletus, in Asia Minor, who was born about 640 years before Christ, transplanted the sciences, and particularly mathematics, from Egypt into Greece. He resided for some time in Egypt, and formed an acquaintance with its priests. He is said to have measured the height of the Pyramids by means of their shadow, and determined the distance of vessels remote from the shore by the principles of Geometry. On his return to Greece he founded what has been called the Ionian school, from Ionia, his native country. To him are attributed various discoveries concerning the circle and the comparison of triangles, and he first discovered that all angles in a semicircle are right angles.

One of the disciples of Thales composed an elementary treatise on Geometry—the earliest on record, and he is said to have invented the gnomon, geographical charts, and sun-dials. Anaxagoras, having been cast into prison on account of his opinions relating to Astronomy, employed his time in attempting to square the circle.

Pythagoras was one of the earliest and most successful cultivators of Geometry. He was born about 580 years before Christ, studied under Thales, and traveled in Egypt and India. On his return he settled in Italy, and there founded one of the most celebrated schools of antiquity. He is said to have discovered that in a right-angled triangle the square of the hypothenuse is equal to the sum of the squares on the two legs. He discovered that the circle has a greater area than any other plane figure having an equal perimeter, and that a sphere has a similar property among solids. He also discovered the properties of the regular solids, and the incommensurability of certain lines. One of the pupils of Pythagoras solved the problem of finding two mean proportionals between two straight lines.

Hippocrates, of the island of Chios, who lived about 400 years before Christ, was one of the best geometers of his time. He was the first who effected the quadrature of a curvilinear space by finding a rectilinear one equal to it. He showed that the *crescent*, bounded by half the circumference of one circle, and one fourth the circumference of another, is equal to an isosceles right-angled triangle whose hypothenuse is the common chord of the two arcs. He also showed that the duplication of the cube depends on the finding of two mean proportionals between two given lines.

One of the most distinguished promoters of science among the Greeks was the celebrated philosopher Plato. He traveled in Egypt and Italy, and, on his return to Greece, made mathematics the basis of his instruction. He put an inscription over the door of his school forbidding any one to enter who did not understand Geometry; and, when questioned concerning the probable employment of the Deity, answered that *he geometrized continually*. Plato is reported to have invented the geometrical analysis, and the conic sections were first studied in his school.

The problem concerning the duplication of the cube acquired its celebrity about the time of Plato, who gave a solution of the problem himself, and it was also resolved by several other geometers. Another celebrated problem which occupied much attention in the school of Plato was the *trisection of an angle*. The geometricians of that school failed, as all others have done, in solving this problem by means of elementary Geometry. While they failed in their main object, their exertions were not thrown away, as they made valuable discoveries regarding the conic sections and other branches of Geometry. Eudoxus, a contemporary of Plato, found the measure of the pyramid and cone, and cultivated the theory of the conic sections.

After the time of Plato, the most remarkable epoch in the history of Geometry was the establishment of the school of Alexandria, about 300 years before Christ. It was here that the celebrated geometer Euclid flourished under the first of the Ptolemies. His native place is not known, but he studied at Athens, under the disciples of Plato, before he settled at Alexandria. It is recorded of Euclid that, when Ptolemy asked him whether there was no easier means of acquiring a knowledge of Geometry than that given in his Elements, he replied, "No, sir; there is no royal road to Geometry." Euclid composed treatises on various branches of the ancient mathematics; but he is best known by his Elements, a work on Geometry and Arithmetic, in thirteen books, under which he has collected all the elementary truths of Geometry which had been found before his time. This work has been translated into the languages of all nations that have made any considerable progress in civilization since it was first published, and has been more generally used for the purposes of teaching than any other work on abstract science that has ever appeared.

Of Euclid's Elements, the first four books treat of the properties of plane figures; the fifth contains the theory of proportion, and the sixth its application to plane figures; the seventh, eighth, ninth, and tenth relate to Arithmetic, and the doctrine of incommensurables; the eleventh and twelfth contain the elements of the geometry of solids, and the thirteenth treats of the five regular solids. Two books more—viz., the fourteenth and fifteenth—on regular solids, have been attributed to Euclid, but are supposed to have been written about two centuries later.

It is only the first six, and the eleventh and twelfth, that are now much used in the schools.

After Euclid comes Archimedes, born at Syracuse about the year 287 B.C. He wrote two books on the sphere and cylinder, containing the discovery that the sphere is two thirds of the circumscribing cylinder, whether we compare their surfaces or solidities. In his book on the measure of the circle, he proves that if the diameter of a circle be reckoned unity, the circumference will be between $3\frac{10}{70}$ and $3\frac{10}{71}$. In his treatise on conoids and spheroids, he compares the area of an ellipse with that of a circle; and he proved that the area of any segment of a parabola cut off by a chord is two thirds of the circumscribing parallelogram.

After Archimedes comes Apollonius of Perga, in Pamphylia, born about 250 B.C. He studied in the Alexandrian school under the successors of Euclid, and so highly esteemed were his discoveries that he acquired the name of the *Great Geometer*. His treatise on the Conic Sections has contributed principally to his celebrity. During the five or six subsequent centuries we find a numerous list of mathematicians, most of whom are chiefly known as cultivators of Astronomy, and some as writers on Geometry. Near the close of the fourth century after Christ, Hypatia, the daughter of Theon, wrote commentaries on Apollonius and Diophantus, and was so learned in Geometry that she was judged worthy to succeed her father in the Alexandrian school. The school of Alexandria ceased in A.D. 640, when that city was taken by the Saracens.

In subsequent centuries the Arabs cultivated Astronomy and Geometry, and, after the revival of learning, the elements of Euclid were first known in Europe through the medium of an Arabic translation. In the fifteenth century, Vieta carried the approximate value of the ratio of the diameter of a circle to its circumference as far as eleven figures, and Adrianus Romanus carried the approximation as far as seventeen decimal figures. In the seventeenth century, Van Ceulen carried this approximation to thirty-five decimal figures.

Albert Girard, a Flemish mathematician in the seventeenth century, was the first who determined the surface of a spherical triangle, or of a polygon bounded by great circles on the sphere. Kepler was the first to introduce the idea of infinity into the language of geometry. He regarded the circle as composed of an infinite number of triangles, having their vertices at the centre; the cone as composed of an infinite number of pyramids, all having the same vertex as the cone.

The application of Algebra to Geometry by Descartes, in the early part of the seventeenth century, produced a complete revolution in this science. By bringing Geometry under the dominion of Algebra, the investigations are freed from that cumbrous formality which, however admirable in the elements of science, and however well it may be calculated to discipline the mind, is powerless in the more advanced researches of science. This application of Algebra has been reduced to a systematic form, constituting a separate branch of science, which is generally called *Analytic Geometry.*

During the present century Geometry has been most successfully cultivated by the French. The treatise on Elementary Geometry which, next to that of Euclid, has been most extensively adopted, is the treatise of Legendre, first published in 1794, and which has lately received important additions and modifications by Blanchet. The present volume follows substantially the order of Blanchet's Legendre, while the form of the demonstrations is modeled after the more logical method of Euclid.

The problem of the duplication of the cube, or its equivalent, the finding of two mean proportionals between two given magnitudes, is supposed to have first called the attention of mathematicians to the conic sections. If four quantities, as A, B, C, D, are in continued proportion, then $A^3 : B^3 :: A : D$; that is, we could find a cube which should have any given ratio to a given cube, provided we could find two mean proportionals between A and D. Thus 24 and 36 are two mean proportionals between 16 and 54. This problem can not be resolved merely by straight lines and circles—the only lines at first admitted into Geometry, and hence it became necessary to inquire what other lines would afford a solution of this and similar problems, and this investigation led to the study of the Conic Sections. We know little more than the names of the early cultivators of this branch of science, among whom are Aristæus, Euclid, Conon, and Archimedes. Archimedes demonstrated that the area of a parabola is two thirds of that of the circumscribing parallelogram; and he also showed what was the ratio of elliptic areas to their circumscribing circles, and of solids formed by the revolution of the different sections to their circumscribing cylinders.

Apollonius of Perga wrote a work on Conic Sections, consisting of eight books; the first four are supposed to comprehend all that was known on the subject before his time, and the remaining books are supposed to have contained his own discoveries. The first seven books of Apollonius's Conics have been preserved, and the eighth has been restored by Dr. Halley from the hints afforded by the account given of it by Pappus, a writer of the fourth century.

In the early ages of science, the Conic Sections were studied merely as a geometrical theory, but the discoveries of modern times have rendered it the most interesting speculation in Pure Geometry. Galileo showed that the path of a body projected obliquely in a vacuum is a parabola, and Kepler discovered that the planetary orbits are ellipses. Newton demonstrated that a body which revolves under the influence of a central force like gravitation, whose intensity decreases as the square of the distance increases, must move in one of the conic sections—that is, either a parabola, an ellipse, or an hyperbola. These discoveries have incorporated the theory of the Conic Sections with those of Astronomy and the other branches of Natural Philosophy.

ELEMENTS OF GEOMETRY.

BOOK I.

GENERAL PRINCIPLES.

Definitions.

1. EVERY material object occupies a limited portion of space. The portion of space which a body occupies, considered separately from the matter of which the body is composed, is called a *Geometrical solid.* The material body which occupies the given space is called a *Physical solid.* A geometrical solid is, therefore, merely the space occupied by a physical solid. In this treatise, only geometrical solids are considered, and they are called simply solids.

A *solid* is, then, a limited portion of space.

2. The *surface* of a solid is the limit or boundary which separates it from the surrounding space.

3. When one surface is cut by another surface, their common section is called a *line.*

4. When two lines cut each other, their common section is called a *point.*

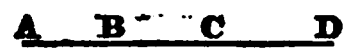
5. Although we may derive the idea of a point from the consideration of lines, the idea of a line from the consideration of surfaces, and the idea of a surface from the consideration of a solid, we may conceive of a surface as independent of the space of which it is the boundary; we may conceive of a line as independent of the surfaces of which it is the common section, and as existing separately in space; and we may conceive of a point as independent of the lines of which it is the common section, and as having only position in space.

6. A solid has extension in all directions; but, for the purpose of measuring its magnitude more conveniently, we consider it as having three specific dimensions, called *length*, *breadth*, and *thickness.*

7. A *surface* has only two dimensions, length and breadth.

A *line* has only one dimension, viz., length.

A *point* has no extension, and therefore neither length, breadth, nor thickness.*

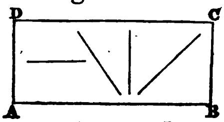


8. A *straight line* is a line which is the shortest path between any two of its points, as ABCD.

9. A *broken line* is a line composed of different straight lines, as ABCDEF.

10. A *curved line*, or simply a *curve*, is a line no portion of which is straight, as ABC.

For the sake of brevity, the word *line* is often used to denote a straight line.



11. A *plane surface*, or simply a *plane*, is a surface in which, if any two points are taken, the straight line which joins them lies wholly in that surface.†

12. A *curved surface* is a surface no portion of which is plane.

13. A *geometrical figure* is any combination of points, lines, surfaces, or solids.

Figures formed by points and lines in a plane are called *plane figures.*

14. *Geometry* is the science which treats of the properties of figures, of their construction, and of their measurement.

15. *Plane geometry* treats of plane figures. *Geometry of space*, or geometry of *three dimensions*, treats of figures all of whose points are not situated in the same plane.

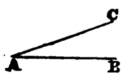
16. When two straight lines meet together, their mutual inclination, or degree of opening, is called an *angle.* The point in which the straight lines meet is called the *vertex* of the angle, and the lines are called the *sides* of the angle.‡

---

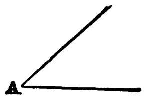
* In geometrical figures or diagrams we are obliged to employ physical lines and points instead of mathematical ones, since the finest line that we can draw has breadth. Our reasoning is not, however, thereby vitiated, because it is conducted on the supposition that the lines have *no* breadth, and nothing in our reasoning depends upon the breadth of the lines in our diagram.

† If two points be taken upon the surface of a ball, the straight line which joins them will lie *within* the ball, and not on its surface. Therefore the surface of a ball is *not* a plane surface.

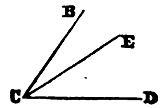
‡ A clear idea of the nature of an angle may be obtained by supposing that one of its sides, as AC, at first coincided with the other side AB, and that it has revolved about the point A (turning about A as one leg of a pair of compasses) until it has reached the position AC. By continuing the revolution, an angle of any magnitude may be formed. It is evident that the magnitude of the angle does not depend upon the length of its sides.



If there is only one angle at a point, it may be denoted by a letter placed at the vertex, as the angle at A.



But when several angles are formed at the same point by different lines, either of the angles may be denoted by three letters, namely, by one letter on each of its sides, together with one at its vertex, which must be written between the other two. Thus the lines CB, CE, CD form three different angles, which are distinguished as BCE, ECD, and BCD.



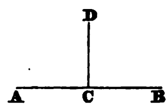
17. Angles are measured by degrees. A *degree* is one of the three hundred and sixty equal parts of the angular space about a point in a plane. (See B. III., Pr. 14.)

18. Angles, like other quantities, may be added, subtracted, multiplied, or divided.

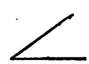
Thus the angle BCD is the sum of the two angles BCE, ECD, and the angle ECD is the difference between the two angles BCD, BCE.

19. When one straight line meets another so as to make two adjacent angles equal, each of these angles is called a *right angle*, and the first line is said to be *perpendicular* to the second.

Thus, if the line CD, meeting the line AB, makes the angles ACD, BCD equal, each is a right angle, and the line CD is perpendicular to AB.



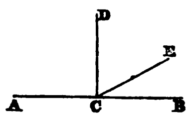
20. An *acute* angle is one which is less than a right angle.



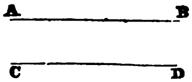
An *obtuse* angle is one which is greater than a right angle.

21. Intersecting lines which are not perpendicular are said to be *oblique* to each other, and angles which are not right angles are sometimes called *oblique*.

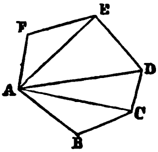
22. When the sum of two angles is equal to a right angle, each is called the *complement* of the other. Thus, if BCD is a right angle, BCE is the complement of DCE, and DCE is the complement of BCE.



23. When the sum of two angles is equal to two right angles, each is called the *supplement* of the other. Thus, if ACE and BCE are together equal to two right angles, then ACE is the supplement of BCE.



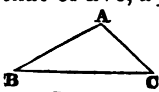
24. *Parallel* straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet, as AB, CD.



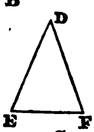
25. A *rectilineal figure*, or *polygon*, is a portion of a plane bounded by straight lines, as ABCDEF. The bounding lines are called the *sides* of the polygon; and the sides, taken together, form the *perimeter* of the polygon.

26. A *diagonal* of a polygon is a line joining the vertices of two angles not adjacent to each other, as AC or AD.

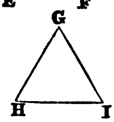
27. The polygon of three sides is the simplest of all, and is called a *triangle;* that of four sides is called a *quadrilateral;* that of five, a *pentagon;* that of six, a *hexagon*, etc.



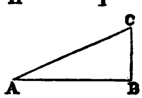
28. A triangle is called *scalene* when no two of its sides are equal, as ABC.



A triangle is called *isosceles* when two of its sides are equal, as DEF.



A triangle is called *equilateral* when its three sides are equal, as GHI.

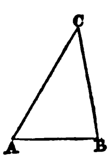


29. A *right-angled* triangle is one which has a right angle, as ABC, which is right-angled at B. The side AC, opposite to the right angle, is called the *hypothenuse.*

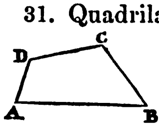
An *obtuse-angled* triangle is one which has an obtuse angle.

An *acute-angled* triangle is one which has three acute angles.

30. The *base* of a triangle is the side upon which it is supposed to stand. Any side may be assumed as the base, but in an isosceles triangle that side is called the base which is not equal to either of the others. When any side AB of a triangle has been adopted as the base, the angle ACB opposite to it is called the *vertical angle.*

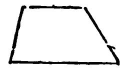


31. Quadrilaterals are divided into classes as follows:



1st. The *trapezium*, having no two sides parallel, as ABCD.

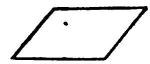
2d. The *trapezoid*, which has two sides parallel.



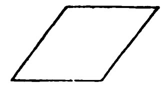
3d. The *parallelogram*, which has two pairs of parallel sides.

32. Parallelograms are divided into classes as follows:

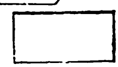
1st. The *rhomboid*, whose angles are not right angles, and its adjacent sides are not necessarily equal.



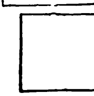
2d. The *rhombus*, which is an equilateral rhomboid.



3d. The *rectangle*, which has all its angles right angles, but all its sides are not necessarily equal.

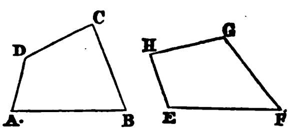


4th. The *square*, which is an equilateral rectangle.

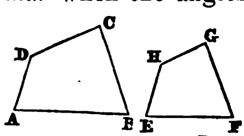


33. An *equilateral* polygon is one which has all its sides equal. An *equiangular* polygon is one which has all its angles equal.

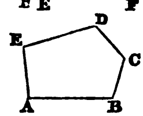
34. Two polygons are *mutually equilateral* when the sides of the one are equal to the corresponding sides of the other, each to each, and arranged in the same order, as ABCD, EFGH. The equal sides are called *homologous* sides, as AB, EF.



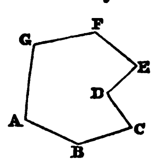
35. Two polygons are *mutually equiangular* when the angles of the one are equal to the corresponding angles of the other, each to each, and arranged in the same order, as ABCD, EFGH. The equal angles are called *homologous* angles, as A and E.



36. A *convex* polygon is such that a straight line, however drawn, can not meet the perimeter of the polygon in more than two points, as ABCDE.



37. A *concave* polygon is such that a straight line may be drawn meeting the perimeter of the polygon in more than two points, as ABCDEFG. The angle D, contained by two re-entrant sides, is called a *re-entrant* angle. All the polygons hereafter considered will be understood to be convex, unless the contrary is stated.



38. An *axiom* is a truth assumed as self-evident.

39. A *theorem* is a truth which becomes evident by a train of reasoning called a *demonstration.*

40. A *problem* is a question proposed which requires a solution.

41. A *postulate* is a problem so simple that it is unnecessary to point out the method of performing it.

42. A *proposition* is a general term for either a theorem or a problem.

43. One proposition is the *converse* of another when the conclusion of the first is made the supposition of the second.

44. A *corollary* is an immediate consequence deduced from one or more propositions.

45. A *scholium* is a remark upon one or more propositions, pointing out their connection, their use, their limitation, or their extension.

46. An *hypothesis* is a supposition made either in the enunciation of a proposition or in the course of a demonstration.

Axioms.

1. Things which are equal to the same thing, or to equals, are equal to one another.

2. If equals, or the same, be added to equals, the wholes are equal.*

3. If equals, or the same, be taken from equals, the remainders are equal.

4. If equals, or the same, be added to unequals, the wholes are unequal.

5. If equals, or the same, be taken from unequals, the remainders are unequal.

6. Things which are doubles of the same, or of equals, are equal to one another.

7. Things which are halves of the same, or of equals, are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than any of its parts.

10. The whole is equal to the sum of all its parts.

---

* When this axiom is applied to geometrical magnitudes, it must be understood to refer simply to equality of areas. It is not designed to assert that when equal triangles are united to equal triangles, the resulting figures will admit of coincidence by superposition.

11. From one point to another only one straight line can be drawn.

12. Two straight lines which intersect one another can not both be parallel to the same straight line.

Explanation of Signs.

For the sake of brevity, it is convenient to employ in Geometry some of the signs of Algebra. The following are those which are most frequently employed:

The sign $=$ denotes that the quantities between which it stands are equal; thus the expression $A=B$ signifies that A is equal to B.

The sign $>$ or $<$ denotes inequality. Thus $A>B$ denotes that A is greater than B; and $A<B$ denotes that A is less than B.

The sign $+$ is called *plus*, and indicates addition; thus $A+B$ represents the sum of the quantities A and B.

The sign $-$ is called *minus*, and indicates subtraction; thus $A-B$ represents what remains after subtracting B from A.

The sign $\times$ indicates multiplication; thus $A\times B$ denotes the product of A by B. Instead of the sign $\times$, a point is sometimes employed; thus $A.B$ is the same as $A\times B$. The same product is also sometimes represented without any intermediate sign, by AB; but this expression should not be employed when there is any danger of confounding it with the line AB.

A parenthesis ( ) indicates that several quantities are to be subjected to the same operation; thus the expression $A\times(B+C-D)$ represents the product of A by the quantity $B+C-D$.

The expression $\frac{A}{B}$ indicates the quotient arising from dividing A by B.

A number placed before a line or a quantity is to be regarded as a multiplier of that line or quantity; thus 3AB denotes that the line AB is taken three times; $\frac{1}{2}A$ denotes the half of A.

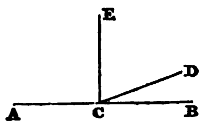
The square of the line AB is denoted by AB^2; its cube by AB^3.

The sign $\surd$ indicates a root to be extracted; thus $\surd 2$ denotes the square root of 2; $\sqrt{A\times B}$ denotes the square root of the product of A and B.

N.B.—*The first six books treat only of plane figures, or figures drawn on a plane surface.*

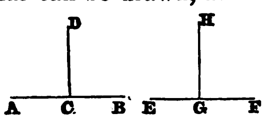
PROPOSITION I. THEOREM.

From a given point in a straight line one perpendicular to that line can be drawn, and but one.



Let AB be a given straight line, and C a given point in it. From the point C one perpendicular can be drawn to the line AB, and only one can be drawn.

Suppose that while one extremity of a straight line remains fixed at C, the line itself turns about this point from the position CB to the position CD. In each of its successive positions it makes two different angles with the line AB; one angle DCB with the portion CB, and another angle ACD with the portion AC. While the line revolves from the position CB around to to the position AC, the angle DCB, which begins from zero, is continually increasing; while the angle ACD, which at first is greater than DCB, is continually decreasing until it becomes zero. The angle DCB, which at first was smaller than ACD, becomes at last greater than ACD. There must, therefore, be one position of the revolving line, as CE, where these two angles are equal; and it is evident that there can be but one such position. Therefore, from a given point in a straight line, one perpendicular can be drawn, and but one.*



Corollary. All right angles are equal to each other. Let the straight line DC be perpendicular to AB, and GH to EF; then will each of the angles ACD, BCD be equal to each of the angles EGH, FGH.

Let the line AB be applied to the line EF so as to coincide with it, and in such a manner that the point C shall fall upon G; then will the line CD take the direction GH; otherwise there would be two perpendiculars to the line AB drawn from the same point C, which, by the preceding Proposition, is impossible. There-

* The words in which a Proposition is expressed are called its *enunciation.* If the enunciation refer to a particular diagram, it is called a *particular* enunciation, otherwise it is a general one.

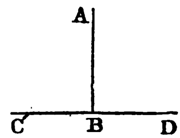
A *demonstration* is a series of arguments which establish the truth of a theorem. The drawing of such lines as may be necessary to a demonstration is called the *construction.*

Under each proposition there is usually given, *first*, the general enunciation; *second*, the particular enunciation; *third*, the construction; and, *fourth*, the demonstration.

fore the line CD must coincide with the line GH, and the angle ACD will be equal to EGH, and BCD to FGH (Axiom 8), and the four angles will be equal to each other (Ax. 1).

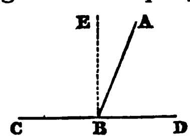
PROPOSITION II. THEOREM.

The angles which one straight line makes with another, upon one side of it, are either two right angles, or are together equal to two right angles.



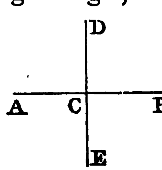
Let the straight line AB make with CD, upon one side of it, the angles ABC, ABD; these are either two right angles, or are together equal to two right angles.

For if the angle ABC is equal to ABD, each of them is a right angle (Def. 19); but if these angles are unequal, suppose the line BE to be drawn from the point B, perpendicular to CD; then will each of the angles CBE, DBE be a right angle. Now the angle CBA is equal to the sum of the two angles CBE, EBA. To each of these equals add the angle ABD; then the sum of the two angles CBA, ABD will be equal to the sum of the three angles CBE, EBA, ABD (Ax. 2).



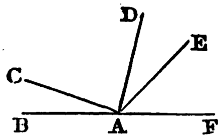
Again, the angle DBE is equal to the sum of the two angles DBA, ABE. Add to each of these equals the angle EBC; then will the sum of the two angles DBE, EBC be equal to the sum of the three angles DBA, ABE, EBC. Now things that are equal to the same thing are equal to each other (Ax. 1); therefore the sum of the angles CBA, ABD is equal to the sum of the angles CBE, EBD. But CBE, EBD are two right angles; therefore ABC, ABD are together equal to two right angles. Therefore, the angles which one straight line, etc.

Cor. 1. If one of the angles ABC, ABD is a right angle, the other is also a right angle.



Cor. 2. If the line DE is perpendicular to AB, conversely, AB is perpendicular to DE.

For, because DE is perpendicular to AB, the angle DCA must be equal to its adjacent angle DCB (Def. 19), and each of them must be a right angle. But since ACD is a right angle, its adjacent angle, ACE, must also be a right angle (Cor. 1). Hence the angle ACE is equal to the angle ACD (Pr. 1, Cor.), and AB is perpendicular to DE.

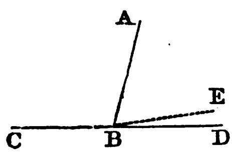


Cor. 3. The sum of all the angles BAC, CAD, DAE, EAF, formed on the same side of the line BF, at a common point A, is equal to two right angles; for their sum is equal to that of the two adjacent angles BAD, DAF, which, by the Proposition, is equal to two right angles.

PROPOSITION III. THEOREM (*Converse of Prop. II.*).

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in one and the same straight line.

At the point B, in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles, ABC, ABD, together equal to two right angles; then will BD be in the same straight line with CB.



For, if BD is not in the same straight line with CB, let BE be in the same straight line with it; then, because the straight line CBE is met by the straight line AB, the angles ABC, ABE are together equal to two right angles (Pr. 2). But, by hypothesis, the angles ABC, ABD are together equal to two right angles; therefore the sum of the angles ABC, ABE is equal to the sum of the angles ABC, ABD. Take away the common angle ABC, and the remaining angle ABE is equal (Ax. 3) to the remaining angle ABD; the less to the greater, which is impossible. Hence BE is not in the same straight line with BC; and in like manner it may be proved that no other can be in the same straight line with it but BD. Therefore, if, at a point, etc.*

* The enunciation of a theorem embraces two parts, an *hypothesis* and a *conclusion*. The hypothesis is a supposition made, and the conclusion is a consequence of the supposition. Prop. 3 might be enunciated thus: *Hypothesis*, if, at a point in a straight line, two other straight lines upon the opposite sides of it make the adjacent angles together equal to two right angles, then, *Conclusion*, these two straight lines are in one and the same straight line.

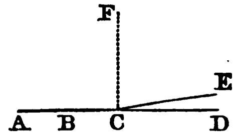
Proposition 3d is the *converse* of the 2d; that is, the conclusion of the 3d is the hypothesis in the 2d.

Proposition 2d may be enunciated thus: *Hypothesis*, if, at a point in a straight line, two other straight lines upon opposite sides form but one straight line, then, *Conclusion*, the two adjacent angles are together equal to two right angles.

Demonstrations are either *direct* or *indirect*. The *direct* demonstration com-

PROPOSITION IV. THEOREM.

Two straight lines, which have two points common, coincide with each other throughout their whole extent, and form but one and the same straight line.



Let there be two straight lines having the points A and B in common; these lines will coincide throughout their whole extent.

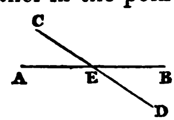
It is plain that the two lines must coincide between A and B, for otherwise there would be two straight lines between A and B, which is impossible (Ax. 11).

Suppose, however, that, on being produced, these lines begin to diverge at the point C, one taking the direction CD, and the other CE. From the point C draw the line CF at right angles with AC; then, since ACD is a straight line, the angle FCD is a right angle (Pr. 2, Cor. 1); and, since ACE is a straight line, the angle FCE is also a right angle; therefore (Pr. 1, Cor.) the angle FCE is equal to the angle FCD, the less to the greater, which is absurd. Therefore two straight lines which have, etc.

PROPOSITION V. THEOREM.

If two straight lines cut one another, the vertical or opposite angles are equal.

Let the two straight lines AB, CD cut one another in the point E; then will the angle AEC be equal to the angle BED, and the angle AED to the angle CEB.



For the angles AEC, AED, which the straight line AE makes with the straight line CD, are together equal to two right angles (Pr. 2); and the angles AED, DEB, which the straight line DE makes with the straight line AB, are also together equal to two right angles; therefore the sum of the two angles AEC, AED is equal to the sum of the two angles AED, DEB. Take away the common angle AED, and

---

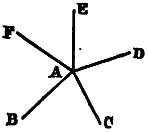
mences with what has been already admitted or proved to be true, and from this deduces a series of other truths, till it finally arrives at the truth to be proved.

In the *indirect* demonstration, or, as it is also called, the *reductio ad absurdum*, a supposition is made which is contrary to the conclusion to be established. On this assumption a demonstration is founded, which leads to a result contrary to some known truth, thus proving the truth of the proposition by showing that the supposition of its contrary leads to an absurd conclusion.

the remaining angle AEC is equal to the remaining angle DEB (Ax. 3).

In the same manner it may be proved that the angle AED is equal to the angle CEB. Therefore, if two straight lines, etc.

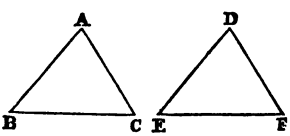
Cor. 1. Hence, if two straight lines cut one another, the four angles formed at the point of intersection are together equal to four right angles.



Cor. 2. If any number of straight lines AB, AC, etc., meet at a point A, the sum of all the angles BAC, CAD, DAE, EAF, FAB, will be equal to four right angles. For if two straight lines are drawn through A perpendicular to each other, the four right angles thus formed will together be equal to the sum of all the angles BAC, CAD, etc., formed about A.

PROPOSITION VI. THEOREM.

If two triangles have two sides, and the included angle of the one equal to two sides and the included angle of the other, each to each, the two triangles will be equal, their third sides will be equal, and their other angles will be equal, each to each.

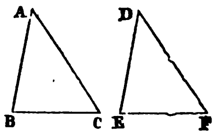


Let ABC, DEF be two triangles, having the side AB equal to DE, and AC to DF, and also the angle A equal to the angle D; then will the triangle ABC be equal to the triangle DEF.

For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB upon DE, the point B will coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC will coincide with DF, because the angle A is equal to the angle D. Hence, also, the point C will coincide with the point F, because AC is equal to DF. But the point B coincides with the point E, therefore the base BC will coincide with the base EF (Ax. 11), and will be equal to it. Hence, also, the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it, and the remaining angles of the one will coincide with the remaining angles of the other, and be equal to them, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE. Therefore, if two triangles, etc.

PROPOSITION VII. THEOREM.

If two triangles have two angles, and the included side of the one equal to two angles and the included side of the other, each to each, the two triangles will be equal, the other sides will be equal each to each, and the third angle of the one to the third angle of the other.

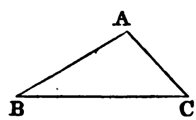


Let ABC, DEF be two triangles having the angle B equal to E, the angle C equal to F, and the included sides BC, EF equal to each other; then will the triangle ABC be equal to the triangle DEF.

For, if the triangle ABC be applied to the triangle DEF, so that the point B may be on E, and the straight line BC upon EF, the point C will coincide with the point F, because BC is equal to EF. Also, since the angle B is equal to the angle E, the side BA will take the direction ED, and therefore the point A will be found somewhere in the line DE. And, because the angle C is equal to the angle F, the line CA will take the direction FD, and the point A will be found somewhere in the line DF; therefore the point A, being found at the same time in the two straight lines DE, DF, must fall at their intersection, D. Hence the two triangles ABC, DEF coincide throughout, and are equal to each other; also, the two sides AB, AC are equal to the two sides DE, DF, each to each, and the angle A to the angle D. Therefore, if two triangles, etc.

PROPOSITION VIII. THEOREM.

Any side of a triangle is less than the sum of the other two.

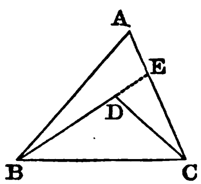


Let ABC be a triangle; any one of its sides is less than the sum of the other two, viz., the side AB is less than the sum of AC and BC; BC is less than the sum of AB and AC; and AC is less than the sum of AB and BC.

For the straight line AB is the shortest path between the points A and B (Def. 8); hence AB is less than the sum of AC and BC. For the same reason, BC is less than the sum of AB and AC, and AC less than the sum of AB and BC. Therefore, any two sides etc.

PROPOSITION IX. THEOREM.

If, from a point within a triangle, two straight lines are drawn to the extremities of either side, their sum will be less than the sum of the other two sides of the triangle.



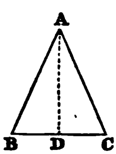
Let the two straight lines BD, CD be drawn from D, a point within the triangle ABC, to the extremities of the side BC; then will the sum of BD and DC be less than the sum of BA, AC, the other two sides of the triangle.

Produce BD until it meets the side AC in E; and, because one side of a triangle is less than the sum of the other two (Pr. 8), the side CD of the triangle CDE is less than the sum of CE and ED. To each of these add DB; then will the sum of CD and BD be less than the sum of CE and EB.

Again, because the side BE of the triangle BAE is less than the sum of BA and AE, if EC be added to each, the sum of BE and EC will be less than the sum of BA and AC. But it has been proved that the sum of BD and DC is less than the sum of BE and EC; much more, then, is the sum of BD and DC less than the sum of BA and AC. Therefore, if from a point, etc.

PROPOSITION X. THEOREM.

The angles at the base of an isosceles triangle are equal to one another.



Let ABC be an isosceles triangle, of which the side AB is equal to AC; then will the angle B be equal to the angle C.

For, conceive the angle BAC to be bisected by the straight line AD;* then, in the two triangles ABD, ACD, two sides AB, AD, and the included angle in the one, are equal to the two sides AC, AD, and the included an-

* Throughout this Treatise we shall assume the possibility of constructing our figures, although the methods of constructing them have not yet been explained. It is not essential to a geometrical demonstration that the precise mode of constructing the figures should be previously given. For the purpose of discovering the properties of figures, we are at liberty to suppose any figure to be constructed, or any line to be drawn, whose existence does not involve an impossibility. We shall show hereafter how the figures employed in these demonstrations may be constructed.

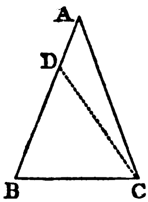
gle in the other; therefore (Pr. 6) the angle B is equal to the angle C. Therefore the angles at the base, etc.

Cor. 1. Hence, also, the line BD is equal to DC, and the angle ADB equal to ADC; consequently, each of these angles is a right angle (Def. 19). Therefore *the line bisecting the vertical angle of an isosceles triangle bisects the base at right angles;* and, conversely, *the line bisecting the base of an isosceles triangle at right angles bisects also the vertical angle.*

Cor. 2. Every equilateral triangle is also equiangular.

PROPOSITION XI. THEOREM (*Converse of Prop. X.*).

If two angles of a triangle are equal to one another, the opposite sides are also equal.



Let ABC be a triangle having the angle ABC equal to the angle ACB; then will the side AB be equal to the side AC.

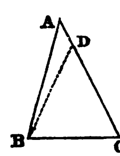
For if AB is not equal to AC, one of them must be greater than the other. Let AB be the greater, and from it cut off DB equal to AC the less, and join CD.

Then, because in the triangles DBC, ACB, DB is equal to AC, and BC is common to both triangles, also, by supposition, the angle DBC is equal to the angle ACB; therefore the triangle DBC is equal to the triangle ACB (Pr. 6), the less to the greater, which is absurd. Hence AB is not unequal to AC, that is, it is equal to it. Therefore, if two angles, etc.

Cor. Hence every equiangular triangle is also equilateral.

PROPOSITION XII. THEOREM.

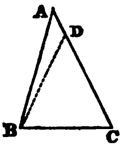
The greater side of every triangle is opposite to the greater angle; and, conversely, the greater angle is opposite to the greater side.



Let ABC be a triangle, having the angle ABC greater than the angle ACB; then will the side AC be greater than the side AB.

Draw the straight line BD, making the angle DBC equal to C; then, in the triangle BCD, the side CD must be equal to BD (Pr. 11). Add AD to each; then will the sum of AD and DC be equal to the sum of AD and DB. But AB is less than the sum of AD and DB (Pr. 8); it is, therefore, less than AC.

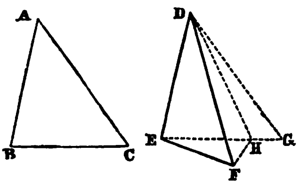
Conversely, if the side AC is greater than the side AB, then will the angle ABC be greater than the angle ACB.



For if ABC is not greater than ACB, it must be either equal to it or less. It is not equal, because then the side AC would be equal to the side AB (Pr. 11), which is contrary to the supposition. Neither is it less, because then the side AC would be less than the side AB, according to the former part of this proposition; hence ABC must be greater than ACB. Therefore the greater side, etc.

PROPOSITION XIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, the base of that which has the greater angle will be greater than the base of the other.



Let ABC, DEF be two triangles, having two sides of the one equal to two sides of the other, viz., AB equal to DE, and AC to DF, but the angle BAC greater than the angle EDF; then will the base BC be greater than the base EF.

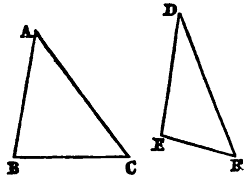
Of the two sides DE, DF, let DE be the side which is not greater than the other; and at the point D, in the straight line DE, make the angle EDG equal to BAC; make DG equal to AC or DF, and join EG.

Because, in the triangles ABC, DEG, AB is equal to DE, and AC to DG; also, the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG (Pr. 6).

Draw the line DH bisecting the angle FDG, and meeting EG in H, and join FH. Now, because the angle FDH is equal to the angle GDH, also DG is equal to DF, and DH is common to the two triangles FDH, GDH, therefore FH is equal to GH (Pr. 6). Adding EH to each of these equals, we have the sum of EH and HF equal to the sum of EH and HG, or EG. But the sum of EH and HF is greater than EF (Pr. 8). Hence EG, or its equal BC, is greater than EF. Therefore, if two triangles, etc.

PROPOSITION XIV. THEOREM (*Converse of Prop. XIII.*).

If two triangles have two sides of the one equal to two sides of the other, each to each, but the bases unequal, the angle contained by the sides of that which has the greater base will be greater than the angle contained by the sides of the other.

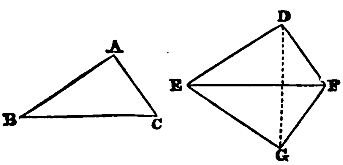


Let ABC, DEF be two triangles having two sides of the one equal to two sides of the other, viz., AB equal to DE, and AC to DF, but the base BC greater than the base EF; then will the angle BAC be greater than the angle EDF.

For if it is not greater, it must be either equal to it, or less. But the angle BAC is not equal to the angle EDF, because then the base BC would be equal to the base EF (Pr. 6), which is contrary to the supposition. Neither is it less, because then the base BC would be less than the base EF (Pr. 13), which is also contrary to the supposition; therefore the angle BAC is not less than the angle EDF, and it has been proved that it is not equal to it; hence the angle BAC must be greater than the angle EDF. Therefore, if two triangles, etc.

PROPOSITION XV. THEOREM.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the three angles will also be equal, each to each, and the triangles themselves will be equal.



Let ABC, DEF be two triangles having the three sides of the one equal to the three sides of the other, viz., AB equal to DE, BC to EF, and AC to DF; then will the three angles also be equal, viz., the angle A to the angle D, the angle B to the angle E, and the angle C to the angle F.

Suppose the triangle ABC to be placed so that its base BC coincides with its equal EF, but so that its vertex A falls on the opposite side of EF from D, as at G. Join DG; and because ED and EG are each equal to AB, they are equal to each other, and the triangle EDG is isosceles; therefore the angle EDG is equal to the angle EGD (Pr. 10).

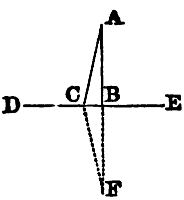
In the same manner it may be shown that the angle FDG is equal to the angle FGD. Therefore, adding equals to equals, the two angles EDG, FDG are together equal to the two angles EGD, FGD; that is, the angle EDF is equal to the angle EGF. But the angle EGF is, by hypothesis, equal to the angle BAC; therefore also the angle BAC is equal to the angle EDF.

Since the two sides AB and AC are equal to the two sides DE and DF, each to each, and their included angles BAC, EDF are also equal, the two triangles ABC, DEF are equal (Pr. 6), and their other angles are equal each to each, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE. Therefore, if two triangles, etc.

Scholium. In equal triangles, the equal angles are opposite to the equal sides; thus the equal angles A and D are opposite to the equal sides BC, EF.

PROPOSITION XVI. THEOREM.

From a given point without a straight line, only one perpendicular can be drawn to that line.



Let A be the given point, and DE the given straight line; from the point A only one perpendicular can be drawn to DE.

For, if possible, let there be drawn two perpendiculars AB, AC. Produce the line AB to F, making BF equal to AB, and join CF.

Then, in the triangles ABC, FBC, because AB is equal to BF, BC is common to both triangles, and the angle ABC is equal to the angle FBC, being both right angles (Pr. 2, Cor. 1); therefore two sides and the included angle of one triangle, are equal to two sides and the included angle of the other triangle; hence the angle ACB is equal to the angle FCB (Pr. 6).

But, since the angle ACB is, by supposition, a right angle, FCB must also be a right angle; and the two adjacent angles BCA, BCF, being together equal to two right angles, the two straight lines AC, AF must form one and the same straight line (Pr. 3); that is, between the two points A and F, two straight lines, ABF, ACF, may be drawn, which is impossible (Ax. 11); hence AB and AC can not both be perpendicular to DE. Therefore, from a point, etc.

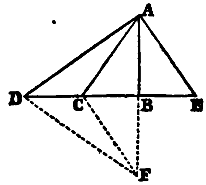
PROPOSITION XVII. THEOREM.

If, from a point without a straight line, a perpendicular be drawn to this line, and oblique lines be drawn to different points:

1*st. The perpendicular will be shorter than any oblique line.*

2*d. Two oblique lines, which meet the proposed line at equal distances from the foot of the perpendicular, will be equal.*

3*d. Of any two oblique lines, that which is further from the perpendicular will be the longer.*



Let DE be the given straight line, and A any point without it. Draw AB perpendicular to DE; draw, also, the oblique lines AC, AD, AE. Produce the line AB to F, making BF equal to AB, and join CF, DF.

First. Because, in the triangles ABC, FBC, AB is equal to BF, BC is common to the two triangles, and the angle ABC is equal to the angle FBC, being both right angles (Pr. 2, Cor. 1); therefore two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle; hence the side CF is equal to the side CA (Pr. 6).

But the straight line ABF is shorter than the broken line ACF (Pr. 8); hence AB, the half of ABF, is shorter than AC, the half of ACF. Therefore the perpendicular AB is shorter than any oblique line, AC.

Secondly. Let AC and AE be two oblique lines which meet the line DE at equal distances from the foot of the perpendicular; they will be equal to each other.

For, in the triangles ABC, ABE, BC is equal to BE, AB is common to the two triangles, and the angle ABC is equal to the angle ABE, being both right angles (Pr. 1, Cor.); therefore two sides and the included angle of one triangle are equal to two sides and the included angle of the other; hence the side AC is equal to the side AE (Pr. 6). Wherefore two oblique lines, equally distant from the perpendicular, are equal.

Thirdly. Let AC, AD be two oblique lines, of which AD is further from the perpendicular than AC; then will AD be longer than AC. For it has already been proved that AC is equal to CF, and in the same manner it may be proved that AD is equal to DF. Now, by Pr. 9, the sum of the two lines AC, CF is less than the sum of the two lines AD, DF. Therefore AC, the half of ACF, is less than AD, the half of ADF; hence the oblique line which is furthest from the perpendicular is the longest. Therefore, if from a point, etc.

Cor. 1. The perpendicular measures the shortest distance of a point from a line, because it is shorter than any oblique line. This shortest distance is frequently called the true distance, or simply the distance.

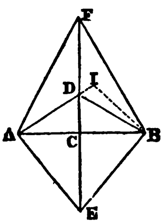
Cor. 2. It is impossible to draw three equal straight lines from the same point to a given straight line.

PROPOSITION XVIII. THEOREM.

If through the middle point of a straight line a perpendicular is drawn to this line:

1*st. Each point in the perpendicular is equally distant from the two extremities of the line.*

2*d. Any point out of the perpendicular is unequally distant from those extremities.*



Let the straight line EF be drawn perpendicular to AB through its middle point, C.

First. Every point of EF is equally distant from the extremities of the line AB; for, since AC is equal to CB, the two oblique lines AD, DB are equally distant from the perpendicular, and are, therefore, equal (Pr. 17).

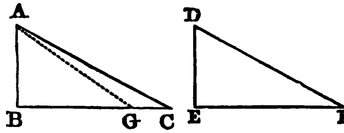
So, also, the two oblique lines AE, EB are equal, and the oblique lines AF, FB are equal; therefore every point of the perpendicular is equally distant from the extremities A and B.

Secondly. Let I be any point out of the perpendicular. Draw the straight lines IA, IB; one of these lines must cut the perpendicular in some point, as D. Join DB; then, by the first case, AD is equal to DB. To each of these equals add ID; then will IA be equal to the sum of ID and DB. Now, in the triangle IDB, IB is less than the sum of ID and DB (Pr. 8); it is, therefore, less than IA; hence every point out of the perpendicular is unequally distant from the extremities A and B.

Cor. If a straight line have two points, each of which is equally distant from the two extremities of a second line, it will be perpendicular to the second line at its middle point.

PROPOSITION XIX. THEOREM.

If two right-angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the triangles are equal.



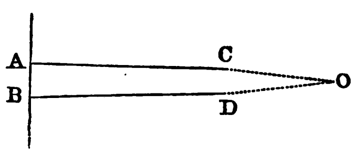
Let ABC, DEF be two right-angled triangles, having the hypothenuse AC and the side AB of the one equal to the hypothenuse DF and side DE of the other; then will the side BC be equal to EF, and the triangle ABC to the triangle DEF.

For if BC is not equal to EF, one of them must be greater than the other. Let BC be the greater, and from it cut off BG equal to EF the less, and join AG.

Then, in the triangles ABG, DEF, because AB is equal to DE, BG is equal to EF, and the angle B equal to the angle E, both of them being right angles, the two triangles are equal (Pr. 6), and AG is equal to DF. But, by hypothesis, AC is equal to DF, and therefore AG is equal to AC. Now the oblique line AC, being further from the perpendicular than AG, is the longer (Pr. 17), and it has been proved to be equal, which is impossible. Hence BC is not unequal to EF; that is, it is equal to it; and the triangle ABC is equal to the triangle DEF (Pr. 15). Therefore, if two right-angled triangles, etc.

PROPOSITION XX. THEOREM.

Two straight lines perpendicular to the same straight line are parallel.

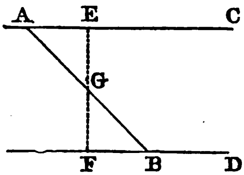


Let the two straight lines AC, BD be both perpendicular to AB; then is AC parallel to BD.

For if these lines are not parallel, being produced, they must meet on one side or the other of AB. Let them be produced, and meet in O; then there will be two perpendiculars, OA, OB, let fall from the same point, on the same straight line, which is impossible (Pr. 16). Therefore two straight lines, etc.

PROPOSITION XXI. THEOREM.

If a straight line meeting two other straight lines makes the interior angles on the same side together equal to two right angles, the two lines are parallel.

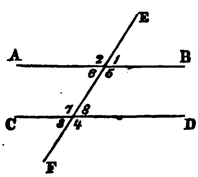


Let the straight line AB, which meets the two straight lines AC, BD, make the interior angles on the same side, BAC, ABD, together equal to two right angles; then is AC parallel to BD.

From G, the middle point of the line AB, draw EFG perpendicular to AC; it will also be perpendicular to BD.

For the sum of the angles ABD and ABF is equal to two right angles (Pr. 2); and, by hypothesis, the sum of the angles ABD and BAC is equal to two right angles. Therefore the sum of ABD and ABF is equal to the sum of ABD and BAC. Take away the common angle ABD, and the remainder, ABF, is equal to BAC; that is, GBF is equal to GAE.

Again, the angle BGF is equal to the angle AGE (Pr. 5); and, by construction, BG is equal to GA; hence the triangles BGF, AGE have two angles and the included side of the one equal to two angles and the included side of the other; they are, therefore, equal (Pr. 7); and the angle BFG is equal to the angle AEG. But AEG is, by construction, a right angle, whence BFG is also a right angle; that is, the two straight lines EC, FD are perpendicular to the same straight line, and are consequently parallel (Pr. 20). Therefore, if a straight line, etc.



Scholium. When two parallel lines AB, CD are cut by a third line EF, called the secant line, the eight angles formed at the points of intersection are named as follows:

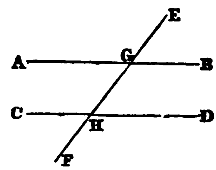
1st. The four angles 1, 2, 3, 4, without the parallel lines, are called *exterior* angles.

2d. The four angles 5, 6, 7, 8, within the parallel lines, are called *interior* angles.

3d. The two angles on opposite sides of the secant line, and not adjacent, are called *alternate* angles, as 1 and 3, or 2 and 4. Also, 5 and 7, or 6 and 8.

PROPOSITION XXII. THEOREM.

If a straight line intersecting two other straight lines makes the alternate angles equal to each other, or makes an exterior angle equal to the interior and remote upon the same side of the secant line, these two lines are parallel.



Let the straight line EF, which intersects the two straight lines AB, CD, make the alternate angles AGH, GHD equal to each other; then AB is parallel to CD.

For, to each of the equal angles AGH, GHD, add the angle HGB; then the sum of AGH and HGB will be equal to the sum of GHD and HGB. But AGH and HGB are equal to two right angles (Pr. 2); therefore GHD and HGB are equal to two right angles; and hence AB is parallel to CD (Pr. 21).

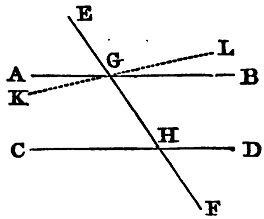
Again, if the exterior angle EGB is equal to the interior and remote angle GHD, then is AB parallel to CD.

For, the angle AGH is equal to the angle EGB (Pr. 5); and, by supposition, EGB is equal to GHD; therefore the angle AGH is equal to the angle GHD, and they are alternate angles; hence, by the first part of the proposition, AB is parallel to CD. Therefore, if a straight line, etc.

PROPOSITION XXIII. THEOREM.

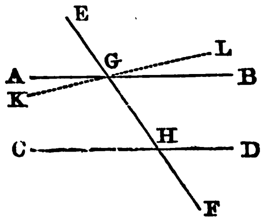
(*Converse of Propositions XXI. and XXII.*)

If a straight line intersect two parallel lines, it makes the alternate angles equal to each other; also, any exterior angle equal to the interior and remote on the same side; and the two interior angles on the same side together equal to two right angles.



Let the straight line EF intersect the two parallel lines AB, CD; the alternate angles AGH, GHD are equal to each other; the exterior angle EGB is equal to the interior and remote angle on the same side, GHD; and the two interior angles on the same side, BGH, GHD, are together equal to two right angles.

For, if AGH is not equal to GHD, through G draw the line KL, making the angle KGH equal to GHD; then KL must be parallel to CD (Pr. 22). But, by supposition, AB is parallel to CD; therefore, through the same point, G, two straight lines have been drawn parallel to CD, which is impossible (Ax. 12). Therefore the angles AGH, GHD are not unequal; that is, they are equal to each other.



Now the angle AGH is equal to EGB (Pr. 5), and AGH has been proved equal to GHD; therefore EGB is also equal to GHD. Add to each of these equals the angle BGH; then will the sum of EGB, BGH be equal to the sum of BGH, GHD. But EGB, BGH are equal to two right angles (Pr. 2); therefore, also, BGH, GHD are equal to two right angles. Therefore, if a straight line, etc.

Cor. 1. If a straight line is perpendicular to one of two parallel lines, it is also perpendicular to the other.

Cor. 2. Of the eight angles formed by a line cutting two parallel lines obliquely, the four acute angles are equal to each other, and the four obtuse angles are also equal to each other.

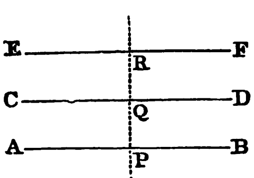
Cor. 3. If two lines, KL and CD, make with EF the two angles KGH, GHC, together less than two right angles, then will KL and CD meet, if sufficiently produced.

For if they do not meet they are parallel (Def. 24). But they are not parallel; for then the angles KGH, GHC would be equal to two right angles.

It is evident that the two lines KL and CD will meet on that side of EF on which the sum of the two angles KGH, GHC is less than two right angles.

PROPOSITION XXIV. THEOREM.

Straight lines which are parallel to the same line are parallel to each other.

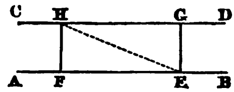


Let the straight lines AB, CD be each of them parallel to the line EF; then will AB be parallel to CD.

For, draw any straight line, as PQR, perpendicular to EF. Then, since AB is parallel to EF, PR, which is perpendicular to EF, will also be perpendicular to AB (Pr. 23, Cor. 1); and, since CD is parallel to EF, PR will also be perpendicular to CD. Hence AB and CD are both perpendicular to the same straight line, and are consequently parallel (Pr. 20). Therefore, straight lines which are parallel, etc.

PROPOSITION XXV. THEOREM.

Two parallel straight lines are every where equally distant from each other.

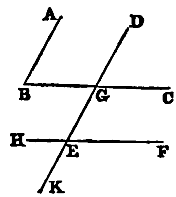


Let AB, CD be two parallel straight lines. From any points, E and F, in one of them, draw the lines EG, FH perpendicular to AB; they will also be perpendicular to CD (Pr. 23, Cor. 1). Join EH; then, because EG and FH are perpendicular to the same straight line AB, they are parallel (Pr. 20); therefore the alternate angles, EHF, HEG, which they make with HE, are equal (Pr. 23).

Again, because AB is parallel to CD, the alternate angles GHE, HEF are also equal. Therefore the triangles HEF, EHG have two angles of the one equal to two angles of the other, each to each, and the side EH, included between the equal angles, common; hence the triangles are equal (Pr. 7); and the line EG, which measures the distance of the parallels at the point E, is equal to the line FH, which measures the distance of the same parallels at the point F. Therefore, two parallel straight lines, etc.

PROPOSITION XXVI. THEOREM.

If two angles have their sides parallel each to each, the two angles will either be equal, or supplements of each other.



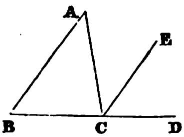
Let AB be parallel to DE, and BC to EF; then the angle ABC will be equal to the angle DEF, and the angle ABC will be the supplement of the angle DEH.

Produce DE, if necessary, until it meets BC in G. Then, because EF is parallel to GC, the angle DEF is equal to DGC (Pr. 23); and, because DG is parallel to AB, the angle DGC is equal to ABC; hence the angle DEF is equal to the angle ABC (Ax. 1). But the angle DEH is the supplement of DEF (Pr. 2). Hence ABC is the supplement of DEH. Therefore, if two angles, etc.

Scholium. Two angles are equal when their sides are not only parallel, but both lie in the same direction, as ABC, DEF; or both lie in opposite directions, as ABC, HEK. They are supplements of each other when their sides are parallel and two of their sides lie in the same direction, while the other two lie in opposite directions, as ABC, DEH.

PROPOSITION XXVII. THEOREM.

If one side of a triangle is produced, the exterior angle is equal to the sum of the two interior and remote angles; and the sum of the three interior angles of every triangle is equal to two right angles.



Let ABC be any plane triangle, and let the side BC be produced to D; then will the exterior angle ACD be equal to the sum of the two interior and remote angles A and B; and the sum of the three angles ABC, BCA, CAB is equal to two right angles.

For, conceive CE to be drawn parallel to the side AB of the triangle; then, because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (Pr. 23).

Again, because AB is parallel to CE, and BD meets them, the exterior angle ECD is equal to the interior and remote angle ABC. But the angle ACE was proved equal to BAC; therefore the whole exterior angle ACD is equal to the two interior and remote angles CAB, ABC (Ax. 2). To each of these equals add the angle ACB; then will the sum of the two angles ACD, ACB be equal to the sum of the three angles ABC, BCA, CAB. But the angles ACD, ACB are equal to two right angles (Pr. 2); hence, also, the angles ABC, BCA, CAB are together equal to two right angles. Therefore, if one side of a triangle, etc.

Cor. 1. If the sum of two angles of a triangle is given, the third may be found by subtracting this sum from two right angles.

Cor. 2. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal, and the triangles are mutually equiangular.

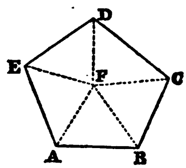
Cor. 3. A triangle can have but one right angle; for if there were two, the third angle would be nothing. Still less can a triangle have more than one obtuse angle.

Cor. 4. In a right-angled triangle, the sum of the two acute angles is equal to one right angle; that is, each of the acute angles is the complement of the other.

Cor. 5. In an equilateral triangle, each of the angles is one third of two right angles, or two thirds of one right angle.

PROPOSITION XXVIII. THEOREM.

All the interior angles of a polygon, together with four right angles, are equal to twice as many right angles as the figure has sides.



Let ABCDE be any polygon; then all its interior angles A, B, C, D, E, together with four right angles, are equal to twice as many right angles as the figure has sides.

For, from any point, F, within it, draw lines FA, FB, FC, etc., to all the angles. The polygon is thus divided into as many triangles as it has sides.

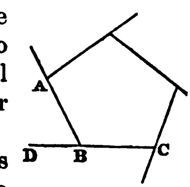
Now the sum of the three angles of each of these triangles is equal to two right angles (Pr. 27); therefore the sum of the angles of all the triangles is equal to twice as many right angles as the polygon has sides. But the same angles are equal to the angles of the polygon, together with the angles at the point F, that is, together with four right angles (Pr. 5, Cor. 2). Therefore the angles of the polygon, together with four right angles, are equal to twice as many right angles as the figure has sides.

Scholium. When this proposition is applied to *concave* polygons (Def. 37), each re-entering angle is to be regarded as greater than two right angles.

Cor. The sum of the angles of a quadrilateral is four right angles; of a pentagon, six right angles; of a hexagon, eight, etc.

PROPOSITION XXIX. THEOREM.

If all the sides of any polygon be produced so as to form an exterior angle at each vertex, the sum of these exterior angles will be equal to four right angles.

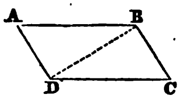


Let all the sides of the polygon ABC, etc., be produced in the same direction; that is, so as to form one exterior angle at each vertex; then will the sum of the exterior angles be equal to four right angles.

For each interior angle ABC, together with its adjacent exterior angle ABD, is equal to two right angles (Pr. 2); therefore the sum of all the interior and exterior angles is equal to twice as many right angles as there are sides of the polygon; that is, they are equal to all the interior angles of the polygon, together with four right angles. Hence the sum of the exterior angles must be equal to four right angles (Ax. 3). Therefore, if all the sides, etc.

PROPOSITION XXX. THEOREM.

The opposite sides and angles of a parallelogram are equal to each other.



Let ABCD be a parallelogram; then will its opposite sides and angles be equal to each other.

Draw the diagonal BD; then, because AB is parallel to CD, and BD meets them, the alternate angles ABD, BDC are equal to each other (Pr. 23).

Also, because AD is parallel to BC, and BD meets them, the alternate angles BDA, DBC are equal to each other. Hence the two triangles ABD, BDC have two angles, ABD, BDA of the one, equal to two angles, BDC, CBD of the other, each to each, and the side BD included between these equal angles common to the two triangles; therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other (Pr. 7), viz., the side AB to the side CD, and AD to BC, and the angle BAD equal to the angle BCD.

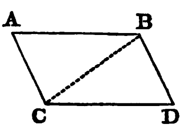
Also, because the angle ABD is equal to the angle BDC, and the angle CBD to the angle BDA, the whole angle ABC is equal to the whole angle ADC. But the angle BAD has been proved equal to the angle BCD; therefore the opposite sides and angles of a parallelogram are equal to each other.

Cor. 1. Two parallels, AB, CD, comprehended between two other parallels, AD, BC, are equal; and the diagonal BD divides the parallelogram into two equal triangles.

Cor. 2. If one angle of a parallelogram is a right angle, all its angles are right angles, and the figure is a rectangle.

PROPOSITION XXXI. THEOREM (*Converse of Prop. XXX.*)

If the opposite sides of a quadrilateral are equal, each to each, the equal sides are parallel, and the figure is a parallelogram.

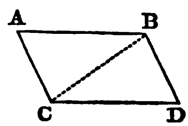


Let ABDC be a quadrilateral, having its opposite sides equal to each other, viz., the side AB equal to CD, and AC to BD; then will the equal sides be parallel, and the figure will be a parallelogram.

Draw the diagonal BC; then the triangles ABC, BCD have all the sides of the one equal to the corresponding sides of the other, each to each; therefore the angle ABC is equal to the angle BCD (Pr. 15), and, consequently, the side AB is parallel to CD (Pr. 22). For a like reason, AC is parallel to BD; hence the quadrilateral ABDC is a parallelogram. Therefore, if the opposite sides, etc.

PROPOSITION XXXII. THEOREM.

If two opposite sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel, and the figure is a parallelogram.



Let ABDC be a quadrilateral, having the sides AB, CD equal and parallel; then will the sides AC, BD be also equal and parallel, and the figure will be a parallelogram.

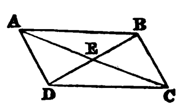
Draw the diagonal BC; then, because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (Pr. 23).

Also, because AB is equal to CD, and BC is common to the two triangles ABC, BCD, the two triangles ABC, BCD have two sides and the included angle of the one equal to two sides and the included angle of the other; therefore the side AC is equal to BD (Pr. 6), and the angle ACB to the angle CBD.

And, because the straight line BC meets the two straight lines AC, BD, making the alternate angles BCA, CBD equal to each other, AC is parallel to BD (Pr. 22); hence the figure ABDC is a parallelogram. Therefore, if two opposite sides, etc.

PROPOSITION XXXIII. THEOREM.

The diagonals of every parallelogram bisect each other.



Let ABCD be a parallelogram, whose diagonals AC, BD intersect each other in E; then will AE be equal to EC, and BE to ED.

Because the alternate angles ABE, EDC are equal (Pr. 23), and also the alternate angles EAB, ECD, the triangles ABE, CDE have two angles in the one equal to two angles in the other, each to each, and the included sides AB, CD are also equal; hence the remaining sides are equal, viz., AE to EC, and DE to EB. Therefore the diagonals of every parallelogram, etc.

Cor. If the side AB is equal to AD, the triangles AEB, AED have all the sides of the one equal to the corresponding sides of the other, and are consequently equal; hence the angle AEB will equal the angle AED, and therefore *the diagonals of a rhombus bisect each other at right angles.*

BOOK II.

RATIO AND PROPORTION.

On the Relation of Magnitudes to Numbers.

1. To *measure* a quantity is to find how many times it contains another quantity of the same kind called the *unit.*

To measure a line is to find how many times it contains another line called the *unit of length*, or the *linear unit.* Thus, when a line is said to be fifteen feet in length, it is to be understood that the line has been compared with the unit of length (one foot), and found to contain it fifteen times.

The number which expresses how many times a quantity contains the unit is called the *numerical measure* of that quantity.

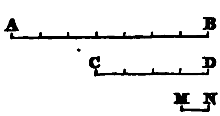
2. *Ratio* is that relation between two quantities which is expressed by the quotient of the first divided by the second. Thus the ratio of 12 to 4 is $\frac{12}{4}$. The ratio of A to B is $\frac{A}{B}$. The two quantities compared together are called the *terms* of the ratio; the first is called the *antecedent*, and the second the *consequent.*

3. To find the ratio of one quantity to another is to find how many times the first contains the second; *i. e.*, it is to measure the first by the second taken as the unit. If B be taken as the unit of measure, the quotient $\frac{A}{B}$ is the numerical value of A expressed in terms of this unit.

The ratio of two quantities is the same as the ratio of their numerical measures. Thus, if p denotes the unit, and if p is contained m times in A, and n times in B, then

$$\frac{A}{B}=\frac{mp}{np}=\frac{m}{n}.$$

4. Two quantities are said to be *commensurable* when there is a third quantity of the same kind which is contained an exact number of times in each. This third quantity is called the *common measure* of the two given quantities.



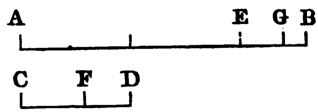
Thus the two lines AB, CD are commensurable if there is a third line, MN, which is contained an exact number of times in each; for example, 7 times in AB, and 4 times in CD.

The ratio of two commensurable quantities can therefore be exactly expressed by a number either whole or fractional. The ratio of AB to CD is $\frac{7}{4}$.

5. Two quantities are said to be *incommensurable* when they have no common measure. Thus the diagonal and side of a square are said to be incommensurable (see B. IV., Pr. 35); also the circumference and diameter of a circle (see B.VI., Pr. 11).

Whether A and B are commensurable or not, their ratio is expressed by $\frac{A}{B}$.

6. *To find the numerical ratio of two given straight lines.* Suppose AB and CD are two straight lines whose numerical ratio is required.



From the greater line, AB, cut off a part equal to the less, CD, as many times as possible; for example, twice with a remainder EB less than CD. From CD cut off a part equal to the remainder EB as often as possible; for example, once with a remainder FD. From the first remainder BE cut off a part equal to FD as often as possible; for example, once with a remainder GB. From the second remainder FD cut off a part equal to the third GB as many times as possible. Continue this process until a remainder is found which is contained an exact number of times in the preceding one. This last remainder will be the common measure of the proposed lines; and, regarding it as the measuring unit, we may easily find the values of the preceding remainders, and at length those of the proposed lines, whence we obtain their ratio in numbers.

For example, if we find GB is contained exactly twice in FD, GB will be the common measure of the two proposed lines: for we have

$$FD=2GB;$$
$$EB=FD+GB=2GB+GB=3GB;$$
$$CD=EB+FD=3GB+2GB=5GB;$$
$$AB=2CD+EB=10GB+3GB=13GB.$$

The ratio of the two lines AB, CD is therefore equal to that of 13GB to 5GB, or $\frac{13}{5}$.

7. It is possible that, however far this operation is continued, we may never find a remainder which is contained an exact number of times in the preceding one. In such a case, the two quan-

tities have no common measure; that is, they are *incommensurable*, and their ratio can not be exactly expressed by any number, whole or fractional.

8. But, although the ratio of incommensurable quantities can not be *exactly* expressed by a number, yet, by taking the measuring unit sufficiently small, a ratio may always be found which shall approach as near as we please to the true ratio.

Suppose $\frac{A}{B}$ denotes the ratio of two incommensurable quantities A and B, and let it be required to obtain a numerical expression of this ratio which shall be correct within an assigned measure of precision, say $\frac{1}{100}$. Let B be divided into 100 equal parts, and suppose A is found to contain 141 of these parts, with a remainder less than one of the parts; then we have

$$\frac{A}{B}=\frac{141}{100} \text{ within } \frac{1}{100};$$

that is, $\frac{141}{100}$ is an approximate value of the ratio $\frac{A}{B}$, within the assumed measure of precision. In the same manner, by dividing B into a greater number of equal parts, the error of our approximate value may be made as small as we please.

9. To generalize this reasoning, let B be divided into n equal parts, and let A contain m of these parts with a remainder less than one of the parts; then we have

$$\frac{A}{B}=\frac{m}{n} \text{ within } \frac{1}{n};$$

and since n may be taken as great as we please, $\frac{1}{n}$ may be made less than any assigned measure of precision, and $\frac{m}{n}$ will be the approximate value of the ratio $\frac{A}{B}$, within the assigned limit.

10. *The ratio of any two magnitudes* A *and* B *is equal to the ratio of two other magnitudes* A′ *and* B′, *when the same number expresses the value of either ratio to the same degree of approximation, however far the approximation may be carried.*

Let $\frac{m}{n}$ represent the approximate value of either ratio, and let B be divided into n equal parts; then A will contain m of these parts plus a remainder which is less than one of the parts; that is, the numerical expression of the ratio $\frac{A}{B}$ will be $\frac{m}{n}$ correct within $\frac{1}{n}$ part. Hence the ratio $\frac{A}{B}$ can not differ from the ratio $\frac{A'}{B'}$ by so much as $\frac{1}{n}$. But the measure $\frac{1}{n}$ may be assumed as small as we please; that is, less than any assignable quantity, however small. Hence $\frac{A}{B}$ can not differ from $\frac{A'}{B'}$ by any assignable quantity, however small; that is, the two ratios are equal to each other.

For an application of this principle, see B. III., Pr. 14.

11. *A proportion is an equality of ratios.* Thus, if the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, the equality

$$\frac{A}{B}=\frac{C}{D}$$

constitutes a proportion. It may be read, the ratio of A to B equals the ratio of C to D; or A is to B as C to D.

A proportion is often written

$$A:B=C:D,$$

or,

$$A:B::C:D,$$

where the notation $A:B$ is equivalent to $A\div B$. The first and last terms of a proportion are called the two *extremes;* the second and third terms are called the two *means.* Of four proportional quantities, the last is called a *fourth proportional* to the other three taken in order.

Since

$$\frac{A}{B}=\frac{C}{D},$$

it is obvious that if A is greater than B, C must be greater than D; that is, if one antecedent is greater than its consequent, the other antecedent must be greater than its consequent; if equal, equal; and if less, less.

12. Three quantities are said to be proportional when the ratio of the first to the second is equal to the ratio of the second to the third; thus, if A, B, and C are in proportion, then

$$A:B::B:C,$$

or,

$$A:B=B:C.$$

In this case the middle term is said to be a *mean proportional* between the other two, and C is called a third proportional to A and B.

13. *Equimultiples* of two magnitudes are the products arising

from multiplying those magnitudes by the same number. Thus 7A, 7B are equimultiples of A and B; so also are mA and mB.

Geometers make use of the following technical terms to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals.

14. *Alternation* is when antecedent is compared with antecedent, and consequent with consequent.

Thus, if $A:B::C:D$,
then, by alternation, $A:C::B:D$.

15. *Inversion* is when the antecedents are made the consequents, and the consequents the antecedents.

Thus, if $A:B::C:D$,
then, inversely, $B:A::D:C$.

16. *Composition* is when the sum of antecedent and consequent is compared either with the antecedent or consequent.

Thus, if $A:B::C:D$,
then, by composition, $A+B:A::C+D:C$,
and $A+B:B::C+D:D$.

17. *Division* is when the difference of antecedent and consequent is compared either with the antecedent or consequent.

Thus, if $A:B::C:D$,
then, by division, $A-B:A::C-D:C$,
and $A-B:B::C-D:D$.

18. When a proportion is said to exist among certain quantities, these quantities are supposed to be represented, or to be capable of being represented by numbers (Art. 3).

If, for example, in the proportion

$$A:B::C:D,$$

A, B, C, D denote lines, we may suppose one of these lines, or a fifth line, if we please, to be taken as a common measure to the whole, and to be regarded as unity; then A, B, C, D will each represent a certain number of units, entire or fractional, commensurable or incommensurable, and the proportion among the lines A, B, C, D becomes a proportion in numbers. Hence the product of two lines A and D may be regarded as the number of linear units contained in A multiplied by the number of linear units contained in D.

In the proportion $A:B::C:D$,
the quantities A and B may be of one kind, as lines, and the quantities C and D of another kind, as surfaces; still, these quantities are to be regarded as represented by numbers. A and B will be

expressed in linear units, C and D in superficial units, and the product of A and D will be a number, as also the product of B and C.

Axioms.

1. Equimultiples of the same or of equal magnitudes are equal to one another.

2. Those magnitudes of which the same or equal magnitudes are equimultiples are equal to one another.

PROPOSITION I. THEOREM.

If four quantities are proportional, the product of the two extremes is equal to the product of the two means.

Let A, B, C, D be the numerical representatives of four proportional quantities, so that $A:B::C:D$; then will $A \times D = B \times C$.

For, since the four quantities are proportional,

$$\frac{A}{B} = \frac{C}{D}.$$

Multiplying each of these equal quantities by B (Ax. 1), we obtain

$$A = \frac{B \times C}{D}.$$

Multiplying each of these last equals by D, we have

$$A \times D = B \times C.$$

Cor. If there are three proportional quantities, the product of the two extremes is equal to the square of the mean.

Thus, if $A:B::B:C$,

then, by this proposition, $A \times C = B \times B$, which is equal to B^2.

PROPOSITION II. THEOREM. (*Converse of Prop. I.*)

If the product of two quantities is equal to the product of two others, the one pair may be made the extremes, and the other the means of a proportion.

Thus, suppose we have given

$$A \times D = B \times C;$$

then will $A:B::C:D$.

For, since $A \times D = B \times C$, dividing each of these equals by D (Ax. 2), we have

$$A = \frac{B \times C}{D}.$$

Dividing each of these last equals by B, we obtain

$$\frac{A}{B} = \frac{C}{D}, \text{ or, } A:B::C:D.$$

PROPOSITION III. THEOREM.

If four quantities are proportional, they are also proportional when taken alternately.

Let A, B, C, D be the numerical representatives of four proportional quantities, so that

$$A : B :: C : D;$$

then, alternately, $A : C :: B : D.$

For, since $A : B :: C : D,$

by Pr. 1, $A \times D = B \times C.$

And, since $A \times D = B \times C,$

by Pr. 2, $A : C :: B : D.$

PROPOSITION IV. THEOREM.

Ratios that are equal to the same ratio are equal to each other.

Let $A : B :: C : D,$

and $A : B :: E : F;$

then will $C : D :: E : F.$

For, since $A : B :: C : D.$

we have $\frac{A}{B} = \frac{C}{D}.$

And, since $A : B :: E : F,$

we have $\frac{A}{B} = \frac{E}{F}.$

But $\frac{C}{D}$ and $\frac{E}{F}$, being severally equal to $\frac{A}{B}$, must be equal to each other, and therefore

$$C : D :: E : F.$$

Cor. If the antecedents of one proportion are equal to the antecedents of another proportion, the consequents are proportional.

If $A : B :: C : D,$

and $A : E :: C : F,$

then will $B : D :: E : F.$

For, by alternation (Pr. 3), the first proportion becomes

$$A : C :: B : D,$$

and the second, $A : C :: E : F.$

Therefore, by this proposition,

$$B : D :: E : F.$$

PROPOSITION V. THEOREM.

If four quantities are proportional, they are also proportional when taken inversely.

Let $A:B::C:D$;
then, inversely, $B:A::D:C$.
For, since $A:B::C:D$,
by Pr. 1, $A\times D=B\times C$,
or, $B\times C=A\times D$;
therefore, by Pr. 2, $B:A::D:C$.

PROPOSITION VI. THEOREM.

If four quantities are proportional, they are also proportional by composition.

Let $A:B::C:D$;
then, by composition, $A+B:A::C+D:C$.
For, since $A:B::C:D$,
by Pr. 1, $B\times C=A\times D$.
To each of these equals add

$$A\times C=A\times C;$$

then $A\times C+B\times C=A\times C+A\times D$,
or, $(A+B)\times C=A\times(C+D)$.
Therefore, by Pr. 2, $A+B:A::C+D:C$.

PROPOSITION VII. THEOREM.

If four quantities are proportional, they are also proportional by division.

Let $A:B::C:D$;
then, by division, $A-B:A::C-D:C$.
For, since $A:B::C:D$,
by Pr. 1, $B\times C=A\times D$.
Subtract each of these equals from $A\times C$;
then, $A\times C-B\times C=A\times C-A\times D$,
or, $(A-B)\times C=A\times(C-D)$.
Therefore, by Pr. 2, $A-B:A::C-D:C$.

PROPOSITION VIII. THEOREM.

If four quantities are proportional, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

Let $A:B::C:D$;

then will $A+B:A-B::C+D:C-D.$

By Pr. 6, $A+B:A::C+D:C;$

and by Pr. 7, $A-B:A::C-D:C.$

By alternation (Pr. 3), these proportions become

$$A+B:C+D::A:C;$$

and $A-B:C-D::A:C.$

Hence, Pr. 4, $A+B:C+D::A-B:C-D;$

or, alternately, $A+B:A-B::C+D:C-D.$

PROPOSITION IX. THEOREM.

If any number of quantities are proportional, any one antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $A:B::C:D::E:F$, etc.;

then will $A:B::A+C+E:B+D+F.$

For, since $A:B::C:D,$

we have $A\times D=B\times C.$

And, since $A:B::E:F,$

we have $A\times F=B\times E.$

To these equals add $A\times B=A\times B,$

and we have

$$A\times B+A\times D+A\times F=A\times B+B\times C+B\times E;$$

or, $A\times(B+D+F)=B\times(A+C+E).$

But $B+D+F$ may be regarded as a single quantity, and $A+C+E$ as a single quantity.

Therefore, by Pr. 2,

$$A:B::A+C+E:B+D+F.$$

PROPOSITION X. THEOREM.

Equimultiples of two quantities have the same ratio as the quantities themselves.

Let A and B be any two quantities of the same kind, and m any number, entire or fractional, we have the equality

$$\frac{A}{B}=\frac{mA}{mB},$$

or, $A:B::mA:mB.$

Cor. If $A:B::C:D,$

then $mA:nB::mC:nD;$

and if $mA:nB::mC:nD,$

then $A:B::C:D;$ that is,

If four magnitudes are proportional, we may multiply the ante-

cedents or the consequents, or divide them by the same quantity, and the results will be proportional.

PROPOSITION XI. THEOREM.

If four quantities are proportional, their squares or cubes are also proportional.

Let $A : B :: C : D$;
then will $A^2 : B^2 :: C^2 : D^2$,
and $A^3 : B^3 :: C^3 : D^3$.

For, since $A : B :: C : D$,
by Pr. 1, $A \times D = B \times C$;
or, multiplying each of these equals by itself (Ax. 1), we have

$$A^2 \times D^2 = B^2 \times C^2;$$

and multiplying these last equals by $A \times D = B \times C$, we have

$$A^3 \times D^3 = B^3 \times C^3.$$

Therefore, by Pr. 2, $A^2 : B^2 :: C^2 : D^2$,
and $A^3 : B^3 :: C^3 : D^3$.

PROPOSITION XII. THEOREM.

If there are two sets of proportional quantities, the products of the corresponding terms are proportional.

Let $A : B :: C : D$,
and $E : F :: G : H$;
then will $A \times E : B \times F :: C \times G : D \times H$.

For, since $A : B :: C : D$,
by Pr. 1, $A \times D = B \times C$.

And, since $E : F :: G : H$,
by Pr. 1, $E \times H = F \times G$.

Multiplying together these equal quantities, we have

$$A \times D \times E \times H = B \times C \times F \times G;$$

or, $(A \times E) \times (D \times H) = (B \times F) \times (C \times G)$;
therefore, by Pr. 2,

$$A \times E : B \times F :: C \times G : D \times H.$$

Cor. If $A : B :: C : D$,
and $B : F :: G : H$,
then $A : F :: C \times G : D \times H$.

For, by the proposition,

$$A \times B : B \times F :: C \times G : D \times H.$$

Also, by Pr. 10, $A \times B : B \times F :: A : F$;
hence, by Pr. 4, $A : F :: C \times G : D \times H$.

C

PROPOSITION XIII. THEOREM.

If three quantities are proportional, the first is to the third as the square of the first to the square of the second.

Thus, if	$A : B :: B : C,$
then	$A : C :: A^2 : B^2.$
For, since,	$A : B :: B : C,$
and	$A : B :: A : B;$
therefore, by Pr. 12,	$A^2 : B^2 :: A \times B : B \times C.$
But, by Pr. 10,	$A \times B : B \times C :: A : C;$
hence, by Pr. 4,	$A : C :: A^2 : B^2.$

BOOK III.

THE CIRCLE, AND THE MEASURE OF ANGLES.

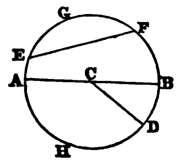
Definitions.

1. A *circle* is a plane figure bounded by a line, all the points of which are equally distant from a point within, called the *centre.*

The line which bounds the circle is called its *circumference.*

2. Any straight line drawn from the centre of the circle to the circumference is called a *radius* of the circle, as CA, CD.

Any straight line drawn through the centre, and terminated each way by the circumference, is called a *diameter*, as AB.



Cor. All the radii of a circle are equal; also all the diameters are equal, and each is double the radius.

3. An *arc* of a circle is any portion of its circumference, as EGF.

The *chord* of an arc is the straight line which joins its two extremities, as EF.

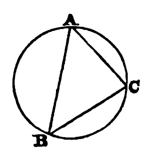
The arc EGF is said to be *subtended* by its chord EF.

4. A *segment* of a circle is the figure included between an arc and its chord, as EGF.

Since the same chord EF subtends two arcs EGF, EHF, to the same chord there correspond two segments EGF, EHF. By the term segment, the smaller of the two is always to be understood, unless the contrary is expressed.

5. A *sector* of a circle is the figure included between an arc and the two radii drawn to the extremities of the arc, as BCD.

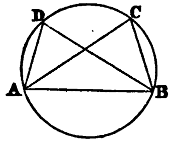
6. A straight line is said to be *inscribed* in a circle when its extremities are on the circumference, as AB.



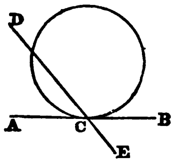
An *inscribed angle* is one whose vertex is on the circumference, and which is formed by two chords, as BAC.

7. A polygon is said to be *inscribed* in a circle when all its angles have their vertices on the circumference, as ABC. The circle is then said to be *described* about the polygon.

8. An angle is said to be *inscribed in a segment* when it is con-

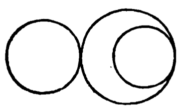


tained by two straight lines drawn from any point in the arc of the segment to the extremities of the subtending chord. Thus the angles ACB, ADB are inscribed in the segment ADCB.

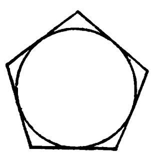


9. A *secant* is a line which cuts the circumference, and lies partly within and partly without the circle, as DE.

10. A straight line is said to *touch* a circle when it meets the circumference, and, being produced, does not cut it, as AB. Such a line is called a *tangent*, and the point in which it meets the circumference is called the *point of contact*, as C.



11. Two circumferences are said to touch one another when they meet, but do not cut one another.

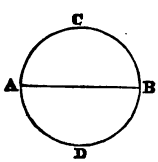


12. A polygon is said to be *described* about a circle when each side of the polygon touches the circumference of the circle.

In this case the circle is said to be inscribed in the polygon.

PROPOSITION I. THEOREM.

Every diameter divides the circle and its circumference into two equal parts.



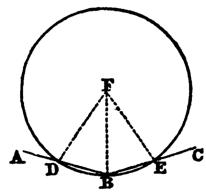
Let ACBD be a circle, and AB its diameter; then will the line AB divide the circle and its circumference into two equal parts.

If the figure ADB be turned about AB, and superposed upon the figure ACB, the curve line ACB must coincide exactly with the curve line ADB. For if any part of the curve ACB were to fall either within or without the curve ADB, there would be points in one or the other unequally distant from the centre, which is contrary to the definition of a circle. Hence the two figures coincide throughout, and are equal in all respects. Therefore every diameter, etc.

PROPOSITION II. THEOREM.

A straight line can not meet the circumference of a circle in more than two points.

For, if it be possible, let the straight line ABC meet the circumference of a circle in three points, DBE. Take F, the centre of the circle, and join FD, FB, FE.

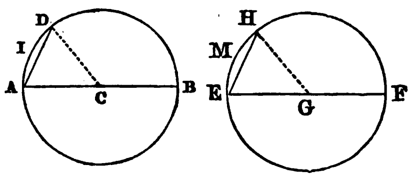


Then, because F is the centre of the circle, the three straight lines FD, FB, FE are all equal to each other. Hence three equal straight lines have been drawn from the same point to the same straight line, which is impossible (B. I., Pr. 17, Cor. 2*). Therefore a straight line, etc.

PROPOSITION III. THEOREM.

In the same circle or in equal circles, equal arcs are subtended by equal chords, and conversely equal chords subtend equal arcs.

Let ADB, EHF be equal circles, and let the arcs AID, EMH also be equal; then will the chord AD be equal to the chord EH.

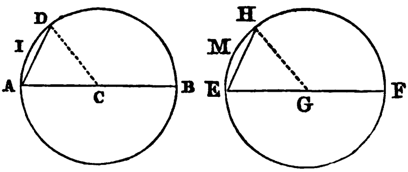


For, the diameter AB being equal to the diameter EF, the semicircle ADB may be applied exactly to the semicircle EHF, and the curve line AIDB will coincide entirely with the curve line EMHF (Pr. 1). But the arc AID is, by hypothesis, equal to the arc EMH; hence the point D will fall on the point H, and therefore the chord AD is equal to the chord EH (Ax. 11, B. I.).

Conversely, if the chord AD is equal to the chord EH, then the arc AID will be equal to the arc EMH.

For, if the radii CD, GH are drawn, the two triangles ACD, EGH will have their three sides equal, each to each, viz., AC to EG, CD to GH, and AD equal to EH; the triangles are consequently equal (B. I., Pr. 15), and the angle ACD is equal to the angle EGH.

* In the references, the Roman numerals denote the Book, and the Arabic numerals indicate the Proposition. Thus, B. I., Pr. 17, means the seventeenth proposition of the first book.

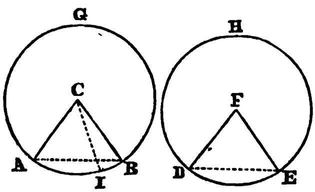


Let, now, the semicircle ADB be applied to the semicircle EHF, so that AC may coincide with EG; then, since the angle ACD is equal to the angle EGH, the radius CD will coincide with the radius GH, and the point D with the point H. Therefore the arc AID must coincide with the arc EMH, and be equal to it.

If the arcs are in the same circle, the demonstration is similar. Therefore, in the same circle, etc.

PROPOSITION IV. THEOREM.

In the same circle or in equal circles, equal angles at the centre are subtended by equal arcs; and, conversely, equal arcs subtend equal angles at the centre.



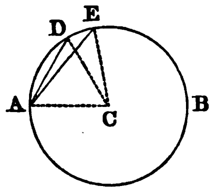
Let AGB, DHE be two equal circles, and let ACB, DFE be equal angles at their centres; then will the arc AB be equal to the arc DE.

Join AB, DE; and, because the circles AGB, DHE are equal, their radii are equal. Therefore the two sides CA, CB are equal to the two sides FD, FE; also, the angle at C is equal to the angle at F; therefore the base AB is equal to the base DE (B. I., Pr. 6). And, because the chord AB is equal to the chord DE, the arc AB must be equal to the arc DE (Pr. 3).

Conversely, if the arc AB is equal to the arc DE, the angle ACB will be equal to the angle DFE. For, if these angles are not equal, one of them is the greater. Let ACB be the greater, and take ACI equal to DFE; then, because equal angles at the centre are subtended by equal arcs, the arc AI is equal to the arc DE. But the arc AB is equal to the arc DE; therefore the arc AI is equal to the arc AB, the less to the greater, which is impossible. Hence the angle ACB is not unequal to the angle DFE, that is, it is equal to it. Therefore, in the same circle, etc.

PROPOSITION V. THEOREM.

In the same circle, or in equal circles, a greater arc is subtended by a greater chord; and, conversely, the greater chord subtends the greater arc, the arcs being both less than a semi-circumference.



In the circle AEB, let the arc AE be greater than the arc AD; then will the chord AE be greater than the chord AD.

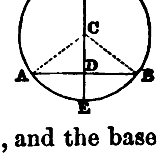
Draw the radii CA, CD, CE. Now, if the arc AE were equal to the arc AD, the angle ACE would be equal to the angle ACD (Pr. 4); hence it is clear that if the arc AE be greater than the arc AD, the angle ACE must be greater than the angle ACD. But the two sides AC, CE of the triangle ACE are equal to the two AC, CD of the triangle ACD, and the angle ACE is greater than the angle ACD; therefore the third side A E is greater than the third side AD (B. I., Pr. 13); hence the chord which subtends the greater arc is the greater.

Conversely, if the chord AE is greater than the chord AD, the arc AE is greater than the arc AD. For, because the two triangles ACE, ACD have two sides of the one equal to two sides of the other, each to each, but the base AE of the one is greater than the base AD of the other, therefore the angle ACE is greater than the angle ACD (B. I., Pr. 14), and hence the arc AE is greater than the arc AD (Pr. 4). Therefore, in the same circle, etc.

Scholium. If the arcs are greater than a semi-circumference, the contrary is true; that is, the greater arc is subtended by a smaller chord.

PROPOSITION VI. THEOREM.

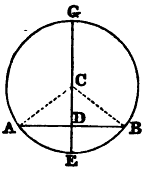
The diameter which is perpendicular to a chord bisects the chord, and also the arc which it subtends.



Let ABG be a circle, of which AB is a chord, and GE a diameter perpendicular to it; the chord AB will be bisected in D, and the arc AEB will be bisected in E.

Draw the radii CA, CB. The two right-angled triangles CDA, CDB have the side AC equal to CB, and CD common; therefore the triangles are equal, and the base AD is equal to the base DB (B. I., Pr. 19).

Secondly. Since the radius AC is equal to CB, and the line CD



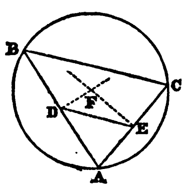
bisects the line AB at right angles, it bisects also the vertical angle ACB (B. I., Pr. 10, Cor. 1). And, since the angle ACE is equal to the angle BCE, the arc AE must be equal to the arc BE (B. III., Pr. 4). Hence the diameter GE, perpendicular to the chord AB, divides the arc subtended by this chord into two equal parts in the point E. Moreover, since the semi-circumference GAE is equal to GBE (B. III., Pr. 1), the arc AG must be equal to BG. Therefore the perpendicular, etc.

Corollary. The centre of the circle, the middle point of the chord AB, and the middle point of the arc AEB subtended by this chord, are three points situated in a straight line perpendicular to the chord. Now two points are sufficient to determine the position of a straight line; therefore any straight line which passes through two of these points will necessarily pass through the third, and be perpendicular to the chord.

Also, *the perpendicular to the chord at its middle point passes through the centre of the circle and through the middle of the arc subtended by the chord.*

PROPOSITION VII. THEOREM.

Through any three points not in the same straight line one circumference may be made to pass, and but one.



Let A, B, C be any three points not in the same straight line; they all lie in the circumference of the same circle. Join AB, AC, and bisect these lines by the perpendiculars DF, EF; DF and EF produced will meet one another.

For, join DE; then, because the angles ADF, AEF are together equal to two right angles, the angles FDE and FED are together less than two right angles; therefore DF and EF will meet if produced (B. I., Pr. 23, Cor. 3). Let them meet in F. Since this point lies in the perpendicular DF, it is equally distant from the two points A and B (B. I., Pr. 18); and, since it lies in the perpendicular EF, it is equally distant from the two points A and C; therefore the three distances FA, FB, FC are all equal; hence the circumference described from the centre F with the radius FA will pass through the three given points A, B, C.

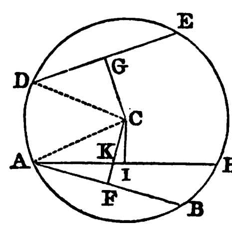
Secondly. No other circumference can pass through the same points. For, if there were a second, its centre could not be out of the line DF, for then it would be unequally distant from A and B (B. I., Pr. 18); neither could it be out of the line FE, for the same reason; therefore it must be on both the lines DF, FE. But two straight lines can not cut each other in more than one point, hence only one circumference can pass through three given points. Therefore, through any three points, etc.

Cor. 1. Two circumferences can not cut each other in more than two points; for, if they had three common points, they would have the same centre, and would coincide with each other.

Cor. 2. The perpendicular drawn from the middle of BC will pass through the point F, since this point is equally distant from B and C; therefore the three straight lines bisecting the three sides of a triangle at right angles meet in the same point.

PROPOSITION VIII. THEOREM.

In the same circle or in equal circles, equal chords are equally distant from the centre; and of two unequal chords, the less is the more remote from the centre.



Let the chords AB, DE, in the circle AB ED, be equal to one another; they are equally distant from the centre. Take C, the centre of the circle, and from it draw CF, CG, perpendiculars to AB, DE.

Join CA, CD; then, because the line CF is perpendicular to the chord AB, it bisects it (Pr. 6). Hence AF is the half of A B; and, for the same reason, DG is the half of DE. But AB is equal to DE, therefore AF is equal to DG (B. I., Ax. 7). Now, in the right-angled triangles ACF, DCG, the hypothenuse AC is equal to the hypothenuse DC, and the side AF is equal to the side DG; therefore the triangles are equal, and CF is equal to C G (B. I., Pr. 19); hence the two equal chords AB, DE are equally distant from the centre.

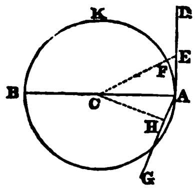
Secondly. Let the chord AH be greater than the chord DE; DE is further from the centre than AH.

For, because the chord AH is greater than the chord DE, the arc ABH is greater than the arc DE (Pr. 5). From the arc AB H cut off a part, AB, equal to DE; draw the chord AB, and let fall CF perpendicular to this chord, and CI perpendicular to AH. It is plain that CF is greater than CK, and CK than CI (B. I., Pr. 17); much more, then, is CF greater than CI. But CF is equal to CG, because the chords AB, DE are equal; hence CG is greater than CI. Therefore, in the same circle, etc.

Cor. Hence the diameter is the longest line that can be inscribed in a circle.

PROPOSITION IX. THEOREM.

A straight line perpendicular to a diameter at its extremity is a tangent to the circumference.



Let ABK be a circle, the centre of which is C, and the diameter AB, and let AD be drawn from A perpendicular to AB; AD will be a tangent to the circumference.

In AD take any point, E, and join CE; then, since CE is an oblique line, it is longer than the perpendicular CA (B. I., Pr. 17).

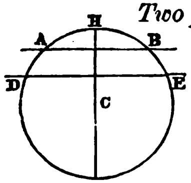
Now CA is equal to CF; therefore CE is greater than CF, and the point E must be without the circle. But E is any point whatever in the line AD; therefore AD has only the point A in common with the circumference, hence it is a tangent (Def. 10). Therefore a straight line, etc.

Cor. 1. Through the same point, A, in the circumference, only one tangent can be drawn. For, if possible, let a second tangent, AG, be drawn; then, since CA can not be perpendicular to A G (B. I., Pr. 1), another line, CH, must be perpendicular to AG, and therefore CH must be less than CA (B. I., Pr. 17); hence the point H falls within the circle, and AH produced will cut the circumference.

Cor. 2. A tangent, AD, to a circle at any point, A, is perpendicular to the diameter drawn to that point. For, since every point of the tangent except A is without the circumference, the radius CA is the shortest line that can be drawn from the point C to the line AD, and is therefore perpendicular to this line (B. I., Pr. 17).

PROPOSITION X. THEOREM.

Two parallels intercept equal arcs on a circumference.



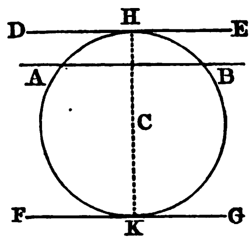
The proposition admits of three cases:

First. When the two parallels are secants, as AB, DE.

Draw the radius CH perpendicular to AB; it will also be perpendicular to DE (B. I., Pr.

23, Cor. 1); therefore the point H will be at the same time the middle of the arc AHB and of the arc DHE (Pr. 6). Hence the arc DH is equal to the arc HE, and the arc AH equal to HB, and therefore the arc AD is equal to the arc BE (B. I., Ax. 3).

Second. When one of the two parallels is a secant and the other a tangent.



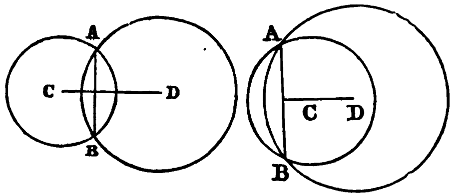
To the point of contact, H, draw the radius CH; it will be perpendicular to the tangent DE (Pr. 9), and also to its parallel AB. But, since CH is perpendicular to the chord AB, the point H is the middle of the arc AHB (Pr. 6); therefore the arcs A H, HB, included between the parallels AB, DE, are equal.

Third. If the two parallels DE, FG are tangents, the one at H, the other at K, draw the parallel secant AB; then, according to the former case, the arc AH is equal to HB, and the arc AK is equal to KB; hence the whole arc HAK is equal to the whole arc HBK (B. I., Ax. 2). It is also evident that each of these arcs is a semi-circumference. Therefore two parallels, etc.

Scholium. The straight line joining the points of contact of two parallel tangents is a diameter.

PROPOSITION XI. · THEOREM.

If two circumferences cut each other, the straight line joining their centres bisects their common chord at right angles.

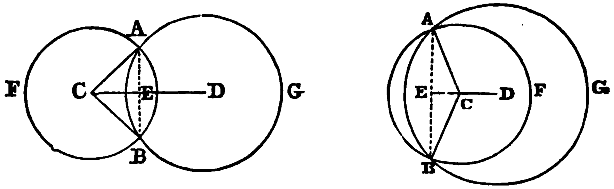


Let two circumferences cut each other in the points A and B; then will the line AB be a common chord to the two circles. Now, if a perpendicular be erected from the middle of this chord, it will pass through C and D, the centres of the two circles (Pr. 6, Cor.). But only one straight line can be drawn through two given points; therefore the straight line which passes through the centres will bisect the common chord at right angles.

PROPOSITION XII. THEOREM.

If two circumferences touch each other, either externally or internally, the distance of their centres must be equal to the sum or difference of their radii.

It is plain that the centres of the circles and the point of contact are in the same straight line; for, if possible, let the point of contact, A, be without the straight line CD.



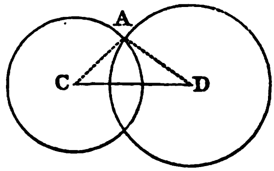
From A let fall upon CD, or CD produced, the perpendicular AE, and produce it to B, making BE equal to AE. Then, in the triangles ACE, BCE, the side AE is equal to EB, CE is common, and the angle AEC is equal to the angle BEC; therefore AC is equal to CB (B. I., Pr. 6), and the point B is in the circumference ABF. In the same manner, it may be shown to be in the circumference ABG, and hence the point B is in both circumferences. Therefore the two circumferences have two points, A and B, in common; that is, they cut each other, which is contrary to the hypothesis. Therefore the point of contact can not be without the line joining the centres; and hence, when the circles touch each other externally, the distance of the centres CD is equal to the sum of the radii CA, DA; and when they touch internally, the distance CD is equal to the difference of the radii CA, DA. Therefore, if two circumferences, etc.

Scholium. If two circumferences touch each other externally or internally, their point of contact is in the straight line joining their centres.

PROPOSITION XIII. THEOREM.

If two circumferences cut each other, the distance between their centres is less than the sum of their radii, and greater than their difference.

Let two circumferences cut each other in the point A. Draw the radii CA, DA; then, because any side of a triangle is less than the sum of the other two (B. I., Pr. 8), CD must be less than the sum of AD and AC. Also, DA must be less than the sum of CD and CA; or, subtracting CA from these unequals (B. I, Ax. 5), CD must be greater than the difference between DA and CA. Therefore, if two circumferences, etc.



Scholium. There may be five different positions of two circles with respect to each other:

1st. When the distance between their centres is greater than the sum of their radii, there can be neither contact nor intersection.

2d. When the distance between their centres is equal to the sum of their radii, the circumferences touch each other externally.

3d. When the distance between their centres is less than the sum of their radii, but greater than their difference, the circumferences intersect.

4th. When the distance between their centres is equal to the difference of their radii, the circumferences touch each other internally.

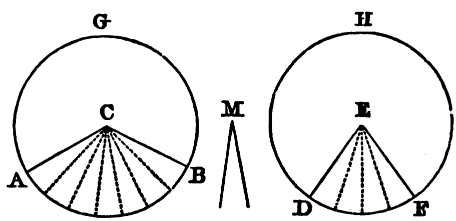
5th. When the distance between their centres is less than the difference of their radii, there can be neither contact nor intersection.

PROPOSITION XIV. THEOREM.

In the same circle, or in equal circles, two angles at the centre have the same ratio as their intercepted arcs.

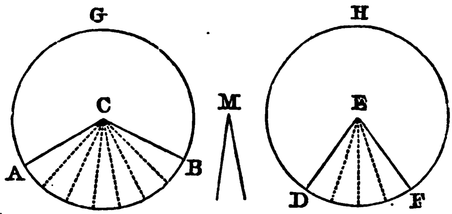
Case first. When the angles are in the ratio of two whole numbers.

Let ABG, DFH be equal circles, and let the angles ACB, DEF at their centres be in the ratio of two whole numbers; then will



the angle ACB : angle DEF :: arc AB : arc DF.

Suppose, for example, that the angles ACB, DEF are to each other as 7 to 4; or, which is the same thing, suppose that the angle M, which may serve as a common measure, is contained seven times in the angle ACB, and four times in the angle DEF. Draw



radii to the several points of division of the arcs. The seven partial angles into which ACB is divided, being each equal to any of the four partial angles into which DEF is divided, the partial arcs will also be equal to each other (Pr. 4), and the entire arc AB will be to the entire arc DF as 7 to 4. Now the same reasoning would apply if, in place of 7 and 4, any whole numbers whatever were employed; therefore, if the ratio of the angles ACB, DEF can be expressed in whole numbers, the arcs AB, DF will be to each other as the angles ACB, DEF.

Case second. When the angles are incommensurable; that is, their ratio can not be expressed exactly in numbers.

Suppose the angle DEF to be divided into any number n of equal parts; then ACB will contain a certain number m of these parts, plus a remainder which is less than one of the parts. The numerical expression of the ratio $\frac{\text{ACB}}{\text{DEF}}$ will be $\frac{m}{n}$, correct within $\frac{1}{n}$ part (B.II., Art. 10). Draw radii to the several points of division of the arcs. The arc DF will be divided into n equal parts, and the arc AB will contain m such parts, plus a remainder which is less than one of the parts. Therefore the numerical expression of the ratio $\frac{\text{AB}}{\text{DF}}$ will also be $\frac{m}{n}$, correct within $\frac{1}{n}$ part. Hence the same number, $\frac{m}{n}$, expresses the value of the ratio $\frac{\text{ACB}}{\text{DEF}}$, and of $\frac{\text{AB}}{\text{DF}}$, however small the parts into which DEF is divided. Therefore these ratios must be absolutely equal; and hence, whatever may be the ratio of the two angles, we have the proportion

$$\text{angle ACB} : \text{angle DEF} :: \text{arc AB} : \text{arc DF}.$$

Therefore, in the same circle, etc.

Scholium. Since the angle at the centre of a circle and the arc intercepted by its sides are so related that when one is increased or diminished, the other is increased or diminished in the same ratio, an angle at the centre is said to be measured by its intercepted arc.

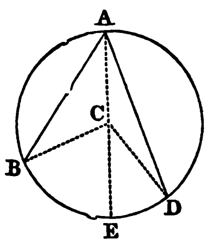
It should, however, be observed that, since angles and arcs are unlike quantities, they are necessarily measured by different units. The most simple unit of measure for angles is the right angle, and the corresponding unit of measure for arcs is a quadrant. An acute angle would accordingly be expressed by some number between 0 and 1; an obtuse angle by some number between 1 and 2.

The unit, however, most commonly employed for angles is an angle equal to $\frac{1}{90}$th part of a right angle, called a *degree.* The corresponding unit of measure for arcs is $\frac{1}{90}$th part of a quadrant, and is also called a degree. An angle or an arc is thus numerically expressed by the unit degree and its subdivisions. A right angle and a quadrant are both expressed by 90 degrees. If an angle is $\frac{4}{5}$ths of a right angle, it is expressed by 72 degrees.

Cor. Since in equal circles sectors are equal when their angles are equal, it follows that *in equal circles sectors are to each other as their arcs.*

PROPOSITION XV. THEOREM.

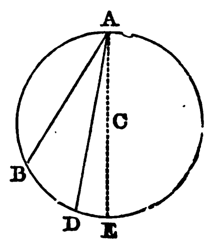
An inscribed angle is measured by half the arc included between its sides.



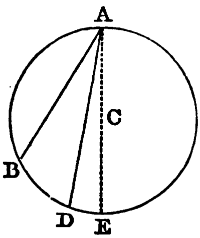
Let BAD be an angle inscribed in the circle BAD. The angle BAD is measured by half the arc BD.

First. Let C, the centre of the circle, be within the angle BAD. Draw the diameter AE, also the radii CB, CD.

Because CA is equal to CB, the angle CAB is equal to the angle CBA (B. I., Pr. 10); therefore the angles CAB, CBA are together double the angle CAB. But the angle BCE is equal (B. I., Pr. 27) to the angles CAB, CBA; therefore, also, the angle BCE is double of the angle BAC. Now the angle BCE, being an angle at the centre, is measured by the arc BE; hence the angle BAE is measured by the half of BE. For the same reason, the angle DAE is measured by half the arc DE. Therefore the whole angle BAD is measured by half the arc BD.



Second. Let C, the centre of the circle, be without the angle BAD. Draw the diameter AE.

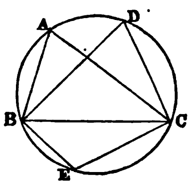


It may be demonstrated, as in the first case, that the angle BAE is measured by half the arc BE, and the angle DAE by half the arc DE; hence their difference, BAD, is measured by half of BD. Therefore, an inscribed angle, etc.

Cor. 1. All the angles BAC, BDC, etc., inscribed in the same segment, are equal, for they are all measured by half the same arc B EC. (See next fig.)

Cor. 2. An angle BCD at the centre of a circle is double of the angle BAD at the circumference, subtended by the same arc.

Cor. 3. Every angle inscribed in a semicircle is a right angle, because it is measured by half a semi-circumference; that is, the fourth part of a circumference.



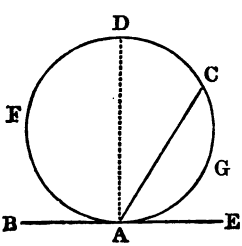
Cor. 4. Every angle inscribed in a segment greater than a semicircle is an acute angle, for it is measured by half an arc less than a semi-circumference.

Every angle inscribed in a segment less than a semicircle in an obtuse angle, for it is measured by half an arc greater than a semi-circumference.

Cor. 5. The opposite angles of an inscribed quadrilateral, ABE C, are supplements of each other; for the angle BAC is measured by half the arc BEC, and the angle BEC is measured by half the arc BAC; therefore the two angles BAC, BEC, taken together, are measured by half the circumference; hence their sum is equal to two right angles.

PROPOSITION XVI. THEOREM.

The angle formed by a tangent and a chord is measured by half the arc included between its sides.



Let the straight line BE touch the circumference ACDF in the point A, and from A let the chord AC be drawn; the angle BAC is measured by half the arc AFC.

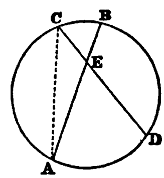
From the point A draw the diameter A D. The angle BAD is a right angle (Pr. 9), and is measured by half the semi-circumference AFD; also, the angle DAC is measured by half the arc DC (Pr. 15); therefore the sum of the angles BAD, DAC is measured by half the entire arc AFDC.

In the same manner, it may be shown that the angle CAE is measured by half the arc AGC, included between its sides.

Cor. The angle BAC is equal to an angle inscribed in the segment AGC, and the angle EAC is equal to an angle inscribed in the segment AFC.

PROPOSITION XVII. THEOREM.

The angle formed by two chords which cut each other is measured by one half the sum of the arcs intercepted between its sides and between the sides of its vertical angle.

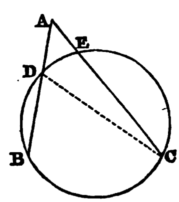


Let AB, CD be two chords which cut each other at E; then will the angle AED be measured by one half the sum of the arcs AD and BC, intercepted between the sides of AED and the sides of its vertical angle BEC.

Join AC; the angle AED is equal to the sum of the angles ACD and BAC (B. I., Pr. 27). But ACD is measured by half the arc AD (B. III., Pr. 15), and the angle BAC is measured by half the arc BC. Therefore AED is measured by half the sum of the arcs AD and BC. Therefore the angle, etc.

PROPOSITION XVIII. THEOREM.

The angle formed by two secants intersecting without the circumference, is measured by one half the difference of the intercepted arcs.



Let AB, AC be two secants which intersect at A; then will the angle BAC be measured by one half the difference of the arcs BC and DE.

Join CD; the angle BDC is equal to the sum of the angles DAC and ACD (B. I., Pr. 27); therefore the angle A is equal to the difference of the angles BDC and ACD. But the angle BDC is measured by one half the arc BC (B. III., Pr. 15), and the angle ACD is measured by one half the arc DE. Therefore the angle A is measured by one half the difference of the arcs BC and DE. Therefore the angle, etc.

BOOK IV.

COMPARISON AND MEASUREMENT OF POLYGONS.

Definitions.

1. The *area* of a figure is its superficial content. The area is expressed numerically by the number of times that the figure contains some other surface which is assumed for its measuring unit; that is, it is the ratio of its surface to that of the unit of surface. A unit of surface is called a *superficial unit.* The most convenient superficial unit is the square, whose side is the linear unit, as a square foot or a square yard.

2. *Equal figures* are such as may be applied the one to the other, so as to coincide throughout. Thus two circles having equal radii are equal; and two triangles having the three sides of the one equal to the three sides of the other, each to each, are also equal.

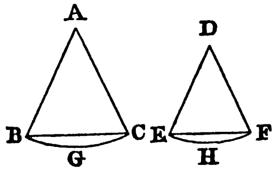
3. *Equivalent figures* are such as contain equal areas. Two figures may be equivalent, however dissimilar. Thus a circle may be equivalent to a square, a triangle to a rectangle, etc.

4. *Similar polygons* are such as have the angles of the one equal to the angles of the other, each to each, and the sides about the equal angles proportional. Sides which have the same position in the two polygons, or which are adjacent to equal angles, are called *homologous.* The equal angles may also be called homologous angles.

Equal polygons are always similar, but similar polygons may be very unequal.

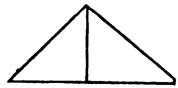
5. Two sides of one polygon are said to be *reciprocally proportional* to two sides of another when one side of the first is to one side of the second as the remaining side of the second is to the remaining side of the first.

6. In different circles, *similar arcs*, *sectors*, or *segments* are those which correspond to equal angles at the centre.

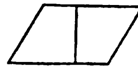


Thus, if the angles A and D are equal, the arc BC will be similar to the arc EF, the sector ABC to the sector DEF, and the segment BGC to the segment EHF.

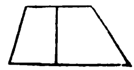
7. The *altitude of a triangle* is the perpendicular let fall from the vertex of an angle on the opposite side, taken as a base, or on the base produced.



8. The *altitude of a parallelogram* is the perpendicular drawn to the base from the opposite side.

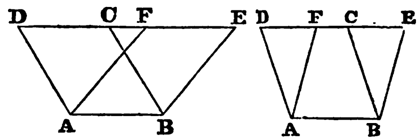


9. The *altitude of a trapezoid* is the perpendicular distance between its parallel sides.



PROPOSITION I. THEOREM.

Parallelograms which have equal bases and equal altitudes are equivalent.



Let the parallelograms ABCD, ABEF be placed so that their equal bases shall coincide with each other. Let AB be the common base; and, since the two parallelograms are supposed to have equal altitudes, their upper bases, DC, FE, will be in the same straight line parallel to AB.

Now, because ABCD is a parallelogram, DC is equal to AB (B. I., Pr. 30). For the same reason, FE is equal to AB, wherefore DC is equal to FE; hence, if DC and FE be taken away from the same line DE, the remainders CE and DF will be equal. But AD is also equal to BC, and AF to BE; therefore the triangles DAF, CBE are mutually equilateral, and consequently equal.

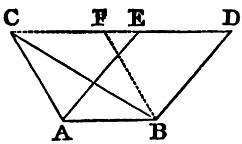
Now if from the quadrilateral ABED we take the triangle ADF, there will remain the parallelogram ABEF; and if from the same quadrilateral we take the triangle BCE, there will remain the parallelogram ABCD. Therefore the two parallelograms ABCD, ABEF, which have the same base and the same altitude, are equivalent.

Cor. Every parallelogram is equivalent to the rectangle which has the same base and the same altitude.

PROPOSITION II. THEOREM.

Every triangle is half of the parallelogram which has the same base and the same altitude.

Let the parallelogram ABDE and the triangle ABC have the



same base, AB, and the same altitude; the triangle is half of the parallelogram.

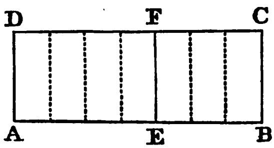
Complete the parallelogram ABFC; then the parallelogram ABFC is equivalent to the parallelogram ABDE, because they have the same base and the same altitude (Pr. 1). But the triangle ABC is half of the parallelogram ABFC (B. I., Pr. 30, Cor. 1), wherefore the triangle ABC is also half of the parallelogram ABDE. Therefore every triangle, etc.

Cor. 1. Every triangle is half of the rectangle which has the same base and altitude.

Cor. 2. Triangles which have equal bases and equal altitudes are equivalent.

PROPOSITION III. THEOREM.

Two rectangles having equal altitudes are to each other as their bases.



Let ABCD, AEFD be two rectangles which have the same altitude AD; they are to each other as their bases AB, AE.

Case first. When the bases are in the ratio of two whole numbers; for example, as 7 to 4. If AB be divided into seven equal parts, AE will contain four of those parts. At each point of division erect a perpendicular to the base; seven partial rectangles will thus be formed, all equal to each other, since they have equal bases and altitudes (Pr. 1). The rectangle ABCD will contain seven partial rectangles, while AEFD will contain four; therefore the rectangle ABCD is to the rectangle AEFD as 7 to 4, or as AB to AE. The same reasoning is applicable to any other ratio than that of 7 to 4; therefore, whenever the ratio of the bases can be expressed in whole numbers, we shall have

ABCD : AEFD :: AB : AE.

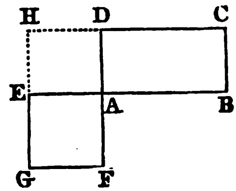
Case second. When the ratio of the bases can not be expressed exactly in numbers, the proposition may be proved by the same method employed in B. III, Pr. 14. Therefore two rectangles, etc.

PROPOSITION IV. THEOREM.

Any two rectangles are to each other as the products of their bases by their altitudes.

Let ABCD, AEGF be two rectangles; the ratio of the rectangle ABCD to the rectangle AEGF is the same with the ratio of the product of AB by AD to the product of AE by AF; that is,

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$



Having placed the two rectangles so that the angles at A are vertical, produce the sides GE, CD till they meet in H. The two rectangles ABCD, AEHD have the same altitude AD; they are, therefore, as their bases AB, AE (Pr. 3).

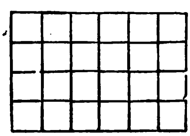
So, also, the rectangles AEHD, AEGF, having the same altitude AE, are to each other as their bases AD, AF. Thus we have the two proportions

$$ABCD : AEHD :: AB : AE,$$
$$AEHD : AEGF :: AD : AF.$$

Hence (B. II., Pr. 12, Cor.),

$$ABCD : AEGF :: AB \times AD : AE \times AF.$$

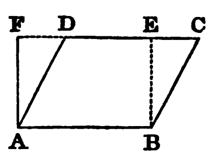
Scholium. Hence we may take as the *measure* of a rectangle the product of its base by its altitude, provided we understand by it the product of two numbers, one of which is the number of linear units contained in the base, and the other the number of linear units contained in the altitude.



Thus, if the base of a rectangle contains 6 inches, and the altitude 4 inches, the rectangle can be divided into 24 squares, each equal to one square inch; that is, its area is represented by 24 square inches. If the base of a second rectangle contains 9 inches, and its altitude 5 inches, its area is represented by 45 square inches, and the ratio of the two rectangles is that of 24 to 45.

PROPOSITION V. THEOREM.

The area of any parallelogram is equal to the product of its base by its altitude.

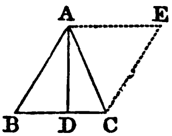


Let ABCD be a parallelogram, AF its altitude, and AB its base; then is its surface measured by the product of AB by AF. For, upon the base AB, construct a rectangle having the altitude AF; the parallelogram ABCD is equivalent to the rectangle ABEF (Pr. 1, Cor.). But the rectangle ABEF is measured by AB × AF (Pr. 4, Sch.); therefore the area of the parallelogram ABCD is equal to AB × AF.

Cor. Parallelograms having equal bases are to each other as their altitudes, and parallelograms having equal altitudes are to each other as their bases; for magnitudes have the same ratio that their equimultiples have (B. II, Pr. 10).

PROPOSITION VI. THEOREM.

The area of a triangle is equal to half the product of its base by its altitude.



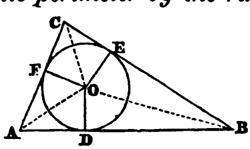
Let ABC be any triangle, BC its base, and AD its altitude; the area of the triangle ABC is measured by half the product of BC by AD.

For, complete the parallelogram ABCE. The triangle ABC is half of the parallelogram ABCE (Pr. 2); but the area of the parallelogram is equal to $BC \times AD$ (Pr. 5); hence the area of the triangle is equal to one half of the product of BC by AD. Therefore the area of a triangle, etc.

Cor. 1. Triangles having equal altitudes are to each other as their bases, and triangles having equal bases are to each other as their altitudes.

Cor. 2. Equivalent triangles whose bases are equal have equal altitudes, and equivalent triangles whose altitudes are equal have equal bases.

Scholium. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle. Let O be the centre of the inscribed circle. From this point let fall the perpendiculars OD, OE, OF upon the sides AB, BC, AC, and draw the lines AO, BO, CO. By this proposition we have triangle $AOB = \frac{1}{2}(AB \times OD)$, triangle $AOC = \frac{1}{2}(AC \times OF)$, and triangle $BOC = \frac{1}{2}(BC \times OE)$. Now the triangle ABC is equivalent to the sum of the triangles AOB, AOC, and BOC, and the three perpendiculars OD, OE, OF are equal to each other.

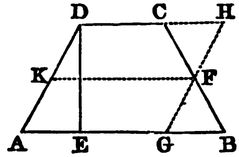


Hence $ABC = \frac{1}{2}(AB + AC + BC)OD.$

PROPOSITION VII. THEOREM.

The area of a trapezoid is equal to the product of its altitude by half the sum of its parallel sides.

Let ABCD be a trapezoid, DE its altitude, AB and CD its parallel sides; its area is measured by half the product of DE by the sum of its sides AB, CD.



Bisect BC in F, and through F draw GH parallel to AD, and produce DC to H.

In the two triangles BFG, CFH the side BF is, by construction, equal to CF, the vertical angles BFG, CFH are equal (B. I., Pr. 5), and the angle FCH is equal to the alternate angle FBG, because CH and BG are parallel (B. I., Pr. 23); therefore the triangle CFH is equal to the triangle BFG.

Now if from the whole figure ABFHD we take away the triangle CFH, there will remain the trapezoid ABCD; and if from the same figure ABFHD we take away the equal triangle BFG, there will remain the parallelogram AGHD. Therefore the trapezoid ABCD is equivalent to the parallelogram AGHD, and is measured by the product of AG by DE.

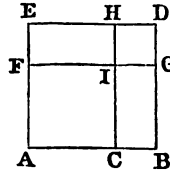
Also, because AG is equal to DH, and BG to CH, therefore the sum of AB and CD is equal to the sum of AG and DH, or twice AG. Hence AG is equal to half the sum of the parallel sides AB, CD; therefore the area of the trapezoid ABCD is equal to the product of the altitude DE by half the sum of the bases AB, CD.

Cor. If through the point F, the middle of BC, we draw FK parallel to the base AB, the point K will also be the middle of AD. For the figure AKFG is a parallelogram, as also DKFH, the opposite sides being parallel. Therefore AK is equal to FG, and DK to HF. But FG is equal to FH, since the triangles BFG, CFH are equal; therefore AK is equal to DK.

Now, since KF is equal to AG, the area of the trapezoid is equal to DE$\times$KF. Hence *the area of a trapezoid is equal to its altitude multiplied by the line which joins the middle points of the sides which are not parallel.*

PROPOSITION VIII. THEOREM.

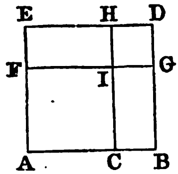
If a straight line is divided into any two parts, the square of the whole line is equivalent to the squares of the two parts, together with twice the rectangle contained by the parts.



Let the straight line AB be divided into any two parts in C; the square on AB is equivalent to the squares on AC, CB, together with twice the rectangle contained by AC, CB; that is,

AB^2, or $(AC+CB)^2=AC^2+CB^2+2AC\times CB$.

Upon AB describe the square ABDE; take



AF equal to AC; through F draw FG parallel to AB, and through C draw CH parallel to AE.

The square ABDE is divided into four parts: the first, ACIF, is the square on AC, since AF was taken equal to AC. The second part, IGDH, is the square on CB; for, because AB is equal to AE, and AC to AF, therefore BC is equal to EF (B. I., Ax. 3).

But, because BCIG is a parallelogram, GI is equal to BC; and because DEFG is a parallelogram, DG is equal to EF (B. I., Pr. 30); therefore HIGD is equal to a square described on BC. If these two parts are taken from the entire square, there will remain the two rectangles BCIG, EFIH, each of which is measured by $AC \times CB$; therefore the whole square on AB is equivalent to the squares on AC and CB, together with twice the rectangle of $AC \times CB$. Therefore, if a straight line, etc.

Cor. The square of any line is equivalent to four times the square of half that line. For, if AC is equal to CB, the four figures AI, CG, FH, ID become equal squares.

Scholium 1. If a and b denote the numbers which represent the two parts of the line AB, this proposition may be expressed algebraically thus: $(a+b)^2=a^2+2ab+b^2$.

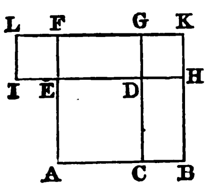
Scholium 2. A rectangle is said to be contained by any two of the straight lines which are about one of the right angles.

PROPOSITION IX. THEOREM.

The square described on the difference of two lines is equivalent to the sum of the squares of the lines, diminished by twice the rectangle contained by the lines.

Let AB, BC be any two lines, and AC their difference; the square described on AC is equivalent to the sum of the squares on AB and CB, diminished by twice the rectangle contained by AB, CB; that is,

$$AC^2, \text{ or } (AB-BC)^2=AB^2+BC^2-2AB \times BC.$$



Upon AB describe the square ABKF; take AE equal to AC; through C draw CG parallel to BK, and through E draw HI parallel to AB, and complete the square EFLI.

Because AB is equal to AF, and AC to AE, therefore CB is equal to EF, and GK to LF. Therefore LG is equal to FK or AB, and hence the two rectangles CBKG, GLID are each measured by $AB \times BC$. If these rectangles are taken from the entire figure ABKLIE, which is equivalent to AB^2+BC^2, there will evidently remain the square ACDE. Therefore the square described, etc.

Scholium. This proposition is expressed algebraically thus:

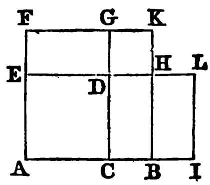
$$(a-b)^2=a^2-2ab+b^2.$$

PROPOSITION X. THEOREM.

The rectangle contained by the sum and difference of two lines is equivalent to the difference of the squares of those lines.

Let AB, BC be any two lines; the rectangle contained by the sum and difference of AB and BC is equivalent to the difference of the squares on AB and BC; that is,

$$(AB+BC)\times(AB-BC)=AB^2-BC^2.$$



Upon AB describe the square ABKF, and upon AC describe the square ACDE; produce AB so that BI shall be equal to BC, and complete the rectangle AILE.

The base AI of the rectangle AILE is the sum of the two lines AB, BC, and its altitude AE is the difference of the same lines; therefore AILE is the rectangle contained by the sum and difference of the lines AB, BC.

But this rectangle is composed of the two parts ABHE and BILH; and the part BILH is equal to the rectangle FGDE, for BH is equal to DE, and BI is equal to EF. Therefore AILE is equivalent to the figure ABHDGF. But ABHDGF is the excess of the square ABKF above the square DHKG, which is the square of BC; therefore

$$(AB+BC)\times(AB-BC)=AB^2-BC^2.$$

Scholium. This proposition is expressed algebraically thus:

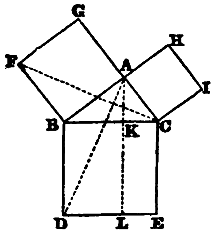
$$(a+b)\times(a-b)=a^2-b^2.$$

PROPOSITION XI. THEOREM.

In any right-angled triangle the square described on the hypothenuse is equivalent to the sum of the squares described on the other two sides.

Let ABC be a right-angled triangle, having the right angle BAC; the square described upon the side BC is equivalent to the sum of the squares upon BA, AC.

On BC describe the square BCED, and on BA, AC, the squares



BG, CH; and through A draw AL parallel to BD, and join AD, FC.

Then, because each of the angles BAC, BAG is a right angle, CA is in the same straight line with AG (B. I., Pr. 3). For the same reason, BA and AH are in the same straight line.

The angle ABD is composed of the angle ABC and the right angle CBD. The angle FBC is composed of the same angle ABC and the right angle ABF; therefore the whole angle ABD is equal to the angle FBC. But AB is equal to BF, being sides of the same square, and BD is equal to BC for the same reason; therefore the triangles ABD, FBC have two sides and the included angle equal; they are therefore equal (B. I., Pr. 6).

But the rectangle BDLK is double of the triangle ABD, because they have the same base BD, and the same altitude BK (Pr. 2, Cor. 1); and the square AF is double of the triangle FBC, for they have the same base BF, and the same altitude AB. Now the doubles of equals are equal to one another (B. I., Ax. 6); therefore the rectangle BDLK is equivalent to the square AF.

In the same manner it may be demonstrated that the rectangle CELK is equivalent to the square AI; therefore the whole square BCED, described on the hypothenuse, is equivalent to the two squares ABFG, ACIH, described on the two other sides; that is,

$$BC^2 = AB^2 + AC^2.$$

Scholium. Tradition has ascribed the discovery of this proposition to Pythagoras (born about 580 B.C.), and hence it is commonly called the *Pythagorean theorem.*

Cor. 1. The square of one of the sides of a right-angled triangle is equivalent to the square of the hypothenuse, diminished by the square of the other side; that is,

$$AB^2 = BC^2 - AC^2.$$

Hence, if the numerical measures of two sides of a right-angled triangle are given, that of the third may be found. For example, if $BC=5$, and $AB=4$, then $AC=$ the square root of $(5^2-4^2)=3$.

Also, if $AC=5$, and $AB=12$, then $BC=13$.

Cor. 2. The square BCED, and the rectangle BKLD, having the same altitude, are to each other as their bases BC, BK (Pr. 3). But the rectangle BKLD is equivalent to the square AF; therefore

$$BC^2 : AB^2 :: BC : BK.$$

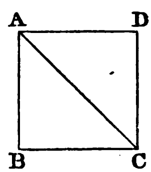
In the same manner, $BC^2 : AC^2 :: BC : KC$.

Therefore (B. II., Pr. 4, Cor.),

$$AB^2 : AC^2 :: BK : KC.$$

That is, in any right-angled triangle, if a line be drawn from the right angle perpendicular to the hypothenuse, *the squares of the two sides are proportional to the adjacent segments of the hypothenuse;* also, *the square of the hypothenuse is to the square of either of the sides as the hypothenuse is to the segment adjacent to that side.*

Cor. 3. Let ABCD be a square, and AC its diagonal; the triangle ABC being right-angled and isosceles, we have



$$AC^2 = AB^2 + BC^2 = 2AB^2;$$

therefore *the square described on the diagonal of a square is double of the square described on a side.*

If we extract the square root of each member of this equation, we shall have $AC = AB\sqrt{2}$; or $AC : AB :: \sqrt{2} : 1$.

The square root of 2 is 1.4142136, correct to seven decimal places. Since the square root of 2 is an incommensurable number, it follows that *the diagonal of a square is incommensurable with its side.*

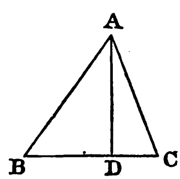
PROPOSITION XII. THEOREM.

In any triangle, the square of the side opposite to an acute angle is less than the squares of the base and of the other side by twice the rectangle contained by the base, and the distance from the acute angle to the foot of the perpendicular let fall from the opposite angle.

Let ABC be any triangle, and the angle at C one of its acute angles, and upon BC let fall the perpendicular AD from the opposite angle; then will

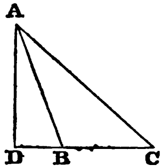
$$AB^2 = BC^2 + AC^2 - 2BC \times CD.$$

First. When the perpendicular falls within the triangle ABC, we have $BD = BC - CD$, and therefore $BD^2 = BC^2 + CD^2 - 2BC \times CD$ (Pr. 9). To each of these equals add AD^2; then $BD^2 + AD^2 = BC^2 + CD^2 + AD^2 - 2BC \times CD$.



But in the right-angled triangle ABD, $BD^2 + AD^2 = AB^2$; and in the triangle ADC, $CD^2 + AD^2 = AC^2$ (Pr. 11); therefore

$$AB^2 = BC^2 + AC^2 - 2BC \times CD.$$



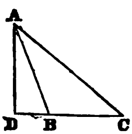
Secondly. When the perpendicular falls without the triangle ABC, we have $BD=CD-BC$, and therefore $BD^2=CD^2+BC^2-2CD\times BC$ (Pr. 9). To each of these equals add AD^2; then $BD^2+AD^2=CD^2+AD^2+BC^2-2CD\times BC$.

But $BD^2+AD^2=AB^2$; and $CD^2+AD^2=AC^2$; therefore $AB^2=BC^2+AC^2-2BC\times CD$.

Scholium. When the perpendicular AD falls upon AB, this proposition reduces to the same as Pr. 11, Cor. 1.

PROPOSITION XIII. THEOREM.

In an obtuse-angled triangle, the square of the side opposite the obtuse angle is greater than the squares of the base and the other side by twice the rectangle contained by the base, and the distance from the obtuse angle to the foot of the perpendicular let fall from the opposite angle on the base produced.



Let ABC be an obtuse-angled triangle, having the obtuse angle ABC, and from the point A let AD be drawn perpendicular to BC produced; the square of AC is greater than the squares of AB, BC by twice the rectangle $BC\times BD$.

For CD is equal to $BC+BD$; therefore $CD^2=BC^2+BD^2+2BC\times BD$ (Pr. 8). To each of these equals add AD^2; then $CD^2+AD^2=BC^2+BD^2+AD^2+2BC\times BD$.

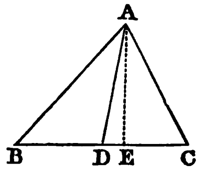
But AC^2 is equal to CD^2+AD^2 (Pr. 11), and AB^2 is equal to BD^2+AD^2; therefore $AC^2=BC^2+AB^2+2BC\times BD$. Therefore, in an obtuse-angled triangle, etc.

Scholium. The right-angled triangle is the only one in which the sum of the squares of two sides is equivalent to the square on the third side; for, if the angle contained by the two sides is acute, the sum of their squares is greater than the square of the opposite side; if obtuse, it is less.

PROPOSITION XIV. THEOREM.

In any triangle, if a straight line is drawn from the vertex to the middle of the base, the sum of the squares of the other two sides is equivalent to twice the square of the bisecting line, together with twice the square of half the base.

Let ABC be a triangle having a line AD drawn from the middle of the base to the opposite angle; the squares of BA and AC are together double of the squares of AD and BD.



From A draw AE perpendicular to BC; then, in the triangle ABD, by Pr. 13,

$$AB^2=AD^2+DB^2+2DB\times DE;$$

and, in the triangle ADC, by Pr. 12,

$$AC^2=AD^2+DC^2-2DC\times DE.$$

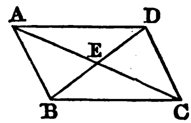
Hence, by adding these equals, and observing that BD=DC, and therefore $BD^2=DC^2$, and $DB\times DE=DC\times DE$, we obtain

$$AB^2+AC^2=2AD^2+2DB^2.$$

Therefore, in any triangle, etc.

PROPOSITION XV. THEOREM.

In every parallelogram, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.



Let ABCD be a parallelogram, of which the diagonals are AC and BD; the sum of the squares of AC and BD is equal to the sum of the squares of AB, BC, CD, DA.

The diagonals AC and BD bisect each other in E (B. I., Pr. 33); therefore, in the triangle ABD (Pr. 14),

$$AB^2+AD^2=2BE^2+2AE^2;$$

and, in the triangle BDC,

$$CD^2+BC^2=2BE^2+2EC^2.$$

Adding these equals, and observing that AE is equal to EC, we have $AB^2+BC^2+CD^2+AD^2=4BE^2+4AE^2$.

But $4BE^2=BD^2$, and $4AE^2=AC^2$ (Pr. 8, Cor.); therefore

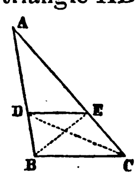
$$AB^2+BC^2+CD^2+AD^2=BD^2+AC^2.$$

Therefore, in every parallelogram, etc.

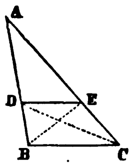
PROPOSITION XVI. THEOREM.

If a straight line be drawn parallel to the base of a triangle, it will cut the other sides proportionally; and if the sides be cut proportionally, the cutting line will be parallel to the base of the triangle.

Let DE be drawn parallel to BC, the base of the triangle AB C; then will AD : DB :: AE : EC.



Join BE and DC; then the triangle BDE is equivalent to the triangle DEC, because they have the same base, DE, and the same altitude, since their vertices B and C are in a line parallel to the base (Pr. 2, Cor. 2).



The triangles ADE, BDE, whose common vertex is E, having the same altitude, are to each other as their bases AD, DB (Pr. 6, Cor. 1); hence

ADE : BDE :: AD : DB.

The triangles ADE, DEC, whose common vertex is D, having the same altitude, are to each other as their bases AE, EC; therefore

ADE : DEC :: AE : EC.

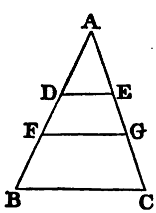
But, since the triangle BDE is equivalent to the triangle DEC, therefore (B. II., Pr. 4),

AD : DB :: AE : EC.

Conversely, let DE cut the sides AB, AC, so that AD : DB :: AE : EC; then DE will be parallel to BC.

For AD : DB :: ADE : BDE (Pr. 6, Cor. 1); and AE : EC :: ADE : DEC; therefore (B. II., Pr. 4), ADE : BDE :: ADE : DEC; that is, the triangles BDE, DEC have the same ratio to the triangle ADE; consequently, the triangles BDE, DEC are equivalent, and, having the same base, DE, their altitudes are equal (Pr. 6, Cor. 2); that is, they are between the same parallels. Therefore, if a straight line, etc.

Cor. 1. Since, by this proposition, AD : DB :: AE : EC; by composition, AD+DB : AD :: AE+EC : AE (B. II., Pr. 6), or AB :: AD :: AC : AE; also, AB : BD :: AC : EC.



Cor. 2. *If two lines be drawn parallel to the base of a triangle, they will divide the other sides proportionally.* For, because FG is drawn parallel to BC, by the preceding proposition, AF : FB :: AG : GC. Also, by the last corollary, because DE is parallel to FG, AF : DF :: AG : EG. Therefore DF : FB :: EG : GC (B. II., Pr. 4, Cor.).

Also, AD : DF :: AE : EG.

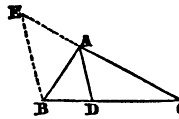
Cor. 3. If any number of lines be drawn parallel to the base of a triangle, the sides will be cut proportionally.

PROPOSITION XVII. THEOREM.

The line which bisects the vertical angle of a triangle divides the base into two segments, which are proportional to the adjacent sides.

Let the angle BAC of the triangle ABC be bisected by the straight line AD; then will

BD : DC :: BA : AC.



Through the point B draw BE parallel to D A, meeting CA produced in E. The triangle A BE is isosceles. For, since AD is parallel to E B, the angle ABE is equal to the alternate angle DAB (B. I., Pr. 23), and the exterior angle CAD is equal to the interior and remote angle AEB. But, by hypothesis, the angle DAB is equal to the angle DAC; therefore the angle ABE is equal to AEB, and the side AE to the side AB (B. I., Pr. 11).

And because AD is parallel to BE, the base of the triangle BC E (Pr. 16), BD : DC :: EA : AC.

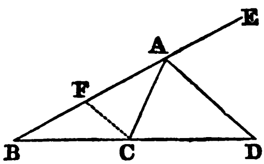
But AE is equal to AB, therefore

BD : DC :: BA : AC.

Therefore, the line, etc.

PROPOSITION XVIII. THEOREM.

The line which bisects the exterior angle of a triangle divides the base produced into segments which are proportional to the adjacent sides.



Let BA, one side of the triangle ABC, be produced to E, and let the exterior angle CAE be bisected by the straight line AD, which meets the base produced at D; then

BD : DC :: BA : AC.

Through C draw CF parallel to AD, meeting AB at F. Then, because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (B. I., Pr. 23). But the angle CAD is, by hypothesis, equal to DAE; therefore DAE is equal to ACF.

Again, because the straight line FAE meets the parallels AD, FC, the exterior angle DAE is equal to the interior and remote angle AFC (B. I., Pr. 23). But DAE has been shown equal to A CF; therefore ACF is equal to AFC, and therefore AF is equal to AC (B. I., Pr. 11).

And because FC is parallel to AD, one of the sides of the triangle ABD, therefore (B. IV., Pr. 16) BD : DC :: BA : AF. But AF is equal to AC; therefore

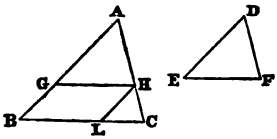
BD : DC :: BA : AC.

Therefore, the line, etc.

Scholium. By the segments of a line we understand the portions into which the line is divided at a given point. So also by the segments of a *line produced* to a given point, we understand the distances between the given point and the extremities of the line.

PROPOSITION XIX. THEOREM.

Two triangles which are mutually equiangular have their homologous sides proportional, and are similar.



Let ABC, DEF be two triangles which are mutually equiangular, having the angle A=D, B=E, and C=F; then the homologous sides will be proportional, and we shall have

AB : DE :: AC : DF :: BC : EF.

Take AG=DE, AH=DF, and join GH. Then the triangles AGH, DEF are equal, since two sides and the included angle in the one are respectively equal to two sides and the included angle in the other (B. I., Pr. 6). Therefore the angle AGH is equal to the angle E. But, by hypothesis, the angle E is equal to the angle B; therefore the angle B is equal to AGH, and therefore GH is parallel to BC (B. I., Pr. 22). Hence (B. IV., Pr. 16) we have AB : AG :: AC : AH.

Draw HL parallel to AB; then BGHL is a parallelogram, and BL is equal to GH.

Also (B. IV., Pr. 16), we have

AC : AH :: BC : BL or GH.

Since these two proportions contain the same ratio AC : AH, we conclude (B. II., Pr. 4)

AB : AG :: AC : AH :: BC : GH,

or, AB : DE :: AC : DF :: BC : EF.

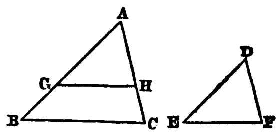
Therefore the triangles ABC, DEF have their homologous sides proportional; hence, by Def. 4, they are similar.

Cor. Two triangles are similar when two angles of the one are respectively equal to two angles of the other, for then the third angles must also be equal (B. I., Pr. 27, Cor. 2).

Scholium. In similar triangles the homologous sides are opposite to the equal angles; thus, the angle ACB being equal to the angle DFE, the side AB is homologous to DE, and so with the other sides.

PROPOSITION XX. THEOREM.

Two triangles which have their homologous sides proportional are mutually equiangular and similar.



Let the triangles ABC, DEF have their sides proportional, so that

BC : EF :: AB : DE :: AC : DF;

then will the triangles have their angles equal, viz., the angle A to the angle D, B equal to E, and C equal to F.

Take AG=DE, AH=DF, and join GH. By hypothesis we have

AB : DE :: AC : DF;

or, substituting for DE and DF their equals AG and AH, we have

AB : AG :: AC : AH.

Therefore GH is parallel to BC (B. IV., Pr. 16), and the triangles ABC, AGH are mutually equiangular. Hence we have

AC : AH :: BC : GH.

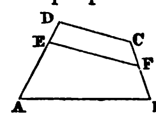
But, by hypothesis, we have

AC : DF :: BC : EF;

and, since AH=DF, we conclude that GH=EF.

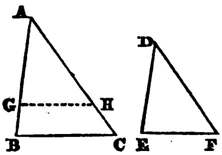
Therefore the triangles AGH, DEF, having the three sides of the one equal to the three sides of the other, are equal, and therefore the angle DEF is equal to AGH, which is equal to ABC; also, the angle DFE is equal to AHG, which is equal to ACB; and the angle D is equal to A. Hence the triangles ABC, DEF are mutually equiangular and similar. Therefore two triangles, etc.

Scholium. It will be seen from the last two propositions that triangles which are mutually equiangular have their homologous sides proportional, and conversely, so that either of these conditions involves the other. This is not true of figures having more than three sides, for in quadrilaterals we may change the angles without changing the sides; or we may change the proportion of the sides without changing the angles. Thus, if we draw EF parallel to DC, the angles of the quadrilateral ABFE are equal to those of the quadrilateral ABCD, but the proportion of the sides is changed. Also, without changing the four sides AB, BC, CD, DA, we may change the angles by moving the point D toward B, or from it.



PROPOSITION XXI. THEOREM.

Two triangles are similar when they have an angle of the one equal to an angle of the other, and the sides including those angles proportional.



Let the triangles ABC, DEF have the angle A of the one equal to the angle D of the other, and let AB : DE :: AC : DF; the triangle ABC is similar to the triangle DEF.

Take AG equal to DE, also AH equal to DF, and join GH. Then the triangles AGH, DEF are equal, since two sides and the included angle in the one are respectively equal to two sides and the included angle in the other (B. I., Pr. 6). But, by hypothesis,

AB : DE :: AC : DF;

therefore AB : AG :: AC : AH;

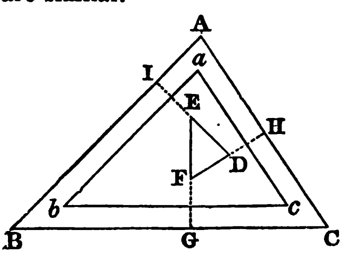
that is, the sides AB, AC, of the triangle ABC, are cut proportionally by the line GH; therefore GH is parallel to BC (Pr. 16).

Hence (B. I., Pr. 23) the angle AGH is equal to ABC, and the triangle AGH is similar to the triangle ABC. But the triangle DEF has been shown to be equal to the triangle AGH; hence the triangle DEF is similar to the triangle ABC. Therefore, two triangles, etc.

PROPOSITION XXII. THEOREM.

Two triangles are similar when they have their homologous sides parallel each to each, or perpendicular each to each.

Let the triangles ABC, *abc*, DEF have their homologous sides parallel each to each, or perpendicular each to each, the triangles are similar.



First. Let the homologous sides be parallel each to each. If the side AB is parallel to *ab*, and BC to *bc*, the angle B is equal to the angle *b* (B. I., Pr. 26); also, if AC is parallel to *ac*, the angle C is equal to the angle *c*; and hence the angle A is equal to the angle *a*. Therefore the triangles ABC, *abc* are equiangular, and consequently similar.

Secondly. Let the homologous sides be perpendicular each to each. Let the side DE be perpendicular to AB, and the side DF to AC. Produce DE to I, and DF to H; then, in the quadrilateral AIDH, the two angles I and H are right angles. But the four angles of a quadrilateral are together equal to four right angles (B. I., Pr. 28, Cor.); therefore the two remaining angles IAH, IDH are together equal to two right angles. But the two angles EDF, IDH are together equal to two right angles (B. I., Pr. 2); therefore the angle EDF is equal to IAH or BAC.

In the same manner, if the side EF is also perpendicular to B C, it may be proved that the angle DFE is equal to C, and, consequently, the angle DEF is equal to B; hence the triangles ABC, DEF are equiangular and similar. Therefore, two triangles, etc.

Scholium. When the sides of the two triangles are parallel to each other, the parallel sides are homologous; but when the sides are perpendicular to each other, the perpendicular sides are homologous. Thus DE is homologous to AB, DF to AC, and EF to BC.

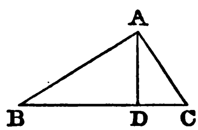
PROPOSITION XXIII. THEOREM.

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypothenuse;

1st. *The triangles on each side of the perpendicular are similar to the whole triangle and to each other.*

2d. *The perpendicular is a mean proportional between the segments of the hypothenuse.*

3d. *Each of the sides is a mean proportional between the hypothenuse and its segment adjacent to that side.*



Let ABC be a right-angled triangle, having the right angle BAC, and from the angle A let AD be drawn perpendicular to the hypothenuse BC.

First. The triangles ABD, ACD are similar to the whole triangle ABC, and to each other.

The triangles BAD, BAC have the common angle B, also the angle BAC equal to BDA, each of them being a right angle, and, therefore, the remaining angle ACB is equal to the remaining angle BAD (B. I., Pr. 27, Cor. 2); therefore the triangles ABC, AB D are equiangular and similar. In like manner, it may be proved that the triangle ADC is equiangular and similar to the triangle ABC; therefore the three triangles ABC, ABD, ACD are equiangular, and similar to each other.

Secondly. The perpendicular AD is a mean proportional between the segments BD, DC of the hypothenuse. For, since the triangle ABD is similar to the triangle ADC, their homologous sides are proportional (Def. 3), and we have

$$BD : AD :: AD : DC.$$

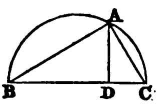
Thirdly. Each of the sides AB, AC is a mean proportional between the hypothenuse and the segment adjacent to that side. For, since the triangle BAD is similar to the triangle BAC, we have

$$BC : BA :: BA : BD.$$

And, since the triangle ABC is similar to the triangle ACD, we have

$$BC : CA :: CA : CD.$$

Therefore, in a right-angled triangle, etc.

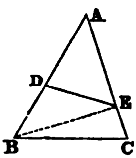


Cor. If from a point A, in the circumference of a circle, two chords AB, AC are drawn to the extremities of the diameter BC, the triangle BAC will be right-angled at A (B. III., Pr. 15, Cor. 3); therefore the perpendicular AD is a mean proportional between BD and DC, the two segments of the diameter; that is,

$$AD^2 = BD \times DC.$$

PROPOSITION XXIV. THEOREM.

Two triangles, having an angle in the one equal to an angle in the other, are to each other as the rectangles of the sides which contain the equal angles.



Let the two triangles ABC, ADE have the angle A in common; then will the triangle ABC be to the triangle ADE as the rectangle $AB \times AC$ is to the rectangle $AD \times AE$.

Join BE. Then the two triangles ABE, ADE, having the common vertex E, have the same altitude, and are to each other as their bases AB, AD (Pr. 6, Cor. 1); therefore $ABE : ADE :: AB : AD$.

Also, the two triangles ABC, ABE, having the common vertex B, have the same altitude, and are to each other as their bases AC, AE; therefore $ABC : ABE :: AC : AE$.

Hence (B. II., Pr. 12, Cor.)

$$ABC : ADE :: AB \times AC : AD \times AE.$$

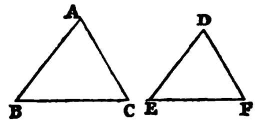
Therefore two triangles, etc.

Cor. 1. If the rectangles of the sides containing the equal angles are equivalent, the triangles will be equivalent.

Cor. 2. Parallelograms which are mutually equiangular are to each other as the rectangles of the sides which contain the equal angles.

PROPOSITION XXV. THEOREM.

Similar triangles are to each other as the squares described on their homologous sides.



Let ABC, DEF be two similar triangles, having the angle A equal to D, the angle B equal to E, and C equal to F; then the triangle ABC is to the triangle DEF as the square on BC is to the square on EF.

By similar triangles, we have (Def. 4)

$$AB : DE :: BC : EF.$$

Also, $$BC : EF :: BC : EF.$$

Multiplying together the corresponding terms of these proportions, we obtain (B. II., Pr. 12),

$$AB \times BC : DE \times EF :: BC^2 : EF^2.$$

But, by Pr. 24,

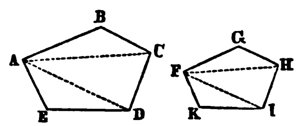
$$ABC : DEF :: AB \times BC : DE \times EF;$$

hence (B. II., Pr. 4) $ABC : DEF :: BC^2 : EF^2$.

Therefore similar triangles, etc.

PROPOSITION XXVI. THEOREM.

Two similar polygons may be divided into the same number of triangles, similar each to each, and similarly situated.



Let ABCDE, FGHIK be two similar polygons; they may be divided into the same number of similar triangles. Join AC, AD, FH, FI.

Because the polygon ABCDE is similar to the polygon FGHIK, the angle B is equal to the angle G (Def. 4), and $AB : BC :: FG : GH$.

And, because the triangles ABC, FGH have an angle in the one equal to an angle in the other, and the sides about these equal angles proportional, they are similar (Pr. 21); therefore the angle BCA is equal to the angle GHF. Also, because the polygons are similar, the whole angle BCD is equal (Def. 4) to the whole angle GHI; therefore the remaining angle ACD is equal to the remaining angle FHI. Now, because the triangles ABC, FGH are similar, $AC : FH :: BC : GH$.

And, because the polygons are similar (Def. 4),

BC : GH :: CD : HI;

whence AC : FH :: CD : HI;

that is, the sides about the equal angles ACD, FHI are proportional; therefore the triangle ACD is similar to the triangle FHI (Pr. 21). For the same reason, the triangle ADE is similar to the triangle FIK; therefore the similar polygons ABCDE, FGHIK are divided into the same number of triangles, which are similar each to each, and similarly situated.

Cor. Conversely, *if two polygons are composed of the same number of triangles, similar each to each, and similarly situated, the polygons are similar.*

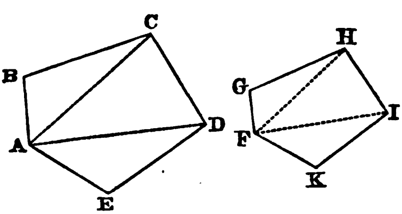
For, because the triangles are similar, the angle ABC is equal to FGH; and because the angle BCA is equal to GHF, and ACD to FHI, therefore the angle BCD is equal to GHI. For the same reason, the angle CDE is equal to HIK, and so on for the other angles. Therefore the two polygons are mutually equiangular.

Moreover, the sides about the equal angles are proportional. For, because the triangles are similar, AB : FG :: BC : GH. Also, BC : GH :: AC : FH, and AC : FH :: CD : HI; hence BC : GH :: CD : HI.

In the same manner, it may be proved that CD : HI :: DE : IK, and so on for the other sides. Therefore the two polygons are similar.

PROPOSITION XXVII. THEOREM.

The perimeters of two similar polygons are to each other as any two homologous sides, and their areas are as the squares of those sides.



Let ABCDE, FGHIK be two similar polygons, and let AB be the side homologous to FG; then the perimeter of ABCDE is to the perimeter of FGHIK as AB is to FG; and the area of ABCDE is to the area of FGHIK as AB^2 is to FG^2.

First. Because the polygon ABCDE is similar to the polygon FGHIK (Def. 4),

AB : FG :: BC : GH :: CD : HI, etc.;

therefore (B. II., Pr. 9) the sum of the antecedents AB+BC+CD, etc., which form the perimeter of the first figure, is to the sum of the consequents FG+GH+HI, etc., which form the perimeter of the second figure, as any one antecedent is to its consequent, or as AB to FG.

Secondly. Because the triangle ABC is similar to the triangle FGH, the triangle ABC : triangle FGH :: $AC^2 : FH^2$ (Pr. 25).

And, because the triangle ACD is similar to the triangle FHI,

$$ACD : FHI :: AC^2 : FH^2.$$

Therefore the triangle ABC : triangle FGH :: triangle ACD : triangle FHI (B. II., Pr. 4).

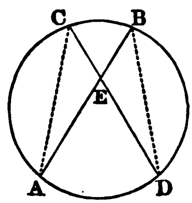
In the same manner, it may be proved that

$$ACD : FHI :: ADE : FIK.$$

Therefore, as the sum of the antecedents ABC+ACD+ADE, or the polygon ABCDE, is to the sum of the consequents FGH+FHI+FIK, or the polygon FGHIK, so is any one antecedent, as ABC, to its consequent FGH; or, as AB^2 to FG^2. Therefore the perimeters, etc.

PROPOSITION XXVIII. THEOREM.

If two chords in a circle cut each other, the rectangle contained by the parts of the one is equivalent to the rectangle contained by the parts of the other.



Let the two chords AB, CD, in the circle ACBD, cut each other in the point E; the rectangle contained by AE, EB is equivalent to the rectangle contained by DE, EC.

Join AC and BD. Then, in the triangles ACE, DBE, the angles at E are equal, being vertical angles (B. I., Pr. 5); the angle A is equal to the angle D, being inscribed in the same segment (B. III., Pr. 15, Cor. 1); therefore the angle C is equal to the angle B. The triangles are consequently similar; and hence (Pr. 19)

$$AE : DE :: EC : EB,$$

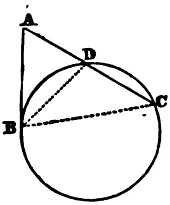
or (B. II., Pr. 1) $\quad AE \times EB :: DE \times EC.$

Therefore, if two chords, etc.

Cor. The parts of two chords which cut each other in a circle are reciprocally proportional; that is, AE : DE :: EC : EB.

PROPOSITION XXIX. THEOREM.

If from a point without a circle a tangent and a secant be drawn, the square of the tangent will be equivalent to the rectangle contained by the whole secant and its external segment.



Let A be any point without the circle BCD, and let AB be a tangent, and AC a secant; then the square of AB is equivalent to the rectangle AD × AC.

Join BD and BC. Then the triangles ABD and ABC are similar, because they have the angle A in common; also, the angle ABD, formed by a tangent and a chord, is measured by half the arc BD (B. III., Pr. 16), and the angle C is measured by half the same arc; therefore the angle ABD is equal to C, and the two triangles ABD, ABC are mutually equiangular, and consequently similar; therefore (Pr. 19)

$$AC : AB :: AB : AD;$$

whence (B. II., Pr. I.) $AB^2 = AC \times AD$.

Therefore, if from a point, etc.

Cor. 1. If from a point without a circle a tangent and a secant be drawn, the tangent will be a mean proportional between the whole secant and its external segment.

Cor. 2. If from a point without a circle two secants be drawn, the rectangle contained by either secant and its external segment will be equivalent to the rectangle contained by the other secant and its external segment; for each of these rectangles is equivalent to the square of the tangent from the same point.

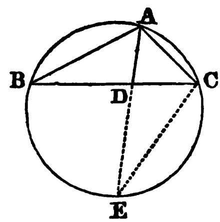
Cor. 3. If from a point without a circle two secants be drawn, the whole secants will be reciprocally proportional to their external segments.

PROPOSITION XXX. THEOREM.

If an angle of a triangle be bisected by a line which cuts the base, the rectangle contained by the sides of the triangle is equivalent to the rectangle contained by the segments of the base, together with the square of the bisecting line.

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; the rectangle BA × AC is equivalent to BD × DC, together with the square of AD.

Describe the circle ACEB about the triangle, and produce AD



to meet the circumference in E, and join EC. Then, because the angle BAD is equal to the angle CAE, and the angle ABD to the angle AEC, for they are in the same segment (B. III., Pr. 15, Cor. 1), the triangles ABD, AEC are mutually equiangular and similar; therefore (Pr. 19)

$$BA : AD :: AE : AC;$$

consequently (B. II., Pr. 1),

$$BA \times AC = AD \times AE.$$

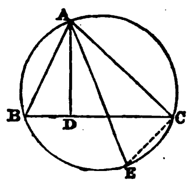
But $AE = AD + DE$; and multiplying each of these equals by AD, we have (Pr. 3) $AD \times AE = AD^2 + AD \times DE$. But $AD \times DE = BD \times DC$ (Pr. 28); hence

$$BA \times AC = BD \times DC + AD^2.$$

Therefore, if an angle, etc.

PROPOSITION XXXI. THEOREM.

In any triangle, the rectangle contained by two sides is equivalent to the rectangle contained by the diameter of the circumscribed circle, and the perpendicular let fall upon the third side from the vertex of the opposite angle.



In the triangle ABC, let AD be drawn perpendicular to BC, and let AE be the diameter of the circumscribed circle; then

$$AB \times AC = AE \times AD.$$

For, drawing EC, the right angle ADB is equal to the angle ACE in a semicircle (B. III., Pr. 15), and the angle B to the angle E in the same segment (B. III., Pr. 15); therefore the triangles ABD, AEC are similar, and we have

$$AB : AE :: AD : AC;$$

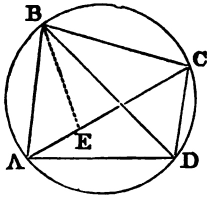
and hence $AB \times AC = AE \times AD.$

Therefore, in any triangle, etc.

PROPOSITION XXXII. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equivalent to the sum of the rectangles of the opposite sides.

Let ABCD be any quadrilateral inscribed in a circle, and let the diagonals AC, BD be drawn; the rectangle $AC \times BD$ is equivalent the sum of the two rectangles $AD \times BC$ and $AB \times CD$.



Draw the straight line BE, making the angle ABE equal to the angle DBC. To each of these equals add the angle EBD; then will the angle ABD be equal to the angle EBC. But the angle BDA is equal to the angle BCE, because they are both in the same segment (B. III., Pr. 15, Cor. 1); hence the triangle ABD is equiangular and similar to the triangle EBC. Therefore we have AD : BD :: CE : BC;

and, consequently, $AD \times BC = BD \times CE$.

Again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC, being angles in the same segment, the triangle ABE is similar to the triangle DBC; and hence

AB : AE :: BD : CD;

consequently, $AB \times CD = BD \times AE$.

Adding together these two results, we obtain

$$AD \times BC + AB \times CD = BD \times CE + BD \times AE,$$

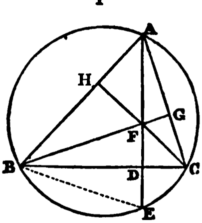
which equals $BD \times (CE + AE)$, or $BD \times AC$.

Therefore the rectangle, etc.

PROPOSITION XXXIII. THEOREM.

The perpendiculars drawn from the three angles of any triangle to the opposite sides intersect one another in the same point.

If the triangle be right angled, it is plain that all the perpendiculars pass through the right angle. But if it be not right angled, let ABC be the triangle, and about it describe a circle. Let B and C be two acute angles; draw ADE perpendicular to BC, meeting the circumference in E. Make DF equal to DE; join BF, and produce it, if necessary, to cut AC, or AC produced, in G; then BG is perpendicular to AC.



Join BE; and, because FD is equal to DE, the angles at D are right angles, and DB is common to the two triangles FDB, EDB, the angle FBD is equal to EBD (B. I., Pr. 6). But CAD, EBD are also equal, because they are in the same segment (B. III., Pr. 15). Therefore CAD is equal to FBD or GBC. But the angle ACB is common to the two triangles ACD, BCG, and therefore the remaining angles ADC, BGC are equal (B. I., Pr. 27). But ADC is a right angle; therefore also BGC is a right angle, and BG is perpendicular to AC.

In the same manner, it may be shown that the straight line CH, drawn through C and F, is perpendicular to AB, and the three perpendiculars all pass through F. Therefore the perpendiculars, etc.

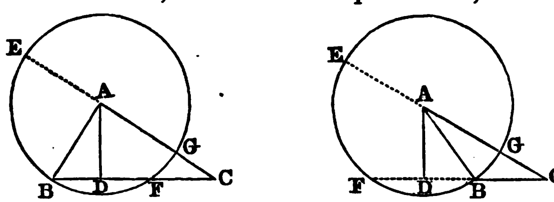
PROPOSITION XXXIV. THEOREM.

If from any angle of a triangle a perpendicular be drawn to the opposite side or base, the rectangle contained by the sum and difference of the other two sides is equivalent to the rectangle contained by the sum and difference of the segments of the base.

Let ABC be any triangle, and let AD be a perpendicular drawn from the angle A on the base BC; then

$$(AC+AB)\times(AC-AB)=(CD+DB)\times(CD-DB).$$

From A as a centre, with a radius equal to AB, the shorter of



the two sides, describe a circumference BFE. Produce AC to meet the circumference in E, and CB, if necessary, to meet it in F.

Then, because AB is equal to AE or AG, CE=AC+AB, the sum of the sides; and CG=AC−AB, the difference of the sides. Also, because BD is equal to DF (B. III., Pr. 6), when the perpendicular falls within the triangle, CF=CD−DF=CD−DB, the difference of the segments of the base. But when the perpendicular falls without the triangle, CF=CD+DF=CD+DB, the sum of the segments of the base.

Now, in either case, the rectangle $CE\times CG$ is equivalent to $CB\times CF$ (Pr. 29, Cor. 2); that is,

$$(AC+AB)\times(AC-AB)=(CD+DB)\times(CD-DB).$$

Therefore, if from any angle, etc.

Cor. If we reduce the preceding equation to a proportion (B. II., Pr. 2), we shall have

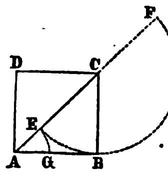
$$CD+DB : AC+AB :: AC-AB : CD-DB;$$

that is, the sum of the segments of the base is to the sum of the two other sides as the difference of the latter is to the difference of the segments of the base.

PROPOSITION XXXV. THEOREM.

The diagonal and side of a square have no common measure.

Let ABCD be a square, and AC its diagonal; AC and AB have no common measure.



In order to find the common measure, if there is one, we must apply CB to CA as often as it is contained in it. For this purpose, from the centre C, with a radius CB, describe the semicircle EBF. We perceive that CB is contained once in AC, with a remainder AE, which remainder must be compared with BC, or its equal AB.

Now, since the angle ABC is a right angle, AB is a tangent to the circumference; and AE : AB :: AB : AF (Prop. 29, Cor. 1). Instead, therefore, of comparing AE with AB, we may substitute the equal ratio of AB to AF. But AB is contained twice in AF, with a remainder AE, which must be again compared with AB. Instead, however, of comparing AE with AB, we may again employ the equal ratio of AB to AF. Hence at each operation we are obliged to compare AB with AF, which leaves a remainder AE; from which we see that the process will never terminate, and therefore there is no common measure between the diagonal and side of a square; that is, there is no line, however small, which is contained an exact number of times in each of them.

The same conclusion was arrived at in Pr. 11, Cor. 3, by a different method.

BOOK V.

PROBLEMS.

Hitherto we have assumed the possibility of constructing our figures, although the methods of constructing them have not yet been explained. For the purpose of discovering the properties of figures, we are at liberty to suppose any figure to be constructed, or any line to be drawn, whose existence does not involve an impossibility. We now proceed to show how the figures employed in these demonstrations may be constructed.

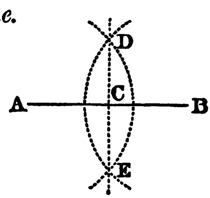
All the constructions of Elementary Geometry are supposed to be effected by means of straight lines and circumferences of circles, these being the only lines treated of in the Elements. A straight line is supposed to be drawn by means of a ruler, and a circle by the aid of a pair of compasses. By means of other curves, which are treated of in Higher Geometry, more difficult problems may be constructed, such as to divide any angle into three equal parts; to find two mean proportionals between two given lines, etc.

Postulates.

1. A straight line may be drawn from any one point to any other point.
2. A terminated straight line may be produced to any length in a straight line.
3. From the greater of two straight lines, a part may be cut off equal to the less.
4. A circumference may be described from any centre and with any radius.

PROBLEM I.

To bisect a given straight line.



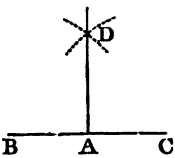
Let AB be the given straight line which it is required to bisect.

From the centre A, with a radius greater than the half of AB, describe an arc of a circle (Postulate 4); and from the centre B, with the same radius, describe another arc intersecting the former in D and E. Through the points of intersection draw the straight line DE (Post. 1); it will bisect AB in C.

For the two points D and E, being each equally distant from the extremities A and B, must both lie in the perpendicular, raised from the middle point of AB (B. I., Pr. 18, Cor.). Therefore the line DE divides the line AB into two equal parts at the point C.

PROBLEM II.

To draw a perpendicular to a straight line from a given point in that line.



Let BC be the given straight line, and A the point given in it; it is required to draw a straight line perpendicular to BC through the given point A.

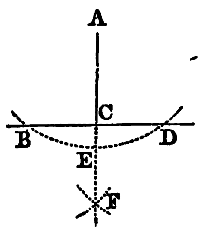
In the straight line BC take any point B, and make AC equal to AB (Post. 3). From B as a centre, with a radius greater than BA, describe an arc of a circle (Post. 4); and from C as a centre, with the same radius, describe another arc intersecting the former in D. Draw AD (Post. 1), and it will be the perpendicular required.

For the points A and D, being equally distant from B and C, must be in a line perpendicular to the middle of BC (B. I., Pr. 18, Cor.). Therefore AD has been drawn perpendicular to BC from the point A.

Scholium. The same construction serves to make a right angle BAD at a given point A, on a given line BC.

PROBLEM III.

To draw a perpendicular to a straight line from a given point without it.



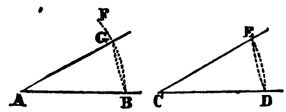
Let BD be a straight line of unlimited length, and let A be a given point without it. It is required to draw a perpendicular to BD from the point A.

Take any point E upon the other side of BD, and from the centre A, with the radius AE, describe the arc BD, cutting the line BCD in the two points B and D. From the points B and D as centres, describe two arcs, as in Prob. 2, cutting each other in F. Join AF, and it will be the perpendicular required.

For the two points A and F are each equally distant from the points B and D; therefore the line AF has been drawn perpendicular to BD (B. I., Pr. 18, Cor.) from the given point A.

PROBLEM IV.

At a given point in a straight line, to make an angle equal to a given angle.



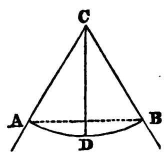
Let AB be the given straight line, A the given point in it, and C the given angle; it is required to make an angle at the point A, in the straight line AB, that shall be equal to the given angle C.

With C as centre, and any radius, describe an arc DE terminating in the sides of the angle; and from the point A as a centre, with the same radius, describe the indefinite arc BF. Draw the chord DE; and from B as a centre, with a radius equal to DE, describe an arc cutting the arc BF in G. Draw AG, and the angle BAG will be equal to the given angle C.

For the two arcs BG, DE are described with equal radii, and they have equal chords; they are, therefore, equal (B. III., Pr. 3). But equal arcs subtend equal angles (B. III., Pr. 4), and hence the angle A has been made equal to the given angle C.

PROBLEM V.

To bisect a given arc or a given angle.



First. Let ADB be the given arc which it is required to bisect.

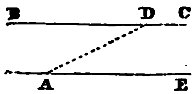
Draw the chord AB, and from the centre C draw CD perpendicular to AB (Prob. 3); it will bisect the arc ADB (B. III., Pr. 6), because CD is a radius perpendicular to a chord.

Secondly. Let ACB be an angle which it is required to bisect. From C as centre, with any radius, describe an arc AB; and, by the first case, draw the line CD bisecting the arc ADB. The line CD will also bisect the angle ACB. For the angles ACD, BCD are equal, being subtended by the equal arcs AD, DB (B. III., Pr. 4).

Scholium. By the same construction, each of the halves AD, DB may be bisected; and thus by successive bisections an arc or angle may be divided into four equal parts, into eight, sixteen, etc.

PROBLEM VI.

Through a given point to draw a straight line parallel to a given line.



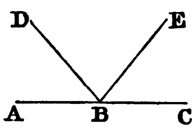
Let A be the given point, and BC the given straight line; it is required to draw through the point A a straight line parallel to BC.

In BC take any point D, and join AD. Then, at the point A, in the straight line AD, make the angle DAE equal to the angle ADB (Prob. 4).

Now, because the straight line AD, which meets the two straight lines BC, AE, makes the alternate angles ADB, DAE equal to each other, AE is parallel to BC (B. I., Pr. 22). Therefore the straight line AE has been drawn through the point A, parallel to the given line BC.

PROBLEM VII.

Two angles of a triangle being given, to find the third angle.

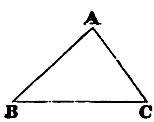


The three angles of every triangle are together equal to two right angles (B. I., Pr. 27). Therefore, draw the indefinite line ABC. At the point B make the angle ABD equal to one of the given angles (Prob. 4), and the angle DBE equal to the other given angle; then will the angle EBC be equal to the third angle of the triangle.

For the three angles ABD, DBE, EBC are together equal to two right angles (B. I., Pr. 2), which is the sum of all the angles of the triangle.

PROBLEM VIII.

Two sides and the included angle of a triangle being given, to construct the triangle.

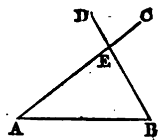


Draw the straight line BC equal to one of the given sides. At the point B make the angle ABC equal to the given angle (Prob. 4), and take AB equal to the other given side. Join AC, and ABC will be the given triangle required. For its sides AB, BC are made equal to the given sides, and the included angle B is made equal to the given angle.

PROBLEM IX.

One side and two angles of a triangle being given, to construct the triangle.

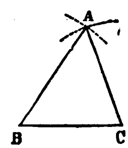
The two given angles will either be both adjacent to the given side, or one adjacent and the other opposite. In the latter case, find the third angle (Prob. 7), and then the two adjacent angles will be known.



Draw the straight line AB equal to the given side; at the point A make the angle BAC equal to one of the adjacent angles, and at the point B make the angle ABD equal to the other adjacent angle. The two lines AC, BD will cut each other in E, and ABE will be the triangle required; for its side AB is equal to the given side, and two of its angles are equal to the given angles.

PROBLEM X.

The three sides of a triangle being given, to construct the triangle.



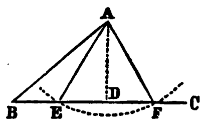
Draw the straight line BC equal to one of the given sides. From the point B as a centre, with a radius equal to one of the other sides, describe an arc of a circle; and from the point C as a centre, with a radius equal to the third side, describe another arc cutting the former in A. Draw AB, AC; then will ABC be the triangle required, because its three sides are equal to the three given straight lines.

Scholium. If one of the given lines was equal to or greater than the sum of the other two, the arcs would not intersect each other, and the problem would be impossible; but the solution will always be possible when each side is less than the sum of the other two.

PROBLEM XI.

Two sides of a triangle and the angle opposite to one of them being given, to construct the triangle.

Draw an indefinite straight line BC. At the point B make the angle ABC equal to the given angle, and make BA equal to that side which is adjacent to the given angle. Then from A as a centre, with a radius equal to the other side, describe an arc cutting BC in the points E and F. Join AE, AF.



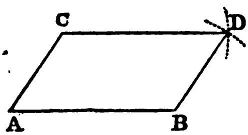
If the points E and F both fall on the same side of the angle B, each of the triangles ABE, ABF will satisfy the given conditions; but if they fall on different sides of B, only one of them, as ABF, will satisfy the conditions, and therefore this will be the triangle required.

If the points E and F coincide with one another, which will happen when AEB is a right angle, there will be only one triangle, ABD, which is the triangle required.

Scholium. If the side opposite the given angle were less than the perpendicular let fall from A upon BC, the problem would be impossible.

PROBLEM XII.

Two adjacent sides of a parallelogram and their included angle being given, to construct the parallelogram.

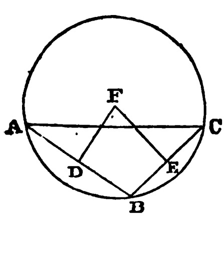


Draw the straight line AB equal to one of the given sides. At the point A make the angle BAC equal to the given angle, and take AC equal to the other given side. From the point C as a centre, with a radius equal to AB, describe an arc, and from the point B as a centre, with a radius equal to AC, describe another arc intersecting the former in D. Draw BD, CD; then will ABDC be the parallelogram required. For, by construction, the opposite sides are equal; therefore the figure is a parallelogram (B. I., Pr. 31), and it is formed with the given sides and the given angle.

Cor. If the given angle is a right angle, the figure will be a rectangle; and if, at the same time, the sides are equal, it will be a square.

PROBLEM XIII.

To find the centre of a given circumference or of a given arc.



Let ABC be the given circumference or arc; it is required to find its centre.

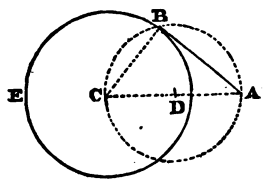
Take any three points in the arc, as A, B, C, and join AB, BC. Bisect AB in D (Prob. I.), and through D draw DF perpendicular to AB (Prob. 2). In the same manner, draw EF perpendicular to BC at its middle point. The perpendiculars DF, EF will meet in a point F equally distant from the points A, B, and C (B. III., Pr. 7), and therefore F is the centre of the circle.

Scholium. By the same construction, a circumference may be made to pass through three given points, A, B, C; and also, a circle may be described about a triangle.

PROBLEM XIV.

Through a given point, to draw a tangent to a given circumference.

First. Let the given point A be without the circle BDE; it is required to draw a tangent to the circumference through the point A.



Find the centre of the circle C, and join AC. Bisect AC in D; and, with D as a centre, and a radius equal to AD, describe a circumference intersecting the given circumference in B. Draw AB, and it will be the tangent required.

Draw the radius CB. The angle ABC, being inscribed in a semicircle, is a right angle (B. III., Pr. 15, Cor. 3). Hence the line AB is a perpendicular at the extremity of the radius CB; it is, therefore, a tangent to the circumference (B. III., Pr. 9).

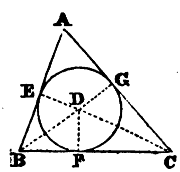
Secondly. If the given point is in the circumference of the circle, as the point B, draw the radius BC, and make BA perpendicular to BC. BA will be the tangent required (B. III., Pr. 9).

Scholium. When the point A lies without the circle, two tangents may always be drawn; for the circumference, whose centre is D, intersects the given circumference in two points.

PROBLEM XV.

To inscribe a circle in a given triangle.

Let ABC be the given triangle; it is required to inscribe a circle in it.



Bisect any two angles B and C by the lines BD, CD, meeting each other in the point D. From the point of intersection, let fall the perpendiculars DE, DF, DG on the three sides of the triangle; these perpendiculars will all be equal.

For, by construction, the angle EBD is equal to the angle FBD; the right angle DEB is equal to the right angle DFB; hence the third angle BDE is equal to the third angle BDF (B.

I., Pr. 27, Cor. 2). Moreover, the side BD is common to the two triangles BDE, BDF, and the angles adjacent to this side are equal; therefore the two triangles are equal, and DE is equal to DF.

For the same reason, DG is equal to DF. Therefore the three straight lines DE, DF, DG are equal to each other; and, if a circumference be described from the centre D, with a radius equal to DE, it will pass through the extremities of the lines DF, DG. It will also touch the straight lines AB, BC, CA, because the angles at the points E, F, G are right angles (B. III., Pr. 9). Therefore the circle EFG is inscribed in the triangle ABC (B. III., Def. 12).

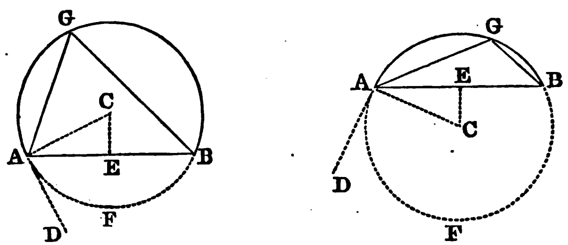
Scholium. The three lines which bisect the angles of a triangle all meet in the same point, viz., the centre of the inscribed circle.

PROBLEM XVI.

Upon a given straight line, to describe a segment of a circle which shall contain a given angle.

Let AB be the given straight line, upon which it is required to describe a segment of a circle containing a given angle.

At the point A, in the straight line AB, make the angle BAD equal to the given angle; and from the point A draw AC perpen-



dicular to AD. Bisect AB in E, and from E draw EC perpendicular to AB. From the point C, where these perpendiculars meet, with a radius equal to AC, describe a circle. Then will AGB be the segment required.

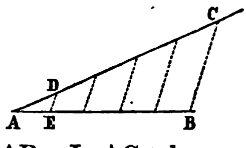
For, since AD is a perpendicular at the extremity of the radius AC, it is a tangent (B. III., Pr. 9), and the angle BAD is measured by half the arc AFB (B. III., Pr. 16). Also, the angle AGB, being an inscribed angle, is measured by half the same arc AFB; hence the angle AGB is equal to the angle BAD, which, by construction, is equal to the given angle. Therefore any angle inscribed in the segment AGB is equal to the given angle.

Scholium. If the given angle was a right angle, the required segment would be a semicircle, described on AB as a diameter.

PROBLEM XVII.

To divide a given straight line into any number of equal parts, or into parts proportional to given lines.

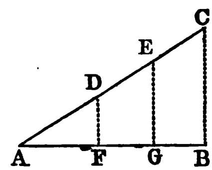
First. Let AB be the given straight line which it is proposed to divide into any number of equal parts, as, for example, five.



From the point A draw the indefinite straight line AC, making any angle with AB. In AC take any point D, and set off AD five times upon AC. Join BC, and draw DE parallel to it; then is AE the fifth part of AB.

For, since ED is parallel to BC, we have AE : AB :: AD : AC (B. IV., Pr. 16). But AD is the fifth part of AC; therefore AE is the fifth part of AB.

Secondly. Let AB be the given straight line, and AC a divided line; it is required to divide AB similarly to AC. Suppose AC to be divided in the points D and E. Place AB, AC so as to contain any angle; join BC, and through the points D, E draw DF, EG parallel to BC. The line AB will be divided into parts proportional to those of AC.

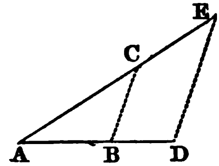


For, because DF and EG are both parallel to CB, we have AD : AF :: DE : FG :: EC : GB (B. IV., Pr. 16, Cor. 2).

PROBLEM XVIII.

To find a fourth proportional to three given lines.

From any point A draw two straight lines AD, AE, containing any angle DAE, and make AB, BD, AC respectively equal to the proposed lines. Join B, C, and through D draw DE parallel to BC; then will CE be the fourth proportional required.



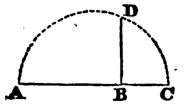
For, because BC is parallel to DE, we have

AB : BD :: AC : CE (B. IV., Pr. 16).

Cor. In the same manner may be found a third proportional to two given lines A and B, for this will be the same as a fourth proportional to the three lines A, B, B.

PROBLEM XIX.

To find a mean proportional between two given lines.



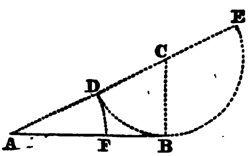
Let AB, BC be the two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line; upon AC describe the semicircle ADC, and from the point B draw BD perpendicular to AC. Then will BD be the mean proportional required.

For the perpendicular BD, let fall from a point in the circumference upon the diameter, is a mean proportional between the two segments of the diameter AB, BC (B. IV., Pr. 23, Cor.), and these segments are equal to the two given lines.

PROBLEM XX.

To divide a given line into two parts such that the greater part may be a mean proportional between the whole line and the other part.



Let AB be the given straight line; it is required to divide it into two parts at the point F, such that AB : AF :: AF : FB.

At the extremity of the line AB erect the perpendicular BC, and make it equal to the half of AB. From C as a centre, with a radius equal to CB, describe a circle. Draw AC cutting the circumference in D, and make AF equal to AD. The line AB will be divided in the point F in the manner required.

For, since AB is a perpendicular to the radius CB at its extremity, it is a tangent (B. III., Pr. 9); and, if we produce AC to E, we shall have AE : AB :: AB : AD (B. IV., Pr. 29). Therefore, by division (B. II., Pr. 7), AE−AB : AB :: AB−AD : AD. But, by construction, AB is equal to DE, and therefore AE−AB is equal to AD or AF, and AB−AD is equal to FB. Hence AF : AB :: FB : AD or AF; and, consequently, by inversion (B. II., Pr. 5),

AB : AF :: AF : FB.

Schol. 1. The line AB is said to be divided in *extreme and mean ratio*. An example of its use may be seen in Book VI., Pr. 5.

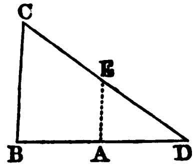
Schol. 2. Let $AB=a$; $AF=AD=AC-CD$. $CD=\frac{a}{2}$.

But $$AC=\sqrt{AB^2+BC^2}=\sqrt{a^2+\frac{a^2}{4}}=\sqrt{\frac{5a^2}{4}}=\frac{a}{2}\sqrt{5}.$$

Therefore $AF=\frac{a}{2}\sqrt{5}-\frac{a}{2}=\frac{a}{2}\times(\sqrt{5}-1).$

PROBLEM XXI.

Through a given point in a given angle, to draw a straight line so that the parts included between the point and the sides of the angle may be equal.



Let A be the given point, and BCD the given angle; it is required to draw through A a line BD, so that BA may be equal to AD.

Through the point A draw AE parallel to BC, and take DE equal to CE. Through the points D and A draw the line BAD; it will be the line required.

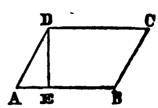
For, because AE is parallel to BC, we have (B. IV., Pr. 16)

DE : EC :: DA : AB.

But DE is equal to EC; therefore DA is equal to AB.

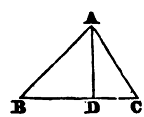
PROBLEM XXII.

To construct a square that shall be equivalent to a given parallelogram or to a given triangle.



First. Let ABCD be the given parallelogram, AB its base, and DE its altitude. Find a mean proportional between AB and DE (Prob. 19), and represent it by X; the square described on X will be equivalent to the given parallelogram ABCD.

For, by construction, AB : X :: X : DE; hence X^2 is equal to AB×DE (B. II., Pr. 1, Cor.). But AB×DE is the measure of the parallelogram, and X^2 is the measure of the square. Therefore the square described on X is equivalent to the given parallelogram ABCD.

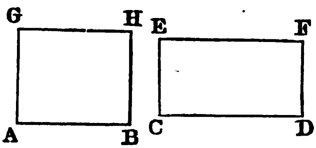


Secondly. Let ABC be the given triangle, BC its base, and AD its altitude. Find a mean proportional between BC and the half of AD, and represent it by Y. Then will the square described on Y be equivalent to the triangle ABC.

For, by construction, BC : Y :: Y : $\frac{1}{2}$ AD; hence Y^2 is equivalent to BC×$\frac{1}{2}$ AD. But BC×$\frac{1}{2}$ AD is the measure of the triangle ABC; therefore the square described on Y is equivalent to the triangle ABC.

PROBLEM XXIII.

Upon a given straight line, to construct a rectangle equivalent to a given rectangle.



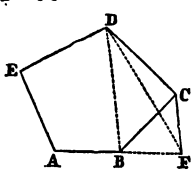
Let AB be the given straight line, and CDFE the given rectangle. It is required to construct on the line AB a rectangle equivalent to CDFE.

Find a fourth proportional (Prob. 18) to the three lines AB, CD, CE, and let AG be that fourth proportional. The rectangle constructed on the lines AB, AG will be equivalent to CDFE.

For, because AB : CD :: CE : AG (B. II., Pr. 1), AB × AG = CD × CE. Therefore the rectangle ABHG is equivalent to the rectangle CDFE, and it is constructed upon the given line AB.

PROBLEM XXIV.

To construct a triangle which shall be equivalent to a given polygon.



Let ABCDE be the given polygon; it is required to construct a triangle equivalent to it.

Draw the diagonal BD, cutting off the triangle BCD. Through the point C draw CF parallel to DB, meeting AB produced in F. Join DF, and the polygon AFDE will be equivalent to the polygon ABCDE.

For the triangles BFD, BCD, being upon the same base BD, and between the same parallels BD, FC, are equivalent. To each of these equals add the polygon ABDE; then will the polygon AFDE be equivalent to the polygon ABCDE; that is, we have found a polygon equivalent to the given polygon, and having the number of its sides diminished by one.

In the same manner, a polygon may be found equivalent to AFDE, and having the number of its sides diminished by one; and, by continuing the process, the number of sides may be at last reduced to three, and a triangle be thus obtained equivalent to the given polygon.

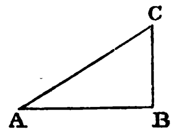
Scholium. By Prob. 22, any triangle may be changed into an equivalent square, and hence a square can always be found equivalent to any given polygon. This operation is called *squaring* the polygon, or finding its *quadrature.*

The problem of the *quadrature of the circle* consists in finding a square equivalent to a circle whose diameter is given.

PROBLEM XXV.

To construct a square equivalent to the sum or difference of two given squares.

First. To make a square equivalent to the sum of two given squares, draw two indefinite lines AB, BC at right angles to each other. Take AB equal to the side of one of the given squares, and BC equal to the side of the other. Join AC; it will be the side of the required square.



For the triangle ABC, being right-angled at B, the square on AC will be equivalent to the sum of the squares upon AB and BC (B. IV., Pr. 11).

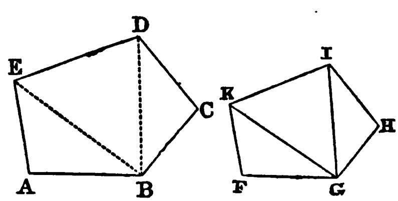
Secondly. To make a square equivalent to the difference of two given squares, draw the lines AB, BC at right angles to each other, and take AB equal to the side of the less square. Then, from A as a centre, with a radius equal to the other side of the square, describe an arc intersecting BC in C; BC will be the side of the square required, because the square of BC is equivalent to the difference of the squares of AC and AB (B. IV., Pr. 11, Cor. 1).

Scholium. In the same manner, a square may be made equivalent to the sum of three or more given squares; for the same construction which reduces two of them to one will reduce three of them to two, and these two to one.

PROBLEM XXVI.

Upon a given straight line, to construct a polygon similar to a given polygon.

Let ABCDE be the given polygon, and FG be the given straight line; it is required, upon the line FG, to construct a polygon similar to ABCDE.



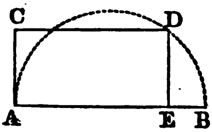
Draw the diagonals BD, BE. At the point F, in the straight line FG, make the angle GFK equal to the angle BAE, and at the point G make the angle FGK equal to the angle ABE. The lines FK, GK will intersect in K, and FGK will be a triangle similar to ABE.

In the same manner, on GK construct the triangle GKI similar to BED, and on GI construct the triangle GIH similar to BDC. The polygon FGHIK will be the polygon required. For these two polygons are composed of the same number of triangles, which are similar to each other, and similarly situated; therefore the polygons are similar (B. IV., Pr. 26, Cor.).

PROBLEM XXVII.

Given the area of a rectangle and the sum of two adjacent sides, to construct the rectangle.

Let AB be a straight line equal to the sum of the sides of the required rectangle.



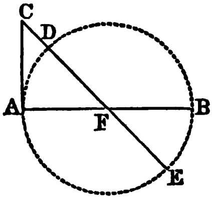
Upon AB as a diameter, describe a semicircle. At the point A erect the perpendicular AC, and make it equal to the side of a square having the given area. Through C draw the line CD parallel to AB, and let it meet the circumference in D, and from D draw DE perpendicular to AB. Then will AE and EB be the sides of the rectangle required.

For (B. IV., Pr. 23, Cor.) the rectangle AE×EB is equivalent to the square of DE or CA, which is, by construction, equivalent to the given area. Also, the sum of the sides AE and EB is equal to the given line AB.

Scholium. The side of the square having the given area must not be greater than the half of AB, for in that case the line CD would not meet the circumference ADB.

PROBLEM XXVIII.

Given the area of a rectangle and the difference of two adjacent sides, to construct the rectangle.



Let AB be a straight line equal to the difference of the sides of the required rectangle.

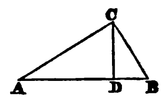
Upon AB as a diameter describe a circle, and at the extremity of the diameter draw the tangent AC equal to the side of a square having the given area. Through the point C and the centre F draw the secant CE; then will CD, CE be the adjacent sides of the rectangle required.

For (B. IV., Pr. 29) the rectangle CD×CE is equivalent to the square of AC, which is, by construction, equivalent to the given area. Also, the difference of the lines CE, CD is equal to DE or AB.

PROBLEM XXIX.

To find two straight lines having the same ratio as the areas of two given polygons.

Since any two polygons can always be transformed into squares, this problem requires us to find two straight lines in the same ratio as two given squares.



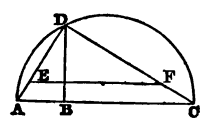
Draw two lines, AC, BC, at right angles with each other, and make AC equal to a side of one of the given squares, and BC equal to a side of the other given square. Join AB, and from C draw CD perpendicular to AB. Then (B. IV., Pr. 11, Cor. 2) we have

$$AD : DB :: AC^2 : CB^2.$$

Therefore AD, DB are in the ratio of the areas of the given polygons.

PROBLEM XXX.

To find a square which shall be to a given square in the ratio of two given straight lines.



Upon a line of indefinite length, take AB equal to one of the given lines, and BC equal to the other line. Upon AC as a diameter describe a semicircle, and at B erect the perpendicular BD, cutting the circumference in D. Join DA, DC; and upon DA, or DA produced, take DE equal to a side of the given square. Through the point E draw EF parallel to AC; then DF is a side of the required square.

For, because EF is parallel to AC (B. IV., Pr. 16), we have

$$DE : DF :: DA : DC;$$

whence (B. II., Pr. 11) $DE^2 : DF^2 :: DA^2 : DC^2$.

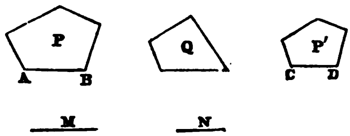
Also, because ADC is a right-angled triangle (B. IV., Pr. 11), we have $DA^2 : DC^2 :: AB : BC$.

Hence $DE^2 : DF^2 :: AB : BC$.

Therefore the square described on DE is to the square described on DF in the ratio of the two given straight lines.

PROBLEM XXXI.

To construct a polygon similar to one given polygon, and equivalent to another given polygon.



Let P and Q be two given polygons. It is required to construct a polygon similar to P, and equivalent to Q.

Find M, the side of a square equivalent to P (Pr. 24, Schol.), and N, the side of a square equivalent to Q. Let AB be one side of P, and let CD be a fourth proportional to the three lines M, N, AB. Upon the side CD homologous to AB, construct the polygon P′ similar to P (Pr. 26); it will be equivalent to the polygon Q.

For (B. IV., Pr. 27) $P : P' : AB^2 : CD^2$.

But, by construction, $AB : CD :: M : N$,

or $AB^2 : CD^2 :: M^2 : N^2$.

Hence $P : P' :: M^2 : N^2$.

But, by construction,

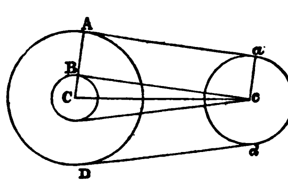
$M^2 = P$, and $N^2 = Q$;

therefore $P : P' :: P : Q$.

Hence $P' = Q$. Therefore the polygon P′ is similar to the polygon P, and equivalent to the polygon Q.

PROBLEM XXXII.

To draw a common tangent to two given circles.



Let C and *c* be the centres of the two given circles. With C as a centre, and a radius CB equal to the difference of the two given radii CA and *ca*, describe a circumference, and from *c* draw a straight line touching the circle CB in the point B (Prob. 14). Join CB, and produce it to meet the given circumference in A. Draw *ca* parallel to CA, and join A*a*. Then A*a* is the common tangent to the two given circles.

For, by the construction, $BC = AC - ac$; and also $BC = AC - AB$; whence $ac = AB$, and AB*ca* is a parallelogram (B. I., Pr. 32). But the angle B is a right angle; therefore this parallelogram is a rectangle, and the angles at A and *a* are right angles. Hence A*a* is a tangent to both circles.

Since two tangents can be drawn from *c* to the circle BC, there are two common tangents to the given circles, viz., A*a* and D*d*.

Scholium. Two other tangents can be drawn to the two given circles, and their points of contact will lie upon opposite sides of the line joining the centres. For this purpose CB must be taken equal to the *sum* of the given radii.

BOOK VI.

REGULAR POLYGONS, AND THE AREA OF THE CIRCLE.

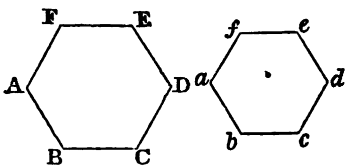
Definition.

A *regular polygon* is a polygon which is both equiangular and equilateral.

An equilateral triangle is a regular polygon of three sides; a square is one of four.

PROPOSITION I. THEOREM.

Regular polygons of the same number of sides are similar figures.



Let ABCDEF, *abcdef* be two regular polygons of the same number of sides; then will they be similar figures.

For, since the two polygons have the same number of sides, they must have the same number of angles. Moreover, the sum of the angles of the one polygon is equal to the sum of the angles of the other (B. I., Pr. 28); and, since the polygons are each equiangular, it follows that the angle A is the same part of the sum of the angles A, B, C, D, E, F, that the angle *a* is of the sum of the angles *a*, *b*, *c*, *d*, *e*, *f*. Therefore the two angles A and *a* are equal to each other. The same is true of the angles B and *b*, C and *c*, etc.

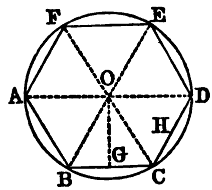
Moreover, since the polygons are regular, the sides AB, BC, CD, etc., are equal to each other (Def.); so, also, are the sides *ab*, *bc*, *cd*, etc. Therefore AB : *ab* :: BC : *bc* :: CD : *cd*, etc. Hence the two polygons have their angles equal, and their homologous sides proportional; they are consequently similar (B. IV., Def. 4). Therefore, regular polygons, etc.

Cor. The perimeters of two regular polygons of the same number of sides are to each other as their homologous sides, and their areas are as the squares of those sides (B. IV., Pr. 27).

Scholium. The magnitude of the angles of a regular polygon is determined by the number of its sides.

PROPOSITION II. THEOREM.

A circle may be described about any regular polygon, and a circle may also be inscribed within it.



Let ABCDEF be any regular polygon; a circle may be described about it, and another may be inscribed within it.

Bisect the angles FAB, ABC by the straight lines AO, BO, and, from the point O in which they meet, draw the lines OC, OD, OE, OF to the other angles of the polygon.

Then, because in the triangles OBA, OBC, AB is, by hypothesis, equal to BC, BO is common to the two triangles, and the included angles OBA, OBC are, by construction, equal to each other; therefore the angle OAB is equal to the angle OCB. But OAB is, by construction, the half of FAB, and FAB is, by hypothesis, equal to DCB; therefore OCB is the half of DCB; that is, the angle BCD is bisected by the line OC. In the same manner, it may be proved that the angles CDE, DEF, EFA are bisected by the straight lines OD, OE, OF.

Now, because the angles OAB, OBA, being halves of equal angles, are equal to each other, OA is equal to OB (B. I., Pr. 11). For the same reason, OC, OD, OE, OF are each of them equal to OA. Therefore a circumference described from the centre O, with a radius equal to OA, will pass through each of the points B, C, D, E, F, and be described about the polygon.

Secondly. A circle may be inscribed within the polygon ABCDEF.

For the sides AB, BC, CD, etc., are equal chords of the same circle; hence they are equally distant from the centre O (B. III., Pr. 8); that is, the perpendiculars OG, OH, etc., are all equal to each other. Therefore, if from O as a centre, with a radius OG, a circumference be described, it will touch the side BC (B. III., Pr. 9), and each of the other sides of the polygon; hence the circle will be inscribed within the polygon. Therefore a circle may be described, etc.

Scholium 1. In regular polygons, the centre of the inscribed and circumscribed circles is also called the centre of the polygon; and the perpendicular from the centre upon one of the sides, that is, the radius of the inscribed circle, is called the *apothegm* of the polygon.

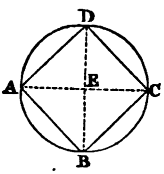
Since all the chords AB, BC, etc., are equal, the angles at the centre, AOB, BOC, etc., are equal; and the value of each may be found by dividing four right angles by the number of sides of the polygon.

The angle at the centre of the inscribed equilateral triangle is $\frac{1}{3}$ of four right angles, or 120°; the angle at the centre of the regular inscribed pentagon is $\frac{1}{5}$ of four right angles, or 72°; the angle at the centre of the regular hexagon is $\frac{1}{6}$ of four right angles, or 60°; the angle at the centre of the regular decagon is $\frac{1}{10}$ of four right angles, or 36°.

Sch. 2. To inscribe a regular polygon of any number of sides in a circle, it is only necessary to divide the circumference into the same number of equal parts; for, if the arcs are equal, the chords AB, BC, CD, etc., will be equal. Hence the triangles AOB, BOC, COD, etc., will also be equal, because they are mutually equilateral; therefore all the angles ABC, BCD, CDE, etc., will be equal, and the figure ABCDEF will be a regular polygon.

PROPOSITION III. PROBLEM.

To inscribe a square in a given circle.



Let ABCD be the given circle; it is required to inscribe a square in it.

Draw two diameters AC, BD at right angles to each other, and join AB, BC, CD, DA.

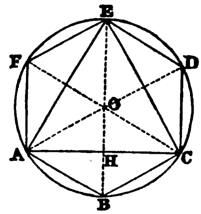
Because the angles AEB, BEC, etc., are equal, the chords AB, BC, etc., are also equal. And because the angles ABC, BCD, etc., are inscribed in semicircles, they are right angles (B. III., Pr. 15, Cor. 2). Therefore ABCD is a square, and it is inscribed in the circle ABCD.

Cor. Since the triangle AEB is right-angled and isosceles, we have the proportion AB : AE :: $\sqrt{2}$: 1 (B. IV., Pr. 11, Cor. 3); therefore *the side of the inscribed square is to the radius, as the square root of* 2 *is to unity.*

PROPOSITION IV. THEOREM.

The side of a regular hexagon is equal to the radius of the circumscribed circle.

Let ABCDEF be a regular hexagon inscribed in a circle whose centre is O; then any side, as AB, will be equal to the radius AO.



Draw the radius BO. Then the angle AOB is the sixth part of four right angles (Pr. 2, Sch. 1), or the third part of two right angles. Also, because the three angles of every triangle are equal to two right angles, the two angles OAB, OBA are together equal to two thirds of two right angles; and since AO is equal to BO, each of these angles is one third of two right angles. Hence the triangle AOB is equiangular, and AB is equal to AO. Therefore the side of a regular hexagon, etc.

Cor. To inscribe a regular hexagon in a given circle, the radius must be applied six times upon the circumference. By joining the alternate angles A, C, E, an equilateral triangle will be inscribed in the circle.

Sch. 1. In the right-angled triangle ACD we have $AC^2=AD^2-DC^2=4AO^2-AO^2=3AO^2$. Whence $AC=AO\sqrt{3}$; that is, *the side of an equilateral triangle is equal to the radius of the circumscribed circle multiplied by the square root of* 3.

Sch. 2. The area of the triangle ACE (B. IV., Pr. 6, Sch.) $=\frac{1}{2}AC\times OH$.

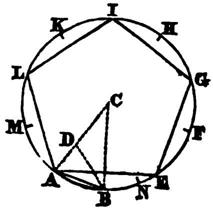
But $$OB=\frac{AC}{\sqrt{3}}=\frac{AC\sqrt{3}}{3}.$$

Therefore $$OH=\frac{AC\sqrt{3}}{6}.$$

Hence the triangle $ACE=\frac{1}{2}AC\times\frac{AC\sqrt{3}}{6}=\frac{AC^2}{4}\sqrt{3}$; that is, *the area of an equilateral triangle is equal to one fourth the square of one of its sides multiplied by the square root of three.*

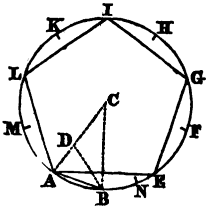
PROPOSITION V. PROBLEM.

To inscribe a regular decagon in a given circle.



Let ABH be the given circle; it is required to inscribe in it a regular decagon.

Take C the centre of the circle; draw the radius AC, and divide it in extreme and mean ratio (B. V., Pr. 20) at the point D. Make the chord AB equal to CD, the greater segment; then will AB be the side of a regular decagon inscribed in the circle.



Join BC, BD. Then, by construction, AC : CD :: CD : AD; but AB is equal to CD; therefore AC : AB :: AB : AD. Hence the triangles ACB, ABD have a common angle A included between proportional sides; they are therefore similar (B. IV., Pr. 21).

And because the triangle ACB is isosceles, the triangle ABD must also be isosceles, and AB is equal to BD. But AB was made equal to CD; hence BD is equal to CD, and the angle DBC is equal to the angle DCB. Therefore the exterior angle ADB, which is equal to the sum of DCB and DBC, must be double of DCB. But the angle ADB is equal to DAB, therefore each of the angles CAB, CBA is double of the angle ACB. Hence the sum of the three angles of the triangle ACB is five times the angle C. But these three angles are equal to two right angles (B. I., Pr. 27); therefore the angle C is the fifth part of two right angles, or the tenth part of four right angles. Hence the arc AB is one tenth of the circumference, and the chord AB is the side of a regular decagon inscribed in the circle.

Scholium. $AB = CD = \frac{AC}{2} \times (\sqrt{5} - 1)$ (see B. V., Pr. 20, Sch. 2); that is, *the side of a regular decagon is equal to half the radius of the circumscribed circle, multiplied by the square root of five, less unity.*

Cor. 1. By joining the alternate angles of the regular decagon, a regular pentagon, AEGIL, may be inscribed in the circle.

Cor. 2. By combining this Proposition with the preceding, a regular pentedecagon may be inscribed in a circle.

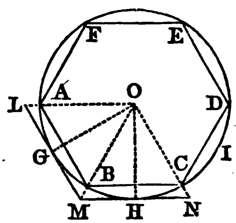
For, let AN be the side of a regular hexagon; then the arc AN will be one sixth of the whole circumference, and the arc AB one tenth of the whole circumference. Hence the arc BN will be $\frac{1}{6} - \frac{1}{10}$ or $\frac{1}{15}$, and the chord of this arc will be the side of a regular pentedecagon.

Scholium. By bisecting the arcs subtended by the sides of any polygon, another polygon of double the number of sides may be inscribed in a circle. Hence the square will enable us to inscribe regular polygons of 8, 16, 32, etc., sides; the hexagon will enable us to inscribe polygons of 12, 24, etc., sides; the decagon will enable us to inscribe polygons of 20, 40, etc., sides; and the pentedecagon, polygons of 30, 60, etc., sides.

The ancient geometricians were unacquainted with any method of inscribing in a circle regular polygons of 7, 9, 11, 13, 14, 17, etc., sides, and for a long time it was believed that these polygons could not be constructed geometrically; but Gauss, a German mathematician, has shown that a regular polygon of 17 sides may be inscribed in a circle by employing straight lines and circles only.

PROPOSITION VI. PROBLEM.

A regular polygon inscribed in a circle being given, to describe a similar polygon about the circle.



Let ABCDEF be a regular polygon inscribed in the circle ABD; it is required to describe a similar polygon about the circle.

Bisect the arc AB in G, and through G draw the tangent LM. Bisect also the arc BC in H, and through H draw the tangent MN, and in the same manner draw tangents to the middle points of the arcs CD, DE, etc. These tangents, by their intersections, will form a circumscribed polygon similar to the one inscribed.

Find O, the centre of the circle, and draw the radii OG, OH. Then, because OG is perpendicular to the tangent LM (B. III., Pr. 9), and also to the chord AB (B. III., Pr. 6, Cor.), the tangent is parallel to the chord (B. I., Pr. 20). In the same manner, it may be proved that the other sides of the circumscribed polygon are parallel to the sides of the inscribed polygon, and therefore the angles of the circumscribed polygon are equal to those of the inscribed one (B. I., Pr. 26).

Since the arcs BG, BH are halves of the equal arcs AGB, BHC, they are equal to each other; that is, the vertex B is at the middle point of the arc GBH.

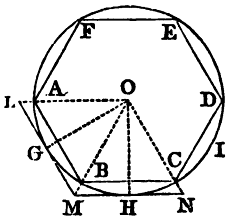
Join OM; the line OM will pass through the point B. For the right-angled triangles OMH, OMG have the hypothenuse OM common, and the side OH equal to OG; therefore the angle GOM is equal to the angle HOM (B. I., Pr. 19), and the line OM passes through the point B, the middle of the arc GBH.

Now, because the triangle OAB is similar to the triangle OLM, and the triangle OBC to the triangle OMN, we have the proportions

$$AB : LM :: BO : MO;$$

also

$$BC : MN :: BO : MO;$$



therefore (B. II., Pr. 4) AB : LM :: BC : MN.

But AB is equal to BC; therefore LM is equal to MN.

In the same manner, it may be proved that the other sides of the circumscribed polygon are equal to each other. Hence this polygon is regular, and similar to the one inscribed.

Cor. 1. Conversely, if the circumscribed polygon is given, and it is required to form the similar inscribed one, draw the lines OL, OM, ON, etc., to the angles of the polygon; these lines will meet the circumference in the points A, B, C, etc. Join these points by the lines AB, BC, CD, etc., and a similar polygon will be inscribed in the circle.

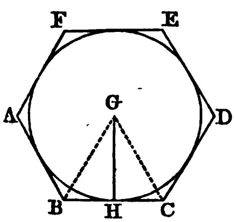
Or we may simply join the points of contact G, H, I, etc., by the chords GH, HI, etc., and there will be formed an inscribed polygon similar to the circumscribed one.

Cor. 2. Hence we can circumscribe about a circle any regular polygon which can be inscribed within it, and conversely.

Cor. 3. A side of the circumscribed polygon MN is equal to twice MH, or MG+MH.

PROPOSITION VII. THEOREM.

The area of a regular polygon is equivalent to the product of its perimeter by half the radius of the inscribed circle.

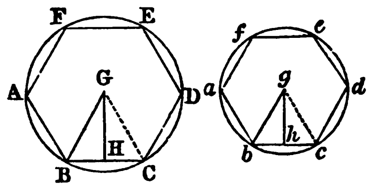


Let ABCDEF be a regular polygon, and G the centre of the inscribed circle. From G draw lines to all the angles of the polygon. The polygon will thus be divided into as many triangles as it has sides; and the common altitude of these triangles is GH, the radius of the circle.

Now the area of the triangle BGC is equal to the product of BC by the half of GH (B. IV., Pr. 6), and so of all the other triangles having their vertices in G. Hence the sum of all the triangles, that is, the surface of the polygon, is equivalent to the product of the sum of the bases AB, BC., etc.; that is, the perimeter of the polygon, multiplied by half of GH, or half the radius of the inscribed circle. Therefore the area of a regular polygon, etc.

PROPOSITION VIII. THEOREM.

The perimeters of two regular polygons of the same number of sides are to each other as the radii of the inscribed or circumscribed circles, and their areas are as the squares of these radii.



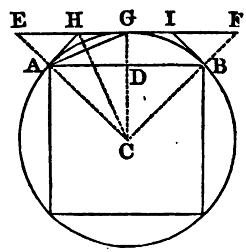
Let ABCDEF, *abcdef* be two regular polygons of the same number of sides; let G and *g* be the centres of the circumscribed circles; and let GH, *gh* be drawn perpendicular to BC and *bc*; then will the perimeters of the polygons be as the radii BG, *bg* of the circumscribed circles; and also as GH, *gh*, the radii of the inscribed circles.

The angle BGC is equal to the angle *bgc* (Pr. 2, Sch. 1), and, since the triangles BGC, *bgc* are isosceles, they are similar. So, also, are the right-angled triangles BGH, *bgh*; and, consequently, BC : *bc* :: BG : *bg* :: GH : *gh*. But the perimeters of the two polygons are to each other as the sides BC, *bc* (Pr. I., Cor.); they are therefore to each other as the radii BG, *bg* of the circumscribed circles; and also as the radii GH, *gh* of the inscribed circles.

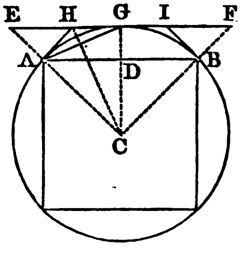
The areas of these polygons are to each other as the squares of the homologous sides BC, *bc* (Pr. 1, Cor.); they are therefore as the squares of BG, *bg*, the radii of the circumscribed circles, or as the squares of GH, *gh*, the radii of the inscribed circles.

PROPOSITION IX. PROBLEM.

The area of a regular inscribed polygon and that of a similar circumscribed polygon being given, to find the areas of regular inscribed and circumscribed polygons having double the number of sides.



Let AB be a side of the given inscribed polygon; EF, parallel to AB, a side of the similar circumscribed polygon, and C the centre of the circle. Draw the chord AG, and it will be the side of the inscribed polygon having double the number of sides. At the points A and B draw tangents, meeting EF in the points H and I; then will HI, which is double of HG, be a side



of the similar circumscribed polygon (Pr. 6, Cor. 1).

Let p represent the inscribed polygon whose side is AB, P the corresponding circumscribed polygon; p' the inscribed polygon having double the number of sides, P′ the similar circumscribed polygon. Then it is plain that the space CAD is the same part of p that CEG is of P; also, CAG of p', and CAHG of P′; for each of these spaces must be repeated the same number of times to complete the polygons to which they severally belong.

First. The triangles ACD, ACG, whose common vertex is A, are to each other as their bases CD, CG; they are also to each other as the polygons p and p'; hence

$$p : p' :: CD : CG.$$

Again, the triangles CGA, CGE, whose common vertex is G, are to each other as their bases CA, CE; they are also to each other as the polygons p' and P; hence

$$p' : P :: CA : CE.$$

But, since AD is parallel to EG, we have CD : CG :: CA : CE; therefore, $p : p' :: p' : P$;
that is, *the polygon* p′ *is a mean proportional between the two given polygons.*

Secondly. The triangles CGH, CHE, having the common altitude CG, are to each other as their bases GH, HE. But, since CH bisects the angle GCE, we have (B. IV., Pr. 17)

$$GH : HE :: CG : CE :: CD : CA, \text{ or } CG :: p : p'.$$

Therefore $$CGH : CHE :: p : p';$$
hence (B. II., Pr. 6)

$$CGH : CGH + CHE, \text{ or } CGE :: p : p + p',$$

or $$2CGH : CGE :: 2p : p + p'.$$

But $$2CGH, \text{ or } CGHA : CGE :: P' : P.$$

Therefore $$P' : P :: 2p : p + p'; \text{ whence } P' = \frac{2pP}{p + p'};$$

that is, *the polygon* P′ *is found by dividing twice the product of the two given polygons by the sum of the two inscribed polygons.*

Hence, by means of the polygons p and P, it is easy to find the polygons p' and P′ having double the number of sides.

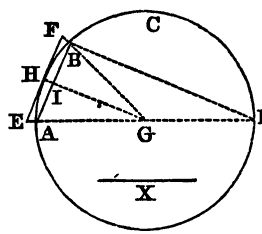
PROPOSITION X. THEOREM.

A circle being given, two similar polygons can always be found, the one described about the circle, and the other inscribed in it, which shall differ from each other by less than any assignable surface.

Let ACD be the given circle, and the square of X any given surface however small; a polygon can be inscribed in the circle ACD, and a similar polygon be described about it, such that the difference between them shall be less than the square of X.

Bisect AC, a fourth part of the circumference; then bisect the half of this fourth, and so continue the bisection until an arc is found whose chord AB is less than X. As this arc must be contained a certain number of times exactly in the whole circumference, if we apply chords AB, BC, etc., each equal to AB, the last will terminate at A, and a regular polygon, ABCD, etc., will be inscribed in the circle.

Next describe a similar polygon about the circle (Pr. 6); the difference of these two polygons will be less than the square of X.



Find the centre G, and draw the diameter AD. Let EF be a side of the circumscribed polygon, and join EG, FG. These lines will pass through the points A and B, as was shown in Pr. 6. Draw GH to the point of contact H; it will bisect AB in I, and be perpendicular to it (B. III., Pr. 6, cor.). Join also BD.

Let P represent the circumscribed polygon, and p the inscribed polygon. Then, because the polygons are similar, they are as the squares of the homologous sides EF and AB (B. IV., Pr. 27); that is, because the triangles EFG, ABG are similar, as the square of EG to the square of AG, that is, of HG.

Again, the triangles EHG, ABD, having their sides parallel to each other, are similar, and therefore

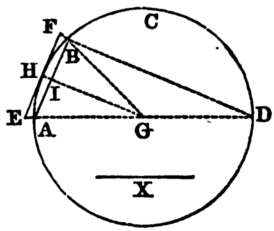
$$EG : HG :: AD : BD.$$

But the polygon P is to the polygon p as the square of EG to the square of HG;

hence $$P : p :: AD^2 : BD^2,$$

and, by division,

$$P : P-p :: AD^2 : AD^2-BD^2, \text{ or } AB^2.$$



But the square of AD is greater than a regular polygon of eight sides described about the circle, because it contains that polygon; and, for the same reason, the polygon of eight sides is greater than the polygon of sixteen, and so on. Therefore P is less than the square of AD, and, consequently (B. II., Def. 11), $P-p$ is less than the square of AB; that is, less than the given square on X. Hence the difference of the two polygons is less than the given surface.

Cor. Since the circle can not be less than any inscribed polygon, nor greater than any circumscribed one, it follows that *a polygon may be inscribed in a circle, and another described about it, each of which shall differ from the circle by less than any assignable surface.*

Scholium. A *variable* quantity is a quantity which assumes successively different values. When the successive values of a variable quantity approach more and more nearly to some constant quantity, so that the difference between the variable and the constant may become less than any assignable quantity, the constant is called the *limit* of the variable. Thus, if we suppose the number of sides of a regular polygon to increase, the magnitude of each angle will also increase; and if the number of sides be made greater than any finite number, each angle of the polygon will approach indefinitely near to two right angles. Here the variable quantity is the angle of the regular polygon, and the *limit* toward which its value continually approaches is two right angles. We see, also, that the circle is the limit to which the inscribed and circumscribed polygons approach when the number of their sides is indefinitely increased. When the number of sides of the polygon is greater than any finite number, the difference between the polygon and circle becomes less than any finite quantity; that is, the circle becomes identical with the inscribed polygon, and also with the circumscribed polygon. *The circle may therefore be regarded as a regular polygon of an infinite number of sides.*

PROPOSITION XI. PROBLEM.

To compute the area of a circle whose radius is unity.

If the radius of a circle be unity, the diameter will be represented by 2, and the area of the circumscribed square will be 4; while that of the inscribed square, being half the circumscribed, is 2.

Now, according to Pr. 9, the area of the inscribed octagon is a mean proportional between the two squares p and P, so that $p'=\sqrt{8}=2.82843$. Also, the circumscribed octagon $P'=\frac{2pP}{p+p'}=\frac{16}{2+\sqrt{8}}=3.31371$.

Having thus obtained the inscribed and circumscribed octagons, we may in the same way determine the polygons having twice the number of sides. We must put $p=2.82843$, and $P=3.31371$, and we shall have $p'=\sqrt{pP}=3.06147$; and $P'=\frac{2pP}{p+p'}=3.18260$.

These polygons of 16 sides will furnish us those of 32, and thus we may proceed until there is no difference between the inscribed and circumscribed polygons, at least for any number of decimal places which may be desired. The following table gives the result of this computation for five decimal places:

Number of Sides.	Inscribed Polygon.	Circumscribed Polygon.
4	2.00000	4.00000
8	2.82843	3.31371
16	3.06147	3.18260
32	3.12145	3.15172
64	3.13655	3.14412
128	3.14033	3.14222
256	3.14128	3.14175
512	3.14151	3.14163
1024	3.14157	3.14160
2048	3.14159	3.14159

Now, as the inscribed polygon can not be greater than the circle, and the circumscribed polygon can not be less than the circle, it is plain that 3.14159 must express the area of a circle, whose radius is unity, correct to five decimal places.

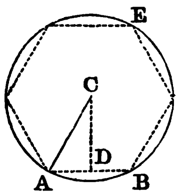
After three bisections of a quadrant of a circle we obtain the inscribed polygon of 32 sides, which differs from the corresponding circumscribed polygon only in the second decimal place. After five bisections we obtain polygons of 128 sides, which differ only in the third decimal place; after nine bisections they agree to five decimal places, but differ in the sixth place; after eighteen bisections they agree to ten decimal places; and thus, by continually bisecting the arcs subtended by the sides of the polygon, new polygons are formed, both inscribed and circumscribed, which agree to a greater number of decimal places.

Vieta, by means of inscribed and circumscribed polygons, carried the approximation to ten places of figures; Van Ceulen carried it to 36 places; Sharp computed the area to 72 places; De Lagny to 128 places; and Dr. Clausen has carried the computation to 250 places of decimals.

By continuing this process of bisection, the difference between the inscribed and circumscribed polygons may be made less than any quantity we can assign, however small.

PROPOSITION XII. THEOREM.

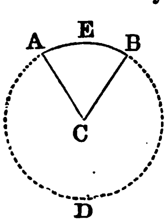
The area of a circle is equal to the product of its circumference by half the radius.



Let ABE be a circle whose centre is C and radius CA: the area of the circle is equal to the product of its circumference by half of CA.

Inscribe in the circle any regular polygon, and from the centre draw CD perpendicular to one of the sides. The area of the polygon will be equal to its perimeter multiplied by half of CD (Pr. 7).

Conceive the number of sides of the polygon to be indefinitely increased by continually bisecting the arcs subtended by the sides, its perimeter will approach more nearly to the circumference of the circle; and, when the number of sides of the polygon is greater than any finite number, the perimeter of the polygon will coincide with the circumference of the circle; the perpendicular CD will become equal to the radius CA, and the area of the polygon will be equal to the area of the circle (Pr. 10, Schol.). Therefore the area of the circle is equal to the product of its circumference by half the radius.



Cor. The area of a sector is equal to the product of its arc by half its radius.

For the sector ACB is to the whole circle ABD as the arc AEB is to the whole circumference ABD (B. III., Pr. 14, Cor.); or, since magnitudes have the same ratio which their equimultiples have (B. II., Pr. 10), as the arc AEB $\times \frac{1}{2}$AC is to the circumference ABD $\times \frac{1}{2}$AC.

But this last expression is equal to the area of the circle; therefore the area of the sector ACB is equal to the product of its arc AEB by half of AC.

PROPOSITION XIII. THEOREM.

The circumferences of circles are to each other as their radii, and their areas are as the squares of their radii.

Let R and r denote the radii of two circles; C and c their circumferences; A and a their areas; then we shall have

$$C : c :: R : r,$$

and

$$A : a :: R^2 : r^2.$$

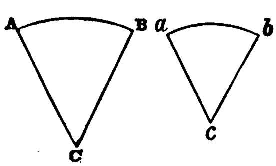
Inscribe within the circles two regular polygons having the same number of sides. Now, whatever be the number of sides of the polygons, their perimeters will be to each other as the radii of the circumscribed circles (Pr. 8). Conceive the arcs subtended by the sides of the polygons to be continually bisected until the number of sides of the polygons becomes indefinitely great, the perimeters of the polygons will approach more nearly to the circumferences of the circles; and when the number of sides of the polygons is greater than any finite number, the perimeters of the polygons will coincide with the circumferences of the circles, and we shall have

$$C : c :: R : r.$$

Again, the areas of the polygons are to each other as the squares of the radii of the circumscribed circles (Pr. 8). But when the number of sides of the polygons is greater than any finite number, the areas of the polygons become equal to the areas of the circles, and we shall have

$$A : a :: R^2 : r^2.$$

Cor. 1. Similar arcs are to each other as their radii, and similar sectors are as the squares of their radii.



For, since the arcs AB, ab are similar, the angle C is equal to the angle c (B. IV., Def. 6). But the angle C is to four right angles as the arc AB is to the whole circumference described with the radius AC (B. III., Pr. 14), and the angle c is to four right angles as the arc ab is to the circumference described with the radius ac. Therefore the arcs AB, ab are to each other as the circumferences of which they form a part. But these circumferences are to each other as AC, ac; therefore

$$arc\ AB : arc\ ab :: AC : ac.$$

For the same reason, the sectors ACB, *acb* are as the entire circles to which they belong, and these are as the squares of their radii; therefore

$$sector\ ACB : sector\ acb :: AC^2 : ac^2.$$

Cor. 2. Let π represent the circumference of a circle whose diameter is unity; also, let D represent the diameter, R the radius, and C the circumference of any other circle; then, since the circumferences of circles are to each other as their diameters,

$$1 : \pi :: 2R : C;$$

therefore $$C = 2\pi R = \pi D;$$

that is, *the circumference of a circle is equal to the product of its diameter by the constant number* π.

Cor. 3. According to Pr. 12, the area of a circle is equal to the product of its circumference by half the radius.

If we put A to represent the area of a circle, then

$$A = C \times \tfrac{1}{2}R = 2\pi R \times \tfrac{1}{2}R = \pi R^2;$$

that is, *the area of a circle is equal to the product of the square of its radius by the constant number* π.

Cor. 4. When R is equal to unity, we have $A = \pi$; that is, π *is equal to the area of a circle whose radius is unity.* According to Pr. 11, π is therefore equal to 3.14159 nearly. This number is represented by π, because it is the first letter of the Greek word which signifies circumference.

EASY EXERCISES ON THE PRECEDING BOOKS.

A few theorems without demonstrations, and problems without solutions, are here subjoined for the exercise of the pupil. They will be found admirably adapted to familiarize the beginner with the preceding principles, and to impart dexterity in their application. No general rule can be given which will be found applicable in all cases, and infallibly lead to the demonstration of a proposed theorem, or the solution of a problem. The following directions may prove of some service:

ANALYSIS OF THEOREMS.

1. Construct a diagram as directed in the enunciation, and assume that the theorem is true.

2. Consider what consequences result from this assumption by combining with it theorems which have been already proved, and which are applicable to the diagram.

3. Examine whether any of these consequences are already known to be *true* or to be *false.*

4. If the assumption of the truth of the proposition lead to some consequence which is inconsistent with any demonstrated truth, the false conclusion thus arrived at indicates the falsehood of the proposition; and by reversing the process of the analysis, it may be demonstrated that the theorem can not be true.

5. If none of the consequences so deduced be *known* to be either true or false, proceed to deduce other consequences from all, or any of these, until a result is obtained which is known to be either true or false.

6. If we thus arrive at some truth which has been previously demonstrated, we then retrace the steps of the investigation pursued in the analysis till they terminate in the theorem which was assumed. This process will constitute the demonstration of the theorem.

ANALYSIS OF PROBLEMS.

1. Construct the diagram as directed in the enunciation, and suppose the solution of the problem to be effected.

2. Study the relations of the lines, angles, triangles, etc., in the diagram, and endeavor to discover the dependence of the assumed solution on some previous theorem or problem in the Geometry.

3. If such can not be found, draw other lines parallel or perpendicular, as the case may seem to require; join given points, or points assumed in the solution, and describe circles if necessary; and then proceed to trace the dependence of the assumed solution on some theorem or problem in Geometry.

4. If we thus arrive at some previously demonstrated or admitted truth, we shall obtain a direct solution of the problem by assuming the last consequence of the analysis as the first step of the process, and proceeding in a contrary order through the several steps of the analysis until the process terminate in the problem required.

GEOMETRICAL EXERCISES ON BOOK I.

THEOREMS.

Prop. 1. The difference between any two sides of a triangle is less than the third side. See Prop. 8.

Prop. 2. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point whatever (except the intersection of the diagonals) to the four angles. See Prop. 8.

Prop. 3. If a straight line which bisects the vertical angle of a triangle also bisects the base, the remaining sides of the triangle are equal to each other.

Demonstration. Produce AD, the bisecting line, making DE= DA; then in the, etc.

Prop. 4. If the base of an isosceles triangle be produced, the exterior angle exceeds one right angle by half the vertical angle. See Prop. 27.

Prop. 5. In any right-angled triangle, the middle point of the hypothenuse is equally distant from the three angles.

Dem. From D, the middle point of the hypothenuse, draw perpendiculars upon the two sides of the triangle; then, etc.

Prop. 6. If, on the sides of a square, at equal distances from the four angles, four points be taken, one on each side, the figure formed by joining those points will also be a square. See Prop. 6.

Prop. 7. The parallelogram whose diagonals are equal is rectangular. See Prop. 32.

Prop. 8. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram. See Props. 6 and 22.

Prop. 9. Any line drawn through the centre of the diagonal of a parallelogram to meet the sides is bisected in that point, and also bisects the parallelogram. See Props. 7 and 29.

Prop. 10. The sum of the three straight lines drawn from any point within a triangle to the three vertices is less than the sum and greater than the half sum of the three sides of the triangle. See Props. 8 and 9.

PROBLEMS.

Prop. 1. On a given line describe an isosceles triangle, each of whose equal sides shall be double of the base.

Solution. Produce the given base AB both ways, making AC = AB = BD. With centre A and radius AD, describe a circle, etc.

Prop. 2. On a given line describe a square, of which the line shall be the diagonal.

Sol. Bisect the given line AB at right angles by DCE, and make CD = CE = CA or CB; then, etc.

Prop. 3. Divide a right angle into three equal angles.

Sol. On one of the sides containing the right angle describe an equilateral triangle, etc.

Prop. 4. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

Sol. The smaller angle is one fourth of a right angle, and its third part is one twelfth of a right angle. May be solved by the method of Prop. 3.

Prop. 5. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the angles on the opposite side.

Sol. May be solved by the method of Prop. 3.

EXERCISES ON BOOK II.

1. Find a third proportional to 8 and 12. *Ans.* 18.
2. Find a fourth proportional to 12, 16, and 39. *Ans.* 52.
3. Find a mean proportional between 24 and 54. *Ans.* 36.
4. If A : B :: C : D, prove that

$$A^2+AB+B^2 : A^2-AB+B^2 :: C^2+CD+D^2 : C^2-CD+D^2.$$

GEOMETRICAL EXERCISES ON BOOK III.

THEOREMS.

Prop. 1. Every chord of a circle is less than the diameter. See B. I., Pr. 7.

Prop. 2. If an arc of a circle be divided into three equal parts by three straight lines drawn from one extremity of the arc, the angle contained by two of the straight lines will be bisected by the third. See B. III., Pr. 15.

Prop. 3. Any two chords of a circle which cut a diameter in the same point, and make equal angles with it, are equal to each other. See B. III., Pr. 17.

Prop. 4. The straight lines which join toward the same parts the extremities of any two chords in a circle equally distant from the centre, are parallel to each other.

Prop. 5. The two straight lines which join the opposite extremities of two parallel chords intersect in a point in that diameter which is perpendicular to the chords.

Prop. 6. If two opposite sides of a quadrilateral figure inscribed in a circle are equal, the other two sides will be parallel.

Prop. 7. All the equal chords in a circle may be touched by another circle.

Prop. 8. The lines bisecting at right angles the sides of a triangle all meet in one point. See B. I., Pr. 18.

Prop. 9. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference. See B. III., Pr. 15, Cor. 2.

Prop. 10. If two circles touch each other externally, and parallel diameters be drawn, the straight line joining the opposite extremities of these diameters will pass through the point of contact. See B. III., Pr. 12, and Pr. 15, Cor. 2.

Prop. 11. The lines which bisect the angles of any parallelogram form a rectangular parallelogram, whose diagonals are parallel to the sides of the former. See B. I., Pr. 27.

Prop. 12. If two opposite sides of a parallelogram be bisected, the lines drawn from the points of bisection to the opposite angles will trisect the diagonal.

PROBLEMS.

Prop. 1. From a given point without a given straight line, draw a line making a given angle with it. See B. V., Pr. 4.

Prop. 2. Through a given point within a circle, draw a chord which shall be bisected in that point. See B. III., Pr. 6.

Prop. 3. Through a given point within a circle, draw the least possible chord. See B. III., Pr. 6.

Prop. 4. Two chords of a circle being given in magnitude and position, describe the circle. See B. III., Pr. 7.

Prop. 5. Describe three equal circles touching one another; and also describe another circle which shall touch them all three.

Sol. Describe an equilateral triangle and bisect its sides.

Prop. 6. How many equal circles can be described around another circle of the same magnitude, touching it and one another?

Prop. 7. With a given radius describe a circle which shall pass through two given points. See B. I., Pr. 18.

Prop. 8. Describe a circle which shall pass through two given points and have its centre in a given line. See B. I., Pr. 18.

Prop. 9. In a given circle inscribe a triangle equiangular to a given triangle. See B. III., Pr. 15.

Prop. 10. From one extremity of a line which can not be produced, draw a line perpendicular to it.

Sol. Take any point C without the given line as a centre, and with a radius equal to the distance of C from the given extremity, describe a circumference, etc.

Prop. 11. Divide a circle into two parts, such that the angle contained in one segment shall equal twice the angle contained in the other.

Sol. Inscribe in the circle an equilateral triangle.

Prop. 12. Divide a circle into two segments, such that the angle contained in one of them shall be five times the angle contained in the other.

Sol. Inscribe in the circle a regular hexagon.

Prop. 13. Describe a circle which shall touch a given circle in a given point, and also touch a given straight line.

Sol. Draw a tangent at A, cutting the given line BC in C; bisect the angle ACB by CD, cutting OA in D, etc.

Prop. 14. With a given radius, describe a circle which shall pass through a given point and touch a given line.

Sol. Draw AC perpendicular to the given line AB, and make it equal to the given radius. Draw CD parallel to AB, etc.

Prop. 15. With a given radius, describe a circle which shall touch a given line, and have its centre in another given line.

Sol. Let AB, AC be the two given lines; from any point C in

AC draw CD perpendicular to AC, and equal to the given radius; through D draw, etc.

GEOMETRICAL EXERCISES ON BOOK IV.

THEOREMS.

Prop. 1. If from any point in the diagonal of a parallelogram lines be drawn to the angles, the parallelogram will be divided into two pairs of equivalent triangles. See B. I., Pr. 32, and B. IV., Pr. 2.

Prop. 2. If the sides of any quadrilateral be bisected, and the points of bisection joined, the included figure will be a parallelogram, and equal in area to half the original figure. See B. IV., Pr. 16.

Prop. 3. Show how the squares in Prop. 11, Book IV., may be dissected, so that the truth of the proposition may be made to appear by superposition of the parts.

Prop. 4. In the figure to Prop. 11, Book IV.,

(*a.*) If BG and CH be joined, those lines will be parallel.

(*b.*) If perpendiculars be let fall from F and I on BC produced, the parts produced will be equal, and the perpendiculars together will be equal to BC.

(*c.*) Join GH, IE, and FD, and prove that each of the triangles so formed is equivalent to the given triangle ABC.

(*d.*) The sum of the squares of GH, IE, and FD will be equal to six times the square of the hypothenuse.

Prop. 5. The square on the base of an isosceles triangle whose vertical angle is a right angle, is equal to four times the area of the triangle.

Prop. 6. If from one of the acute angles of a right-angled triangle a straight line be drawn bisecting the opposite side, the square upon that line will be less than the square upon the hypothenuse by three times the square upon half the line bisected.

Prop. 7. In a right-angled triangle, the square on either of the two sides containing the right angle is equal to the rectangle contained by the sum and difference of the other sides.

Prop. 8. In any triangle, if a perpendicular be drawn from the vertex to the base, the difference of the squares upon the sides is equal to the difference of the squares upon the segments of the base.

Prop. 9. The squares of the diagonals of any quadrilateral figure are together double the squares of the two lines joining the middle points of the opposite sides.

Sol. Compare this Prop. with Prop. 2 above.

Prop. 10. If one side of a right-angled triangle is double the other, the perpendicular from the vertex upon the hypothenuse will divide the hypothenuse into parts which are in the ratio of 1 to 4.

Prop. 11. If two circles intersect, the common chord produced will bisect the common tangent.

Prop. 12. The tangents to a circle at the extremities of any chord contain an angle which is twice the angle contained by the same chord and a diameter drawn from either of the extremities.

Prop. 13. If two circles cut each other, and if from any given point in the straight line produced which joins their intersections two tangents be drawn, one to each circle, they will be equal to one another.

Prop. 14. If from a point without a circle two tangents be drawn, the straight line which joins the point of contact will be bisected at right angles by a line drawn from the centre to the point without the circle.

PROBLEMS.

Prop. 1. Trisect a given straight line, and hence divide an equilateral triangle into nine equal parts.

Sol. On the given line describe an equilateral triangle; bisect two of its angles, and from the point of intersection of the bisecting lines draw lines parallel to the sides of the triangle, etc.

Prop. 2. Inscribe a circle in a given rhombus.

Sol. Draw the diagonals of the rhombus, etc.

Prop. 3. Describe a circle whose circumference shall pass through one angle and touch two sides of a given square.

Sol. Divide the given angle into four equal parts, etc.

Prop. 4. In a given square, inscribe an equilateral triangle having its vertex in the middle of a side of the square.

Sol. From the middle of a side as centre, with a radius equal to one side of the square, describe a circle, etc.

Prop. 5. In a given square, inscribe an equilateral triangle having its vertex in one angle of the square.

Sol. On two adjacent sides of the square, describe equilateral triangles exterior to the square, and join their vertices with the remote vertex of the square, etc.

Prop. 6. If the sides of a triangle are in the ratio of the numbers 2, 4, and 5, show whether it will be acute-angled or obtuse-angled.

Prop. 7. Given the area and hypothenuse of a right-angled triangle, to construct the triangle.

Sol. On half the hypothenuse describe a rectangle equal to the given area, etc.

Prop. 8. Bisect a triangle by a line drawn from a given point in one of the sides.

Sol. Let D be the given point in the side AB, and A the angle nearest to D. Bisect BC in E, and draw AF parallel to DE, etc.

Prop. 9. To a circle of given radius draw two tangents which shall contain an angle equal to a given angle.

Prop. 10. Construct a triangle, having given one side, the angle opposite to it, and the ratio of the other two sides.

Sol. On the given base BC describe a segment containing the given angle; draw DE perpendicular to BC at its middle point, and cutting the remaining segment in E; divide BC in F in the given ratio; join EF, etc.

Prop. 11. Construct a triangle, having given the perimeter and the angles of the triangle.

Sol. On the line which is equal to the perimeter of the required triangle describe a triangle having its angles equal to the given angles. Bisect the angles at the base, etc.

Prop. 12. Upon a given base describe a right-angled triangle, having given the perpendicular from the right angle upon the hypothenuse.

Sol. Draw any straight line, and erect DC perpendicular to it and equal to the given perpendicular. With centre C and radius equal to the given base, describe a circle cutting the first line in B. At C draw, etc.

Prop. 13. Construct a triangle, having given one angle, a side opposite to it, and the sum of the other two sides.

Sol. On the given side AB describe a segment containing half the given angle, in which segment inscribe AC equal to the given sum. Make the angle CBD equal to BCA, etc.

Prop. 14. Construct a triangle, having given one angle, an adjacent side, and the sum of the other two sides.

Sol. Make BC the given base, B the given angle, and BD equal to the sum of the two sides; make the angle DCA equal to CDA, etc.

Prop. 15. Inscribe a square in a given right-angled isosceles triangle.

Sol. Trisect the hypothenuse, etc.

NUMERICAL EXERCISES.

1. If the base and perpendicular of a triangle be 78 and 43 yards respectively, what is the area? *Ans.* 1677 square yards.

2. Given the hypothenuse of a right-angled triangle equal to 260 feet, and one of the legs equal to 224 feet, to find the other leg. *Ans.* 132 feet.

3. Given the legs of a right-angled triangle equal to 765 and 408 yards respectively, to compute the length of the perpendicular from the right angle to the hypothenuse. *Ans.* 360 yards.

4. If the sides of a triangle are 845, 910, and 975 respectively, what are the lengths of the segments into which they are severally divided by the perpendiculars from the opposite angles?

Ans. $\begin{cases} 350, \\ 495, \end{cases} \begin{cases} 325, \\ 585, \end{cases} \begin{cases} 429, \\ 546. \end{cases}$

5. Given the hypothenuse and one leg of a right-angled triangle equal to 353 and 272, to find the remaining leg without squaring the given numbers. *Ans.* 225.

6. If the base of a triangle be 210, and the other sides 135 and 105, what is the length of the straight line drawn from the vertical angle to the point of bisection of the base? *Ans.* 60.

7. If two adjacent sides and one of the diagonals of a parallelogram be 245, 315, and 280, what is the length of the other diagonal? *Ans.* 490.

8. Given the sides of a triangle equal to 147, 119, and 70 yards respectively, to compute the area. *Ans.* 4116 square yards.

9. If a chord of a circular arc 16 inches in length be divided into two parts of 7 and 9 inches respectively by another chord, what is the length of the latter, one of its segments being 3 inches? *Ans.* 24 inches.

10. If the chord of an arc be 720 feet, and the chord of its half be 369 feet, what is the diameter of the circle?

Ans. 1681 feet.

11. If from a point without a circle two secants be drawn whose external segments are 8 inches and 7 inches, while the internal segment of the latter is 17 inches, what is the internal segment of the former? *Ans.* 13 inches.

12. From a point without a circular pond two tangents to the

circumference are drawn, forming with each other an angle of an equilateral triangle, and the length of each tangent is 18 rods, what is the diameter? *Ans.* $12\sqrt{3}=20.7846$ rods.

13. If the sides of a triangle are 39, 42, and 45 inches respectively, what is the radius of the inscribed circle?

Ans. 12 inches.

14. Given the legs of a right-angled triangle equal to 455 and 1092 respectively, to compute the segments into which the hypothenuse is divided by the perpendicular from the right angle, and to compute also the perpendicular.

Ans. The segments are 175 and 1008, and the perpendicular 420.

15. If the base of a triangle be 246, and the other sides 250 and 160 respectively, what is the length of the line bisecting the vertical angle? *Ans.* 160.

16. If two similar fields together contain 518 square rods, what are their separate contents, their homologous sides being as 5 to 7? *Ans.* 175 and 343 square rods.

17. If the sides of a triangle are 104, 112, and 120 respectively, what is the radius of the circumscribed circle? *Ans.* 65.

18. If the base of a triangle be 54, and the other sides 75 and 48 respectively, what is the length of the external segment of the base made by a straight line bisecting the exterior angle at the vertex? *Ans.* 96.

19. Two chords on opposite sides of the centre of a circle are parallel, and one of them has a length of 48, and the other of 14 inches, the distance between them being 31 inches; what is the diameter of the circle? *Ans.* 50 inches.

20. Two parallel chords on the same side of the centre of a circle whose diameter is 50 inches are measured, and found to be the one 48 and the other 14 inches; what is their distance apart?

Ans. 17 inches.

21. The area of a rectangle is 18 square feet, and its base is 4.62 feet; what is its altitude?

22. The base of one rectangle is 6 feet and altitude 5 feet; the base of another rectangle is 4 feet and altitude 3 feet; what is the ratio of the two rectangles?

GEOMETRICAL EXERCISES ON BOOK VI.

THEOREMS.

Prop. 1. The square inscribed in a circle is equal to half the square described about the same circle.

Prop. 2. Any number of triangles having the same base and the same vertical angle may be circumscribed by one circle.

Prop. 3. If an equilateral triangle be inscribed in a circle, each of its sides will cut off one fourth part of the diameter drawn through the opposite angle.

Prop. 4. The circle inscribed in an equilateral triangle has the same centre with the circle described about the same triangle, and the diameter of one is double that of the other.

Prop. 5. If an equilateral triangle be inscribed in a circle, and the arcs cut off by two of its sides be bisected, the line joining the points of bisection will be trisected by the sides.

Prop. 6. The side of an equilateral triangle inscribed in a circle is to the radius as the square root of 3 is to unity.

Prop. 7. The sum of the perpendiculars let fall from any point within an equilateral triangle upon the sides is equal to the perpendicular let fall from one of the angles upon the opposite side.

Prop. 8. If two circles be described, one without and the other within a right-angled triangle, the sum of their diameters will be equal to the sum of the sides containing the right angle.

Prop. 9. If a circle be inscribed in a right-angled triangle, the sum of the two sides containing the right angle will exceed the hypothenuse by a line equal to the diameter of the inscribed circle.

Prop. 10. The square inscribed in a semicircle is to the square inscribed in the entire circle as 2 to 5.

Prop. 11. The square inscribed in a semicircle is to the square inscribed in a quadrant of the same circle as 8 to 5.

Prop. 12. The area of an equilateral triangle inscribed in a circle is equal to half that of the regular hexagon inscribed in the same circle.

Prop. 13. The square of the side of an equilateral triangle inscribed in a circle is triple the square of the side of the regular hexagon inscribed in the same circle.

Prop. 14. The area of a regular hexagon inscribed in a circle is three fourths of the regular hexagon circumscribed about the same circle.

Prop. 15. The triangle, square, and hexagon are the only regular polygons by which the angular space about a point can be completely filled up.

PROBLEMS.

Prop. 1. Trisect a given circle by dividing it into three equal sectors.

Prop. 2. The centre of a circle being given, find two opposite points in the circumference by means of a pair of compasses only.

Prop. 3. Divide a right angle into five equal parts.

Prop. 4. Inscribe a square in a given segment of a circle.

Prop. 5. Having given the difference between the diagonal and side of a square, describe the square.

Prop. 6. Inscribe a square in a given quadrant.

Prop. 7. Inscribe a circle in a given quadrant.

Prop. 8. Describe a circle touching three given straight lines.

Prop. 9. Within a given circle describe six equal circles touching each other and also the given circle, and show that the interior circle which touches them all is equal to each of them.

Prop. 10. Within a given circle describe eight equal circles touching each other and the given circle.

Prop. 11. Inscribe a regular hexagon in a given equilateral triangle.

Prop. 12. Upon a given straight line describe a regular octagon.

NUMERICAL EXERCISES.

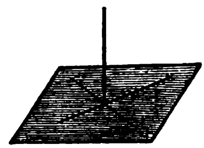
1. What is the circumference of a circle whose diameter is 28?
2. What is the diameter of a circle whose circumference is 50?
3. What is the area of a circle whose diameter is 19?
4. What is the area of a circle whose circumference is 30?
5. What is the area of a quadrant of a circle whose radius is 11?
6. What is the diameter of a circle whose area is 40?
7. What is the circumference of a circle whose area is 35?
8. What is the circumference of the earth, supposing it to be a circle whose diameter is 7912 miles?
9. What is the circumference of a circle whose area is 27.45 square rods?
10. What is the area of a sector whose arc is one sixth of the circumference in a circle whose radius is 17 inches?

GEOMETRY OF SPACE.

BOOK VII.

PLANES AND SOLID ANGLES.

Definitions.



1. A STRAIGHT line is *perpendicular to a plane* when it is perpendicular to every straight line which it meets in that plane.

Conversely, the plane in this case is perpendicular to the line.

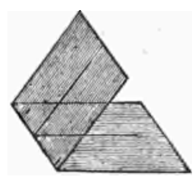
The *foot* of the perpendicular is the point in which it meets the plane.

2. A straight line is *parallel to a plane* when it can not meet the plane, though produced ever so far.

Conversely, the plane in this case is parallel to the line.

3. Two *planes are parallel* to each other when they can not meet, though produced ever so far in every direction.

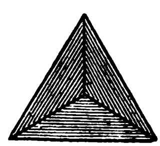
4. The *angle contained by two planes* which meet one another is the angle contained by two lines drawn from any point in the line of their common section, at right angles to that line, one in each of the planes.



This angle may be acute, right, or obtuse.

If it is a right angle, the two planes are perpendicular to each other.

5. A *solid angle* is the angular space contained by more than two planes which meet at the same point, and not lying in the same plane.



To represent a plane in a diagram, we are obliged to take a limited portion of it; but the planes treated of in this Book are supposed to be indefinite in extent.

PROPOSITION I. THEOREM.

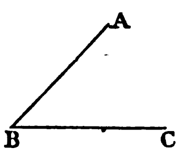
One part of a straight line can not be in a plane, and another part without it.

For, from the definition of a plane (B. I., Def. 11), when a straight line has two points common with a plane, it lies wholly in that plane.

Scholium. To discover whether a surface is plane, we apply a straight line in different directions to this surface, and see if it touches throughout its whole extent.

PROPOSITION II. THEOREM.

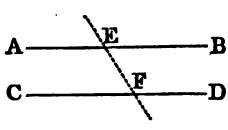
Any two straight lines which cut each other are in one plane, and determine its position.



Let the two straight lines AB, BC cut each other in B; then will AB, BC be in the same plane.

Conceive a plane to pass through the straight line BC, and let this plane be turned about BC until it pass through the point A. Then, because the points A and B are situated in this plane, the straight line AB lies in it (B. I., Def. 11). Hence the position of the plane is determined by the condition of its containing the two lines AB, BC; for if it is turned in either direction about BC, it will cease to contain the point A. Therefore, any two straight lines, etc.

Cor. 1. A triangle ABC, or three points A, B, C, not in the same straight line, determine the position of a plane.

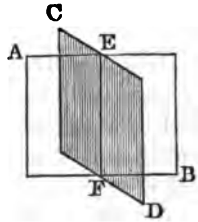


Cor. 2. Two parallel lines AB, CD determine the position of a plane. For, if the line EF be drawn, the plane of the two straight lines AE, EF will be the same as that of the parallels AB, CD; and it has already been proved that two straight lines which cut each other determine the position of a plane.

PROPOSITION III. THEOREM.

If two planes cut each other, their common section is a straight line.

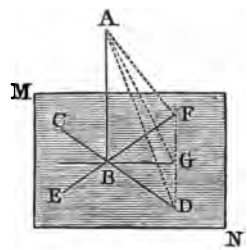
Let the two planes AB, CD cut each other, and let E, F be two points in their common section. From E to F draw the straight line EF. Then, since the points E and F are in the plane AB, the straight line EF which joins them must lie wholly in that plane (B. I., Def. 11). For the same reason, EF must lie wholly in the plane CD. Therefore the straight line EF is common to the two planes AB, CD; that is, it is their common section. Hence, if two planes, etc.



PROPOSITION IV. THEOREM.

If a straight line be perpendicular to each of two straight lines at their point of intersection, it will be perpendicular to the plane in which these lines are.

Let the straight line AB be perpendicular to each of the straight lines CD, EF which intersect at B; AB will also be perpendicular to the plane MN which passes through these lines.



Through B draw any line BG, in the plane MN; let G be any point of this line, and through G draw DGF, so that DG shall be equal to GF (B. V., Pr. 21). Join AD, AG, and AF.

Then, since the base DF of the triangle DBF is bisected in G, we shall have (B. IV., Pr. 14),

$$BD^2+BF^2=2BG^2+2GF^2.$$

Also, in the triangle DAF,

$$AD^2+AF^2=2AG^2+2GF^2.$$

Subtracting the first equation from the second, we have

$$AD^2-BD^2+AF^2-BF^2=2AG^2-2BG^2.$$

But, because ABD is a right-angled triangle,

$$AD^2-BD^2=AB^2;$$

and, because ABF is a right-angled triangle,

$$AF^2-BF^2=AB^2.$$

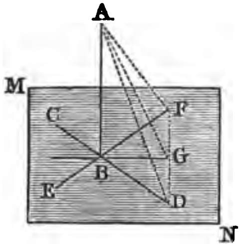
Therefore, substituting these values in the former equation, we have $\quad AB^2+AB^2=2AG^2-2BG^2;$

whence $\quad AB^2=AG^2-BG^2,$

or $\quad AG^2=AB^2+BG^2.$

Wherefore ABG is a right angle (B. IV., Pr. 13, Sch.); that is, AB is perpendicular to the straight line BG. In like manner, it may be proved that AB is perpendicular to any other straight line passing through B in the plane MN; hence it is perpendicular to the plane MN (Def. 1). Therefore, if a straight line, etc.

Scholium. Hence it appears not only that a straight line *may be* perpendicular to every straight line which passes through its foot in a plane, but that it always *must be* so whenever it is perpendicular to two lines in the plane, which shows that the first definition involves no impossibility.



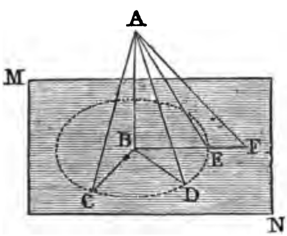
Cor. 1. The perpendicular AB is shorter than any oblique line AD; it therefore measures the true distance of the point A from the plane MN.

Cor. 2. Through a given point B in a plane, only one perpendicular can be drawn to this plane. For, if there could be two perpendiculars, suppose a plane to pass through them, whose intersection with the plane MN is BG; then these two perpendiculars would both be at right angles to the line BG, at the same point and in the same plane, which is impossible (B. I., Pr. 1).

It is also impossible, from a given point without a plane, to let fall two perpendiculars upon the plane. For, suppose AB, AG to be two such perpendiculars; then the triangle ABG will have two right angles, which is impossible (B. I., Pr. 27, Cor. 3).

PROPOSITION V. THEOREM.

Oblique lines drawn from a point to a plane, at equal distances from the perpendicular, are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the longer.



Let the straight line AB be drawn perpendicular to the plane MN; and let AC, AD, AE be oblique lines drawn from the point A, equally distant from the perpendicular; also, let AF be more remote from the perpendicular than AE; then will the lines AC, AD, AE all be equal to each other, and AF be longer than AE.

For, since the angles ABC, ABD, ABE are right angles, and BC, BD, BE are equal, the triangles ABC, ABD, ABE have two sides and the included angle equal; therefore the third sides AC, AD, AE are equal to each other.

So, also, since the distance BF is greater than BE, it is plain that the oblique line AF is longer than AE (B. I., Pr. 17).

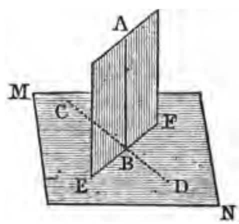
Cor. All the equal oblique lines AC, AD, AE, etc., terminate in the circumference CDE, which is described from B, the foot of the perpendicular, as a centre.

If, then, it is required to draw a straight line perpendicular to the plane MN, from a point A without it, take three points in the plane C, D, E, equally distant from A, and find B, the centre of the circle which passes through these points. Join AB, and it will be the perpendicular required.

Scholium. The angle AEB is called *the inclination of the line* AE to the plane MN. All the lines AC, AD, AE, etc., which are equally distant from the perpendicular, have the same inclination to the plane, because all the angles ACB, ADB, AEB, etc., are equal.

PROPOSITION VI. THEOREM.

If a straight line is perpendicular to a plane, every plane which passes through that line is perpendicular to the first-mentioned plane.



Let the straight line AB be perpendicular to the plane MN; then will every plane which passes through AB be perpendicular to the plane MN.

Suppose any plane, as AE, to pass through AB, and let EF be the common section of the planes AE, MN. In the plane MN, through the point B, draw CD perpendicular to the common section EF.

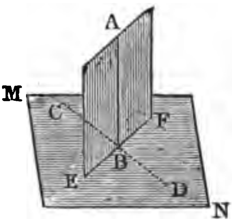
Then, since the line AB is perpendicular to the plane MN, it must be perpendicular to each of the two straight lines CD, EF (Def. 1). But the angle ABD, formed by the two perpendiculars BA, BD, to the common section EF, measures the angle of the two planes AE, MN (Def. 4), and, since this is a right angle, the two planes must be perpendicular to each other. Therefore, if a straight line, etc.

Scholium. When three straight lines, as AB, CD, EF, are perpendicular to each other, each of these lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

PROPOSITION VII. THEOREM.

If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their common section will be perpendicular to the other plane.

Let the plane AE be perpendicular to the plane MN, and let the line AB be drawn in the plane AE perpendicular to the common section EF; then will AB be perpendicular to the plane MN.



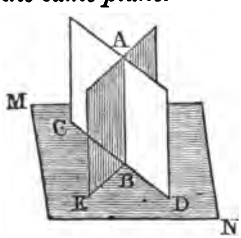
For in the plane MN, draw CD through the point B perpendicular to EF. Then, because the planes AE and MN are perpendicular, the angle ABD is a right angle. Hence the line AB is perpendicular to the two straight lines CD, EF at their point of intersection; it is consequently perpendicular to their plane MN (Pr. 4). Therefore, if two planes, etc.

Cor. If the plane AE is perpendicular to the plane MN, and if from any point B, in their common section, we erect a perpendicular to the plane MN, this perpendicular will be in the plane AE.

For if not, then we may draw from the same point a straight line AB in the plane AE perpendicular to EF, and this line, according to the Proposition, will be perpendicular to the plane MN. Therefore there would be two perpendiculars to the plane MN, drawn from the same point, which is impossible (Pr. 4, Cor. 2).

PROPOSITION VIII. THEOREM.

If two planes which cut one another are each of them perpendicular to a third plane, their common section is perpendicular to the same plane.



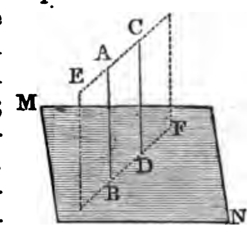
Let the two planes AE, AD be each of them perpendicular to a third plane MN, and let AB be the common section of the first two planes; then will AB be perpendicular to the plane MN.

For, from the point B, erect a perpendicular to the plane MN. Then, by the Corollary of the last Proposition, this line must be situated both in the plane AD and in the plane AE; hence it is their common section AB. Therefore, if two planes, etc.

PROPOSITION IX. THEOREM.

Two straight lines which are perpendicular to the same plane are parallel to each other.

Let the two straight lines AB, CD be each of them perpendicular to the same plane MN; then will AB be parallel to CD.



In the plane MN, draw the straight line BD, joining the points B and D. Through the lines AB, BD pass the plane EF; it will be perpendicular to the plane MN (Pr. 6); also, the line CD will lie in this plane, because it is perpendicular to MN (Pr. 7, Cor.).

Now, because AB and CD are both perpendicular to the plane MN, they are perpendicular to the line BD in that plane; and, since AB, CD are both perpendicular to the same line BD, and lie in the same plane, they are parallel to each other (B. I., Pr. 20). Therefore, two straight lines, etc.

Cor. 1. If one of two parallel lines be perpendicular to a plane, the other will be perpendicular to the same plane. If AB is perpendicular to the plane MN, then (Pr. 6) the plane EF will be perpendicular to MN. Also, AB is perpendicular to BD; and if CD is parallel to AB, it will be perpendicular to BD, and therefore (Pr. 7) it is perpendicular to the plane MN.

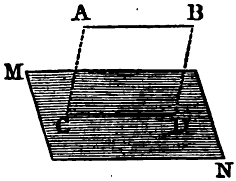
Cor. 2. Two straight lines parallel to the same straight line are parallel to each other. For, suppose a plane to be drawn perpendicular to any one of them; then the other two, being parallel to the first, will be perpendicular to the same plane, by the preceding Corollary; hence, by the Proposition, they will be parallel to each other.

The three straight lines are supposed not to be in the same plane; for in this case the Proposition has been already demonstrated.

PROPOSITION X. THEOREM.

If a straight line, without a given plane, be parallel to a straight line in the plane, it will be parallel to the plane.

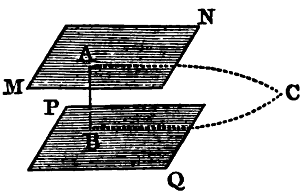
Let the straight line AB be parallel to the straight line CD, in the plane MN; then will it be parallel to the plane MN.



Through the parallels AB, CD suppose a plane ABDC to pass. If the line AB can meet the plane MN, it must meet it in some point of the line CD, which is the common intersection of the two planes. But AB can not meet CD, since they are parallel; hence it can not meet the plane MN; that is, AB is parallel to the plane MN (Def. 2). Therefore, if a straight line, etc.

PROPOSITION XI. THEOREM.

Two planes which are perpendicular to the same straight line are parallel to each other.



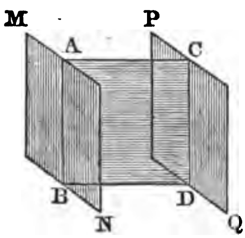
Let the planes MN, PQ be perpendicular to the line AB; then will they be parallel to each other.

For, if they are not parallel, they will meet if produced. Let them be produced and meet in C. Join AC, BC.

Now the line AB, which is perpendicular to the plane MN, is perpendicular to the line AC drawn through its foot in that plane. For the same reason, AB is perpendicular to BC. Therefore CA and CB are two perpendiculars let fall from the same point C upon the same straight line AB, which is impossible (B. I., Pr. 16). Hence the planes MN, PQ can not meet when produced; that is, they are parallel to each other. Therefore two planes, etc.

PROPOSITION XII. THEOREM.

If two parallel planes are cut by a third plane, their common sections with it are parallel.

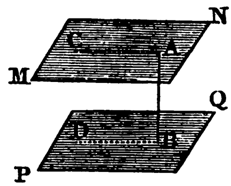


Let the parallel planes MN, PQ be cut by the plane ABDC, and let their common sections with it be AB, CD; then will AB be parallel to CD.

For the two lines AB, CD are in the same plane, viz., in the plane ABDC which cuts the planes MN, PQ; and if these lines were not parallel, they would meet when produced; therefore the planes MN, PQ would also meet, which is impossible, because they are parallel. Hence the lines AB, CD are parallel. Therefore, if two parallel planes, etc.

PROPOSITION XIII. THEOREM.

If two planes are parallel, a straight line which is perpendicular to one of them is also perpendicular to the other.



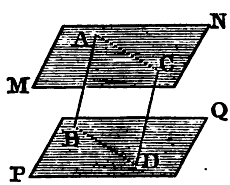
Let the two planes MN, PQ be parallel, and let the straight line AB be perpendicular to the plane MN; AB will also be perpendicular to the plane PQ.

Through the point B draw any line BD in the plane PQ, and through the lines AB, BD suppose a plane to pass intersecting the plane MN in AC. The two lines AC, BD will be parallel (Pr. 12).

But the line AB, being perpendicular to the plane MN, is perpendicular to the straight line AC, which meets it in that plane; it must, therefore, be perpendicular to its parallel BD (B. I., Pr. 23, Cor. 1). But BD is any line drawn through B in the plane PQ; and, since AB is perpendicular to any line drawn through its foot in the plane PQ, it must be perpendicular to the plane PQ (Def. 1). Therefore, if two planes, etc.

PROPOSITION XIV. THEOREM.

Parallel straight lines included between two parallel planes are equal.



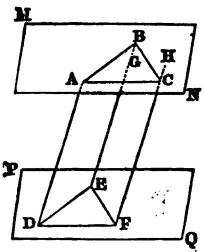
Let AB, CD be two parallel straight lines included between two parallel planes MN, PQ; then will AB be equal to CD.

Through the two parallel lines AB, CD, suppose a plane ABDC to pass, intersecting the parallel planes in AC and BD. The lines AC, BD will be parallel to each other (Pr. 12). But AB is, by supposition, parallel to CD; therefore the figure ABDC is a parallelogram, and, consequently, AB is equal to CD (B. I., Pr. 30). Therefore parallel straight lines, etc.

Cor. Hence *two parallel planes are every where equally distant;* for if AB, CD are perpendicular to the plane MN, they will be perpendicular to the parallel plane PQ (Pr. 13), and, being both perpendicular to the same plane, they will be parallel to each other (Pr. 9), and consequently equal.

PROPOSITION XV. THEOREM.

If two angles not in the same plane have their sides parallel to each other and similarly situated, these angles will be equal, and their planes will be parallel.



Let the two angles ABC, DEF, lying in different planes MN, PQ, have their sides parallel each to each and similarly situated; then will the angle ABC be equal to the angle DEF, and the plane MN be parallel to the plane PQ.

Take AB equal to DE, and BC equal to EF, and join AD, BE, CF, AC, DF. Then, because AB is equal and parallel to DE, the figure ABED is a parallelogram (B. I., Pr. 32), and AD is equal and parallel to BE.

For the same reason, CF is equal and parallel to BE. Consequently, AD and CF, being each of them equal and parallel to BE, are parallel to each other (Pr. 9, Cor. 2), and also equal; therefore AC is also equal and parallel to DF (B. I., Pr. 32). Hence the triangles ABC, DEF are mutually equilateral, and the angle ABC is equal to the angle DEF (B. I., Pr. 15).

Also, the plane ABC is parallel to the plane DEF. For, if they are not parallel, suppose a plane to pass through A parallel to DEF, and let it meet the straight lines BE, CF in the points G and H. Then the three lines AD, GE, HF will be equal (Pr. 14). But the three lines AD, BE, CF have already been proved to be equal; hence BE is equal to GE, and CF is equal to HF, which is absurd; consequently, the plane ABC must be parallel to the plane DEF. Therefore, if two angles, etc.

Cor. 1. If two parallel planes MN, PQ are met by two other planes ABED, BCFE, the angles formed by the intersections of the parallel planes will be equal. For the section AB is parallel to the section DE (Pr. 12), and BC is parallel to EF; therefore, by the Proposition, the angle ABC is equal to the angle DEF.

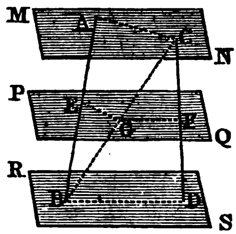
Cor. 2. If three straight lines AD, BE, CF, not situated in the same plane, are equal and parallel, the triangles ABC, DEF, formed by joining the extremities of these lines, will be equal, and their planes will be parallel.

For, since AD is equal and parallel to BE, the figure ABED is a parallelogram; hence the side AB is equal and parallel to DE.

For the same reason, the sides BC and EF are equal and parallel, as also the sides AC and DF. Consequently, the two triangles ABC, DEF are equal, and, according to the Proposition, their planes are parallel.

PROPOSITION XVI. THEOREM.

If two straight lines are cut by three parallel planes, their corresponding segments are proportional.



Let the straight lines AB, CD be cut by the parallel planes MN, PQ, RS in the points A, E, B, C, F, D; then we shall have the proportion

AE : EB :: CF : FD.

Draw the line BC meeting the plane PQ in G, and join AC, BD, EG, GF.

Then, because the two parallel planes MN, PQ are cut by the plane ABC, the common sections AC, EG are parallel (Pr. 12). Also, because the two parallel planes PQ, RS are cut by the plane BCD, the common sections BD, GF are parallel. Now, because EG is parallel to AC, a side of the triangle ABC (B. IV., Pr. 16), we have

AE : EB :: CG : GB.

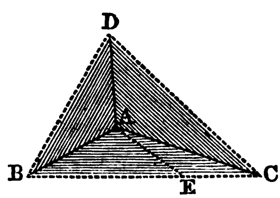
Also, because GF is parallel to BD, one side of the triangle BCD, we have CG : GB :: CF : FD;

hence (B. II., Pr. 4) AE : EB :: CF : FD.

Therefore, if two straight lines, etc.

PROPOSITION XVII. THEOREM.

If a solid angle is contained by three plane angles, the sum of any two of these angles is greater than the third.



Let the solid angle at A be contained by the three plane angles BAC, CAD, DAB; any two of these angles will be greater than the third.

If these three angles are all equal to each other, it is plain that any two of them must be greater than the third.

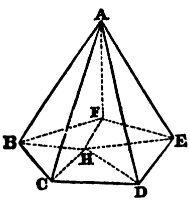
But if they are not equal, let BAC be that angle which is not less than either of the other two, and is greater than one of them, BAD. Then, at the point A, make the angle BAE equal to the angle BAD; take AE equal to AD; through E draw the line BEC, cutting AB, AC in the points B and C, and join DB, DC.

Now, because, in the two triangles BAD, BAE, AD is equal to AE, AB is common to both, and the angle BAD is equal to the angle BAE; therefore the base BD is equal to the base BE (B. I., Pr. 6). Also, because the sum of the lines BD, DC is greater than BC (B. I., Pr. 8), and BD is proved equal to BE, a part of BC, therefore the remaining line DC is greater than EC.

Now, in the two triangles CAD, CAE, because AD is equal to AE, AC is common; but the base CD is greater than the base CE, therefore the angle CAD is greater than the angle CAE (B. I., Pr. 14). But, by construction, the angle BAD is equal to the angle BAE; therefore the two angles BAD, CAD are together greater than BAE, CAE, that is, than the angle BAC. Now BAC is not less than either of the angles BAD, CAD; hence BAC, with either of them, is greater than the third. Therefore, if a solid angle, etc.

PROPOSITION XVIII. THEOREM.

The plane angles which contain any solid angle are together less than four right angles.



Let A be a solid angle contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; these angles are together less than four right angles.

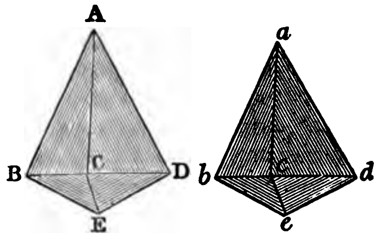
Let the planes which contain the solid angle at A be cut by another plane, forming the polygon BCDEF, and from any point H within this polygon draw the lines HB, HC, HD, HE, HF.

Now, because the solid angle at B is contained by three plane angles, any two of which are greater than the third (Pr. 17), the two angles ABC, ABF are greater than the angle FBC. For the same reason, the two angles ACB, ACD are greater than the angle BCD, and so with the other angles of the polygon BCDEF. Hence the sum of all the angles at the bases of the triangles having the common vertex A is greater than the sum of all the angles at the bases of the triangles whose vertex is H. But the sum of all the angles of the triangles whose vertex is A is equal to the sum of the angles of the same number of triangles whose vertex is H. Therefore the sum of the angles at A is less than the sum of the angles at H; that is, less than four right angles. Therefore the plane angles, etc.

Scholium. This demonstration supposes that the solid angle is convex; that is, that the plane of neither of the faces, if produced, would cut the solid angle. If it were otherwise, the sum of the plane angles would no longer be limited, and might be of any magnitude.

PROPOSITION XIX. THEOREM.

If two solid angles are contained by three plane angles which are equal each to each, the planes of the equal angles will be equally inclined to each other.

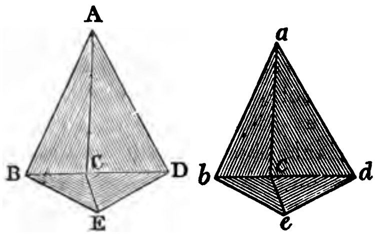


Let A and *a* be two solid angles contained by three plane angles which are equal each to each, viz., the angle BAC equal to *bac*, the angle CAD to *cad*, and BAD equal to *bad;* then will the inclination of the planes ABC, ABD be equal to the inclination of the planes *abc*, *abd*.

In the line AC, the common section of the planes ABC, ACD, take any point C, and through C let a plane BCE pass perpendicular to AB, and another plane CDE perpendicular to AD. Also, take *ac* equal to AC, and through *c* let a plane *bce* pass perpendicular to *ab*, and another plane *cde* perpendicular to *ad*.

Now, since the line AB is perpendicular to the plane BCE, it is perpendicular to every straight line which it meets in that plane; hence ABC and ABE are right angles. For the same reason, *abc* and *abe* are right angles. Now, in the triangles ABC, *abc*, the angle BAC is, by hypothesis, equal to *bac*, and the angles ABC, *abc* are right angles; therefore the angles ACB, *acb* are equal. But the side AC was made equal to the side *ac;* hence the two triangles are equal (B. I., Pr. 7); that is, the side AB is equal to *ab*, and BC to *bc*. In the same manner, it may be proved that AD is equal to *ad*, and CD to *cd*.

We can now prove that the quadrilateral ABED is equal to the quadrilateral *abed*. For, let the angle BAD be placed upon the equal angle *bad*, then the point B will fall upon the point *b*, and the point D upon the point *d;* because AB is equal to *ab*, and AD to *ad*. At the same time, BE, which is perpendicular to



AB, will fall upon *be*, which is perpendicular to *ab;* and, for a similar reason, DE will fall upon *de*. Hence the point E will fall upon *e*, and we shall have BE equal to *be*, and DE equal to *de*.

Now, since the plane BCE is perpendicular to the line AB, it is perpendicular to the plane ABD which passes through AB (Pr. 6). For the same reason, CDE is perpendicular to the same plane; hence CE, their common section, is perpendicular to the plane ABD (Pr. 8).

In the same manner, it may be proved that *ce* is perpendicular to the plane *abd*. Now, in the triangles BCE, *bce*, the angles BEC, *bec* are right angles, the hypothenuse BC is equal to the hypothenuse *bc*, and the side BE is equal to *be;* hence the two triangles are equal, and the angle CBE is equal to the angle *cbe*. But the angle CBE is the inclination of the planes ABC, ABD (Def. 4), and the angle *cbe* is the inclination of the planes *abc*, *abd;* hence these planes are equally inclined to each other. Therefore, if two solid angles, etc.

Scholium 1. The angle CBE is not, properly speaking, the inclination of the planes ABC, ABD, except when the perpendicular CE falls upon the same side of AB as AD does. If it fall upon the other side of AB, then the angle between the two planes will be obtuse, and this angle, together with the angle B of the triangle CBE, will make two right angles. But in this case, the angle between the two planes *abc*, *abd* will also be obtuse, and this angle, together with the angle *b* of the triangle *cbe*, will also make two right angles. And, since the angle B is always equal to the angle *b*, the inclination of the two planes ABC, ABD will always be equal to that of the planes *abc*, *abd*.

Scholium 2. If two solid angles are contained by three plane angles which are equal each to each, and *similarly situated*, the angles will be equal, and will coincide when applied the one to the other.

For we have proved that the quadrilateral ABED will coincide with its equal *abed*. Now, because the triangle BCE is equal to the triangle *bce*, the line CE, which is perpendicular to the plane ABED, is equal to the line *ce*, which is perpendicular to the plane *abed*. And, since only one perpendicular can be drawn to a plane from the same point (Pr. 4, Cor. 2), the lines CE, *ce* must coincide with each other, and the point C coincide with the point *c*. Hence the two solid angles must coincide throughout.

It should, however, be observed, that the two solid angles do not admit of superposition unless the three equal plane angles are *similarly situated* in both cases. For if the perpendiculars CE, *ce* lay on opposite sides of the planes ABED, *abed*, the two solid angles could not be made to coincide. Nevertheless, the Proposition will always hold true, that the planes containing the equal angles are equally inclined to each other.

BOOK VIII.

POLYEDRONS.

Definitions.

1. A *polyedron* is a geometrical solid bounded by planes. The polygons formed by the mutual intersection of the bounding planes are called the *faces* of the polyedron.

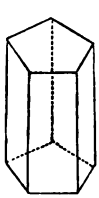
2. The least number of planes that can form a polyedron is four, for it requires at least three planes to form a solid angle, and it requires a fourth plane to inclose a finite portion of space, or to form a solid. A polyedron of four faces is called a *tetraedron;* one of six faces a *hexaedron;* one of eight faces an *octaedron;* one of twelve faces a *dodecaedron;* and one of twenty faces an *icosaedron.*

3. The common intersection of two adjacent faces of a polyedron is called an *edge* of the polyedron. A *diagonal* of a polyedron is a straight line which joins any two of its vertices not lying in the same face.

4. *Similar* polyedrons are such as have all their solid angles equal each to each, and are contained by the same number of similar polygons similarly placed.

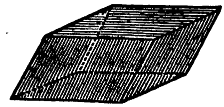
5. A *regular* polyedron is one whose solid angles are all equal to each other, and whose faces are all equal and regular polygons.

6. A *prism* is a polyedron having two faces which are equal and parallel polygons, and the others are parallelograms. The equal and parallel polygons are called the *bases* of the prism; the other faces, taken together, form its *lateral* or *convex surface.* The intersections of the lateral faces are called the *lateral edges* of the prism. The *altitude* of a prism is the perpendicular distance between the planes of its bases.

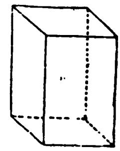


7. A *right prism* is one whose lateral edges are all perpendicular to the planes of its bases. An *oblique prism* is one whose lateral edges are oblique to the planes of its bases.

8. A prism is *triangular, quadrangular, pentagonal, hexagonal,* etc., according as its base is a triangle, a quadrilateral, a pentagon, a hexagon, etc.

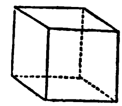


9. A *parallelopiped* is a prism whose bases are parallelograms. It is therefore a polyedron, all of whose faces are parallelograms.

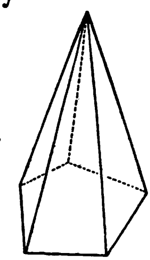


10. A *right* parallelopiped is a parallelopiped whose lateral edges are perpendicular to the planes of its bases. Hence its lateral faces are all rectangles, but its bases may be either rhomboids or rectangles.

A *rectangular* parallelopiped is a right parallelopiped whose bases are rectangles. Hence it is a parallelopiped all of whose faces are rectangles.



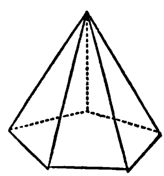
11. A *cube* is a rectangular parallelopiped whose six faces are all squares.



12. A *pyramid* is a polyedron bounded by a polygon called its *base*, and three or more triangles meeting in a point without the polygon called the *vertex* of the pyramid. The triangular faces taken together constitute its *lateral* or *convex* surface.

13. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base produced, if necessary.

14. A *triangular* pyramid is one whose base is a triangle; a *quadrangular* pyramid is one whose base is a quadrilateral, etc. A triangular pyramid is a tetraedron, and any one of its faces may be taken as its base.



15. A *regular* pyramid is one whose base is a regular polygon, and the perpendicular drawn from its vertex to its base passes through the centre of the base. This perpendicular is called the *axis* of the pyramid.

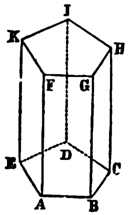
The *slant height* of a regular pyramid is the perpendicular from the vertex to one side of the polygon which forms its base.

16. A *frustum* of a pyramid is a portion of the pyramid included between its base and a section made by a plane parallel to the base. The *altitude* of a frustum is the perpendicular distance between the two parallel planes.

17. The *volume* of a polyedron is the numerical measure of its magnitude, referred to some other polyedron as the unit. The polyedron adopted as the unit is called the *unit of volume.*

PROPOSITION I. THEOREM.

The lateral surface of a right prism is equal to the product of the perimeter of its base by its altitude.



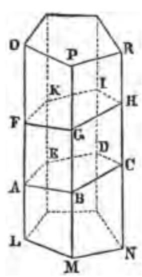
Let ABCDE–K be a right prism; then will its lateral surface be equal to the perimeter of its base (viz., AB+BC+CD+DE+EA) multiplied by its altitude AF.

For the lateral surface of the prism is equal to the sum of the parallelograms AG, BH, CI, etc. Now the area of the parallelogram AG is measured by the product of its base AB by its altitude AF (B. IV., Pr. 4, Sch.). The area of the parallelogram BH is measured by BC × BG; the area of CI is measured by CD × CH, and so of the others. But the lines AF, BG, CH, etc., are all equal to each other (B. VII., Pr. 14), and each is equal to the altitude of the prism. Also, the lines AB, BC, CD, etc., taken together, form the perimeter of the base of the prism. Therefore the sum of these parallelograms, or the lateral surface of the prism, is equal to the product of the perimeter of its base by its altitude.

Cor. If two right prisms have the same altitude, their convex surfaces will be to each other as the perimeters of their bases.

PROPOSITION II. THEOREM.

Sections of a prism made by parallel planes are equal polygons.



Let the prism LR be cut by the parallel planes AC, FH; then will the sections ABCDE, FGHIK be equal polygons.

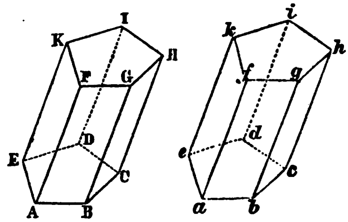
Since AB and FG are the intersections of two parallel planes, with a third plane LMPO, they are parallel. The lines AF, BG are also parallel, being edges of the prism; therefore ABGF is a parallelogram, and AB is equal to FG. For the same reason, BC is equal and parallel to GH, CD to IH, DE to IK, and AE to FK.

Because the sides of the angle ABC are parallel to those of FGH, and are similarly situated, the angle ABC is equal to FGH (B. VII., Pr. 15). In like manner, it may be proved that the angle BCD is equal to the angle GHI, and so of the rest. Therefore the polygons ABCDE, FGHIK being mutually equilateral, and also mutually equiangular, are equal.

Cor. Any section of a prism made by a plane parallel to the base is equal to the base.

PROPOSITION III. THEOREM.

Two prisms are equal when they have a solid angle contained by three faces which are equal each to each, and similarly situated.



Let AI, *ai* be two prisms having the faces which contain the solid angle B equal to the faces which contain the solid angle *b*; viz., the base ABCDE to the base *abcde*, the parallelogram AG to the parallelogram *ag*, and the parallelogram BH to the parallelogram *bh*; then will the prism AI be equal to the prism *ai*.

Let the prism AI be applied to the prism *ai*, so that the equal bases AD and *ad* may coincide, the point A falling upon *a*, B upon *b*, and so on.

And because the three plane angles which contain the solid angle B are equal to the three plane angles which contain the solid angle *b*, and these planes are similarly situated, the solid angles B and *b* are equal (B. VII., Pr. 19, Sch. 2). Hence the edge BG will coincide with its equal *bg*, and the point G will coincide with the point *g*.

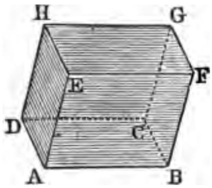
Now, because the parallelograms AG and *ag* are equal, the side GF will fall upon its equal *gf*; and, for the same reason, GH will fall upon *gh*. Hence the plane of the base FGHIK will coincide with the plane of the base *fghik* (B. VII., Pr. 2). But, since the upper bases are equal to their corresponding lower bases, they are equal to each other; therefore the base FI will coincide throughout with *fi*; viz., HI with *hi*, IK with *ik*, and KF with *kf*; hence the lateral faces of the two prisms will coincide each with each, and the prisms coincide throughout, and are equal to each other. Therefore, two prisms, etc.

Cor. Two right prisms, which have equal bases and equal altitudes, are equal.

For, since the side AB is equal to *ab*, and the altitude BG to *bg*, the rectangle ABGF is equal to the rectangle *abgf*. So, also, the rectangle BGHC is equal to the rectangle *bghc*; hence the three faces which contain the solid angle B are equal to the three faces which contain the solid angle *b*; consequently the two prisms are equal.

PROPOSITION IV. THEOREM.

The opposite faces of a parallelopiped are equal and parallel.



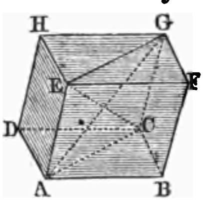
Let ABGH be a parallelopiped; then will its opposite faces be equal and parallel.

From the definition of a parallelopiped (Def. 9), the bases AC, EG are equal and parallel; and it remains to be proved that the same is true of any two opposite faces, as AH, BG.

Now, because AC is a parallelogram, the side AD is equal and parallel to BC. For the same reason, AE is equal and parallel to BF; hence the angle DAE is equal to the angle CBF (B. VII., Pr. 15), and the plane DAE is parallel to the plane CBF. Therefore also the parallelogram AH is equal to the parallelogram BG. In the same manner, it may be proved that the opposite faces AF and DG are equal and parallel. Therefore, the opposite faces, etc.

Cor. 1. Since a parallelopiped is a solid contained by six faces, of which the opposite ones are equal and parallel, any face may be assumed as the base of a parallelopiped.

Cor. 2. *The four diagonals of a parallelopiped bisect each other.*



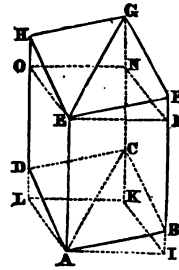
Draw any two diagonals AG, EC; they will bisect each other.

Since AE is equal and parallel to CG, the figure AEGC is a parallelogram, and therefore the diagonals AG, EC bisect each other (B. I., Pr. 33). In the same manner, it may be proved that the two diagonals BH and DF bisect each other; and hence the four diagonals mutually bisect each other in a point which may be regarded as the centre of the parallelopiped.

PROPOSITION V. THEOREM.

If a parallelopiped be cut by a plane passing through the diagonals of two opposite faces, it will be divided into two equivalent triangular prisms.

Let AG be a parallelopiped, and AC, EG the diagonals of the opposite parallelograms BD, FH. Now, because AE, CG are



each of them parallel to BF, they are parallel to each other; therefore the diagonals AC, EG are in the same plane with AE, CG; and the plane AEGC divides the solid AG into two equivalent prisms.

Through the vertices A and E draw the planes AIKL, EMNO perpendicular to AE, meeting the other edges of the parallelopiped in the points I, K, L, and in M, N, O. The sections AIKL, EMNO are equal, because they are formed by planes perpendicular to the same straight line, and consequently parallel (Pr. 2). They are also parallelograms, because AI, KL, two opposite sides of the same section, are the intersections of two parallel planes ABFE, DCGH, by the same plane.

For the same reason, the figure ALOE is a parallelogram; so, also, are AIME, IKNM, KLON, the other lateral faces of the solid AIKL–EMNO; hence this solid is a prism (Def. 6); and it is a right prism, because AE is perpendicular to the plane of its base. But the right prism AN is divided into two equal prisms ALK–N, AIK–N; for the bases of these prisms are equal, being halves of the same parallelogram AIKL, and they have the common altitude AE; they are therefore equal (Pr. 3, Cor.).

Now, because AEHD, AEOL are parallelograms, the sides DH, LO, being equal to AE, are equal to each other. Take away the common part DO, and we have DL equal to HO. For the same reason, CK is equal to GN.

Conceive now that ENO, the base of the solid ENGHO, is placed on AKL, the base of the solid AKCDL; then, the point O falling on L, and N on K, the lines HO, GN will coincide with their equals DL, CK, because they are perpendiculars to the same plane. Hence the two solids coincide throughout, and are equal to each other. To each of these equals add the solid ADC–N; then will the oblique prism ADC–G be equivalent to the right prism ALK–N.

In the same manner, it may be proved that the oblique prism ABC–G is equivalent to the right prism AIK–N. But the two right prisms have been proved to be equal; hence the two oblique prisms ADC–G, ABC–G are equivalent to each other. Therefore, if a parallelopiped, etc.

Cor. Every triangular prism is half of a parallelopiped having the same solid angle, and the same edges AB, BC, BF.

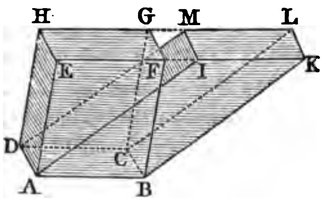
Scholium. The triangular prisms into which the oblique parallelopiped is divided can not be made to coincide, because the plane angles about the corresponding solid angles are not similarly situated.

PROPOSITION VI. THEOREM.

Parallelopipeds upon the same base and of the same altitude are equivalent.

Case first. When their upper bases are between the same parallel lines.

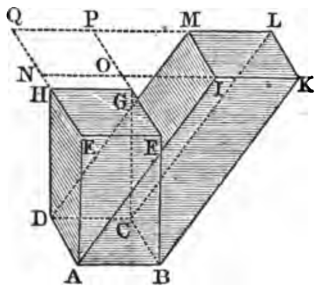
Let the parallelopipeds AG, AL have the base AC common, and let their opposite bases EG, IL be in the same plane, and between the same parallels EK, HL; then will the solid AG be equivalent to the solid AL.



Because AF, AK are parallelograms, EF and IK are each equal to AB, and therefore equal to each other. Hence, if EF and IK be taken away from the same line EK, the remainders EI and FK will be equal. Therefore the triangle AEI is equal to the triangle BFK.

Also, the parallelogram EM is equal to the parallelogram FL, and AH to BG. Hence the solid angles at E and F are contained by three faces which are equal to each other and similarly situated; therefore the prism AEI–M is equal to the prism BFK–L (Pr. 3).

Now if from the whole solid AL we take the prism AEI–M, there will remain the parallelopiped AL; and if from the same solid AL we take the prism BFK–L, there will remain the parallelopiped AG. Hence the parallelopipeds AL, AG are equivalent to one another.



Case second. When their upper bases are not between the same parallel lines.

Let the parallelopipeds AG, AL have the same base AC and the same altitude; then will their opposite bases EG, IL be in the same plane. Also, since the sides EF and

IK are equal and parallel to AB, they are equal and parallel to each other. For the same reason, FG is equal and parallel to KL.

Produce the sides EH, FG, as also IK, LM, and let them meet in the points N, O, P, Q; the figure NOPQ is a parallelogram equal to each of the bases EG, IL; and, consequently, equal to ABCD, and parallel to it.

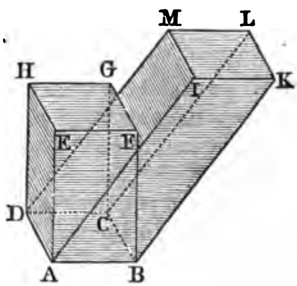
Conceive now a third parallelopiped AP, having AC for its lower base, and NP for its upper base. The solid AP will be equivalent to the solid AG by the first Case, because they have the same lower base, and their upper bases are in the same plane and between the same parallels, EQ, FP. For the same reason, the solid AP is equivalent to the solid AL; hence the solid AG is equivalent to the solid AL. Therefore parallelopipeds, etc.

PROPOSITION VII. THEOREM.

Any parallelopiped is equivalent to a rectangular parallelopiped having the same altitude and an equivalent base.

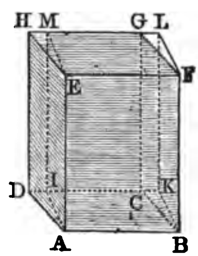
Let AL be any parallelopiped; it is equivalent to a right parallelopiped having the same altitude and an equivalent base.

From the points A, B, C, D draw AE, BF, CG, DH perpendicular to the plane of the lower base, meeting the plane of the upper base in the points E, F, G, H. Join EF, FG, GH, HE; there will thus be formed the parallelopiped AG, equivalent to AL (Pr. 6); and its lateral faces AF, BG, CH, DE are rectangles.



If the base ABCD is also a rectangle, AG will be a rectangular parallelopiped, and it is equivalent to the parallelopiped AL.

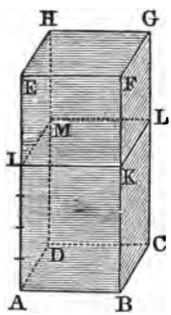
But if ABCD is not a rectangle, from A and B draw AI, BK perpendicular to CD, and from E and F draw EM, FL perpendicular to GH, and join IM, KL. The solid ABKI–M will be a rectangular parallelopiped. For, by construction, the bases ABKI and EFLM are rectangles; so, also, are the lateral faces, because the edges AE, BF, KL, IM are perpendicular to the plane of the base. Therefore the solid AL



is a rectangular parallelopiped. But the two parallelopipeds AG, AL may be regarded as having the same base AF, and the same altitude AI; they are therefore equivalent. But the parallelopiped AG is equivalent to the first supposed parallelopiped; hence this parallelopiped is equivalent to the rectangular parallelopiped AL, having the same altitude, and an equivalent base. Therefore any parallelopiped etc.

PROPOSITION VIII. THEOREM.

Two rectangular parallelopipeds having the same base are to each other as their altitudes.



Let AG, AL be two rectangular parallelopipeds having the same base ABCD; then will they be to each other as their altitudes AE, AI.

Case first. When the altitudes are in the ratio of two whole numbers.

Suppose the altitudes AE, AI are in the ratio of two whole numbers; for example, as seven to four. Divide AE into seven equal parts; AI will contain four of those parts. Through the several points of division let planes be drawn parallel to the base; these planes will divide the solid AG into seven small parallelopipeds, all equal to each other, having equal bases and equal altitudes. The bases are equal, because every section of a prism parallel to the base is equal to the base (Pr. 2, Cor.); the altitudes are equal, for these altitudes are the equal divisions of the edge AE. But of these seven equal parallelopipeds, AL contains four; hence the solid AG is to the solid AL as seven to four, or as the altitude AE is to the altitude AI.

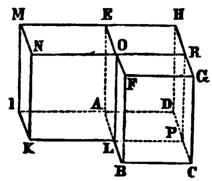
Case second. When the altitudes are not in the ratio of two whole numbers; that is, are incommensurable, the demonstration will be similar to that given in B. III., Pr. 14. Therefore two rectangular parallelopipeds, etc.

PROPOSITION IX. THEOREM.

Two rectangular parallelopipeds having the same altitude are to each other as their bases.

Let AG, AN be two rectangular parallelopipieds having the same altitude AE; then will they be to each other as their bases; that is, *solid* AG : *solid* AN :: *base* ABCD : *base* AIKL.

Place the two solids so that their surfaces may have the common angle BAE; produce the plane LKNO till it meets the plane DCGH in the line PR; a third parallelopiped AR will thus be formed, which may be compared with each of the parallelopipeds AG, AN.



The two solids AG, AR, having the same base AEHD, are to each other as their altitudes AB, AL (Pr. 8); and the two solids AR, AN, having the same base ALOE, are to each other as their altitudes AD, AI. Hence we have the two proportions *solid* AG : *solid* AR :: AB : AL;

solid AR : *solid* AN :: AD : AI.

Hence (B. II., Pr. 12, Cor.)

solid AG : *solid* AN :: AB × AD : AL × AI.

But AB × AD is the measure of the base ABCD (B. IV., Pr. 4, Sch.); and AL × AI is the measure of the base AIKL; hence

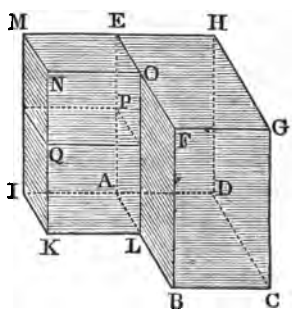
solid AG : *solid* AN :: *base* ABCD : *base* AIKL.

Therefore two rectangular parallelopipeds, etc.

PROPOSITION X. THEOREM.

Any two rectangular parallelopipeds are to each other as the products of their bases by their altitudes.

Let AG, AQ be two rectangular parallelopipeds, of which the bases are the rectangles ABCD, AIKL, and the altitudes the perpendiculars AE, AP; then will the solid AG be to the solid AQ as the product of ABCD by AE is to the product of AIKL by AP.

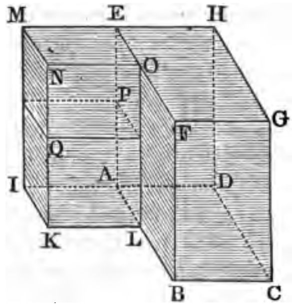


Place the two solids so that their surfaces may have the common angle BAE; produce the planes necessary to form the third parallelopiped AN, having the same base with AQ, and the same altitude with AG. Then, by the last Proposition, we shall have

solid AG : *solid* AN :: ABCD : AIKL.

But the two parallelopipeds AN, AQ, having the same base AIKL, are to each other as their altitudes AE, AP (Pr. 8); hence we have *solid* AN : *solid* AQ :: AE : AP.

Comparing these two proportions (B. II., Pr. 12, Cor.), we have



solid AG : *solid* AQ :: ABCD × AE : AIKL × AP.

If, instead of the base ABCD, we put its equal AB × AD, and instead of AIKL, we put its equal AI × AL, we shall have

solid AG : *solid* AQ :: AB × AD × AE : AI × AL × AP.

Therefore any two rectangular parallelopipeds, etc.

Scholium. Hence a rectangular parallelopiped is measured by the product of its base and altitude, or the product of its three dimensions.

It should be remembered that, by the product of two or more lines, we understand the product of the numbers which represent those lines; and these numbers depend upon the linear unit employed, which may be assumed at pleasure. If we take a foot as the unit of measure, then the number of feet in the length of the base, multiplied by the number of feet in its breadth, will give the number of square feet in the base. If we multiply this product by the number of feet in the altitude, it will give the number of cubic feet in the parallelopiped. If we take an inch as the unit of measure, we shall obtain in the same manner the number of cubic inches in the parallelopiped.

PROPOSITION XI. THEOREM.

The volume of a prism is measured by the product of its base by its altitude.

For any parallelopiped is equivalent to a rectangular parallelopiped, having the same altitude and an equivalent base (Pr. 7). But the volume of the latter is measured by the product of its base by its altitude; therefore the volume of the former is also measured by the product of its base by its altitude.

Now a triangular prism is half of a parallelopiped having the same altitude and a double base (Pr. 5). But the volume of the latter is measured by the product of its base by its altitude; hence a triangular prism is measured by the product of its base by its altitude.

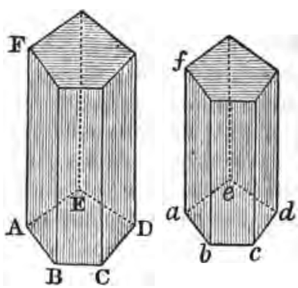
But any prism can be divided into as many triangular prisms of the same altitude as there are triangles in the polygon which forms its base.

Also, the volume of each of these triangular prisms is measured by the product of its base by its altitude; and, since they all have the same altitude, the sum of these prisms will be measured by the sum of the triangles which form the bases, multiplied by the common altitude. Therefore the volume of any prism is measured by the product of its base by its altitude.

Cor. If two prisms have equal altitudes, the products of the bases by the altitudes will be as the bases (B. II., Pr. 10); hence *prisms having equal altitudes are to each other as their bases.* For the same reason, *prisms having equivalent bases are to each other as their altitudes; and any two prisms are to each other as the products of their bases and altitudes.*

PROPOSITION XII. THEOREM.

Similar prisms are to each other as the cubes of their homologous edges.



Let ABCDE–F, *abcde–f* be two similar prisms; then will the prism AD–F be to the prism *ad–f* as AB^3 to ab^3, or as AF^3 to af^3.

For the solids are to each other as the products of their bases and altitudes (Pr. 11, Cor.); that is, as ABCDE $\times$ AF to $abcde \times af$. But, since the prisms are similar, the bases are similar figures, and are to each other as the squares of their homologous sides; that is, as AB^2 to ab^2. Therefore we have

$$\textit{solid}\ FD : \textit{solid}\ fd :: AB^2 \times AF : ab^2 \times af.$$

But, since BF and *bf* are similar figures, their homologous sides are proportional; that is,

$$AB : ab :: AF : af;$$

whence (B. II., Pr. 11)

$$AB^2 : ab^2 :: AF^2 : af^2.$$

Also,

$$AF : af :: AF : af.$$

Therefore (B. II., Pr. 12),

$$AB^2 \times AF : ab^2 \times af :: AF^3 : af^3 :: AB^3 : ab^3.$$

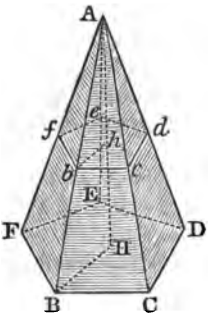
Hence (B. II., Pr. 4) we have

$$\textit{solid}\ FD : \textit{solid}\ fd :: AB^3 : ab^3 :: AF^3 : af^3.$$

Therefore similar prisms, etc.

PROPOSITION XIII. THEOREM.

If a pyramid be cut by a plane parallel to its base,
1st. *The edges and the altitude will be divided proportionally.*
2d. *The section will be a polygon similar to the base.*



Let A–BCDEF be a pyramid cut by a plane *bcdef* parallel to its base, and let AH be its altitude; then will the edges AB, AC, AD, etc., with the altitude AH, be divided proportionally in *b*, *c*, *d*, *e*, *f*, *h*, and the section *bcdef* will be similar to BCDEF.

First. Since the planes FBC, *fbc* are parallel, their sections FB, *fb*, with a third plane AFB, are parallel (B. VII., Pr. 12); therefore the triangles AFB, A*fb* are similar, and we have the proportion

$$AF : Af :: AB : Ab.$$

For the same reason, $AB : Ab :: AC : Ac$,
and so for the other edges. Therefore the edges AB, AC, etc., are cut proportionally in *b*, *c*, etc. Also, since BH and *bh* are parallel, we have $AH : Ah :: AB : Ab$.

Secondly. Because *fb* is parallel to FB, *bc* to BC, *cd* to CD, etc., the angle *fbc* is equal to FBC (B. VII., Pr. 15), the angle *bcd* is equal to BCD, and so on. Moreover, since the triangles AFB, A*fb* are similar, we have $FB : fb :: AB : Ab$.
And because the triangles ABC, A*bc* are similar, we have

$$AB : Ab :: BC : bc.$$

Therefore, by equality of ratios (B. II., Pr. 4),

$$FB : fb :: BC : bc.$$

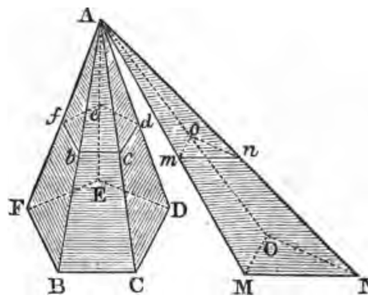
For the same reason,

$$BC : bc :: CD : cd, \text{ and so on.}$$

Therefore the polygons BCDEF, *bcdef* have their angles equal each to each, and their homologous sides proportional; hence they are similar. Therefore, if a pyramid, etc.

Cor. 1. *If two pyramids having the same altitude, and their bases situated in the same plane, are cut by a plane parallel to their bases, the sections will be to each other as the bases.*

Let A–BCDEF, A–MNO be two pyramids having the same altitude, and their bases situated in the same plane; if these pyramids are cut by a plane parallel to the bases, the sections *bcdef*, *mno* will be to each other as the bases BCDEF, MNO.



For, since the polygons BCDEF, *bcdef* are similar, their surfaces are as the squares of the homologous sides BC, *bc* (B. IV., Pr. 27). But, by the preceding Proposition,

$$BC : bc :: AB : Ab.$$

Therefore

$$BCDEF : bcdef :: AB^2 : Ab^2.$$

For the same reason,

$$MNO : mno :: AM^2 : Am^2.$$

But, since *bcdef* and *mno* are in the same plane, we have

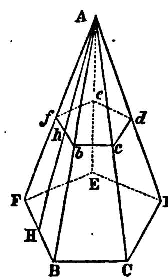
$$AB : Ab :: AM : Am \text{ (B. VII., Pr. 16)};$$

consequently, $BCDEF : bcdef :: MNO : mno.$

Cor. 2. If the bases BCDEF, MNO are equivalent, the sections *bcdef*, *mno* will also be equivalent.

PROPOSITION XIV. THEOREM.

The lateral surface of a regular pyramid is equal to the product of the perimeter of its base by half its slant height.

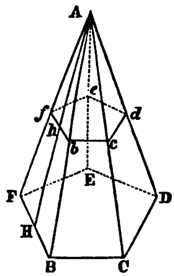


Let A–BDE be a regular pyramid whose base is the polygon BCDEF, and its slant height AH; then will its lateral surface be equal to the perimeter BC+CD+DE, etc., multiplied by half of AH.

The triangles AFB, ABC, ACD, etc., are all equal, for the sides FB, BC, CD, etc., are all equal (Def. 15); and, since the oblique lines AF, AB, AC, etc., are all at equal distances from the perpendicular, they are equal to each other (B. VII., Pr. 5). Hence the altitudes of these several triangles are equal.

But the area of the triangle AFB is equal to FB multiplied by half of AH; and the same is true of the other triangles ABC, ACD, etc. Hence the sum of the triangles is equal to the sum of the bases FB, BC, CD, DE, EF multiplied by half the common altitude AH; that is, the lateral surface of the pyramid is equal to the perimeter of its base multiplied by half the slant height.

Cor. 1. *The lateral surface of a frustum of a regular pyramid is equal to the sum of the perimeters of its two bases multiplied by half its slant height.*

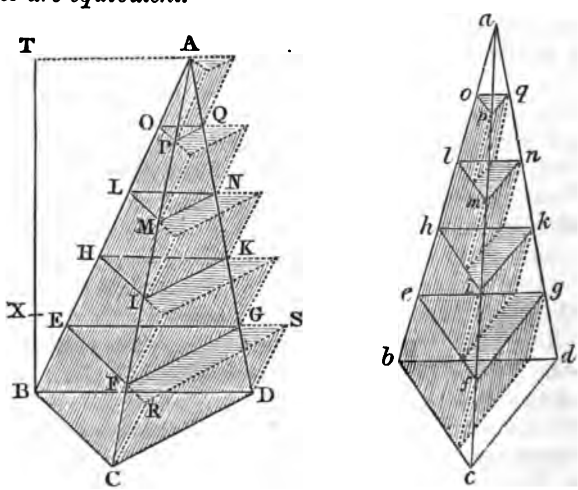


Each side of a frustum of a regular pyramid, as FBbf, is a trapezoid (Pr. 13). Now the area of this trapezoid is equal to the sum of its parallel sides FB, fb, multiplied by half its altitude Hh (B. IV., Pr. 7). But the altitude of each of these trapezoids is the same; therefore the area of all the trapezoids, or the lateral surface of the frustum, is equal to the sum of the perimeters of the two bases multiplied by half the slant height.

Cor. 2. If the frustum is cut by a plane parallel to the bases, and at equal distances from them, this plane must bisect the edges Bb, Cc, etc. (B. IV., Pr. 16); and the area of each trapezoid is equal to its altitude multiplied by the line which joins the middle points of its two inclined sides (B. IV., Pr. 7, Cor.). Hence *the lateral surface of a frustum of a pyramid is equal to its slant height multiplied by the perimeter of a section at equal distances between the two bases.*

PROPOSITION XV. THEOREM.

Two triangular pyramids having equivalent bases and equal altitudes are equivalent.



Let A–BCD, a–bcd be two triangular pyramids having equivalent bases BCD, bcd, supposed to be situated in the same plane, and having the common altitude TB; then will the pyramid A–BCD be equivalent to the pyramid *a–bcd*.

For, if they are not equivalent, let the pyramid A–BCD be the greater, and suppose it to exceed the pyramid *a–bcd* by a prism whose base is BCD, and altitude BX.

Divide the altitude BT into equal parts, each less than BX; and through the several points of division let planes be made to pass parallel to the base BCD, making the sections EFG, *efg* equivalent to each other (Pr. 13, Cor. 2); also, HIK equivalent to *hik*, etc.

From the point C draw the straight line CR parallel to BE, meeting EF produced in R; and from D draw DS parallel to BE, meeting EG in S. Join RS, and it is plain that the solid BCD–ERS is a prism lying partly without the pyramid.

In the same manner, upon the triangles EFG, HIK, etc., taken as bases, construct exterior prisms, having for edges the parts EH, HL, etc., of the line AB. In like manner, on the bases *efg*, *hik*, *lmn*, etc., in the second pyramid, construct interior prisms, having for edges the corresponding parts of *ab*.

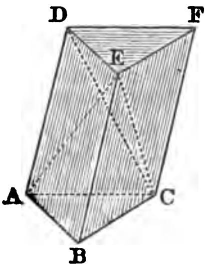
It is plain that the sum of all the exterior prisms of the pyramid A–BCD is greater than this pyramid; and also, that the sum of all the interior prisms of the pyramid *a–bcd* is smaller than this pyramid. Hence the difference between the sum of all the exterior prisms and the sum of all the interior ones must be greater than the difference between the two pyramids themselves.

Now, beginning with the bases BCD, *bcd*, the second exterior prism EFG–H is equivalent to the first interior prism *efg–b*, because their bases are equivalent, and they have the same altitude. For the same reason, the third exterior prism HIK–L, and the second interior prism *hik–e* are equivalent; the fourth exterior and the third interior, and so on to the last in each series. Hence all the exterior prisms of the pyramid A–BCD, excepting the first prism BCD–E, have equivalent corresponding ones in the interior prisms of the pyramid *a–bcd*.

Therefore the prism BCD–E is the difference between the sum of all the exterior prisms of the pyramid A–BCD, and the sum of all the interior prisms of the pyramid *a–bcd*. But the difference between these two sets of prisms has been proved to be greater than that of the two pyramids; hence the prism BCD–E is greater than the prism BCD–X, which is impossible, for they have the same base BCD, and the altitude of the first is less than BX, the altitude of the second. Hence the pyramids A–BCD, *a–bcd* are not unequal in volume; that is, they are equivalent to each other. Therefore, triangular pyramids, etc.

PROPOSITION XVI. THEOREM.

Any triangular pyramid is the third part of a triangular prism having the same base and the same altitude.



Let E–ABC be a triangular pyramid, and ABC–DEF a triangular prism having the same base and the same altitude; then will the pyramid be one third of the prism.

Cut off from the prism the pyramid E–ABC by the plane EAC; there will remain the solid E–ACFD, which may be considered as a quadrangular pyramid whose vertex is E, and whose base is the parallelogram ACFD. Draw the diagonal CD, and through the points C, D, E pass a plane, dividing the quadrangular pyramid into two triangular ones E–ACD, E–CDF.

Then, because ACFD is a parallelogram, of which CD is the diagonal, the triangle ACD is equal to the triangle CDF. Therefore the pyramid, whose base is the triangle ACD, and vertex the point E, is equivalent to the pyramid whose base is the triangle CDF, and vertex the point E. But the latter pyramid is equivalent to the pyramid E–ABC, for they have equal bases, viz., the triangles ABC, DEF, and the same altitude, viz., the altitude of the prism ABC–DEF. Therefore the three pyramids E–ABC, E–ACD, E–CDF, are equivalent to each other, and they compose the whole prism ABC–DEF; hence the pyramid EABC is the third part of the prism which has the same base and the same altitude.

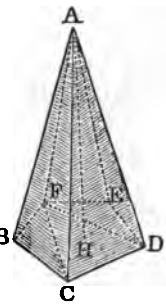
Cor. The volume of a triangular pyramid is measured by the product of its base by one third of its altitude.

PROPOSITION XVII. THEOREM.

The volume of any pyramid is measured by the product of its base by one third of its altitude.

Let A–BCDEF be any pyramid, whose base is the polygon BCDEF, and altitude AH; then will the volume of the pyramid be measured by BCDEF $\times \frac{1}{3}$AH.

Divide the polygon BCDEF into triangles by the diagonals CF, DF, and let planes pass through these lines and the vertex A; they will divide the polygonal pyramid A–BCDEF into triangular pyramids, all having the same altitude AH.



But each of these pyramids is measured by the product of its base by one third of its altitude (Pr. 16, Cor.); hence the sum of the triangular pyramids, or the polygonal pyramid A–BCDEF, will be measured by the sum of the triangles BCF, CDF, DEF, or the polygon BCDEF, multiplied by one third of AH. Therefore every pyramid is measured by the product of its base by one third of its altitude.

Cor. 1. Every pyramid is one third of a prism having the same base and altitude.

Cor. 2. Pyramids having equal altitudes are to each other as their bases; pyramids having equivalent bases are to each other as their altitudes; and any two pyramids are to each other as the products of their bases by their altitudes.

Cor. 3. Similar pyramids are to each other as the cubes of their homologous edges.

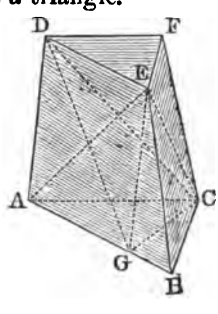
Scholium. The volume of any polyedron may be found by dividing it into pyramids, by planes passing through its vertices.

PROPOSITION XVIII. THEOREM.

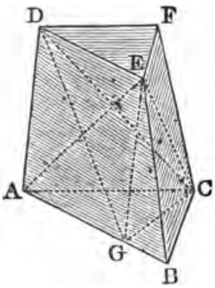
A frustum of a pyramid is equivalent to the sum of three pyramids having the same altitude as the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them.

Case first. When the base of the frustum is a triangle.

Let ABC–DEF be a frustum of a triangular pyramid. If a plane be made to pass through the points A, C, E, it will cut off the pyramid E–ABC, whose altitude is the altitude of the frustum, and its base is ABC, the lower base of the frustum.



Pass another plane through the points C, D, E; it will cut off the pyramid C–DEF, whose altitude is that of the frustum, and its base is DEF, the upper base of the frustum.



To find the magnitude of the remaining pyramid E–ACD, draw EG parallel to AD; join CG, DG. Then, because the two triangles AGC, DEF have the angles at A and D equal to each other, we have (B. IV., Pr. 24)

AGC : DEF :: AG × AC : DE × DF,

:: AC : DF, because AG is equal to DE.

Also (B. IV., Pr. 6, Cor. 1),

ACB : ACG :: AB : AG or DE.

But, because the triangles ABC, DEF are similar (Pr. 13), we have

AB : DE :: AC : DF.

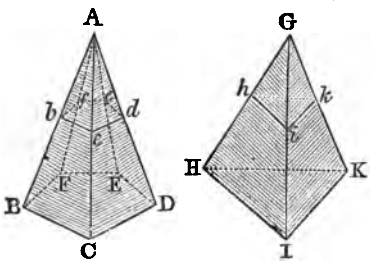
Therefore (B. II., Pr. 4)

ACB : ACG :: ACG : DEF;

that is, the triangle ACG is a mean proportional between ACB and DEF, the two bases of the frustum.

Now the pyramid E–ACD is equivalent to the pyramid G–ACD, because it has the same base and the same altitude; for EG is parallel to AD, and consequently parallel to the plane ACD. But the pyramid G–ACD has the same altitude as the frustum, and its base ACG is a mean proportional between the two bases of the frustum.

Case second. When the base of the frustum is any polygon.



Let BCDEF–*bcdef* be a frustum of any pyramid.

Let G–HIK be a triangular pyramid having the same altitude and an equivalent base with the pyramid A–BCDEF, and from it let a frustum HIK–*hik* be cut off, having the same altitude with the frustum BCDEF–*bcdef*.

The entire pyramids are equivalent (Pr. 17), and the small pyramids A–*bcdef*, G–*hik* are also equivalent, for their altitudes are equal, and their bases are equivalent (Pr. 13, Cor. 2). Hence the two frustums are equivalent, and they have the same altitude, with equivalent bases. But the frustum HIK–*hik* has been proved to be equivalent to the sum of three pyramids, each having the same altitude as the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them. Hence the same must be true of the frustum of any pyramid. Therefore a frustum of a pyramid, etc.

Scholium. If V denotes the volume of the frustum, B its lower base, b its upper base, and h its altitude, this proposition is expressed by the formula

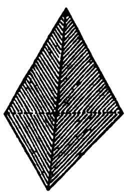
$$V=\tfrac{1}{3}h(B+b+\sqrt{B\times b}).$$

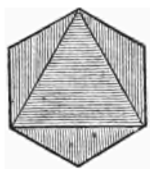
PROPOSITION XIX. THEOREM.

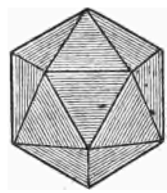
There can be but five regular polyedrons.

Since the faces of a regular polyedron are regular polygons, they must consist of equilateral triangles, of squares, of regular pentagons, or polygons of a greater number of sides.

First. If the faces are equilateral triangles, each solid angle of the polyedron may be contained by three of these triangles, form-



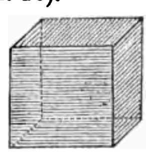




ing the *tetraedron;* or by four, forming the *octaedron;* or by five, forming the *icosaedron.*

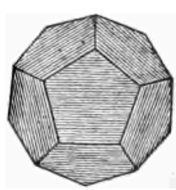
No other regular polyedron can be formed with equilateral triangles; for six angles of these triangles amount to four right angles, and can not form a solid angle (B. VII., Pr. 18).

Secondly. If the faces are squares, their angles may be united three and three, forming the *hexaedron* or cube.



Four angles of squares amount to four right angles, and can not form a solid angle.

Thirdly. If the faces are regular pentagons, their angles may be united three and three, forming the regular *dodecaedron.* Four angles of a regular pentagon are greater than four right angles, and can not form a solid angle.

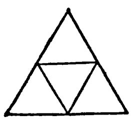


Fourthly. A regular polyedron can not be formed with regular hexagons, for three angles of a regular hexagon amount to four right angles. Three angles of a regular heptagon amount to more than four right angles; and the same is true of any polygon having a greater number of sides.

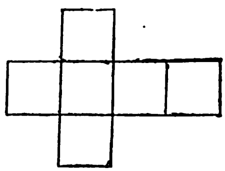
Hence there can be but five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

Scholium. Models of the regular polyedrons may be easily obtained as follows: Let the figures represented below be accurately drawn on card-board and cut out entire. At the lines separating two adjacent polygons let the card-board be cut half through; the edges of the several polygons in each figure may then be brought together so as to represent a regular polyedron, and they may be secured in their place by gluing the edges.

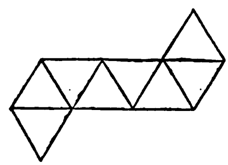
Tetraedron.



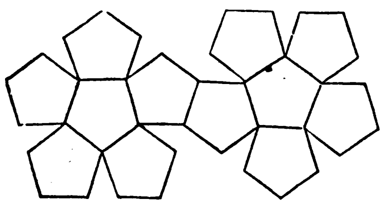
Hexaedron.



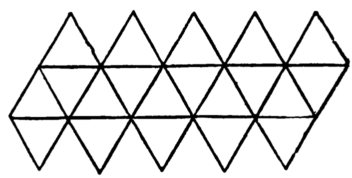
Octaedron.



Dodecaedron.

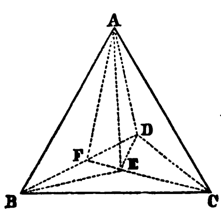


Icosaedron.



PROPOSITION XX. PROBLEM.

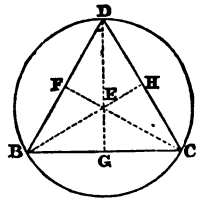
To compute the volume of a regular tetraedron.



Let A–BCD be a regular tetraedron; it is required to determine its volume.

From one angle, A, let fall the perpendicular AE upon the opposite face BCD. By Def. 5, the faces of the tetraedron are all equal triangles, therefore AB, AC, AD are equal to each other. Hence they are equally distant from the perpendicular (B. VII., Pr. 5, Cor.); that is, E is the centre of a circle described about the equilateral triangle BCD. The area of the triangle BCD is equal to $\frac{BC^2}{4}\sqrt{3}$ (B. VI., Pr. 4, Sch. 2).

Since EF is one half of EC (B. VI., Pr. 4), it is one third of FC or AF. Then, in the triangle AEF, we have (preceding figure)



$$AE^2=AF^2-FE^2=AF^2-\tfrac{1}{9}AF^2=\tfrac{8}{9}AF^2.$$

Also, $\quad AF^2=CF^2=\tfrac{3}{4}BC^2.$

Therefore $AE^2=\tfrac{8}{9}\times\tfrac{3}{4}BC^2=\tfrac{2}{3}BC^2$;

or, $\quad AE=BC\sqrt{\tfrac{2}{3}}.$

Hence the volume of the tetraedron is equal

to $$\frac{BC^2}{4}\sqrt{3}\times\tfrac{1}{3}BC\sqrt{\tfrac{2}{3}}=\tfrac{1}{12}BC^3\sqrt{2};$$

that is, *the volume of a regular tetraedron is equal to the cube of a linear edge multiplied by one twelfth the square root of two.*

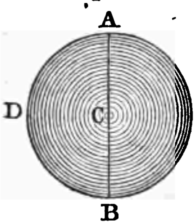
Cor. The entire surface of the tetraedron is equal to four times the area of the triangle BCD; or $BC^2\sqrt{3}$; that is, *the surface of a regular tetraedron is equal to the square of a linear edge multiplied by the square root of three.*

BOOK IX.

SPHERICAL GEOMETRY.

Definitions.

1. A *sphere* is a solid bounded by a curved surface, all the points of which are equally distant from a point within called the *centre*.



A sphere may be conceived to be described by the revolution of a semicircle ADB about its diameter AB, which remains unmoved.

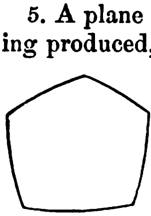
2. A radius of a sphere is a straight line drawn from the centre to any point of the surface. A *diameter* is any straight line drawn through the centre, and terminated each way by the surface.

All the radii of a sphere are equal; all the diameters are also equal, and each double of the radius.

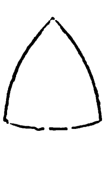
3. It will be shown (Prop. 1) that every section of a sphere made by a plane is a circle. A *great circle* is a section made by a plane which passes through the centre of the sphere. A *small circle* is a section made by a plane which does not pass through the centre.

4. The *poles* of a circle of a sphere are the extremities of that diameter of the sphere which is perpendicular to the plane of the circle.

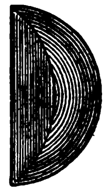
5. A plane *touches* a sphere when it meets the sphere, but, being produced, does not cut it.



6. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles, each of which is less than a semi-circumference. These arcs are called the sides of the polygon; and the angles which their planes make with each other are the angles of the polygon.

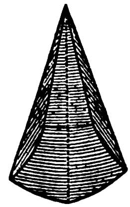


7. A *spherical triangle* is a spherical polygon of three sides. It is called *right-angled*, *isosceles*, or *equilateral* in the same cases as a plane triangle.

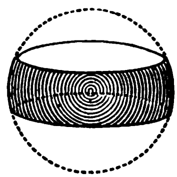


8. A *lune* is a portion of the surface of a sphere included between two semi-circumferences of great circles having a common diameter.

9. A *spherical ungula* or *wedge* is a portion of a sphere included between the halves of two great circles, and has the lune for its *base.*



10. A *spherical pyramid* is a portion of a sphere included between the planes of a solid angle whose vertex is at the centre. The *base* of the pyramid is the spherical polygon intercepted by those planes.

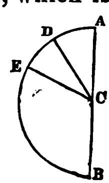


11. A *zone* is a portion of the surface of a sphere included between two parallel planes.

12. A *spherical segment* is a portion of a sphere included between two parallel planes.

13. The *bases* of the segment are the sections of the sphere made by the parallel planes; the *altitude* of the segment or zone is the distance between the planes. One of the two planes may *touch* the sphere, in which case the segment has but one base.

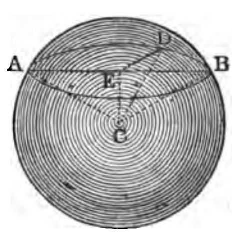
14. When a semicircle, revolving about its diameter, describes a sphere, any sector of the semicircle describes a solid, which is called a *spherical sector.*



Thus, when the semicircle AEB, revolving about its diameter AB, describes a sphere, any circular sector, as ACD or DCE, describes a spherical sector.

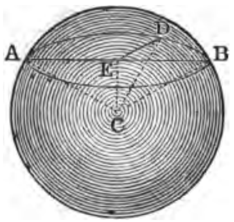
PROPOSITION I. THEOREM.

Every section of a sphere made by a plane is a circle.



Let ABD be a section made by a plane in a sphere whose centre is C. From the point C draw CE perpendicular to the plane ABD; and draw lines CA, CB, CD, etc., to different points of the curve ABD which bounds the section.

The oblique lines CA, CB, CD are equal, because they are radii of the sphere; there-



fore they are equally distant from the perpendicular CE (B. VII., Pr. 5, Cor.). Hence all the lines EA, EB, ED are equal; and, consequently, the section ABD is a circle, of which E is the centre. Therefore every section, etc.

Cor. 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; hence all great circles of a sphere are equal to each other.

Cor. 2. Any two great circles of a sphere bisect each other; for, since they have the same centre, their common section is a diameter of both, and therefore bisects both.

Cor. 3. Every great circle divides the sphere and its surface into two equal parts. For if the two parts are separated and applied to each other, base to base, with their convexities turned the same way, the two surfaces must coincide; otherwise there would be points in these surfaces unequally distant from the centre.

Cor. 4. The centre of a small circle and that of the sphere are in a straight line perpendicular to the plane of the small circle.

Cor. 5. The circle which is farthest from the centre is the least; for the greater the distance CE, the less is the chord AB, which is the diameter of the small circle ABD.

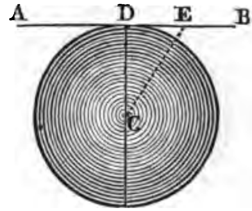
Cor. 6. An arc of a great circle may be made to pass through any two points on the surface of a sphere; for the two given points, together with the centre of the sphere, make three points which are necessary to determine the position of a plane. If, however, the two given points were situated at the extremities of a diameter, these two points and the centre would then be in one straight line, and any number of great circles might be made to pass through them.

PROPOSITION II. THEOREM.

A plane perpendicular to a diameter at its extremity touches the sphere.

Let ADB be a plane perpendicular to the diameter DC at its extremity D, then the plane ADB touches the sphere at the point D.

Let E be any other point in the plane ADB, and join DE, CE. Because CD is perpendicular to the plane ADB, it is perpendicu-

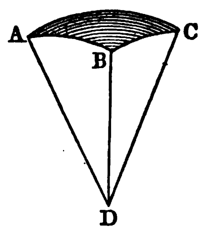


lar to the line AB (B. VII., Def. 1); hence the angle CDE is a right angle, and the line CE is greater than CD. Consequently, the point E lies without the sphere. Hence the plane ADB has only the point D in common with the sphere; it therefore touches the sphere (Def. 5). Therefore a plane, etc.

Cor. In the same manner, it may be proved that two spheres touch each other when the distance between their centres is equal to the sum or difference of their radii, in which case the centres and the point of contact lie in one straight line.

PROPOSITION III. THEOREM.

Any side of a spherical triangle is less than the sum of the other two.

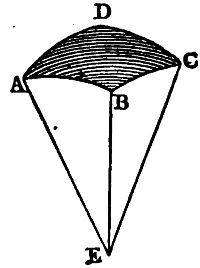


Let ABC be a spherical triangle; then any side, as AC, is less than the sum of the other two, AB and BC.

Let D be the centre of the sphere, and draw the radii AD, BD, CD. Conceive the planes ADB, BDC, CDA to be drawn, forming a solid angle at D. The angles ADB, BDC, CDA will be measured by AB, BC, CA, the sides of the spherical triangle ABC. But when a solid angle is formed by three plane angles, any one of them is less than the sum of the other two (B. VII., Pr. 17); hence any one of the arcs AB, BC, CA must be less than the sum of the other two. Therefore any side, etc.

PROPOSITION IV. THEOREM.

The sum of the sides of a spherical polygon is less than the circumference of a great circle.

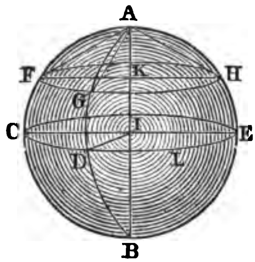


Let ABCD be any spherical polygon; then will the sum of the sides AB, BC, CD, DA be less than the circumference of a great circle.

Let E be the centre of the sphere, and join AE, BE, CE, DE. The solid angle at E is contained by the plane angles AEB, BEC, CED, DEA, which together are less than four right angles (B. VII., Pr. 18). Hence the sides AB, BC, CD, DA, which are the measures of these angles, are together less than four quadrants described with the radius AE; that is, than the circumference of a great circle. Therefore the sum of the sides, etc.

PROPOSITION V. THEOREM.

All the points in the circumference of a circle of the sphere are equally distant from each of its poles.



Let FGH be any circle of the sphere, and AB any diameter of the sphere which is perpendicular to its plane; then, by the definition (4), A and B are the poles of the circle FGH.

Since AB is perpendicular to the plane of the circle FGH, it passes through K, the centre of that circle (Pr. 1, Cor. 4). Hence, if we draw the oblique lines AF, AG, AH, these lines will be equally distant from the perpendicular AK, and are therefore equal to each other (B. VII., Pr. 5). Hence all the points of the circumference FGH are equally distant from the pole A. For a similar reason, they are equally distant from the pole B. Therefore all the points, etc.

Cor. 1. All the arcs of great circles drawn from a pole of a circle to points in its circumference are equal. For the chords AF, AG, AH are all equal, and therefore the arcs AF, AG, AH are equal.

Cor. 2. The arc of a great circle AD, drawn from the pole to the circumference of another great circle CDE, is a quadrant, for this arc is the measure of the right angle AID.

Cor. 3. If the distance of the point A from each of the points C and D is equal to a quadrant, the point A will be the pole of the arc CD. For, since the arcs AC, AD are quadrants, the angles AIC, AID are right angles; therefore the diameter AB is perpendicular to each of the lines CI, DI, and is consequently perpendicular to the plane of the arc CD (B. VII., Pr. 4); hence it is the pole of the arc CD.

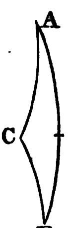
Cor. 4. To find the pole of an arc of a great circle, as CD, at each of the extremities C and D draw the arcs of great circles CA and DA perpendicular to CD; the point of intersection of these arcs will be the pole required.

Scholium. Arcs of circles may be drawn upon the surface of a sphere with the same ease as upon a plane surface. Thus, by revolving the arc AF around the pole A, the point F will describe the small circle FGH; and by revolving the quadrant AC around the pole A, the extremity C will describe the great circle CDE.

If it is required to draw an arc of a great circle through two points C and D on the surface of the sphere, then, from the points C and D as centres, with a radius equal to a quadrant, describe two arcs intersecting in A. The point A will be the pole of the great circle required; and if from A as a centre, with a radius equal to a quadrant, we describe a circle CDE, it will be a great circle passing through C and D.

PROPOSITION VI. THEOREM.

The shortest path from one point to another on the surface of a sphere is the smaller of the two arcs of a great circle, joining the two given points.



Let A and B be any two points on the surface of a sphere, and let ADB be the arc of a great circle which joins them; then will the line ADB be the shortest path from A to B on the surface of the sphere.

For, if possible, let the shortest path from A to B pass through C, a point situated out of the arc of a great circle ADB. Draw AC, CB, arcs of great circles, and take BD equal to BC.

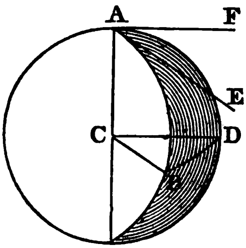
By Prop. 3, the arc ADB is less than AC+CB. Subtracting the equal arcs BD and BC, there will remain AD less than AC. Now the shortest path from B to C, whether it be an arc of a circle or some other line, is equal to the shortest path from B to D; for, by revolving BC around B, the point C may be made to coincide with D, and thus the shortest path from B to C must coincide with the shortest path from B to D. But the shortest path from A to B was supposed to pass through C; hence the shortest path from A to C can not be greater than the shortest path from A to D.

Now the arc AD has been proved to be less than AC; and therefore, if AC be revolved about A until the point C falls on the arc ADB, the point C will fall between D and B. Hence the shortest path from C to A must be greater than the shortest path from D to A; but it has just been proved not to be greater, which is absurd. Consequently, no point of the shortest path from A to B can be out of the arc of a great circle ADB. Therefore the shortest path, etc.

PROPOSITION VII. THEOREM.

The angle formed by two arcs of great circles is equal to the angle formed by the tangents of those arcs at the point of their intersection, and is measured by the arc of a great circle described from its vertex as a pole, and included between its sides.

Let BAD be an angle formed by two arcs of great circles; then will it be equal to the angle EAF formed by the tangents

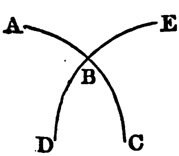


of these arcs at the point A; and it is measured by the arc DB described from the vertex A as a pole.

For the tangent AE, drawn in the plane of the arc AB, is perpendicular to the radius AC (B. III., Pr. 9); also, the tangent AF, drawn in the plane of the arc AD, is perpendicular to the same radius AC. Hence the angle EAF is equal to the angle of the planes ACB, ACD (B. VII., Def. 4), which is the same as that of the arcs AB, AD.

Also, if the arcs AB, AD are each equal to a quadrant, the lines CB, CD will be perpendicular to AC, and the angle BCD will be equal to the angle of the planes ACB, ACD; hence the arc BD measures the angle of the planes, or the angle BAD.

Cor. 1. Angles of spherical triangles may be compared with each other by means of arcs of great circles described from their vertices as poles, and included between their sides; and thus an angle can easily be made equal to a given angle.



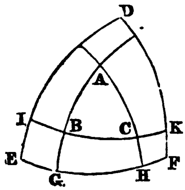
Cor. 2. If two arcs of great circles AC, DE cut each other, the vertical angles ABE, DBC are equal; for each is equal to the angle formed by the two planes ABC, DBE. Also, the two adjacent angles ABD, DBC are together equal to two right angles.

PROPOSITION VIII. THEOREM.

If from the vertices of a given spherical triangle, as poles, arcs of great circles are described, these arcs, by their intersection, form a second triangle, whose vertices are poles of the sides of the given triangle.

Let ABC be a spherical triangle; and from the points A, B, C as poles, let great circles be described intersecting each other in

D, E, and F; then will the points D, E, and F be the poles of the sides of the triangle ABC.



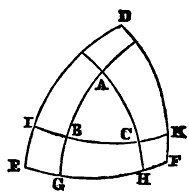
For, because the point A is the pole of the arc EF, the distance from A to E is a quadrant. Also, because the point C is the pole of the arc DE, the distance from C to E is a quadrant. Hence the point E is at a quadrant's distance from each of the points A and C; it is therefore the pole of the arc AC (Pr. 5, Cor. 3). In the same manner, it may be proved that D is the pole of the arc BC, and F the pole of the arc AB.

Scholium. The triangle DEF is called the *polar triangle* of ABC; and so, also, ABC is the polar triangle of DEF.

Since all great circles intersect each other in two points, the arcs DE, EF, DF, if produced, will form three other triangles; but the triangle which is taken as the polar triangle is the central one, whose vertex D, homologous to A, is on the same side of BC as the vertex A; and so of the other vertices.

PROPOSITION IX. THEOREM.

In two polar triangles, each angle of either triangle is measured by the supplement of the side lying opposite to it in the other triangle.



Let DEF be a spherical triangle, ABC its polar triangle, then will the side EF be the supplement of the arc which measures the angle A, and the side BC is the supplement of the arc which measures the angle D.

Produce the sides AB, AC, if necessary, until they meet EF in G and H. Then, because the point A is the pole of the arc GH, the angle A is measured by the arc GH (Pr. 7).

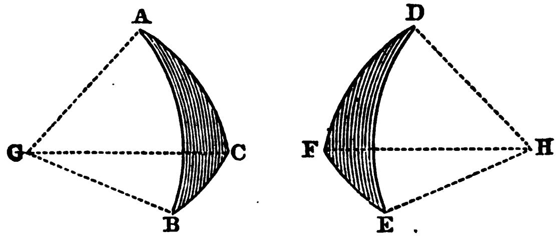
Also, because E is the pole of the arc AH, the arc EH is a quadrant; and because F is the pole of AG, the arc FG is a quadrant. Hence EH and GF, or EF and GH, are together equal to a semi-circumference. Therefore EF is the supplement of GH, which measures the angle A.

So, also, DF is the supplement of the arc which measures the angle B, and DE is the supplement of the arc which measures the angle C. In the same manner, it can be shown that each angle of the triangle DEF is measured by the supplement of the side lying opposite to it in the triangle ABC. Therefore in two polar triangles, etc.

PROPOSITION X. THEOREM.

If two triangles on equal spheres are mutually equilateral, they are mutually equiangular.

Let ABC, DEF be two triangles on equal spheres, having the side AB equal to DE, AC to DF, and BC to EF; then will the angles also be equal each to each.



Let the centres of the spheres be G and H, and draw the radii GA, GB, GC, HD, HE, HF. A solid angle may be conceived as formed at G by the three plane angles AGB, AGC, BGC; and another solid angle at H by the three plane angles DHE, DHF, EHF. Then, because the arcs AB, DE are equal, the angles AGB, DHE, which are measured by these arcs, are equal. For the same reason, the angles AGC, DHF are equal to each other; and, also, BGC equal to EHF. Hence G and H are two solid angles contained by three equal plane angles; therefore the planes of these equal angles are equally inclined to each other (B. VII., Pr. 19). That is, the angles of the triangle ABC are equal to those of the triangle DEF, viz., the angle ABC to the angle DEF, BAC to EDF, and ACB to DFE.

Scholium. It should be observed that the two triangles ABC, DEF do not admit of superposition unless the three sides are *similarly situated* in both cases. Triangles which are mutually equilateral, but can not be applied to each other so as to coincide, are called *symmetrical* triangles.

PROPOSITION XI. THEOREM.

If two triangles on equal spheres are mutually equiangular, they are mutually equilateral.

Denote by A and B two spherical triangles which are mutually equiangular, and by P and Q their polar triangles.

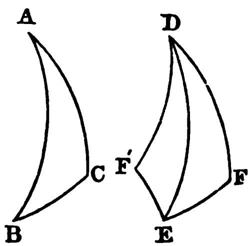
Since the sides of P and Q are the supplements of the arcs which measure the angles of A and B (Pr. 9), P and Q must be mutually equilateral. Also, because P and Q are mutually equilateral, they must be mutually equiangular (Pr. 10). But the sides of A and B are the supplements of the arcs which measure the angles of P and Q, and, therefore, A and B are mutually equilateral.

PROPOSITION XII. THEOREM.

If two triangles on equal spheres have two sides and the included angle of the one equal to two sides and the included angle of the other each to each, their third sides will be equal, and their other angles will be equal each to each.

Let ABC, DEF be two triangles having the side AB equal to DE, AC equal to DF, and the angle BAC equal to the angle EDF; then will the side BC be equal to EF, the angle ABC to DEF, and ACB to DFE.

If the equal sides in the two triangles are similarly situated, the triangle ABC may be applied to the triangle DEF in the same manner as in plane triangles (B. I., Pr. 6), and the two triangles will coincide throughout. Therefore all the parts of the one triangle will be equal to the corresponding parts of the other triangle.

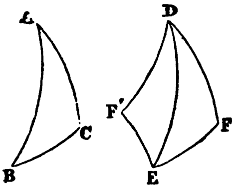


But if the equal sides in the two triangles are not similarly situated, then construct the triangle DF′E symmetrical with DFE, having DF′ equal to DF, and EF′ equal to EF. The two triangles DEF′, DEF, being mutually equilateral, are also mutually equiangular (Pr. 10).

Now the triangle ABC may be applied to the triangle DEF′ so as to coincide throughout, and hence all the parts of the one triangle will be equal to the corresponding parts of the other triangle. Therefore the side BC, being equal to EF′, is also equal to EF; the angle ABC, being equal to DEF′, is also equal to DEF; and the angle ACB, being equal to DF′E, is also equal to DFE. Therefore, if two triangles, etc.

PROPOSITION XIII. THEOREM.

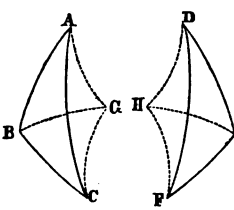
If two triangles on equal spheres have two angles and the included side of the one equal to two angles and the included side of the other each to each, their third angles will be equal, and their other sides will be equal each to each.



If the two triangles ABC, DEF have the angle BAC equal to the angle EDF, the angle ABC equal to DEF, and the included side AB equal to DE, the triangle ABC can be placed upon the triangle DEF, or upon its symmetrical triangle DEF′, so as to coincide. Hence the remaining parts of the triangle ABC will be equal to the remaining parts of the triangle DEF; that is, the side AC will be equal to DF, BC to EF, and the angle ACB to the angle DFE. Therefore, if two triangles, etc.

PROPOSITION XIV. THEOREM.

If two triangles on equal spheres are mutually equilateral, they are equivalent.



Let ABC, DEF be two triangles which have the three sides of the one equal to the three sides of the other each to each, viz., AB to DE, AC to DF, and BC to EF; then will the triangle ABC be equivalent to the triangle DEF.

Let G be the pole of the small circle passing through the three points A, B, C; draw the great circle arcs GA, GB, GC; these arcs will be equal to each other (Pr. 5). At the point E make the angle DEH equal to the angle ABG; make the arc EH equal to the arc BG, and join DH, FH.

Because, in the triangles ABG, DEH, the sides DE, EH are equal to the sides AB, BG, and the included angle DEH is equal to ABG, the arc DH is equal to AG, and the angle DHE equal to AGB (Pr. 12).

Now, because the triangles ABC, DEF are mutually equilateral, they are mutually equiangular (Pr. 10); hence the angle ABC is equal to the angle DEF. Subtracting the equal angles ABG, DEH, the remainder GBC will be equal to the remainder HEF.

Moreover, the sides BG, BC are equal to the sides EH, EF; hence the arc HF is equal to the arc GC, and the angle EHF to the angle BGC (Pr. 13).

Now the triangle DEH may be applied to the triangle ABG so as to coincide. For, place DH upon its equal BG, and HE upon its equal AG, they will coincide, because the angle DHE is equal to the angle AGB; therefore the two triangles coincide throughout, and have equal surfaces.

For the same reason, the surface HEF is equal to the surface GBC, and the surface DFH to the surface ACG. Hence

$$ABG+GBC-ACG=DEH+EHF-DFH;$$

or,

$$ABC=DEF;$$

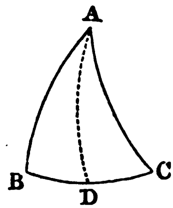
that is, the two triangles ABC, DEF are equivalent. Therefore, if two triangles, etc.

Scholium. The poles G and H might be situated within the triangles ABC, DEF, in which case it would be necessary to add the three triangles ABG, GBC, ACG to form the triangle ABC, and also to add the three triangles DEH, EHF, DFH to form the triangle DEF, otherwise the demonstration would be the same as above.

Cor. If two triangles on equal spheres are mutually equiangular, they are equivalent. They are also equivalent if they have two sides and the included angle of the one equal to two sides and the included angle of the other each to each, or two angles and the included side of the one equal to two angles and the included side of the other.

PROPOSITION XV. THEOREM.

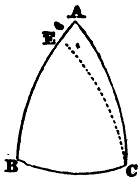
In an isosceles spherical triangle, the angles opposite the equal sides are equal; and, conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.



Let ABC be a spherical triangle having the side AB equal to AC; then will the angle ABC be equal to the angle ACB.

From the point A draw the arc AD to the middle of the base BC. Then, in the two triangles ABD, ACD, the side AB is equal to AC, BD is equal to DC, and the side AD is common; hence the angle ABD is equal to the angle ACD (Pr. 11).

Conversely. Let the angle B be equal to the angle C; then will the side AC be equal to the side AB.



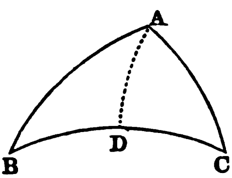
For if the two sides are not equal to each other, let AB be the greater; take BE equal to AC, and join EC.

Then, in the triangles EBC, ACB, the two sides BE, BC are equal to the two sides CA, CB, and the included angles EBC, ACB are equal; hence the angle ECB is equal to the angle ABC (Pr. 13). But, by hypothesis, the angle ABC is equal to ACB; hence ECB is equal to ACB, which is absurd. Therefore AB is not greater than AC; and, in the same manner, it can be proved that it is not less; it is, consequently, equal to AC. Therefore, in an isosceles spherical triangle, etc.

Cor. The angle BAD is equal to the angle CAD, and the angle ADB to the angle ADC; therefore each of the last two angles is a right angle. Hence *the arc drawn from the vertex of an isosceles spherical triangle to the middle of the base is perpendicular to the base, and also bisects the vertical angle.*

PROPOSITION XVI. THEOREM.

In a spherical triangle, the greater side is opposite the greater angle, and conversely.



Let ABC be a spherical triangle having the angle A greater than the angle B; then will the side BC be greater than the side AC.

Draw the arc AD, making the angle BAD equal to B. Then, in the triangle ABD, we shall have AD equal to DB (Pr. 15); that is, BC is equal to the sum of AD and DC. But AD and DC are together greater than AC (Pr. 2); hence BC is greater than AC.

Conversely. If the side BC is greater than AC, then will the angle A be greater than the angle B.

For if the angle A is not greater than B, it must be equal to it, or less. It is not equal; for then the side BC would be equal to AC (Pr. 15), which is contrary to the hypothesis. Neither can it be less, for then the side BC would be less than AC by the first case, which is also contrary to the hypothesis. Hence the angle BAC is greater than the angle ABC. Therefore, in a spherical triangle, etc.

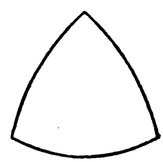
PROPOSITION XVII. THEOREM.

The sum of the angles of a spherical triangle is greater than two, and less than six right angles.

Let A, B, and C be the angles of a spherical triangle. The arcs which measure the angles A, B, and C, together with the three sides of the polar triangle, are equal to three semi-circumferences (Pr. 9). But the three sides of the polar triangle are less than two semi-circumferences (Pr. 4); hence the arcs which measure the angles A, B, and C are greater than one semi-circumference, and, therefore, the angles A, B, and C are greater than two right angles.

Also, because each angle of a spherical triangle is less than two right angles, the sum of the three angles must be less than six right angles.

Cor. A spherical triangle may have two, or even three right angles; also two, or even three obtuse angles.

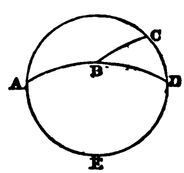


If a triangle have three right angles, each of its sides will be a quadrant, and the triangle is called a *tri-rectangular* triangle. The tri-rectangular triangle is contained eight times in the surface of the sphere.*

---

* In all the preceding propositions, it has been supposed, in conformity with Def. 6, that spherical triangles always have each of their sides less than a semi-circumference, in which case their angles are always less than two right angles.

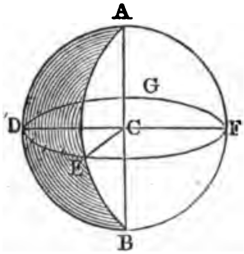
It should, however, be remarked, that there are spherical triangles of which certain sides are greater than a semi-circumference, and certain angles greater than two right angles. For if we produce the side AC so as to form an entire circumference, ACDE, the part which remains, after taking from the surface of the hemisphere the triangle ABC, is a new triangle, which may also be designated by ABC, and the sides of which are AB, BC, CDEA. Here we see that the side CDEA is greater than the semi-circumference DEA, and, at the same time, the opposite angle ABC exceeds two right angles by the quantity CBD.



Triangles whose sides and angles are so large have been excluded by the definition, because their solution always reduces itself to that of triangles embraced in the definition. Thus, if we know the sides and angles of the triangle ABC, we shall know immediately the sides and angles of the triangle of the same name, which is the remainder of the surface of the hemisphere.

PROPOSITION XVIII. THEOREM.

The area of a lune is to the surface of the sphere as the angle of the lune is to four right angles.



Let ADBE be a lune, upon a sphere whose centre is C, and the diameter AB; then will the area of the lune be to the surface of the sphere as the angle DCE to four right angles, or as the arc DE to the circumference of a great circle.

First. When the ratio of the arc to the circumference can be expressed in whole numbers.

Suppose the ratio of DE to DEFG to be as 4 to 25. Now, if if we divide the circumference DEFG in 25 equal parts, DE will contain 4 of those parts. If we join the pole A and the several points of division by arcs of great circles, there will be formed on the hemisphere ADEFG 25 triangles, all equal to each other, being mutually equilateral. The entire sphere will contain 50 of these small triangles, and the lune ADBE 8 of them. Hence the area of the lune is to the surface of the sphere as 8 to 50, or as 4 to 25; that is, as the arc DE to the circumference.

Secondly. When the ratio of the arc to the circumference can not be expressed in whole numbers, it may be proved, as in B. III., Pr. 14, that the lune is still to the surface of the sphere as the angle of the lune to four right angles.

Cor. 1. On equal spheres, two lunes are to each other as the angles included between their planes.

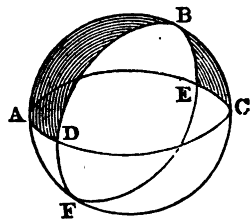
Cor. 2. We have seen that the entire surface of the sphere is equal to eight tri-rectangular triangles (Pr. 17, Cor.). If the area of the tri-rectangular triangle be represented by T, the surface of the sphere will be represented by 8T. Also, if we take the right angle for unity, and represent the angle of the lune by A, we shall have the proportion, *area of the lune* : 8T :: A : 4.

Hence the area of the lune is equal to $\frac{8A \times T}{4}$, or $2A \times T$.

Cor. 3. The spherical ungula, comprehended by the planes ADB, AEB, is to the entire sphere as the angle DCE is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence, in equal spheres, two ungulas are to each other as the angles included between their planes.

PROPOSITION XIX. THEOREM.

If two great circles intersect each other on the surface of a hemisphere, the sum of the opposite triangles thus formed is equivalent to a lune whose angle is equal to the inclination of the two circles.



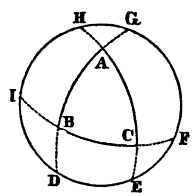
Let the great circles ABC, DBE intersect each other on the surface of the hemisphere BADCE; then will the sum of the opposite triangles ABD, CBE be equivalent to a lune whose angle is CBE.

For, produce the arcs BC, BE till they meet in F; then will BCF be a semi-circumference, as also ABC. Subtracting BC from each, we shall have CF equal to AB. For the same reason, EF is equal to DB, and CE is equal to AD. Hence the two triangles ABD, CFE are mutually equilateral; they are, therefore, equivalent (Pr. 15).

But the two triangles CBE, CFE compose the lune BCFE, whose angle is CBE; hence the sum of the triangles ABD, CBE is equivalent to the lune whose angle is CBE. Therefore, if two great circles, etc.

PROPOSITION XX. THEOREM.

The area of a spherical triangle is measured by the excess of the sum of its angles above two right angles multiplied by the tri-rectangular triangle.



Let ABC be any spherical triangle; its surface is measured by the sum of its angles A, B, C diminished by two right angles, and multiplied by the tri-rectangular triangle.

Produce the sides of the triangle ABC until they meet the great circle DEG drawn without the triangle. The two triangles ADE, AGH are together equal to the lune whose angle is A (Pr. 19); and this lune is measured by $2A \times T$ (Pr. 18, Cor. 2). Hence we have

$$ADE + AGH = 2A \times T.$$

For the same reason, $BFG + BDI = 2B \times T$;

also, $$CHI + CEF = 2C \times T.$$

But the sum of these six triangles exceeds the surface of the hemisphere by twice the triangle ABC, and the hemisphere is represented by $4T$; hence we have

$$4T+2ABC=2A\times T+2B\times T+2C\times T;$$

or, dividing by 2, and then subtracting 2T from each of these equals, we have

$$ABC=A\times T+B\times T+C\times T-2T,$$

or

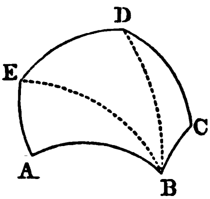
$$ABC=(A+B+C-2)\times T.$$

Hence every spherical triangle is measured by the sum of its angles diminished by two right angles, and multiplied by the tri-rectangular triangle.

Cor. If the sum of the three angles of a triangle is equal to three right angles, its surface will be equal to the tri-rectangular triangle; if the sum is equal to four right angles, the surface of the triangle will be equal to two tri-rectangular triangles; if the sum is equal to five right angles, the surface will be equal to three tri-rectangular triangles, etc.

PROPOSITION XXI. THEOREM.

The area of a spherical polygon is measured by the sum of its angles, diminished by as many times two right angles as it has sides less two, multiplied by the tri-rectangular triangle.



Let ABCDE be any spherical polygon. From the vertex B draw the arcs BD, BE to the opposite angles; the polygon will be divided into as many triangles as it has sides minus two.

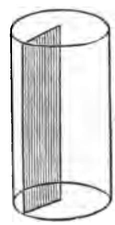
But the surface of each triangle is measured by the sum of its angles minus two right angles, multiplied by the tri-rectangular triangle. Also, the sum of all the angles of the triangles is equal to the sum of all the angles of the polygon; hence the surface of the polygon is measured by the sum of its angles, diminished by as many times two right angles as it has sides less two, multiplied by the tri-rectangular triangle.

Cor. If the polygon has five sides, and the sum of its angles is equal to seven right angles, its surface will be equal to the tri-rectangular triangle; if the sum is equal to eight right angles, its surface will be equal to two tri-rectangular triangles; if the sum is equal to nine right angles, the surface will be equal to three tri-rectangular triangles, etc.

BOOK X.

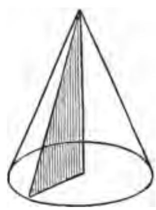
MEASUREMENT OF THE THREE ROUND BODIES.

Definitions.



1. A *cylinder* is a solid described by the revolution of a rectangle about one of its sides, which remains fixed. The *bases* of the cylinder are the circles described by the two revolving opposite sides of the rectangle.

2. The *axis* of a cylinder is the fixed straight line about which the rectangle revolves. The opposite side of the rectangle describes the *lateral* or *convex surface.*



3. A *cone* is a solid described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. The *base* of the cone is the circle described by that side containing the right angle which revolves.

4. The *axis* of a cone is the fixed straight line about which the triangle revolves. The hypothenuse of the triangle describes the *lateral* or *convex surface.* The *side* of the cone is the distance from the vertex to the circumference of the base.

5. A *frustum* of a cone is the part of a cone next the base, cut off by a plane parallel to the base.

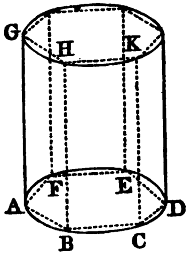
6. *Similar* cones and cylinders are those which have their axes and the diameters of their bases proportionals.

PROPOSITION I. THEOREM.

The convex surface of a cylinder is equal to the product of its altitude by the circumference of its base.

Let ACE–G be a cylinder whose base is the circle ACE, and altitude AG; then will its convex surface be equal to the product of AG by the circumference ACE.

In the circle ACE inscribe the regular polygon ABCDEF, and upon this polygon let a right prism be constructed of the same altitude with the cylinder.



The edges AG, BH, CK, etc., of the prism, being perpendicular to the plane of the base, will be contained in the convex surface of the cylinder. The convex surface of this prism is equal to the product of its altitude by the perimeter of its base (B. VIII., Pr. 1).

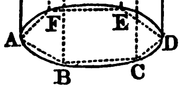
Let, now, the arcs subtended by the sides AB, BC, etc., be bisected, and the number of sides of the polygon be indefinitely increased, its perimeter will approach the circumference of the circle; and when the number of sides of the polygon becomes greater than any finite number, its perimeter will become equal to the circumference of the circle (B. VI., Pr. 10), and the convex surface of the prism will become equal to the convex surface of the cylinder.

But, whatever be the number of sides of the prism, its convex surface is equal to the product of its altitude by the perimeter of its base; hence the convex surface of the cylinder is equal to the product of its altitude by the circumference of its base.

Cor. If H represent the altitude of a cylinder, and R the radius of its base, the circumference of the base will be represented by $2\pi R$ (B. VI., Pr. 13, Cor. 2), and the convex surface of the cylinder by $2\pi RH$.

PROPOSITION II. THEOREM.

The volume of a cylinder is equal to the product of its base by its altitude.



Let ACE–G be a cylinder whose base is the circle ACE, and altitude AG; its volume is equal to the product of its base by its altitude.

In the circle ACE inscribe the regular polygon ABCDEF, and upon this polygon let a right prism be constructed of the same altitude with the cylinder. The volume of this prism is equal to the product of its base by its altitude (B. VIII., Pr. 11).

Let, now, the number of sides of the polygon be indefinitely increased; its area will become equal to that of the circle, and the volume of the prism becomes equal to that of the cylinder. But, whatever be the number of sides of the prism, its volume is equal to the product of its base by its altitude;

hence the volume of a cylinder is equal to the product of its base by its altitude.

Cor. 1. If H represent the altitude of a cylinder, and R the radius of its base, the area of the base will be represented by πR^2 (B. VI., Pr. 13, Cor. 3), and the volume of the cylinder will be $\pi R^2 H$.

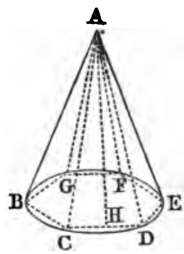
Cor. 2. Cylinders of the same altitude are to each other as their bases, and cylinders of the same base are to each other as their altitudes.

Cor. 3. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases.

For the bases are as the squares of their diameters; and, since the cylinders are similar, the diameters of their bases are as their altitudes (Def. 6). Therefore the bases are as the squares of the altitudes, and hence the products of the bases by the altitudes, or the cylinders themselves, will be as the cubes of the altitudes.

PROPOSITION III. THEOREM.

The convex surface of a cone is equal to the product of half its side by the circumference of its base.



Let A–BCDEFG be a cone whose base is the circle BDEG, and its side AB; then will its convex surface be equal to the product of half its side by the circumference of the circle BDF.

In the circle BDF inscribe the regular polygon BCDEFG, and upon this polygon let a regular pyramid be constructed having A for its vertex. The edges of this pyramid will lie in the convex surface of the cone. From A draw AH perpendicular to CD, one of the sides of the polygon. The convex surface of the pyramid is equal to the product of half the slant height AH by the perimeter of its base (B. VIII., Pr. 14).

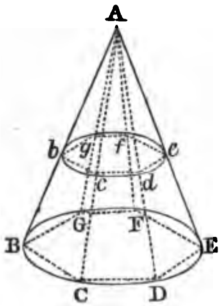
Let, now, the arcs subtended by the sides BC, CD, etc., be bisected, and the number of sides of the polygon be indefinitely increased, its perimeter will become equal to the circumference of the circle, the slant height AH becomes equal to the side of the cone AB, and the convex surface of the pyramid becomes equal to the convex surface of the cone.

But, whatever be the number of faces of the pyramid, its convex surface is equal to the product of half its slant height by the perimeter of its base; hence the convex surface of the cone is equal to the product of half its side by the circumference of its base.

Cor. If S represent the side of a cone, and R the radius of its base, then the circumference of the base will be represented by $2\pi R$, and the convex surface of the cone by $2\pi R \times \frac{1}{2}S$, or πRS.

PROPOSITION IV. THEOREM.

The convex surface of a frustum of a cone is equal to the product of its side by half the sum of the circumferences of its two bases.



Let BDF–*bdf* be a frustum of a cone whose bases are BDF, *bdf*, and B*b* its side; its convex surface is equal to the product of B*b* by half the sum of the circumferences BDF, *bdf*.

Complete the cone A–BDF to which the frustum belongs, and in the circle BDF inscribe the regular polygon BCDEFG, and upon this polygon let a regular pyramid be constructed having A for its vertex. Then will BDF–*bdf* be a frustum of a regular pyramid, whose convex surface is equal to the product of its slant height by half the sum of the perimeters of its two bases (B. VIII., Pr. 14, Cor. 1).

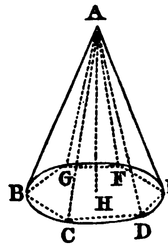
Let, now, the number of sides of the polygon be indefinitely increased, its perimeter will become equal to the circumference of the circle, and the convex surface of the pyramid will become equal to the convex surface of the cone. But, whatever be the number of faces of the pyramid, the convex surface of its frustum is equal to the product of its slant height by half the sum of the perimeters of its two bases. Hence the convex surface of a frustum of a cone is equal to the product of its side by half the sum of the circumferences of its two bases.

Cor. It was proved (B. VIII., Pr. 14, Cor. 2) that the convex surface of a frustum of a pyramid is equal to the product of its slant height by the perimeter of a section at equal distances between its two bases; hence *the convex surface of a frustum of a cone is equal to the product of its side by the circumference of a section at equal distances between the two bases.*

PROPOSITION V. THEOREM.

The volume of a cone is equal to one third of the product of its base by its altitude.

Let A–BCDF be a cone whose base is the circle BCDEFG, and AH its altitude; the volume of the cone will be equal to one third of the product of the base BCDF by the altitude AH.



In the circle BDF inscribe a regular polygon BCDEFG, and construct a pyramid whose base is the polygon BDF, and having its vertex in A. The volume of this pyramid is equal to one third of the product of the polygon BCDEFG by its altitude AH (B. VIII., Pr. 17).

Let, now, the number of sides of the polygon be indefinitely increased; its area will become equal to the area of the circle, and the volume of the pyramid will become equal to the volume of the cone. But, whatever be the number of faces of the pyramid, its volume is equal to one third of the product of its base by its altitude; hence the volume of the cone is equal to one third of the product of its base by its altitude.

Cor. 1. Since a cone is one third of a cylinder having the same base and altitude, it follows that cones of equal altitudes are to each other as their bases; cones of equal bases are to each other as their altitudes; and similar cones are as the cubes of their altitudes, or as the cubes of the diameters of their bases.

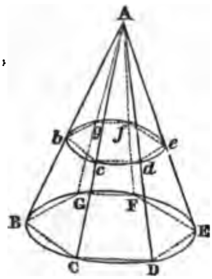
Cor. 2. If H represent the altitude of a cone, and R the radius of its base, the volume of the cone will be represented by $\pi R^2 \times \frac{1}{3}H$, or $\frac{1}{3}\pi R^2 H$.

PROPOSITION VI. THEOREM.

A frustum of a cone is equivalent to the sum of three cones having the same altitude with the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them.

Let BDF–*bdf* be any frustum of a cone. Complete the cone to which the frustum belongs, and in the circle BDF inscribe the regular polygon BCDEFG, and upon this polygon let a regular pyramid be constructed having its vertex in A.

Then will BCDEFG–*bcdefg* be a frustum of a regular pyramid whose volume is equal to three pyramids having the same alti-

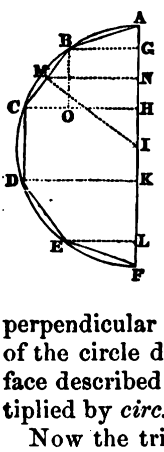


tude with the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them (B. VIII., Pr. 18).

Let, now, the number of sides of the polygon be indefinitely increased, its area will become equal to the area of the circle, and the frustum of the pyramid will become the frustum of a cone. Hence the frustum of a cone is equivalent to the sum of three cones having the same altitude with the frustum, and whose bases are the lower base of the frustum, its upper base, and a mean proportional between them.

PROPOSITION VII. THEOREM.

The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.



Let ABDF be the semicircle by the revolution of which the sphere is described. Inscribe in the semicircle a regular semi-polygon ABCDEF, and from the points B, C, D, E let fall the perpendiculars BG, CH, DK, EL upon the diameter AF.

If, now, the polygon be revolved about AF, the lines AB, EF will describe the convex surface of two cones, and BC, CD, DE will describe the convex surface of frustums of cones.

From the centre I draw IM perpendicular to BC; also draw MN perpendicular to AF, and BO perpendicular to CH. Let *circ.* MN represent the circumference of the circle described by the revolution of MN. Then the surface described by the revolution of BC will be equal to BC multiplied by *circ.* MN (Pr. 4, Cor.).

Now the triangles IMN, BCO are similar, since their sides are perpendicular to each other (B. IV., Pr. 22); whence

$$BC : BO \text{ or } GH :: IM : MN,$$
$$:: \textit{circ.}\, IM : \textit{circ.}\, MN.$$

Hence (B. II., Pr. 1)

$$BC \times \textit{circ.}\, MN = GH \times \textit{circ.}\, IM.$$

Therefore the surface described by BC is equal to the altitude GH multiplied by *circ.* IM, or the circumference of the inscribed circle.

In like manner, it may be proved that the surface described by CD is equal to the altitude HK multiplied by the circumference of the inscribed circle; and the same may be proved of the other sides. Hence the entire surface described by ABCDEF is equal to the circumference of the inscribed circle multiplied by the sum of the altitudes AG, GH, HK, KL, and LF; that is, the axis of the polygon.

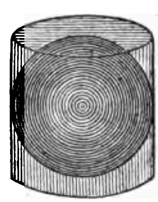
Let, now, the arcs AB, BC, etc., be bisected, and the number of sides of the polygon be indefinitely increased, its perimeter will coincide with the circumference of the semicircle, and the perpendicular IM will become equal to the radius of the sphere; that is, the circumference of the inscribed circle will become the circumference of a great circle. Hence the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.

Cor. 1. *The area of a zone is equal to the product of its altitude by the circumference of a great circle.*

For the surface described by the lines BC, CD is equal to the altitude GK multiplied by the circumference of the inscribed circle. But when the number of sides of the polygon is indefinitely increased, the perimeter BC+CD becomes the arc BCD, and the inscribed circle becomes a great circle. Hence the area of the zone produced by the revolution of BCD is equal to the product of its altitude GK by the circumference of a great circle.

Cor. 2. The area of a great circle is equal to the product of its circumference by half the radius (B. VI., Pr. 12), or one fourth of the diameter; hence *the surface of a sphere is equivalent to four of its great circles.*

Cor. 3. *The surface of a sphere is equal to the convex surface of the circumscribed cylinder.*



For the latter is equal to the product of its altitude by the circumference of its base. But its base is equal to a great circle of the sphere, and its altitude to the diameter; hence the convex surface of the cylinder is equal to the product of its diameter by the circumference of a great circle, which is also the measure of the surface of a sphere.

Cor. 4. Two zones upon equal spheres are to each other as their altitudes, and any zone is to the surface of its sphere as the altitude of the zone is to the diameter of the sphere.

Cor. 5. Let R denote the radius of a sphere, D its diameter, C the circumference of a great circle, and S the surface of a sphere; then we shall have

$$C=2\pi R, \text{ or } \pi D \text{ (B. VI, Pr. 13, Cor. 2).}$$

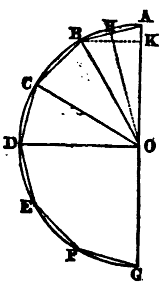
Also $$S=2\pi R\times 2R=4\pi R^2, \text{ or } \pi D^2.$$

If H represents the altitude of a zone, its area will be

$$2\pi RH.$$

PROPOSITION VIII. THEOREM.

The volume of a sphere is equal to one third the product of its surface by the radius.



Let ACEG be the semicircle by the revolution of which the sphere is described. Inscribe in the semicircle a regular semi-polygon ABCDEFG, and draw the radii BO, CO, DO, etc.

The solid described by the revolution of the polygon ABCDEFG about AG is composed of the solids formed by the revolution of the triangles ABO, BCO, CDO, etc., about AG.

First. To find the value of the solid formed by the revolution of the triangle ABO.

From O draw OH perpendicular to AB, and from B draw BK perpendicular to AO. The two triangles ABK, BKO, in their revolution about AO, will describe two cones having a common base, viz., the circle whose radius is BK.

The solid described by the triangle ABO will then be represented by $\frac{1}{3}\pi R^2H$, or $\frac{1}{3}\pi BK^2\times AO$ (Prop. 5, Cor. 2).

But, by similar triangles,

$$BK:BA::HO:AO;$$

therefore $$BK\times AO=HO\times AB;$$

or, multiplying by $\frac{\pi}{3}BK$, we have

$$\tfrac{1}{3}\pi BK^2\times AO=\tfrac{1}{3}HO\times\pi AB\times BK.$$

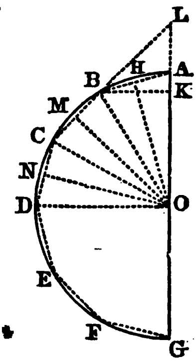
But the surface described by $AB=\pi AB\times BK$ (Prop. 3, Cor.).

Hence the solid described by the triangle ABO is equal to $\frac{1}{3}HO\times$ the surface described by AB.

Secondly. To find the value of the solid formed by the revolution of the triangle BCO.

Produce BC until it meets AG produced in L. It is evident, from the preceding demonstration, that the solid described by the triangle LCO is equal to

$$\tfrac{1}{3}OM\times \text{ surface described by LC;}$$



and the solid described by the triangle LBO is equal to

$\frac{1}{3}$OM$\times$ surface described by LB;

hence the solid described by the triangle BCO is equal to

$\frac{1}{3}$OM$\times$ surface described by BC.

In the same manner, it may be proved that the solid described by the triangle CDO is equal to $\frac{1}{3}$ON$\times$ surface described by CD, and so on for the other triangles. But the perpendiculars OH, OM, ON, etc., are all equal; hence the solid described by the polygon ABC DEFG is equal to the surface described by the perimeter of the polygon multiplied by $\frac{1}{3}$OH.

Let, now, the number of sides of the polygon be indefinitely increased, the perpendicular OH will become the radius OA, the perimeter ACEG will become the semi-circumference ADG, and the solid described by the polygon becomes a sphere; hence the volume of a sphere is equal to one third of the product of its surface by the radius.

Cor. 1. *The volume of a spherical sector is equal to the product of the zone which forms its base by one third of the radius of the sphere.*

For the solid described by the revolution of BCDO is equal to the surface described by BC+CD multiplied by $\frac{1}{3}$OM.

But when the number of sides of the polygon is indefinitely increased, the perpendicular OM becomes the radius OB, the quadrilateral BCDO becomes the sector BDO, and the solid described by the revolution of BCDO becomes a spherical sector. Hence the volume of a spherical sector is equal to the product of the zone which forms its base by one third of the radius of the sphere.

Cor. 2. Let R represent the radius of a sphere, D its diameter, S its surface, and V its volume; then we shall have

$$S=4\pi R^2, \text{ or } \pi D^2 \text{ (Pr. 7, Cor. 5).}$$

Also $$V=\tfrac{1}{3}R\times S=\tfrac{4}{3}\pi R^3, \text{ or } \tfrac{1}{6}\pi D^3;$$

hence *the volumes of spheres are to each other as the cubes of their radii.*

If we put H to represent the altitude of the zone which forms the base of a sector, then the volume of the sector will be represented by $2\pi RH\times\frac{1}{3}R=\frac{2}{3}\pi R^2H$.

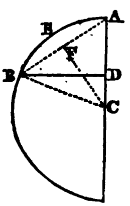
Cor. 3. *Every sphere is two thirds of the circumscribed cylinder*

For, since the base of the circumscribed cylinder is equal to a great circle, and its altitude to a diameter, the volume of the cylinder is equal to a great circle multiplied by the diameter (Pr. 2).

But the volume of a sphere is equal to four great circles multiplied by one third of the radius, or one great circle multiplied by $\frac{4}{3}$ of the radius, or $\frac{2}{3}$ of the diameter. Hence a sphere is two thirds of the circumscribed cylinder.

PROPOSITION IX. THEOREM.

A spherical segment with one base is equivalent to half of a cylinder having the same base and altitude, plus a sphere whose diameter is the altitude of the segment.



Let BD be the radius of the base of the segment, AD its altitude, and let the segment be generated by the revolution of the circular half segment AEBD about the axis AC. Join CB, and from the centre C draw CF perpendicular to AB.

The solid generated by the revolution of the segment AEB is equal to the difference of the solids generated by the sector ACBE and the triangle ACB. Now the solid generated by the sector ACBE is equal to

$$\tfrac{2}{3}\pi CB^2 \times AD \text{ (Pr. 8, Cor. 2).}$$

And the solid generated by the triangle ACB, by Pr. 8, is equal to $\frac{1}{3}$CF multiplied by the convex surface described by AB, which is $2\pi CF \times AD$ (Pr. 7), making, for the solid generated by the triangle ACB, $\frac{2}{3}\pi CF^2 \times AD$.

Therefore the solid generated by the segment AEB is equal to

$$\tfrac{2}{3}\pi AD \times (CB^2 - CF^2),$$

or

$$\tfrac{2}{3}\pi AD \times BF^2;$$

that is,

$$\tfrac{1}{6}\pi AD \times AB^2,$$

because $CB^2 - CF^2$ is equal to BF^2, and BF^2 is equal to one fourth of AB^2.

Now the cone generated by the triangle ABD is equal to

$$\tfrac{1}{3}\pi AD \times BD^2 \text{ (Pr. 5, Cor. 2).}$$

Therefore the spherical segment in question, which is the sum of the solids described by AEB and ABD, is equal to

$$\tfrac{1}{6}\pi AD(2BD^2 + AB^2);$$

that is,

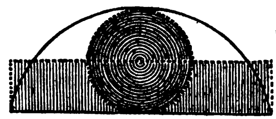
$$\tfrac{1}{6}\pi AD(3BD^2 + AD^2),$$

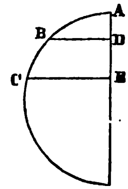
because AB^2 is equal to $BD^2 + AD^2$.

This expression may be separated into the two parts

$$\tfrac{1}{2}\pi AD \times BD^2, \text{ and } \tfrac{1}{6}\pi AD^3.$$

The first part represents the volume of a cylinder having the same base with the segment and half its altitude (Pr. 2); the other part represents a sphere, of which AD is the diameter (Pr. 8, Cor. 2). Therefore a spherical segment, etc.





Cor. The volume of the spherical segment of two bases generated by the revolution of BCED about the axis AE may be found by subtracting that of the segment of one base generated by ABD from that of the segment of one base generated by ACE.

EXERCISES ON THE PRECEDING PRINCIPLES.

1. What is the entire surface of a triangular prism whose base is an equilateral triangle, having each of its sides equal to 17 inches, and its altitude 5 feet?

2. What is the entire surface of a regular triangular pyramid whose slant height is 15 feet, and each side of the base 4 feet?

3. What is the convex surface of the frustum of a square pyramid whose slant height is 14 feet, each side of the lower base being $3\frac{1}{2}$ feet, and each side of the upper base $2\frac{1}{2}$ feet?

4. What is the volume of a triangular prism whose height is 12 feet, and the three sides of its base 4, 5, and 6 feet?

5. What is the volume of a triangular pyramid whose altitude is 25 feet, and each side of the base 4 feet?

6. What is the volume of a piece of timber whose bases are squares, each side of the lower base being 14 inches, and each side of the upper base 12 inches, the altitude being 25 feet?

7. What is the entire surface of a cylinder whose altitude is 17 feet, and the diameter of its base 3 feet?

8. What is the entire surface of a cone whose side is 24 feet, and the diameter of its base 5 feet?

9. What is the entire surface of a frustum of a cone whose side is 18 feet, and the radii of the bases 5 feet and 4 feet?

10. What is the volume of a cylinder whose altitude is 16 feet, and the circumference of its base 5 feet?

11. What is the volume of a cone whose altitude is 13 feet, and the circumference of its base 7 feet?

12. What is the volume of a frustum of a cone whose altitude is 22 feet, the circumference of its lower base 18 feet, and that of the upper base 14 feet?

13. What is the surface of a sphere, the circumference of its great circle being 40 feet?

14. What is the area of the surface of the earth, supposing it to be a sphere whose diameter is 7912 miles?

15. What is the convex surface of a zone whose altitude is 13 inches, upon a sphere whose diameter is 40 inches?

16. What is the volume of a sphere whose diameter is 17 inches?

17. What is the volume of the earth, supposing it to be a sphere whose diameter is 7912 miles?

18. What is the volume of a spherical segment with one base, the diameter of the sphere being 12 feet, and the altitude of the segment 3 feet?

19. What is the surface of a regular tetraedron whose edge is 7 feet?

20. What is the volume of a regular tetraedron whose edge is 9 feet?

21. What is the edge of a regular tetraedron whose volume is 20 cubic feet?

22. The base of a rectangular parallelopiped is 3.42 feet by 4.36 feet, and its volume is 100 cubic feet; what is its altitude?

23. The volume of a parallelopiped is 366.4 cubic feet, and its altitude is 23.4 feet; what is the area of its base?

24. The sides of the base of a tetraedron are 13, 15, and 17 feet, and its altitude is 11 feet; required its volume.

25. What is the volume of a frustum of a regular triangular pyramid having a side of one base equal to 4 feet, and a side of the other base 3 feet, and the lateral edge equal to $3\frac{1}{2}$ feet?

26. The volume of a sphere is 1870 cubic feet; required its radius.

27. The edge of a cube is 30 inches; required the volume of the circumscribing sphere.

28. The side of a right cone is 22 feet, and its altitude 15 feet; required its lateral surface.

29. A stone obelisk has the form of a regular quadrangular pyramid, having a side of its base equal to 4 feet, and its slant height 13 feet. The density of the stone is 2.5 times that of water; what is its weight, assuming that a cubic foot of water weighs $62\frac{1}{2}$ pounds.

30. Supposing the earth to be a sphere, and that a quadrant is equal to 32,800,000 feet, it is required to determine the radius of the earth, the area of its surface, its volume, and its weight, the mean density of the earth being 4.5 times that of water.

CONIC SECTIONS.

THERE are three curves whose properties are extensively applied in Astronomy and many other branches of Natural Philosophy, which, being the sections of a cone made by a plane in different positions, are called the *Conic Sections*. These are

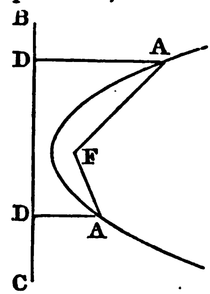
The Parabola,
The Ellipse, and
The Hyperbola.

PARABOLA.

Definitions.

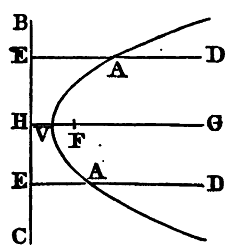
1. A *parabola* is a plane curve, every point of which is equally distant from a given fixed point and a given straight line.

2. The fixed point is called the *focus* of the parabola, and the given straight line is called the *directrix*.



Thus, if a straight line BC, and a point without it, F, be given in position in a plane, and the point A be supposed to move in such a manner that AF, its distance from the given point, is always equal to AD, its perpendicular distance from the given line, the point A will describe a curve called a parabola.

3. Any straight line perpendicular to the directrix, terminated at one extremity by the parabola, and produced indefinitely within the curve, is called a *diameter*.



The *vertex* of a diameter is the point in which it meets the parabola.

Thus, through any point of the curve, as A, draw a line DE perpendicular to the directrix BC; AD is a diameter of the parabola, and the point A is the vertex of this diameter.

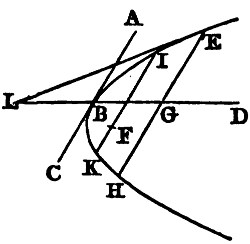
4. The *axis* of the parabola is the diameter which passes through the focus, and the vertex of the axis is called the *principal vertex.*

Thus, through the focus F draw GH perpendicular to the directrix; GV is the axis of the parabola, and the point V, where the axis meets the curve, is called the principal vertex of the parabola, or simply the vertex.

It is evident, from Def. 1, that FV=VH; that is, a perpendicular drawn from the focus to the directrix is bisected at the vertex of the axis.

5. A *tangent* to the parabola is a straight line which meets the curve in one point only, and every where else falls without the curve.

6. An *ordinate* to a diameter is a straight line drawn from any point of the curve to meet that diameter, and is parallel to the tangent at its vertex.



Thus, let AC be a tangent to the parabola at B, the vertex of the diameter BD, and from any point E of the curve draw EGH parallel to AC; then is EG an ordinate to the diameter BD.

It is proved in Prop. 12 that EG is equal to GH; hence the entire line EH is sometimes called a *double ordinate.*

7. An *abscissa* is the part of a diameter intercepted between its vertex and an ordinate.

Thus BG is the abscissa of the diameter BD corresponding to the ordinate EG, and also to the point E of the curve.

8. A *subtangent* is that part of a diameter produced which is included between a tangent and an ordinate drawn from the point of contact.

Thus, let EL, a tangent to the curve at E, meet the diameter BD in the point L, and let the ordinate EG meet the same diameter in G; then LG is the subtangent of BD corresponding to the point E.

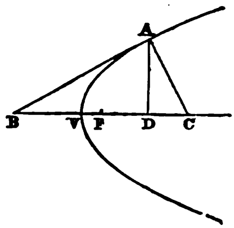
9. The *parameter* of a diameter is the double ordinate which passes through the focus.

Thus, through the focus F draw IK parallel to AC, which touches the curve at the vertex of the diameter BD; then is IK the parameter of the diameter BD.

10. The parameter of the axis is called the principal parameter, or *latus rectum.*

11. A *normal* is a line drawn perpendicular to a tangent from the point of contact, and terminated by the axis.

12. A *subnormal* is the part of the axis included between the normal and an ordinate drawn from the same point of the curve.

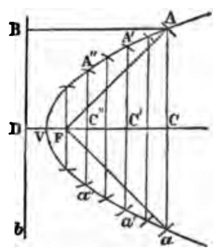


Thus, let AB be a tangent to the parabola at any point A. From A draw AC perpendicular to AB, and draw AD an ordinate to the axis VC; then AC is the normal, and DC is the subnormal corresponding to the point A.*

PROPOSITION I. PROBLEM.

The focus and directrix of a parabola being given, to describe the curve.

FIRST METHOD. *By points.*



Let F be the focus, and B*b* the directrix of a parabola. Through F draw DC perpendicular to B*b*, and bisect FD in V; then, since DV=VF, V is a point on the curve, and CV is the axis of the parabola.

To find other points of the curve, draw any number of lines A*a*, A′*a*′, A″*a*″, etc., perpendicular to CD; then, with the distances DC, DC′, DC″, etc., as radii, and the focus F as a centre, describe arcs intersecting the perpendiculars in A, A′, A″, etc. The points A, A′, A″, etc., in which the arcs cut the perpendiculars, are points of the curve.

For FA=DC=AB (Def. 1).

We may thus determine as many points on the curve as we please, and the curve line which passes through all the points V, A, A′, A″, etc., will be the parabola whose focus is F, and directrix B*b*.

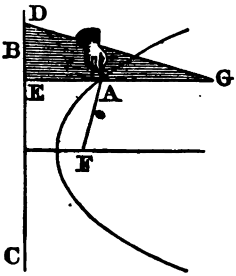
Cor. The same radius determines two points of the curve, one above and one below the axis; and, since AF=*a*F, FC is common

* The *subtangent* is so called because it is below the tangent, being limited by the tangent and ordinate to the point of contact. The *subnormal* is so called because it is below the normal, being limited by the normal and ordinate. The subtangent and subnormal may be regarded as the projections of the tangent and normal upon a diameter.

to the two triangles AFC, aFC, and the angles at C are right angles; therefore AC$=a$C; that is, every straight line terminated by the curve, and perpendicular to the axis, is bisected by it; and, consequently, the parabola consists of two equal branches similarly situated with respect to the axis.

Moreover, since the radius FA is always greater than FC, the arc described with F as a centre will always intersect the corresponding perpendicular, and there is therefore no limit to the distance to which the curve may extend on both sides of the axis.

SECOND METHOD. *By continuous motion.*



Let BC be a ruler whose edge coincides with the directrix of the parabola, and let DEG be a square. Take a thread equal in length to EG, and attach one extremity of it at G, and the other at the focus F. Then slide the side of the square DE along the ruler BC, and at the same time keep the thread continually stretched by means of the point of a pencil A in contact with the square; the pencil will describe one part of the required parabola. For, in every position of the square,

$$AF+AG=AE+AG;$$

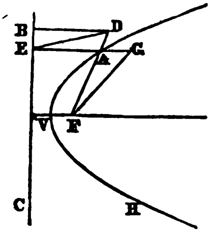
and hence $$AF=AE;$$

that is, the point A is always equally distant from the focus F and the directrix BC.

If the square be turned over and moved on the other side of the point F, the other part of the same parabola may be described.

PROPOSITION II. THEOREM.

The distance of any point without the parabola from the focus is greater than its distance from the directrix; and the distance of any point within the parabola from the focus is less than its distance from the directrix.



Let AVH be a parabola, of which F is the focus, and BC the directrix; and let D be a point without the curve, that is, on the same side of the curve as the directrix. Then, if DF be joined, and BD be drawn perpendicular to BC, DF will be greater than DB.

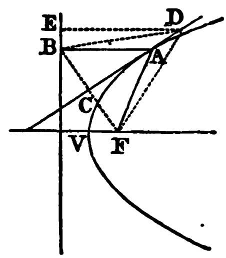
For, as DF necessarily cuts the curve, let A be the point of section. Draw AE perpendicular to the directrix, and join DE. Then, because A is a point in the parabola, AE=AF (Def. 1); therefore DF=DA+AE; but DA+AE is greater than DE (B. I., Pr. 8), and therefore still greater than DB (B. I., Pr. 17). Therefore DF is greater than DB.

Again, let G be a point within the parabola. Then GF, a line drawn to the focus, is less than GE, a perpendicular to the directrix. The perpendicular GE necessarily cuts the curve; let A be the point of section, and join AF. Then AF=AE (Def. 1), and GA+AF=GE. But GF is less than GA+AF, therefore GF is less than GE.

Cor. A point is without or within the parabola according as its distance from the focus is greater or less than its distance from the directrix.

PROPOSITION III. THEOREM.

The straight line which bisects the angle contained by two lines drawn from the same point in the curve, the one to the focus and the other perpendicular to the directrix, is a tangent to the parabola at that point.



Let A be any point of the parabola AV, from which draw the line AF to the focus, and AB perpendicular to the directrix, and draw AC bisecting the angle BAF; AC is a tangent to the curve at the point A.

Let D be any other point in the line AC, from which draw DB, DF. Also draw DE perpendicular to the directrix, and join BF. Since, in the two triangles, ACB, ACF, AF is equal to AB (Def. 1), AC is common to both triangles, and the angle CAB is, by supposition, equal to the angle CAF; therefore CB is equal to CF, and the angle ACB to the angle ACF.

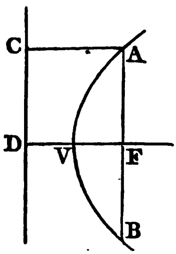
Again, in the two triangles DCB, DCF, because BC is equal to CF, the side DC is common to both triangles, and the angle DCB is equal to the angle DCF; therefore DB is equal to DF. But DB is greater than DE (B. I., Pr. 17); therefore the distance of the point D from the focus is greater than its distance from the directrix; hence that point is without the parabola (Pr. 2, Cor.). Therefore every point of the line DC, except A, is without the curve; that is, DC is a tangent to the curve at A (Def. 5).

Cor. 1. Since the angle BAF continually increases as the point A moves toward V, and at V becomes equal to two right angles, *the tangent at the principal vertex is perpendicular to the axis.* The tangent at the vertex V is called the *vertical tangent.*

Cor. 2. Since an ordinate to any diameter is parallel to the tangent at its vertex, an ordinate to the axis is perpendicular to the axis.

PROPOSITION IV. THEOREM.

The latus rectum is equal to four times the distance from the focus to the vertex.



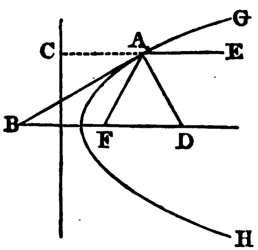
Let AVB be a parabola, of which F is the focus, and V the principal vertex; then the latus rectum AFB will be equal to four times FV.

Let CD be the directrix, and let AC be drawn perpendicular to it; then, according to Def. 1, AF is equal to AC or DF, because ACDF is a parallelogram. But DV is equal to VF; that is, DF is equal to twice VF. Hence AF is equal to twice VF. In the same manner, it may be proved that BF is equal to twice VF; consequently, AB is equal to four times VF. Therefore the latus rectum, etc.

PROPOSITION V. THEOREM.

If a tangent to the parabola cut the axis produced, the points of contact and of intersection are equally distant from the focus.

Let AB be a tangent to the parabola GAH at the point A, and let it cut the axis produced in B; also, let AF be drawn to the focus; then will the line AF be equal to BF.



Draw AC perpendicular to the directrix; then, since AC is parallel to BF, the angle BAC is equal to ABF. But the angle BAC is equal to BAF (Pr. 3); hence the angle ABF is equal to BAF, and, consequently, AF is equal to BF. Therefore, if a tangent, etc.

Cor. 1. Let the normal AD be drawn. Then, because BAD is a right angle, it is equal to the sum of the two angles ABD, ADB, or to the sum of the two angles BAF, ADB. Take away the common angle BAF, and we have the angle DAF equal to ADF. Hence the line AF is equal to FD. Therefore, *if a circle be described with the centre* F *and radius* FA, *it will pass through the three points* B, A, D.

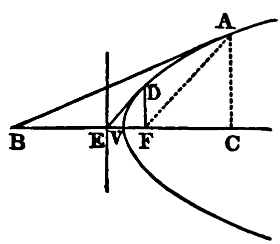
Cor. 2. *The normal bisects the angle made by the diameter at the point of contact with the line drawn from that point to the focus.*

For, because BD is parallel to CE, the alternate angles ADF, DAE are equal. But the angle ADF has been proved equal to DAF; hence the angles DAF, DAE are equal to each other.

Scholium. It is a law in Optics that the angle made by a ray of reflected light with a perpendicular to the reflecting surface is equal to the angle which the incident ray makes with the same perpendicular. Hence, if GAH represent a polished surface whose figure is that produced by the revolution of a parabola about its axis, a ray of light falling upon it in the direction EA would be reflected to F. The same would be true of all rays parallel to the axis. Hence the point F, in which all the rays would intersect each other, is called the *focus*, or *burning point.*

PROPOSITION VI. THEOREM.

The subtangent to the axis is bisected by the vertex.



Let AB be a tangent to the parabola ADV at the point A, and AC an ordinate to the axis; then will BC be the subtangent, and it will be bisected at the vertex V.

For BF is equal to AF (Pr. 5), and AF is equal to CE, which is the distance of the point A from the directrix.

That is, BF=CE.

But FV=EV.

Therefore the remainder BV=the remainder CV.

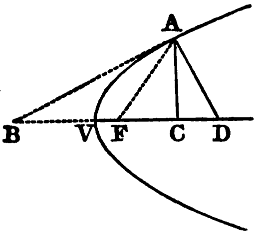
Cor. 1. Hence the tangent at D, the extremity of the latus rectum, meets the axis in E, the same point with the directrix. For, by Def. 8, EF is the subtangent corresponding to the tangent DE.

Cor. 2. Hence, if it is required to draw a tangent to the curve at a given point A, draw the ordinate AC to the axis. Make BV equal to VC; join the points B, A, and the line BA will be the tangent required.

PROPOSITION VII. THEOREM.

The subnormal is equal to half the latus rectum.

Let AB be a tangent to the parabola AV at the point A; let



AC be the ordinate, and AD the normal from the point of contact; then CD is the subnormal, and is equal to half the latus rectum.

For the distance of the point A from the focus is equal to its distance from the directrix, which is equal to VF+VC, or 2VF+FC; that is,

$$FA=2VF+FC,$$

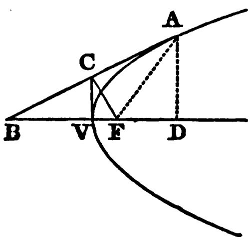
But $FA=FD$ (Pr. 5, Cor. 1).

Hence $FD=2VF+FC.$

Taking away the common part FC, the remainder $CD=2VF$, which is equal to half the latus rectum (Pr. 4).

PROPOSITION VIII. THEOREM.

If a perpendicular be drawn from the focus to any tangent, the point of intersection will be in the vertical tangent.



Let AB be any tangent to the parabola AV, and FC a perpendicular let fall from the focus upon AB; join VC; then will the line VC be a tangent to the curve at the vertex V.

Draw the ordinate AD to the axis. Since FA is equal to FB (Pr. 5), and FC is drawn perpendicular to AB, it divides the triangle AFB into two equal parts, and therefore AC is equal to BC. But BV is equal to VD (Pr. 6); hence $BC:CA::BV:VD,$

and therefore CV is parallel to AD (B. IV., Pr. 16). But AD is perpendicular to the axis BD; hence CV is also perpendicular to the axis, and is a tangent to the curve at the point V (Pr. 3, Cor. 1). Therefore, if a perpendicular, etc.

Cor. 1. Because the triangles FVC, FCA are similar, we have

$$FV:FC::FC:FA;$$

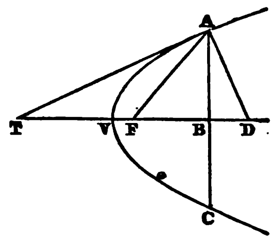
that is, *the perpendicular from the focus upon any tangent is a mean proportional between the distances of the focus from the vertex and from the point of contact.*

Cor. 2. From Cor. 1 we have $FC^2=FV\times FA$.

But FV remains constant for the same parabola; therefore *the square of the perpendicular from the focus to any tangent varies as the distance from the focus to the point of contact.*

PROPOSITION IX. THEOREM.

The square of an ordinate to the axis is equal to the product of the latus rectum by the corresponding abscissa.



Let AVC be a parabola, and A any point of the curve. From A draw the ordinate AB; then is the square of AB equal to the product of VB by the latus rectum.

Draw the tangent AT and the normal AD. Since TAD is a right angle, and AB perpendicular to TD,

$AB^2 = TB \times BD$ (B. IV., Pr. 23).

But $TB = 2VB$ (Pr. 6),

and $BD = 2VF$ (Pr. 7).

Therefore $AB^2 = 4VB \times VF$,

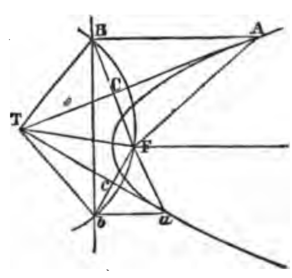
or $= VB \times$ the latus rectum (Pr. 4).

Cor. 1. Since the latus rectum is constant for the same parabola, *the squares of ordinates to the axis are to each other as their corresponding abscissas.*

Cor. 2. The preceding demonstration is equally applicable to ordinates on either side of the axis; hence AB is equal to BC, and AC is called a *double ordinate.* The curve is composed of two branches of unlimited extent, which recede continually from the axis as well as from the directrix.

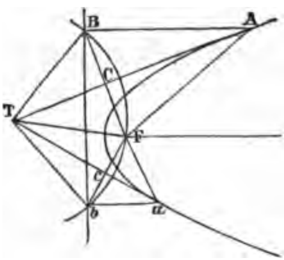
PROPOSITION X. THEOREM.

If two tangents to the parabola intersect each other, and lines be drawn from the focus to the points of contact and to the point of intersection, the two triangles thus formed will be similar to each other.



Let two lines which touch the parabola at A and *a* intersect each other at T; from the focus draw FA, FT, and F*a*; the two triangles TFA, TF*a* are similar.

Draw AB and *ab* perpendicular to the directrix B*b*, and join TB, T*b*, and BF. The two triangles ACB, ACF are equal to each other, since AB is equal to AF, AC is common to the two tri-



angles, and the angle CAB is equal to CAF (Pr. 3); therefore the angles at C are right angles, and BC is equal to CF.

Also, the two triangles TCB, TCF are equal, since BC is equal to CF, TC is common to both triangles, and the angles at C are equal; therefore TF is equal to TB.

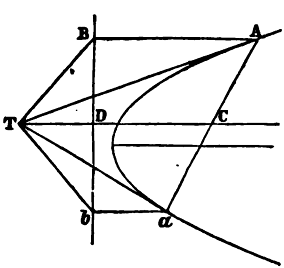
In the same manner, it may be proved that TF is equal to T*b*, the angle FT*a* is equal to *b*T*a*, and a circle described from the centre T, with radius TF, will pass through B and *b*.

The angle FB*b* is equal to the angle CAB, since each is the complement of ABC; also, the angle BAC is equal to FAC (Pr. 3); therefore the angle FAC is equal to FB*b*. But the angle FB*b* is half the angle FT*b* (B. III., Pr. 15, Cor. 2), and is therefore equal to the angle FT*a*. Therefore the angle FAT is equal to the angle FT*a*.

In the same manner, it may be proved that the angle ATF is equal to F*a*T. Therefore the remaining angle TFA is equal to the angle TF*a*, and the triangle AFT is similar to the triangle *a*FT.

PROPOSITION XI. THEOREM.

If two tangents to a parabola be drawn at the extremities of a chord, the diameter which passes through their point of intersection will bisect the chord.



Let two lines which touch the parabola at A and *a* intersect each other at T, and from T let TC be drawn perpendicular to the directrix B*b*, meeting the chord A*a* in C; then A*a* will be bisected in C.

Draw AB, *ab* perpendicular to the directrix; join TB, T*b*, and let TC meet B*b* in D.

The two triangles TDB, TD*b* are equal, since TB is equal to T*b* (Pr. 10), TD is common to the two triangles, and the angles at D are right angles; therefore BD is equal to *b*D.

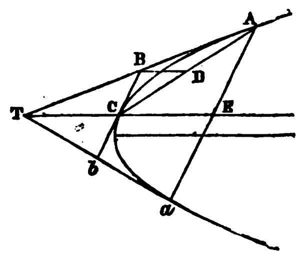
Because the lines AB, CD, ab are parallel, we have

$$AC : Ca :: BD : Db.$$

But $BD = Db$; therefore $AC = Ca$; that is, Aa is bisected in C.

PROPOSITION XII. THEOREM.

If two tangents to a parabola be drawn at the extremities of a chord, and a diameter be drawn through their point of intersection, the tangent at its vertex will be parallel to the chord.



If from a point T two tangents TA, Ta be drawn to a parabola, and TC be drawn parallel to the axis, meeting the parabola in C, the tangent BCb will be parallel to the chord Aa.

Let the tangent BCb meet TA, Ta in B and b. Join AC, and draw BD parallel to the axis, meeting AC in D.

Because BD is parallel to TC, we have

$$TB : BA :: CD : DA.$$

But $CD = DA$ (Pr. 11); therefore $TB = BA$.

For the same reason, $Tb = ba$.

Therefore $TB : BA :: Tb : ba$, and Bb is parallel to Aa (B. IV., Pr. 16).

Cor. 1. Since AE is parallel to the tangent BC, it is an ordinate to the diameter CE; and since Aa is bisected in E (Pr. 11), Aa is a double ordinate to CE. Hence *every diameter bisects its double ordinates.*

Cor. 2. Since BC is parallel to AE, we have

$$TC : CE :: TB : BA.$$

But $TB = BA$; therefore $TC = CE$; that is, *the subtangent upon any diameter is bisected at the vertex of that diameter.*

PROPOSITION XIII. THEOREM.

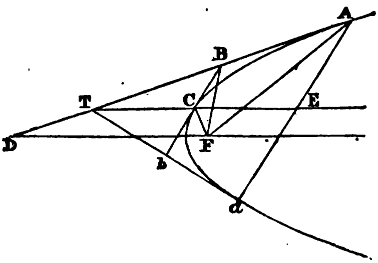
The square of an ordinate to any diameter is equal to four times the product of the corresponding abscissa by the distance from the vertex of that diameter to the focus.

Let AE be an ordinate to the diameter CE; then

$$AE^2 = 4CE \times CF.$$

Produce AE to meet the parabola in a, and draw the tangents TA, Ta, meeting CE produced in the point T (Pr. 12). Let the tangent at C meet TA in B, and join FA, FB, and FC.

Now, since from the point B two tangents BA, BC are drawn



to the parabola, the triangle BCF is similar to the triangle BFA (Pr. 10); therefore the angle CBF is equal to BAF. But BAF is equal to BDF (Pr. 3), which equals BTC; therefore the angle CBF is equal to BTC. Also, the angle FCb is equal to TCb; therefore their supplements are equal; that is, FCB is equal to BCT. Therefore the remaining angle BFC is equal to the remaining angle CBT, and the triangle BCF is similar to BCT. Hence CF : CB :: CB : CT,

or $\qquad CB^2 = CT \times CF = CE \times CF$ (Pr. 12, Cor. 2).

Also, since AE is parallel to BC, we have

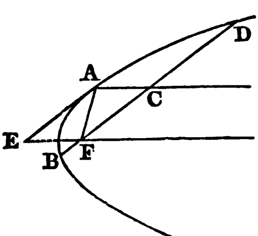
$$AE : BC :: ET : CT.$$

But $\qquad ET = 2CT$ (Pr. 12, Cor. 2); therefore $AE = 2BC$;

and $$AE^2 = 4BC^2 = 4CE \times CF.$$

PROPOSITION XIV. THEOREM.

The parameter of any diameter is equal to four times the distance from its vertex to the focus.



Let BAD be a parabola, of which F is the focus, AC is any diameter, and BD its parameter; then is BD equal to four times AF.

Draw the tangent AE; then, since AEFC is a parallelogram, AC is equal to EF, which is equal to AF (Pr. 4).

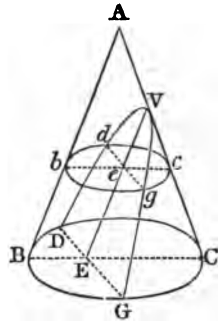
Now, by Pr. 13, BC^2 is equal to $4AF \times AC$; that is, to $4AF^2$. Hence BC is equal to twice AF, and BD is equal to four times AF. Therefore the parameter of any diameter, etc.

Cor. Hence *the square of an ordinate to any diameter is equal to the product of its parameter by the corresponding abscissa* (Pr. 13).

PROPOSITION XV. THEOREM.

If a cone be cut by a plane parallel to its side, the section is a parabola.

Let ABGCD be a cone cut by a plane VDG parallel to the slant side AB; then will the section DVG be a parabola.



Let ABC be a plane section through the axis of the cone, and perpendicular to the plane VDG; then VE, which is their common section, will be parallel to AB (B. VII., Pr. 12). Let *bgcd* be a plane parallel to the base of the cone; the intersection of this plane with the cone will be a circle.

Since the plane ABC divides the cone into two equal parts, BC is a diameter of the circle BGCD, and *bc* is a diameter of the circle *bgcd*. Let DEG, *deg* be the common sections of the plane VDG with the planes BGCD, *bgcd* respectively. Then DG is perpendicular to the plane ABC (B. VII., Pr. 8), and, consequently, to the lines VE, BC. For the same reason, *dg* is perpendicular to the two lines VE, *bc*.

Now, since *be* is parallel to BE, and *b*B to *e*E, the figure *b*BE*e* is a parallelogram, and *be* is equal to BE. But, because the triangles V*ec*, VEC are similar, we have

$$ec : \mathrm{EC} :: \mathrm{V}e : \mathrm{VE};$$

and, multiplying the first and second terms of this proportion by the equals *be* and BE, we have

$$be \times ec : \mathrm{BE} \times \mathrm{EC} :: \mathrm{V}e : \mathrm{VE}.$$

But, since *bc* is a diameter of the circle *bgcd*, and *de* is perpendicular to *bc* (B. IV., Pr. 23, Cor.), $be \times ec = de^2$.

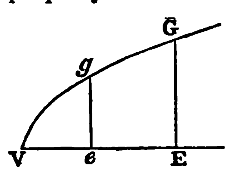
For the same reason, $\mathrm{BE} \times \mathrm{EC} = \mathrm{DE}^2$.

Substituting these values of $be \times ec$, and $\mathrm{BE} \times \mathrm{EC}$ in the preceding proportion, we have

$$de^2 : \mathrm{DE}^2 :: \mathrm{V}e : \mathrm{VE};$$

that is, the squares of the ordinates are to each other as the corresponding abscissas, and hence the curve is a parabola whose axis is VE (Pr. 9, Cor. 1). Hence the parabola is called a *conic section*, as mentioned on page 203.

Schol. 1. The conclusion that DVG is a parabola would not be legitimate unless it was proved that the property that "the squares of the ordinates are to each other as the corresponding abscissas" is *peculiar* to the parabola. That such is the case appears from the fact that, when the axis and one point of a parabola are given, this property will determine the position of every other point of the curve. Thus, let VE be the axis of a parabola,



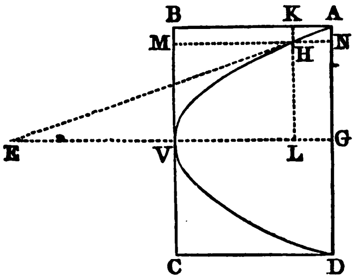
and g any point of the curve, from which draw the ordinate ge. Take any other point in the axis, as E, and make GE of such a length that $\mathrm{V}e : \mathrm{VE} :: ge^2 : \mathrm{GE}^2$.

Since the first three terms of this proportion are given, the fourth is determined, and the same proportion will determine any number of points of the curve.

Schol. 2. AB, AC, the sides of the cone, may be conceived to be indefinitely extended, until the height of the cone ABC is infinite. If the plane DVG be also indefinitely extended, the two branches of the parabola DVG will extend to an infinite distance from V, and will also recede to an infinite distance from the axis, as stated in Prop. 9, Cor. 2.

PROPOSITION XVI. THEOREM.

A segment of a parabola cut off by a double ordinate to the axis is two thirds of its circumscribing rectangle.



Let AVD be a segment of a parabola cut off by the straight line AD perpendicular to the axis. Through V draw the tangent BC; also, draw AB, CD parallel to the axis; then will the parabolic segment AVD be two thirds of the rectangle ABCD.

Let H be a point of the curve near to A, and through A and H draw the secant line AHE. Also, through H draw KL perpendicular, and MN parallel to the axis.

The area of the trapezoid AHLG is equal to $\frac{1}{2}(\mathrm{AG}+\mathrm{HL})\mathrm{HN}$, (B. IV., Pr. 7); and the area of the trapezoid ABMH is equal to $\frac{1}{2}(\mathrm{AB}+\mathrm{MH})\mathrm{AN}$. Hence we have

$$\mathrm{AHLG} : \mathrm{ABMH} :: (\mathrm{AG}+\mathrm{HL})\mathrm{HN} : (\mathrm{AB}+\mathrm{MH})\mathrm{AN},$$
$$:: (\mathrm{AG}+\mathrm{HL})\mathrm{EG} : (\mathrm{AB}+\mathrm{MH})\mathrm{AG},$$

because $\mathrm{EG} : \mathrm{AG} :: \mathrm{HN} : \mathrm{AN}$.

If, now, we suppose the point H to move toward A, the secant line AHE will approach the position of a tangent to the curve at A, and will coincide with the tangent when H coincides with A. When this takes place, AG will be equal to HL, and AB to MH; also, EG will be double of VG or AB (Pr. 6). We shall then have

$$\frac{\mathrm{AHLG}}{\mathrm{ABMH}} = \frac{2\mathrm{AG.EG}}{2\mathrm{AB.AG}} = \frac{\mathrm{EG}}{\mathrm{AB}} = 2.$$

Hence the portion of the parabola included between two ordinates indefinitely near is double of the corresponding portion of the external space ABV. The same may be proved for every point of the curve, and hence the whole space AVG is double the space ABV. Whence AVG is two thirds of ABVG, and the parabolic segment AVD is two thirds of the circumscribing rectangle ABCD. Therefore a segment, etc.

EXERCISES ON THE PARABOLA.

1. The diameter of the circle described about the triangle AVB is equal to 5FV. (See fig., Pr. 4.)

2. If from the point D, DE be drawn at right angles to FA, then AE is equal to 2VF. (See fig., Pr. 7.)

3. If the triangle ADF is equilateral, then AF is equal to the latus rectum. (See fig., Pr. 7.)

4. If AB is a common tangent to a parabola, and the circle described on the latus rectum as a diameter, prove that AF and BF make equal angles with the latus rectum.

5. If the tangent AC meets the directrix in G, prove that AC. $AG=AF^2$, and that $AC.CG=AF.FV$. (See fig., Pr. 3.)

6. If AE be drawn at right angles to AV, meeting the axis in E, then CE is equal to 4VF. (See fig., Pr. 7.)

7. The tangent at any point of a parabola meets the directrix and latus rectum produced in points equally distant from the focus.

8. Prove that $BC=CD$, and that $BA.BC=BF.BD$. (See fig., Pr. 8.)

9. If a circle be described about the triangle AFC, the tangent to it from V is equal to one half AC. (See fig., Pr. 7.)

10. If the ordinate of a point A bisect the subnormal of a point B, the ordinate of A is equal to the normal of B.

11. If from any point on the tangent to a parabola a line be drawn touching the parabola, the angle between this line and the line to the focus from the same point is constant.

12. If the diameter AC meets the directrix in G, and the chord drawn through the focus parallel to the tangent at A in C, prove that $AC=AG$. (See fig., Pr. 14.)

13. Required the area of a segment of a parabola cut off by a chord 15 inches in length, perpendicular to the axis, the corresponding abscissa of the axis being 21 inches.

14. An ordinate to the axis of a parabola is 9 inches, and the corresponding abscissa is 10 inches; required the latus rectum.

15. An ordinate to a diameter of a parabola is 12 inches, and the corresponding abscissa is 5 inches; required the parameter of that diameter.

16. The latus rectum of a parabola is 20 inches; required the area of the segment cut off by a double ordinate to the axis when the corresponding abscissa is 30 inches.

17. The latus rectum of a parabola is 9. What is the ordinate to the axis corresponding to the abscissa 4?

18. The latus rectum of a parabola is 10 inches. Find the ordinate to the axis corresponding to that point of the curve from which, if a tangent and normal be drawn, they will form with the axis a triangle whose area is 36 inches.

19. The latus rectum of a parabola is 15, and a tangent is drawn through the point whose ordinate to the axis is 4. Determine where the tangent line meets the axis produced.

20. The latus rectum of a parabola is 12, and a tangent is drawn through the point whose ordinate to the axis is 7. Determine where the normal line passing through the same point meets the axis.

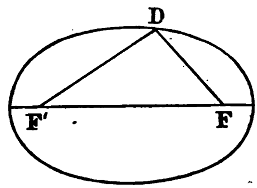
ELLIPSE.

Definitions.

1. An *ellipse* is a plane curve traced out by a point which moves in such a manner that the *sum* of its distances from two fixed points is always the same.

2. The two fixed points are called the *foci* of the ellipse.

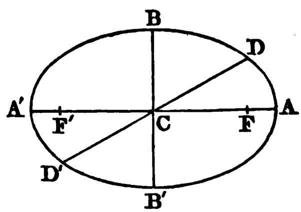
Thus, if F and F′ are two fixed points, and if the point D moves about F in such a manner that the sum of its distances from F and F′ is always the same, the point D will describe an ellipse, of which F and F′ are the foci.



3. The *centre* of the ellipse is the middle point of the straight line joining the foci.

4. The *eccentricity* is the distance from either focus to the centre.

Thus, let F and F′ be the foci of the ellipse ABA′B′. Draw the line FF′, and bisect it in C. The point C is the centre of the ellipse, and CF or CF′ is the eccentricity.



5. A *diameter* is any straight line passing through the centre, and terminated on both sides by the curve.

6. The extremities of a diameter are called its *vertices*.

Thus, through C draw any straight line DD′ terminated by the curve; DD′ is a diameter of the ellipse; D and D′ are the vertices of that diameter.

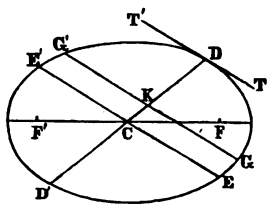
7. The *major axis* is the diameter which passes through the foci.

8. The *minor axis* is the diameter which is perpendicular to the major axis.

Thus, produce the line FF′ to meet the curve in A and A′, and through C draw BB′ perpendicular to AA′; then is AA′ the major axis, and BB′ the minor axis.

9. A *tangent* to an ellipse is a straight line which meets the curve in one point only, and every where else falls without it.

10. An *ordinate* to a diameter is a straight line drawn from any point of the curve to the diameter, and is parallel to the tangent at one of its vertices.



Thus, let DD′ be any diameter, and TT′ a tangent to the ellipse at D. From any point G of the curve draw GKG′ parallel to TT′, and cutting DD′ in K; then is GK an ordinate to the diameter DD′. It is proved in Pr. 7 that the tangents at D and D′ are parallel.

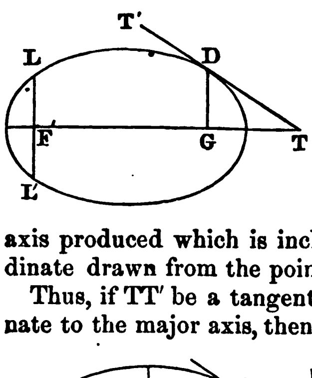
It is proved in Pr. 21, Cor. 1, that GK is equal to G′K; hence the entire line GG′ is called a *double ordinate.*

11. Each of the parts into which a diameter is divided by an ordinate is called an *abscissa.*

Thus, DK and D′K are the abscissas of the diameter DD′ corresponding to the ordinate GK, or to the point G.

12. One diameter is said to be *conjugate* to another when it is parallel to the ordinates of the other diameter.

Thus, draw the diameter EE′ parallel to GK, an ordinate to the diameter DD′, in which case it will, of course, be parallel to the tangent TT′; then is the diameter EE′ conjugate to DD′.

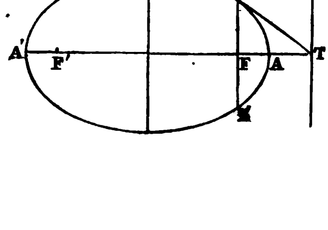


13. The *latus rectum* is the double ordinate to the major axis which passes through one of the foci.

Thus, through the focus F′ draw LL′, a double ordinate to the major axis; it will be the latus rectum of the ellipse.

14. A *subtangent* is that part of an axis produced which is included between a tangent and the ordinate drawn from the point of contact.

Thus, if TT′ be a tangent to the curve at D, and DG an ordinate to the major axis, then GT is the corresponding subtangent.



15. The *directrix* of an ellipse is a straight line perpendicular to the major axis produced, and intersecting it in the same point with the tangent drawn through one extremity of the latus rectum.

Thus, if LT be a tangent drawn

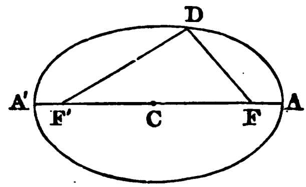
through one extremity of the latus rectum LL′, meeting the axis produced in T, and GT be drawn through the point of intersection perpendicular to the axis, it will be the directrix of the ellipse.

The ellipse has two directrices, one corresponding to the focus F, and the other to the focus F′.

PROPOSITION I. THEOREM.

The sum of the two lines drawn from any point of an ellipse to the foci is equal to the major axis.

Let ADA′ be an ellipse, of which F, F′ are the foci, AA′ is the major axis, and D any point of the curve; then will DF+DF′ be equal to AA′.



For, by Def. 1, the sum of the distances of any point of the curve from the foci is equal to a given line. Now, when the point D arrives at A, FA+F′A, or 2AF+FF′ is equal to the given line. And when D is at A′, FA′+F′A′, or 2A′F′+FF′ is equal to the same line. Hence

$$2AF+FF'=2A'F'+FF';$$

consequently, AF is equal to A′F′.

Hence DF+DF′, which is equal to AF+AF′, must be equal to AA′. Therefore the sum of the two lines, etc.

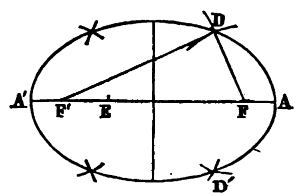
Cor. The major axis is bisected in the centre. For, by Def. 3, CF is equal to CF′; and we have just proved that AF is equal to A′F′; therefore AC is equal to A′C.

PROPOSITION II. PROBLEM.

The major axis and foci of an ellipse being given, to describe the curve.

First method. *By points.*

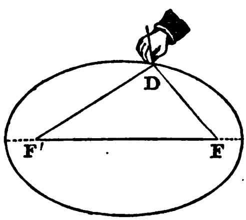
Let AA′ be the major axis, and F, F′ the foci of an ellipse. Take E any point between the foci, and from F and F′ as centres, with the distances AE, A′E as radii, describe two circles cutting each other in the point D; D will be a point on the ellipse. For, join FD, F′D; then DF+DF′=EA+EA′=AA′; and, at whatever point between the foci E is taken, the sum of DF and DF′ will be equal to AA′. Hence, by Def. 1, D is a point on the curve; and, in the



same manner, any number of points in the ellipse may be determined.

Cor. The same circles determine two points of the curve D and D′, one above and one below the major axis. It is also evident that these two points are equally distant from the axis; that is, the ellipse is symmetrical with respect to its major axis, and is bisected by it.

SECOND METHOD. *By continuous motion.*

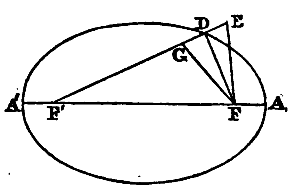


Take a thread equal in length to the major axis of the ellipse, and fasten one of its extremities at F, the other at F′. Then let a pencil be made to glide along the thread, so as to keep it always stretched; the curve described by the point of the pencil will be an ellipse. For in every position of the pencil the sum of the distances DF, DF′ will be the same, viz., equal to the entire length of the string.

Scholium. The ellipse is evidently a continuous and closed curve.

PROPOSITION III. THEOREM.

The sum of two lines drawn from any point without the ellipse to the foci is greater than the major axis; and the sum of two lines drawn from any point within the ellipse to the foci is less than the major axis.



Let ADA′ be an ellipse, of which F, F′ are the foci, and AA′ the major axis; and let E be a point without the ellipse. Join EF, EF′; the sum of EF and EF′ will be greater than AA′.

Let EF′, which must meet the ellipse, meet it in D; then DE+EF is greater than DF (B. I., Pr. 8). Adding DF′ to these unequals, we have EF+EF′ greater than DF+DF′; that is, than AA′.

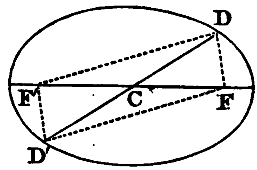
Again, let G be a point within the ellipse; then GF+GF′ will be less than AA′.

Let F′G, which must meet the curve if produced beyond G, meet it in D, and join DF. The line GF is less than DG+DF (B. I., Pr. 8). Adding GF′ to these unequals, we have GF+GF′ less than DF+DF′; that is, less than AA′. Therefore the sum, etc.

Cor. A point is without or within the ellipse according as the sum of two lines drawn from it to the foci is greater or less than the major axis.

PROPOSITION IV. THEOREM.

Every diameter of an ellipse is bisected in the centre.

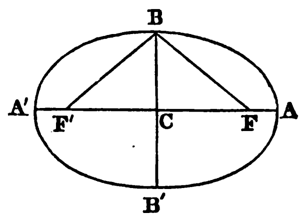


Let D be any point of an ellipse; join DF, DF', and FF'. Complete the parallelogram DFD'F', and join DD'.

Now, because the opposite sides of a parallelogram are equal, the sum of DF and DF' is equal to the sum of D'F and D'F'; hence D' is a point in the ellipse. But the diagonals of a parallelogram bisect each other; therefore FF' is bisected in C; that is, C is the centre of the ellipse, and DD' is a diameter bisected in C. Therefore every diameter, etc.

PROPOSITION V. THEOREM.

The distance from either focus to the extremity of the minor axis is equal to half the major axis.



Let F and F' be the foci of an ellipse, AA' the major axis, and BB' the minor axis; draw the straight lines BF, BF'; then BF, BF' are each equal to AC.

In the two right-angled triangles BCF, BCF', CF is equal to CF', and BC is common to both triangles; hence BF is equal to BF'. But BF+BF' is equal to 2AC (Pr. 1); consequently, BF and BF' are each equal to AC. Therefore the distance, etc.

Cor. 1. *Half the minor axis is a mean proportional between the parts into which either focus divides the major axis.*

For BC^2 is equal to BF^2-FC^2 (B. IV., Pr. 11), which is equal to AC^2-FC^2 (Pr. 5). Hence (B. IV., Pr. 10)

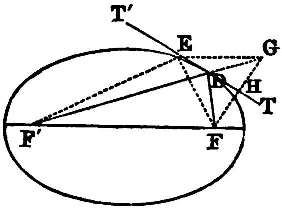
$$BC^2=(AC+FC)\times(AC-FC)$$
$$=AF'\times AF;\text{ and, therefore,}$$
$$AF:BC::BC:FA'.$$

Cor. 2. *The square of the eccentricity is equal to the difference of the squares of the semi-axes.*

For FC^2 is equal to BF^2-BC^2, which is equal to AC^2-BC^2.

PROPOSITION VI. THEOREM.

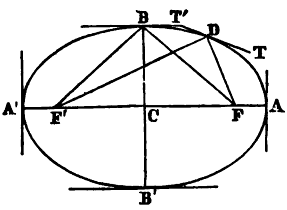
A tangent to the ellipse makes equal angles with straight lines drawn from the point of contact to the foci.



Let F, F′ be the foci of an ellipse, and D any point of the curve; if through the point D the line TT′ be drawn, making the angle TDF equal to T′DF′, then will TT′ be a tangent to the ellipse at D.

Let E be any point in the line TT′ different from D. Produce F′D to G, making DG equal to DF, and join EF, EF′, EG and FG.

Now, in the two triangles DFH, DGH, because DF is equal to DG, DH is common to both triangles, and the angle FDH is, by supposition, equal to F′DT′, which is equal to the vertical angle GDH; therefore HF is equal to HG, and the angle DHF is equal to the angle DHG. Hence the line TT′ is perpendicular to FG at its middle point; and, therefore, EF is equal to EG. Hence EF+EF′ is equal to EG+EF′. But EG+EF′ is greater than GF′; that is, greater than FD+F′D, which is equal to the major axis of the ellipse; therefore EF+EF′ is greater than the major axis, and hence the point E is without the ellipse (Pr. 3, Cor.). Therefore every point of the line TT′ except D is without the curve; that is, TT′ is a tangent to the curve at D.

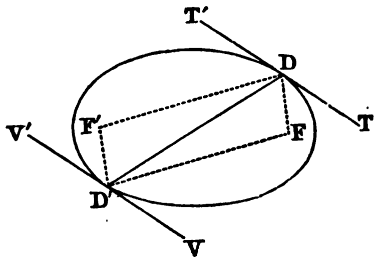


Cor. 1. As the point D moves toward A, each of the angles FDT, F′DT′ increases, and at A becomes a right angle. Hence the tangents at the vertices of the major axis are perpendicular to that axis. Also, since the angle FBC is equal to F′BC (Pr. 5), the tangents at the vertices of the minor axis are perpendicular to that axis, and hence an ordinate to either axis is perpendicular to that axis.

Cor. 2. If TT′ represent a plane mirror, a ray of light proceeding from F in the direction FD would be reflected in the direction DF′, making the angle of reflection equal to the angle of incidence. And, since the ellipse may be regarded as coinciding with a tangent at the point of contact, if rays of light proceed from one focus of a polished concave surface whose figure is that produced by the revolution of an ellipse about its major axis, they will all be reflected to the other focus. For this reason, the points F, F′ are called the *foci*, or burning points.

PROPOSITION VII. THEOREM.

Tangents to the ellipse at the vertices of any diameter are parallel to each other.



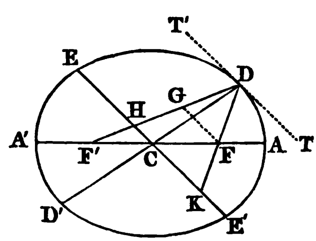
Let DD′ be any diameter of an ellipse, and TT′, VV′ tangents to the curve at the points D, D′; then will they be parallel to each other.

Join DF, DF′, D′F, D′F′; then, by the preceding Proposition, the angle FDT is equal to F′DT′, and the angle FD′V is equal to F′D′V′. But, by Pr. 4, DFD′F′ is a parallelogram; and, since the opposite angles of a parallelogram are equal, the angle FDF′ is equal to FD′F′; therefore the angle FDT is equal to F′D′V′ (B. I., Pr. 2). Also, since FD is parallel to F′D′, the angle FDD′ is equal to F′D′D; hence the whole angle D′DT is equal to DD′V′; and, consequently, TT′ is parallel to VV′. Therefore tangents, etc.

Cor. If tangents are drawn through the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

PROPOSITION VIII. THEOREM.

If from the vertex of any diameter straight lines are drawn through the foci, meeting the conjugate diameter, the part intercepted by the conjugate is equal to half the major axis.



Let EE′ be a diameter conjugate to DD′, and let the lines DF, DF′ be drawn, and produced, if necessary, so as to meet EE′ in H and K; then will DH or DK be equal to AC.

Draw FG parallel to EE′ or TT′. Then the angle DGF is equal to the alternate angle F′DT′, and the angle DFG is equal to FDT. But the angles FDT, F′DT′ are equal to each other (Pr. 7); hence the angles DGF, DFG are equal to each other, and DG is equal to DF.

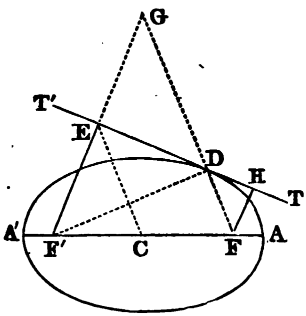
Also, because CH is parallel to FG, and CF is equal to CF′, therefore HG must be equal to HF′.

Hence FD+F′D is equal to 2DG+2GH or 2DH. But FD+F′D is equal to 2AC. Therefore 2AC is equal to 2DH, or AC is equal to DH.

Also, the angle DHK is equal to DKH, and hence DK is equal to DH or AC. Therefore, if from the vertex, etc.

PROPOSITION IX. THEOREM.

Perpendiculars drawn from the foci upon a tangent to the ellipse meet the tangent in the circumference of a circle whose diameter is the major axis.



Let TT′ be a tangent to the ellipse at D, and from F′ draw F′E perpendicular to T′T; the point E will be in the circumference of a circle described upon AA′ as a diameter.

Join CE, FD, F′D, and produce F′E to meet FD produced in G.

Then, in the two triangles DEF′, DEG, because DE is common to both triangles, the angles at E are equal, being right angles; also, the angle EDF′ is equal to FDT (Pr. 6), which is equal to the vertical angle EDG; therefore DF′ is equal to DG, and EF′ is equal to EG.

Also, because F′E is equal to EG, and F′C is equal to CF, CE must be parallel to FG, and, consequently, equal to half of FG.

But, since DG has been proved equal to DF′, FG is equal to FD+DF′, which is equal to AA′. Hence CE is equal to half of AA′ or AC; and a circle described with C as a centre, and radius CA, will pass through the point E.

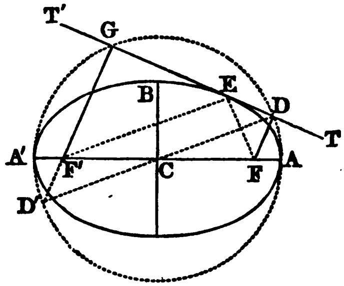
The same may be proved of a perpendicular let fall upon TT′ from the focus F. Therefore perpendiculars, etc.

Cor. CE is parallel to DF; and, if CH be joined, CH will be parallel to DF′.

PROPOSITION X. THEOREM.

The product of the perpendiculars let fall from the foci upon a tangent is equal to the square of half the minor axis.

Let TT′ be a tangent to the ellipse at any point E, and let the



perpendiculars FD, F'G be drawn from the foci; then will the product of FD by F'G be equal to the square of BC.

On AA' as a diameter, describe a circle; it will pass through the points D and G (Pr. 9).

Produce GF' to meet the circle in D', and join DD'; then, since the angle at G is a right angle, DD' passes through the centre C. Because FD and D'G are perpendicular to the same straight line, they are parallel to each other, and the alternate angles CFD, CF'D' are equal. Also, the vertical angles DCF, D'CF' are equal, and CF is equal to CF'. Therefore DF is equal to D'F'; hence DF×GF' is equal to D'F'×GF', which is equal to A'F'×F'A (B. IV., Pr. 28), which is equal to BC^2 (Pr. 5, Cor. 1).

Cor. The triangles FDE, F'GE are similar; hence

FD : F'G :: FE : F'E;

that is, *perpendiculars let fall from the foci upon a tangent are to each other as the distances of the point of contact from the foci.*

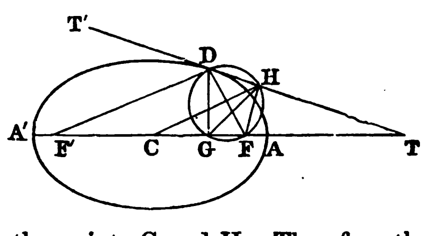
PROPOSITION XI. THEOREM.

If a tangent and ordinate be drawn from the same point of an ellipse, meeting either axis produced, half of that axis will be a mean proportional between the distances of the two intersections from the centre.

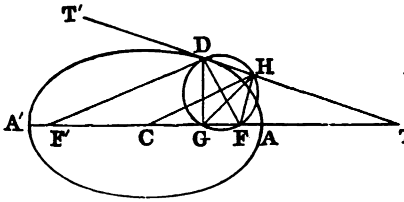
1st. *For the major axis.*

Let TT' be a tangent to the ellipse, and DG an ordinate to the major axis from the point of contact; then we shall have

CT : CA :: CA : CG.



From F draw FH perpendicular to TT'; join DF, DF', CH and GH. Then, by Pr. 9, Cor., CH is parallel to DF'. Also, since DGF, DHF are both right angles, a circle described on DF as a diameter will pass through the points G and H. Therefore the angle HGF is equal to the angle HDF (B. III., Pr. 15, Cor. 1),



which is equal to T'DF' or DHC. Hence the angles CGH and CHT, which are the supplements of HGF and DHC are equal; and, since the angle C is common to the two triangles CGH, CHT, these triangles are equiangular, and we have

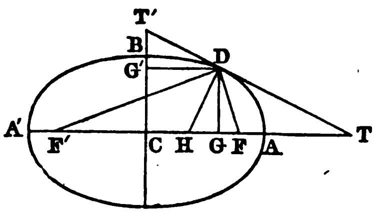
$$CT : CH :: CH : CG.$$

But CH is equal to CA (Pr. 9); therefore

$$CT : CA :: CA : CG.$$

2d. *For the minor axis.*

Let the tangent TT' meet the minor axis in T', and let DG' be an ordinate to the minor axis from the point of contact; then we shall have $CT' : CB :: CB : CG'$.



Draw DH perpendicular to TT', and it will bisect the angle FDF' (Pr. 6). Hence

$$HF' : HF :: DF' : DF$$
$$:: TF' : TF \text{ (Pr. 10, Cor.)}.$$

Therefore (B. II., Pr. 8)

$$2CF : 2CH :: 2CT : 2CF.$$

Whence $CT \times CH = CF^2$.

But we have proved that

$$CT \times CG = CA^2.$$

Subtracting the former from the latter, we have

$$CT \times GH = CA^2 - CF^2 = CB^2.$$

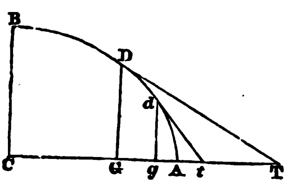
Because the triangles DGH and CTT' are similar, we have

$$CT : CT' :: DG : GH.$$

Whence $\quad CT \times GH = CT' \times DG = CT' \times CG'$.

Therefore $\quad CT' \times CG' = CB^2$,

or $\quad CT' : CB :: CB : CG'$.



Cor. By this Proposition,

$$CA^2 = CG.CT.$$

If a second ordinate *dg*, and tangent *dt*, be drawn, we shall also have

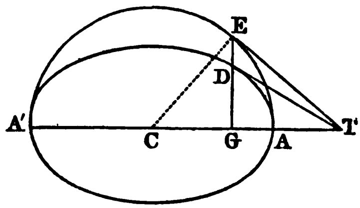
$$CA^2 = Cg.Ct.$$

Whence $\quad CG.CT = Cg.Ct$,

or, $\quad GC : Cg :: Ct : CT$.

PROPOSITION XII. THEOREM.

The subtangent of an ellipse is equal to the corresponding subtangent of the circle described upon its major axis.



Let AEA′ be a circle described on AA′, the major axis of an ellipse, and from any point E in the circle draw the ordinate EG, cutting the ellipse in D. Draw DT touching the ellipse at D, and join ET; then will ET be a tangent to the circle at E.

Join CE. Then, by the last Proposition,

$$CT : CA :: CA : CG;$$

or, because CA is equal to CE,

$$CT : CE :: CE : CG.$$

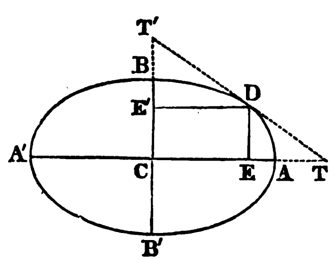
Hence the triangles CET, CGE, having the angle at C common, and the sides about this angle proportional, are similar (B. IV., Pr. 21). Therefore the angle CET, being equal to the angle CGE, is a right angle; that is, the line ET is perpendicular to the radius CE, and is, consequently, a tangent to the circle (B. III., Pr. 9). Hence GT is the subtangent corresponding to each of the tangents DT and ET. Therefore the subtangent, etc.

Cor. A similar property may be proved of a tangent to the ellipse meeting the minor axis.

PROPOSITION XIII. THEOREM.

The square of either axis is to the square of the other as the rectangle of the abscissas of the former is to the square of their ordinate.

1st. *For the major axis.*



Let DE be an ordinate to the major axis from the point D; then we shall have

$$CA^2 : CB^2 :: AE \times EA' : DE^2.$$

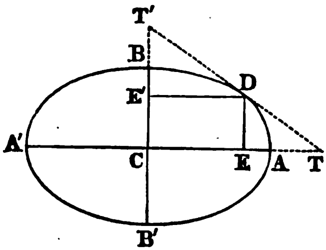
Draw TT′ a tangent to the ellipse at D; then, by Pr. 11,

$$CT : CA :: CA : CE.$$

Hence (B. II., Pr. 13)

$$CT : CE :: CA^2 : CE^2;$$

and by division (B. II., Pr. 7),



$$CT : ET :: CA^2 : CA^2 - CE^2. \quad (1)$$

Again, by Pr. 11,

$$CT' : CB :: CB : CE' \text{ or } DE.$$

Hence $CT' : DE :: CB^2 : DE^2$.

But, by similar triangles,

$$CT' : DE :: CT : ET;$$

Therefore

$$CT : ET :: CB^2 : DE^2. \quad (2)$$

Comparing proportions (1) and (2), we have $CA^2 : CA^2 - CE^2 :: CB^2 : DE^2$.

But $CA^2 - CE^2$ is equal to $AE \times EA'$ (B. IV., Pr. 10).

Hence $CA^2 : CB^2 :: AE \times EA' : DE^2$.

2d. *For the minor axis.*

Let DE' be an ordinate to the minor axis; then we shall have

$$CB^2 : CA^2 :: BE' \times E'B' : DE'^2.$$

We have already proved that

$$CA^2 : CA^2 - CE^2 :: CB^2 : DE^2 (= CE'^2);$$

therefore, by division,

$$CA^2 : CE^2 :: CB^2 : CB^2 - CE'^2;$$

or $CB^2 : CA^2 :: CB^2 - CE'^2 : DE'^2$.

But $CB^2 - CE'^2$ is equal to $BE' \times E'B'$ (B. IV., Pr. 10).

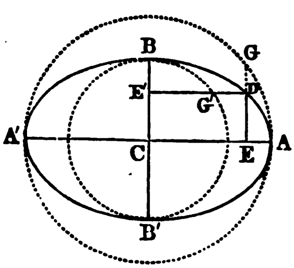
Hence $CB^2 : CA^2 :: BE' \times E'B' : DE'^2$.

Cor. 1. $CA^2 : CB^2 :: CA^2 - CE^2 : DE^2$.

Cor. 2. The squares of the ordinates to either axis are to each other as the rectangles of their abscissas.

PROPOSITION XIV. THEOREM.

If a circle be described on either axis, then any ordinate in the circle is to the corresponding ordinate in the ellipse as the axis of that ordinate is to the other axis.



Let a circle be described on AA' as a diameter, and let DE, an ordinate to the axis, be produced to meet the circle in G; then

$$GE : DE :: AC : BC.$$

For (Pr. 13)

$$AC^2 : BC^2 :: AE \times EA' : DE^2.$$

But $AE \times EA'$ is equal to GE^2 (B. IV., Pr. 23, Cor.)

Therefore $AC^2 : BC^2 :: GE^2 : DE^2$,

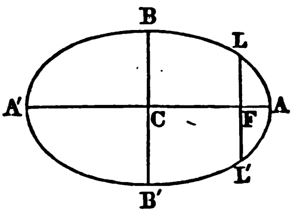
or $AC : BC :: GE : DE$.

Also, if a circle be described on BB′ as a diameter, and the ordinate DE′ be drawn meeting the circle in G′, then

$$G'E' : DE' :: BC : AC.$$

PROPOSITION XV. THEOREM.

The latus rectum is a third proportional to the major and minor axes.



Let LL′ be a double ordinate to the major axis passing through the focus F; then we shall have

$$AA' : BB' :: BB' : LL'.$$

Because LF is an ordinate to the major axis,

$$AC^2 : BC^2 :: AF \times FA' : LF^2 \text{ (Pr. 13).}$$
$$:: BC^2 : LF^2 \text{ (Pr. 5, Cor. 1).}$$

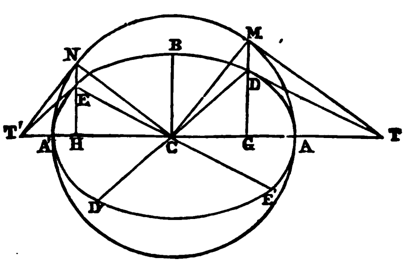
Hence $AC : BC :: BC : LF,$

or $AA' : BB' :: BB' : LL'.$

Therefore the latus rectum, etc.

PROPOSITION XVI. THEOREM.

If one diameter of an ellipse is conjugate to another, and if from the vertices of these two diameters ordinates be drawn to either axis, the sum of the squares of these ordinates will be equal to the square of half the other axis.



Let the diameter EE′ be conjugate to DD′; and let DG and EH, ordinates to the major axis, be drawn from their vertices; in which case CG and CH will be equal to the ordinates of the minor axis drawn from the same points; then we shall have

$$CG^2 + CH^2 = CA^2;$$

and $$DG^2 + EH^2 = CB^2.$$

Upon AA′ as a diameter describe the circle AMA′, and produce DG and EH to cut the circumference in M and N. Draw the tangents at D and M, which will meet each other in T, in the axis produced (Pr. 12). Join CM and CN.

Since DT is parallel to EC, the triangles DTG and ECH are similar, and therefore

$$CH : GT :: EH : DG$$
$$:: NH : MG. \quad \text{By Pr. 14.}$$

Hence the triangle NHC is similar to MGT, and it is also similar to MCG (B. IV., Pr. 23). But the hypothenuse $CM = CN$; therefore $MG = CH$; and, consequently,

$$CG^2 + CH^2 = CG^2 + GM^2 = CM^2 = CA^2.$$

Secondly. By Pr. 14,

$$AC^2 : BC^2 :: NH^2 : EH^2$$
$$:: MG^2 : DG^2$$
$$:: NH^2 + MG^2 : EH^2 + DG^2 \text{ (B. II., Pr. 6).}$$

But $NH^2 + MG^2 = NH^2 + CH^2 = CN^2 = AC^2$;

therefore $EH^2 + DG^2 = BC^2$.

Therefore, if one diameter, etc.

Cor. 1. Since $CG^2 = NH^2$, we have

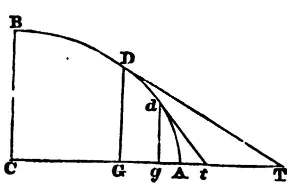
$$AC^2 : BC^2 :: CG^2 : EH^2.$$

Cor. 2. If one diameter of an ellipse is conjugate to another, the second is conjugate to the first. For if the tangent ET′ be drawn, it will be parallel to DD′.

Draw NT′; it will be tangent to the circle at N, and the triangle NT′H will be similar to NHC; that is, to CGM.

Hence $T'H : CG :: NH : MG$
$$:: EH : DG.$$

Therefore the triangles ET′H and DCG are similar, and ET′ is parallel to CD.



Cor. 3. Since

$$CA^2 : CB^2 :: MG^2 : DG^2,$$

and

$$MG^2 = CG.GT \text{ (B. IV., Pr. 23, Cor.),}$$

we have $CA^2 : CB^2 :: CG.GT : DG^2$.

If a second ordinate *dg*, and tangent *dt* be drawn, we shall have

$$CA^2 : CB^2 :: Cg.gt : dg^2.$$

Hence $CG.GT : Cg.gt :: DG^2 : dg^2$.

PROPOSITION XVII. THEOREM.

The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.

Let DD′, EE′ be any two conjugate diameters; then we shall have $DD'^2 + EE'^2 = AA'^2 + BB'^2$.

Draw DG, EH ordinates to the major axis. Then, by the preceding Proposition, $CG^2 + CH^2 = CA^2$, and $DG^2+EH^2=CB^2$.

Hence

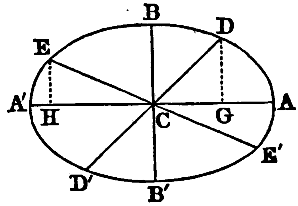
$CG^2+DG^2+CH^2+EH^2=CA^2+CB^2$,

or $CD^2+CE^2=CA^2+CB^2$;

that is

$DD'^2+EE'^2=AA'^2+BB'^2$.

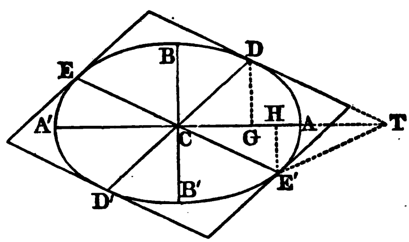
Therefore the sum of the squares, etc.



PROPOSITION XVIII. THEOREM.

The parallelogram formed by drawing tangents through the vertices of two conjugate diameters is equal to the rectangle of the axes.

Let DED'E' be a parallelogram formed by drawing tangents to the ellipse through the vertices of two conjugate diameters DD', EE'; its area is equal to $AA' \times BB'$.



Let the tangent at D meet the major axis produced in T; join E'T, and draw the ordinates DG, E'H.

Then, by Pr. 16, Cor. 1, we have

$CA^2 : CB^2 :: CG^2 : E'H^2$,

or $CA : CB :: CG : E'H$.

But $CT : CA :: CA : CG$ (Pr. 11);

hence $CT : CB :: CA : E'H$,

or $CA \times CB$ is equal to $CT \times E'H$,

which is equal to twice the triangle CE'T, or the parallelogram DE'; since the triangle and parallelogram have the same base CE', and are between the same parallels.

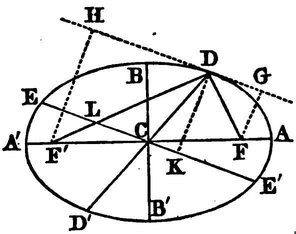
Hence $4CA \times CB$ or $AA' \times BB'$ is equal to 4DE', or the parallelogram DED'E'. Therefore the parallelogram, etc.

PROPOSITION XIX. THEOREM.

If from the vertex of any diameter straight lines are drawn to the foci, their product is equal to the square of half the conjugate diameter.

Let DD', EE' be two conjugate diameters, and from D let lines be drawn to the foci; then will $FD \times F'D$ be equal to EC^2.

Draw a tangent to the ellipse at D, and upon it let fall the perpendiculars FG, F'H; draw, also, DK perpendicular to EE'.



Then, because the triangles DFG, DLK, DF'H are similar, we have

$$FD : FG :: DL : DK.$$

Also, $F'D : F'H :: DL : DK.$

Whence (B. II., Pr. 12)

$$FD \times F'D : FG \times F'H :: DL^2 : DK^2. \quad (1)$$

But, by Pr. 18,

$$AC \times BC = EC \times DK;$$

whence

$$AC \text{ or } DL : DK :: EC : BC,$$

and $$DL^2 : DK^2 :: EC^2 : BC^2. \quad (2)$$

Comparing proportions (1) and (2), we have

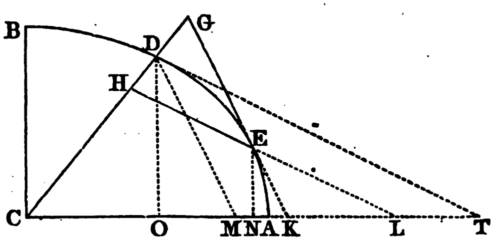
$$FD \times F'D : FG \times F'H :: EC^2 : BC^2.$$

But $FG \times F'H$ is equal to BC^2 (Pr. 10); hence $FD \times F'D$ is equal to EC^2. Therefore, if from the vertex, etc.

PROPOSITION XX. THEOREM.

If a tangent and ordinate be drawn from the same point of an ellipse to any diameter, half of that diameter will be a mean proportional between the distances of the two intersections from the centre.

Let a tangent EG and an ordinate EH be drawn from the same point E of an ellipse, meeting the diameter CD produced; then we shall have $CG : CD :: CD : CH.$



Produce EG and EH to meet the major axis in K and L; draw DT a tangent to the curve at the point D, and draw DM parallel to GK. Also, draw the ordinates EN, DO.

By similar triangles we have

$$OM : NK :: DO : EN,$$

and also $$OT : NL :: DO : EN.$$

Multiplying together the terms of these proportions (B. II., Pr. 12), we have

$OM.OT :: NK.NL :: DO^2 : EN^2 :: CO.OT : CN.NK$ (Pr. 16, Cor. 3). Omitting the factor OT in the antecedents, and NK in the consequents of this proportion (B. II., Pr. 10, Cor.), we have

$$OM : NL :: CO : CN,$$

and, by composition, $CO : CN :: CM : CL.$

But, by Pr. 11, Cor., $CO : CN :: CK : CT.$

Whence $CK : CM :: CT : CL.$

But $CK : CM :: CG : CD,$

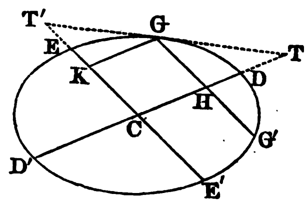
and $CT : CL :: CD : CH;$

hence $CG : CD :: CD : CH.$

Therefore, if a tangent, etc.

PROPOSITION XXI. THEOREM.

The square of any diameter is to the square of its conjugate as the rectangle of its abscissas is to the square of their ordinate.



Let DD′, EE′ be two conjugate diameters, and GH an ordinate to DD′; then

$$DD'^2 : EE'^2 :: DH \times HD' : GH^2.$$

Draw TT′ a tangent to the curve at the point G, and draw GK an ordinate to EE′. Then, by Pr. 20,

$$CT : CD :: CD : CH,$$

and $CD^2 : CH^2 :: CT : CH$ (B. II., Pr. 13);

whence, by division,

$$CD^2 : CD^2 - CH^2 :: CT : HT. \quad (1)$$

Also, by Pr. 20, $CT' : CE :: CE : CK,$

and $CE^2 : CK^2 :: CT' : CK$ or $GH,$

$$:: CT : HT. \quad (2)$$

Comparing proportions (1) and (2), we have

$$CD^2 : CE^2 :: CD^2 - CH^2 : CK^2 \text{ or } GH^2,$$

or $DD'^2 : EE'^2 :: DH \times HD' : GH^2.$

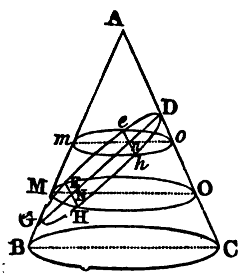
Therefore the square, etc.

Cor. 1. In the same manner, it may be proved that $DD'^2 : EE'^2 :: DH \times HD' : G'H^2$; hence GH is equal to G′H, or *every diameter bisects all chords parallel to the tangents at its vertices.*

Cor. 2. The squares of the ordinates to any diameter are to each other as the rectangles of their abscissas.

PROPOSITION XXII. THEOREM.

If *a cone be cut by a plane, making an angle with the base less than that made by the side of the cone, the section is an ellipse.*



Let ABC be a cone cut by a plane DEGH, making an angle with the base less than that made by the side of the cone; the section D*e*EGH*h* is an ellipse.

Let ABC be a section through the axis of the cone, and perpendicular to the plane DEGH. Let EMHO, *emho* be circular sections parallel to the base; then EH, the intersection of the planes DEGH, EMHO will be perpendicular to the plane ABC, and, consequently, to each of the lines DG, MO. So, also, *eh* will be perpendicular to DG and *mo*.

Now, because the triangles DNO, D*no* are similar, as also the triangles GMN, G*mn*, we have the proportions

$$NO : no :: DN : Dn,$$

and

$$MN : mn :: NG : nG.$$

Hence, by B. II., Pr. 12,

$$MN \times NO : mn \times no :: DN \times NG : Dn \times nG.$$

But, since MO is a diameter of the circle EMHO, and EN is perpendicular to MO, we have (B. IV., Pr. 23, Cor.)

$$MN \times NO = EN^2.$$

For the same reason, $mn \times no = en^2$.

Substituting these values of $MN \times NO$ and $mn \times no$ in the preceding proportion, we have

$$EN^2 : en^2 :: DN \times NG : Dn \times nG;$$

that is, the squares of the ordinates to the diameter DG are to each other as the products of the corresponding abscissas. Therefore the curve is an ellipse (Pr. 13, Cor. 2), whose major axis is DG. Hence the ellipse is called a *conic section*, as mentioned on page 203.

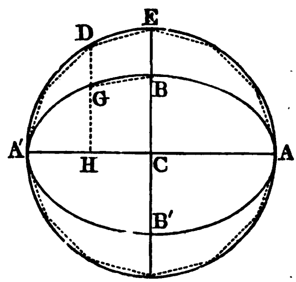
Scholium. The conclusion that the curve DEGH is an ellipse would not be legitimate unless the property above demonstrated were peculiar to the ellipse. That such is the case appears from the fact that when the major axis and one point of an ellipse are given, this property will determine the position of every other point of the curve, in the same manner as was shown in the corresponding Proposition for the parabola, p. 215.

PROPOSITION XXIII. THEOREM.

The area of an ellipse is a mean proportional between the two circles described on its axes.

Let AA′ be the major axis of an ellipse ABA′B′. On AA′ as a diameter describe a circle; inscribe in the circle any regular polygon AEDA′, and from the vertices E, D, etc., of the polygon draw perpendiculars to AA′. Join the points B, G, etc., in which these perpendiculars intersect the ellipse, and there will be inscribed in the ellipse a polygon of an equal number of sides.

Now the area of the trapezoid CEDH is equal to $(CE+DH)\times\frac{CH}{2}$; and the area of the trapezoid CBGH is equal to $(CB+GH)\times\frac{CH}{2}$. These trapezoids are to each other as CE+DH to CB+GH, or as AC to BC (Pr. 14).



In the same manner, it may be proved that each of the trapezoids composing the polygon inscribed in the circle is to the corresponding trapezoid of the polygon inscribed in the ellipse as AC to BC. Hence the entire polygon inscribed in the circle is to the polygon inscribed in the ellipse as AC to BC.

Since this proportion is true, whatever be the number of sides of the polygons, it will be true when the number is indefinitely increased; in which case one of the polygons coincides with the circle, and the other with the ellipse. Hence we have

area of circle : *area of ellipse* :: AC : BC.

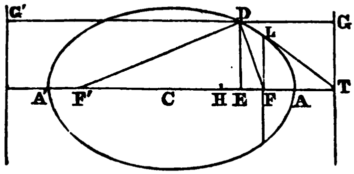
But the area of the circle is represented by πAC^2; hence the area of the ellipse is equal to $\pi AC\times BC$, which is a mean proportional between the two circles described on the axes.

PROPOSITION XXIV. THEOREM.

The distance of any point in an ellipse from either focus is to its distance from the corresponding directrix as the eccentricity to half the major axis.

Let D be any point in the ellipse; let DF, DF′ be drawn to the two foci, and DG, DG′ perpendicular to the directrices; then

DF : DG :: DF′ : DG′ :: CF : CA.



Draw DE perpendicular to the major axis, and take H, a point in the axis, so that AH=DF, and consequently HA′=DF′; then CH is half the difference between A′H and AH, or DF′ and DF, and CE is half the difference between FE and F′E. By B. IV., Pr. 34,

$$FF' : DF'+DF :: DF'-DF : F'E-FE.$$

Dividing each term by two, we have

$$CF : CA :: CH : CE.$$

But, by Pr. 11, $CA : CT :: CF : CA.$

Therefore $CA : CT :: CH : CE.$

Hence (B. II., Pr. 7)

$$CA-CH : CT-CE :: CA : CT,$$

or $AH : ET :: CA : CT :: CF : CA;$

that is, $DF : DG :: CF : CA.$

In the same manner, it may be proved that

$$DF' : DG' :: CF : CA.$$

EXERCISES ON THE ELLIPSE.

1. If a series of ellipses be described having the same major axis, the tangents at the extremities of their latera recta will all meet the minor axis in the same point.

2. The foci of an ellipse being given, it is required to describe an ellipse touching a given straight line.

3. If the angle FBF′ be a right angle, prove that $CA^2=2CB^2$. (See fig., Pr. 5.)

4. If a circle be described touching the major axis in one focus, and passing through one extremity of the minor axis, AC will be a mean proportional between BC and the diameter of this circle. (See fig., Pr. 5.)

5. If, on the two axes of an ellipse as diameters, circles be described, and a line be drawn through the centre cutting the larger circle in H and H′, and the smaller circle in K and K′, then $HK.H'K=CF^2$. (See fig., Pr. 14.)

6. If DG produced meet the tangent at the extremity of the latus rectum in K, then KG=DF. (See fig., Pr. 11.)

7. A tangent to the ellipse makes a greater angle with a line drawn from the point of contact to one of the foci than with the perpendicular on the directrix. (See fig., Pr. 24.)

8. If from C one line be drawn parallel, and another perpendicular to the tangent at D, they inclose a part of DF′ equal to DF. (See fig., Pr. 9.)

9. If the tangent at the vertex A cut any two conjugate diameters in T and t, then $AT.At = BC^2$. (See fig., Pr. 16.)

10. What is the area of an ellipse whose axes are 46 and 34 feet?

11. An ordinate to the major axis of an ellipse is 7 inches, and the corresponding abscissas are 5 and 20 inches; required the latus rectum.

12. The latus rectum of an ellipse is 11 inches, and the major axis 26 inches; required the area of the ellipse.

13. The eccentricity of an ellipse is 10 inches, and its latus rectum 12 inches; required the area of the ellipse.

14. Supposing a meridional section of the earth to be an ellipse whose major axis is 7926 miles, and its minor axis 7900 miles, what is the area of the section?

15. What is the latus rectum of the terrestrial ellipse, and what is its eccentricity?

16. What is the distance of the directrix of the terrestrial ellipse from the nearest vertex of the major axis?

17. If the axes of an ellipse are 60 and 100 feet, what is the radius of a circle described to touch the curve, when its centre is in the major axis at the distance of 16 feet from the centre of the ellipse? *Ans.* 27.495 feet.

18. If the axes of an ellipse are 60 and 80 feet, what are the areas of the two segments into which it is divided by a line perpendicular to the major axis at the distance of 10 feet from the centre? *Ans.* 1291.27 and 2478.65 feet.

19. The minor axis of an ellipse is 8 inches, the latus rectum 5 inches, and an ordinate of 3 inches is drawn to the major axis; determine where the tangent line drawn through the extremity of this ordinate meets the major axis produced.

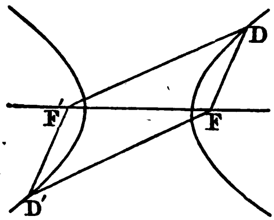
20. Determine where the tangent line in the last example meets the minor axis produced.

HYPERBOLA.

Definitions.

1. An *hyperbola* is a plane curve traced out by a point which moves in such a manner that the *difference* of its distances from two fixed points is always the same.

2. The two fixed points are called the *foci* of the hyperbola.

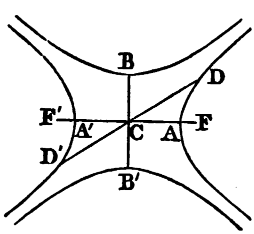


Thus, if F and F′ are two fixed points, and if the point D moves about F in such a manner that the difference of its distances from F and F′ is always the same, the point D will describe an hyperbola, of which F and F′ are the foci.

If the point D′ moves about F′ in such a manner that D′F−D′F′ is always equal to DF′−DF, the point D′ will describe a second branch of the curve similar to the first. The two branches are called *branches* of the hyperbola.

3. The *centre* of the hyperbola is the middle point of the straight line joining the foci.

4. The *eccentricity* is the distance from either focus to the centre.



Thus, let F and F′ be the foci of an hyperbola. Draw the line FF′, and bisect it in C. The point C is the centre of the hyperbola, and CF or CF′ is the eccentricity.

5. A *diameter* is any straight line passing through the centre, and terminated on both sides by opposite branches of an hyperbola.

6. The extremities of a diameter are called its *vertices*.

Thus, through C draw any straight line DD′ terminated by the opposite curves; DD′ is a diameter of the hyperbola; D and D′ are the vertices of that diameter.

7. The *transverse* axis is the diameter which, when produced, passes through the foci.

8. The *conjugate axis* is a line drawn through the centre perpendicular to the transverse axis, and terminated by the circum-

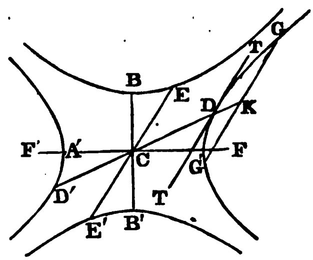
ference described from one of the vertices of the transverse axis as a centre, and with a radius equal to the eccentricity.

Thus, through C draw BB′ perpendicular to AA′, and with A as a centre, and with CF as a radius, describe a circumference cutting this perpendicular in B and B′; then AA′ is the transverse axis, and BB′ the conjugate axis.

If, on BB′ as a transverse axis, opposite branches of an hyperbola are described, having AA′ as their conjugate axis, this hyperbola is said to be *conjugate* to the former.

9. A tangent to an hyperbola is a straight line which meets the curve in one point only, and every where else falls without it.

10. An *ordinate* to a diameter is a straight line drawn from any point of the curve to meet the diameter produced, and is parallel to the tangent at one of its vertices.



Thus, let DD′ be any diameter, and TT′ a tangent to the hyperbola at D. From any point G of the curve draw GKG′ parallel to TT′, and cutting DD′ produced in K; then is GK an ordinate to the diameter DD′.

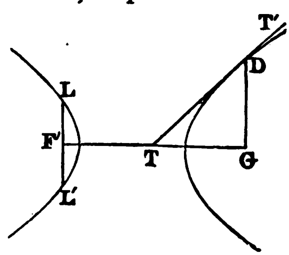
It is proved in Pr. 21, Cor. 1, that GK is equal to G′K; hence the entire line GG′ is called a *double ordinate.*

11. The parts of the diameter produced, intercepted between its vertices and an ordinate, are called its *abscissas.*

Thus, DK and D′K are the abscissas of the diameter DD′ corresponding to the ordinate GK.

12. When the ordinates of a diameter of an hyperbola are parallel to a diameter of the conjugate hyperbola, the latter diameter is said to be *conjugate* to the former.

Thus, draw the diameter EE′ parallel to GK, an ordinate to the diameter DD′, in which case it will, of course, be parallel to the tangent TT′; then is the diameter EE′ conjugate to DD′.



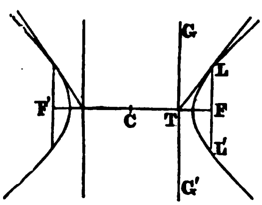
13. The *latus rectum* is the double ordinate to the transverse axis which passes through one of the foci.

Thus, through the focus F′ draw LL′, a double ordinate to the transverse axis; it will be the latus rectum of the hyperbola.

14. A *subtangent* is that part of an axis produced which is included between a tangent and the ordinate drawn from the point of contact.

Thus, if TT′ be a tangent to the curve at D, and DG an ordinate to the transverse axis, then GT is the corresponding subtangent.

15. The *directrix* of an hyperbola is a straight line perpendicular to the transverse axis, and intersecting it in the same point with the tangent to the curve at one extremity of the latus rectum.

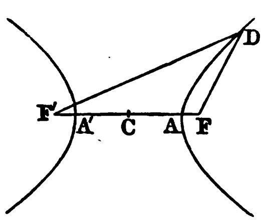


Thus, if LT be a tangent drawn through one extremity of the latus rectum LL′, meeting the axis in T, and, through the point of intersection, GG′ be drawn perpendicular to the axis, it will be the directrix of the hyperbola.

The hyperbola has two directrices, one corresponding to the focus F, and the other to the focus F′.

PROPOSITION I. THEOREM.

The difference of the two lines drawn from any point of an hyperbola to the foci is equal to the transverse axis.



Let F and F′ be the foci of two opposite hyperbolas, AA′ the transverse axis, and D any point of the curve; then will DF′−DF be equal to AA′.

For, by Def. 1, the difference of the distances of any point of the curve from the foci is equal to a given line. Now when the point D arrives at A, F′A−FA, or AA′+F′A′−FA, is equal to the given line. And when D is at A′, FA′−F′A′, or AA′+AF−A′F′, is equal to the same line. Hence AA′+AF−A′F′=AA′+F′A′−FA,

or 2AF=2A′F′;

that is, AF is equal to A′F′.

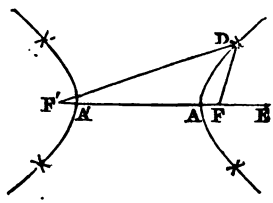
Hence DF′−DF, which is equal to AF′−AF, must be equal to AA′. Therefore the difference of the two lines, etc.

Cor. The transverse axis is bisected in the centre. For, by Def. 3, CF is equal to CF′; and we have just proved that AF is equal to A′F′; therefore AC is equal to A′C.

PROPOSITION II. PROBLEM.

The transverse axis and foci of an hyperbola being given, to describe the curve.

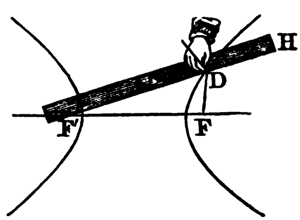
First method. *By points.*



Let AA′ be the transverse axis, and F, F′ the foci of an hyperbola. In the transverse axis AA′ produced, take any point E, and from F and F′ as centres, with the distances AE, A′E as radii, describe two circles cutting each other in the point D; D will be a point in the hyperbola. For, join FD, F′D; then DF′−DF=EA′−EA=AA′; and at whatever point of the transverse axis produced E is taken, the difference between DF′ and DF will be equal to AA′. Hence, by Def. 1, D is a point on the curve; and, in the same manner, any number of points in the hyperbola may be determined. In a similar manner the opposite branch may be constructed.

Cor. The same circles determine two points of the curve D and D′, one above and one below the transverse axis. It is also evident that these two points are equally distant from the axis; that is, the hyperbola is symmetrical with respect to its transverse axis.

Second method. *By continuous motion.*



Take a ruler longer than the distance FF′, and fasten one of its extremities at the point F′. Take a thread shorter than the ruler, and fasten one end of it at F, and the other to the end H of the ruler. Then move the ruler HDF′ about the point F′, while the thread is kept constantly stretched by a pencil pressed against the ruler; the curve described by the point of the pencil will be a portion of an hyperbola. For, in every position of the ruler, the difference of the lines DF, DF′ will be the same, viz., the difference between the length of the ruler and the length of the string.

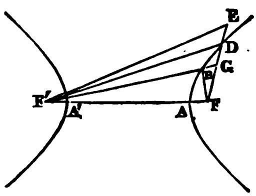
If the ruler be turned, and move on the other side of the point F, the other part of the same branch may be described.

Also, if one end of the ruler be fixed in F, and that of the thread in F′, the opposite branch may be described.

It is evident that each portion of each branch will extend to an indefinitely great distance from the foci and centre.

PROPOSITION III. THEOREM.

The difference of the two lines drawn to the foci from any point without the hyperbola is less than the transverse axis, and the difference of the two lines drawn to the foci from any point within the hyperbola is greater than the transverse axis.



Let F and F′ be the foci of an hyperbola; let AA′ be the transverse axis, and E any point without the curve. Join EF, EF′; the difference of EF′ and EF will be less than AA′.

Let F be the focus nearest to E; the line EF must cut the curve in some point D; then EF′ is less than ED+DF′ (B. I., Pr. 8). Subtracting EF, or ED+DF, from these unequals, we have EF′−EF less than DF′−DF; that is, than AA′.

Again, let G be a point within either branch of the hyperbola, and let F be the nearer focus; then F′G will cut the nearer branch of the curve in H. Join FH; then FG<HG+HF. Subtract each from F′G, and we have

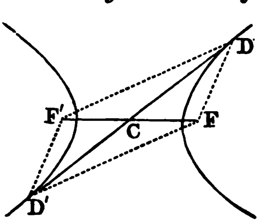
$$F'G-FG>F'G-HG-HF, \text{ which equals } F'H-FH;$$

that is, $F'G-FG>AA'$.

Cor. A point is without or within the hyperbola according as the difference of two lines drawn from it to the foci is less or greater than the transverse axis.

PROPOSITION IV. THEOREM.

Every diameter of an hyperbola is bisected in the centre.

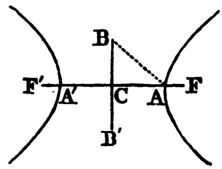


Let D be any point of an hyperbola; join DF, DF′, and FF′. Complete the parallelogram DFD′F′, and join DD′.

Now, because the opposite sides of a parallelogram are equal, the difference between DF and DF′ is equal to the difference between D′F and D′F′; hence D′ is a point in the opposite branch of the hyperbola. But the diagonals of a parallelogram bisect each other; therefore FF′ is bisected in C; that is, C is the centre of the hyperbola, and DD′ is a diameter bisected in C. Therefore every diameter, etc.

PROPOSITION V. THEOREM.

Half the conjugate axis is a mean proportional between the distances from one of the foci to the vertices of the transverse axis.



Let F and F' be the foci of an hyperbola, AA' the transverse axis, and BB' the conjugate axis; then will BC be a mean proportional between AF and A'F.

Join AB. Now BC^2 is equal to $AB^2 - AC^2$, which is equal to $FC^2 - AC^2$ (Def. 8). Hence (B. IV., Pr. 10)

$$BC^2 = (FC - AC) \times (FC + AC)$$
$$= AF \times A'F;$$

and hence $AF : BC :: BC : A'F$.

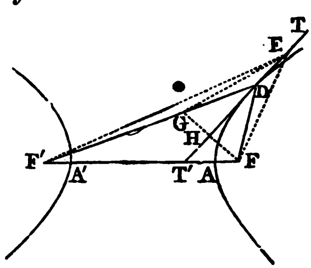
Cor. 1. *The square of the eccentricity is equal to the sum of the squares of the semi-axes.*

For FC^2 is equal to AB^2 (Def. 8), which is equal to $AC^2 + BC^2$.

Cor. 2. The eccentricity of an hyperbola and of its conjugate are equal, and a circle described from C as a centre and CF as a radius will pass through the four foci of the two hyperbolas.

PROPOSITION VI. THEOREM.

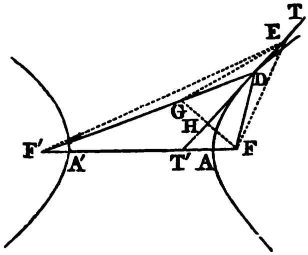
A tangent to the hyperbola bisects the angle contained by lines drawn from the point of contact to the foci.



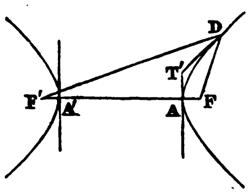
Let F, F' be the foci of an hyperbola, and D any point of the curve; if, through the point D, the line TT' be drawn bisecting the angle FDF', then will TT' be a tangent to the hyperbola at D.

Let E be any point in the line TT' different from D, and let F be the focus nearest to E. On DF' take DG equal to DF, and join EF, EF', EG, and FG.

Now, in the two triangles DFH, DGH, because DF is equal to DG, DH is common to both triangles, and the angle FDH is, by supposition, equal to GDH; therefore HF is equal to HG, and the angle DHF is equal to the angle DHG. Hence the line TT' is perpendicular to FG at its middle point, and therefore EF is equal to EG.



Hence EF′−EF is equal to EF′−EG. But EF′−EG is less than GF′ (B. I., Pr. 8); that is, less than the difference of DF′ and DF, which is equal to AA′; therefore EF′−EF is less than the transverse axis, and hence the point E is without the hyperbola (Pr. 3, Cor.). Therefore every point of the line TT′ except D is without the curve; that is, TT′ is a tangent to the curve at D.

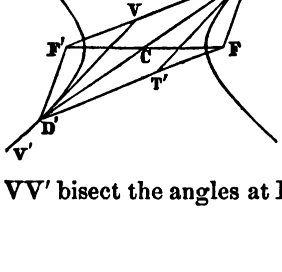


Cor. 1. As the point D moves toward A, each of the angles FDT′, F′DT′ increases, and at A becomes a right angle. Hence the tangents at the vertices of the transverse axis are perpendicular to that axis.

Cor. 2. If TT′ represent a plane mirror, a ray of light proceeding from F in the direction FD would be reflected in a line which, if produced backward, would pass through F′, making the angle of reflection equal to the angle of incidence. And, since the hyperbola may be regarded as coinciding with a tangent at the point of contact, if rays of light proceed from one focus of a polished surface whose figure, whether concave or convex, is that produced by the revolution of an hyperbola about its transverse axis, they will be reflected in lines diverging from the other focus. For this reason, the points F, F′ are called the *foci.*

PROPOSITION VII. THEOREM.

Tangents to the hyperbola at the vertices of any diameter are parallel to each other.



Let DD′ be any diameter of an hyperbola, and TT′, VV′ tangents to the curve at the points D, D′; then will they be parallel to each other.

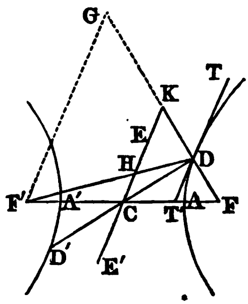
Join DF, DF′, D′F, D′F′. Then, by Pr. 4, FDF′D′ is a parallelogram; and, since the opposite angles of a parallelogram are equal, the angle FDF′ is equal to FD′F′. But the tangents TT′, VV′ bisect the angles at D and D′ (Pr. 6); hence the angle F′DT′.

or its alternate angle FT′D, is equal to FD′V. But FT′D is the exterior angle opposite to FD′V; hence TT′ is parallel to VV′. Therefore tangents, etc.

Cor. If tangents are drawn through the vertices of any two diameters, whether of the same or of conjugate hyperbolas, they will form a parallelogram.

PROPOSITION VIII. THEOREM.

If through the vertex of any diameter straight lines are drawn from the foci, meeting the conjugate diameter, the part intercepted by the conjugate is equal to half of the transverse axis.



Let EE′ be a diameter conjugate to DD′, and let the lines DF, DF′ be drawn, and produced, if necessary, so as to meet EE′ in H and K; then will DH or DK be equal to AC.

Draw F′G parallel to EE′ or TT′, meeting FD produced in G. Then the angle DGF′ is equal to the exterior angle FDT′, and the angle DF′G is equal to the alternate angle F′DT′. But the angles FDT′, F′DT′ are equal to each other (Pr. 6); hence the angles DGF′, DF′G are equal to each other, and DG is equal to DF′. Also, because CK is parallel to F′G, and CF is equal to CF′, therefore FK must be equal to KG.

Hence F′D—FD is equal to GD—FD or GF—2DF; that is, 2KF—2DF or 2DK. But F′D—FD is equal to 2AC. Therefore 2AC is equal to 2DK, or AC is equal to DK.

Also, the angle DHK is equal to DKH, and hence DH is equal to DK or AC. Therefore, if through the vertex, etc.

PROPOSITION IX. THEOREM.

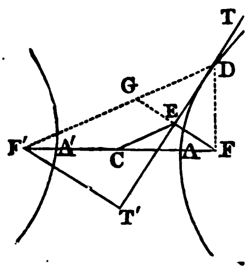
Perpendiculars drawn from the foci upon a tangent to the hyperbola meet the tangent in the circumference of a circle whose diameter is the transverse axis.

Let TT′ be a tangent to the hyperbola at D, and from F draw FE perpendicular to TT′; the point E will be in the circumference of a circle described upon AA′ as a diameter.

Join CE, FD, F′D, and produce FE to meet F′D in G.

Then, in the two triangles DEF, DEG, because DE is common to both triangles, the angles at E are equal, being right angles;

also, the angle EDF is equal to EDG (Pr. 6); therefore DF is equal to DG, and EF to EG.



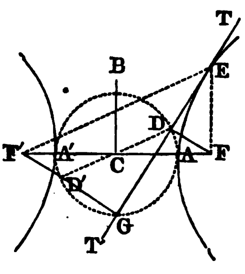
Also, because FE is equal to EG, and CF is equal to CF′, CE must be parallel to F′G, and, consequently, equal to half of F′G.

But, since DG has been proved equal to DF, F′G is equal to F′D−FD, which is equal to AA′. Hence CE is equal to half of AA′ or AC, and a circle described with C as a centre, and radius CA, will pass through the point E.

The same may be proved of a perpendicular let fall upon TT′ from the focus F′. Therefore perpendiculars, etc.

PROPOSITION X. THEOREM.

The product of the perpendiculars from the foci upon a tangent is equal to the square of half the conjugate axis.



Let TT′ be a tangent to the hyperbola at any point E, and let the perpendiculars FD, F′G be drawn from the foci; then will the product of FD by F′G be equal to the square of BC.

On AA′ as a diameter describe a circle; it will pass through the points D and G (Pr. 9). Let GF′ meet the circle in D′, and join DD′; then, since the angle at G is a right angle, DD′ passes through the centre C. Because FD and F′G are perpendicular to the same straight line TT′, they are parallel to each other, and the alternate angles CFD, CF′D′ are equal. Also, the vertical angles DCF, D′CF′ are equal, and CF is equal to CF′. Therefore DF is equal to D′F′; hence DF × GF′ is equal to D′F′ × GF′, which is equal to A′F′ × F′A (B. IV., Pr. 29, Cor. 2), which is equal to BC^2 (Pr. 5).

Cor. The triangles FDE, F′GE are similar; hence

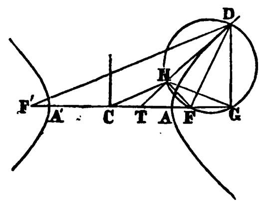
$$FD : F'G :: FE : F'E;$$

that is, *perpendiculars let fall from the foci upon a tangent are to each other as the distances of the point of contact from the foci.*

PROPOSITION XI. THEOREM.

If a tangent and ordinate be drawn from the same point of an hyperbola, meeting either axis produced, half of that axis will be a mean proportional between the distances of the two intersections from the centre.

1st. *For the transverse axis.*



Let DT be a tangent to the hyperbola, and DG an ordinate to the transverse axis from the point of contact; then we shall have CT : CA :: CA : CG.

From F draw FH perpendicular to DT, and join DF, DF′, CH, and GH. Then, by Pr. 9, CH is parallel to DF′. Also, since DGF, DHF are both right angles, a circle described on DF as a diameter will pass through the points G and H. Therefore the angle CGH or FGH is equal to the angle HDF (B. III., Pr. 15, Cor. 1), which is equal to F′DT or CHT. That is, the angle CGH is equal to CHT; and, since the angle C is common to the two triangles CGH, CHT, these triangles are equiangular, and we have

CT : CH :: CH : CG.

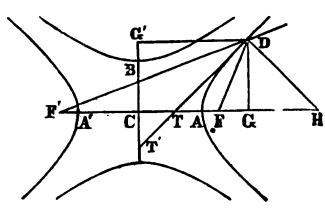
But CH is equal to CA (Pr. 9); therefore

CT : CA :: CA : CG.

2d. *For the conjugate axis.*

Let the tangent DTT′ meet the conjugate axis in T′, and let DG′ be an ordinate to the conjugate axis from the point of contact; then we shall have

CT′ : CB :: CB : CG′.



Draw DH perpendicular to DT, and it will bisect the exterior angle of the triangle FDF′. Hence (B. IV., Pr. 18)

HF′ : HF :: DF′ : DF
:: TF′ : TF.

Therefore (B. II., Pr. 8)

2CF : 2CH :: 2CT : 2CF.

Whence $CT \times CH = CF^2$.

But we have proved that $CT \times CG = CA^2$.

Subtracting the latter from the former, we have

$$CT \times GH = CF^2 - CA^2 = CB^2.$$

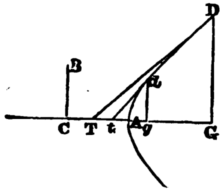
Because the triangles DGH and CTT′ are similar, we have

$$CT : CT' :: DG : GH.$$

Whence $CT \times GH = CT' \times DG = CT' \times CG'$.

Therefore $CT' \times CG' = CB^2$,

or $CT' : CB :: CB : CG'$.



Cor. By this Proposition,

$$CA^2 = CG \times CT.$$

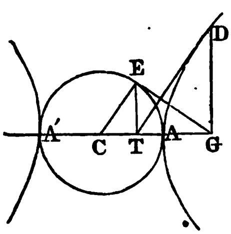
If a second ordinate *dg*, and tangent *dt* be drawn, we shall also have $CA^2 = Cg.Ct$.

Whence $CG \times CT = Cg.Ct$,

or $CT : Ct :: Cg : CG$.

PROPOSITION XII. THEOREM.

The subtangent of an hyperbola is equal to the corresponding subtangent of the circle described upon its transverse axis.



Let AEA′ be a circle described on AA′, the transverse axis of an hyperbola, and from any point E in the circle draw the ordinate ET. Through T draw the line DT touching the hyperbola in D, and from the point of contact draw the ordinate DG. Join GE; then will GE be a tangent to the circle at E.

Join CE. Then, by the last Proposition,

$$CT : CA :: CA : CG;$$

or, because CA is equal to CE,

$$CT : CE :: CE : CG.$$

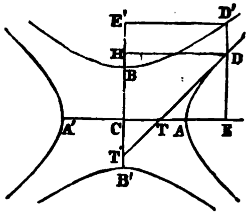
Hence the triangles CET, CGE, having the angle at C common, and the sides about this angle proportional, are similar (B. IV., Pr. 21). Therefore the angle CEG, being equal to the angle CTE, is a right angle; that is, the line GE is perpendicular to the radius CE, and is, consequently, a tangent to the circle (B. III., Pr. 9). Hence GT is the subtangent corresponding to each of the tangents DT and EG. Therefore the subtangent, etc.

PROPOSITION XIII. THEOREM.

The square of the transverse axis is to the square of the conjugate as the rectangle of the abscissas of the former is to the square of their ordinate.

Let DE be an ordinate to the transverse axis from the point D; then we shall have

$CA^2 : CB^2 :: AE \times EA' : DE^2.$



Draw DTT′ a tangent to the hyperbola at D; then, by Pr. 11,

$$CT : CA :: CA : CE.$$

Hence (B. II., Pr. 13)

$$CT : CE :: CA^2 : CE^2;$$

and, by division (B. II., Pr. 7),

$$CT : ET :: CA^2 : CE^2 - CA^2. \quad (1)$$

Again, by Pr. 11, $CT' : CB :: CB : CH$ or DE.

Hence $CT' : DE :: CB^2 : DE^2.$

But, by similar triangles,

$$CT' : DE :: CT : ET;$$

therefore $CT : ET :: CB^2 : DE^2. \quad (2)$

Comparing proportions (1) and (2), we have

$$CA^2 : CE^2 - CA^2 :: CB^2 : DE^2.$$

But $CE^2 - CA^2$ is equal to $AE \times EA'$ (B. IV., Pr. 10).

Hence $CA^2 : CB^2 :: AE \times EA' : DE^2.$

Cor. 1. $CA^2 : CB^2 :: CE^2 - CA^2 : DE^2.$

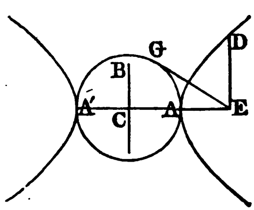
Cor. 2. The squares of the ordinates to the transverse axis are to each other as the rectangles of their abscissas.

Cor. 3. Produce DE to meet the conjugate hyperbola in D′, and draw D′E′ at right angles to CE′; then, since the conjugate hyperbola is described with BB′ as transverse axis and AA′ as conjugate axis, we shall have

$$CB^2 : CA^2 :: CE'^2 - CB^2 : D'E'^2.$$

PROPOSITION XIV. THEOREM.

If a circle be described on the transverse axis of an hyperbola, an ordinate to this axis is to a tangent to the circle drawn from the foot of the ordinate as the conjugate axis is to the transverse.



Let a circle be described on AA′ as a diameter; draw the ordinate DE, and from E draw EG tangent to the circle; then $ED : EG :: BC : AC.$

For, by Pr. 13,

$$ED^2 : AE \times EA' :: CB^2 : CA^2.$$

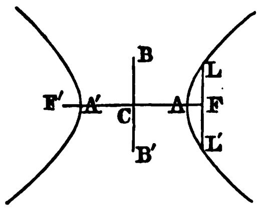
But $AE \times EA'$ is equal to EG^2 (B. IV., Pr. 29).

Therefore $ED^2 : EG^2 :: CB^2 : CA^2;$

or $ED : EG :: CB : CA.$

PROPOSITION XV. THEOREM.

The latus rectum is a third proportional to the transverse and conjugate axes.



Let LL′ be a double ordinate to the transverse axis passing through the focus F; then we shall have

$$AA' : BB' :: BB' : LL'.$$

Because LF is an ordinate to the transverse axis,

$$AC^2 : BC^2 :: AF \times FA' : LF^2 \text{ (Pr. 13)}$$
$$:: BC^2 : LF^2 \text{ (Pr. 5)}.$$

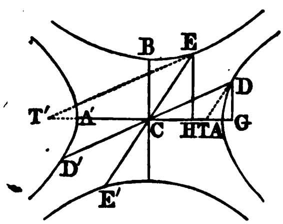
Hence $AC : BC :: BC : LF,$

or $AA' : BB' :: BB' : LL'.$

Therefore the latus rectum, etc.

PROPOSITION XVI. THEOREM.

If a diameter of the hyperbola is conjugate to a diameter of the conjugate hyperbola, and if ordinates be drawn to either axis from the vertices of the two diameters, the difference of their squares will be equal to the square of half the other axis.



Let DD′ be a diameter of an hyperbola, and DT a tangent at the point D; and let EE′ be a diameter of the conjugate hyperbola parallel to DT. Let DG and EH be ordinates to the axis AA′; then we shall have $CG^2 - CH^2 = CA^2$,

and $EH^2 - DG^2 = CB^2$.

Through E draw the tangent ET′; then, by Pr. 13, Cor. 3,

$$CA^2 : CB^2 :: CH^2 : EH^2 - CB^2,$$

and, by composition,

$$CA^2 + CH^2 : EH^2 :: CA^2 : CB^2$$
$$:: CG^2 - CA^2 : DG^2 \text{ (Pr. 13, Cor. 1)}.$$

But $CA^2 + CH^2 = CH.CT' + CH^2 = CH.HT'$ (Pr. 11),

and $CG^2 - CA^2 = CG^2 - CG.CT = CG.GT.$

Hence $CH.HT' : CG.GT :: EH^2 : DG^2$

$$:: CH^2 : GT^2, \text{ by sim. triangles.}$$

Hence, B. II., Pr. 10, Cor.,

$$HT' : CG :: CH : GT :: EH : DG.$$

Therefore the triangles EHT′ and DGC are similar, and ET′ is parallel to DD′. Hence the triangles ECT′ and DCT are similar, and we have $CT : CT' :: GT : CH.$

But $CT : CT' :: CH : CG$ (Pr. 11, Cor.).

Hence $GT : CH :: CH : CG,$

or $CH^2 = CG.GT.$

Subtract each of these equals from CG^2, and we have

$$CG^2 - CH^2 = CG^2 - CG.GT = CG.CT = CA^2.$$

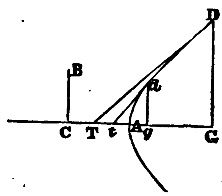
Also, since ET′ is parallel to DD′, the diameter DD′ is conjugate to EE′, and we have $EH^2 - DG^2 = CB^2.$

Therefore, if a diameter, etc.

Cor. 1. $CA^2 + CH^2 = CG^2;$

hence $CA^2 : CB^2 :: CG^2 : EH^2.$

Cor. 2. If a diameter of an hyperbola is conjugate to a diameter of the conjugate hyperbola, the second diameter is conjugate to the first; for it has been proved that if EE′ be parallel to the tangent DT, DD′ will be parallel to the tangent ET′.



Cor. 3. $CG^2 - CA^2 = CG.GT;$

hence $CA^2 : CB^2 :: CG \times GT : DG^2.$

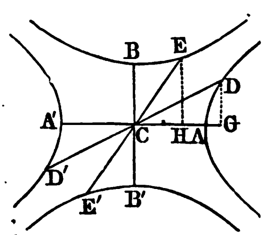
If a second ordinate dg, and tangent dt be drawn, we shall have

$$CA^2 : CB^2 :: Cg \times gt : dg^2.$$

Hence $CG \times GT : Cg \times gt :: DG^2 : dg^2.$

PROPOSITION XVII. THEOREM.

The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.



Let DD′, EE′ be any two conjugate diameters; then we shall have

$$DD'^2 - EE'^2 = AA'^2 - BB'^2.$$

Draw DG, EH ordinates to the transverse axis. Then, by the preceding Proposition, $CG^2 - CH^2 = CA^2,$

and $EH^2 - DG^2 = CB^2.$

Hence

$$CG^2 + DG^2 - CH^2 - EH^2 = CA^2 - CB^2,$$

or $CD^2 - CE^2 = CA^2 - CB^2;$

that is, $DD'^2 - EE'^2 = AA'^2 - BB'^2.$

Therefore the difference of the squares, etc.

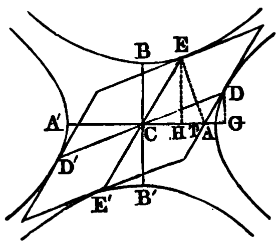
PROPOSITION XVIII. THEOREM.

The parallelogram formed by drawing tangents through the vertices of two conjugate diameters is equal to the rectangle of the axes.

Let DED′E′ be a parallelogram formed by drawing tangents to the conjugate hyperbolas through the vertices of two conjugate diameters DD′, EE′; its area is equal to AA′ × BB′.

Let the tangent at D meet the transverse axis in T; join ET, and draw the ordinates DG, EH.

Then, by Pr. 16, Cor. 1, we have $CA^2 : CB^2 :: CG^2 : EH^2$,

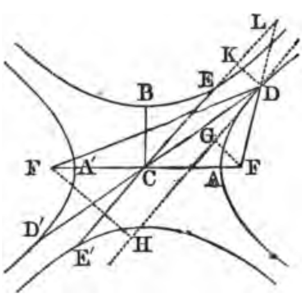


or CA : CB :: CG : EH.
But CT : CA :: CA : CG (Pr. 11);
hence CT : CB :: CA : EH,
or CA × CB is equal to CT × EH, which is equal to twice the triangle CTE, or the parallelogram DE; since the triangle and parallelogram have the same base CE, and are between the same parallels.

Hence 4CA × CB or AA′ × BB′ is equal to 4DE, or the parallelogram DED′E′ Therefore the parallelogram, etc.

PROPOSITION XIX. THEOREM.

If from the vertex of any diameter straight lines are drawn to the foci, their product is equal to the square of half the conjugate diameter.



Let DD′, EE′ be two conjugate diameters, and from D let lines be drawn to the foci; then will FD × F′D be equal to EC^2.

Draw a tangent to the hyperbola at D, and upon it let fall the perpendiculars FG, F′H; draw, also, DK perpendicular to EE′.

Then, because the triangles DFG, DLK, DF′H are similar, we have

FD : FG :: DL : DK.

Also, F′D : F′H :: DL : DK.

Whence (B. II., Pr. 12)

$$FD \times F'D : FG \times F'H :: DL^2 : DK^2. \qquad (1)$$

But, by Pr. 18, $AC \times BC = EC \times DK$;
whence AC or $DL : DK :: EC : BC$,
and $DL^2 : DK^2 :: EC^2 : BC^2$. (2)
Comparing proportions (1) and (2), we have

$$FD \times F'D : FG \times F'H :: EC^2 : BC^2.$$

But $FG \times F'H$ is equal to BC^2 (Pr. 10); hence $FD \times F'D$ is equal to EC^2. Therefore, if from the vertex, etc.

PROPOSITION XX. THEOREM.

If a tangent and ordinate be drawn from the same point of an hyperbola to any diameter, half of that diameter will be a mean proportional between the distances of the two intersections from the centre.

Let a tangent EG, and an ordinate EH, be drawn from the same point E of an hyperbola, meeting the diameter CD produced; then we shall have

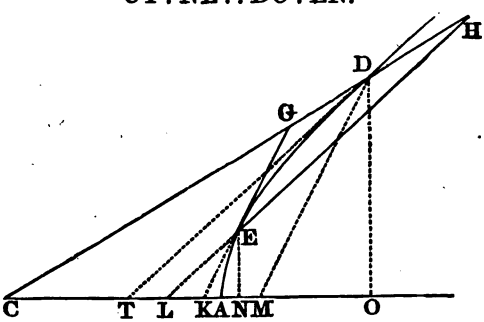
$$CG : CD :: CD : CH.$$

Produce GE and HE to meet the transverse axis in K and L; draw DT a tangent to the curve at the point D, and draw DM parallel to GK. Also draw the ordinates EN, DO.

By similar triangles we have

$$OM : NK :: DO : EN,$$

and also $OT : NL :: DO : EN$.



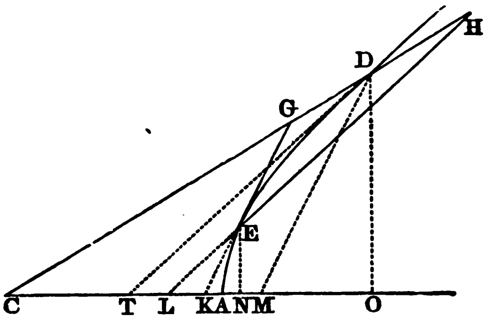
Multiplying together the terms of these proportions (B. II., Pr. 12), we have

$$OM \times OT :: NK \times NL :: DO^2 : EN^2 :: CO \times OT : CN \times NK$$

(Pr. 16, Cor. 3).

Omitting the factor OT in the antecedents, and NK in the consequents of this proportion (B. II., Pr. 10, Cor.), we have

$$OM : NL :: CO : CN,$$



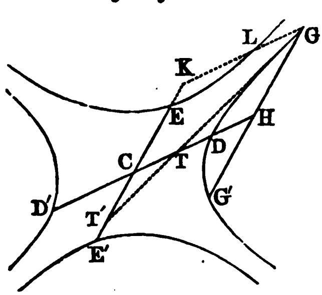
and, by division, CO : CN : : CM : CL.
But, by Pr. 11, Cor., CO : CN : : CK : CT.
Whence CK : CM : : CT : CL.
But CK : CM : : CG : CD,
and CT : CL : : CD : CH;
hence CG : CD : : CD : CH.

Therefore, if a tangent, etc.

Cor. If a tangent to the hyperbola meet a conjugate diameter, and from the point of contact an ordinate be drawn to that diameter, it may be proved that half of that diameter is a mean proportional between the distances of the two intersections from the centre.

PROPOSITION XXI. THEOREM.

The square of any diameter is to the square of its conjugate as the rectangle of its abscissas is to the square of their ordinate.



Let DD′, EE′ be two conjugate diameters, and GH an ordinate to DD′; then

$DD'^2 : EE'^2 :: DH \times HD' : GH^2$.

Draw GTT′ a tangent to the curve at the point G, and draw GK an ordinate to EE′. Then, by Pr. 20,

CT : CD : : CD : CH,

and $CD^2 : CH^2 :: CT : CH$

(B. II., Pr. 13),

whence, by division, $CD^2 : CH^2 - CD^2 :: CT : HT$. (1)

Also, by Pr. 20, Cor., CT′ : CE : : CE : CK,

and $CE^2 : CK^2 :: CT' : CK$ or GH,

$:: CT : HT$. (2)

Comparing proportions (1) and (2), we have

$$CD^2 : CE^2 :: CH^2 - CD^2 : CK^2 \text{ or } GH^2,$$

or

$$DD'^2 : EE'^2 :: DH \times HD' : GH^2.$$

Therefore the square, etc.

Cor. 1. In the same manner, it may be proved that $DD'^2 : EE'^2 :: DH \times HD' : G'H^2$; hence GH is equal to G'H, or *every diameter bisects all chords parallel to the tangents at its vertices.*

Cor. 2. The squares of the ordinates to any diameter are to each other as the rectangles of their abscissas.

Scholium. If DD' be produced beyond D', and ordinates be drawn in the opposite branch of the hyperbola, all the propositions which refer to the ordinates of the diameter DD' will apply indiscriminately to ordinates of either or both branches.

Thus, let DD' be produced to *h*, and draw the ordinate *gh*; then, by Cor. 2, $DH.D'H : Dh.D'h :: GH^2 : gh^2$.

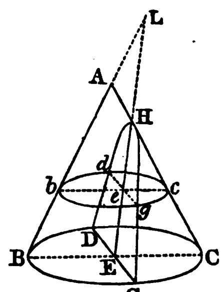
Also, produce EE' beyond E' to *k*, and draw the ordinate *kl*; then

$$EK.E'K : Ek.E'k :: KL^2 : kl^2.$$

PROPOSITION XXII. THEOREM.

If a cone be cut by a plane not passing through the vertex, and making an angle with the base greater than that made by the side of the cone, the section is an hyperbola.

Let ABC be a cone cut by a plane DGH, not passing through the vertex, and making an angle with the base greater than that made by the side of the cone, the section DHG is an hyperbola.



Let ABC be a section through the axis of the cone, and perpendicular to the plane HDG. Let *bgcd* be a section made by a plane parallel to the base of the cone; then DE, the intersection of the planes HDG, BGCD, will be perpendicular to the plane ABC, and, consequently, to each of the lines BC, HE. So, also, *de* will be perpendicular to *bc* and HE. Let AB and HE be produced to meet in L.

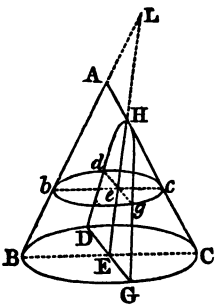
Now, because the triangles LBE, L*be* are similar, as also the triangles HEC, H*ec*, we have the proportions

$$BE : be :: EL : eL,$$

$$EC : ec :: HE : He.$$

Hence, by B. II., Pr. 12,

$$BE \times EC : be \times ec :: HE \times EL : He \times eL.$$



But, since BC is a diameter of the circle BGCD, and DE is perpendicular to BC, we have (B. IV., Pr. 23, Cor.)

$$BE \times EC = DE^2.$$

For the same reason,

$$be \times ec = de^2.$$

Substituting these values of BE×EC and $be \times ec$ in the preceding proportion, we have

$$DE^2 : de^2 :: HE \times EL : He \times eL;$$

that is, the squares of the ordinates to the diameter HE are to each other as the products of the corresponding abscissas. Therefore the curve DHG is an hyperbola (Pr. 13, Cor. 2) whose transverse axis is LH. Hence the hyperbola is called a *conic section*, as mentioned on page 203.

Schol. 1. The conclusion that the curve DHG is an hyperbola would not be legitimate unless the property above demonstrated were peculiar to the hyperbola. That such is the case appears from the fact that, when the transverse axis and one point of an hyperbola are given, this property will determine the position of every other point of the curve in the same manner as shown in the corresponding Proposition for the parabola, p. 215.

It will be noticed that this property of the hyperbola differs from the corresponding property of the ellipse in this particular, that the ordinate of the hyperbola falls upon the axis *produced*, while in the ellipse it falls upon the axis itself.

Schol. 2. The surface of the cone may be regarded as extending indefinitely below the base BGC, and hence the curve will extend indefinitely in the same direction.

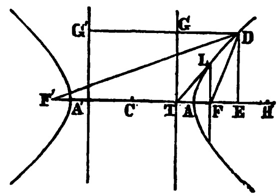
The surface of the cone is described by the motion of the line AB (B. X., Def. 3). If the portion of AB produced toward L be regarded as describing a second portion of the conical surface, the intersection of the plane DHGE with this second portion will be the opposite branch of the hyperbola DHG.

PROPOSITION XXIII. THEOREM.

The distance of any point in an hyperbola from either focus is to its distance from the corresponding directrix as the eccentricity to half the transverse axis.

Let D be any point in the hyperbola; let DF, DF′ be drawn to the two foci, and DGG′ perpendicular to the directrices; then

DF : DG :: DF′ : DG′ :: CF : CA.



Draw DE perpendicular to the transverse axis, and take H a point in the axis, so that AH=DF, and, consequently, HA′=DF′; then CH is half the sum of AH and A′H, or DF and DF′; and CE is half the sum of FE and F′E.

By B. IV., Pr. 34,

FF′ : DF′−DF : : DF′+DF : F′E+FE.

Dividing each of these equals by two, we have

CF : CA : : CH : CE.

By Pr. 11, CF : CA : : CA : CT.

Therefore CH : CE : : CA : CT.

Hence (B. II., Pr. 7)

CH−CA : CE−CT : : CA : CT;

or AH : ET : : CA : CT : : CF : CA;

that is, DF : DG : : CF : CA.

In the same manner, it may be proved that

DF′ : DG′ : : CF : CA.

Scholium 1. We have seen that, in the parabola, the distance of any point of the curve from the focus is equal to its distance from the directrix, while in the ellipse and hyperbola these distances are in the ratio of the eccentricity to half the major or transverse axis. In the ellipse the eccentricity is less than the semi-major axis, while in the hyperbola it is greater than the semi-transverse axis. In each of these three curves the two distances have to each other a constant ratio. In the parabola this ratio is unity; in the ellipse it is less than unity; while in the hyperbola it is greater than unity.

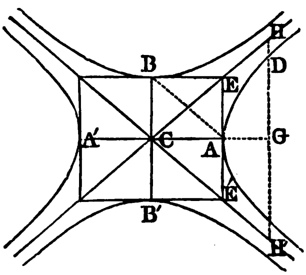
Scholium 2. Astronomers generally regard the semi-major axis of a planetary orbit as unity, in which case the eccentricity of the ellipse will be less than unity. If we regard the semi-transverse axis of an hyperbola as unity, its eccentricity will be greater than unity. The parabola may be regarded as an ellipse whose major axis is infinite, and in which the eccentricity is equal to the semi-major axis; that is, the eccentricity is unity. In Astronomy, therefore, the eccentricity of a parabola is considered as unity; that of an ellipse is less than unity; and that of an hyperbola is greater than unity. In each case the value of the eccentricity expresses the ratio of the distances of any point of the curve from the focus and directrix.

OF THE ASYMPTOTES.

Definition. If tangents to two conjugate hyperbolas be drawn through the vertices of the axes, the diagonals of the rectangle so formed, being indefinitely produced, are called *asymptotes* to the hyperbolas.

PROPOSITION XXIV. THEOREM.

If an ordinate to the transverse axis be produced to meet the asymptotes, the rectangles of the segments into which it is divided by the curve will be equal to the square of half the conjugate axis.



Let AA′, BB′ be the axes of two conjugate hyperbolas, and through the vertices A, A′, B, B′ let tangents to the curve be drawn, and let CE, CE′, the diagonals of the rectangle thus formed, be indefinitely produced, they will be asymptotes to the curves.

From any point D of one of the curves draw the ordinate DG to the transverse axis, and produce it to meet CE in H, and CE′ in H′. Then, from Pr. 13, Cor. 1, we shall have $\quad CA^2 : CB^2(=AE^2) :: CG^2-CA^2 : DG^2$

$:: CG^2 : GH^2$, by similar triangles.

Hence $\quad CG^2 : GH^2 :: CG^2-CA^2 : DG^2$,

and by division,

$$CG^2 : GH^2 :: CA^2 : GH^2-DG^2, \text{ or as } CA^2 : AE^2.$$

Since the antecedents of this proportion are equal to each other, the consequents must be equal; that is,

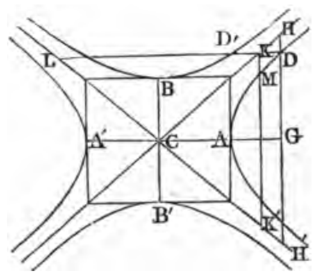
$$AE^2 \text{ or } BC^2 \text{ is equal to } GH^2-DG^2,$$

which is equal to $\quad HD \times DH'$ (B. IV., Pr. 10).

Cor. 1. Since the rectangle contained by HD and DH′ remains constant, while HDH′ is removed from C, and the line DH′ consequently increases, DH must diminish; and, by taking H sufficiently far from C, DH may be made less than any assignable magnitude. The line CH, therefore, approaches nearer and nearer to the hyperbola the farther it is produced, though it never actually reaches it at any finite distance from C. When the distance of H from C becomes infinitely great, DH becomes less than any assignable quantity, and *the asymptote may therefore be considered as a tangent to the curve at a point infinitely distant from the centre.*

The asymptote CH′, in the same manner, approaches nearer and nearer to the other branch of the hyperbola the farther it is produced.

Cor. 2. The line AB, joining the vertices of the two axes, is bisected by one asymptote, and is parallel to the other.



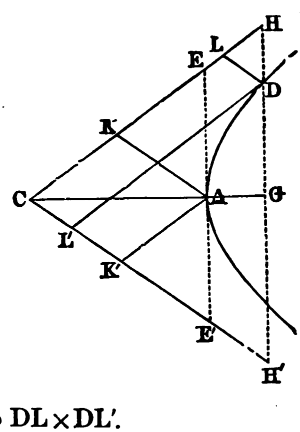
Cor. 3. If DL be drawn perpendicular to the conjugate axis, and meet the asymptotes in K and L, and the conjugate hyperbola in D′, it may also be proved that $CA^2 = D'K \times D'L$. The asymptote CH, therefore, continually approaches the conjugate hyperbola, and becomes tangent to it at an infinite distance from the centre.

Cor. 4. If KK′ be drawn parallel to HH′, then $KM \times MK' = HD \times DH'$, for each of them is equal to BC^2; that is, if two ordinates to the transverse axis be produced to meet the asymptotes, the rectangles of the segments into which these lines are divided by the curve are equal to each other.

PROPOSITION XXV. THEOREM.

All the parallelograms formed by drawing lines from any point of an hyperbola parallel to the asymptotes are equal to each other.

Let CH, CH′ be the asymptotes of an hyperbola; let the lines AK, DL be drawn parallel to CH′, and the lines AK′, DL′ parallel to CH; then will the parallelogram CLDL′ be equal to the parallelogram CKAK′.



Through the points A and D draw EE′, HH′ perpendicular to the transverse axis; then, because the triangles AEK, DHL are similar, as also the triangles AE′K′, DH′L′, we have the proportions

$$AK : AE :: DL : DH.$$

Also, $AK' : AE' :: DL' : DH'$.

Hence (B. II., Pr. 12)

$$AK \times AK' : AE \times AE' :: DL \times DL' : DH \times DH'.$$

But, by Pr. 24, Cor. 4, the consequents of this proportion are equal to each other; hence $AK \times AK'$ is equal to $DL \times DL'$.

But the parallelograms CA, CD, being equiangular, are as the rectangles of the sides which contain the equal angles (B. IV., Pr. 24, Cor. 2); hence the parallelogram CD is equal to the parallelogram CA.

EXERCISES ON THE HYPERBOLA.

1. In an hyperbola, the tangents at the vertices of the transverse axis will meet the asymptotes in the circumference of the circle described on FF′ as a diameter.

2. If DM be drawn parallel to CG (fig., Pr. 14), meeting the transverse axis in M, then ME=BC.

3. If an hyperbola and an ellipse have the same foci, they cut one another at right angles.

4. If DG (fig. 2d, Pr. 11) be the ordinate of a point D, and GK be drawn parallel to AD to meet CD in K, then AK is parallel to the tangent at D.

5. If from any point of the hyperbola lines be drawn parallel to, and terminating in the asymptotes, the parallelogram so formed will be equal to one eighth of the rectangle described on the axes.

6. An ordinate to the transverse axis of an hyperbola is 43 inches, and the corresponding abscissas are 30 and 85 inches; required the latus rectum.

7. If the axes of an hyperbola are 65 and 54 inches, what is the radius of a circle described to touch the curve, when its centre is in the transverse axis produced, at the distance of 112 inches from the centre of the hyperbola?

8. If the axes of an hyperbola are 65 and 54 inches, what is its latus rectum, and what is the position of its directrix?

9. The conjugate axis of an hyperbola is 52 inches, the latus rectum 42 inches, and an ordinate of 36 inches is drawn to the transverse axis; determine where the tangent line drawn through the extremity of this ordinate meets the transverse axis.

10. Determine where the tangent line in the last example meets the conjugate axis.

PLANE TRIGONOMETRY.

1. TRIGONOMETRY is that branch of Mathematics which teaches how to determine the several parts of a triangle by means of others that are given. In a more enlarged sense, it embraces the investigation of the relations of angles in general.

Plane Trigonometry treats of plane angles and triangles; Spherical Trigonometry treats of spherical triangles.

2. In every triangle there are *six parts:* three sides and three angles. These parts are so related to each other that when any three of them are given, provided one of them is a side, the remaining parts can be determined.

3. In order to subject angles to computation, they must be expressed by numbers. The units by which angles are expressed are the *degree*, *minute*, and *second*, designated by the characters °, ′, ″.

A *degree* is the 90th part of a right angle, or the 360th part of the whole angular space about a point. A right angle is expressed by 90°; two right angles by 180°; and the whole angular space about a point by 360°.

A *minute* is an angle equal to the 60th part of a degree. Therefore one degree$=60'$.

A *second* is an angle equal to the 60th part of a minute. Therefore one minute$=60''$.

Angles less than a second are expressed as decimal parts of a second. Thus $\frac{1}{7}$th of four right angles will be expressed by

$$51^\circ\ 25'\ 42.''86.$$

4. Since angles at the centre of a circle are proportional to the arcs intercepted between their sides, these arcs may be taken as the measures of the angles. An angle may therefore be measured by the number of *units of arc* intercepted on the circumference.

The units of arc are also the degree, minute, and second. They are the arcs which subtend angles of a degree, a minute, and a second respectively at the centre. The quadrant is therefore expressed by 90°; the semi-circumference by 180°; and the whole circumference by 360°.

The radius of the circle employed in measuring angles is arbitrary, and, for convenience, is generally taken as *unity*. When this is not done, it is denoted by its initial letter R.

5. The circumference of a circle whose diameter is unity is 3.14159. If the radius be unity, the semi-circumference, or an arc of 180°, will be 3.14159. Hence the length of an arc of 1° will be 0.01745; and the length of an arc of 1′ will be 0.00029, etc.

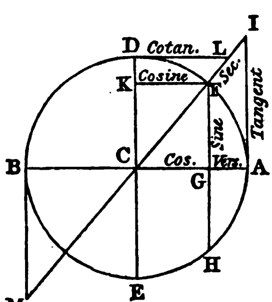
6. The *complement* of an arc or angle is the remainder obtained by subtracting the arc or angle from 90°. Thus the complement of 25° 15′ is 64° 45′. Since the two acute angles of a right-angled triangle are together equal to a right angle, each of them must be the complement of the other.

In general, if we represent any arc by A, its complement is 90°−A. Hence, if an arc exceeds 90°, its complement must be negative. Thus the complement of 113° 15′ is −23° 15′. See Art. 79.

7. The *supplement* of an arc or angle is the remainder obtained by subtracting the arc or angle from 180°. Thus the supplement of 25° 15′ is 154° 45′. Since in every plane triangle the sum of the three angles is 180°, either angle is the supplement of the sum of the other two.

In general, if we represent any arc by A, its supplement is 180°−A. Hence, if an arc is greater than 180°, its supplement must be negative. Thus the supplement of 200° is −20°.

8. *The sine of an arc is the perpendicular let fall from one extremity of the arc upon the diameter passing through the other extremity.*



Thus FG is the sine of the arc AF, or of the angle ACF.

Every sine is half the chord of double the arc. Thus the sine FG is the half of FH, which is the chord of the arc FAH, double of FA. The chord which subtends the sixth part of the circumference, or the chord of 60°, is equal to the radius (Geom., B. VI., Pr. 4); hence the sine of 30° is equal to half of the radius.

9. *The tangent of an arc is the line which touches the circle at one extremity of the arc, and is limited by a line drawn from the centre through the other extremity.*

Thus AI is the tangent of the arc AF, or of the angle ACF.

10. *The secant of an arc is the line drawn from the centre of the circle through one extremity of the arc, and is limited by the tangent drawn through the other extremity.*

Thus CI is the secant of the arc AF, or of the angle ACF.

In the preceding definitions of sine, tangent, and secant, the radius of the circle has been assumed as unity. In a circle of any other radius, we must suppose these lines to be divided by that radius.

11. The *cosine* of an arc is the sine of the complement of that arc.

Thus the arc DF, being the complement of AF, FK, or its equal CG, is the sine of the arc DF, or the cosine of the arc AF.

The *cotangent* of an arc is the tangent of the complement of that arc. Thus DL is the tangent of the arc DF, or the cotangent of the arc AF.

The *cosecant* of an arc is the secant of the complement of that arc. Thus CL is the secant of the arc DF, or the cosecant of the arc AF.

In general, if we represent any angle by A,

$$\text{cos. } A = \text{ sine } (90^\circ - A);$$
$$\text{cot. } A = \text{tang. } (90^\circ - A);$$
$$\text{cosec. } A = \text{ sec. } (90^\circ - A).$$

Since in a right-angled triangle either of the acute angles is the complement of the other, the sine, tangent, and secant of one of these angles is the cosine, cotangent, and cosecant of the other.

12. The *versed sine* of an arc is that part of the diameter intercepted between the extremity of the arc and the foot of the sine.

Thus GA is the versed sine of the arc AF, or of the angle ACF.

The versed sine of an acute angle ACF is equal to the radius *minus* the cosine CG. The versed sine of an obtuse angle BCF is equal to radius *plus* the cosine CG; that is, to BG.

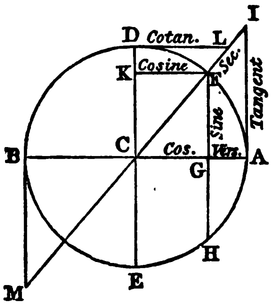
13. The sine, tangent, and secant of any arc are equal to the sine, tangent, and secant of its *supplement.*

Thus FG is the sine of the arc AF, or of its supplement BDF.

AI, the tangent of the arc AF, is equal to BM, the tangent of the arc BDF.

And CI, the secant of the arc AF, is equal to CM, the secant of the arc BDF.

14. *Fundamental formulæ.* The relations of the sine, cosine, etc., to each other may be derived from the proportions of the



sides of similar triangles. Thus the triangles CGF, CAI, CDL being similar, we have

1. CG : GF :: CA : AI; that is, representing the arc by A, and the radius of the circle by R, we have

$$\cos. A : \sin. A :: R : \text{tang. } A.$$

Whence $\text{tang. } A = \frac{R \sin. A}{\cos. A}$.

2. CG : CF :: CA : CI; that is,

$$\cos. A : R :: R : \sec. A.$$

Whence $\sec. A = \frac{R^2}{\cos. A}$.

3. GF : CG :: CD : DL; that is, sin. A : cos. A :: R : cot. A.

Whence $\cot. A = \frac{R \cos. A}{\sin. A}$.

4. GF : CF :: CD : CL; that is, sin. A : R :: R : cosec. A.

Whence $\text{cosec. } A = \frac{R^2}{\sin. A}$.

5. AI : AC :: CD : DL; that is, tang. A : R :: R : cot. A.

Whence $\text{tang. } A = \frac{R^2}{\cot. A}$.

The preceding formulæ will be frequently referred to hereafter.

15. *Given the sine of an angle, to find the cosine, tangent,* etc.

In the right-angled triangle CGF, we find $CG^2+GF^2=CF^2$; that is, $\sin.^2 A+\cos.^2 A=R^2$, where $\sin.^2 A$ signifies "the square of the sine of A." When radius is taken as unity, we have

$$\cos. A=\sqrt{1-\sin.^2 A}=\sqrt{(1+\sin. A)(1-\sin. A)}.$$

When the sine and cosine of an angle have been determined, the tangent may be found by Eq. 1, Art. 14,

$$\text{tang. } A=\frac{\sin. A}{\cos. A},$$

and the cotangent by Eq. 3, Art. 14,

$$\cot. A=\frac{\cos. A}{\sin. A}.$$

Also the secant by Eq. 2, Art. 14,

$$\sec. A=\frac{1}{\cos. A},$$

and the cosecant by Eq. 4, Art. 14,

$$\text{cosec. } A=\frac{1}{\sin. A}$$

Hence we see that if we had a table of sines for every degree and minute of the quadrant, we could easily obtain the cosines, tangents, cotangents, etc.

Ex. 1. Compute the cosine, tangent, etc., of 30°.

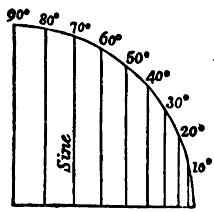
Ex. 2. Given the tangent of 20°, equal to 0.364, to find the secant of 20°. Find also the sine, etc., of the same angle.

Ex. 3. The tangent of 45° is unity. Compute the sine and secant of 45°.

Ex. 4. The sine of 40° is 0.643. Compute the cosine, tangent, etc.

16. A table of *natural sines, tangents*, etc., is a table giving the lengths of those lines for different angles in a circle whose radius is unity.

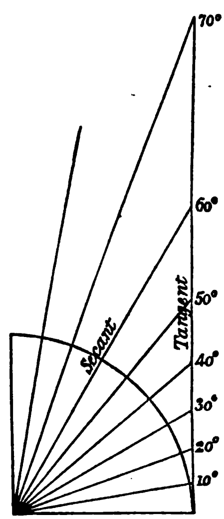
Thus, if we describe a circle with a radius of one inch, and divide the circumference into equal parts of ten degrees, we shall find that



sine 10°	=0.174	sine 50°	=0.766
" 20	=0.342	" 60	=0.866
" 30	=0.500	" 70	=0.940
" 40	=0.643	" 80	=0.985
" 45	=0.707	" 90	=1.000

If we draw the tangents of the same arcs, we shall find that

tangent 10°	=0.176	tangent 50°	=1.192
" 20	=0.364	" 60	=1.732
" 30	=0.577	" 70	=2.747
" 40	=0.839	" 80	=5.671
" 45	=1.000	" 90	=infinite.



Also, if we draw the secants of the same arcs, we shall find that

secant 10°	=1.015	secant 50°	=1.556
" 20	=1.064	" 60	=2.000
" 30	=1.155	" 70	=2.924
" 40	=1.305	" 80	=5.759
" 45	=1.414	" 90	=infinite.

17. The following table, pages 268–9, gives the sines and tangents between 0° and 90° for every ten minutes to four places of figures. For angles less than 45°, look for the degrees in the first vertical column, and for the minutes at the top of one of the six following columns; and for angles greater than 45°, look

	0′	10′	20′	30′	40′	50′		0′	10′	20′	30′	40′	50′
0°	0000	0029	0058	0087	0116	0145	45°	7071	7092	7112	7132	7153	7173
1	0175	0204	0233	0262	0291	0320	46	7193	7214	7234	7254	7274	7294
2	0349	0378	0407	0436	0465	0494	47	7314	7333	7353	7373	7392	7412
3	0523	0552	0581	0610	0640	0669	48	7431	7451	7470	7490	7509	7528
4	0698	0727	0756	0785	0814	0843	49	7547	7566	7585	7604	7623	7642
5	0872	0901	0929	0958	0987	1016	50	7660	7679	7698	7716	7735	7753
6	1045	1074	1103	1132	1161	1190	51	7771	7790	7808	7826	7844	7862
7	1219	1248	1276	1305	1334	1363	52	7880	7898	7916	7934	7951	7969
8	1392	1421	1449	1478	1507	1536	53	7986	8004	8021	8039	8056	8073
9	1564	1593	1622	1650	1679	1708	54	8090	8107	8124	8141	8158	8175
10	1736	1765	1794	1822	1851	1880	55	8192	8208	8225	8241	8258	8274
11	1908	1937	1965	1994	2022	2051	56	8290	8307	8323	8339	8355	8371
12	2079	2108	2136	2164	2193	2221	57	8387	8403	8418	8434	8450	8465
13	2250	2278	2306	2334	2363	2391	58	8480	8496	8511	8526	8542	8557
14	2419	2447	2476	2504	2532	2560	59	8572	8587	8601	8616	8631	8646
15	2588	2616	2644	2672	2700	2728	60	8660	8675	8689	8704	8718	8732
16	2756	2784	2812	2840	2868	2896	61	8746	8760	8774	8788	8802	8816
17	2924	2952	2979	3007	3035	3062	62	8829	8843	8857	8870	8883	8897
18	3090	3118	3145	3173	3201	3228	63	8910	8923	8936	8949	8962	8975
19	3256	3283	3311	3338	3365	3393	64	8988	9001	9013	9026	9038	9051
20	3420	3448	3475	3502	3529	3557	65	9063	9075	9088	9100	9112	9124
21	3584	3611	3638	3665	3692	3719	66	9135	9147	9159	9171	9182	9194
22	3746	3773	3800	3827	3854	3881	67	9205	9216	9228	9239	9250	9261
23	3907	3934	3961	3987	4014	4041	68	9272	9283	9293	9304	9315	9325
24	4067	4094	4120	4147	4173	4200	69	9336	9346	9356	9367	9377	9387
25	4226	4253	4279	4305	4331	4358	70	9397	9407	9417	9426	9436	9446
26	4384	4410	4436	4462	4488	4514	71	9455	9465	9474	9483	9492	9502
27	4540	4566	4592	4617	4643	4669	72	9511	9520	9528	9537	9546	9554
28	4695	4720	4746	4772	4797	4823	73	9563	9572	9580	9588	9596	9605
29	4848	4874	4899	4924	4950	4975	74	9613	9621	9628	9636	9644	9652
30	5000	5025	5050	5075	5100	5125	75	9659	9667	9674	9681	9689	9696
31	5150	5175	5200	5225	5250	5275	76	9703	9710	9717	9724	9730	9737
32	5299	5324	5348	5373	5398	5422	77	9744	9750	9757	9763	9769	9775
33	5446	5471	5495	5519	5544	5568	78	9781	9787	9793	9799	9805	9811
34	5592	5616	5640	5664	5688	5712	79	9816	9822	9827	9833	9838	9843
35	5736	5760	5783	5807	5831	5854	80	9848	9853	9858	9863	9868	9872
36	5878	5901	5925	5948	5972	5995	81	9877	9881	9886	9890	9894	9899
37	6018	6041	6065	6088	6111	6134	82	9903	9907	9911	9914	9918	9922
38	6157	6180	6202	6225	6248	6271	83	9925	9929	9932	9936	9939	9942
39	6293	6316	6338	6361	6383	6406	84	9945	9948	9951	9954	9957	9959
40	6428	6450	6472	6494	6517	6539	85	9962	9964	9967	9969	9971	9974
41	6561	6583	6604	6626	6648	6670	86	9976	9978	9980	9981	9983	9985
42	6691	6713	6734	6756	6777	6799	87	9986	9988	9989	9990	9992	9993
43	6820	6841	6862	6884	6905	6926	88	9994	9995	9996	9997	9997	9998
44	6947	6967	6988	7009	7030	7050	89	9998	9999	9999	unity	unity	unity
	0′	10′	20′	30′	40′	50′		0′	10′	20′	30′	40′	50′

	0′	10′	20′	30′	40′	50′		0′	10′	20′	30′	40′	50′
0°	0000	0029	0058	0087	0116	0145	45°	1.000	1.006	1.012	1.018	1.024	1.030
1	0175	0204	0233	0262	0291	0320	46	1.036	1.042	1.048	1.054	1.060	1.066
2	0349	0378	0407	0437	0466	0495	47	1.072	1.079	1.085	1.091	1.098	1.104
3	0524	0553	0582	0612	0641	0670	48	1.111	1.117	1.124	1.130	1.137	1.144
4	0699	0729	0758	0787	0816	0846	49	1.150	1.157	1.164	1.171	1.178	1.185
5	0875	0904	0934	0963	0992	1022	50	1.192	1.199	1.206	1.213	1.220	1.228
6	1051	1080	1110	1139	1169	1198	51	1.235	1.242	1.250	1.257	1.265	1.272
7	1228	1257	1287	1317	1346	1376	52	1.280	1.288	1.295	1.303	1.311	1.319
8	1405	1435	1465	1495	1524	1554	53	1.327	1.335	1.343	1.351	1.360	1.368
9	1584	1614	1644	1673	1703	1733	54	1.376	1.385	1.393	1.402	1.411	1.419
10	1763	1793	1823	1853	1883	1914	55	1.428	1.437	1.446	1.455	1.464	1.473
11	1944	1974	2004	2035	2065	2095	56	1.483	1.492	1.501	1.511	1.520	1.530
12	2126	2156	2186	2217	2247	2278	57	1.540	1.550	1.560	1.570	1.580	1.590
13	2309	2339	2370	2401	2432	2462	58	1.600	1.611	1.621	1.632	1.643	1.653
14	2493	2524	2555	2586	2617	2648	59	1.664	1.675	1.686	1.698	1.709	1.720
15	2679	2711	2742	2773	2805	2836	60	1.732	1.744	1.756	1.767	1.780	1.792
16	2867	2899	2931	2962	2994	3026	61	1.804	1.816	1.829	1.842	1.855	1.868
17	3057	3089	3121	3153	3185	3217	62	1.881	1.894	1.907	1.921	1.935	1.949
18	3249	3281	3314	3346	3378	3411	63	1.963	1.977	1.991	2.006	2.020	2.035
19	3443	3476	3508	3541	3574	3607	64	2.050	2.066	2.081	2.097	2.112	2.128
20	3640	3673	3706	3739	3772	3805	65	2.145	2.161	2.177	2.194	2.211	2.229
21	3839	3872	3906	3939	3973	4006	66	2.246	2.264	2.282	2.300	2.318	2.337
22	4040	4074	4108	4142	4176	4210	67	2.356	2.375	2.394	2.414	2.434	2.455
23	4245	4279	4314	4348	4383	4417	68	2.475	2.496	2.517	2.539	2.560	2.583
24	4452	4487	4522	4557	4592	4628	69	2.605	2.628	2.651	2.675	2.699	2.723
25	4663	4699	4734	4770	4806	4841	70	2.747	2.773	2.798	2.824	2.850	2.877
26	4877	4913	4950	4986	5022	5059	71	2.904	2.932	2.960	2.989	3.018	3.047
27	5095	5132	5169	5206	5243	5280	72	3.078	3.108	3.140	3.172	3.204	3.237
28	5317	5354	5392	5430	5467	5505	73	3.271	3.305	3.340	3.376	3.412	3.450
29	5543	5581	5619	5658	5696	5735	74	3.487	3.526	3.566	3.606	3.647	3.689
30	5773	5812	5851	5890	5930	5969	75	3.732	3.776	3.821	3.867	3.914	3.962
31	6009	6048	6088	6128	6168	6208	76	4.011	4.061	4.113	4.165	4.219	4.275
32	6249	6289	6330	6371	6412	6453	77	4.331	4.390	4.449	4.511	4.574	4.638
33	6494	6536	6577	6619	6661	6703	78	4.705	4.773	4.843	4.915	4.989	5.066
34	6745	6787	6830	6873	6916	6959	79	5.145	5.226	5.309	5.396	5.485	5.576
35	7002	7046	7089	7133	7177	7221	80	5.671	5.769	5.871	5.976	6.084	6.197
36	7265	7310	7355	7400	7445	7490	81	6.314	6.435	6.561	6.691	6.827	6.968
37	7536	7581	7627	7673	7720	7766	82	7.115	7.269	7.429	7.596	7.770	7.953
38	7813	7860	7907	7954	8002	8050	83	8.144	8.345	8.556	8.777	9.010	9.255
39	8098	8146	8195	8243	8292	8342	84	9.514	9.788	10.08	10.39	10.71	11.06
40	8391	8441	8491	8541	8591	8642	85	11.43	11.83	12.25	12.71	13.20	13.73
41	8693	8744	8796	8847	8899	8952	86	14.30	14.92	15.60	16.35	17.17	18.07
42	9004	9057	9110	9163	9217	9271	87	19.08	20.21	21.47	22.90	24.54	26.43
43	9325	9380	9435	9490	9545	9601	88	28.64	31.24	34.37	38.19	42.96	49.10
44	9657	9713	9770	9827	9884	9942	89	57.29	68.75	85.94	114.6	171.9	343.8
	0′	10′	20′	30′	40′	50′		0′	10′	20′	30′	40′	50′

for the degrees in the eighth vertical column, and for the minutes at the top of one of the six following columns. Upon the same horizontal line with the degrees, and under the given number of minutes at the top of the page, will be found the sine or tangent required. Since the radius of the circle is supposed to be unity, the sine of every arc below 90° is less than unity. The sines are expressed in decimal parts of radius; and, although the decimal point is not written in the table, it must always be prefixed. Thus the sine of 25° 10′ is 0.4253;
" 51 30 is 0.7826.
So also the tangent of 31° 40′ is 0.6168;
" 65 20 is 2.1770.

If the cosine of an angle is required, we must look for the sine of the complement of that angle. Thus
the cosine of 16° 40′ is the sine of 73° 20′, or 0.9580;
" 67 20 " 22 40, or 0.3854.

The cotangents are found in the same manner.

It is not necessary to extend the tables beyond a quadrant, because the sine of an angle is equal to that of its supplement, Art. 13.

Thus the sine of 116° 10′ is the same as the sine of 63° 50′.
cosine of 132 40 " sine of 42 40;
tangent of 143 20 " tangent of 36 40;
cotangent of 151 50 " tangent of 61 50.

18. If a sine is required for an angle containing a number of minutes not given in the table, it must be found by *interpolation*. This interpolation is based upon the assumption that the differences of the sines are proportional to the differences of the angles; and, although this assumption is not strictly correct, the error is generally so small that it may be neglected. Thus
the sine of 40° 20′ is 0.6472;
" 40 30 is 0.6494.

The difference of the sines corresponding to ten minutes of arc is .0022, which is called the *tabular difference*.

The correction for 1′ is therefore .00022; for 2′ it is .00044; for 3′ it is .00066, etc.

As the tables only extend to four decimal places, we omit the fifth decimal, and, when the fraction omitted exceeds a half, we increase the preceding figure by unity. Thus we find
the sine of 40° 21′ is 0.6474;
" 40 22 0.6476;
" 40 23 0.6479, etc.

Thus we see that the correction for the odd minutes is found by *multiplying the tabular difference by the number of minutes, and dividing the product by* 10.

In this manner we find

the sine of 27° 17′ is 0.4584;
cosine of 45 23 is 0.7024;
the tangent of 63 32 is 2.0090;
cotangent of 81 48 is 0.1441.

19. *To find the number of degrees and minutes belonging to a given sine or tangent.*

If the given sine is found exactly in the table, the corresponding degrees will be found in the first or eighth vertical column, and the minutes at the top of the page. But when the given number is not found exactly in the table, look for the sine or tangent which is next less than the proposed one, and take out the corresponding degrees and minutes. The additional minutes may be found by reversing the process described in the preceding article.

Find the difference between the given number and the one *next less* in the table; *multiply this difference by* 10, *and divide the result by the tabular difference. The quotient will be the additional minutes required.*

Ex. Required the arc whose sine is 0.5060.

The next less sine in the table is 0.5050, which corresponds to 30° 20′. The difference between this sine and the given sine is .0010, which, multiplied by 10 and divided by the tabular difference .0025, gives 4, the additional minutes required. The required arc is therefore 30° 24′.

In the same manner we find

the arc whose tangent is 1.750 is 60° 15′.

If the arc corresponding to a cosine or a cotangent is required, first find the arc corresponding to the same number regarded as a sine or tangent, and take the complement of this arc. Thus

the arc whose cosine is 0.8264 is 34° 16′;
" cotangent is 0.7146 is 54° 27′.

LOGARITHMS.

20. Logarithms are numbers designed to diminish the labor of multiplication and division by substituting in their stead addition and subtraction. All numbers are regarded as powers of some one number, which is called the *base* of the system; and *the* *exponent of the power to which the base must be raised in order to be equal to a given number is called the logarithm of that number.*

The base of the common system of logarithms (called, from their inventor, Briggs's Logarithms) is the number 10. Hence all numbers are to be regarded as powers of 10. Thus, since

$10^0=1$ we have logarithm of 1 $=0$;
$10^1=10$ " 10 $=1$;
$10^2=100$ " 100 $=2$;
$10^3=1000$ " 1000 $=3$, etc.

Whence it appears that in Briggs's system the logarithm of any number between 1 and 10 is some number between 0 and 1; that is, it is a fraction less than unity, and is generally expressed as a decimal. The logarithm of any number between 10 and 100 is some number between 1 and 2; that is, it is equal to 1 plus a decimal. The logarithm of any number between 100 and 1000 is some number between 2 and 3; that is, it is equal to 2 plus a decimal; and so on.

21. The same principle may be extended to *fractions* by means of negative exponents. Thus, since

$10^{-1}=\frac{1}{10}$, or 0.1, -1 is the logarithm of 0.1;
$10^{-2}=\frac{1}{100}$, or 0.01, -2 " 0.01;
$10^{-3}=\frac{1}{1000}$, or 0.001, -3 " 0.001;
$10^{-4}=\frac{1}{10000}$, or 0.0001, -4 " 0.0001, etc.

Hence it appears that the logarithm of any number between 1 and 0.1 is some number between 0 and -1, or may be represented by -1 plus a decimal. The logarithm of any number between 0.1 and 0.01 is some number between -1 and -2, or may be represented by -2 plus a decimal. The logarithm of any number between 0.01 and 0.001 is some number between -2 and -3, or may be represented by -3 plus a decimal, and so on.

22. Hence we see that the logarithms of most numbers must consist of two parts, an integral part and a decimal part. The integral part is called the *characteristic* or *index* of the logarithm. The characteristic may always be determined by the following

RULE.

The characteristic of the logarithm of any number is equal to the number of places by which the first significant figure of that number is removed from the unit's place; and is positive when this figure is to the left of the unit's place, negative when it is to the right, and zero when it is in the unit's place.

Thus the characteristic of the logarithm of 397 is +2, and that of 4673 is +3, while the characteristic of the logarithm of 0.0046 is −3.

23. Since powers of the same quantity are multiplied by adding their exponents, *the logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.* Also, since powers of the same quantity are divided by subtracting their exponents, *the logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by that of the divisor.*

Since the logarithm of 10 is 1, if a number be multiplied or divided by 10, its logarithm will be increased or diminished by 1, the decimal part remaining unchanged. Hence

The decimal part of the logarithm of any number is the same as that of the number multiplied or divided by 10, 100, 1000, etc.

Thus, if we denote the decimal part of the logarithm of 3456 by m, we shall have

logarithm of 3456 $=3+m$;	logarithm of .3456 $=-1+m$;
" 345.6 $=2+m$;	" .03456 $=-2+m$;
" 34.56 $=1+m$;	" .003456 $=-3+m$;
" 3.456 $=0+m$;	" .0003456 $=-4+m$.

Table of Logarithms.

24. A table of logarithms usually contains the logarithms of the entire series of natural numbers from 1 up to 10,000, and the larger tables extend to 100,000 or more. In the smaller tables the logarithms are usually given to five or six decimal places; the larger tables extend to seven, and sometimes eight or more places.

In the accompanying table, the logarithms of the first 100 numbers are given, with their characteristics; but for all other numbers, only the decimal part of the logarithm is given, while the characteristic is left to be supplied according to the rule in Art. 22.

To find the Logarithm of any Number between 1 *and* 100.

25. Look on the first page of the accompanying table, along the column of numbers under N., for the given number, and against it, in the next column, will be found the logarithm, with its characteristic. Thus

opposite 13 is 1.113943, which is the logarithm of 13;
" 65 is 1.812913, " " 65.

To find the Logarithm of any Number consisting of three Figures.

Look on one of the pages of the table from 322 to 342, along the left-hand column, marked N., for the given number, and against it, in the column headed 0, will be found the decimal part of its logarithm. To this the characteristic must be prefixed, according to the rule in Art. 22. Thus

the logarithm of 347, from page 330, will be found, 2.540329;
" " 871, " 340, " 2.940018.

As the first two figures of the decimal are the same for several successive numbers in the table, they are not repeated for each logarithm separately, but are left to be supplied. Thus the decimal part of the logarithm of 339 is .530200. The first two figures of the decimal remain the same up to 347; they are therefore omitted in the table, and are to be supplied.

To find the Logarithm of any Number consisting of four Figures.

Find the three left-hand figures in the column marked N., as before, and the fourth figure at the head of one of the other columns. Opposite to the first three figures, and in the column under the fourth figure, will be found four figures of the logarithm, to which two figures from the column headed 0 are to be prefixed, as in the former case. The characteristic must be supplied according to Art. 22. Thus

the logarithm of 3456 is 3.538574;
" " 8765 is 3.942752.

In several of the columns headed 1, 2, 3, etc., small dots are found in the place of figures. This is to show that the two figures which are to be prefixed from the first column have changed, and they are to be taken from the horizontal line directly *below*. The place of the dots is to be supplied with ciphers. Thus

the logarithm of 2045 is 3.310693;
" " 9777 is 3.990206.

The two leading figures from the column 0 must also be taken from the horizontal line below, if any dots have been passed over on the same horizontal line. Thus

the logarithm of 1628 is 3.211654.

To find the Logarithm of any Number containing more than four Figures.

26. By inspecting the table, we shall find that the differences of the logarithms are nearly proportional to the differences of their corresponding numbers. Thus

the logarithm of 7250 is 3.860338;
" " 7251 is 3.860398;
" " 7252 is 3.860458;
" " 7253 is 3.860518.

Here the difference between the successive logarithms, called *the tabular difference*, is constantly 60, corresponding to a difference of unity in the natural numbers. If, then, we suppose the differences of the logarithms to be proportional to the differences of their corresponding numbers (as they are nearly), a difference of 0.1 in the numbers should correspond to a difference of 6 in the logarithms; a difference of 0.2 in the numbers should correspond to a difference of 12 in the logarithms, etc. Hence

the logarithm of 7250.1 must be 3.860344;
" " 7250.2 " 3.860350;
" " 7250.3 " 3.860356.

In order to facilitate the computation, the tabular difference is inserted on page 338 in the column headed D., and the proportional part for the fifth figure of the natural number is given at the bottom of the page. Thus, when the tabular difference is 60, the corrections for .1, .2, .3, etc., are seen to be 6, 12, 18, etc.

If the given number was 72501, the characteristic of its logarithm would be 4, but the decimal part would be the same as for 7250.1.

If it were required to find the correction for a sixth figure in the natural number, it is readily obtained from the Proportional Parts in the table. The correction for a figure in the sixth place must be one tenth of the correction for the same figure if it stood in the fifth place. Thus, if the correction for .5 is 30, the correction for .05 is obviously 3.

Required the logarithm of 452789.

The logarithm of 452700 is 5.655810.

The tabular difference is 96.

Accordingly, the correction for the fifth figure, 8, is 77, and for the sixth figure, 9, is 8.6, or 9 nearly. Adding these corrections to the number before found, we obtain 5.655896.

The preceding logarithms do not pretend to be perfectly exact, but only the nearest numbers limited to six decimal places. Accordingly, when the fraction which is omitted exceeds half a unit in the sixth decimal place, the last figure must be increased by unity.

Required the logarithm of 8765432.

The logarithm of 8765000 is	6.942752
Correction for the fifth figure, 4,	20
" " sixth figure, 3,	1.5
" " seventh figure 2,	0.1
Therefore the logarithm of 8765432 is	6.942774.

Required the logarithm of 234567.

The logarithm of 234500 is	5.370143
Correction for the fifth figure, 6,	111
" " sixth figure, 7,	13
Therefore the logarithm of 234567 is	5.370267.

To find the Logarithm of a Decimal Fraction.

27. According to Art. 23, the decimal part of the logarithm of any number is the same as that of the number multiplied or divided by 10, 100, 1000, etc. Hence, for a decimal fraction, we find the logarithm as if the figures were integers, and prefix the characteristic according to the rule of Art. 22.

EXAMPLES.

The logarithm of	345.6	is 2.538574;
" "	87.65	is 1.942752;
" "	2.345	is 0.370143;
" "	.1234	is $\bar{1}$.091315;
" "	.005678	is $\bar{3}$.754195.

The minus sign is here placed *over* the characteristic, to show that *that* alone is negative, while the decimal part of the logarithm is positive.

To find the Logarithm of a Vulgar Fraction.

28. We may reduce the vulgar fraction to a decimal, and find its logarithm by the preceding article; or, since the value of a fraction is equal to the quotient of the numerator divided by the denominator, we may, according to Art. 23, *subtract the logarithm of the denominator from that of the numerator;* the difference will be the logarithm of the fraction.

Ex. 1. Find the logarithm of $\frac{3}{16}$, or 0.1875.

From the logarithm of 3,	0.477121,
Take the logarithm of 16,	1.204120.
Leaves the logarithm of $\frac{3}{16}$, or .1875,	$\bar{1}.273001$.

Ex. 2. The logarithm of $\frac{4}{55}$ is $\bar{2}.861697$.

Ex. 3. The logarithm of $\frac{123}{876}$ is $\bar{1}.147401$.

To find the Natural Number corresponding to any Logarithm.

29. Look in the table, in the column headed 0, for the first two figures of the logarithm, neglecting the characteristic; the other four figures are to be looked for in the same column, or in one of the nine following columns; and if they are exactly found, the first three figures of the corresponding number will be found opposite to them in the column headed N., and the fourth figure will be found at the top of the page. This number must be made to correspond with the characteristic of the given logarithm by pointing off decimals or annexing ciphers. Thus
the natural number belonging to the log. 4.370143 is 23450;
" " " " 1.538574 is 34.56.

If the decimal part of the logarithm can not be exactly found in the table, look for the *nearest less* logarithm, and take out the four figures of the corresponding natural number as before; the additional figures may be obtained by means of the Proportional Parts at the bottom of the page.

Required the number belonging to the logarithm 4.368399.

On page 328 we find the next less logarithm .368287.

The four corresponding figures of the natural number are 2335. Their logarithm is less than the one proposed by 112. The tabular difference is 186; and, by referring to the bottom of page 328, we find that, with a difference of 186, the figure corresponding to the proportional part 112 is 6. Hence the five figures of the natural number are 23356; and, since the characteristic of the proposed logarithm is 4, these five figures are all integral.

Required the number belonging to the logarithm 5.345678.

The next less logarithm in the table is	345570.
Their difference is	108.

The first four figures of the natural number are 2216.

With the tabular difference 196, the fifth figure, corresponding to 108, is seen to be 5, with a remainder of 10. To find the sixth figure corresponding to this remainder 10, we may multiply it by

10, making 100, and search for 100 in the same line of Proportional Parts. We see that a difference of 100 would give us 5 in the fifth place of the natural number. Therefore a difference of 10 must give us 5 in the sixth place of the natural number. Hence the required number is 221655.

In the same manner we find

the number corresponding to the log.	3.538672	is	3456.78;
" " "	1.994605	is	98.7654;
" " "	$\bar{1}.647817$	is	.444444.

MULTIPLICATION BY LOGARITHMS.

30. According to Art. 23, the logarithm of the product of two or more factors is equal to the sum of the logarithms of those factors. Hence, for multiplication by logarithms, we have the following

RULE.

Add the logarithms of the factors; the sum will be the logarithm of their product.

Ex. 1. Required the product of 57.98 by 18.

The logarithm of	57.98	is 1.763278
" "	18	is 1.255273
The logarithm of the product	1043.64	is 3.018551.

Ex. 2. Required the product of 397.65 by 43.78.

Ans. 17409.117.

Ex. 3. Required the continued product of 54.32, 6543, and 12.345.

The word *sum* in the preceding rule is to be understood in its algebraic sense; therefore, if any of the characteristics of the logarithms are *negative*, we must take the difference between their sum and that of the positive characteristics, and prefix the sign of the greater. It should be remembered that the decimal part of the logarithm is invariably positive; hence that which is carried from the decimal part to the characteristic must be considered positive.

Ex. 4. Multiply 0.00563 by 17.

The logarithm of	0.00563	is $\bar{3}.750508$
" "	17	is 1.230449
Product, 0.09571, whose logarithm		is $\bar{2}.980957$.

Ex. 5. Multiply 0.3854 by 0.0576.

The logarithm of 0.3854 is $\bar{1}.585912$
" " 0.0576 is $\bar{2}.760422$

Product 0.022199, whose logarithm is $\bar{2}.346334$.

Ex. 6. Multiply 0.007853 by 0.00476. *Ans.* 0.00003738.

Ex. 7. Find the continued product of 11.35, 0.072, and 0.017.

31. The logarithm of a *negative* number is an imaginary quantity. If, therefore, it is required to multiply negative numbers by means of logarithms, we must multiply the equal positive numbers, and give to the product the sign required by the rule of signs in Multiplication. To distinguish the negative sign of a natural number from the negative characteristic of a logarithm, we append the letter n to the logarithm of a negative factor. Thus for -56 we write the logarithm 1.748188 n.

Ex. 8. Multiply 53.46 by -29.47.

The logarithm of 53.46 is 1.728029
For -29.47 we write the logarithm 1.469380 n.
Product, -1575.47, log. 3.197409 n.

Ex. 9. Find the continued product of 372.1, $-.0054$, and -175.6.

Ex. 10. Find the continued product of -0.137, -7.689, and $-.0376$.

DIVISION BY LOGARITHMS.

32. According to Art. 23, the logarithm of the quotient of one number divided by another is equal to the difference of the logarithms of those numbers. Hence, for division by logarithms, we have the following

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.

Ex. 1. Required the quotient of 888.7 divided by 42.24.

The logarithm of 888.7 is 2.948755
" " 42.24 is 1.625724

The quotient is 21.039, whose log. is 1.323031.

Ex. 2. Required the quotient of 3807.6 divided by 13.7.
Ans. 277.927.

The word *difference*, in the preceding rule, is to be understood in its algebraic sense; therefore, if the characteristic of one of the logarithms is negative, or the lower one is greater than the upper, we must change the sign of the subtrahend, and proceed as in addition. If unity is carried from the decimal part, this must be considered as positive, and must be united with the characteristic before its sign is changed.

Ex. 3. Required the quotient of 56.4 divided by 0.00015.

The logarithm of	56.4 is	1.751279
" "	0.00015 is	$\bar{4}$.176091

The quotient is 376000, whose logarithm is 5.575188.

This result may be verified in the same way as subtraction in common arithmetic. The remainder, added to the subtrahend, should be equal to the minuend. This precaution should always be observed when there is any doubt with regard to the sign of the result.

Ex. 4. Required the quotient of .8692 divided by 42.258.

Ex. 5. Required the quotient of .74274 divided by .00928.

The logarithm of	0.74274 is	$\bar{1}$.870837
" "	0.00928 is	$\bar{3}$.967548

The quotient is 80.037, whose logarithm is 1.903289.

Ex. 6. Required the quotient of 24.934 divided by .078541.

If the divisor or dividend, or both, be *negative*, we perform the division by logarithms by using the equal positive numbers, and prefixing to the quotient the sign required by the rule of signs in Algebra.

Ex. 7. Required the quotient of −79.54 divided by 0.08321.

Ex. 8. Required the quotient of −0.4753 divided by −36.74.

INVOLUTION BY LOGARITHMS.

33. It is proved in Algebra, Art. 398, that the logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power. Hence, to involve a number by logarithms, we have the following

RULE.

Multiply the logarithm of the number by the exponent of the power required.

Ex. 1. Required the square of 428.

The logarithm of 428 is 2.631444
2
Square, 183184, log. 5.262888.

Ex. 2. Required the 20th power of 1.06.

The logarithm of 1.06 is 0.025306

$$\frac{\times\ 20}{}$$

20th power, 3.2071, log. 0.506120.

Ex. 3. Required the 5th power of 2.846.

It should be remembered that what is carried from the decimal part of the logarithm is positive, whether the characteristic is positive or negative.

Ex. 4. Required the cube of .07654.

The logarithm of .07654 is $\bar{2}.883888$

3

Cube, .0004484, log. $\bar{4}.651664$.

Ex. 5. Required the fourth power of 0.09874.

Ex. 6. Required the seventh power of 0.8952.

EVOLUTION BY LOGARITHMS.

34. It is proved in Algebra, Art. 399, that the logarithm of any root of a number is equal to the logarithm of that number divided by the index of the root. Hence, to extract the root of a number by logarithms, we have the following

RULE.

Divide the logarithm of the number by the index of the root required.

Ex. 1. Required the cube root of 482.38.

The logarithm of 482.38 is 2.683389.

Dividing by 3, we have 0.894463, which corresponds to 7.842, which is therefore the root required.

Ex. 2. Required the 100th root of 365. *Ans.* 1.0608.

When the characteristic of the logarithm is negative, and is not divisible by the given divisor, we may increase the characteristic by any number which will make it exactly divisible, provided we prefix an equal positive number to the decimal part of the logarithm.

Ex. 3. Required the seventh root of 0.005846.

The logarithm of 0.005846 is $\bar{3}.766859$, which may be written $\bar{7}+4.766859$.

Dividing by 7, we have $\bar{1}.680980$, which is the logarithm of .4797, which is, therefore, the root required.

This result may be verified by multiplying $\bar{1}.680980$ by 7; the result will be found to be $\bar{3}.766860$.

Ex. 4. Required the fifth root of 0.08452.

Ex. 5. Required the tenth root of 0.007815.

PROPORTION BY LOGARITHMS.

35. The fourth term of a proportion is found by multiplying together the second and third terms, and dividing by the first. Hence, to find the fourth term of a proportion by logarithms,

Add the logarithms of the second and third terms, and from their sum subtract the logarithm of the first term.

Ex. 1. Find a fourth proportional to 72.34, 2.519, and 357.48.

Ans. 12.448.

36. When one logarithm is to be subtracted from another, it is sometimes more convenient to convert the subtraction into an addition, which may be done by first subtracting the given logarithm from 10, adding the difference tc the other logarithm, and afterward rejecting the 10.

The difference between a given logarithm and 10 is called its *complement*; and this is easily taken from the table by beginning at the left hand, subtracting each figure from 9, except the last significant figure on the right, which must be subtracted from 10.

To subtract one logarithm from another is the same as to add its complement, and then reject 10 from the result. For $a-b$ is equivalent to $10-b+a-10$.

To work a proportion, then, by logarithms, we must

Add the complement of the logarithm of the first term to the logarithms of the second and third terms.

The characteristic must afterward be diminished by 10.

Ex. 1. Find a fourth proportional to 6853, 489, and 38750.

The complement of the logarithm of	6853 is	6.164119
The logarithm of	489 is	2.689309
" "	38750 is	4.588272
The fourth term is 2765, whose logarithm	is	3.441700.

One advantage of using the complement of the first term in working a proportion by logarithms is, that it enables us to exhibit the operation in a more compact form.

Ex. 2. Find a fourth proportional to 73.84, 658.3, and 4872.

Ans.

Ex. 3. Find a fourth proportional to 5.745, 781.2, and 54.27.

LOGARITHMIC SINES AND TANGENTS.

37. When the natural sines, tangents, etc., are used in proportions, it is necessary to perform the tedious operations of multiplication and division. It is therefore generally preferable to employ the *logarithms* of the sines; and, for convenience, these numbers are arranged in a separate table, called *logarithmic sines*, etc. Thus

the natural sine of 32° 30′ is 0.5373.

Its logarithm, found from page 335, is $\bar{1}.730217$.

The characteristic of the logarithm is *negative*, as must be the case with all the sines, since they are less than unity. To avoid the introduction of negative numbers in the table, we increase the characteristic by 10, making 9.730217, and this is the number found on page 376 for the logarithmic sine of 32° 30′. The radius of the table of logarithmic sines is therefore sometimes regarded as 10,000,000,000, whose logarithm is 10.

The accompanying table contains the logarithmic sines and tangents for every degree and minute of the quadrant.

38. *To find the logarithmic sine, cosine, etc., of a given arc or angle.* If the angle be less than 45°, find the degrees at the top of the page, and the minutes in the left vertical column, marked M.; then, in the column marked *sine* at the top, and opposite to the minutes, will be found the logarithmic sine of the given arc; in the column marked *cosine*, and opposite to the minutes, will be found the cosine of the given arc, etc.

Thus, on page 371, we find

the log. sine		of 27° 38′ is	9.666342;
	cosine	"	9.947401;
	tangent	"	9.718940;
	cotangent	"	10.281060.

If the angle be greater than 45°, find the degrees at the bottom of the page, and the minutes in the vertical column on the right; then, in the column marked sine at the bottom, and opposite to the minutes, will be found the logarithmic sine of the given arc, etc.

It will be seen that the angle found by taking the degrees at the top of the page, and the minutes from the first vertical column on the left, is the complement of the angle found by taking the corresponding minutes upon the same horizontal line from the vertical column on the right, and the degrees at the bottom of the page. Thus, on page 371, having found 27° 38′, follow the horizontal line containing the minutes to the right vertical column, and we find 22′ with 62° at the bottom of the page; and we see that 62° 22′ is the complement of 27° 38′. Now the sine of 27° 38′ is the cosine of 62° 22′; and the cosine of 27° 38′ is the sine of 62° 22′. This fact is indicated in the table, where the column marked sine at the top is marked cosine at the bottom; and the column marked tangent at the top is marked cotangent at the bottom.

On page 379 we find

the log. sine	of 54° 43′	is	9.911853;
cosine	"		9.761642;
tangent	"		10.150210;
cotangent	"		9.849790.

39. If a sine is required for an arc consisting of degrees, minutes, and *seconds*, we must make an allowance for the seconds in the same manner as was directed in the case of logarithms, Art. 26; for within certain limits the differences of the logarithmic sines are proportional to the differences of the corresponding arcs. Thus the log. sine of 24° 15′ is 9.613545;
" 25 16 is 9.613825.

The difference of the log. sines corresponding to one minute of arc, or 60″, is .000280; or 280 if we regard the sixth decimal place as units. The proportional part for 1″ is found by dividing the tabular difference by 60, which in this case gives 4.67; that is, the allowance for 100″ would be 467; and this is the number given on page 368, in the column with the title D. 100″, upon the horizontal line between 15′ and 16′. The correction for any number of seconds will be found by multiplying the proportional part for 1″ by the number of seconds; or *multiplying the corresponding number in the column marked D. by the number of seconds, and rejecting the last two figures of the product.*

Required the log. sine of 32° 45′ 37″.

On page 376 the corresponding number in the column marked D. is 327. Multiplying this by 37, and rejecting the last two figures of the product, we obtain 121, which is the correction for 37″. Adding this to the sine of 32° 45′, we find

the log. sine of 32° 45′ 37″ is 9.733298.

In a similar manner we find the tangent of an arc consisting of degrees, minutes, and seconds; and so also for cosines and cotangents, except that the correction for the seconds is to be *sub-*

tracted instead of *added*, because the cosines decrease while the arcs increase.

The column marked D. between the tangents and cotangents answers for each of these columns, because by Eq. 5, Art. 14, tang. $A \times \text{cot.} A = R^2$; that is, log. tang. A+log. cot. A=20; and it will be observed that the sum of any two numbers on the same horizontal line in these two columns is equal to 20. Hence the difference for 1″ is the same in both columns.

Examples. The log. sine of 37° 24′ 13″ is 9.783493;
log. cosine of 48 32 29 is 9.820910;
the log. tangent of 62° 45′ 31″ is 10.288325;
log. cotangent of 81 17 58 is 9.184781.

40. For arcs not exceeding half a degree, the sine and tangent may be found more conveniently, and in general more accurately, as in the following examples: for in so small an arc the sine and tangent do not differ from the arc by so much as a unit in the sixth decimal place, and hence *the sine of a small arc may be assumed as equal to the sine of* 1″ *multiplied by the number of seconds in the arc.*

Ex. 1. Required the log. sine of 23″.87.

The log. sine of 1″ is	4.685575
log. of 23.87 is	1.377852
The log. sine of 23.″87 is	6.063427.

Ex. 2. Required the log. tangent of 5′ 37.″5.

The log. tangent of 1″ is	4.685575
log. of 337.5 is	2.528274
The log. tangent of 5′ 37″.5 is	7.213849.

For arcs not exceeding 7′ this method will give the log. sine or tangent correct to six decimal places; and for arcs not exceeding one degree, the error is quite small.

41. It is not necessary to extend the tables beyond 90°, because the sine of an angle is equal to that of its supplement, Art. 13. Thus the log. sine of 126° 17′ 24″ is 9.906352;
log. cosine of 132 29 53 is 9.829667;
log. tangent of 158 42 12 is 9.590860;
log. cotangent of 147 51 38 is 10.201862.

42. The secants and cosecants are omitted in this table, since they are easily derived from the sines and cosines. We have found, Art. 14, Eq. 2, secant $= \frac{R^2}{\text{cosine}}$; or, taking the logarithms, we have

$$\text{log. secant} = 2. \text{log. R} - \text{log. cosine};$$

$$\text{log. secant} = 20 - \text{log. cosine.}$$

Also,
$$\text{cosecant} = \frac{R^2}{\text{sine}};$$

$$\text{or log. cosecant} = 20 - \text{log. sine; that is,}$$

The logarithmic secant is found by subtracting the logarithmic cosine from 20; *and the logarithmic cosecant is found by subtracting the logarithmic sine from* 20.

Thus we have found the logarithmic sine of 37° 24′ 13″ to be 9.783493.

Hence the logarithmic cosecant of 37° 24′ 13″ is 10.216507.
The logarithmic cosine of 48° 32′ 29″ is 9.820910.
Hence the logarithmic secant of 48° 32′ 29″ is 10.179090.

43. *To find the arc corresponding to a given logarithmic sine or tangent.*

If the given number is found exactly in the table, then, when the appropriate title is found at the top of the column, the degrees will be found at the top of the page, and the minutes in the vertical column on the left; but if the title is found at the bottom of the column, the degrees will be found at the bottom of the page, and the minutes in the vertical column on the right.

But when the given number is not found exactly in the table, look for the sine or tangent which is next *less* than the one proposed, and take out the corresponding degrees and minutes. Find also the difference between this tabular number and the number proposed; annex two ciphers, and *divide the result by the corresponding number in the column D. The quotient will be the required number of seconds*, to be added to the degrees and minutes before found.

Example. Find the arc whose log. sine is 9.750000.
The next less sine in the table is 9.749987.
The arc corresponding to which is 34° 13′.

The difference between its sine and the one proposed is 13. Annexing two ciphers, and dividing by 309 (the corresponding number in column D.), we obtain 4 nearly. Hence the required arc is 34° 13′ 4″.

In the same manner we find the arc corresponding to log. tangent 10.250000 to be 60° 38′ 57″.

If a cosine or cotangent is required, we must look for the number in the table which is next *greater* than the one proposed, and then proceed as for a sine or tangent. Thus

the arc whose cosine is 9.602000 is $66° 25' 31''$;
" cotangent is 10.300000 is 26 37 10.

44. For arcs not exceeding half a degree, it will be most convenient to reverse the method of Art. 40. For this purpose subtract the log. sine of $1''$ from the given log. sine, and the remainder will be the logarithm of the number of seconds in the arc.

Required the arc whose log. sine is	7.000000
Subtracting the log. sine of $1''$	4.685575
we have	2.314425,

which is the log. of 206.26.

Hence the required arc is $3' 26''.26$.

Required the arc whose log. tangent is	7.500000
Subtracting the log. tangent of $1''$	4.685575
we have	2.814425,

which is the log. of 652.27.

Hence the required arc is $10' 52''.27$.

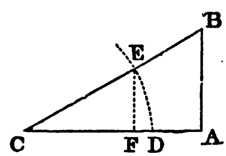
SOLUTION OF RIGHT-ANGLED TRIANGLES.

THEOREM I.

45. *In any right-angled triangle, radius is to the hypothenuse as the sine of either acute angle is to the opposite side, or the cosine of either acute angle to the adjacent side.*

Let the triangle CAB be right-angled at A; then will

R : CB :: sin. C : BA :: cos. C : CA.



From the point C as a centre, with a radius equal to the radius of the tables, describe the arc DE, and on AC let fall the perpendicular EF. Then EF will be the sine, and CF the cosine of the angle C.

Because the triangles CAB, CFE are similar, we have

CE : CB :: EF : BA,

or R : CB :: sin. C : BA.

Also, CE : CB :: CF : CA,

or R : CB :: cos. C : CA.

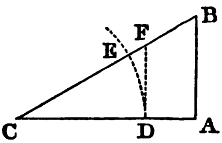
Cor. If radius be taken as unity, we shall have

AB=CB sin. C, and AC=CB cos. C.

Hence, *in any right-angled triangle, either of the sides which contain the right angle is equal to the product of the hypothenuse by the sine of the angle opposite to that side, or by the cosine of the acute angle adjacent to that side.*

THEOREM II.

46. *In any right-angled triangle, radius is to either side as the tangent of the adjacent acute angle is to the opposite side, or the secant of the same angle to the hypothenuse.*



Let the triangle CAB be right-angled at A; then will

R : CA :: tang. C : AB :: sec. C : CB.

From the point C as a centre, with a radius equal to the radius of the tables, describe the arc DE, and from the point D draw DF perpendicular to CA. Then DF will be the tangent, and CF the secant of the angle C.

Because the triangles CAB, CDF are similar, we have

CD : CA :: DF : AB,

or R : CA :: tang. C : AB.

Also, CD : CA :: CF : CB,

or R : CA :: sec. C : CB.

Cor. If radius be taken as unity, we shall have

AB = AC tang. C, and BC = AC sec. C.

Hence, *in any right-angled triangle, either of the sides which contain the right angle is equal to the product of the other side by the tangent of the angle which is opposite to the first side; and the hypothenuse is equal to the product of either side by the secant of the acute angle adjacent to that side.*

47. In every plane triangle there are *six* parts: three sides and three angles. Of these, any three being given, provided one of them is a side, the others may be determined. In a right-angled triangle, one of the six parts, viz., the right angle, is always given; and if one of the acute angles is given, the other is, of course, known. Hence the number of parts to be considered in a right-angled triangle is reduced to *four*, any two of which being given, the others may be found.

It is desirable to have appropriate names by which to designate each of the parts of a triangle. One of the sides adjacent to the right angle being called the base, the other side adjacent to the right angle may be called the perpendicular. The three sides will then be called the hypothenuse, base, and perpendicular. The base and perpendicular are sometimes called the legs of the triangle. Of the two acute angles, that which is adjacent to the base may be called the angle at the base, and the other the angle at the perpendicular.

We may, therefore, have four cases, according as there are given,

1. The hypothenuse and the angles;
2. The hypothenuse and a leg;
3. One leg and the angles; or,
4. The two legs.

All these cases may be solved by the two preceding theorems.

CASE I.

48. *Given the hypothenuse and the angles, to find the base and perpendicular.*

This case is solved by Theorem I.

Radius : *hypothenuse* :: *sine of the angle at the base* : *perpendicular;*

Radius : *hypothenuse* :: *cosine of the angle at the base* : *base.*

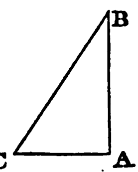
Ex. 1. Given the hypothenuse 275, and the angle at the base 57° 20′, to find the base and perpendicular.

The natural sine of 57° 20′ is .8418.

" cosine " .5398.

Hence 1 : 275 :: .8418 : 231.5 = AB.

1 : 275 :: .5398 : 148.4 = AC.



The computation is here made by natural numbers. If we work the proportion by logarithms, we shall have

	radius,	10.000000
	is to the hypothenuse 275,	2.439333
	as the sine of C 57° 20′,	9.925222
	to the perpendicular 231.50,	2.364555.
Also,	radius,	10.000000
	is to the hypothenuse 275,	2.439333
	as the cosine of C 57° 20′,	9.732193
	to the base 148.43,	2.171526.

Ex. 2. Given the hypothenuse 67.43, and the angle at the perpendicular 38° 43′, to find the base and perpendicular.

Ans. The base is 42.175, and perpendicular 52.612.

The student should work the examples both by natural numbers and by logarithms until he has made himself perfectly familiar with both methods. He may then employ either method, as may appear to him most expeditious.

CASE II.

49. *Given the hypothenuse and one leg, to find the angles and the other leg.*

This case is solved by Theorem I.

Hypothenuse : *radius* :: *base* : *cosine of the angle at the base.*

Radius : *hypothenuse* :: *sine of the angle at the base* : *perpendicular.*

When the perpendicular is given, perpendicular must be substituted for base in this proportion.

Ex. 1. Given the hypothenuse 54.32, and the base 32.11, to find the angles and the perpendicular.

By natural numbers we have

54.32 : 1 :: 32.11 : cos. C.

Also 1 : 54.32 :: sin. C : AB.

By logarithms,

54.32,	1.734960
is to radius,	10.000000
as 32.11,	1.506640
is to cos. 53° 45′ 47″,	9.771680.

That is, the angle C=53° 45′ 47″, and therefore the angle B=36° 14′ 13″.

Also

radius,	10.000000
is to 54.32,	1.734960
as sine 53° 45′ 47″,	9.906647
is to 43.813, the perpendicular	1.641607.

Ex. 2. Given the hypothenuse 332.49, and the perpendicular 98.399, to find the angles and the base.

Ans. The angles are 17° 12′ 51″ and 72° 47′ 9″; the base, 317.6.

CASE III.

50. *Given one leg and the angles, to find the other leg and hypothenuse.*

This case may be solved by Theorem II.

Radius : *base* :: *tangent of the angle at the base* : *the perpendicular.*
:: *secant of the angle at the base* : *hypothenuse.*

When the perpendicular is given, perpendicular must be substituted for base in this proportion.

This case may also be solved by Theorem I.

sin. B : base :: sin. C : perpendicular;
:: radius : hypothenuse.

Ex. 1. Given the base 222, and the angle at the base 25° 15′, to find the perpendicular and hypothenuse.

By natural numbers we have

1 : 222 :: tang. 25° 15′ : perpendicular.

Also sin. 64° 45′ : 222 :: radius : hypothenuse.

By logarithims,

	radius,	10.000000
	is to 222,	2.346353
	as tang. 25° 15′,	9.673602
	is to 104.70, the perpendicular,	2.019955.
Also	sin. 64° 45′,	9.956387
	is to 222,	2.346353
	as radius,	10.000000.
	is to 245.45, the hypothenuse,	2.389966.

Ex. 2. Given the perpendicular 125, and the angle at the perpendicular 51° 19′, to find the hypothenuse and base.

Ans. Hypothenuse, 199.99; base, 156.12.

CASE IV.

51. *Given the two legs, to find the angles and hypothenuse.*

This case is solved by Theorem II.

Base : *radius* :: *perpendicular* : *tangent of the angle at the base.*

Radius : *base* :: *secant of the angle at the base* : *hypothenuse.*

When the angles have been found, the hypothenuse may be found by Theorem I.

sin. C : AB :: radius : BC.

Ex. 1. Given the base 123, and perpendicular 765, to find the angles and hypothenuse.

By natural numbers we have

123 : 1 :: 765 : tang. C;

sin. C : 765 :: 1 : hypothenuse.

By logarithms,

	123,	2.089905
	is to radius,	10.000000
	as 765,	2.883661
	is to tang. 80° 51′ 57″,	0.793756.
Also	sin. 80° 51′ 57″,	9.994458
	is to 765,	2.883661
	as radius,	10.000000
	is to 774.82, hypothenuse,	2.889203.

Ex. 2. Given the base 53, and perpendicular 67, to find the angles and hypothenuse.

Ans. The angles are $51° 39' 16''$, and $38° 20' 44''$; hypothenuse, 85.428.

Examples for Practice.

1. Given the base 777, and perpendicular 345, to find the hypothenuse and angles.

This example, it will be seen, falls under Case IV.

2. Given the hypothenuse 324, and the angle at the base $48° 17'$, to find the base and perpendicular.

3. Given the perpendicular 543, and the angle at the base $72° 45'$, to find the hypothenuse and base.

4. Given the hypothenuse 666, and base 432, to find the angles and perpendicular.

5. Given the base 634, and the angle at the base $53° 27'$, to find the hypothenuse and perpendicular.

6. Given the hypothenuse 1234, and perpendicular 555, to find the base and angles.

7. Suppose the radius of the earth to be 3963 miles, and that it subtends an angle of $57' 2''.3$ at the moon, what is the distance of the moon from the earth?

8. Suppose that when the moon's distance from the earth is 238,885 miles, its apparent semi-diameter is $15' 33''.5$, what is its diameter in miles?

9. Suppose the radius of the earth to be 3963 miles, and that it subtends an angle of $8''.9$ at the sun, what is the distance of the sun from the earth?

10. Suppose that the sun's distance from the earth is 92,000,000 miles, and that its apparent semi-diameter is $16' 1''.8$, what is its diameter in miles?

52. When two sides of a right-angled triangle are given, the third may be found by means of the property that the square of the hypothenuse is equal to the sum of the squares of the other two sides.

Hence, representing the hypothenuse, base, and perpendicular by the initial letters of these words, we have

$$h=\sqrt{b^2+p^2};\ b=\sqrt{h^2-p^2};\ p=\sqrt{h^2-b^2}.$$

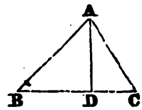
Ex. 1. If the base is 2720, and the perpendicular 3104, what is the hypothenuse? *Ans.* 4127.1.

Ex. 2. If the hypothenuse is 514, and the perpendicular 432, what is the base?

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

THEOREM I.

53. *In any plane triangle, the sines of the angles are proportional to the opposite sides.*



Let ABC be any triangle, and from one of its angles, as A, let AD be drawn perpendicular to the opposite side BC. There may be two cases.

First. If the perpendicular falls within the triangle, because the triangle ABD is right-angled at D, we have

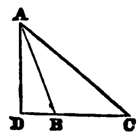
R : sin. B :: AB : AD; whence $R \times AD = \sin. B \times AB$.

For a similar reason,

R : sin. C :: AC : AD; whence $R \times AD = \sin. C \times AC$.

Therefore $\sin. B \times AB = \sin. C \times AC$;

or, sin. B : sin. C :: AC : AB.



Second. If the perpendicular falls without the triangle, we have in the triangle ABD, as before,

R : sin. ABD :: AB : AD.

Also, in the triangle ACD,

R : sin. C :: AC : AD;

whence sin. ABD : sin. C :: AC : AB.

But, since ABD is the supplement of ABC, their sines are equal, Art. 13.

Therefore sin. ABC : sin. C :: AC : AB.

THEOREM II.

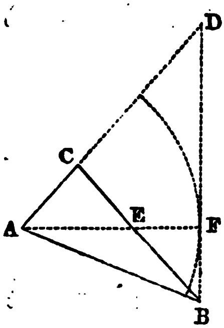
54. *In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Let ABC be any triangle; then will

$$CB+CA : CB-CA :: \text{tang.} \frac{A+B}{2} : \text{tang.} \frac{A-B}{2}.$$

Produce AC to D, making CD equal to CB, and join DB. Take CE equal to CA; draw AE, and produce it to F. Then AD is the *sum* of CB and CA, and BE is their *difference.*

The sum of the two angles CAE, CEA is equal to the sum of CAB, CBA, each being the supplement of ACB (Geom., B. I., Pr. 27). But, since CA is equal to CE, the angle CAE is equal to the angle CEA; therefore CAE is the *half sum* of the angles CAB,



CBA. Also, if from the greater of the two angles CAB, CBA there be taken their half sum, the remainder, FAB, will be their *half difference* (Algebra, p. 89).

Since CD is equal to CB, the angle ADF is equal to the angle EBF; also, the angle CAE is equal to AEC, which is equal to the vertical angle BEF. Therefore the two triangles DAF, BEF are mutually equiangular; hence the two angles at F are equal, and AF is perpendicular to DB.

If, then, AF be made radius, DF will be the tangent of DAF, and BF will be the tangent of BAF. But, by similar triangles, we have

$$AD : BE :: DF : BF; \text{ that is,}$$

$$CB + CA : CB \dot{-} CA :: \text{tang.} \frac{A+B}{2} : \text{tang.} \frac{A \dot{-} B}{2}.$$

THEOREM III.

55. *If from any angle of a triangle a perpendicular be drawn to the opposite side or base, the sum of the segments of the base is to the sum of the two other sides as the difference of those sides is to the difference of the segments of the base.*

For demonstration, see Geometry, B. IV., Pr. 34, Cor.

56. In every plane triangle three parts must be given to enable us to determine the others, and of the given parts one at least must be a side. For, if the angles only are given, these might belong to an infinite number of different triangles. In solving oblique-angled triangles four different cases may therefore be presented. There may be given,

1. Two angles and a side;
2. Two sides and an angle opposite one of them;
3. Two sides and the included angle; or,
4. The three sides.

We shall represent the three angles of the proposed triangle by A, B, C, and the sides opposite them respectively by a, b, c.

CASE I.

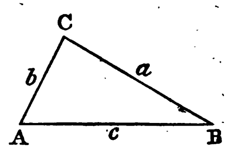
57. *Given two angles and a side, to find the third angle and the other two sides.*

To find the third angle, add the given angles together, and subtract their sum from 180°.

The required sides may be found by Theorem I. The proportion will be,

The sine of the angle opposite the given side : the given side :: the sine of the angle opposite the required side : the required side.

Ex. 1. In the triangle ABC, there are given the angle A, 57° 15′, the angle B, 35° 30′, and the side c, 364, to find the other parts.



The sum of the given angles, subtracted from 180°, leaves 87° 15′ for the angle C. Then, to find the side a, we say,

sin. C : c :: sin. A : a.

By natural numbers,

.9988 : 364 :: .8410 : 306.49 = a.

This proportion is most easily worked by logarithms, thus:

As the sine of the angle C,	87° 15′, comp.	0.000500
Is to the side c,	364,	2.561101
So is the sine of the angle A,	57° 15′,	9.924816
To the side a,	306.49,	2.486417.

To find the side b, we have, sin. C : c :: sin. B : b.

By natural numbers,

9988 : 364 :: .5807 : 211.62 = b.

The work by logarithms is as follows:

sin. C,	87° 15′,	comp.	0.000500
: c,	364,		2.561101
:: sin. B,	35° 30′,		9.763954
: b,	211.62,		2.325555.

Ex. 2. In the triangle ABC, there are given the angle A, 49° 25′, the angle C, 63° 48′, and the side c, 275, to find the other parts. *Ans.* B = 66° 47′; a = 232.766; b = 281.67.

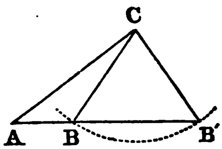
CASE II.

58. *Given two sides and an angle opposite one of them, to find the third side and the remaining angles.*

One of the required angles is found by Theorem I. The proportion is,

The side opposite the given angle : the sine of that angle :: the other given side : the sine of the opposite angle.

The third angle is found by subtracting the sum of the other two from 180°; and the third side is found as in Case I.



If the side BC, opposite the given angle A, is shorter than the other given side AC, the solution will be *ambiguous*; that is, two different triangles ABC, AB′C may be formed, each of which will satisfy the conditions of the problem.

The numerical result is also ambiguous, for the fourth term of the first proportion is a sine of an angle. But this may be the sine either of the *acute* angle AB′C, or of its supplement, the obtuse angle ABC (Art. 13). In practice, however, there will generally be some circumstance to determine whether the required angle is acute or obtuse. If the side opposite the given angle is longer than the other given side, there can be no ambiguity, for B will fall on B′A produced, and the triangle ABC will no longer be one solution of the problem. This is always the case when the given angle is obtuse.

Ex. 1. In a triangle ABC, there are given AC, 458, BC, 307, and the angle A, 28° 45′, to find the other parts.

To find the angle B;

BC : sin. A :: AC : sin. B.

By natural numbers,

307 : .4810 :: 458 : .7176, sin. B, the arc corresponding to which is 45° 51′, or 134° 9′.

This proportion is most easily worked by logarithms, thus:

BC,	307,	comp.	7.512862
: sin. A,	28° 45′,		9.682135
:: AC,	458,		2.660865
: sin. B,	45° 51′ 14″, or 134° 8′ 46″,		9.855862.

The angle ABC is 134° 8′ 46,″ and the angle AB′C, 45° 51′ 14″. Hence the angle ACB is 17° 6′ 14″, and the angle ACB′, 105° 23′ 46″.

To find the side AB;

sin. A : CB :: sin. ACB. AB.

By logarithms,

sin. A,	28° 45′,	comp.	0.317865
: CB,	307,		2.487138
:: sin. ACB,	17° 6′ 14″,		9.468502
: AB,	187.72,		2.273505.

To find the side AB′;

sin. A : CB′ :: sin. ACB′ : AB′.

By logarithms,

sin. A,	28° 45′,	comp. 0.317865
: CB′,	307,	2.487138
:: sin. ACB′,	105° 23′ 46″,	9.984128
: AB′,	615.36,	2.789131

Ex. 2. In a triangle ABC, there are given AB, 532, BC, 358, and the angle C, 107° 40′, to find the other parts.

Ans. A=39° 52′ 52″; B=32° 27′ 8″; AC=299.6.

In this example there is no ambiguity, because the given angle is obtuse.

CASE III.

59. *Given two sides and the included angle, to find the third side and the remaining angles.*

The *sum* of the required angles is found by subtracting the given angle from 180°. The *difference* of the required angles is then found by Theorem II. Half the difference added to half the sum gives the greater angle, and, subtracted, gives the less angle. The third side is then found by Theorem I.

Ex. 1. In the triangle ABC, the angle A is given 53° 8′; the side c, 420, and the side b, 535, to find the remaining parts.

The sum of the angles B+C=180°−53° 8′=126° 52′. Half their sum is 63° 26′.

Then, by Theorem II.,

535+420 : 535−420 :: tang. 63° 26′ : tang. 13° 32′ 25″,

which is half the difference of the two required angles.

Hence the angle B is 76° 58′ 25″, and the angle C, 49° 53′ 35″.

To find the side a;

sin. C : c :: sin. A : a=439.32.

Ex. 2. Given the side c, 176, a, 133, and the included angle B, 73°, to find the remaining parts.

Ans. b=187.022, the angle C, 64° 9′ 3″, and A, 42° 50′ 57″.

CASE IV.

60. *Given the three sides, to find the angles.*

Let fall a perpendicular upon the longest side from the opposite angle, dividing the given triangle into two right-angled triangles. The two segments of the base may be found by Theorem III. There will then be given the hypothenuse and one side of a right-angled triangle to find the angles.

Ex. 1. In the triangle ABC, the side a is 261, the side b, 345, and c, 395. What are the angles?

Let fall the perpendicular CD upon AB.

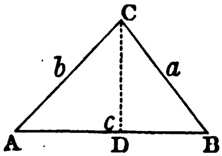
Then, by Theorem III.,

$$AB : AC+CB :: AC-CB : AD-DB;$$

or

$$395 : 606 :: 84 : 128.87.$$

Half the difference of the segments added to half their sum gives the greater segment, and subtracted gives the less segment.



Therefore AD is 261.935, and BD, 133.065.

Then, in each of the right-angled triangles ACD, BCD we have given the hypothenuse and base, to find the angles by Case II. of right-angled triangles. Hence

$$AC : R :: AD : \cos. A = 40° 36' 13'';$$

$$BC : R :: BD : \cos. B = 59° 20' 52''.$$

Therefore the angle $C = 80° 2' 55''$.

Ex. 2. If the three sides of a triangle are 150, 140, and 130, what are the angles?

Ans. 67° 22′ 48″, 59° 29′ 23″, and 53° 7′ 49″.

Examples for Practice.

1. Given two sides of a triangle, 478 and 567, and the included angle, 47° 30′, to find the remaining parts.

2. Given the angle A, 56° 34′, the opposite side, a, 735, and the side b, 576, to find the remaining parts.

3. Given the angle A, 65° 40′, the angle B, 74° 20′, and the side a, 275, to find the remaining parts.

4. Given the three sides, 742, 657, and 379, to find the angles.

5. Given the angle A, 116° 32′, the opposite side, a, 492, and the side c, 295, to find the remaining parts.

6. Given the angle C, 56° 18′, the opposite side, c, 184, and the side b, 219, to find the remaining parts.

This problem admits of two answers.

7. Given the angle B, 68° 35′ 27″, the angle C, 44° 48′ 47″, and the side c, 479, to find A, a, and b.

8. Given the angle A, 67° 23′ 56″, the side a, 1486.73, and the side b, 2073.22, to find B, C, and c.

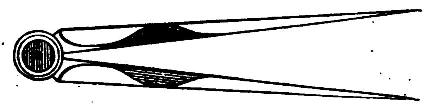
9. Given the angle C, 66° 3′ 27″, the side a, 897, and the side b, 571, to find A, B, and c.

10. Given $a=2251$, $b=738$, and $c=830$, to find A, B, and C.

INSTRUMENTS USED IN DRAWING.

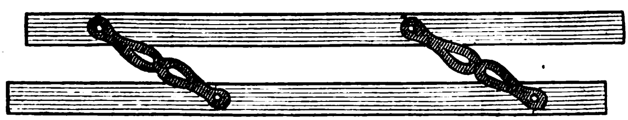
61. The following are some of the most important instruments used in drawing.

I. The *dividers* consist of two legs, revolving upon a pivot at one extremity. The joints should be composed of two different metals, of unequal hardness: one part, for example, of steel, and the other of brass or silver, in order that they may move upon each other with greater freedom. The points should be of



tempered steel, and, when the dividers are closed, they should meet with great exactness. The dividers are often furnished with various appendages, which are exceedingly convenient in drawing. Sometimes one of the legs is furnished with an adjusting screw, by which a slow motion may be given to one of the points, in which case they are called *hair compasses.* It is also useful to have a movable leg, which may be removed at pleasure, and other parts fitted to its place; as, for example, a long beam for drawing large circles, a pencil-point for drawing circles with a pencil, an ink-point for drawing black circles, etc.

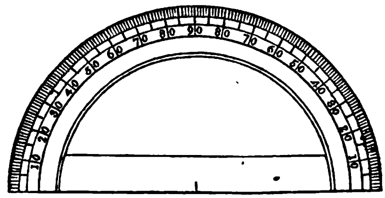
62. II. The *parallel rule* consists of two flat rules, made of wood or ivory, and connected together by two cross-bars of equal length, and parallel to each other. This instrument is useful for drawing a line parallel to a given line, through a given point.



For this purpose, place the edge of one of the flat rules against the given line, and move the other rule until its edge coincides with the given point. A line drawn along its edge will be parallel to the given line.

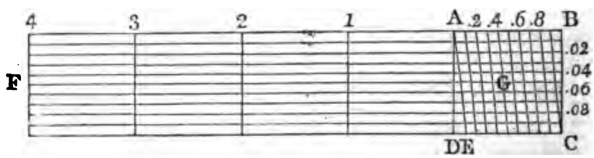
63. III. The *protractor* is used to lay down or to measure angles. It consists of a semicircle, usually of brass, and is divided into degrees, and sometimes smaller portions, the centre of the circle being indicated by a small notch.

To lay down an angle with the protractor, draw a base line,



and apply to it the edge of the protractor, so that its centre shall fall at the angular point. Count the degrees contained in the proposed angle on the limb of the circle, and mark the extremity of the arc with a fine dot. Remove the instrument, and through the dot draw a line to the angular point; it will give the angle required. In a similar manner, the inclination of any two lines may be measured with the protractor.

64. IV. The *plane scale* is a ruler, frequently two feet in length, containing a line of *equal parts*, *chords*, *sines*, *tangents*, etc. For a scale of equal parts, a line is divided into inches and tenths of an inch, or half inches and twentieths. When smaller fractions are required, they are obtained by means of the *diagonal scale*, which is constructed in the following manner. Describe a square inch, ABCD, and divide each of its sides into ten equal parts.



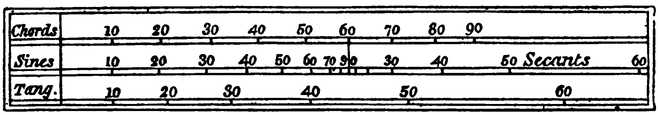
Draw diagonal lines from the first point of division on the upper line to the second on the lower; from the second on the upper line to the third on the lower, and so on. Draw, also, other lines parallel to AB, through the points of division of BC. Then, in the triangle ADE, the base, DE, is one tenth of an inch; and, since the line AD is divided into ten equal parts, and through the points of division lines are drawn parallel to the base, forming nine smaller triangles, the base of the least is one tenth of DE, that is, .01 of an inch; the base of the second is .02 of an inch; the third, .03, and so on. Thus the diagonal scale furnishes us *hundredths* of an inch.

To take off from the scale a line of given length, as, for example, 4.45 inches, place one foot of the dividers at F, on the sixth horizontal line, and extend the other foot to G, the fifth diagonal line.

A half inch or less is frequently subdivided in the same manner

65. A *line of chords*, commonly marked CHO., is found on most plane scales, and is useful in setting off angles. To form this line, describe a circle with any convenient radius, and divide the circumference into degrees. Let the length of the chords for every degree of the quadrant be determined and laid off on a scale: this is called a line of chords.

Since the chord of 60° is equal to radius, in order to lay down

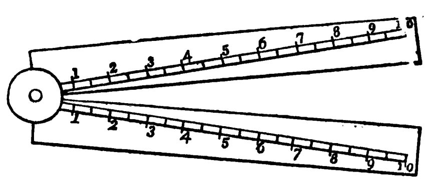


an angle, we take from the scale the chord of 60°, and with this radius describe an arc of a circle. Then take from the scale the chord of the given angle, and set it off upon the former arc. Through these two points of division draw lines to the centre of the circle, and they will contain the required angle.

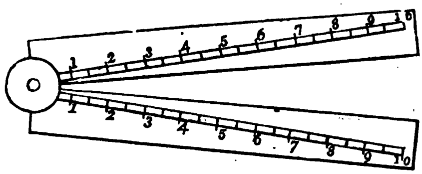
The line of sines, commonly marked SIN., exhibits the lengths of the sines to every degree of the quadrant, to the same radius as the line of chords. The line of tangents and the line of secants are constructed in the same manner. Since the sine of 90° is equal to radius, and the secant of 0° is the same, the graduation on the line of secants begins where the line of sines ends.

On the back side of the plane scale are often found lines representing the logarithms of numbers, sines, tangents, etc. This is called Gunter's Scale.

66. V. The *Sector* is a very convenient instrument in drawing. It is generally made of ivory or brass, and consists of two equal arms, movable about a pivot as a centre, having several scales drawn on the faces, some single, others double. The



single scales are like those upon a common Gunter's Scale. The double scales are those which proceed from the centre, each being laid twice on the same face of the instrument, viz., once on each leg. The double scales are a scale of lines marked Lin., or L.; the scale of chords, sines, etc. On each arm of the sector there is a diagonal line, which diverges from the central point like the radius of a circle, and these diagonal lines are divided into equal parts.



The advantage of the sector is to enable us to draw a line upon paper to any scale; as, for example, a scale of 6 feet to the inch. For this purpose, take an inch with the dividers from the scale of inches; then, placing one foot of the dividers at 6 on one arm of the sector, open the sector until the other foot reaches to the same number on the other arm. Now, regarding the lines on the sector as the sides of a triangle, of which the line measured from 6 on one arm to 6 on the other arm is the base, it is plain that if any other line be measured across the angle of the sector, the bases of the triangles thus formed will be proportional to their sides. Therefore a line of 7 feet will be represented by the distance from 7 to 7, and similarly for other lines.

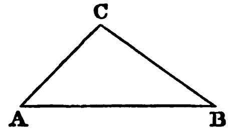
The sector also contains a line of *chords*, arranged like the line of equal parts already mentioned. Two lines of chords are drawn, one on each arm of the sector, diverging from the central point. This double line of chords is more convenient than the single one upon the plane scale, because it furnishes chords to *any radius.* If it be required to lay down any angle, as, for example, an angle of 25°, describe a circle with any convenient radius. Open the sector so that the distance from 60 to 60, on the line of chords, shall be equal to this radius. Then, preserving the same opening of the sector, place one foot of the dividers upon the division 25 on one scale, and extend the other foot to the same number upon the other scale: this distance will be the chord of 25 degrees, which must be set off upon the circle first described.

The lines of sines, tangents, etc., are arranged in the same manner.

67. By means of the instruments now enumerated, all the cases in Plane Trigonometry may be solved mechanically, without the use of tables, and without any arithmetical process. The sides and angles which are *given* are laid down according to the preceding directions, and the *required* parts are then measured from the same scale. The student will do well to exercise himself upon the following problems:

I. *Given the angles and one side of a triangle,* to find, by construction, the other two sides.

Draw an indefinite straight line, and from the scale of equal parts lay off a portion, AB, equal to the given side. From each extremity lay off an angle equal to one of the adjacent angles by means of a protractor or a scale of chords. Extend the two lines till they intersect, and measure their lengths upon the same scale of equal parts which was used in laying off the base.



Ex. 1. Given the angle A, 45° 30′, the angle B, 35° 20′, and the side AB, 432 rods, to construct the triangle, and find the lengths of the sides AC and BC.

The triangle ABC may be constructed of any dimensions whatever; all which is essential is that its angles be made equal to the given angles. We may construct the triangle upon a scale of 100 rods to an inch, in which case the side AB will be represented by 4.32 inches; or we may construct it upon a scale of 200 rods to an inch; that is, 100 rods to a half inch, which is very conveniently done from a scale on which a half inch is divided like that described in Art. 64; or we may use any other scale at pleasure. It should, however, be remembered, that the required sides must be measured upon the *same* scale as the given sides.

Ex. 2. Given the angle A, 48°, the angle C, 113°, and the side AC, 795, to construct the triangle.

II. *Given two sides of a triangle and an angle opposite one of them*, to find the other two parts.

Draw the side which is adjacent to the given angle. From one end of it lay off the given angle, and extend a line indefinitely for the required side. From the other extremity of the first side, with the remaining given side for radius, describe an arc cutting the indefinite line. The point of intersection will determine the third angle of the triangle. The side and angles required may then be measured.

Ex. 1. Given the angle A, 74° 45′, the side AC, 432, and the side BC, 475, to construct the triangle, and find the other parts.

Ex. 2. Given the angle A, 105°, the side BC, 498, and the side AC, 375, to construct the triangle.

III. *Given two sides of a triangle and the included angle*, to find the other parts.

Draw one of the given sides. From one end of it lay off the given angle, and draw the other given side, making the required angle with the first side. Then connect the extremities of the two sides, and there will be formed the triangle required. The side and angles required may then be measured.

Ex. 1. Given the angle A, 37° 25′, the side AC, 675, and the side AB, 417, to construct the triangle, and find the other parts.

Ex. 2. Given the angle A, 75°, the side AC, 543, and the side AB, 721, to construct the triangle.

IV. *Given the three sides of a triangle*, to find the angles.

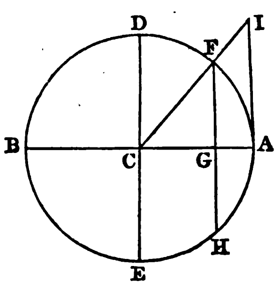
Draw one of the sides as a base; and from one extremity of the base, with a radius equal to the second side, describe an arc of a circle. From the other end of the base, with a radius equal to the third side, describe a second arc intersecting the former; the point of intersection will be the third angle of the triangle.

Ex. 1. Given AB, 678, AC, 598, and BC, 435, to find the angles.

Ex. 2. Given the three sides 476, 287, and 354, to find the angles.

Sines, Tangents, etc., of Arcs of any Magnitude.

68. In a plane triangle each angle is less than 180°, and the sines, tangents, etc., of the angles of such a triangle are the sines, etc., of angles less than 180°, or of arcs less than a semi-circumference. Frequently, however, especially in Astronomy, we have occasion to consider arcs greater than a semi-circumference, or even than an entire circumference. Thus the moon, in its motion about the earth, describes an entire revolution in less than 30 days, and in the course of a year completes more than twelve revolutions; that is, its apparent angular motion through the heavens exceeds 4000 degrees.



Suppose the line CF, starting from the position CA, to revolve about the point C, in the direction of the arc AFD; when it arrives at CD it will have described an angular magnitude of 90°; when it arrives at CB it will have described an angular magnitude of 180°; at CE, 270°; and at CA again, 360°. If it continue its revolution, when it arrives again at CD, it will have described an angular magnitude of 450°; and thus we may have an angular magnitude of any number of degrees, and we may have arcs equal to or greater than one, two, or more circumferences.

69. For convenience, we draw two diameters, AB, DE, at right angles to each other, and suppose one of them to occupy a horizontal position, and the other a vertical position. Then ACD is called the *first* quadrant, DCB the *second* quadrant, BCE the *third* quadrant, and ECA the *fourth* quadrant; that is, the first quadrant is above the horizontal diameter and on the right of the vertical diameter; the second quadrant is above the horizontal diameter and on the left of the vertical; and so on. We propose now to consider the values of the sines, tangents, etc., for arcs of any magnitude.

70. *Sines, etc., of* 0° *and* 90°. When the line CF coincides with CA, that is, when the arc AF is zero, the sine is zero, and the cosine is equal to the radius of the circle. As the point F advances toward D, the sine increases and the cosine decreases; when F arrives at D, the sine is equal to the radius, and the cosine becomes zero.

The tangent begins with zero at A, and increases with the arc. As the point F approaches D, the tangent increases very rapidly; and when the difference between the arc and 90° is less than any assignable quantity, the tangent is greater than any assignable quantity. Hence the tangent of 90° is said to be *infinite.*

Since the cotangent of an arc is equal to the tangent of its complement, the cotangent is infinite at A, and zero at D.

The secant begins with radius at A, increases through the first quadrant, and becomes infinite at D. The cosecant is infinite at A, and equal to radius at D. Hence we have

$\sin. 0° = \cos. 90° = 0$;	$\cot. 0° = \text{tang}. 90° = \infty$;
$\cos. 0° = \sin. 90° = 1$;	$\sec. 0° = \text{cosec}. 90° = 1$;
$\text{tang}. 0° = \cot. 90° = 0$;	$\text{cosec}. 0° = \sec. 90° = \infty$.

71. *Sine, etc., of* 180°. As the point F advances from D toward B, the sine diminishes and becomes zero at B; that is, the sine of 180° is zero. During the motion through the second quadrant the cosine increases, and becomes equal to radius at B.

In the motion through the second quadrant the tangent is at first infinitely great, being drawn from A downward to meet the secant, and it rapidly diminishes till at B it is reduced to zero. The secant also diminishes in the second quadrant, till at B it becomes CA, or radius. Hence we have

$\sin. 180° = \text{tang}. 180° = 0$;	$\cot. 180° = \text{cosec}. 180° = \infty$.
$\cos. 180° = \sec. 180° = 1$;	

72. *Sine, etc., of* 270°, 360°, *etc.* During the motion through the third quadrant the sine again increases, and becomes equal to radius at E; the tangent and secant, which are now AI and CI, also increase, and become infinite at E.

When the line FC, in its motion about C, has revolved through 360°, it comes again into coincidence with AC. Hence the sine, tangent, etc., of 360° are the same as those of 0°.

The same reasoning shows that the sine, tangent, etc., of 450° are the same as those of 90°; the sine of 540° is the same as that of 180°, etc.

If C represent an entire circumference, or 360°, and A any arc whatever, we shall have

$$\sin. A = \sin.(C+A) = \sin.(2\,C+A) = \sin.(3\,C+A), \text{ etc.}$$

The same is true of the cosine, tangent, etc.; that is, *the sine, tangent, etc., of an arc which exceeds* 360°, *is the same as those of the excess above* 360°, *and so also for any multiple of* 360°. In fact, since the sine is drawn from one *end* of an arc perpendicular to a diameter through the other *end*, two arcs that have the same extremities must have the same sine; and so of the tangent, etc.

Values of the Sines, Cosines, etc., of certain Arcs or Angles.

73. *Sine, etc., of* 30° *and* 60°. By Art. 8, the sine of 30° is equal to half radius; and if we call radius unity, we have

$$\sin. 30° = \cos. 60° = \tfrac{1}{2}.$$

Also, since $\cos. A = \sqrt{R^2 - \sin.^2 A}$, Art. 15, we have

$$\sin. 60° = \cos. 30° = \sqrt{1-\tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3}.$$

Since $\text{tang. } A = \dfrac{\sin. A}{\cos. A}$, Art. 15, we have

$$\text{tang. } 30° = \cot. 60° = \frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}.$$

Since $\cot. A = \dfrac{R^2}{\text{tang. } A}$, Art. 14, we have $\cot. 30° = \text{tang. } 60° = \sqrt{3}$.

Since $\sec. A = \dfrac{R^2}{\cos. A}$, we have $\sec. 30° = \text{cosec. } 60° = \dfrac{2}{\sqrt{3}} = \tfrac{2}{3}\sqrt{3}$.

Since $\text{cosec. } A = \dfrac{R^2}{\sin. A}$, Art. 14, we have $\text{cosec. } 30° = \sec. 60° = 2$.

74. *Sine, etc., of* 45°. Since sin. 45° = cos. 45°; and $\sin.^2 A + \cos.^2 A = R^2$, Art. 15, we have

$$\sin.^2 45° + \sin.^2 45° = 1. \quad \text{Hence } \sin.^2 45° = \tfrac{1}{2},$$

and

$$\sin. 45° = \cos. 45° = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2}.$$

Also, $\text{tang. } 45° = \text{cot. } 45° = \frac{\text{sin. } 45°}{\text{cos. } 45°} = 1,$

and $\text{sec. } 45° = \text{cosec. } 45° = \frac{1}{\text{sin. } 45°} = \sqrt{2}.$

75. *Algebraic signs of the trigonometrical functions.* If we attribute proper algebraic signs to the trigonometrical functions, the formulæ which have been demonstrated for arcs less than 180° will apply also to arcs greater than 180°. For this purpose we adopt the general principle that *lines measured in opposite directions from a fixed line must have opposite signs.* It is also convenient to assume that in the first quadrant the sines and cosines are both positive.

76. In the first and second quadrants the sines are measured *upward* from the horizontal diameter AB, while in the third and fourth quadrants they are measured *downward.* Hence, regarding the sines as positive in the first quadrant, they will also be positive in the second quadrant, but negative in the third and fourth.

In the first and fourth quadrants the cosine extends to the *right* from the vertical diameter DE, but in the second and third quadrants to the *left.* Hence the cosines are positive in the first and fourth quadrants, but negative in the second and third.

77. The signs of the tangents are derived from those of the sines and cosines. For $\text{tang.} = \frac{\text{R. sin.}}{\text{cos.}}$ (Art. 14). Hence, when the sine and cosine have like algebraic signs, the tangent will be positive; when unlike, negative. That is, the tangent is positive in the first and third quadrants, and negative in the second and fourth.

Also, $\text{cotangent} = \frac{R^2}{\text{tang.}}$ (Art. 14); hence the tangent and cotangent have always the same sign.

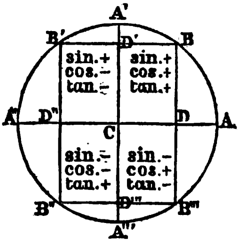
We have seen that $\text{sec.} = \frac{R^2}{\text{cos.}}$; hence the secant must have the same sign as the cosine.

Also, $\text{cosec.} = \frac{R^2}{\text{sin.}}$; hence the cosecant must have the same sign as the sine.

The same results are obtained from the figure; for the tangent is drawn from A *upward* for an arc ending in the first or third quadrant, and *downward* for one ending in the second or fourth.

The cotangent is drawn from A′ to the right for an arc ending in the first or third quadrant, and to the left for the second and fourth.

The secant is positive when drawn from the centre *through* the end of the arc; that is, for an arc ending in the first or fourth quadrant; and negative when drawn from the centre *away from* the end of the arc; that is, for the second or third quadrant. So also for the cosecant.

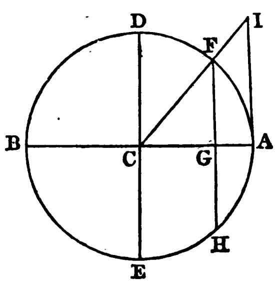


The accompanying figure may assist the student to retain in memory the algebraic signs of the different trigonometrical lines.

78. The preceding results are exhibited in the following tables, which should be made perfectly familiar:

	First quad.	*Second* quad.	*Third* quad.	*Fourth* quad.
Sine and cosecant,	+	+	−	−
Cosine and secant,	+	−	−	+
Tangent and cotangent,	+	−	+	−

	0°	90°	180°	270°	360°
Sine,	0	+R	0	−R	0
Cosine,	+R	0	−R	0	+R
Tangent,	0	∞	0	∞	0
Cotangent,	∞	0	∞	0	∞
Secant,	+R	∞	−R	∞	+R
Cosecant,	∞	+R	∞	−R	∞



79. *Negative arcs.* We generally consider those arcs as positive which are estimated from A in the direction ADBE. If, then, an arc were estimated in the direction AEBD, it should be considered as negative; that is, if the arc AF be considered positive, AH must be considered negative.

Now, wherever a plus arc may end, the equal minus arc will end upon the opposite side of the horizontal diameter AB, and in the same vertical line. The sines will evidently be equal, but one will be plus, and the other minus. Thus sin. AH = −sin. AF, and sin. AEF = −sin. ADH.

and universally sin. (−A) = −sin. A.

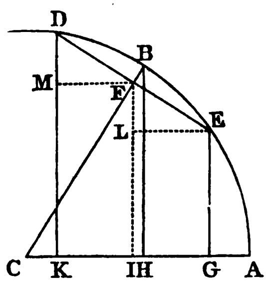
In like manner, cos. (−A) = cos. A.

Hence, also, dividing, tang. (−A) = −tang. A,

and cot. (−A) = −cot. A.

TRIGONOMETRICAL FORMULÆ.

80. *Expressions for the sine and cosine of the sum and difference of two arcs.*



Let AB and BD represent any two given arcs; take BE equal to BD: it is required to find an expression for the sine of AD, the sum, and of AE, the difference of these arcs.

Put AB $=a$, and BD $=b$; then AD $=a+b$, and AE $=a-b$. Draw the chord DE, and the radius CB, which may be represented by R. Since DB is, by construction, equal to BE, DF is equal to FE, and therefore DE is perpendicular to CB. Let fall the perpendiculars EG, BH, FI, and DK upon AC, and draw EL, FM parallel to AC.

Because the triangles BCH, FCI are similar, we have

CB : CF :: BH : FI; or R : cos. b :: sin. a : FI.

Therefore, $$\text{FI} = \frac{\sin. a \cos. b}{\text{R}}.$$

Also, CB : CF :: CH : CI; or R : cos. b :: cos. a : CI.

Therefore, $$\text{CI} = \frac{\cos. a \cos. b}{\text{R}}.$$

The triangles DFM, CBH, having their sides perpendicular each to each, are similar, and give the proportions

CB : DF :: CH : DM; or R : sin. b :: cos. a : DM.

Hence $$\text{DM} = \frac{\cos. a \sin. b}{\text{R}}.$$

Also, CB : DF :: BH : FM; or R : sin. b :: sin. a : FM.

Hence $$\text{FM} = \frac{\sin. a \sin. b}{\text{R}}.$$

But FI + DM = DK = sin. $(a+b)$;

and CI − FM = CK = cos. $(a+b)$.

Also, FI − FL = EG = sin. $(a-b)$;

and CI + EL = CG = cos. $(a-b)$.

Hence
$$\sin.(a+b)=\frac{\sin. a \cos. b+\cos. a \sin. b}{R}; \quad (1)$$
$$\cos.(a+b)=\frac{\cos. a \cos. b-\sin. a \sin. b}{R}; \quad (2)$$
$$\sin.(a-b)=\frac{\sin. a \cos. b-\cos. a \sin. b}{R}; \quad (3)$$
$$\cos.(a-b)=\frac{\cos. a \cos. b+\sin. a \sin. b}{R}. \quad (4)$$

These four equations express important geometrical theorems. The last of them may be stated as follows: *The product of radius and the cosine of the difference between two arcs is equal to the sum of the product of the sines and the product of the cosines of those arcs.*

81. *Expressions for the sine and cosine of a double arc.*

If, in the formulas of the preceding article, we make $b=a$, the first and second will become
$$\sin. 2a=\frac{2 \sin. a \cos. a}{R},$$
$$\cos. 2a=\frac{\cos.^2 a-\sin.^2 a}{R}.$$

Making radius equal to unity, and substituting the values of sin. a, cos. a, etc., from Art. 14, we obtain
$$\sin. 2a=\frac{2 \text{ tang. } a}{1+\text{tang.}^2 a},$$
$$\cos. 2a=\frac{1-\text{tang.}^2 a}{1+\text{tang.}^2 a}.$$

82. *Expressions for the sine and cosine of half a given arc.*

If we put $\frac{1}{2}a$ for a in the preceding equations, we obtain
$$\sin. a=\frac{2 \sin. \frac{1}{2}a \cos. \frac{1}{2}a}{R},$$
$$\cos. a=\frac{\cos.^2 \frac{1}{2}a-\sin.^2 \frac{1}{2}a}{R}.$$

We may also find the sine and cosine of $\frac{1}{2}a$ in terms of a.

Since the sum of the squares of the sine and cosine is equal to the square of radius, we have
$$\cos.^2 \tfrac{1}{2}a+\sin.^2 \tfrac{1}{2}a=R^2.$$

And, from the preceding equation,
$$\cos.^2 \tfrac{1}{2}a-\sin.^2 \tfrac{1}{2}a=R \cos. a.$$

If we subtract one of these from the other, we have
$$2 \sin.^2 \tfrac{1}{2}a=R^2-R \cos. a.$$

And, adding the same equations, we have

$$2\cos.^2\tfrac{1}{2}a = R^2 + R\cos. a.$$

Hence
$$\sin.\tfrac{1}{2}a = \sqrt{\tfrac{1}{2}R^2 - \tfrac{1}{2}R\cos. a};$$
$$\cos.\tfrac{1}{2}a = \sqrt{\tfrac{1}{2}R^2 + \tfrac{1}{2}R\cos. a}.$$

83. *Expressions for the products of sines and cosines.*

By adding and subtracting the formulas of Art. 80, we obtain

$$\sin.(a+b) + \sin.(a-b) = \frac{2}{R}\sin. a\cos. b.$$

$$\sin.(a+b) - \sin.(a-b) = \frac{2}{R}\cos. a\sin. b;$$

$$\cos.(a+b) + \cos.(a-b) = \frac{2}{R}\cos. a\cos. b;$$

$$\cos.(a-b) - \cos.(a+b) = \frac{2}{R}\sin. a\sin. b.$$

If, in these formulas, we make $a+b=A$, and $a-b=B$; that is, $a=\frac{1}{2}(A+B)$, and $b=\frac{1}{2}(A-B)$, we shall have

$$\sin. A + \sin. B = \frac{2}{R}\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B); \quad (1)$$

$$\sin. A - \sin. B = \frac{2}{R}\sin.\tfrac{1}{2}(A-B)\cos.\tfrac{1}{2}(A+B); \quad (2)$$

$$\cos. A + \cos. B = \frac{2}{R}\cos.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B); \quad (3)$$

$$\cos. B - \cos. A = \frac{2}{R}\sin.\tfrac{1}{2}(A+B)\sin.\tfrac{1}{2}(A-B). \quad (4)$$

These four equations express important geometrical theorems. The first of them may be stated as follows: *The sum of the sines of any two arcs is equal to twice the sine of half the sum of the arcs multiplied by the cosine of half their difference*, radius being unity.

84. *Theorems relating to the sum and difference of two arcs.*

Dividing formula (1) by (2), Art. 83, and considering that $\frac{\sin. a}{\cos. a} = \frac{\text{tang.}\, a}{R}$ (Art. 14), we have

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}(A-B)\cos.\tfrac{1}{2}(A+B)} = \frac{\text{tang.}\tfrac{1}{2}(A+B)}{\text{tang.}\tfrac{1}{2}(A-B)},$$

that is,

The sum of the sines of two arcs or angles is to their difference as the tangent of half the sum of those arcs is to the tangent of half their difference.

Since the sides of a plane triangle are as the sines of their opposite angles (Art. 53), it follows, from the preceding theorem, that the sum of any two sides of a plane triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.

This is the same as Theorem II., Art. 54, which is here demonstrated by a more general method.

Dividing formula (3) by (4), and considering that $\frac{\text{cos.}}{\text{sin.}}=\frac{\text{cot.}}{\text{R}}=\frac{\text{R}}{\text{tang.}}$ (Art. 14), we have

$$\frac{\text{cos. A}+\text{cos. B}}{\text{cos. B}-\text{cos. A}}=\frac{\text{cos. }\frac{1}{2}(\text{A}+\text{B})\ \text{cos. }\frac{1}{2}(\text{A}-\text{B})}{\text{sin. }\frac{1}{2}(\text{A}+\text{B})\ \text{sin. }\frac{1}{2}(\text{A}-\text{B})}=\frac{\text{cot. }\frac{1}{2}(\text{A}+\text{B})}{\text{tang. }\frac{1}{2}(\text{A}-\text{B})};$$

that is,

The sum of the cosines of two arcs is to their difference as the cotangent of half the sum of those arcs is to the tangent of half their difference.

From the first formula of Art. 82, by substituting A+B for *a*, we have

$$\text{sin. }(\text{A}+\text{B})=\frac{2\ \text{sin. }\frac{1}{2}(\text{A}+\text{B})\times\text{cos. }\frac{1}{2}(\text{A}+\text{B})}{\text{R}}$$

Dividing formula (1), Art. 83, by this, we obtain

$$\frac{\text{sin. A}+\text{sin. B}}{\text{sin. }(\text{A}+\text{B})}=\frac{\text{sin. }\frac{1}{2}(\text{A}+\text{B})\ \text{cos. }\frac{1}{2}(\text{A}-\text{B})}{\text{sin. }\frac{1}{2}(\text{A}+\text{B})\ \text{cos. }\frac{1}{2}(\text{A}+\text{B})}=\frac{\text{cos. }\frac{1}{2}(\text{A}-\text{B})}{\text{cos. }\frac{1}{2}(\text{A}+\text{B})};$$

that is,

The sum of the sines of two arcs is to the sine of their sum as the cosine of half the difference of those arcs is to the cosine of half their sum.

If we divide equation (1), Art. 80, by equation (3), we shall have

$$\frac{\text{sin. }(a+b)}{\text{sin. }(a-b)}=\frac{\text{sin. }a\ \text{cos. }b+\text{sin. }b\ \text{cos. }a}{\text{sin. }a\ \text{cos. }b-\text{sin. }b\ \text{cos. }a}.$$

By dividing both numerator and denominator of the second member by cos. a cos. b, and substituting $\frac{\text{tang.}}{\text{R}}$ for $\frac{\text{sin.}}{\text{cos.}}$, we obtain

$$\frac{\text{sin. }(a+b)}{\text{sin. }(a-b)}=\frac{\text{tang. }a+\text{tang. }b}{\text{tang. }a-\text{tang. }b};$$

that is,

The sine of the sum of two arcs is to the sine of their difference as the sum of the tangents of those arcs is to the difference of the tangents.

From equation (3), Art. 80, by dividing each member by cos. a cos. b, we obtain

$$\frac{\sin.(a-b)}{\cos. a \cos. b}=\frac{\sin. a \cos. b-\sin. b \cos. a}{\text{R} \cos. a \cos. b}=\frac{\text{tang}. a-\text{tang}. b}{\text{R}^2};$$

that is,

The sine of the difference of two arcs is to the product of their cosines as the difference of their tangents is to the square of radius

85. *Expressions for the tangents of arcs.*

If we take the expression $\text{tang}. (a+b)=\frac{\text{R} \sin. (a+b)}{\cos. (a+b)}$ (Art. 14), and substitute for sin. $(a+b)$ and cos. $(a+b)$ their values given in Art. 80, we shall find

$$\text{tang}. (a+b)=\frac{\text{R} (\sin. a \cos. b+\sin. b \cos. a)}{\cos. a \cos. b-\sin. a \sin. b}.$$

But $\sin. a=\frac{\cos. a \text{ tang}. a}{\text{R}}$, and $\sin. b=\frac{\cos. b \text{ tang}. b}{\text{R}}$ (Art. 14).

If we substitute these values in the preceding equation, and divide all the terms by cos. a cos. b, we shall have

$$\text{tang}. (a+b)=\frac{\text{R}^2 (\text{tang}. a+\text{tang}. b)}{\text{R}^2-\text{tang}. a \text{ tang}. b}.$$

In like manner we shall find

$$\text{tang}. (a-b)=\frac{\text{R}^2 (\text{tang}. a-\text{tang}. b)}{\text{R}^2+\text{tang}. a \text{ tang}. b}.$$

Suppose $b=a$, then

$$\text{tang}. 2a=\frac{2\text{R}^2 \text{ tang}. a}{\text{R}^2-\text{tang}.^2 a}.$$

Suppose $b=2a$, then

$$\text{tang}. 3a=\frac{\text{R}^2 (\text{tang}. a+\text{tang}. 2a)}{\text{R}^2-\text{tang}. a \text{ tang}. 2a}.$$

In the same manner we find

$$\cot. (a+b)=\frac{\cot. a \cot. b-\text{R}^2}{\cot. b+\cot. a},$$

$$\cot. (a-b)=\frac{\cot. a \cot. b+\text{R}^2}{\cot. b-\cot. a}.$$

86. *Formula for an angle of a triangle when the three sides are given.*

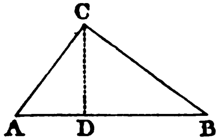
When the three sides of a triangle are given, the angles may be found by the formula

$$\sin. \tfrac{1}{2}\text{A}=\text{R}\sqrt{\frac{(\text{S}-b)(\text{S}-c)}{bc}},$$

where S represents half the sum of the sides a, b, and c.

Demonstration.

Let ABC be any triangle; then (*Geom.*, B. IV., Pr. 12)



$$BC^2=AB^2+AC^2-2AB\times AD.$$

Hence $$AD=\frac{AB^2+AC^2-BC^2}{2AB}.$$

But in the right-angled triangle ACD we have

$$R:AC::\cos. A:AD.$$

Hence $$\cos. A=\frac{R\times AD}{AC};$$

or, by substituting the value of AD, we have

$$\cos. A=R\times\frac{AB^2+AC^2-BC^2}{2AB\times AC}.$$

Let a, b, c represent the sides opposite the angles A, B, C; then

$$\cos. A=R\times\frac{b^2+c^2-a^2}{2bc}.$$

This equation expresses the following theorem: In every plane triangle *the cosine of either of the angles is equal to the sum of the squares of the adjacent sides, diminished by the square of the opposite side, and divided by twice the product of the adjacent sides*, radius being unity.

This formula is not well adapted to computation by logarithms, but may be rendered suitable by the following transformation:

By Art. 82, we have $2\sin.^2\frac{1}{2}A=R^2-R\cos. A.$

Substituting for cos. A its value given above, we obtain

$$2\sin.^2\tfrac{1}{2}A=R^2-R^2\times\frac{b^2+c^2-a^2}{2bc}=R^2\times\frac{2bc+a^2-b^2-c^2}{2bc},$$

$$=\frac{R^2\times(a+b-c)\,(a+c-b)}{2bc}.$$

Put $S=\frac{1}{2}(a+b+c)$, and we obtain, after reduction,

$$\sin.\tfrac{1}{2}A=R\sqrt{\frac{(S-b)\,(S-c)}{bc}}.$$

In the same manner we find

$$\sin.\tfrac{1}{2}B=R\sqrt{\frac{(S-a)\,(S-c)}{ac}}.$$

$$\sin.\tfrac{1}{2}C=R\sqrt{\frac{(S-a)\,(S-b)}{ab}};$$

that is, in every plane triangle *the square of the cosine of half* *either of the angles is equal to the product of the excess of the semiperimeter over the two adjacent sides divided by the product of those sides*, radius being unity.

Ex. 1. What are the angles of a plane triangle whose sides are 432, 543, and 654?

Here $S=814.5$; $S-b=382.5$; $S-c=271.5$.

log. 382.5	2.582631
log. 271.5	2.433770
log. b, 432	comp. 7.364516
log. c, 543	comp. 7.265200
	2)19.646117
sin. $\frac{1}{2}$A, 41° 42′ 36$\frac{1}{2}$″.	9.823058.

Angle $A=83° 25' 13''$.

In a similar manner we find the angle $B=41° 0' 39''$, and the angle $C=55° 34' 8''$.

Ex. 2. What are the angles of a plane triangle whose sides are 245, 219, and 91?

Ex. 3. What are the angles of a plane triangle whose sides are 538, 475, and 647?

87. *On the computation of a table of sines, cosines, etc.*

In computing a table of sines and cosines, we begin with finding the sine and cosine of *one minute*, and thence deduce the sines and cosines of larger arcs. The sine of so small an angle as one minute is nearly equal to the corresponding arc. The radius being taken as unity, the semi-circumference is known to be 3.14159. This, being divided successively by 180 and 60, gives .0002908882 for the arc of one minute, which may be regarded as the sine of one minute.

$$\text{The cosine of } 1'=\sqrt{1-\sin.^2}=0.9999999577.$$

The sines of very small angles are nearly proportional to the angles themselves. We might then obtain several other sines by direct proportion. This method will give the sines correct to five decimal places, as far as two degrees. By the following method they may be obtained with greater accuracy for the entire quadrant.

By Art. 83 we have, by transposition,

$$\sin. (a+b)=2 \sin. a \cos. b-\sin. (a-b),$$
$$\cos. (a+b)=2 \cos. a \cos. b-\cos. (a-b).$$

If we make $a=b$, $2b$, $3b$, etc., successively, we shall have

$$\sin. 2b=2 \sin. b \cos. b;$$
$$\sin. 3b=2 \sin. 2b \cos. b-\sin. b;$$

$$\sin. 4b = 2 \sin. 3b \cos. b - \sin. 2b,$$
etc., etc.
$$\cos. 2b = 2 \cos. b \cos. b - 1;$$
$$\cos. 3b = 2 \cos. 2b \cos. b - \cos. b;$$
$$\cos. 4b = 2 \cos. 3b \cos. b - \cos. 2b,$$
etc., etc.

Whence, making $b = 1'$, we have

$$\sin. 2' = 2 \sin. 1' \cos. 1' = .000582;$$
$$\sin. 3' = 2 \sin. 2' \cos. 1' - \sin. 1' = .000873;$$
$$\sin. 4' = 2 \sin. 3' \cos. 1' - \sin. 2' = .001164,$$
etc., etc.
$$\cos. 2' = 2 \cos. 1' \cos. 1' - 1 = 0.999999;$$
$$\cos. 3' = 2 \cos. 2' \cos. 1' - \cos. 1' = 0.999999;$$
$$\cos. 4' = 2 \cos. 3' \cos. 1' - \cos. 2' = 0.999999,$$
etc., etc.

The table of tangents may be computed from the sines and cosines by the formula $\text{tang. } A = \frac{\sin. A}{\cos. A}$. The rule is, *divide each sine by the corresponding cosine.*

The secants are computed by the formula $\sec. A = \frac{1}{\cos. A}$; or, the rule, *divide unity by each cosine.*

The cotangents and cosecants are computed by the formulas $\cot. = \frac{1}{\text{tang.}}$, and $\text{cosec.} = \frac{1}{\text{sine}}$.

The logarithmic tables are formed by taking the logarithms of the numbers in the tables computed as above, and adding 10 to each index.

88. *Formulæ of verification.* In so extended a work as the computation of the sines and cosines of all angles from 0° to 90°, it is necessary from time to time to verify the accuracy of the results by independent computations. For this purpose we employ special formulæ for the values of the sines and cosines of certain angles. The sines and cosines of 30°, 45°, and 60° have been given in Arts. 73 and 74. The sines and cosines of other angles may be found by means of the preceding formulas. By means of the Equations of Art. 82, from the cosine of any angle we can find the sine and cosine of its half; hence from the cosine of 45° we can find the sine and cosine of 22° 30′; and from these, the sine and cosine of 11° 15′. Also, from cos. 30°, we can find the sine and cosine of 15°, 7° 30′, and 3° 45′. If the values of the sines of these angles agree with the values obtained by the process of Art. 87, the whole work may be presumed to be correct.

Examples for Practice.

Prob. 1. Given the three sides of a triangle, 627, 718.9, and 1140, to find the angles.

Ans. 29° 44′ 2″, 34° 39′ 26″, and 115° 36′ 32″.

Prob. 2. In the triangle ABC, the angle A is given 89° 45′ 43″ the side AB 654, and the side AC 460, to find the remaining parts.

Ans. BC=798; the angle B=35° 12′ 1″, and the angle C=55° 2′ 16″.

Prob. 3. In the triangle ABC, the angle A is given 56° 12′ 45″, the side BC 2597.84, and the side AC 3084.33, to find the remaining parts. *Ans.* B=80° 39′ 40″, C=43° 7′ 35″, c=2136.8;
or, B=99 20 20, C=24 26 55, c=1293.8.

Prob. 4. In the triangle ABC, the angle A is given 44° 13′ 24″, the angle B 55° 59′ 58″, and the side AC 368, to find the remaining parts.

Ans. C=79° 46′ 38″, AB=436.844, and BC=309.595.

Prob. 5. In a right-angled triangle, if the sum of the hypothenuse and base be 3409 feet, and the angle at the base 53° 12′ 14″, what is the perpendicular? *Ans.* 1707.2 feet.

Prob. 6. In a right-angled triangle, if the difference of the hypothenuse and base be 169.9 yards, and the angle at the base 42° 36′ 12″, what is the length of the perpendicular?

Ans. 435.732 yards.

Prob. 7. In a right-angled triangle, if the sum of the base and perpendicular be 123.7 feet, and the angle at the base 58° 19′ 32″, what is the length of the hypothenuse? *Ans.* 89.889 feet.

Prob. 8. In a right-angled triangle, if the difference of the base and perpendicular be 12 yards, and the angle at the base 38° 1′ 8″, what is the length of the hypothenuse? *Ans.* 69.81 yards.

Prob. 9. A May-pole 50 feet 11 inches high, at a certain time will cast a shadow 98 feet 6 inches long; what, then, is the breadth of a river which runs within 20 feet 6 inches of the foot of a steeple 300 feet 8 inches high, if the steeple at the same time throws its shadow 30 feet 9 inches beyond the stream?

Ans. 530 feet 5 inches.

Prob. 10. A ladder 40 feet long may be so placed that it shall reach a window 33 feet from the ground on one side of the street, and by turning it over. without moving the foot out of its place, it will do the same by a window 21 feet high on the other side. Required the breadth of the street. *Ans.* 56.649 feet.

Prob. 11. A May-pole, whose top was broken off by a blast of wind, struck the ground at the distance of 15 feet from the foot of the pole; what was the height of the whole May-pole, supposing the length of the broken piece to be 39 feet?

Ans. 75 feet.

Prob. 12. How must three trees, A, B, C, be planted, so that the angle at A may be double the angle at B, the angle at B double the angle at C, and a line of 400 yards may just go round them?

Sol. Assume AB=1, and compute the corresponding values of AC and BC.

Ans. AB=79.225, AC=142.758, and BC=178.017 yards.

Prob. 13. The town B is half way between the towns A and C, and the towns B, C, and D are equidistant from each other. What is the ratio of the distance AB to AD?

Ans. As unity to $\sqrt{3}$.

Prob. 14. There are two columns left standing upright in the ruins of Persepolis; the one is 66 feet above the plain, and the other 48. In a straight line between them stands an ancient statue, the head of which is 100 feet from the summit of the higher, and 84 feet from the top of the lower column, the base of which measures just 74 feet to the centre of the figure's base. Required the distance between the tops of the two columns.

Ans. 156.68 feet.

Prob. 15. Prove that $\text{tang.}\,(45°-b)=\frac{1-\text{tang.}\,b}{1+\text{tang.}\,b}$.

Prob. 16. One angle of a triangle is 45°, and the perpendicular from this angle upon the opposite base divides the base into two parts, which are in the ratio of 2 to 3. What are the parts into which the vertical angle is divided by this perpendicular?

Sol. Let x=the larger angle; then

$$\text{tang.}\,(45°-a)=\tfrac{2}{3}\,\text{tang.}\,a=\frac{1-\text{tang.}\,a}{1+\text{tang.}\,a},$$

which can be solved as an equation of the second degree.

Ans. 18° 26′ 6″, and 26° 33′ 54″.

Prob. 17. Prove that $\sin. 3b=3 \sin. b-4 \sin.^3 b$.

Prob. 18. One side of a triangle is 25, another is 22, and the angle contained by these two sides is one half of the angle opposite the side 25. What is the value of the included angle?

Sol. $\frac{\sin. 3x}{\sin. 2x} = .88 = \frac{3 \sin. x - 4 \sin.^3 x}{2 \sin. x \cos. x} = \frac{3 - 4 \sin.^2 x}{2 \cos. x} = \frac{3 - 4 \sin.^2 x}{2\sqrt{1 - \sin.^2 x}}$, which can be solved as an equation of the second degree.

Ans. 39° 58′ 51″.

Prob. 19. One side of a triangle is 25, another is 22, and the angle contained by these two sides is one half of the angle opposite the side 22. What is the value of the included angle?

Sol. Like the preceding. *Ans.* 30° 46′ 38″.

Prob. 20. Two sides of a triangle are in the ratio of 11 to 9, and the opposite angles have the ratio of 3 to 1. What are those angles?

Sol. $3 \sin. x - 4 \sin.^3 x : \sin. x :: 11 : 9.$

Ans. The sine of the smaller of the two angles is $\frac{2}{3}$, and of the greater $\frac{22}{27}$; the angles are 41° 48′ 37″, and 125° 25′ 51″.

Prob. 21. One side of a triangle is 15, and the difference of the two other sides is 6; also, the angle included between the first side and the greater of the two others is 60°. What is the length of the side opposite to this angle? *Ans.* 57.

Prob. 22. One side of a triangle is 15, and the difference of the two other sides is 6; also, the angle opposite to the greater of the two latter sides is 60°. What is the length of said side?

Ans. 13.

Prob. 23. One side of a triangle is 15, and the opposite angle is 60°; also, the difference of the two other sides is 6. What are the lengths of those sides? *Ans.* 11.0712, and 17.0712.

Prob. 24. The perimeter of a triangle is 100; the perpendicular let fall from one of the angles upon the opposite base is 30, and the angle at one end of this base is 50°. What is the length of the base? *Ans.* 30.388.

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857332	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

N.B.—In the following table, commencing with page 322, the two leading figures in the first column of logarithms are to be prefixed to all the numbers of the same horizontal line in the next nine columns; but when a point (.) occurs, its place is to be supplied by a cipher, and the two leading figures are to be taken from the next lower line.

The logarithms of the first 100 numbers are given with their characteristics; but for all other numbers the decimal part only of the logarithm is given, and the characteristic is to be supplied by the usual rule.

The last column of each page shows the difference between the successive logarithms on the same horizontal line; and on the lower portion of each page are given the Proportional Parts for a fifth figure in the natural number.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	.300	.724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	.361	.775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	.195	.600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	.207	.602	.998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	.380	.766	1153	1538	1924	2309	2694	386

N.	0 (Differences)	1 (Proportional Parts)	2	3	4	5	6	7	8	9	D.
	434	43	87	130	174	217	260	304	347	391	
	433	43	87	130	173	217	260	303	346	390	
	432	43	86	130	173	216	259	302	346	389	
	431	43	86	129	172	216	259	302	345	388	
	430	43	86	129	172	215	258	301	344	387	
	429	43	86	129	172	215	257	300	343	386	
	428	43	86	128	171	214	257	300	342	385	
	427	43	85	128	171	214	256	299	342	384	
	426	43	85	128	170	213	256	298	341	383	
	425	43	85	128	170	213	255	298	340	383	
	424	42	85	127	170	212	254	297	339	382	
	423	42	85	127	169	212	254	296	338	381	
	422	42	84	127	169	211	253	295	338	380	
	421	42	84	126	168	211	253	295	337	379	
	420	42	84	126	168	210	252	294	336	378	
	419	42	84	126	168	210	251	293	335	377	
	418	42	84	125	167	209	251	293	334	376	
	417	42	83	125	167	209	250	292	334	375	
	416	42	83	125	166	208	250	291	333	374	
	415	42	83	125	166	208	249	291	332	374	
	414	41	83	124	166	207	248	290	331	373	
	413	41	83	124	165	207	248	289	330	372	
	412	41	82	124	165	206	247	288	330	371	
	411	41	82	123	164	206	247	288	329	370	
	410	41	82	123	164	205	246	287	328	369	
	409	41	82	123	164	205	245	286	327	368	
	408	41	82	122	163	204	245	286	326	367	
	407	41	81	122	163	204	244	285	326	366	
	406	41	81	122	162	203	244	284	325	365	
	405	41	81	122	162	203	243	284	324	365	
	404	40	81	121	162	202	242	283	323	364	
	403	40	81	121	161	202	242	282	322	363	
	402	40	80	121	161	201	241	281	322	362	
	401	40	80	120	160	201	241	281	321	361	
	400	40	80	120	160	200	240	280	320	360	
	399	40	80	120	160	200	239	279	319	359	
	398	40	80	119	159	199	239	279	318	358	
	397	40	79	119	159	199	238	278	318	357	
	396	40	79	119	158	198	238	277	317	356	
	395	40	79	119	158	198	237	277	316	356	
	394	39	79	118	158	197	236	276	315	355	
	393	39	79	118	157	197	236	275	314	354	
	392	39	78	118	157	196	235	274	314	353	
	391	39	78	117	156	196	235	274	313	352	
	390	39	78	117	156	195	234	273	312	351	
	389	39	78	117	156	195	233	272	311	350	
	388	39	78	116	155	194	233	272	310	349	
	387	39	77	116	155	194	232	271	310	348	
	386	39	77	116	154	193	232	270	309	347	

N.	0	1	2	3	4	5	6	7	8	9	D.
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	.320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	..38	.407	.776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	9181	9543	9904	.266	.626	.987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	.258	.611	.963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	..26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	385	39	77	116	154	193	231	270	308	347	
	384	38	77	115	154	192	230	269	307	346	
	383	38	77	115	153	192	230	268	306	345	
	382	38	76	115	153	191	229	267	306	344	
	381	38	76	114	152	191	229	267	305	343	
	380	38	76	114	152	190	228	266	304	342	
	379	38	76	114	152	190	227	265	303	341	
	378	38	76	113	151	189	227	265	302	340	
	377	38	75	113	151	189	226	264	302	339	
	376	38	75	113	150	188	226	263	301	338	
	375	38	75	113	150	188	225	263	300	338	
	374	37	75	112	150	187	224	262	299	337	
	373	37	75	112	149	187	224	261	298	336	
	372	37	74	112	149	186	223	260	298	335	
	371	37	74	111	148	186	223	260	297	334	
	370	37	74	111	148	185	222	259	296	333	
	369	37	74	111	148	185	221	258	295	332	
	368	37	74	110	147	184	221	258	294	331	
	367	37	73	110	147	184	220	257	294	330	
	366	37	73	110	146	183	220	256	293	329	
	365	37	73	110	146	183	219	256	292	329	
	364	36	73	109	146	182	218	255	291	328	
	363	36	73	109	145	182	218	254	290	327	
	362	36	72	109	145	181	217	253	290	326	
	361	36	72	108	144	181	217	253	289	325	
	360	36	72	108	144	180	216	252	288	324	
	359	36	72	108	144	180	215	251	287	323	
	358	36	72	107	143	179	215	251	286	322	
	357	36	71	107	143	179	214	250	286	321	
	356	36	71	107	142	178	214	249	285	320	
	355	36	71	107	142	178	213	249	284	320	
	354	35	71	106	142	177	212	248	283	319	
	353	35	71	106	141	177	212	247	282	318	
	352	35	70	106	141	176	211	246	282	317	
	351	35	70	105	140	176	211	246	281	316	
	350	35	70	105	140	175	210	245	280	315	
	349	35	70	105	140	175	209	244	279	314	
	348	35	70	104	139	174	209	244	278	313	
	347	35	69	104	139	174	208	243	278	312	
	346	35	69	104	138	173	208	242	277	311	
	345	35	69	104	138	173	207	242	276	311	
	344	34	69	103	138	172	206	241	275	310	
	343	34	69	103	137	172	206	240	274	309	
	342	34	68	103	137	171	205	239	274	308	
	341	34	68	102	136	171	205	239	273	307	
	340	34	68	102	136	170	204	238	272	306	
	339	34	68	102	136	170	203	237	271	305	

N.	0	1	2	3	4	5	6	7	8	9	D.
128	107210	7549	7888	8227	8565	8903	9241	9579	9916	.253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	.245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	..12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	.194	.508	.822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	.142	.449	.756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	.168	.469	.769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	339	34	68	102	136	170	203	237	271	305	
	338	34	68	101	135	169	203	237	270	304	
	337	34	67	101	135	169	202	236	270	303	
	336	34	67	101	134	168	202	235	269	302	
	335	34	67	101	134	168	201	235	268	302	
	334	33	67	100	134	167	200	234	267	301	
	333	33	67	100	133	167	200	233	266	300	
	332	33	66	100	133	166	199	232	266	299	
	331	33	66	99	132	166	199	232	265	298	
	330	33	66	99	132	165	198	231	264	297	
	329	33	66	99	132	165	197	230	263	296	
	328	33	66	98	131	164	197	230	262	295	
	327	33	65	98	131	164	196	229	262	294	
	326	33	65	98	130	163	196	228	261	293	
	325	33	65	98	130	163	195	228	260	293	
	324	32	65	97	130	162	194	227	259	292	
	323	32	65	97	129	162	194	226	258	291	
	322	32	64	97	129	161	193	225	258	290	
	321	32	64	96	128	161	193	225	257	289	
	320	32	64	96	128	160	192	224	256	288	
	319	32	64	96	128	160	191	223	255	287	
	318	32	64	95	127	159	191	223	254	286	
	317	32	63	95	127	159	190	222	254	285	
	316	32	63	95	126	158	190	221	253	284	
	315	32	63	95	126	158	189	221	252	284	
	314	31	63	94	126	157	188	220	251	283	
	313	31	63	94	125	157	188	219	250	282	
	312	31	62	94	125	156	187	218	250	281	
	311	31	62	93	124	156	187	218	249	280	
	310	31	62	93	124	155	186	217	248	279	
	309	31	62	93	124	155	185	216	247	278	
	308	31	62	92	123	154	185	216	246	277	
	307	31	61	92	123	154	184	215	246	276	
	306	31	61	92	122	153	184	214	245	275	
	305	31	61	92	122	153	183	214	244	275	
	304	30	61	91	122	152	182	213	243	274	
	303	30	60	91	121	152	182	212	242	273	
	302	30	60	91	121	151	181	211	242	272	
	301	30	60	90	120	151	181	211	241	271	
	300	30	60	90	120	150	180	210	240	270	
	299	30	60	90	120	150	179	209	239	269	
	298	30	60	89	119	149	179	209	238	268	
	297	30	59	89	119	149	178	208	238	267	

N.	0	1	2	3	4	5	6	7	8	9	D.
147	167317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	.126	.413	.699	.986	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	..51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	..29	.303	.577	.850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	..51	.319	.586	.853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	296	30	59	89	118	148	178	207	237	266	
	295	30	59	89	118	148	177	207	236	266	
	294	29	59	88	118	147	176	206	235	265	
	293	29	59	88	117	147	176	205	234	264	
	292	29	58	88	117	146	175	204	234	263	
	291	29	58	87	116	146	175	204	233	262	
	290	29	58	87	116	145	174	203	232	261	
	289	29	58	87	116	145	173	202	231	260	
	288	29	58	86	115	144	173	202	230	259	
	287	29	57	86	115	144	172	201	230	258	
	286	29	57	86	114	143	172	200	229	257	
	285	29	57	86	114	143	171	200	228	257	
	284	28	57	85	114	142	170	199	227	256	
	283	28	57	85	113	142	170	198	226	255	
	282	28	56	85	113	141	169	197	226	254	
	281	28	56	84	112	141	169	197	225	253	
	280	28	56	84	112	140	168	196	224	252	
	279	28	56	84	112	140	167	195	223	251	
	278	28	56	83	111	139	167	195	222	250	
	277	28	55	83	111	139	166	194	222	249	
	276	28	55	83	110	138	166	193	221	248	
	275	28	55	83	110	138	165	193	220	248	
	274	27	55	82	110	137	164	192	219	247	
	273	27	55	82	109	137	164	191	218	246	
	272	27	54	82	109	136	163	190	218	245	
	271	27	54	81	108	136	163	190	217	244	
	270	27	54	81	108	135	162	189	216	243	
	269	27	54	81	108	135	161	188	215	242	
	268	27	54	80	107	134	161	188	214	241	
	267	27	53	80	107	134	160	187	214	240	
	266	27	53	80	106	133	160	186	213	239	
	265	27	53	80	106	133	159	186	212	239	
	264	26	53	79	106	132	158	185	211	238	
	263	26	53	79	105	132	158	184	210	237	
	262	26	52	79	105	131	157	183	210	236	
	261	26	52	78	104	131	157	183	209	235	
	260	26	52	78	104	130	156	182	208	234	
	259	26	52	78	104	130	155	181	207	233	
	258	26	52	77	103	129	155	181	206	232	
	257	26	51	77	103	129	154	180	206	231	

N.	0	1	2	3	4	5	6	7	8	9	D.
169	227887	8144	8400	8657	8913	9170	9426	9682	9938	.193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	..50	.300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	.176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	.213	.446	.679	.912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	8754	8982	9211	9439	9667	9895	.123	.351	.578	.806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	257	26	51	77	103	129	154	180	206	231	
	256	26	51	77	102	128	154	179	205	230	
	255	26	51	77	102	128	153	179	204	230	
	254	25	51	76	102	127	152	178	203	229	
	253	25	51	76	101	127	152	177	202	228	
	252	25	50	76	101	126	151	176	202	227	
	251	25	50	75	100	126	151	176	201	226	
	250	25	50	75	100	125	150	175	200	225	
	249	25	50	75	100	125	149	174	199	224	
	248	25	50	74	99	124	149	174	198	223	
	247	25	49	74	99	124	148	173	198	222	
	246	25	49	74	98	123	148	172	197	221	
	245	25	49	74	98	123	147	172	196	221	
	244	24	49	73	98	122	146	171	195	220	
	243	24	49	73	97	122	146	170	194	219	
	242	24	48	73	97	121	145	169	194	218	
	241	24	48	72	96	121	145	169	193	217	
	240	24	48	72	96	120	144	168	192	216	
	239	24	48	72	96	120	143	167	191	215	
	238	24	48	71	95	119	143	167	190	214	
	237	24	47	71	95	119	142	166	190	213	
	236	24	47	71	94	118	142	165	189	212	
	235	24	47	71	94	118	141	165	188	212	
	234	23	47	70	94	117	140	164	187	211	
	233	23	47	70	93	117	140	163	186	210	
	232	23	46	70	93	116	139	162	186	209	
	231	23	46	69	92	116	139	162	185	208	
	230	23	46	69	92	115	138	161	184	207	
	229	23	46	69	92	115	137	160	183	206	
	228	23	46	68	91	114	137	160	182	205	
	227	23	45	68	91	114	136	159	182	204	
	226	23	45	68	90	113	136	158	181	203	
	225	23	45	68	90	113	135	158	180	203	
	224	22	45	67	90	112	134	157	179	202	
	223	22	45	67	89	112	134	156	178	201	

N.	0	1	2	3	4	5	6	7	8	9	D.
196	292256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	.161	.378	.595	.813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	..56	.268	.481	.693	.906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	...8	.211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	..47	.246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	..54	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	222	22	44	67	89	111	133	155	178	200	
	221	22	44	66	88	111	133	155	177	199	
	220	22	44	66	88	110	132	154	176	198	
	219	22	44	66	88	110	131	153	175	197	
	218	22	44	65	87	109	131	153	174	196	
	217	22	43	65	87	109	130	152	174	195	
	216	22	43	65	86	108	130	151	173	194	
	215	22	43	65	86	108	129	151	172	194	
	214	21	43	64	86	107	128	150	171	193	
	213	21	43	64	85	107	128	149	170	192	
	212	21	42	64	85	106	127	148	170	191	
	211	21	42	63	84	106	127	148	169	190	
	210	21	42	63	84	105	126	147	168	189	
	209	21	42	63	84	105	125	146	167	188	
	208	21	42	62	83	104	125	146	166	187	
	207	21	41	62	83	104	124	145	166	186	
	206	21	41	62	82	103	124	144	165	185	
	205	21	41	62	82	103	123	144	164	185	
	204	20	41	61	82	102	122	143	163	184	
	203	20	41	61	81	102	122	142	162	183	
	202	20	40	61	81	101	121	141	162	182	
	201	20	40	60	80	101	121	141	161	181	
	200	20	40	60	80	100	120	140	160	180	
	199	20	40	60	80	100	119	139	159	179	
	198	20	40	59	79	99	119	139	158	178	
	197	20	39	59	79	99	118	138	158	177	
	196	20	39	59	78	98	118	137	157	176	
	195	20	39	59	78	98	117	137	156	176	
	194	19	39	58	78	97	116	136	155	175	
	193	19	39	58	77	97	116	135	154	174	
	192	19	38	58	77	96	115	134	154	173	

N.	0	1	2	3	4	5	6	7	8	9	D.
227	356026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	..25	.215	.404	.593	.783	.972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	.143	.328	.513	.698	.883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	..30	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	..51	.228	.405	.582	.759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	..20	.192	.365	.538	.711	.883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	.102	.271	.440	.609	.777	.946	1114	1283	1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166

N.	0 (Differences)	1 (Proportional Parts)	2	3	4	5	6	7	8	9	D.
	192	19	38	58	77	96	115	134	154	173	
	191	19	38	57	76	96	115	134	153	172	
	190	19	38	57	76	95	114	133	152	171	
	189	19	38	57	76	95	113	132	151	170	
	188	19	38	56	75	94	113	132	150	169	
	187	19	37	56	75	94	112	131	150	168	
	186	19	37	56	74	93	112	130	149	167	
	185	19	37	56	74	93	111	130	148	167	
	184	18	37	55	74	92	110	129	147	166	
	183	18	37	55	73	92	110	128	146	165	
	182	18	36	55	73	91	109	127	146	164	
	181	18	36	54	72	91	109	127	145	163	
	180	18	36	54	72	90	108	126	144	162	
	179	18	36	54	72	90	107	125	143	161	
	178	18	36	53	71	89	107	125	142	160	
	177	18	35	53	71	89	106	124	142	159	
	176	18	35	53	70	88	106	123	141	158	
	175	18	35	53	70	88	105	123	140	158	
	174	17	35	52	70	87	104	122	139	157	
	173	17	35	52	69	87	104	121	138	156	
	172	17	34	52	69	86	103	120	138	155	
	171	17	34	51	68	86	103	120	137	154	
	170	17	34	51	68	85	102	119	136	153	
	169	17	34	51	68	85	101	118	135	152	
	168	17	34	50	67	84	101	118	134	151	
	167	17	33	50	67	84	100	117	134	150	
	166	17	33	50	66	83	100	116	133	149	

N.	0	1	2	3	4	5	6	7	8	9	D.
262	418301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	.121	.286	.451	.616	.781	.945	1110	1275	1439	
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	
269	9752	9914	..75	.236	.398	.559	.720	.881	1042	1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	.122	.279	.437	.594	.752	
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	..95	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	.146	.296	.447	.597	.748	
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	
295	9822	9969	.116	.263	.410	.557	.704	.851	.998	1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	
301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	165	17	33	50	66	83	99	116	132	149	
	164	16	33	49	66	82	98	115	131	148	
	163	16	33	49	65	82	98	114	130	147	
	162	16	32	49	65	81	97	113	130	146	
	161	16	32	48	64	81	97	113	129	145	
	160	16	32	48	64	80	96	112	128	144	
	159	16	32	48	64	80	95	111	127	143	
	158	16	32	47	63	79	95	111	126	142	
	157	16	31	47	63	79	94	110	126	141	
	156	16	31	47	62	78	94	109	125	140	
	155	16	31	47	62	78	93	109	124	140	
	154	15	31	46	62	77	92	108	123	139	
	153	15	31	46	61	77	92	107	122	138	
	152	15	30	46	61	76	91	106	122	137	
	151	15	30	45	60	76	91	106	121	136	
	150	15	30	45	60	75	90	105	120	135	
	149	15	30	45	60	75	89	104	119	134	
	148	15	30	44	59	74	89	104	118	133	
	147	15	29	44	59	74	88	103	118	132	
	146	15	29	44	58	73	88	102	117	131	
	145	15	29	44	58	73	87	102	116	131	
	144	14	29	43	58	72	86	101	115	130	

N.	0	1	2	3	4	5	6	7	8	9	D.
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	
309	9958	..99	.239	.380	.520	.661	.801	.941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	
316	9687	9824	9962	..99	.236	.374	.511	.648	.785	.922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	
323	9203	9337	9471	9606	9740	9874	...9	.143	.277	.411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	..90	.221	.353	.484	.615	.745	.876	1007	
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	..72	128
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	
340	1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	144	14	29	43	58	72	86	101	115	130	
	143	14	29	43	57	72	86	100	114	129	
	142	14	28	43	57	71	85	99	114	128	
	141	14	28	42	56	71	85	99	113	127	
	140	14	28	42	56	70	84	98	112	126	
	139	14	28	42	56	70	83	97	111	125	
	138	14	28	41	55	69	83	97	110	124	
	137	14	27	41	55	69	82	96	110	123	
	136	14	27	41	54	68	82	95	109	122	
	135	14	27	41	54	68	81	95	108	122	
	134	13	27	40	54	67	80	94	107	121	
	133	13	27	40	53	67	80	93	106	120	
	132	13	26	40	53	66	79	92	106	119	
	131	13	26	39	52	66	79	92	105	118	
	130	13	26	39	52	65	78	91	104	117	
	129	13	26	39	52	65	77	90	103	116	
	128	13	26	38	51	64	77	90	102	115	
	127	13	25	38	51	64	76	89	102	114	
	126	13	25	38	50	63	76	88	101	113	

N.	0	1	2	3	4	5	6	7	8	9	D.
345	537819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	..79	.204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	
351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	.106	
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	
359	5004	5215	5336	5457	5578	5699	5820	5940	6061	6182	
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	
363	9907	..26	.146	.265	.385	.504	.624	.743	.863	.982	
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	..76	.193	.309	.426	
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	9784	9898	..12	.126	.241	.355	.469	.583	.697	.811	
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	
389	9950	..61	.173	.284	.396	.507	.619	.730	.842	.953	
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	

N.	0 (Differences)	1 (Proportional Parts)	2	3	4	5	6	7	8	9	D.
	125	13	25	38	50	63	75	88	100	113	
	124	12	25	37	50	62	74	87	99	112	
	123	12	25	37	49	62	74	86	98	111	
	122	12	24	37	49	61	73	85	98	110	
	121	12	24	36	48	61	73	85	97	109	
	120	12	24	36	48	60	72	84	96	108	
	119	12	24	36	48	60	71	83	95	107	
	118	12	24	35	47	59	71	83	94	106	
	117	12	23	35	47	59	70	82	94	105	
	116	12	23	35	46	58	70	81	93	104	
	115	12	23	35	46	58	69	81	92	104	
	114	11	23	34	46	57	68	80	91	103	
	113	11	23	34	45	57	68	79	90	102	
	112	11	22	34	45	56	67	78	90	101	
	111	11	22	33	44	56	67	78	89	100	

N.	0	1	2	3	4	5	6	7	8	9	D.
392	593286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	.101	.210	.319	.428	.537	.646	.755	.864	
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	
407	9594	9701	9808	9914	..21	.128	.234	.341	.447	.554	
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	..32	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	
419	2214	2318	2421	2525	2628	2782	2835	2939	3042	3146	
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	
426	9410	9512	9613	9715	9817	9919	..21	.123	.224	.326	
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	99
436	9486	9586	9686	9785	9885	9984	..84	.183	.283	.382	
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	111	11	22	33	44	56	67	78	89	100	
	110	11	22	33	44	55	66	77	88	99	
	109	11	22	33	44	55	65	76	87	98	
	108	11	22	32	43	54	65	76	86	97	
	107	11	21	32	43	54	64	75	86	96	
	106	11	21	32	42	53	64	74	85	95	
	105	11	21	32	42	53	63	74	84	95	
	104	10	21	31	42	52	62	73	83	94	
	103	10	21	31	41	52	62	72	82	93	
	102	10	20	31	41	51	61	71	82	92	
	101	10	20	30	40	51	61	71	81	91	
	100	10	20	30	40	50	60	70	80	90	
	99	10	20	30	40	50	59	69	79	89	

N.	0	1	2	3	4	5	6	7	8	9	D.
441	644439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	..16	.113	.210	
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
457	9916	..11	.106	.201	.296	.391	.486	.581	.676	.771	
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	
460	2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
467	9317	9410	9503	9596	9689	9782	9875	9967	..60	.153	
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	
478	9428	9519	9610	9700	9791	9882	9973	..63	.154	.245	
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
489	9309	9398	9486	9575	9664	9753	9841	9930	..19	.107	
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88

N.	0 (Differences.)	1 (Proportional Parts.)	2	3	4	5	6	7	8	9	D.
	98	10	20	29	39	49	59	69	78	88	
	97	10	19	29	39	49	58	68	78	87	
	96	10	19	29	38	48	58	67	77	86	
	95	10	19	29	38	48	57	67	76	86	
	94	9	19	28	38	47	56	66	75	85	
	93	9	19	28	37	47	56	65	74	84	
	92	9	18	28	37	46	55	64	74	83	
	91	9	18	27	36	46	55	64	73	82	
	90	9	18	27	36	45	54	63	72	81	
	89	9	18	27	36	45	53	62	71	80	
	88	9	18	26	35	44	53	62	70	79	

N.	0	1	2	3	4	5	6	7	8	9	D.
492	691965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	
500	8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	
501	9838	9924	..11	..98	.184	.271	.358	.444	.531	.617	
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	..33	
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	..77	
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	
531	5095	5176	52[illegible]	5340	5422	5503	5585	5667	5748	5830	
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	
537	9974	..55	.136	.217	.298	.378	.459	.540	.621	.702	
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Differences. 88	Propor. Parts. 9	18	26	35	44	53	62	70	79	
	87	9	17	26	35	44	52	61	70	78	
	86	9	17	26	34	43	52	60	69	77	
	85	9	17	26	34	43	51	60	68	77	
	84	8	17	25	34	42	50	59	67	76	
	83	8	17	25	33	42	50	58	66	75	
	82	8	16	25	33	41	49	57	66	74	
	81	8	16	24	32	41	49	57	65	73	
	80	8	16	24	32	40	48	56	64	72	

N.	0	1	2	3	4	5	6	7	8	9	D.
545	736397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	
549	9572	9651	9731	9810	9889	9968	..47	.126	.205	.284	
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	
562	9736	9814	9891	9968	..45	.123	.200	.277	.354	.431	
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	
575	9668	9743	9819	9894	9970	..45	.121	.196	.272	.347	75
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	..42	
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	

N.	0 (Differences.)	1 (Propor. Parts.)	2	3	4	5	6	7	8	9	D.
	79	8	16	24	32	40	47	55	63	71	
	78	8	16	23	31	39	47	55	62	70	
	77	8	15	23	31	39	46	54	62	69	
	76	8	15	23	30	38	46	53	61	68	
	75	8	15	23	30	38	45	53	60	68	
	74	7	15	22	30	37	44	52	59	67	
	73	7	15	22	29	37	44	51	58	66	
	72	7	14	22	29	36	43	50	58	65	

N.	0	1	2	3	4	5	6	7	8	9	D.
599	777427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	
602	9596	9669	9741	9813	9885	9957	..29	.101	.173	.245	
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	
616	9581	9651	9722	9792	9863	9933	...4	..74	.144	.215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6643	6713	6782	6852	6921	6990	7060	7129	7198	
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	
645	9560	9627	9694	9762	9829	9896	9964	..31	..98	.165	
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Differ. 72	Pr. Parts 7	14	22	29	36	43	50	58	65	
	71	7	14	21	28	36	43	50	57	64	
	70	7	14	21	28	35	42	49	56	63	
	69	7	14	21	28	35	41	48	55	62	
	68	7	14	20	27	34	41	48	54	61	
	67	7	13	20	27	34	40	47	54	60	

N.	0	1	2	3	4	5	6	7	8	9	D.
655	816241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	
660	9544	9610	9676	9741	9807	9873	9939	...4	..70	.136	
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
676	9947	..11	..75	.139	.204	.268	.332	.396	.460	.525	
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5961	5754	5817	5881	5944	6007	6071	6134	6197	6261	
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	..43	
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	
700	5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Differ. 66	Pr. Parts 7	13	20	26	33	40	46	53	59	
	65	7	13	20	26	33	39	46	52	59	
	64	6	13	19	26	32	38	45	51	58	
	63	6	13	19	25	32	38	44	50	57	
	62	6	12	19	25	31	37	43	50	56	
	61	6	12	18	24	31	37	43	49	55	

N.	0	1	2	3	4	5	6	7	8	9	D.
711	851870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	
724	9739	9799	9859	9918	9978	..38	..98	.158	.218	.278	
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	
741	9818	9877	9935	9994	..53	.111	.170	.228	.287	.345	
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	
758	9669	9726	9784	9841	9898	9956	..13	..70	.127	.185	
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Differ. 61	Pr. Parts 6	12	18	24	31	37	43	49	55	
	60	6	12	18	24	30	36	42	48	54	
	59	6	12	18	24	30	35	41	47	53	
	58	6	12	17	23	29	35	41	46	52	
	57	6	11	17	23	29	34	40	46	51	
	56	6	11	17	22	28	34	39	45	50	

N.	0	1	2	3	4	5	6	7	8	9	D.
767	884795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	
776	9862	9918	9974	..30	..86	.141	.197	.253	.309	.365	
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	
794	9821	9875	9930	9985	..39	..94	.149	.203	.258	.312	
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	..37	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Diff. 55	P. P. 6	11	17	22	28	33	39	44	50	
	54	5	11	16	22	27	32	38	43	49	
	53	5	11	16	21	27	32	37	42	48	
	52	5	10	16	21	26	31	36	42	47	

N.	0	1	2	3	4	5	6	7	8	9	D.
825	916454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	
831	9601	9653	9706	9758	9810	9862	9914	9967	..19	..71	
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	
851	9930	9981	..32	..83	.134	.185	.236	.287	.338	.389	
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	

N.	0	1	2	3	4	5	6	7	8	9	D.
	Diff. 52	P. P. 5	10	16	21	26	31	36	42	47	
	51	5	10	15	20	26	31	36	41	46	
	50	5	10	15	20	25	30	35	40	45	
	49	5	10	15	20	25	29	34	39	44	

N.	0	1	2	3	4	5	6	7	8	9	D.
883	945961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	
891	9878	9926	9975	..24	..73	.121	.170	.219	.267	.316	
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	
900	4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	
912	9995	..42	..90	.138	.185	.233	.280	.328	.376	.423	
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	
933	9882	9928	9975	..21	..68	.114	.161	.207	.254	.300	
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	
N.	0	1	2	3	4	5	6	7	8	9	D.
	Diff. 48	P.P. 5	10	14	19	24	29	34	38	43	
	Diff. 47	P.P. 5	9	14	19	24	28	33	38	42	
	Diff. 46	P.P. 5	9	14	18	23	28	32	37	41	

N.	0	1	2	3	4	5	6	7	8	9	D.
942	974051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	..28	..72	.117	.161	.206	.250	.294	
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

N.	0	1	2	3	4	5	6	7	8	9	D.
	Diff. 46	P. P. 5	9	14	18	23	28	32	37	41	
	45	5	9	14	18	23	27	32	36	41	
	44	4	9	13	18	22	26	31	35	40	
	43	4	9	13	17	22	26	30	34	39	

LOGARITHMIC

SINES AND TANGENTS

FOR EVERY DEGREE AND MINUTE OF THE QUADRANT.

N.B.—The minutes in the left-hand column of each page, increasing downward, belong to the degrees at the top; and those in the right-hand column, increasing upward, belong to the degrees below.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	—Infinite		10.000000		—Infinite		Infinite.	60
1	6.463726	501717	000000	00	6.463726	501717	13.536274	59
2	764756	293485	000000	00	764756	293485	235244	58
3	940847	208231	000000	00	940847	208231	059153	57
4	7.065786	161517	000000	00	7.065786	161517	12.934214	56
5	162696	131968	000000	00	162696	131969	837304	55
6	241877	111578	9.999999	01	241878	111578	758122	54
7	308824	96653	999999	01	308825	99653	691175	53
8	366816	85254	999999	01	366817	85254	633183	52
9	417968	76263	999999	01	417970	76263	582030	51
10	463726	68988	999998	01	463727	68988	536273	50
11	7.505118	62981	9.999998	01	7.505120	62981	12.494880	49
12	542906	57936	999997	01	542909	57937	457091	48
13	577668	53641	999997	01	577672	53642	422328	47
14	609853	49938	999996	01	609857	49939	390143	46
15	639816	46714	999996	01	639820	46715	360180	45
16	667845	43881	999995	01	667849	43882	332151	44
17	694173	41372	999995	01	694179	41373	305821	43
18	718997	39135	999994	01	719003	39136	280997	42
19	742478	37127	999993	01	742484	37128	257516	41
20	764754	35315	999993	01	764761	35136	235239	40
21	7.785943	33672	9.999992	01	7.785951	33673	12.214049	39
22	806146	32175	999991	01	806155	32176	193845	38
23	825451	30805	999990	01	825460	30806	174540	37
24	843934	29547	999989	02	843944	29549	156056	36
25	861662	28388	999989	02	861674	28390	138326	35
26	878695	27317	999988	02	878708	27318	121292	34
27	895085	26323	999987	02	895099	26325	104901	33
28	910879	25399	999986	02	910894	25401	089106	32
29	926119	24538	999985	02	926134	24540	073866	31
30	940842	23733	999983	02	940858	23735	059142	30
31	7.955082	22980	9.999982	02	7.955100	22981	12.044900	29
32	968870	22273	999981	02	968889	22275	031111	28
33	982233	21608	999980	02	982253	21610	017747	27
34	995198	20981	999979	02	995219	20983	004781	26
35	8.007787	20390	999977	02	8.007809	20392	11.992191	25
36	020021	19831	999976	02	020044	19833	979956	24
37	031919	19302	999975	02	031945	19305	968055	23
38	043501	18801	999973	02	043527	18803	956473	22
39	054781	18325	999972	02	054809	18327	945191	21
40	065776	17872	999971	02	065806	17874	934194	20
41	8.076500	17441	9.999969	02	8.076531	17444	11.923469	19
42	086965	17031	999968	02	086997	17034	913003	18
43	097183	16639	999966	02	097217	16642	902783	17
44	107167	16265	999964	03	107203	16268	892797	16
45	116926	15908	999963	03	116963	15910	883037	15
46	126471	15566	999961	03	126510	15568	873490	14
47	135810	15238	999959	03	135851	15241	864149	13
48	144953	14924	999958	03	144996	14927	855004	12
49	153907	14622	999956	03	153952	14625	846048	11
50	162681	14333	999954	03	162727	14336	837273	10
51	8.171280	14054	9.999952	03	8.171328	14057	11.828672	9
52	179713	13786	999950	03	179763	13790	820237	8
53	187985	13529	999948	03	188036	13532	811964	7
54	196102	13280	999946	03	196156	13284	803844	6
55	204070	13041	999944	03	204126	13044	795874	5
56	211895	12810	999942	04	211953	12814	788047	4
57	219581	12587	999940	04	219641	12590	780359	3
58	227134	12372	999938	04	227195	12376	772805	2
59	234557	12164	999936	04	234621	12168	765379	1
60	241855	11963	999934	04	241921	11967	758079	0
	Cosine.		Sine.		Cotang.		Tang.	M.

89 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	8.241855	11963	9.999934	04	8.241921	11967	11.758079	60
1	249033	11768	999932	04	249102	11772	750898	59
2	256094	11580	999929	04	256165	11584	743835	58
3	263042	11398	999927	04	263115	11402	736885	57
4	269881	11221	999925	04	269956	11225	730044	56
5	276614	11050	999922	04	276691	11054	723309	55
6	283243	10883	999920	04	283323	10887	716677	54
7	289773	10722	999918	04	289856	10726	710144	53
8	296207	10565	999915	04	296292	10570	703708	52
9	302546	10413	999913	04	302634	10418	697366	51
10	308794	10266	999910	04	308884	10270	691116	50
11	8.314954	10122	9.999907	04	8.315046	10126	11.684954	49
12	321027	9982	999905	04	321122	9987	678878	48
13	327016	9847	999902	04	327114	9851	672886	47
14	332924	9714	999899	05	333025	9719	666975	46
15	338753	9586	999897	05	338856	9590	661144	45
16	344504	9460	999894	05	344610	9465	655390	44
17	350181	9338	999891	05	350289	9343	649711	43
18	355783	9219	999888	05	355895	9224	644105	42
19	361315	9103	999885	05	361430	9108	638570	41
20	366777	8990	999882	05	366895	8995	633105	40
21	8.372171	8880	9.999879	05	8.372292	8885	11.627708	39
22	377499	8772	999876	05	377622	8777	622378	38
23	382762	8667	999873	05	382889	8672	617111	37
24	387962	8564	999870	05	388092	8570	611908	36
25	393101	8464	999867	05	393234	8470	606766	35
26	398179	8366	999864	05	398315	8371	601685	34
27	403199	8271	999861	05	403338	8276	596662	33
28	408161	8177	999858	05	408304	8182	591696	32
29	413068	8086	999854	05	413213	8091	586787	31
30	417919	7996	999851	06	418068	8002	581932	30
31	8.422717	7909	9.999848	06	8.422869	7914	11.577131	29
32	427462	7823	999844	06	427618	7829	572382	28
33	432156	7740	999841	06	432315	7745	567685	27
34	436800	7657	999838	06	436962	7663	563038	26
35	441394	7577	999834	06	441560	7583	558440	25
36	445941	7499	999831	06	446110	7505	553890	24
37	450440	7422	999827	06	450613	7428	549387	23
38	454893	7346	999824	06	455070	7352	544930	22
39	459301	7273	999820	06	459481	7279	540519	21
40	463665	7200	999816	06	463849	7206	536151	20
41	8.467985	7129	9.999813	06	8.468172	7135	11.531828	19
42	472263	7060	999809	06	472454	7066	527546	18
43	476498	6991	999805	06	476693	6998	523307	17
44	480693	6924	999801	06	480892	6931	519108	16
45	484848	6859	999797	06	485050	6865	514950	15
46	488963	6794	999794	07	489170	6801	510830	14
47	493040	6731	999790	07	493250	6738	506750	13
48	497078	6669	999786	07	497293	6676	502707	12
49	501080	6608	999782	07	501298	6615	498702	11
50	505045	6548	999778	07	505267	6555	494733	10
51	8.508974	6489	9.999774	07	8.509200	6496	11.490800	9
52	512867	6431	999769	07	513098	6439	486902	8
53	516726	6375	999765	07	516961	6382	483039	7
54	520551	6319	999761	07	520790	6326	479210	6
55	524343	6264	999757	07	524586	6272	475414	5
56	528102	6211	999753	07	528349	6218	471651	4
57	531828	6158	999748	07	532080	6165	467920	3
58	535523	6106	999744	07	535779	6113	464221	2
59	539186	6055	999740	07	539447	6062	460553	1
60	542819	6004	999735	07	543084	6012	456916	0
	Cosine.		Sine.		Cotang.		Tang.	M.

88 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	8.542819	6004	9.999735	07	8.543084	6012	11.456916	60
1	546422	5955	999731	07	546691	5962	453309	59
2	549995	5906	999726	08	550268	5914	449732	58
3	553539	5858	999722	08	553817	5866	446183	57
4	557054	5811	999717	08	557336	5819	442664	56
5	560540	5765	999713	08	560828	5773	439172	55
6	563999	5719	999708	08	564291	5727	435709	54
7	567431	5674	999704	08	567727	5682	432273	53
8	570836	5630	999699	08	571137	5638	428863	52
9	574214	5587	999694	08	574520	5595	425480	51
10	577566	5544	999689	08	577877	5552	422123	50
11	8.580892	5502	9.999685	08	8.581208	5510	11.418792	49
12	584193	5460	999680	08	584514	5468	415486	48
13	587469	5419	999675	08	587795	5427	412205	47
14	590721	5379	999670	08	591051	5387	408949	46
15	593948	5339	999665	08	594283	5347	405717	45
16	597152	5300	999660	08	597492	5308	402508	44
17	600332	5261	999655	08	600677	5270	399323	43
18	603489	5223	999650	08	603839	5232	396161	42
19	606623	5186	999645	09	606978	5194	393022	41
20	609734	5149	999640	09	610094	5158	389906	40
21	8.612823	5112	9.999635	09	8.613189	5121	11.386811	39
22	615891	5077	999629	09	616262	5085	383738	38
23	618937	5041	999624	09	619313	5050	380687	37
24	621962	5006	999619	09	622343	5015	377657	36
25	624965	4972	999614	09	625352	4981	374648	35
26	627948	4938	999608	09	628340	4947	371660	34
27	630911	4904	999603	09	631308	4913	368692	33
28	633854	4871	999597	09	634256	4880	365744	32
29	636776	4839	999592	09	637184	4848	362816	31
30	639680	4806	999586	09	640093	4816	359907	30
31	8.642563	4775	9.999581	09	8.642982	4784	11.357018	29
32	645428	4743	999575	09	645853	4753	354147	28
33	648274	4712	999570	09	648704	4722	351296	27
34	651102	4682	999564	09	651537	4691	348463	26
35	653911	4652	999558	10	654352	4661	345648	25
36	656702	4622	999553	10	657149	4631	342851	24
37	659475	4592	999547	10	659928	4602	340072	23
38	662230	4563	999541	10	662689	4573	337311	22
39	664968	4535	999535	10	665433	4544	334567	21
40	667689	4506	999529	10	668160	4516	331840	20
41	8.670393	4479	9.999524	10	8.670870	4488	11.329130	19
42	673080	4451	999518	10	673563	4461	326437	18
43	675751	4424	999512	10	676239	4434	323761	17
44	678405	4397	999506	10	678900	4407	321100	16
45	681043	4370	999500	10	681544	4380	318456	15
46	683665	4344	999493	10	684172	4354	315828	14
47	686272	4318	999487	10	686784	4328	313216	13
48	688863	4292	999481	10	689381	4303	310619	12
49	691438	4267	999475	10	691963	4277	308037	11
50	693998	4242	999469	10	694529	4252	305471	10
51	8.696543	4217	9.999463	11	8.697081	4228	11.302919	9
52	699073	4193	999456	11	699617	4203	300383	8
53	701589	4168	999450	11	702139	4179	297861	7
54	704090	4144	999443	11	704646	4155	295354	6
55	706577	4121	999437	11	707140	4132	292860	5
56	709049	4097	999431	11	709618	4108	290382	4
57	711507	4074	999424	11	712083	4085	287917	3
58	713952	4051	999418	11	714534	4062	285466	2
59	716383	4029	999411	11	716972	4040	283028	1
60	718800	4006	999404	11	719396	4017	280604	0
	Cosine.		Sine.		Cotang.		Tang.	M.

87 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	8.718800	4006	9.999404	11	8.719396	4017	11.280604	60
1	721204	3984	999398	11	721806	3995	278194	59
2	723595	3962	999391	11	724204	3974	275796	58
3	725972	3941	999384	11	726588	3952	273412	57
4	728337	3919	999378	11	728959	3931	271041	56
5	730688	3898	999371	11	731317	3910	268683	55
6	733027	3877	999364	11	733663	3889	266337	54
7	735354	3857	999357	11	735996	3868	264004	53
8	737667	3836	999350	12	738317	3848	261683	52
9	739969	3816	999343	12	740626	3827	259374	51
10	742259	3796	999336	12	742922	3807	257078	50
11	8.744536	3776	9.999329	12	8.745207	3788	11.254793	49
12	746802	3756	999322	12	747479	3768	252521	48
13	749055	3737	999315	12	749740	3749	250260	47
14	751297	3717	999308	12	751989	3729	248011	46
15	753528	3698	999301	12	754227	3710	245773	45
16	755747	3680	999294	12	756453	3692	243547	44
17	757955	3661	999287	12	758668	3673	241332	43
18	760151	3642	999279	12	760872	3655	239128	42
19	762337	3624	999272	12	763065	3636	236935	41
20	764511	3606	999265	12	765246	3618	234754	40
21	8.766675	3588	9.999257	12	8.767417	3600	11.232583	39
22	768828	3570	999250	12	769578	3583	230422	38
23	770970	3553	999242	12	771727	3565	228273	37
24	773101	3535	999235	13	773866	3548	226134	36
25	775223	3518	999227	13	775995	3531	224005	35
26	777333	3501	999220	13	778114	3514	221886	34
27	779434	3484	999212	13	780222	3497	219778	33
28	781524	3467	999205	13	782320	3480	217680	32
29	783605	3451	999197	13	784408	3464	215592	31
30	785675	3434	999189	13	786486	3447	213514	30
31	8.787736	3418	9.999181	13	8.788554	3431	11.211446	29
32	789787	3402	999174	13	790613	3415	209387	28
33	791828	3386	999166	13	792662	3399	207338	27
34	793859	3370	999158	13	794701	3383	205299	26
35	795881	3354	999150	13	796731	3368	203269	25
36	797894	3339	999142	13	798752	3352	201248	24
37	799897	3323	999134	13	800763	3337	199237	23
38	801892	3308	999126	13	802765	3322	197235	22
39	803876	3293	999118	13	804758	3307	195242	21
40	805852	3278	999110	14	806742	3292	193258	20
41	8.807819	3263	9.999102	14	8.808717	3277	11.191283	19
42	809777	3249	999094	14	810683	3262	189317	18
43	811726	3234	999086	14	812641	3248	187359	17
44	813667	3219	999077	14	814589	3233	185411	16
45	815599	3205	999069	14	816529	3219	183471	15
46	817522	3191	999061	14	818461	3205	181539	14
47	819436	3177	999053	14	820384	3191	179616	13
48	821343	3163	999044	14	822298	3177	177702	12
49	823240	3149	999036	14	824205	3163	175795	11
50	825130	3135	999027	14	826103	3150	173897	10
51	8.827011	3122	9.999019	14	8.827992	3136	11.172008	9
52	828884	3108	999010	14	829874	3123	170126	8
53	830749	3095	999002	14	831748	3109	168252	7
54	832607	3082	998993	14	833613	3096	166387	6
55	834456	3069	998984	14	835471	3083	164529	5
56	836297	3056	998976	15	837321	3070	162679	4
57	838130	3043	998967	15	839163	3057	160837	3
58	839956	3030	998958	15	840998	3045	159002	2
59	841774	3017	998950	15	842825	3032	157175	1
60	843585	3005	998941	15	844644	3019	155356	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	8.843585	3005	9.998941	15	8.844644	3019	11.155356	60
1	845387	2992	998932	15	846455	3007	153545	59
2	847183	2980	998923	15	848260	2995	151740	58
3	848971	2968	998914	15	850057	2983	149943	57
4	850751	2955	998905	15	851846	2970	148154	56
5	852525	2943	998896	15	853628	2958	146372	55
6	854291	2931	998887	15	855403	2946	144597	54
7	856049	2919	998878	15	857171	2935	142829	53
8	857801	2908	998869	15	858932	2923	141068	52
9	859546	2896	998860	15	860686	2911	139314	51
10	861283	2884	998851	15	862433	2900	137567	50
11	8.863014	2873	9.998841	15	8.864173	2888	11.135827	49
12	864738	2861	998832	15	865906	2877	134094	48
13	866455	2850	998823	16	867632	2866	132368	47
14	868165	2839	998813	16	869351	2854	130649	46
15	869868	2828	998804	16	871064	2843	128936	45
16	871565	2817	998795	16	872770	2832	127230	44
17	873255	2806	998785	16	874469	2821	125531	43
18	874938	2795	998776	16	876162	2811	123838	42
19	876615	2784	998766	16	877849	2800	122151	41
20	878285	2773	998757	16	879529	2789	120471	40
21	8.879949	2763	9.998747	16	8.881202	2779	11.118798	39
22	881607	2752	998738	16	882869	2768	117131	38
23	883258	2742	998728	16	884530	2758	115470	37
24	884903	2731	998718	16	886185	2747	113815	36
25	886542	2721	998708	16	887833	2737	112167	35
26	888174	2711	998699	16	889476	2727	110524	34
27	889801	2700	998689	16	891112	2717	108888	33
28	891421	2690	998679	16	892742	2707	107258	32
29	893035	2680	998669	17	894366	2697	105634	31
30	894643	2670	998659	17	895984	2687	104016	30
31	8.896246	2660	9.998649	17	8.897596	2677	11.102404	29
32	897842	2651	998639	17	899203	2667	100797	28
33	899432	2641	998629	17	900803	2658	099197	27
34	901017	2631	998619	17	902398	2648	097602	26
35	902596	2622	998609	17	903987	2639	096013	25
36	904169	2612	998599	17	905570	2629	094430	24
37	905736	2603	998589	17	907147	2620	092853	23
38	907297	2593	998578	17	908719	2610	091281	22
39	908853	2584	998568	17	910285	2601	089715	21
40	910404	2575	998558	17	911846	2592	088154	20
41	8.911949	2566	9.998548	17	8.913401	2583	11.086599	19
42	913488	2556	998537	17	914951	2574	085049	18
43	915022	2547	998527	17	916495	2565	083505	17
44	916550	2538	998516	17	918034	2556	081966	16
45	918073	2529	998506	18	919568	2547	080432	15
46	919591	2521	998495	18	921096	2538	078904	14
47	921103	2512	998485	18	922619	2529	077381	13
48	922610	2503	998474	18	924136	2521	075864	12
49	924112	2494	998464	18	925649	2512	074351	11
50	925609	2486	998453	18	927156	2504	072844	10
51	8.927100	2477	9.998442	18	8.928658	2495	11.071342	9
52	928587	2469	998431	18	930155	2487	069845	8
53	930068	2460	998421	18	931647	2478	068353	7
54	931544	2452	998410	18	933134	2470	066866	6
55	933015	2443	998399	18	934616	2462	065384	5
56	934481	2435	998388	18	936093	2453	063907	4
57	935942	2427	998377	18	937565	2445	062435	3
58	937398	2419	998366	18	939032	2437	060968	2
59	938850	2411	998355	18	940494	2429	059506	1
60	940296	2403	9 8344	18	941952	2421	058048	0
	Cosine.		Sine.		Cotang.		Tang.	M.

85 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	8.940296	2403	9.998344	18	8.941952	2421	11.058048	60
1	941738	2394	998333	19	943404	2413	056596	59
2	943174	2387	998322	19	944852	2405	055148	58
3	944606	2379	998311	19	946295	2397	053705	57
4	946034	2371	998300	19	947734	2390	052266	56
5	947456	2363	998289	19	949168	2382	050832	55
6	948874	2355	998277	19	950597	2374	049403	54
7	950287	2348	998266	19	952021	2367	047979	53
8	951696	2340	998255	19	953441	2359	046559	52
9	953100	2332	998243	19	954856	2351	045144	51
10	954499	2325	998232	19	956267	2344	043733	50
11	8.955894	2317	9.998220	19	8.957674	2336	11.042326	49
12	957284	2310	998209	19	959075	2329	040925	48
13	958670	2302	998197	19	960473	2322	039527	47
14	960052	2295	998186	19	961866	2314	038134	46
15	961429	2288	998174	19	963255	2307	036745	45
16	962801	2280	998163	19	964639	2300	035361	44
17	964170	2273	998151	19	966019	2293	033981	43
18	965534	2266	998139	20	967394	2286	032606	42
19	966893	2259	998128	20	968766	2279	031234	41
20	968249	2252	998116	20	970133	2271	029867	40
21	8.969600	2245	9.998104	20	8.971496	2265	11.028504	39
22	970947	2238	998092	20	972855	2257	027145	38
23	972289	2231	998080	20	974209	2251	025791	37
24	973628	2224	998068	20	975560	2244	024440	36
25	974962	2217	998056	20	976906	2237	023094	35
26	976293	2210	998044	20	978248	2230	021752	34
27	977619	2203	998032	20	979586	2224	020414	33
28	978941	2197	998020	20	980921	2217	019079	32
29	980259	2190	998008	20	982251	2210	017749	31
30	981573	2183	997996	20	983577	2204	016423	30
31	8.982883	2177	9.997984	20	8.984899	2197	11.015101	29
32	984189	2170	997972	20	986217	2191	013783	28
33	985491	2164	997959	20	987532	2184	012468	27
34	986789	2157	997947	21	988842	2178	011158	26
35	988083	2151	997935	21	990149	2171	009851	25
36	989374	2144	997922	21	991451	2165	008549	24
37	990660	2138	997910	21	992750	2159	007250	23
38	991943	2131	997897	21	994045	2152	005955	22
39	993222	2125	997885	21	995337	2146	004663	21
40	994497	2119	997872	21	996624	2140	003376	20
41	8.995768	2113	9.997860	21	8.997908	2134	11.002092	19
42	997036	2106	997847	21	999188	2127	000812	18
43	998299	2100	997835	21	9.000465	2121	10.999535	17
44	999560	2094	997822	21	001738	2115	998262	16
45	9.000816	2088	997809	21	003007	2109	996993	15
46	002069	2082	997797	21	004272	2103	995728	14
47	003318	2076	997784	21	005534	2097	994466	13
48	004563	2070	997771	21	006792	2091	993208	12
49	005805	2064	997758	21	008047	2085	991953	11
50	007044	2058	997745	21	009298	2080	990702	10
51	9.008278	2052	9.997732	22	9.010546	2074	10.989454	9
52	009510	2046	997719	22	011790	2068	988210	8
53	010737	2040	997706	22	013031	2062	986969	7
54	011962	2035	997693	22	014268	2056	985732	6
55	013182	2029	997680	22	015502	2051	984498	5
56	014400	2023	997667	22	016732	2045	983268	4
57	015613	2017	997654	22	017959	2039	982041	3
58	016824	2012	997641	22	019183	2034	980817	2
59	018031	2006	997628	22	020403	2028	979597	1
60	019235	2001	997614	22	021620	2023	978380	0
	Cosine.		Sine.		Cotang.		Tang.	M.

84 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.019235	2001	9.997614	22	9.021620	2023	10.978380	60
1	020435	1995	997601	22	022834	2018	977166	59
2	021632	1990	997588	22	024044	2012	975956	58
3	022825	1984	997574	22	025251	2007	974749	57
4	024016	1979	997561	22	026455	2001	973545	56
5	025203	1973	997547	22	027655	1996	972345	55
6	026386	1968	997534	23	028852	1990	971148	54
7	027567	1962	997520	23	030046	1985	969954	53
8	028744	1957	997507	23	031237	1980	968763	52
9	029918	1952	997493	23	032425	1974	967575	51
10	031089	1947	997480	23	033609	1969	966391	50
11	9.032257	1941	9.997466	23	9.034791	1964	10.965209	49
12	033421	1936	997452	23	035969	1959	964031	48
13	034582	1931	997439	23	037144	1954	962856	47
14	035741	1926	997425	23	038316	1949	961684	46
15	036896	1920	997411	23	039485	1943	960515	45
16	038048	1915	997397	23	040651	1938	959349	44
17	039197	1910	997383	23	041813	1933	958187	43
18	040342	1905	997369	23	042973	1928	957027	42
19	041485	1900	997355	23	044130	1923	955870	41
20	042625	1895	997341	23	045284	1918	954716	40
21	9.043762	1890	9.997327	23	9.046434	1913	10.953566	39
22	044895	1885	997313	24	047582	1908	952418	38
23	046026	1880	997299	24	048727	1904	951273	37
24	047154	1875	997285	24	049869	1899	950131	36
25	048279	1870	997271	24	051008	1894	948992	35
26	049400	1865	997257	24	052144	1889	947856	34
27	050519	1860	997242	24	053277	1884	946723	33
28	051635	1856	997228	24	054407	1879	945593	32
29	052749	1851	997214	24	055535	1875	944465	31
30	053859	1846	997199	24	056659	1870	943341	30
31	9.054966	1841	9.997185	24	9.057781	1865	10.942219	29
32	056071	1836	997170	24	058900	1861	941100	28
33	057172	1832	997156	24	060016	1856	939984	27
34	058271	1827	997141	24	061130	1851	938870	26
35	059367	1822	997127	24	062240	1847	937760	25
36	060460	1818	997112	24	063348	1842	936652	24
37	061551	1813	997098	24	064453	1838	935547	23
38	062639	1809	997083	25	065556	1833	934444	22
39	063724	1804	997068	25	066655	1829	933345	21
40	064806	1799	997053	25	067752	1824	932248	20
41	9.065885	1795	9.997039	25	9.068846	1820	10.931154	19
42	066962	1790	997024	25	069938	1815	930062	18
43	068036	1786	997009	25	071027	1811	928973	17
44	069107	1781	996994	25	072113	1806	927887	16
45	070176	1777	996979	25	073197	1802	926803	15
46	071242	1773	996964	25	074278	1798	925722	14
47	072306	1768	996949	25	075356	1793	924644	13
48	073366	1764	996934	25	076432	1789	923568	12
49	074424	1760	996919	25	077505	1785	922495	11
50	075480	1755	996904	25	078576	1780	921424	10
51	9.076533	1751	9.996889	25	9.079644	1776	10.920356	9
52	077583	1747	996874	25	080710	1772	919290	8
53	078631	1742	996858	25	081773	1768	918227	7
54	079676	1738	996843	26	082833	1764	917167	6
55	080719	1734	996828	26	083891	1759	916109	5
56	081759	1730	996812	26	084947	1755	915053	4
57	082797	1725	996797	26	086000	1751	914000	3
58	083832	1721	996782	26	087050	1747	912950	2
59	084864	1717	996766	26	088098	1743	911902	1
60	085894	1713	996751	26	089144	1739	910856	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.085894	1713	9.996751	26	9.089144	1739	10.910856	60
1	086922	1709	996735	26	090187	1735	909813	59
2	087947	1705	996720	26	091228	1731	908772	58
3	088970	1701	996704	26	092266	1727	907734	57
4	089990	1697	996688	26	093302	1723	906698	56
5	091008	1693	996673	26	094336	1719	905664	55
6	092024	1689	996657	26	095367	1715	904633	54
7	093037	1685	996641	26	096395	1711	903605	53
8	094047	1681	996625	26	097422	1707	902578	52
9	095056	1677	996610	26	098446	1703	901554	51
10	096062	1673	996594	26	099468	1699	900532	50
11	9.097065	1669	9.996578	27	9.100487	1695	10.899513	49
12	098066	1665	996562	27	101504	1692	898496	48
13	099065	1661	996546	27	102519	1688	897481	47
14	100062	1657	996530	27	103532	1684	896468	46
15	101056	1653	996514	27	104542	1680	895458	45
16	102048	1650	996498	27	105550	1676	894450	44
17	103037	1646	999482	27	106556	1673	893444	43
18	104025	1642	999465	27	107559	1669	892441	42
19	105010	1638	996449	27	108560	1665	891440	41
20	105992	1634	996433	27	109559	1662	890441	40
21	9.106973	1631	9.996417	27	9.110556	1658	10.889444	39
22	107951	1627	996400	27	111551	1654	888449	38
23	108927	1623	996384	27	112543	1651	887457	37
24	109901	1620	996368	27	113533	1647	886467	36
25	110873	1616	996351	27	114521	1643	885479	35
26	111842	1612	996335	27	115507	1640	884493	34
27	112809	1609	996318	28	116491	1636	883509	33
28	113774	1605	996302	28	117472	1633	882528	32
29	114737	1601	996285	28	118452	1629	881548	31
30	115698	1598	996269	28	119429	1625	880571	30
31	9.116656	1594	9.996252	28	9.120404	1622	10.879596	29
32	117613	1591	996235	28	121377	1618	878623	28
33	118567	1587	996219	28	122348	1615	877652	27
34	119519	1584	996202	28	123317	1612	876683	26
35	120469	1580	996185	28	124284	1608	875716	25
36	121417	1577	996168	28	125249	1605	874751	24
37	122362	1573	996151	28	126211	1601	873789	23
38	123306	1570	996134	28	127172	1598	872828	22
39	124248	1566	996117	28	128130	1594	871870	21
40	125187	1563	996100	28	129087	1591	870913	20
41	9.126125	1559	9.996083	28	9.130041	1588	10.869959	19
42	127060	1556	996066	28	130994	1584	869006	18
43	127993	1552	999049	29	131944	1581	868056	17
44	128925	1549	996032	29	132893	1578	867107	16
45	129854	1546	996015	29	133839	1574	866161	15
46	130781	1542	995998	29	134784	1571	865216	14
47	131706	1539	995980	29	135726	1568	864274	13
48	132630	1536	995963	29	136667	1564	863333	12
49	133551	1532	995946	29	137605	1561	862395	11
50	134470	1529	995928	29	138542	1558	861458	10
51	9.135387	1526	9.995911	29	9.139476	1555	10.860524	9
52	136303	1522	995894	29	140409	1552	859591	8
53	137216	1519	995876	29	141340	1548	858660	7
54	138128	1516	995859	29	142269	1545	857731	6
55	139037	1513	995841	29	143196	1542	856804	5
56	139944	1510	995823	29	144121	1539	855879	4
57	140850	1506	995806	29	145044	1536	854956	3
58	141754	1503	995788	29	145966	1533	854034	2
59	142655	1500	995771	30	146885	1530	853115	1
60	143555	1497	995753	30	147803	1526	852197	0
	Cosine.		Sine.		Cotang.		Tang.	M.

82 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.143555	1497	9.995753	30	9.147803	1526	10.852197	60
1	144453	1494	995735	30	148718	1523	851282	59
2	145349	1491	995717	30	149632	1520	850368	58
3	146243	1487	995699	30	150544	1517	849456	57
4	147136	1484	995681	30	151454	1514	848546	56
5	148026	1481	995664	30	152363	1511	847637	55
6	148915	1478	995646	30	153269	1508	846731	54
7	149802	1475	995628	30	154174	1505	845826	53
8	150686	1472	995610	30	155077	1502	844923	52
9	151569	1469	995591	30	155978	1499	844022	51
10	152451	1466	995573	30	156877	1496	843123	50
11	9.153330	1463	9.995555	30	9.157775	1493	10.842225	49
12	154208	1460	995537	30	158671	1490	841329	48
13	155083	1457	995519	30	159565	1487	840435	47
14	155957	1454	995501	30	160457	1484	839543	46
15	156830	1451	995482	31	161347	1481	838653	45
16	157700	1448	995464	31	162236	1479	837764	44
17	158569	1445	995446	31	163123	1476	836877	43
18	159435	1442	995427	31	164008	1473	835992	42
19	160301	1439	995409	31	164892	1470	835108	41
20	161164	1436	995390	31	165774	1467	834226	40
21	9.162025	1433	9.995372	31	9.166654	1464	10.833346	39
22	162885	1430	995353	31	167532	1461	832468	38
23	163743	1427	995334	31	168409	1459	831591	37
24	164600	1425	995316	31	169284	1456	830716	36
25	165454	1422	995297	31	170157	1453	829843	35
26	166307	1419	995278	31	171029	1450	828971	34
27	167159	1416	995260	31	171899	1447	828101	33
28	168008	1413	995241	31	172767	1445	827233	32
29	168856	1410	995222	31	173634	1442	826366	31
30	169702	1408	995203	32	174499	1439	825501	30
31	9.170547	1405	9.995184	32	9.175362	1436	10.824638	29
32	171389	1402	995165	32	176224	1434	823776	28
33	172230	1399	995146	32	177084	1431	822916	27
34	173070	1397	995127	32	177942	1428	822058	26
35	173908	1394	995108	32	178799	1426	821201	25
36	174744	1391	995089	32	179655	1423	820345	24
37	175578	1388	995070	32	180508	1420	819492	23
38	176411	1386	995051	32	181360	1418	818640	22
39	177242	1383	995032	32	182211	1415	817789	21
40	178072	1380	995013	32	183059	1412	816941	20
41	9.178900	1377	9.994993	32	9.183907	1410	10.816093	19
42	179726	1375	994974	32	184752	1407	815248	18
43	180551	1372	994955	32	185597	1404	814403	17
44	181374	1369	994935	32	186439	1402	813561	16
45	182196	1367	994916	32	187280	1399	812720	15
46	183016	1364	994896	33	188120	1397	811880	14
47	183834	1362	994877	33	188958	1394	811042	13
48	184651	1359	994857	33	189794	1392	810206	12
49	185466	1356	994838	33	190629	1389	809371	11
50	186280	1354	994818	33	191462	1386	808538	10
51	9.187092	1351	9.994798	33	9.192294	1384	10.807706	9
52	187903	1349	994779	33	193124	1381	806876	8
53	188712	1346	994759	33	193953	1379	806047	7
54	189519	1343	994739	33	194780	1376	805220	6
55	190325	1341	994720	33	195606	1374	804394	5
56	191130	1338	994700	33	196430	1371	803570	4
57	191933	1336	994680	33	197253	1369	802747	3
58	192734	1333	994660	33	198074	1367	801926	2
59	193534	1331	994640	33	198894	1364	801106	1
60	194332	1328	994620	33	199713	1362	800287	0
	Cosine.		Sine.		Cotang.		Tang.	M.

81 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.194332	1328	9.994620	33	9.199713	1362	10.800287	60
1	195129	1326	994600	33	200529	1359	799471	59
2	195925	1323	994580	33	201345	1357	798655	58
3	196719	1321	994560	34	202159	1354	797841	57
4	197511	1318	994540	34	202971	1352	797029	56
5	198302	1316	994519	34	203782	1350	796218	55
6	199091	1313	994499	34	204592	1347	795408	54
7	199879	1311	994479	34	205400	1345	794600	53
8	200666	1309	994459	34	206207	1342	793793	52
9	201451	1306	994438	34	207013	1340	792987	51
10	202234	1304	994418	34	207817	1338	792183	50
11	9.203017	1301	9.994398	34	9.208619	1335	10.791381	49
12	203797	1299	994377	34	209420	1333	790580	48
13	204577	1297	994357	34	210220	1331	789780	47
14	205354	1294	994336	34	211018	1328	788982	46
15	206131	1292	994316	34	211815	1326	788185	45
16	206908	1289	994295	34	212611	1324	787389	44
17	207679	1287	994274	34	213405	1322	786595	43
18	208452	1285	994254	35	214198	1319	785802	42
19	209222	1282	994233	35	214989	1317	785011	41
20	209992	1280	994212	35	215780	1315	784220	40
21	9.210760	1278	9.994191	35	9.216568	1312	10.783432	39
22	211526	1275	994171	35	217356	1310	782644	38
23	212291	1273	994150	35	218142	1308	781858	37
24	213055	1271	994129	35	218926	1306	781074	36
25	213818	1269	994108	35	219710	1303	780290	35
26	214579	1266	994087	35	220492	1301	779508	34
27	215338	1264	994066	35	221272	1299	778728	33
28	216097	1262	994045	35	222052	1297	777948	32
29	216854	1259	994024	35	222830	1295	777170	31
30	217609	1257	994003	35	223607	1292	776393	30
31	9.218363	1255	9.993982	35	9.224382	1290	10.775618	29
32	219116	1253	993960	35	225156	1288	774844	28
33	219868	1251	993939	35	225929	1286	774071	27
34	220618	1248	993918	35	226700	1284	773300	26
35	221367	1246	993897	36	227471	1282	772529	25
36	222115	1244	993875	36	228239	1280	771761	24
37	222861	1242	993854	36	229007	1277	770993	23
38	223606	1240	993832	36	229773	1275	770227	22
39	224349	1237	993811	36	230539	1273	769461	21
40	225092	1235	993789	36	231302	1271	768698	20
41	9.225833	1233	9.993768	36	9.232065	1269	10.767935	19
42	226573	1231	993746	36	232826	1267	767174	18
43	227311	1229	993725	36	233586	1265	766414	17
44	228048	1227	993703	36	234345	1263	765655	16
45	228784	1224	993681	36	235103	1261	764897	15
46	229518	1222	993660	36	235859	1259	764141	14
47	230252	1220	993638	36	236614	1256	763386	13
48	230984	1218	993616	36	237368	1254	762632	12
49	231715	1216	993594	36	238120	1252	761880	11
50	232444	1214	993572	37	238872	1250	761128	10
51	9.233172	1212	9.993550	37	9.239622	1248	10.760378	9
52	233899	1210	993528	37	240371	1246	759629	8
53	234625	1208	993506	37	241118	1244	758882	7
54	235349	1205	993484	37	241865	1242	758135	6
55	236073	1203	993462	37	242610	1240	757390	5
56	236795	1201	993440	37	243354	1238	756646	4
57	237515	1199	993418	37	244097	1236	755903	3
58	238235	1197	993396	37	244839	1234	755161	2
59	238953	1195	993374	37	245579	1232	754421	1
60	239670	1193	993351	37	246319	1230	753681	0
	Cosine.		Sine.		Cotang.		Tang.	M.

80 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.239670	1193	9.993351	37	9.246319	1230	10.753681	60
1	240386	1191	993329	37	247057	1228	752943	59
2	241101	1189	993307	37	247794	1226	752206	58
3	241814	1187	993284	37	248530	1224	751470	57
4	242526	1185	993262	37	249264	1223	750736	56
5	243237	1183	993240	37	249998	1221	750002	55
6	243947	1181	993217	38	250730	1219	749270	54
7	244656	1179	993195	38	251461	1217	748539	53
8	245363	1177	993172	38	252191	1215	747809	52
9	246069	1175	993149	38	252920	1213	747080	51
10	246775	1173	993127	38	253648	1211	746352	50
11	9.247478	1171	9.993104	38	9.254374	1209	10.745626	49
12	248181	1169	993081	38	255100	1207	744900	48
13	248883	1167	993059	38	255824	1205	744176	47
14	249583	1165	993036	38	256547	1203	743453	46
15	250282	1164	993013	38	257269	1202	742731	45
16	250980	1162	992990	38	257990	1200	742010	44
17	251677	1160	992967	38	258710	1198	741290	43
18	252373	1158	992944	38	259429	1196	740571	42
19	253067	1156	992921	38	260146	1194	739854	41
20	253761	1154	992898	38	260863	1192	739137	40
21	9.254453	1152	9.992875	38	9.261578	1191	10.738422	39
22	255144	1150	992852	39	262292	1189	737708	38
23	255834	1148	992829	39	263005	1187	736995	37
24	256523	1146	992806	39	263717	1185	736283	36
25	257211	1145	992783	39	264428	1183	735572	35
26	257898	1143	992759	39	265138	1181	734862	34
27	258583	1141	992736	39	265847	1180	734153	33
28	259268	1139	992713	39	266555	1178	733445	32
29	259951	1137	992690	39	267261	1176	732739	31
30	260633	1135	992666	39	267967	1174	732033	30
31	9.261314	1133	9.992643	39	9.268671	1172	10.731329	29
32	261994	1132	992619	39	269375	1171	730625	28
33	262673	1130	992596	39	270077	1169	729923	27
34	263351	1128	992572	39	270779	1167	729221	26
35	264027	1126	992549	39	271479	1165	728521	25
36	264703	1124	992525	39	272178	1164	727822	24
37	265377	1122	992501	39	272876	1162	727124	23
38	266051	1121	992478	40	273573	1160	726427	22
39	266723	1119	992454	40	274269	1158	725731	21
40	267395	1117	992430	40	274964	1157	725036	20
41	9.268065	1115	9.992406	40	9.275658	1155	10.724342	19
42	268734	1114	992382	40	276351	1153	723649	18
43	269402	1112	992359	40	277043	1152	722957	17
44	270069	1110	992335	40	277734	1150	722266	16
45	270735	1108	992311	40	278424	1148	721576	15
46	271400	1106	992287	40	279113	1147	720887	14
47	272064	1105	992263	40	279801	1145	720199	13
48	272726	1103	992239	40	280488	1143	719512	12
49	273388	1101	992214	40	281174	1141	718826	11
50	274049	1100	992190	40	281858	1140	718142	10
51	9.274708	1098	9.992166	40	9.282542	1138	10.717458	9
52	275367	1096	992142	40	283225	1136	716775	8
53	276025	1094	992118	41	283907	1135	716093	7
54	276681	1093	992093	41	284588	1133	715412	6
55	277337	1091	992069	41	285268	1132	714732	5
56	277991	1089	992044	41	285947	1130	714053	4
57	278645	1088	992020	41	286624	1128	713376	3
58	279297	1086	991996	41	287301	1127	712699	2
59	279948	1084	991971	41	287977	1125	712023	1
60	280599	1082	991947	41	288652	1123	711348	0
	Cosine.		Sine.		Cotang.		Tang.	M.

79 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.280599	1082	9.991947	41	9.288652	1123	10.711348	60
1	281248	1081	991922	41	289326	1122	710674	59
2	281897	1079	991897	41	289999	1120	710001	58
3	282544	1077	991873	41	290671	1119	709329	57
4	283190	1076	991848	41	291342	1117	708658	56
5	283836	1074	991823	41	292013	1115	707987	55
6	284480	1072	991799	41	292682	1114	707318	54
7	285124	1071	991774	41	293350	1112	706650	53
8	285766	1069	991749	41	294017	1111	705983	52
9	286408	1068	991724	42	294684	1109	705316	51
10	287048	1066	991699	42	295349	1107	704651	50
11	9.287688	1064	9.991674	42	9.296013	1106	10.703987	49
12	288326	1063	991649	42	296677	1104	703323	48
13	288964	1061	991624	42	297339	1103	702661	47
14	289600	1059	991599	42	298001	1101	701999	46
15	290236	1058	991574	42	298662	1100	701338	45
16	290870	1056	991549	42	299322	1098	700678	44
17	291504	1055	991524	42	299980	1097	700020	43
18	292137	1053	991498	42	300638	1095	699362	42
19	292768	1051	991473	42	301295	1094	698705	41
20	293399	1050	991448	42	301951	1092	698049	40
21	9.294029	1048	9.991422	42	9.302607	1091	10.697393	39
22	294658	1047	991397	42	303261	1089	696739	38
23	295286	1045	991372	42	303914	1088	696086	37
24	295913	1044	991346	43	304567	1086	695433	36
25	296539	1042	991321	43	305218	1084	694782	35
26	297164	1040	991295	43	305869	1083	694131	34
27	297788	1039	991270	43	306519	1082	693481	33
28	298412	1037	991244	43	307168	1080	692832	32
29	299034	1036	991218	43	307816	1079	692184	31
30	299655	1034	991193	43	308463	1077	691537	30
31	9.300276	1033	9.991167	43	9.309109	1076	10.690891	29
32	300895	1031	991141	43	309754	1074	690246	28
33	301514	1030	991115	43	310399	1073	689601	27
34	302132	1028	991090	43	311042	1071	688958	26
35	302748	1027	991064	43	311685	1070	688315	25
36	303364	1025	991038	43	312327	1068	687673	24
37	303979	1024	991012	43	312968	1067	687032	23
38	304593	1022	990986	43	313608	1065	686392	22
39	305207	1021	990960	43	314247	1064	685753	21
40	305819	1019	990934	44	314885	1063	685115	20
41	9.306430	1018	9.990908	44	9.315523	1061	10.684477	19
42	307041	1016	990882	44	316159	1060	683841	18
43	307650	1015	990855	44	316795	1058	683205	17
44	308259	1013	990829	44	317430	1057	682570	16
45	308867	1012	990803	44	318064	1055	681936	15
46	309474	1010	990777	44	318697	1054	681303	14
47	310080	1009	990750	44	319330	1053	680670	13
48	310685	1007	990724	44	319961	1051	680039	12
49	311289	1006	990697	44	320592	1050	679408	11
50	311893	1004	990671	44	321222	1048	678778	10
51	9.312495	1003	9.990645	44	9.321851	1047	10.678149	9
52	313097	1001	990618	44	322479	1046	677521	8
53	313698	1000	990591	44	323106	1044	676894	7
54	314297	998	990565	44	323733	1043	676267	6
55	314897	997	990538	44	324358	1042	675642	5
56	315495	996	990511	45	324983	1040	675017	4
57	316092	994	990485	45	325607	1039	674393	3
58	316689	993	990458	45	326231	1037	673769	2
59	317284	991	990431	45	326853	1036	673147	1
60	317879	990	990404	45	327475	1035	672525	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100".	Cosine.	D.	Tang.	D.100".	Cotang.	
0	9.317879	990	9.990404	45	9.327475	1035	10.672525	60
1	318473	989	990378	45	328095	1033	671905	59
2	319066	987	990351	45	328715	1032	671285	58
3	319658	986	990324	45	329334	1031	670666	57
4	320249	984	990297	45	329953	1029	670047	56
5	320840	983	990270	45	330570	1028	669430	55
6	321430	982	990243	45	331187	1027	668813	54
7	322019	980	990215	45	331803	1025	668197	53
8	322607	979	990188	45	332418	1024	667582	52
9	323194	977	990161	45	333033	1023	666967	51
10	323780	976	990134	45	333646	1021	666354	50
11	9.324366	975	9.990107	46	9.334259	1020	10.665741	49
12	324950	973	990079	46	334871	1019	665129	48
13	325534	972	990052	46	335482	1017	664518	47
14	326117	970	990025	46	336093	1016	663907	46
15	326700	969	989997	46	336702	1015	663298	45
16	327281	968	989970	46	337311	1014	662689	44
17	327862	966	989942	46	337919	1012	662081	43
18	328442	965	989915	46	338527	1011	661473	42
19	329021	964	989887	46	339133	1010	660867	41
20	329599	962	989860	46	339739	1008	660261	40
21	9.330176	961	9.989832	46	9.340344	1007	10.659656	39
22	330753	960	989804	46	340948	1006	659052	38
23	331329	958	989777	46	341552	1005	658448	37
24	331903	957	989749	46	342155	1003	657845	36
25	332478	956	989721	46	342757	1002	657243	35
26	333051	954	989693	46	343358	1001	656642	34
27	333624	953	989665	47	343958	1000	656042	33
28	334195	952	989637	47	344558	998	655442	32
29	334767	950	989610	47	345157	997	654843	31
30	335337	949	989582	47	345755	996	654245	30
31	9.335906	948	9.989553	47	9.346353	995	10.653647	29
32	336475	947	989525	47	346949	993	653051	28
33	337043	945	989497	47	347545	992	652455	27
34	337610	944	989469	47	348141	991	651859	26
35	338176	943	989441	47	348735	990	651265	25
36	338742	941	989413	47	349329	988	650671	24
37	339307	940	989385	47	349922	987	650078	23
38	339871	939	989356	47	350514	986	649486	22
39	340434	938	989328	47	351106	985	648894	21
40	340996	936	989300	47	351697	984	648303	20
41	9.341558	935	9.989271	47	9.352287	982	10.647713	19
42	342119	934	989243	47	352876	981	647124	18
43	342679	932	989214	48	353465	980	646535	17
44	343239	931	989186	48	354053	979	645947	16
45	343797	930	989157	48	354640	978	645360	15
46	344355	929	989128	48	355227	976	644773	14
47	344912	927	989100	48	355813	975	644187	13
48	345469	926	989071	48	356398	974	643602	12
49	346024	925	989042	48	356982	973	643018	11
50	346579	924	989014	48	357566	972	642434	10
51	9.347134	922	9.988985	48	9.358149	971	10.641851	9
52	347687	921	988956	48	358731	969	641269	8
53	348240	920	988927	48	359313	968	640687	7
54	348792	919	988898	48	359893	967	640107	6
55	349343	918	988869	48	360474	966	639526	5
56	349893	916	988840	48	361053	965	638947	4
57	350443	915	988811	48	361632	964	638368	3
58	350992	914	988782	49	362210	962	637790	2
59	351540	913	988753	49	362787	961	637213	1
60	352088	911	988724	49	363364	960	636636	0
	Cosine.		Sine.		Cotang.		Tang.	M.

77 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.352088	911	9.988724	49	9.363364	960	10.636636	60
1	352635	910	988695	49	363940	959	636060	59
2	353181	909	988666	49	364515	958	635485	58
3	353726	908	988636	49	365090	957	634910	57
4	354271	907	988607	49	365664	956	634336	56
5	354815	905	988578	49	366237	954	633763	55
6	355358	904	988548	49	366810	953	633190	54
7	355901	903	988519	49	367382	952	632618	53
8	356443	902	988489	49	367953	951	632047	52
9	356984	901	988460	49	368524	950	631476	51
10	357524	900	988430	49	369094	949	630906	50
11	9.358064	898	9.988401	49	9.369663	948	10.630337	49
12	358603	897	988371	49	370232	947	629768	48
13	359141	896	988342	49	370799	945	629201	47
14	359678	895	988312	50	371367	944	628633	46
15	360215	894	988282	50	371933	943	628067	45
16	360752	892	988252	50	372499	942	627501	44
17	361287	891	988223	50	373064	941	626936	43
18	361822	890	988193	50	373629	940	626371	42
19	362356	889	988163	50	374193	939	625807	41
20	362889	888	988133	50	374756	938	625244	40
21	9.363422	887	9.988103	50	9.375319	937	10.624681	39
22	363954	886	988073	50	375881	936	624119	38
23	364485	884	988043	50	376442	935	623558	37
24	365016	883	988013	50	377003	933	622997	36
25	365546	882	987983	50	377563	932	622437	35
26	366075	881	987953	50	378122	931	621878	34
27	366604	880	987922	50	378681	930	621319	33
28	367131	879	987892	50	379239	929	620761	32
29	367659	878	987862	50	379797	928	620203	31
30	368185	876	987832	51	380354	927	619646	30
31	9.368711	875	9.987801	51	9.380910	926	10.619090	29
32	369236	874	987771	51	381466	925	618534	28
33	369761	873	987740	51	382020	924	617980	27
34	370285	872	987710	51	382575	923	617425	26
35	370808	871	987679	51	383129	922	616871	25
36	371330	870	987649	51	383682	921	616318	24
37	371852	869	987618	51	384234	920	615766	23
38	372373	868	987588	51	384786	919	615214	22
39	372894	866	987557	51	385337	918	614663	21
40	373414	865	987526	51	385888	917	614112	20
41	9.373933	864	9.987496	51	9.386438	916	10.613562	19
42	374452	863	987465	51	386987	915	613013	18
43	374970	862	987434	51	387536	914	612464	17
44	375487	861	987403	51	388084	913	611916	16
45	376003	860	987372	52	388631	912	611369	15
46	376519	859	987341	52	389178	910	610822	14
47	377035	858	987310	52	389724	909	610276	13
48	377549	857	987279	52	390270	908	609730	12
49	378063	856	987248	52	390815	907	609185	11
50	378577	855	987217	52	391360	906	608640	10
51	9.379089	854	9.987186	52	9.391903	905	10.608097	9
52	379601	852	987155	52	392447	904	607553	8
53	380113	851	987124	52	392989	903	607011	7
54	380624	850	987092	52	393531	902	606469	6
55	381134	849	987061	52	394073	901	605927	5
56	381643	848	987030	52	394614	900	605386	4
57	382152	847	986998	52	395154	899	604846	3
58	382661	846	986967	52	395694	898	604306	2
59	383168	845	986936	52	396233	897	603767	1
60	383675	844	986904	52	396771	897	603229	0
	Cosine.		Sine.		Cotang.		Tang.	M.

76 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.383675	844	9.986904	52	9.396771	897	10.603229	60
1	384182	843	986873	53	397309	896	602691	59
2	384687	842	986841	53	397846	895	602154	58
3	385192	841	986809	53	398383	894	601617	57
4	385697	840	986778	53	398919	893	601081	56
5	386201	839	986746	53	399455	892	600545	55
6	386704	838	986714	53	399990	891	600010	54
7	387207	837	986683	53	400524	890	599476	53
8	387709	836	986651	53	401058	889	598942	52
9	388210	835	986619	53	401591	888	598409	51
10	388711	834	986587	53	402124	887	597876	50
11	9.389211	833	9.986555	53	9.402656	886	10.597344	49
12	389711	832	986523	53	403187	885	596813	48
13	390210	831	986491	53	403718	884	596282	47
14	390708	830	986459	53	404249	883	595751	46
15	391206	829	986427	53	404778	882	595222	45
16	391703	828	986395	54	405308	881	594692	44
17	392199	827	986363	54	405836	880	594164	43
18	392695	826	986331	54	406364	879	593636	42
19	393191	825	986299	54	406892	878	593108	41
20	393685	824	986266	54	407419	877	592581	40
21	9.394179	823	9.986234	54	9.407945	876	10.592055	39
22	394673	822	986202	54	408471	876	591529	38
23	395166	821	986169	54	408996	875	591004	37
24	395658	820	986137	54	409521	874	590479	36
25	396150	819	986104	54	410045	873	589955	35
26	396641	818	986072	54	410569	872	589431	34
27	397132	817	986039	54	411092	871	588908	33
28	397621	816	986007	54	411615	870	588385	32
29	398111	815	985974	54	412137	869	587863	31
30	398600	814	985942	54	412658	863	587342	30
31	9.399088	813	9.985909	55	9.413179	867	10.586821	29
32	399575	812	985876	55	413699	866	586301	28
33	400062	811	985843	55	414219	865	585781	27
34	400549	810	985811	55	414738	865	585262	26
35	401035	809	985778	55	415257	864	584743	25
36	401520	808	985745	55	415775	863	584225	24
37	402005	807	985712	55	416293	862	583707	23
38	402489	806	985679	55	416810	861	583190	22
39	402972	805	985646	55	417326	860	582674	21
40	403455	804	985613	55	417842	859	582158	20
41	9.403938	803	9.985580	55	9.418358	858	10.581642	19
42	404420	802	985547	55	418873	857	581127	18
43	404901	801	985514	55	419387	857	580613	17
44	405382	800	985480	55	419901	856	580099	16
45	405862	799	985447	55	420415	855	579585	15
46	406341	798	985414	56	420927	854	579073	14
47	406820	797	985381	56	421440	853	578560	13
48	407299	796	985347	56	421952	852	578048	12
49	407777	795	985314	56	422463	851	577537	11
50	408254	795	985280	56	422974	850	577026	10
51	9.408731	794	9.985247	56	9.423484	850	10.576516	9
52	409207	793	985213	56	423993	849	576007	8
53	409682	792	985180	56	424503	848	575497	7
54	410157	791	985146	56	425011	847	574989	6
55	410632	790	985113	56	425519	846	574481	5
56	411106	789	985079	56	426027	845	573973	4
57	411579	788	985045	56	426534	844	573466	3
58	412052	787	985011	56	427041	844	572959	2
59	412524	786	984978	56	427547	843	572453	1
60	412996	785	984944	56	428052	842	571948	0
	Cosine.		Sine.		Cotang.		Tang.	M.

75 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.412996	785	9.984944	57	9.428052	842	10.571948	60
1	413467	784	984910	57	428558	841	571442	59
2	413938	784	984876	57	429062	840	570938	58
3	414408	783	984842	57	429566	839	570434	57
4	414878	782	984808	57	430070	838	569930	56
5	415347	781	984774	57	430573	838	569427	55
6	415815	780	984740	57	431075	837	568925	54
7	416283	779	984706	57	431577	836	568423	53
8	416751	778	984672	57	432079	835	567921	52
9	417217	777	984638	57	432580	834	567420	51
10	417684	776	984603	57	433080	833	566920	50
11	9.418150	775	9.984569	57	9.433580	833	10.566420	49
12	418615	775	984535	57	434080	832	565920	48
13	419079	774	984500	57	434579	831	565421	47
14	419544	773	984466	57	435078	830	564922	46
15	420007	772	984432	57	435576	829	564424	45
16	420470	771	984397	58	436073	828	563927	44
17	420933	770	984363	58	436570	828	563430	43
18	421395	769	984328	58	437067	827	562933	42
19	421857	768	984294	58	437563	826	562437	41
20	422318	767	984259	58	438059	825	561941	40
21	9.422778	767	9.984224	58	9.438554	824	10.561446	39
22	423238	766	984190	58	439048	824	560952	38
23	423697	765	984155	58	439543	823	560457	37
24	424156	764	984120	58	440036	822	559964	36
25	424615	763	984085	58	440529	821	559471	35
26	425073	762	984050	58	441022	820	558978	34
27	425530	761	984015	58	441514	820	558486	33
28	425987	761	983981	58	442006	819	557994	32
29	426443	760	983946	58	442497	818	557503	31
30	426899	759	983911	58	442988	817	557012	30
31	9.427354	758	9.983875	58	9.443479	816	10.556521	29
32	427809	757	983840	59	443968	816	556032	28
33	428263	756	983805	59	444458	815	555542	27
34	428717	755	983770	59	444947	814	555053	26
35	429170	755	983735	59	445435	813	554565	25
36	429623	754	983700	59	445923	813	554077	24
37	430075	753	983664	59	446411	812	553589	23
38	430527	752	983629	59	446898	811	553102	22
39	430978	751	983594	59	447384	810	552616	21
40	431429	750	983558	59	447870	809	552130	20
41	9.431879	750	9.983523	59	9.448356	809	10.551644	19
42	432329	749	983487	59	448841	808	551159	18
43	432778	748	983452	59	449326	807	550674	17
44	433226	747	983416	59	449810	806	550190	16
45	433675	746	983381	59	450294	806	549706	15
46	434122	745	983345	59	450777	805	549223	14
47	434569	745	983309	60	451260	804	548740	13
48	435016	744	983273	60	451743	803	548257	12
49	435462	743	983238	60	452225	803	547775	11
50	435908	742	983202	60	452706	802	547294	10
51	9.436353	741	9.983166	60	9.453187	801	10.546813	9
52	436798	740	983130	60	453668	800	546332	8
53	437242	740	983094	60	454148	800	545852	7
54	437686	739	983058	60	454628	799	545372	6
55	438129	738	983022	60	455107	798	544893	5
56	438572	737	982986	60	455586	797	544414	4
57	439014	736	982950	60	456064	797	543936	3
58	439456	736	982914	60	456542	796	543458	2
59	439897	735	982878	60	457019	795	542981	1
60	440338	734	982842	60	457496	794	542504	0
	Cosine.		Sine.		Cotang.		Tang.	M.

74 Degrees.

M.	Sine.	D.100".	Cosine.	D.	Tang.	D.100".	Cotang.	
0	9.440338	734	9.982842	60	9.457496	794	10.542504	60
1	440778	733	982805	60	457973	794	542027	59
2	441218	732	982769	61	458449	793	541551	58
3	441658	731	982733	61	458925	792	541075	57
4	442096	731	982696	61	459400	791	540600	56
5	442535	730	982660	61	459875	791	540125	55
6	442973	729	982624	61	460349	790	539651	54
7	443410	728	982587	61	460823	789	539177	53
8	443847	728	982551	61	461297	788	538703	52
9	444284	727	982514	61	461770	788	538230	51
10	444720	726	982477	61	462242	787	537758	50
11	9.445155	725	9.982441	61	9.462715	786	10.537285	49
12	445590	724	982404	61	463186	786	536814	48
13	446025	724	982367	61	463658	785	536342	47
14	446459	723	982331	61	464128	784	535872	46
15	446893	722	982294	61	464599	783	535401	45
16	447326	721	982257	61	465069	783	534931	44
17	447759	720	982220	62	465539	782	534461	43
18	448191	720	982183	62	466008	781	533992	42
19	448623	719	982146	62	466477	781	533523	41
20	449054	718	982109	62	466945	780	533055	40
21	9.449485	717	9.982072	62	9.467413	779	10.532587	39
22	449915	717	982035	62	467880	778	532120	38
23	450345	716	981998	62	468347	778	531653	37
24	450775	715	981961	62	468814	777	531186	36
25	451204	714	981924	62	469280	776	530720	35
26	451632	713	981886	62	469746	776	530254	34
27	452060	713	981849	62	470211	775	529789	33
28	452488	712	981812	62	470676	774	529324	32
29	452915	711	981774	62	471141	774	528859	31
30	453342	710	981737	62	471605	773	528395	30
31	9.453768	710	9.981700	62	9.472069	772	10.527931	29
32	454194	709	981662	63	472532	771	527468	28
33	454619	708	981625	63	472995	771	527005	27
34	455044	707	981587	63	473457	770	526543	26
35	455469	707	981549	63	473919	769	526081	25
36	455893	706	981512	63	474381	769	525619	24
37	456316	705	981474	63	474842	768	525158	23
38	456739	704	981436	63	475303	767	524697	22
39	457162	704	981399	63	475763	767	524237	21
40	457584	703	981361	63	476223	766	523777	20
41	9.458006	702	9.981323	63	9.476683	765	10.523317	19
42	458427	701	981285	63	477142	765	522858	18
43	458848	701	981247	63	477601	764	522399	17
44	459268	700	981209	63	478059	763	521941	16
45	459688	699	981171	63	478517	763	521483	15
46	460108	698	981133	63	478975	762	521025	14
47	460527	698	981095	64	479432	761	520568	13
48	460946	697	981057	64	479889	761	520111	12
49	461364	696	981019	64	480345	760	519655	11
50	461782	696	980981	64	480801	759	519199	10
51	9.462199	695	9.980942	64	9.481257	759	10.518743	9
52	462616	694	980904	64	481712	758	518288	8
53	463032	693	980866	64	482167	757	517833	7
54	463448	693	980827	64	482621	757	517379	6
55	463864	692	980789	64	483075	756	516925	5
56	464279	691	980750	64	483529	755	516471	4
57	464694	690	980712	64	483982	755	516018	3
58	465108	690	980673	64	484435	754	515565	2
59	465522	689	980635	64	484887	753	515113	1
60	465935	688	980596	64	485339	753	514661	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100".	Cosine.	D.	Tang.	D.100".	Cotang.	
0	9.465935	688	9.980596	64	9.485339	753	10.514661	60
1	466348	688	980558	64	485791	752	514209	59
2	466761	687	980519	65	486242	751	513758	58
3	467173	686	980480	65	486693	751	513307	57
4	467585	685	980442	65	487143	750	512857	56
5	467996	685	980403	65	487593	750	512407	55
6	468407	684	980364	65	488043	749	511957	54
7	468817	683	980325	65	488492	748	511508	53
8	469227	683	980286	65	488941	748	511059	52
9	469637	682	980247	65	489390	747	510610	51
10	470046	681	980208	65	489838	746	510162	50
11	9.470455	681	9.980169	65	9.490286	746	10.509714	49
12	470863	680	980130	65	490733	745	509267	48
13	471271	679	980091	65	491180	744	508820	47
14	471679	678	980052	65	491627	744	508373	46
15	472086	678	980012	65	492073	743	507927	45
16	472492	677	979973	65	492519	743	507481	44
17	472898	676	979934	66	492965	742	507035	43
18	473304	676	979895	66	493410	741	506590	42
19	473710	675	979855	66	493854	741	506146	41
20	474115	674	979816	66	494299	740	505701	40
21	9.474519	674	9.979776	66	9.494743	740	10.505257	39
22	474923	673	979737	66	495186	739	504814	38
23	475327	672	979697	66	465630	738	504370	37
24	475730	672	979658	66	496073	738	503927	36
25	476133	671	979618	66	496515	737	503485	35
26	476536	670	979579	66	496957	736	503043	34
27	476938	669	979539	66	497399	736	502601	33
28	477340	669	979499	66	497841	735	502159	32
29	477741	668	979459	66	498282	734	501718	31
30	478142	667	979420	66	498722	734	501278	30
31	9.478542	667	9.979380	66	9.499163	733	10.500837	29
32	478942	666	979340	67	499603	733	500397	28
33	479342	665	979300	67	500042	732	499958	27
34	479741	665	979260	67	500481	731	499519	26
35	480140	664	979220	67	500920	731	499080	25
36	480539	663	979180	67	501359	730	498641	24
37	480937	663	979140	67	501797	730	498203	23
38	481334	662	979100	67	502235	729	497765	22
39	481731	661	979059	67	502672	728	497328	21
40	482128	661	979019	67	503109	728	496891	20
41	9.482525	660	9.978979	67	9.503546	727	10.496454	19
42	482921	659	978939	67	503982	727	496018	18
43	483316	659	978898	67	504418	726	495582	17
44	483712	658	978858	67	504854	725	495146	16
45	484107	657	978817	67	505289	725	494711	15
46	484501	657	978777	67	505724	724	494276	14
47	484895	656	978737	68	506159	724	493841	13
48	485289	655	978696	68	506593	723	493407	12
49	485682	655	978655	68	507027	723	492973	11
50	486075	654	978615	68	507460	722	492540	10
51	9.486467	654	9.978574	68	9.507893	721	10.492107	9
52	486860	653	978533	68	508326	721	491674	8
53	487251	652	978493	68	508759	720	491241	7
54	487643	652	978452	68	509191	720	490809	6
55	488034	651	978411	68	509622	719	490378	5
56	488424	650	978370	68	510054	718	489946	4
57	488814	650	978329	68	510485	718	489515	3
58	489204	649	978288	68	510916	717	489084	2
59	489593	648	978247	68	511346	717	488654	1
60	489982	648	978206	68	511776	716	488224	0
	Cosine.		Sine.		Cotang.		Tang.	M.

72 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.489982	648	9.978206	68	9.511776	716	10.488224	60
1	490371	647	978165	68	512206	716	487794	59
2	490759	647	978124	69	512635	715	487365	58
3	491147	646	978083	69	513064	714	486936	57
4	491535	645	978042	69	513493	714	486507	56
5	491922	645	978001	69	513921	713	486079	55
6	492308	644	977959	69	514349	713	485651	54
7	492695	643	977918	69	514777	712	485223	53
8	493081	643	977877	69	515204	712	484796	52
9	493466	642	977835	69	515631	711	484369	51
10	493851	641	977794	69	516057	710	483943	50
11	9.494236	641	9.977752	69	9.516484	710	10.483516	49
12	494621	640	977711	69	516910	709	483090	48
13	495005	640	977669	69	517335	709	482665	47
14	495388	639	977628	69	517761	708	482239	46
15	495772	638	977586	69	518186	708	481814	45
16	496154	638	977544	70	518610	707	481390	44
17	496537	637	977503	70	519034	707	480966	43
18	496919	636	977461	70	519458	706	480542	42
19	497301	636	977419	70	519882	705	480118	41
20	497682	635	977377	70	520305	705	479695	40
21	9.498064	635	9.977335	70	9.520728	704	10.479272	39
22	498444	634	977293	70	521151	704	478849	38
23	498825	633	977251	70	521573	703	478427	37
24	499204	633	977209	70	521995	703	478005	36
25	499584	632	977167	70	522417	702	477583	35
26	499963	632	977125	70	522838	702	477162	34
27	500342	631	977083	70	523259	701	476741	33
28	500721	630	977041	70	523680	701	476320	32
29	501099	630	976999	70	524100	700	475900	31
30	501476	629	976957	70	524520	699	475480	30
31	9.501854	628	9.976914	71	9.524940	699	10.475060	29
32	502231	628	976872	71	525359	698	474641	28
33	502607	627	976830	71	525778	698	474222	27
34	502984	627	976787	71	526197	697	473803	26
35	503360	626	976745	71	526615	697	473385	25
36	503735	625	976702	71	527033	696	472967	24
37	504110	625	976660	71	527451	696	472549	23
38	504485	624	976617	71	527868	695	472132	22
39	504860	624	976574	71	528285	695	471715	21
40	505234	623	976532	71	528702	694	471298	20
41	9.505608	622	9.976489	71	9.529119	694	10.470881	19
42	505981	622	976446	71	529535	693	470465	18
43	506354	621	976404	71	529951	693	470049	17
44	506727	621	976361	71	530366	692	469634	16
45	507099	620	976318	71	530781	691	469219	15
46	507471	619	976275	72	531196	691	468804	14
47	507843	619	976232	72	531611	690	468389	13
48	508214	618	976189	72	532025	690	467975	12
49	508585	618	976146	72	532439	689	467561	11
50	508956	617	976103	72	532853	689	467147	10
51	9.509326	617	9.976060	72	9.533266	688	10.466734	9
52	509696	616	976017	72	533679	688	466321	8
53	510065	615	975974	72	534092	687	465908	7
54	510434	615	975930	72	534504	687	465496	6
55	510803	614	975887	72	534916	686	465084	5
56	511172	614	975844	72	535328	686	464672	4
57	511540	613	975800	72	535739	685	464261	3
58	511907	612	975757	72	536150	685	463850	2
59	512275	612	975714	72	536561	684	463439	1
60	512642	611	975670	72	536972	684	463028	0
	Cosine.		Sine.		Cotang.		Tang.	M.

71 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.512642	611	9.975670	73	9.536972	684	10.463028	60
1	513009	611	975627	73	537382	683	462618	59
2	513375	610	975583	73	537792	683	462208	58
3	513741	609	975539	73	538202	682	461798	57
4	514107	609	975496	73	538611	682	461389	56
5	514472	608	975452	73	539020	681	460980	55
6	514837	608	975408	73	539429	681	460571	54
7	515202	607	975365	73	539837	680	460163	53
8	515566	607	975321	73	540245	680	459755	52
9	515930	606	975277	73	540653	679	459347	51
10	516294	606	975233	73	541061	679	458939	50
11	9.516657	605	9.975189	73	9.541468	678	10.458532	49
12	517020	604	975145	73	541875	678	458125	48
13	517382	604	975101	73	542281	677	457719	47
14	517745	603	975057	73	542688	677	457312	46
15	518107	603	975013	74	543094	676	456906	45
16	518468	602	974969	74	543499	676	456501	44
17	518829	602	974925	74	543905	675	456095	43
18	519190	601	974880	74	544310	675	455690	42
19	519551	600	974836	74	544715	674	455285	41
20	519911	600	974792	74	545119	674	454881	40
21	9.520271	599	9.974748	74	9.545524	673	10.454476	39
22	520631	599	974703	74	545928	673	454072	38
23	520990	598	974659	74	546331	672	453669	37
24	521349	598	974614	74	546735	672	453265	36
25	521707	597	974570	74	547138	671	452862	35
26	522066	597	974525	74	547540	671	452460	34
27	522424	596	974481	74	547943	670	452057	33
28	522781	596	974436	74	548345	670	451655	32
29	523138	595	974391	75	548747	669	451253	31
30	523495	594	974347	75	549149	669	450851	30
31	9.523852	594	9.974302	75	9.549550	668	10.450450	29
32	524208	593	974257	75	549951	668	450049	28
33	524564	593	974212	75	550352	667	449648	27
34	524920	592	974167	75	550752	667	449248	26
35	525275	592	974122	75	551153	666	448847	25
36	525630	591	974077	75	551552	666	448448	24
37	525984	591	974032	75	551952	666	448048	23
38	526339	590	973987	75	552351	665	447649	22
39	526693	589	973942	75	552750	665	447250	21
40	527046	589	973897	75	553149	664	446851	20
41	9.527400	588	9.973852	75	9.553548	664	10.446452	19
42	527753	588	973807	75	553946	663	446054	18
43	528105	587	973761	75	554344	663	445656	17
44	528458	587	973716	76	554741	662	445259	16
45	528810	586	973671	76	555139	662	444861	15
46	529161	586	973625	76	555536	661	444464	14
47	529513	585	973580	76	555933	661	444067	13
48	529864	585	973535	76	556329	660	443671	12
49	530215	584	973489	76	556725	660	443275	11
50	530565	584	973444	76	557121	659	442879	10
51	9.530915	583	9.973398	76	9.557517	659	10.442483	9
52	531265	583	973352	76	557913	659	442087	8
53	531614	582	973307	76	558308	658	441692	7
54	531963	581	973261	76	558703	658	441297	6
55	532312	581	973215	76	559097	657	440903	5
56	532661	580	973169	76	559491	657	440509	4
57	533009	580	973124	76	559885	656	440115	3
58	533357	579	973078	76	560279	656	439721	2
59	533704	579	973032	77	560673	655	439327	1
60	534052	578	972986	77	561066	655	438934	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.534052	578	9.972986	77	9.561066	655	10.438934	60
1	534399	578	972940	77	561459	654	438541	59
2	534745	577	972894	77	561851	654	438149	58
3	535092	577	972848	77	562244	654	437756	57
4	535438	576	972802	77	562636	653	437364	56
5	535783	576	972755	77	563028	653	436972	55
6	536129	575	972709	77	563419	652	436581	54
7	536474	575	972663	77	563811	652	436189	53
8	536818	574	972617	77	564202	651	435798	52
9	537163	574	972570	77	564593	651	435407	51
10	537507	573	972524	77	564983	650	435017	50
11	9.537851	573	9.972478	77	9.565373	650	10.434627	49
12	538194	572	972431	77	565763	649	434237	48
13	538538	571	972385	78	566153	649	433847	47
14	538880	571	972338	78	566542	649	433458	46
15	539223	570	972291	78	566932	648	433068	45
16	539565	570	972245	78	567320	648	432680	44
17	539907	569	972198	78	567709	647	432291	43
18	540249	569	972151	78	568098	647	431902	42
19	540590	568	972105	78	568486	646	431514	41
20	540931	568	972058	78	568873	646	431127	40
21	9.541272	567	9.972011	78	9.569261	646	10.430739	39
22	541613	567	971964	78	569648	645	430352	38
23	541953	566	971917	78	570035	645	429965	37
24	542293	566	971870	78	570422	644	429578	36
25	542632	565	971823	78	570809	644	429191	35
26	542971	565	971776	78	571195	643	428805	34
27	543310	564	971729	79	571581	643	428419	33
28	543649	564	971682	79	571967	643	428033	32
29	543987	563	971635	79	572352	642	427648	31
30	544325	563	971588	79	572738	642	427262	30
31	9.544663	562	9.971540	79	9.573123	641	10.426877	29
32	545000	562	971493	79	573507	641	426493	28
33	545338	561	971446	79	573892	640	426108	27
34	545674	561	971398	79	574276	640	425724	26
35	546011	560	971351	79	574660	640	425340	25
36	546347	560	971303	79	575044	639	424956	24
37	546683	559	971256	79	575427	639	424573	23
38	547019	559	971208	79	575810	638	424190	22
39	547354	558	971161	79	576193	638	423807	21
40	547689	558	971113	79	576576	637	423424	20
41	9.548024	557	9.971066	80	9.576959	637	10.423041	19
42	548359	557	971018	80	577341	637	422659	18
43	548693	556	970970	80	577723	636	422277	17
44	549027	556	970922	80	578104	636	421896	16
45	549360	555	970874	80	578486	635	421514	15
46	549693	555	970827	80	578867	635	421133	14
47	550026	555	970779	80	579248	634	420752	13
48	550359	554	970731	80	579629	634	420371	12
49	550692	554	970683	80	580009	634	419991	11
50	551024	553	970635	80	580389	633	419611	10
51	9.551356	553	9.970586	80	9.580769	633	10.419231	9
52	551687	552	970538	80	581149	632	418851	8
53	552018	552	970490	80	581528	632	418472	7
54	552349	551	970442	80	581907	632	418093	6
55	552680	551	970394	80	582286	631	417714	5
56	553010	550	970345	81	582665	631	417335	4
57	553341	550	970297	81	583044	630	416956	3
58	553670	549	970249	81	583422	630	416578	2
59	554000	549	970200	81	583800	630	416200	1
60	554329	548	970152	81	584177	629	415823	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.554329	548	9.970152	81	9.584177	629	10.415823	60
1	554658	548	970103	81	584555	629	415445	59
2	554987	547	970055	81	584932	628	415068	58
3	555315	547	970006	81	585309	628	414691	57
4	555643	546	969957	81	585686	627	414314	56
5	555971	546	969909	81	586062	627	413938	55
6	556299	545	969860	81	586439	627	413561	54
7	556626	545	969811	81	586815	626	413185	53
8	556953	544	969762	81	587190	626	412810	52
9	557280	544	969714	81	587566	625	412434	51
10	557606	544	969665	82	587941	625	412059	50
11	9.557932	543	9.969616	82	9.588316	625	10.411684	49
12	558258	543	969567	82	588691	624	411309	48
13	558583	542	969518	82	589066	624	410934	47
14	558909	542	969469	82	589440	623	410560	46
15	559234	541	969420	82	589814	623	410186	45
16	559558	541	969370	82	590188	623	409812	44
17	559883	540	969321	82	590562	622	409438	43
18	560207	540	969272	82	590935	622	409065	42
19	560531	539	969223	82	591308	622	408692	41
20	560855	539	969173	82	591681	621	408319	40
21	9.561178	538	9.969124	82	9.592054	621	10.407946	39
22	561501	538	969075	82	592426	620	407574	38
23	561824	537	969025	82	592799	620	407201	37
24	562146	537	968976	83	593171	620	406829	36
25	562468	537	968926	83	593542	619	406458	35
26	562790	536	968877	83	593914	619	406086	34
27	563112	536	968827	83	594285	618	405715	33
28	563433	535	968777	83	594656	618	405344	32
29	563755	535	968728	83	595027	618	404973	31
30	564075	534	968678	83	595398	617	404602	30
31	9.564396	534	9.968628	83	9.595768	617	10.404232	29
32	564716	533	968578	83	596138	616	403862	28
33	565036	533	968528	83	596508	616	403492	27
34	565356	532	968479	83	596878	616	403122	26
35	565676	532	968429	83	597247	615	402753	25
36	565995	532	968379	83	597616	615	402384	24
37	566314	531	968329	83	597985	615	402015	23
38	566632	531	968278	84	598354	614	401646	22
39	566951	530	968228	84	598722	614	401278	21
40	567269	530	968178	84	599091	613	400909	20
41	9.567587	529	9.968128	84	9.599459	613	10.400541	19
42	567904	529	968078	84	599827	613	400173	18
43	568222	528	968027	84	600194	612	399806	17
44	568539	528	967977	84	600562	612	399438	16
45	568856	528	967927	84	600929	612	399071	15
46	569172	527	967876	84	601296	611	398704	14
47	569488	527	967826	84	601663	611	398337	13
48	569804	526	967775	84	602029	610	397971	12
49	570120	526	967725	84	602395	610	397605	11
50	570435	525	967674	84	602761	610	397239	10
51	9.570751	525	9.967624	84	9.603127	609	10.396873	9
52	571066	524	967573	85	603493	609	396507	8
53	571380	524	967522	85	603858	609	396142	7
54	571695	524	967471	85	604223	608	395777	6
55	572009	523	967421	85	604588	608	395412	5
56	572323	523	967370	85	604953	607	395047	4
57	572636	522	967319	85	605317	607	394683	3
58	572950	522	967268	85	605682	607	394318	2
59	573263	521	967217	85	606046	606	393954	1
60	573575	521	967166	85	606410	606	393590	0
	Cosine.		Sine.		Cotang.		Tang.	M.

68 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.573575	521	9.967166	85	9.606410	606	10.393590	60
1	573888	520	967115	85	606773	606	393227	59
2	574200	520	967064	85	607137	605	392863	58
3	574512	520	967013	85	607500	605	392500	57
4	574824	519	966961	85	607863	605	392137	56
5	575136	519	966910	85	608225	604	391775	55
6	575447	518	966859	86	608588	604	391412	54
7	575758	518	966808	86	608950	603	391050	53
8	576069	517	966756	86	609312	603	390688	52
9	576379	517	966705	86	609674	603	390326	51
10	576689	517	966653	86	610036	602	389964	50
11	9.576999	516	9.966602	86	9.610397	602	10.389603	49
12	577309	516	966550	86	610759	602	389241	48
13	577618	515	966499	86	611120	601	388880	47
14	577927	515	966447	86	611480	601	388520	46
15	578236	514	966395	86	611841	601	388159	45
16	578545	514	966344	86	612201	600	387799	44
17	578853	514	966292	86	612561	600	387439	43
18	579162	513	966240	86	612921	600	387079	42
19	579470	513	966188	86	613281	599	386719	41
20	579777	512	966136	87	613641	599	386359	40
21	9.580085	512	9.966085	87	9.614000	598	10.386000	39
22	580392	511	966033	87	614359	598	385641	38
23	580699	511	965981	87	614718	598	385282	37
24	581005	511	965929	87	615077	597	384923	36
25	581312	510	965876	87	615435	597	384565	35
26	581618	510	965824	87	615793	597	384207	34
27	581924	509	965772	87	616151	596	383849	33
28	582229	509	965720	87	616509	596	383491	32
29	582535	509	965668	87	616867	596	383133	31
30	582840	508	965615	87	617224	595	382776	30
31	9.583145	508	9.965563	87	9.617582	595	10.382418	29
32	583449	507	965511	87	617939	595	382061	28
33	583754	507	965458	87	618295	594	381705	27
34	584058	506	965406	88	618652	594	381348	26
35	584361	506	965353	88	619008	594	380992	25
36	584665	506	965301	88	619364	593	380636	24
37	584968	505	965248	88	619720	593	380280	23
38	585272	505	965195	88	620076	593	379924	22
39	585574	504	965143	88	620432	592	379568	21
40	585877	504	965090	88	620787	592	379213	20
41	9.586179	504	9.965037	88	9.621142	592	10.378858	19
42	586482	503	964984	88	621497	591	378503	18
43	586783	503	964931	88	621852	591	378148	17
44	587085	502	964879	88	622207	591	377793	16
45	587386	502	964826	88	622561	590	377439	15
46	587688	501	964773	88	622915	590	377085	14
47	587989	501	964720	88	623269	590	376731	13
48	588289	501	964666	89	623623	589	376377	12
49	588590	500	964613	89	623976	589	376024	11
50	588890	500	964560	89	624330	589	375670	10
51	9.589190	499	9.964507	89	9.624683	588	10.375317	9
52	589489	499	964454	89	625036	588	374964	8
53	589789	499	964400	89	625388	588	374612	7
54	590088	498	964347	89	625741	587	374259	6
55	590387	498	964294	89	626093	587	373907	5
56	590686	497	964240	89	626445	587	373555	4
57	590984	497	964187	89	626797	586	373203	3
58	591282	497	964133	89	627149	586	372851	2
59	591580	496	964080	89	627501	586	372499	1
60	591878	496	964026	89	627852	585	372148	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.591878	496	9.964026	89	9.627852	585	10.372148	60
1	592176	495	963972	89	628203	585	371797	59
2	592473	495	963919	90	628554	585	371446	58
3	592770	495	963865	90	628905	584	371095	57
4	593067	494	963811	90	629255	584	370745	56
5	593363	494	963757	90	629606	584	370394	55
6	593659	493	963704	90	629956	583	370044	54
7	593955	493	963650	90	630306	583	369694	53
8	594251	493	963596	90	630656	583	369344	52
9	594547	492	963542	90	631005	582	368995	51
10	594842	492	963488	90	631355	582	368645	50
11	9.595137	491	9.963434	90	9.631704	582	10.368296	49
12	595432	491	963379	90	632053	581	367947	48
13	595727	491	963325	90	632402	581	367598	47
14	596021	490	963271	90	632750	581	367250	46
15	596315	490	963217	90	633099	580	366901	45
16	596609	489	963163	91	633447	580	366553	44
17	596903	489	963108	91	633795	580	366205	43
18	597196	489	963054	91	634143	579	365857	42
19	597490	488	962999	91	634490	579	365510	41
20	597783	488	962945	91	634838	579	365162	40
21	9.598075	488	9.962890	91	9.635185	578	10.364815	39
22	598368	487	962836	91	635532	578	364468	38
23	598660	487	962781	91	635879	578	364121	37
24	598952	486	962727	91	636226	577	363774	36
25	599244	486	962672	91	636572	577	363428	35
26	599536	486	962617	91	636919	577	363081	34
27	599827	485	962562	91	637265	577	362735	33
28	600118	485	962508	91	637611	576	362389	32
29	600409	484	962453	91	637956	576	362044	31
30	600700	484	962398	92	638302	576	361698	30
31	9.600990	484	9.962343	92	9.638647	575	10.361353	29
32	601280	483	962288	92	638992	575	361008	28
33	601570	483	962233	92	639337	575	360663	27
34	601860	482	962178	92	639682	574	360318	26
35	602150	482	962123	92	640027	574	359973	25
36	602439	482	962067	92	640371	574	359629	24
37	602728	481	962012	92	640716	573	359284	23
38	603017	481	961957	92	641060	573	358940	22
39	603305	481	961902	92	641404	573	358596	21
40	603594	480	961846	92	641747	573	358253	20
41	9.603882	480	9.961791	92	9.642091	572	10.357909	19
42	604170	479	961735	92	642434	572	357566	18
43	604457	479	961680	93	642777	572	357223	17
44	604745	479	961624	93	643120	571	356880	16
45	605032	478	961569	93	643463	571	356537	15
46	605319	478	961513	93	643806	571	356194	14
47	605606	478	961458	93	644148	570	355852	13
48	605892	477	961402	93	644490	570	355510	12
49	606179	477	961346	93	644832	570	355168	11
50	606465	476	961290	93	645174	569	354826	10
51	9.606751	476	9.961235	93	9.645516	569	10.354484	9
52	607036	476	961179	93	645857	569	354143	8
53	607322	475	961123	93	646199	569	353801	7
54	607607	475	961067	93	646540	568	353460	6
55	607892	475	961011	93	646881	568	353119	5
56	608177	474	960955	93	647222	568	352778	4
57	608461	474	960899	94	647562	567	352438	3
58	608745	473	960843	94	647903	567	352097	2
59	609029	473	960786	94	648243	567	351757	1
60	609313	473	960730	94	648583	566	351417	0
	Cosine.		Sine.		Cotang.		Tang.	M.

66 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.609313	473	9.960730	94	9.648583	566	10.351417	60
1	609597	472	960674	94	648923	566	351077	59
2	609880	472	960618	94	649263	566	350737	58
3	610164	472	960561	94	649602	566	350398	57
4	610447	471	960505	94	649942	565	350058	56
5	610729	471	960448	94	650281	565	349719	55
6	611012	470	960392	94	650620	565	349380	54
7	611294	470	960335	94	650959	564	349041	53
8	611576	470	960279	94	651297	564	348703	52
9	611858	469	960222	94	651636	564	348364	51
10	612140	469	960165	94	651974	564	348026	50
11	9.612421	469	9.960109	95	9.652312	563	10.347688	49
12	612702	468	960052	95	652650	563	347350	48
13	612983	468	959995	95	652988	563	347012	47
14	613264	468	959938	95	653326	562	346674	46
15	613545	467	959882	95	653663	562	346337	45
16	613825	467	959825	95	654000	562	346000	44
17	614105	466	959768	95	654337	562	345663	43
18	614385	466	959711	95	654674	561	345326	42
19	614665	466	959654	95	655011	561	344989	41
20	614944	465	959596	95	655348	561	344652	40
21	9.615223	465	9.959539	95	9.655684	560	10.344316	39
22	615502	465	959482	95	656020	560	343980	38
23	615781	464	959425	95	656356	560	343644	37
24	616060	464	959368	96	656692	560	343308	36
25	616338	464	959310	96	657028	559	342972	35
26	616616	463	959253	96	657364	559	342636	34
27	616894	463	959195	96	657699	559	342301	33
28	617172	463	959138	96	658034	558	341966	32
29	617450	462	959080	96	658369	558	341631	31
30	617727	462	959023	96	658704	558	341296	30
31	9.618004	461	9.958965	96	9.659039	558	10.340961	29
32	618281	461	958908	96	659373	557	340627	28
33	618558	461	958850	96	659708	557	340292	27
34	618834	460	958792	96	660042	557	339958	26
35	619110	460	958734	96	660376	556	339624	25
36	619386	460	958677	96	660710	556	339290	24
37	619662	459	958619	96	661043	556	338957	23
38	619938	459	958561	97	661377	556	338623	22
39	620213	459	958503	97	661710	555	338290	21
40	620488	458	958445	97	662043	555	337957	20
41	9.620763	458	9.958387	97	9.662376	555	10.337624	19
42	621038	458	958329	97	662709	554	337291	18
43	621313	457	958271	97	663042	554	336958	17
44	621587	457	958213	97	663375	554	336625	16
45	621861	457	958154	97	663707	554	336293	15
46	622135	456	958096	97	664039	553	335961	14
47	622409	456	958038	97	664371	553	335629	13
48	622682	455	957979	97	664703	553	335297	12
49	622956	455	957921	97	665035	553	334965	11
50	623229	455	957863	97	665366	552	334634	10
51	9.623502	454	9.957804	98	9.665698	552	10.334302	9
52	623774	454	957746	98	666029	552	333971	8
53	624047	454	957687	98	666360	551	333640	7
54	624319	453	957628	98	666691	551	333309	6
55	624591	453	957570	98	667021	551	332979	5
56	624863	453	957511	98	667352	551	332648	4
57	625135	452	957452	98	667682	550	332318	3
58	625406	452	957393	98	668013	550	331987	2
59	625677	452	957335	98	668343	550	331657	1
60	625948	451	957276	98	668673	550	331327	0
	Cosine.		Sine.		Cotang.		Tang.	M.

65 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.625948	451	9.957276	98	9.668673	550	10.331327	60
1	626219	451	957217	98	669002	549	330998	59
2	626490	451	957158	98	669332	549	330668	58
3	626760	450	957099	98	669661	549	330339	57
4	627030	450	957040	98	669991	548	330009	56
5	627300	450	956981	99	670320	548	329680	55
6	627570	449	956921	99	670649	548	329351	54
7	627840	449	956862	99	670977	548	329023	53
8	628109	449	956803	99	671306	547	328694	52
9	628378	448	956744	99	671635	547	328365	51
10	628647	448	956684	99	671963	547	328037	50
11	9.628916	448	9.956625	99	9.672291	547	10.327709	49
12	629185	447	956566	99	672619	546	327381	48
13	629453	447	956506	99	672947	546	327053	47
14	629721	447	956447	99	673274	546	326726	46
15	629989	446	956387	99	673602	546	326398	45
16	630257	446	956327	99	673929	545	326071	44
17	630524	446	956268	99	674257	545	325743	43
18	630792	445	956208	100	674584	545	325416	42
19	631059	445	956148	100	674911	545	325089	41
20	631326	445	956089	100	675237	544	324763	40
21	9.631593	444	9.956029	100	9.675564	544	10.324436	39
22	631859	444	955969	100	675890	544	324110	38
23	632125	444	955909	100	676217	543	323783	37
24	632392	443	955849	100	676543	543	323457	36
25	632658	443	955789	100	676869	543	323131	35
26	632923	443	955729	100	677194	543	322806	34
27	633189	442	955669	100	677520	542	322480	33
28	633454	442	955609	100	677846	542	322154	32
29	633719	442	955548	100	678171	542	321829	31
30	633984	441	955488	100	678496	542	321504	30
31	9.634249	441	9.955428	101	9.678821	541	10.321179	29
32	634514	441	955368	101	679146	541	320854	28
33	634778	440	955307	101	679471	541	320529	27
34	635042	440	955247	101	679795	541	320205	26
35	635306	440	955186	101	680120	540	319880	25
36	635570	439	955126	101	680444	540	319556	24
37	635834	439	955065	101	680768	540	319232	23
38	636097	439	955005	101	681092	540	318908	22
39	636360	438	954944	101	681416	539	318584	21
40	636623	438	954883	101	681740	539	318260	20
41	9.636886	438	9.954823	101	9.682063	539	10.317937	19
42	637148	437	954762	101	682387	539	317613	18
43	637411	437	954701	101	682710	538	317290	17
44	637673	437	954640	102	683033	538	316967	16
45	637935	436	954579	102	683356	538	316644	15
46	638197	436	954518	102	683679	538	316321	14
47	638458	436	954457	102	684001	537	315999	13
48	638720	435	954396	102	684324	537	315676	12
49	638981	435	954335	102	684646	537	315354	11
50	639242	435	954274	102	684968	537	315032	10
51	9.639503	434	9.954213	102	9.685290	536	10.314710	9
52	639764	434	954152	102	685612	536	314388	8
53	640024	434	954090	102	685934	536	314066	7
54	640284	433	954029	102	686255	536	313745	6
55	640544	433	953968	102	686577	535	313423	5
56	640804	433	953906	102	686898	535	313102	4
57	641064	432	953845	102	687219	535	312781	3
58	641324	432	953783	103	687540	535	312460	2
59	641583	432	953722	103	687861	534	312139	1
60	641842	432	953660	103	688182	534	311818	0
	Cosine.		Sine.		Cotang.		Tang.	M.

64 Degrees.

M.	Sine.	D.100".	Cosine.	D.	Tang.	D.100".	Cotang.	
0	9.641842	432	9.953660	108	9.688182	534	10.311818	60
1	642101	431	953599	103	688502	534	311498	59
2	642360	431	953537	103	688823	534	311177	58
3	642618	431	953475	103	689143	534	310857	57
4	642877	430	953413	103	689463	533	310537	56
5	643135	430	953352	103	689783	533	310217	55
6	643393	430	953290	103	690103	533	309897	54
7	643650	429	953228	103	690423	533	309577	53
8	643908	429	953166	103	690742	532	309258	52
9	644165	429	953104	103	691062	532	308938	51
10	644423	428	953042	103	691381	532	308619	50
11	9.644680	428	9.952980	104	9.691700	532	10.308300	49
12	644936	428	952918	104	692019	531	307981	48
13	645193	427	952855	104	692338	531	307662	47
14	645450	427	952793	104	692656	531	307344	46
15	645706	427	952731	104	692975	531	307025	45
16	645962	426	952669	104	693293	530	306707	44
17	646218	426	952606	104	693612	530	306388	43
18	646474	426	952544	104	693930	530	306070	42
19	646729	426	952481	104	694248	530	305752	41
20	646984	425	952419	104	694566	529	305434	40
21	9.647240	425	9.952356	104	9.694883	529	10.305117	39
22	647494	425	952294	104	695201	529	304799	38
23	647749	424	952231	104	695518	529	304482	37
24	648004	424	952168	105	695836	529	304164	36
25	648258	424	952106	105	696153	528	303847	35
26	648512	423	952043	105	696470	528	303530	34
27	648766	423	951980	105	696787	528	303213	33
28	649020	423	951917	105	697103	528	302897	32
29	649274	422	951854	105	697420	527	302580	31
30	649527	422	951791	105	697736	527	302264	30
31	9.649781	422	9.951728	105	9.698053	527	10.301947	29
32	650034	422	951665	105	698369	527	301631	28
33	650287	421	951602	105	698685	526	301315	27
34	650539	421	951539	105	699001	526	300999	26
35	650792	421	951476	105	699316	526	300684	25
36	651044	420	951412	106	699632	526	300368	24
37	651297	420	951349	106	699947	526	300053	23
38	651549	420	951286	106	700263	525	299737	22
39	651800	419	951222	106	700578	525	299422	21
40	652052	419	951159	106	700893	525	299107	20
41	9.652304	419	9.951096	106	9.701208	525	10.298792	19
42	652555	418	951032	106	701523	524	298477	18
43	652806	418	950968	106	701837	524	298163	17
44	653057	418	950905	106	702152	524	297848	16
45	653308	418	950841	106	702466	524	297534	15
46	653558	417	950778	106	702781	523	297219	14
47	653808	417	950714	106	703095	523	296905	13
48	654059	417	950650	106	703409	523	296591	12
49	654309	416	950586	106	703722	523	296278	11
50	654558	416	950522	107	704036	523	295964	10
51	9.654808	416	9.950458	107	9.704350	522	10.295650	9
52	655058	415	950394	107	704663	522	295337	8
53	655307	415	950330	107	704976	522	295024	7
54	655556	415	950266	107	705290	522	294710	6
55	655805	415	950202	107	705603	521	294397	5
56	656054	414	950138	107	705916	521	294084	4
57	656302	414	950074	107	706228	521	293772	3
58	656551	414	950010	107	706541	521	293459	2
59	656799	413	949945	107	706854	521	293146	1
60	657047	413	949881	107	707166	520	292834	0
	Cosine.		Sine.		Cotang.		Tang.	M.

63 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.657047	413	9.949881	107	9.707166	520	10.292834	60
1	657295	413	949816	107	707478	520	292522	59
2	657542	412	949752	107	707790	520	292210	58
3	657790	412	949688	108	708102	520	291898	57
4	658037	412	949623	108	708414	519	291586	56
5	658284	412	949558	108	708726	519	291274	55
6	658531	411	949494	108	709037	519	290963	54
7	658778	411	949429	108	709349	519	290651	53
8	659025	411	949364	108	709660	519	290340	52
9	659271	410	949300	108	709971	518	290029	51
10	659517	410	949235	108	710282	518	289718	50
11	9.659763	410	9.949170	108	9.710593	518	10.289407	49
12	660009	410	949105	108	710904	518	289096	48
13	660255	409	949040	108	711215	518	288785	47
14	660501	409	948975	108	711525	517	288475	46
15	660746	409	948910	108	711836	517	288164	45
16	660991	408	948845	109	712146	517	287854	44
17	661236	408	948780	109	712456	517	287544	43
18	661481	408	948715	109	712766	516	287234	42
19	661726	407	948650	109	713076	516	286924	41
20	661970	407	948584	109	713386	516	286614	40
21	9.662214	407	9.948519	109	9.713696	516	10.286304	39
22	662459	407	948454	109	714005	516	285995	38
23	662703	406	948388	109	714314	515	285686	37
24	662946	406	948323	109	714624	515	285376	36
25	663190	406	948257	109	714933	515	285067	35
26	663433	405	948192	109	715242	515	284758	34
27	663677	405	948126	109	715551	515	284449	33
28	663920	405	948060	109	715860	514	284140	32
29	664163	405	947995	110	716168	514	283832	31
30	664406	404	947929	110	716477	514	283523	30
31	9.664648	404	9.947863	110	9.716785	514	10.283215	29
32	664891	404	947797	110	717093	514	282907	28
33	665133	403	947731	110	717401	513	282599	27
34	665375	403	947665	110	717709	513	282291	26
35	665617	403	947600	110	718017	513	281983	25
36	665859	403	947533	110	718325	513	281675	24
37	666100	402	947467	110	718633	512	281367	23
38	666342	402	947401	110	718940	512	281060	22
39	666583	402	947335	110	719248	512	280752	21
40	666824	401	947269	110	719555	512	280445	20
41	9.667065	401	9.947203	110	9.719862	512	10.280138	19
42	667305	401	947136	111	720169	511	279831	18
43	667546	401	947070	111	720476	511	279524	17
44	667786	400	947004	111	720783	511	279217	16
45	668027	400	946937	111	721089	511	278911	15
46	668267	400	946871	111	721396	511	278604	14
47	668506	399	946804	111	721702	510	278298	13
48	668746	399	946738	111	722009	510	277991	12
49	668986	399	946671	111	722315	510	277685	11
50	669225	399	946604	111	722621	510	277379	10
51	9.669464	398	9.946538	111	9.722927	510	10.277073	9
52	669703	398	946471	111	723232	509	276768	8
53	669942	398	946404	111	723538	509	276462	7
54	670181	397	946337	111	723844	509	276156	6
55	670419	397	946270	112	724149	509	275851	5
56	670658	397	946203	112	724454	509	275546	4
57	670896	397	946136	112	724760	508	275240	3
58	671134	396	946069	112	725065	508	274935	2
59	671372	396	946002	112	725370	508	274630	1
60	671609	396	945935	112	725674	508	274326	0
	Cosine.		Sine.		Cotang.		Tang.	M.

62 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.671609	396	9.945935	112	9.725674	508	10.274326	60
1	671847	396	945868	112	725979	508	274021	59
2	672084	395	945800	112	726284	507	273716	58
3	672321	395	945733	112	726588	507	273412	57
4	672558	395	945666	112	726892	507	273108	56
5	672795	394	945598	112	727197	507	272803	55
6	673032	394	945531	112	727501	507	272499	54
7	673268	394	945464	113	727805	506	272195	53
8	673505	394	945396	113	728109	506	271891	52
9	673741	393	945328	113	728412	506	271588	51
10	673977	393	945261	113	728716	506	271284	50
11	9.674213	393	9.945193	113	9.729020	506	10.270980	49
12	674448	393	945125	113	729323	505	270677	48
13	674684	392	945058	113	729626	505	270374	47
14	674919	392	944990	113	729929	505	270071	46
15	675155	392	944922	113	730233	505	269767	45
16	675390	391	944854	113	730535	505	269465	44
17	675624	391	944786	113	730838	504	269162	43
18	675859	391	944718	113	731141	504	268859	42
19	676094	391	944650	113	731444	504	268556	41
20	676328	390	944582	114	731746	504	268254	40
21	9.676562	390	9.944514	114	9.732048	504	10.267952	39
22	676796	390	944446	114	732351	504	267649	38
23	677030	390	944377	114	732653	503	267347	37
24	677264	389	944309	114	732955	503	267045	36
25	677498	389	944241	114	733257	503	266743	35
26	677731	389	944172	114	733558	503	266442	34
27	677964	388	944104	114	733860	503	266140	33
28	678197	388	944036	114	734162	502	265838	32
29	678430	388	943967	114	734463	502	265537	31
30	678663	388	943899	114	734764	502	265236	30
31	9.678895	387	9.943830	114	9.735066	502	10.264934	29
32	679128	387	943761	114	735367	502	264633	28
33	679360	387	943693	115	735668	501	264332	27
34	679592	387	943624	115	735969	501	264031	26
35	679824	386	943555	115	736269	501	263731	25
36	680056	386	943486	115	736570	501	263430	24
37	680288	386	943417	115	736870	501	263130	23
38	680519	386	943348	115	737171	500	262829	22
39	680750	385	943279	115	737471	500	262529	21
40	680982	385	943210	115	737771	500	262229	20
41	9.681213	385	9.943141	115	9.738071	500	10.261929	19
42	681443	384	943072	115	738371	500	261629	18
43	681674	384	943003	115	738671	500	261329	17
44	681905	384	942934	115	738971	499	261029	16
45	682135	384	942864	116	739271	499	260729	15
46	682365	383	942795	116	739570	499	260430	14
47	682595	383	942726	116	739870	499	260130	13
48	682825	383	942656	116	740169	499	259831	12
49	683055	383	942587	116	740468	498	259532	11
50	683284	382	942517	116	740767	498	259233	10
51	9.683514	382	9.942448	116	9.741066	498	10.258934	9
52	683743	382	942378	116	741365	498	258635	8
53	683972	382	942308	116	741664	498	258336	7
54	684201	381	942239	116	741962	498	258038	6
55	684430	381	942169	116	742261	497	257739	5
56	684658	381	942099	116	742559	497	257441	4
57	684887	380	942029	116	742858	497	257142	3
58	685115	380	941959	117	743156	497	256844	2
59	685343	380	941889	117	743454	497	256546	1
60	685571	380	941819	117	743752	496	256248	0
	Cosine.		Sine.		Cotang.		Tang.	M.

61 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.685571	380	9.941819	117	9.743752	496	10.256248	60
1	685799	379	941749	117	744050	496	255950	59
2	686027	379	941679	117	744348	496	255652	58
3	686254	379	941609	117	744645	496	255355	57
4	686482	379	941539	117	744943	496	255057	56
5	686709	378	941469	117	745240	496	254760	55
6	686936	378	941398	117	745538	495	254462	54
7	687163	378	941328	117	745835	495	254165	53
8	687389	378	941258	117	746132	495	253868	52
9	687616	377	941187	117	746429	495	253571	51
10	687843	377	941117	118	746726	495	253274	50
11	9.688069	377	9.941046	118	9.747023	494	10.252977	49
12	688295	377	940975	118	747319	494	252681	48
13	688521	376	940905	118	747616	494	252384	47
14	688747	376	940834	118	747913	494	252087	46
15	688972	376	940763	118	748209	494	251791	45
16	689198	376	940693	118	748505	494	251495	44
17	689423	375	940622	118	748801	493	251199	43
18	689648	375	940551	118	749097	493	250903	42
19	689873	375	940480	118	749393	493	250607	41
20	690098	375	940409	118	749689	493	250311	40
21	9.690323	374	9.940338	118	9.749985	493	10.250015	39
22	690548	374	940267	118	750281	493	249719	38
23	690772	374	940196	119	750576	492	249424	37
24	690996	374	940125	119	750872	492	249128	36
25	691220	373	940054	119	751167	492	248833	35
26	691444	373	939982	119	751462	492	248538	34
27	691668	373	939911	119	751757	492	248243	33
28	691892	373	939840	119	752052	491	247948	32
29	692115	372	939768	119	752347	491	247653	31
30	692339	372	939697	119	752642	491	247358	30
31	9.692562	372	9.939625	119	9.752937	491	10.247063	29
32	692785	371	939554	119	753231	491	246769	28
33	693008	371	939482	119	753526	491	246474	27
34	693231	371	939410	119	753820	490	246180	26
35	693453	371	939339	120	754115	490	245885	25
36	693676	370	939267	120	754409	490	245591	24
37	693898	370	939195	120	754703	490	245297	23
38	694120	370	939123	120	754997	490	245003	22
39	694342	370	939052	120	755291	490	244709	21
40	694564	369	938980	120	755585	489	244415	20
41	9.694786	369	9.938908	120	9.755878	489	10.244122	19
42	695007	369	938836	120	756172	489	243828	18
43	695229	369	938763	120	756465	489	243535	17
44	695450	368	938691	120	756759	489	243241	16
45	695671	368	938619	120	757052	489	242948	15
46	695892	368	938547	120	757345	488	242655	14
47	696113	368	938475	121	757638	488	242362	13
48	696334	367	938402	121	757931	488	242069	12
49	696554	367	938330	121	758224	488	241776	11
50	696775	367	938258	121	758517	488	241483	10
51	9.696995	367	9.938185	121	9.758810	488	10.241190	9
52	697215	367	938113	121	759102	487	240898	8
53	697435	366	938040	121	759395	487	240605	7
54	697654	366	937967	121	759687	487	240313	6
55	697874	366	937895	121	759979	487	240021	5
56	698094	366	937822	121	760272	487	239728	4
57	698313	365	937749	121	760564	487	239436	3
58	698532	365	937676	121	760856	486	239144	2
59	698751	365	937604	121	761148	486	238852	1
60	698970	365	937531	122	761439	486	238561	0
	Cosine.		Sine.		Cotang.		Tang.	M.

60 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.698970	365	9.937531	122	9.761439	486	10.238561	60
1	699189	364	937458	122	761731	486	238269	59
2	699407	364	937385	122	762023	486	237977	58
3	699626	364	937312	122	762314	486	237686	57
4	699844	364	937238	122	762606	485	237394	56
5	700062	363	937165	122	762897	485	237103	55
6	700280	363	937092	122	763188	485	236812	54
7	700498	363	937019	122	763479	485	236521	53
8	700716	363	936946	122	763770	485	236230	52
9	700933	362	936872	122	764061	485	235939	51
10	701151	362	936799	122	764352	484	235648	50
11	9.701368	362	9.936725	122	9.764643	484	10.235357	49
12	701585	362	936652	123	764933	484	235067	48
13	701802	361	936578	123	765224	484	234776	47
14	702019	361	936505	123	765514	484	234486	46
15	702236	361	936431	123	765805	484	234195	45
16	702452	361	936357	123	766095	484	233905	44
17	702669	360	936284	123	766385	483	233615	43
18	702885	360	936210	123	766675	483	233325	42
19	703101	360	936136	123	766965	483	233035	41
20	703317	360	936062	123	767255	483	232745	40
21	9.703533	359	9.935988	123	9.767545	483	10.232455	39
22	703749	359	935914	123	767834	483	232166	38
23	703964	359	935840	123	768124	482	231876	37
24	704179	359	935766	124	768414	482	231586	36
25	704395	359	935692	124	768703	482	231297	35
26	704610	358	935618	124	768992	482	231008	34
27	704825	358	935543	124	769281	482	230719	33
28	705040	358	935469	124	769571	482	230429	32
29	705254	358	935395	124	769860	481	230140	31
30	705469	357	935320	124	770148	481	229852	30
31	9.705683	357	9.935246	124	9.770437	481	10.229563	29
32	705898	357	935171	124	770726	481	229274	28
33	706112	357	935097	124	771015	481	228985	27
34	706326	356	935022	124	771303	481	228697	26
35	706539	356	934948	124	771592	481	228408	25
36	706753	356	934873	125	771880	480	228120	24
37	706967	356	934798	125	772168	480	227832	23
38	707180	355	934723	125	772457	480	227543	22
39	707393	355	934649	125	772745	480	227255	21
40	707606	355	934574	125	773033	480	226967	20
41	9.707819	355	9.934499	125	9.773321	480	10.226679	19
42	708032	354	934424	125	773608	480	226392	18
43	708245	354	934349	125	773896	479	226104	17
44	708458	354	934274	125	774184	479	225816	16
45	708670	354	934199	125	774471	479	225529	15
46	708882	354	934123	125	774759	479	225241	14
47	709094	353	934048	125	775046	479	224954	13
48	709306	353	933973	126	775333	479	224667	12
49	709518	353	933898	126	775621	478	224379	11
50	709730	353	933822	126	775908	478	224092	10
51	9.709941	352	9.933747	126	9.776195	478	10.223805	9
52	710153	352	933671	126	776482	478	223518	8
53	710364	352	933596	126	776768	478	223232	7
54	710575	352	933520	126	777055	478	222945	6
55	710786	351	933445	126	777342	478	222658	5
56	710997	351	933369	126	777628	477	222372	4
57	711208	351	933293	126	777915	477	222085	3
58	711419	351	933217	126	778201	477	221799	2
59	711629	351	933141	126	778488	477	221512	1
60	711839	350	933066	127	778774	477	221226	0
	Cosine.		Sine.		Cotang.		Tang.	M.

59 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.711839	350	9.933066	127	9.778774	477	10.221226	60
1	712050	350	932990	127	779060	477	220940	59
2	712260	350	932914	127	779346	477	220654	58
3	712469	350	932838	127	779632	476	220368	57
4	712679	349	932762	127	779918	476	220082	56
5	712889	349	932685	127	780203	476	219797	55
6	713098	349	932609	127	780489	476	219511	54
7	713308	349	932533	127	780775	476	219225	53
8	713517	348	932457	127	781060	476	218940	52
9	713726	348	932380	127	781346	475	218654	51
10	713935	348	932304	127	781631	475	218369	50
11	9.714144	348	9.932228	127	9.781916	475	10.218084	49
12	714352	347	932151	128	782201	475	217799	48
13	714561	347	932075	128	782486	475	217514	47
14	714769	347	931998	128	782771	475	217229	46
15	714978	347	931921	128	783056	475	216944	45
16	715186	347	931845	128	783341	474	216659	44
17	715394	346	931768	128	783626	474	216374	43
18	715602	346	931691	128	783910	474	216090	42
19	715809	346	931614	128	784195	474	215805	41
20	716017	346	931537	128	784479	474	215521	40
21	9.716224	345	9.931460	128	9.784764	474	10.215236	39
22	716432	345	931383	128	785048	474	214952	38
23	716639	345	931306	128	785332	473	214668	37
24	716846	345	931229	129	785616	473	214384	36
25	717053	345	931152	129	785900	473	214100	35
26	717259	344	931075	129	786184	473	213816	34
27	717466	344	930998	129	786468	473	213532	33
28	717673	344	930921	129	786752	473	213248	32
29	717879	344	930843	129	787036	473	212964	31
30	718085	343	930766	129	787319	472	212681	30
31	9.718291	343	9.930688	129	9.787603	472	10.212397	29
32	718497	343	930611	129	787886	472	212114	28
33	718703	343	930533	129	788170	472	211830	27
34	718909	343	930456	129	788453	472	211547	26
35	719114	342	930378	129	788736	472	211264	25
36	719320	342	930300	130	789019	472	210981	24
37	719525	342	930223	130	789302	472	210698	23
38	719730	342	930145	130	789585	471	210415	22
39	719935	341	930067	130	789868	471	210132	21
40	720140	341	929989	130	790151	471	209849	20
41	9.720345	341	9.929911	130	9.790434	471	10.209566	19
42	720549	341	929833	130	790716	471	209284	18
43	720754	341	929755	130	790999	471	209001	17
44	720958	340	929677	130	791281	471	208719	16
45	721162	340	929599	130	791563	470	208437	15
46	721366	340	929521	130	791846	470	208154	14
47	721570	340	929442	130	792128	470	207872	13
48	721774	339	929364	131	792410	470	207590	12
49	721978	339	929286	131	792692	470	207308	11
50	722181	339	929207	131	792974	470	207026	10
51	9.722385	339	9.929129	131	9.793256	470	10.206744	9
52	722588	339	929050	131	793538	469	206462	8
53	722791	338	928972	131	793819	469	206181	7
54	722994	338	928893	131	794101	469	205899	6
55	723197	338	928815	131	794383	469	205617	5
56	723400	338	928736	131	794664	469	205336	4
57	723603	337	928657	131	794946	469	205054	3
58	723805	337	928578	131	795227	469	204773	2
59	724007	337	928499	131	795508	469	204492	1
60	724210	337	928420	132	795789	468	204211	0
	Cosine.		Sine.		Cotang.		Tang.	M.

58 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.724210	337	9.928420	132	9.795789	468	10.204211	60
1	724412	337	928342	132	796070	468	203930	59
2	724614	336	928263	132	796351	468	203649	58
3	724816	336	928183	132	796632	468	203368	57
4	725017	336	928104	132	796913	468	203087	56
5	725219	336	928025	132	797194	468	202806	55
6	725420	335	927946	132	797474	468	202526	54
7	725622	335	927867	132	797755	468	202245	53
8	725823	335	927787	132	798036	467	201964	52
9	726024	335	927708	132	798316	467	201684	51
10	726225	335	927629	132	798596	467	201404	50
11	9.726426	334	9.927549	133	9.798877	467	10.201123	49
12	726626	334	927470	133	799157	467	200843	48
13	726827	334	927390	133	799437	467	200563	47
14	727027	334	927310	133	799717	467	200283	46
15	727228	334	927231	133	799997	466	200003	45
16	727428	333	927151	133	800277	466	199723	44
17	727628	333	927071	133	800557	466	199443	43
18	727828	335	926991	133	800836	466	199164	42
19	728027	333	926911	133	801116	466	198884	41
20	728227	332	926831	133	801396	466	198604	40
21	9.728427	332	9.926751	133	9.801675	466	10.198325	39
22	728626	332	926671	133	801955	466	198045	38
23	728825	332	926591	134	802234	465	197766	37
24	729024	332	926511	134	802513	465	197487	36
25	729223	331	926431	134	802792	465	197208	35
26	729422	331	926351	134	803072	465	196928	34
27	729621	331	926270	134	803351	465	196649	33
28	729820	331	926190	134	803630	465	196370	32
29	730018	331	926110	134	803909	465	196091	31
30	730217	330	926029	134	804187	465	195813	30
31	9.730415	330	9.925949	134	9.804466	464	10.195534	29
32	730613	330	925868	134	804745	464	195255	28
33	730811	330	925788	134	805023	464	194977	27
34	731009	329	925707	134	805302	464	194698	26
35	731206	329	925626	135	805580	464	194420	25
36	731404	329	925545	135	805859	464	194141	24
37	731602	329	925465	135	806137	464	193863	23
38	731799	329	925384	135	806415	464	193585	22
39	731996	328	925303	135	806693	463	193307	21
40	732193	328	925222	135	806971	463	193029	20
41	9.732390	328	9.925141	135	9.807249	463	10.192751	19
42	732587	328	925060	135	807527	463	192473	18
43	732784	328	924979	135	807805	463	192195	17
44	732980	327	924897	135	808083	463	191917	16
45	733177	327	924816	135	808361	463	191639	15
46	733373	327	924735	136	808638	463	191362	14
47	733569	327	924654	136	808916	462	191084	13
48	733765	327	924572	136	809193	462	190807	12
49	733961	326	924491	136	809471	462	190529	11
50	734157	326	924409	136	809748	462	190252	10
51	9.734353	326	9.924328	136	9.810025	462	10.189975	9
52	734549	326	924246	136	810302	462	189698	8
53	734744	326	924164	136	810580	462	189420	7
54	734939	325	924083	136	810857	462	189143	6
55	735135	325	924001	136	811134	461	188866	5
56	735330	325	923919	136	811410	461	188590	4
57	735525	325	923837	136	811687	461	188313	3
58	735719	324	923755	137	811964	461	188036	2
59	735914	324	923673	137	812241	461	187759	1
60	736109	324	923591	137	812517	461	187483	0
	Cosine.		Sine.		Cotang.		Tang.	M.

57 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.736109	324	9.923591	137	9.812517	461	10.187483	60
1	736303	324	923509	137	812794	461	187206	59
2	736498	324	923427	137	813070	461	186930	58
3	736692	323	923345	137	813347	460	186653	57
4	736886	323	923263	137	813623	460	186377	56
5	737080	323	923181	137	813899	460	186101	55
6	737274	323	923098	137	814176	460	185824	54
7	737467	323	923016	137	814452	460	185548	53
8	737661	322	922933	137	814728	460	185272	52
9	737855	322	922851	138	815004	460	184996	51
10	738048	322	922768	138	815280	460	184720	50
11	9.738241	322	9.922686	138	9.815555	460	10.184445	49
12	738434	322	922603	138	815831	459	184169	48
13	738627	321	922520	138	816107	459	183893	47
14	738820	321	922438	138	816382	459	183618	46
15	739013	321	922355	138	816658	459	183342	45
16	739206	321	922272	138	816933	459	183067	44
17	739398	321	922189	138	817209	459	182791	43
18	739590	320	922106	138	817484	459	182516	42
19	739783	320	922023	138	817759	459	182241	41
20	739975	320	921940	138	818035	459	181965	40
21	9.740167	320	9.921857	139	9.818310	458	10.181690	39
22	740359	320	921774	139	818585	458	181415	38
23	740550	319	921691	139	818860	458	181140	37
24	740742	319	921607	139	819135	458	180865	36
25	740934	319	921524	139	819410	458	180590	35
26	741125	319	921441	139	819684	458	180316	34
27	741316	319	921357	139	819959	458	180041	33
28	741508	318	921274	139	820234	458	179766	32
29	741699	318	921190	139	820508	457	179492	31
30	741889	318	921107	139	820783	457	179217	30
31	9.742080	318	9.921023	139	9.821057	457	10.178943	29
32	742271	318	920939	140	821332	457	178668	28
33	742462	317	920856	140	821606	457	178394	27
34	742652	317	920772	140	821880	457	178120	26
35	742842	317	920688	140	822154	457	177846	25
36	743033	317	920604	140	822429	457	177571	24
37	743223	317	920520	140	822703	457	177297	23
38	743413	316	920436	140	822977	456	177023	22
39	743602	316	920352	140	823251	456	176749	21
40	743792	316	920268	140	823524	456	176476	20
41	9.743982	316	9.920184	140	9.823798	456	10.176202	19
42	744171	316	920099	140	824072	456	175928	18
43	744361	315	920015	141	824345	456	175655	17
44	744550	315	919931	141	824619	456	175381	16
45	744739	315	919846	141	824893	456	175107	15
46	744928	315	919762	141	825166	456	174834	14
47	745117	315	919677	141	825439	455	174561	13
48	745306	314	919593	141	825713	455	174287	12
49	745494	314	919508	141	825986	455	174014	11
50	745683	314	919424	141	826259	455	173741	10
51	9.745871	314	9.919339	141	9.826532	455	10.173468	9
52	746060	314	919254	141	826805	455	173195	8
53	746248	313	919169	141	827078	455	172922	7
54	746436	313	919085	141	827351	455	172649	6
55	746624	313	919000	142	827624	455	172376	5
56	746812	313	918915	142	827897	455	172103	4
57	746999	313	918830	142	828170	454	171830	3
58	747187	312	918745	142	828442	454	171558	2
59	747374	312	918659	142	828715	454	171285	1
60	747562	312	918574	142	828987	454	171013	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.747562	312	9.918574	142	9.828987	454	10.171013	60
1	747749	312	918489	142	829260	454	170740	59
2	747936	312	918404	142	829532	454	170468	58
3	748123	311	918318	142	829805	454	170195	57
4	748310	311	918233	142	830077	454	169923	56
5	748497	311	918147	142	830349	454	169651	55
6	748683	311	918062	143	830621	453	169379	54
7	748870	311	917976	143	830893	453	169107	53
8	749056	310	917891	143	831165	453	168835	52
9	749243	310	917805	143	831437	453	168563	51
10	749429	310	917719	143	831709	453	168291	50
11	9.749615	310	9.917634	143	9.831981	453	10.168019	49
12	749801	310	917548	143	832253	453	167747	48
13	749987	309	917462	143	832525	453	167475	47
14	750172	309	917376	143	832796	453	167204	46
15	750358	309	917290	143	833068	452	166932	45
16	750543	309	917204	143	833339	452	166661	44
17	750729	309	917118	144	833611	452	166389	43
18	750914	309	917032	144	833882	452	166118	42
19	751099	308	916946	144	834154	452	165846	41
20	751284	308	916859	144	834425	452	165575	40
21	9.751469	308	9.916773	144	9.834696	452	10.165304	39
22	751654	308	916687	144	834967	452	165033	38
23	751839	308	916600	144	835238	452	164762	37
24	752023	307	916514	144	835509	452	164491	36
25	752208	307	916427	144	835780	451	164220	35
26	752392	307	916341	144	836051	451	163949	34
27	752576	307	916254	144	836322	451	163678	33
28	752760	307	916167	145	836593	451	163407	32
29	752944	306	916081	145	836864	451	163136	31
30	753128	306	915994	145	837134	451	162866	30
31	9.753312	306	9.915907	145	9.837405	451	10.162595	29
32	753495	306	915820	145	837675	451	162325	28
33	753679	306	915733	145	837946	451	162054	27
34	753862	305	915646	145	838216	451	161784	26
35	754046	305	915559	145	838487	450	161513	25
36	754229	305	915472	145	838757	450	161243	24
37	754412	305	915385	145	839027	450	160973	23
38	754595	305	915297	145	839297	450	160703	22
39	754778	304	915210	146	839568	450	160432	21
40	754960	304	915123	146	839838	450	160162	20
41	9.755143	304	9.915035	146	9.840108	450	10.159892	19
42	755326	304	914948	146	840378	450	159622	18
43	755508	304	914860	146	840648	450	159352	17
44	755690	304	914773	146	840917	450	159083	16
45	755872	303	914685	146	841187	449	158813	15
46	756054	303	914598	146	841457	449	158543	14
47	756236	303	914510	146	841727	449	158273	13
48	756418	303	914422	146	841996	449	158004	12
49	756600	303	914334	146	842266	449	157734	11
50	756782	302	914246	147	842535	449	157465	10
51	9.756963	302	9.914158	147	9.842805	449	10.157195	9
52	757144	302	914070	147	843074	449	156926	8
53	757326	302	913982	147	843343	449	156657	7
54	757507	302	913894	147	843612	449	156388	6
55	757688	301	913806	147	843882	449	156118	5
56	757869	301	913718	147	844151	448	155849	4
57	758050	301	913630	147	844420	448	155580	3
58	758230	301	913541	147	844689	448	155311	2
59	758411	301	913453	147	844958	448	155042	1
60	758591	301	913365	147	845227	448	154773	0
	Cosine.		Sine.		Cotang.		Tang.	M.

55 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.758591	301	9.913365	147	9.845227	448	10.154773	60
1	758772	300	913276	148	845496	448	154504	59
2	758952	300	913187	148	845764	448	154236	58
3	759132	300	913099	148	846033	448	153967	57
4	759312	300	913010	148	846302	448	153698	56
5	759492	300	912922	148	846570	448	153430	55
6	759672	299	912833	148	846839	448	153161	54
7	759852	299	912744	148	847108	447	152892	53
8	760031	299	912655	148	847376	447	152624	52
9	760211	299	912566	148	847644	447	152356	51
10	760390	299	912477	148	847913	447	152087	50
11	9.760569	299	9.912388	148	9.848181	447	10.151819	49
12	760748	298	912299	149	848449	447	151551	48
13	760927	298	912210	149	848717	447	151283	47
14	761106	298	912121	149	848986	447	151014	46
15	761285	298	912031	149	849254	447	150746	45
16	761464	298	911942	149	849522	447	150478	44
17	761642	297	911853	149	849790	446	150210	43
18	761821	297	911763	149	850057	446	149943	42
19	761999	297	911674	149	850325	446	149675	41
20	762177	297	911584	149	850593	446	149407	40
21	9.762356	297	9.911495	149	9.850861	446	10.149139	39
22	762534	296	911405	149	851129	446	148871	38
23	762712	296	911315	150	851396	446	148604	37
24	762889	296	911226	150	851664	446	148336	36
25	763067	296	911136	150	851931	446	148069	35
26	763245	296	911046	150	852199	446	147801	34
27	763422	296	910956	150	852466	446	147534	33
28	763600	295	910866	150	852733	446	147267	32
29	763777	295	910776	150	853001	445	146999	31
30	763954	295	910686	150	853268	445	146732	30
31	9.764131	295	9.910596	150	9.853535	445	10.146465	29
32	764308	295	910506	150	853802	445	146198	28
33	764485	294	910415	151	854069	445	145931	27
34	764662	294	910325	151	854336	445	145664	26
35	764838	294	910235	151	854603	445	145397	25
36	765015	294	910144	151	854870	445	145130	24
37	765191	294	910054	151	855137	445	144863	23
38	765367	294	909963	151	855404	445	144596	22
39	765544	293	909873	151	855671	445	144329	21
40	765720	293	909782	151	855938	444	144062	20
41	9.765896	293	9.909691	151	9.856204	444	10.143796	19
42	766072	293	909601	151	856471	444	143529	18
43	766247	293	909510	151	856737	444	143263	17
44	766423	293	909419	152	857004	444	142996	16
45	766598	292	909328	152	857270	444	142730	15
46	766774	292	909237	152	857537	444	142463	14
47	766949	292	909146	152	857803	444	142197	13
48	767124	292	909055	152	858069	444	141931	12
49	767300	292	908964	152	858336	444	141664	11
50	767475	291	908873	152	858602	444	141398	10
51	9.767649	291	9.908781	152	9.858868	443	10.141132	9
52	767824	291	908690	152	859134	443	140866	8
53	767999	291	908599	152	859400	443	140600	7
54	768173	291	908507	152	859666	443	140334	6
55	768348	291	908416	153	859932	443	140068	5
56	768522	290	908324	153	860198	443	139802	4
57	768697	290	908233	153	860464	443	139536	3
58	768871	290	908141	153	860730	443	139270	2
59	769045	290	908049	153	860995	443	139005	1
60	769219	290	907958	153	861261	443	138739	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.769219	290	9.907958	153	9.861261	443	10.138739	60
1	769393	289	907866	153	861527	443	138473	59
2	769566	289	907774	153	861792	443	138208	58
3	769740	289	907682	153	862058	442	137942	57
4	769913	289	907590	153	862323	442	137677	56
5	770087	289	907498	153	862589	442	137411	55
6	770260	289	907406	154	862854	442	137146	54
7	770433	288	907314	154	863119	442	136881	53
8	770606	288	907222	154	863385	442	136615	52
9	770779	288	907129	154	863650	442	136350	51
10	770952	288	907037	154	863915	442	136085	50
11	9.771125	288	9.906945	154	9.864180	442	10.135820	49
12	771298	288	906852	154	864445	442	135555	48
13	771470	287	906760	154	864710	442	135290	47
14	771643	287	906667	154	864975	442	135025	46
15	771815	287	906575	154	865240	441	134760	45
16	771987	287	906482	154	865505	441	134495	44
17	772159	287	906389	155	865770	441	134230	43
18	772331	286	906296	155	866035	441	133965	42
19	772503	286	906204	155	866300	441	133700	41
20	772675	286	906111	155	866564	441	133436	40
21	9.772847	286	9.906018	155	9.866829	441	10.133171	39
22	773018	286	905925	155	867094	441	132906	38
23	773190	286	905832	155	867358	441	132642	37
24	773361	285	905739	155	867623	441	132377	36
25	773533	285	905645	155	867887	441	132113	35
26	773704	285	905552	155	868152	441	131848	34
27	773875	285	905459	156	868416	440	131584	33
28	774046	285	905366	156	868680	440	131320	32
29	774217	285	905272	156	868945	440	131055	31
30	774388	284	905179	156	869209	440	130791	30
31	9.774558	284	9.905085	156	9.869473	440	10.130527	29
32	774729	284	904992	156	869737	440	130263	28
33	774899	284	904898	156	870001	440	129999	27
34	775070	284	904804	156	870265	440	129735	26
35	775240	284	904711	156	870529	440	129471	25
36	775410	283	904617	156	870793	440	129207	24
37	775580	283	904523	156	871057	440	128943	23
38	775750	283	904429	157	871321	440	128679	22
39	775920	283	904335	157	871585	440	128415	21
40	776090	283	904241	157	871849	439	128151	20
41	9.776259	283	9.904147	157	9.872112	439	10.127888	19
42	776429	282	904053	157	872376	439	127624	18
43	776598	282	903959	157	872640	439	127360	17
44	776768	282	903864	157	872903	439	127097	16
45	776937	282	903770	157	873167	439	126833	15
46	777106	282	903676	157	873430	439	126570	14
47	777275	281	903581	157	873694	439	126306	13
48	777444	281	903487	158	873957	439	126043	12
49	777613	281	903392	158	874220	439	125780	11
50	777781	281	903298	158	874484	439	125516	10
51	9.777950	281	9.903203	158	9.874747	439	10.125253	9
52	778119	281	903108	158	875010	439	124990	8
53	778287	280	903014	158	875273	439	124727	7
54	778455	280	902919	158	875537	438	124463	6
55	778624	280	902824	158	875800	438	124200	5
56	778792	280	902729	158	876063	438	123937	4
57	778960	280	902634	158	876326	438	123674	3
58	779128	280	902539	158	876589	438	123411	2
59	779295	279	902444	159	876852	438	123148	1
60	779463	279	902349	159	877114	438	122886	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.779463	279	9.902349	159	9.877114	438	10.122886	60
1	779631	279	902253	159	877377	438	122623	59
2	779798	279	902158	159	877640	438	122360	58
3	779966	279	902063	159	877903	438	122097	57
4	780133	279	901967	159	878165	438	121835	56
5	780300	278	901872	159	878428	438	121572	55
6	780467	278	901776	159	878691	438	121309	54
7	780634	278	901681	159	878953	437	121047	53
8	780801	278	901585	159	879216	437	120784	52
9	780968	278	901490	160	879478	437	120522	51
10	781134	278	901394	160	879741	437	120259	50
11	9.781301	277	9.901298	160	9.880003	437	10.119997	49
12	781468	277	901202	160	880265	437	119735	48
13	781634	277	901106	160	880528	437	119472	47
14	781800	277	901010	160	880790	437	119210	46
15	781966	277	900914	160	881052	437	118948	45
16	782132	277	900818	160	881314	437	118686	44
17	782298	276	900722	160	881577	437	118423	43
18	782464	276	900626	160	881839	437	118161	42
19	782630	276	900529	160	882101	437	117899	41
20	782796	276	900433	161	882363	437	117637	40
21	9.782961	276	9.900337	161	9.882625	436	10.117375	39
22	783127	276	900240	161	882887	436	117113	38
23	783292	275	900144	161	883148	436	116852	37
24	783458	275	900047	161	883410	436	116590	36
25	783623	275	899951	161	883672	436	116328	35
26	783788	275	899854	161	883934	436	116066	34
27	783953	275	899757	161	884196	436	115804	33
28	784118	275	899660	161	884457	436	115543	32
29	784282	274	899564	161	884719	436	115281	31
30	784447	274	899467	162	884980	436	115020	30
31	9.784612	274	9.899370	162	9.885242	436	10.114758	29
32	784776	274	899273	162	885504	436	114496	28
33	784941	274	899176	162	885765	436	114235	27
34	785105	274	899078	162	886026	436	113974	26
35	785269	273	898981	162	886288	436	113712	25
36	785433	273	898884	162	886549	435	113451	24
37	785597	273	898787	162	886811	435	113189	23
38	785761	273	898689	162	887072	435	112928	22
39	785925	273	898592	162	887333	435	112667	21
40	786089	273	898494	163	887594	435	112406	20
41	9.786252	272	9.898397	163	9.887855	435	10.112145	19
42	786416	272	898299	163	888116	435	111884	18
43	786579	272	898202	163	888378	435	111622	17
44	786742	272	898104	163	888639	435	111361	16
45	786906	272	898006	163	888900	435	111100	15
46	787069	272	897908	163	889161	435	110839	14
47	787232	271	897810	163	889421	435	110579	13
48	787395	271	897712	163	889682	435	110318	12
49	787557	271	897614	163	889943	435	110057	11
50	787720	271	897516	164	890204	435	109796	10
51	9.787883	271	9.897418	164	9.890465	434	10.109535	9
52	788045	271	897320	164	890725	434	109275	8
53	788208	270	897222	164	890986	434	109014	7
54	788370	270	897123	164	891247	434	108753	6
55	788532	270	897025	164	891507	434	108493	5
56	788694	270	896926	164	891768	434	108232	4
57	788856	270	896828	164	892028	434	107972	3
58	789018	270	896729	164	892289	434	107711	2
59	789180	270	896631	164	892549	434	107451	1
60	789342	269	896532	165	892810	434	107190	0
	Cosine.		Sine.		Cotang.		Tang.	M.

52 Degrees.

M.	Sine.	D.100".	Cosine.	D.	Tang.	D.100".	Cotang.	
0	9.789342	269	9.896532	165	9.892810	434	10.107190	60
1	789504	269	896433	165	893070	434	106930	59
2	789665	269	896335	165	893331	434	106669	58
3	789827	269	896236	165	893591	434	106409	57
4	789988	269	896137	165	893851	434	106149	56
5	790149	269	896038	165	894111	434	105889	55
6	790310	268	895939	165	894372	434	105628	54
7	790471	268	895840	165	894632	434	105368	53
8	790632	268	895741	165	894892	433	105108	52
9	790793	268	895641	165	895152	433	104848	51
10	790954	268	895542	165	895412	433	104588	50
11	9.791115	268	9.895443	166	9.895672	433	10.104328	49
12	791275	267	895343	166	895932	433	104068	48
13	791436	267	895244	166	896192	433	103808	47
14	791596	267	895145	166	896452	433	103548	46
15	791757	267	895045	166	896712	433	103288	45
16	791917	267	894945	166	896971	433	103029	44
17	792077	267	894846	166	897231	433	102769	43
18	792237	266	894746	166	897491	433	102509	42
19	792397	266	894646	166	897751	433	102249	41
20	792557	266	894546	166	898010	433	101990	40
21	9.792716	266	9.894446	167	9.898270	433	10.101730	39
22	792876	266	894346	167	898530	433	101470	38
23	793035	266	894246	167	898789	433	101211	37
24	793195	266	894146	167	899049	433	100951	36
25	793354	265	894046	167	899308	432	100692	35
26	793514	265	893946	167	899568	432	100432	34
27	793673	265	893846	167	899827	432	100173	33
28	793832	265	893745	167	900087	432	099913	32
29	793991	265	893645	167	900346	432	099654	31
30	794150	265	893544	167	900605	432	099395	30
31	9.794308	264	9.893444	168	9.900864	432	10.099136	29
32	794467	264	893343	168	901124	432	098876	28
33	794626	264	893243	168	901383	432	098617	27
34	794784	264	893142	168	901642	432	098358	26
35	794942	264	893041	168	901901	432	098099	25
36	795101	264	892940	168	902160	432	097840	24
37	795259	263	892839	168	902420	432	097580	23
38	795417	263	892739	168	902679	432	097321	22
39	795575	263	892638	168	902938	432	097062	21
40	795733	263	892536	168	903197	432	096803	20
41	9.795891	263	9.892435	169	9.903456	432	10.096544	19
42	796049	263	892334	169	903714	431	096286	18
43	796206	263	892233	169	903973	431	096027	17
44	796364	262	892132	169	904232	431	095768	16
45	796521	262	892030	169	904491	431	095509	15
46	796679	262	891929	169	904750	431	095250	14
47	796836	262	891827	169	905008	431	094992	13
48	796993	262	891726	169	905267	431	094733	12
49	797150	262	891624	169	905526	431	094474	11
50	797307	261	891523	169	905785	431	094215	10
51	9.797464	261	9.891421	170	9.906043	431	10.093957	9
52	797621	261	891319	170	906302	431	093698	8
53	797777	261	891217	170	906560	431	093440	7
54	797934	261	891115	170	906819	431	093181	6
55	798091	261	891013	170	907077	431	092923	5
56	798247	261	890911	170	907336	431	092664	4
57	798403	260	890809	170	907594	431	092406	3
58	798560	260	890707	170	907853	431	092147	2
59	798716	260	890605	170	908111	431	091889	1
60	798872	260	890503	171	908369	430	091631	0
	Cosine.		Sine.		Cotang.		Tang.	M.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.798872	260	9.890503	171	9.908369	430	10.091631	60
1	799028	260	890400	171	908628	430	091372	59
2	799184	260	890298	171	908886	430	091114	58
3	799339	259	890195	171	909144	430	090856	57
4	799495	259	890093	171	909402	430	090598	56
5	799651	259	889990	171	909660	430	090340	55
6	799806	259	889888	171	909918	430	090082	54
7	799962	259	889785	171	910177	430	089823	53
8	800117	259	889682	171	910435	430	089565	52
9	800272	259	889579	171	910693	430	089307	51
10	800427	258	889477	172	910951	430	089049	50
11	9.800582	258	9.889374	172	9.911209	430	10.088791	49
12	800737	258	889271	172	911467	430	088533	48
13	800892	258	889168	172	911725	430	088275	47
14	801047	258	889064	172	911982	430	088018	46
15	801201	258	888961	172	912240	430	087760	45
16	801356	257	888858	172	912498	430	087502	44
17	801511	257	888755	172	912756	430	087244	43
18	801665	257	888651	172	913014	430	086986	42
19	801819	257	888548	172	913271	429	086729	41
20	801973	257	888444	173	913529	429	086471	40
21	9.802128	257	9.888341	173	9.913787	429	10.086213	39
22	802282	257	888237	173	914044	429	085956	38
23	802436	256	888134	173	914302	429	085698	37
24	802589	256	888030	173	914560	429	085440	36
25	802743	256	887926	173	914817	429	085183	35
26	802897	256	887822	173	915075	429	084925	34
27	803050	256	887718	173	915332	429	084668	33
28	803204	256	887614	173	915590	429	084410	32
29	803357	255	887510	173	915847	429	084153	31
30	803511	255	887406	174	916104	429	083896	30
31	9.803664	255	9.887302	174	9.916362	429	10.083638	29
32	803817	255	887198	174	916619	429	083381	28
33	803970	255	887093	174	916877	429	083123	27
34	804123	255	886989	174	917134	429	082866	26
35	804276	255	886885	174	917391	429	082609	25
36	804428	254	886780	174	917648	429	082352	24
37	804581	254	886676	174	917906	429	082094	23
38	804734	254	886571	174	918163	429	081837	22
39	804886	254	886466	175	918420	428	081580	21
40	805039	254	886362	175	918677	428	081323	20
41	9.805191	254	9.886257	175	9.918934	428	10.081066	19
42	805343	253	886152	175	919191	428	080809	18
43	805495	253	886047	175	919448	428	080552	17
44	805647	253	885942	175	919705	428	080295	16
45	805799	253	885837	175	919962	428	080038	15
46	805951	253	885732	175	920219	428	079781	14
47	806103	253	885627	175	920476	428	079524	13
48	806254	253	885522	175	920733	428	079267	12
49	806406	252	885416	176	920990	428	079010	11
50	806557	252	885311	176	921247	428	078753	10
51	9.806709	252	9.885205	176	9.921503	428	10.078497	9
52	806860	252	885100	176	921760	428	078240	8
53	807011	252	884994	176	922017	428	077983	7
54	807163	252	884889	176	922274	428	077726	6
55	807314	252	884783	176	922530	428	077470	5
56	807465	251	884677	176	922787	428	077213	4
57	807615	251	884572	176	923044	428	076956	3
58	807766	251	884466	176	923300	428	076700	2
59	807917	251	884360	177	923557	428	076443	1
60	808067	251	884254	177	923814	428	076186	0
	Cosine.		Sine.		Cotang.		Tang.	M.

50 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.808067	251	9.884254	177	9.923814	428	10.076186	60
1	808218	251	884148	177	924070	427	075930	59
2	808368	251	884042	177	924327	427	075673	58
3	808519	250	883936	177	924583	427	075417	57
4	808669	250	883829	177	924840	427	075160	56
5	808819	250	883723	177	925096	427	074904	55
6	808969	250	883617	177	925352	427	074648	54
7	809119	250	883510	177	925609	427	074391	53
8	809269	250	883404	178	925865	427	074135	52
9	809419	249	883297	178	926122	427	073878	51
10	809569	249	883191	178	926378	427	073622	50
11	9.809718	249	9.883084	178	9.926634	427	10.073366	49
12	809868	249	882977	178	926890	427	073110	48
13	810017	249	882871	178	927147	427	072853	47
14	810167	249	882764	178	927403	427	072597	46
15	810316	249	882657	178	927659	427	072341	45
16	810465	248	882550	178	927915	427	072085	44
17	810614	248	882443	178	928171	427	071829	43
18	810763	248	882336	179	928427	427	071573	42
19	810912	248	882229	179	928684	427	071316	41
20	811061	248	882121	179	928940	427	071060	40
21	9.811210	248	9.882014	179	9.929196	427	10.070804	39
22	811358	248	881907	179	929452	427	070548	38
23	811507	247	881799	179	929708	427	070292	37
24	811655	247	881692	179	929964	427	070036	36
25	811804	247	881584	179	930220	426	069780	35
26	811952	247	881477	179	930475	426	069525	34
27	812100	247	881369	180	930731	426	069269	33
28	812248	247	881261	180	930987	426	069013	32
29	812396	247	881153	180	931243	426	068757	31
30	812544	246	881046	180	931499	426	068501	30
31	9.812692	246	9.880938	180	9.931755	426	10.068245	29
32	812840	246	880830	180	932010	426	067990	28
33	812988	246	880722	180	932266	426	067734	27
34	813135	246	880613	180	932522	426	067478	26
35	813283	246	880505	180	932778	426	067222	25
36	813430	246	880397	180	933033	426	066967	24
37	813578	245	880289	181	933289	426	066711	23
38	813725	245	880180	181	933545	426	066455	22
39	813872	245	880072	181	933800	426	066200	21
40	814019	245	879963	181	934056	426	065944	20
41	9.814166	245	9.879855	181	9.934311	426	10.065689	19
42	814313	245	879746	181	934567	426	065433	18
43	814460	245	879637	181	934822	426	065178	17
44	814607	244	879529	181	935078	426	064922	16
45	814753	244	879420	181	935333	426	064667	15
46	814900	244	879311	182	935589	426	064411	14
47	815046	244	879202	182	935844	426	064156	13
48	815193	244	879093	182	936100	426	063900	12
49	815339	244	878984	182	936355	426	063645	11
50	815485	244	878875	182	936611	426	063389	10
51	9.815632	243	9.878766	182	9.936866	425	10.063134	9
52	815778	243	878656	182	937121	425	062879	8
53	815924	243	878547	182	937377	425	062623	7
54	816069	243	878438	182	937632	425	062368	6
55	816215	243	878328	183	937887	425	062113	5
56	816361	243	878219	183	938142	425	061858	4
57	816507	243	878109	183	938398	425	061602	3
58	816652	242	877999	183	938653	425	061347	2
59	816798	242	877890	183	938908	425	061092	1
60	816943	242	877780	183	939163	425	060837	0
	Cosine.		Sine.		Cotang.		Tang.	M.

49 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.816943	242	9.877780	183	9.939163	425	10.060837	60
1	817088	242	877670	183	939418	425	060582	59
2	817233	242	877560	183	939673	425	060327	58
3	817379	242	877450	183	939928	425	060072	57
4	817524	242	877340	184	940183	425	059817	56
5	817668	241	877230	184	940439	425	059561	55
6	817813	241	877120	184	940694	425	059306	54
7	817958	241	877010	184	940949	425	059051	53
8	818103	241	876899	184	941204	425	058796	52
9	818247	241	876789	184	941459	425	058541	51
10	818392	241	876678	184	941713	425	058287	50
11	9.818536	241	9.876568	184	9.941968	425	10.058032	49
12	818681	240	876457	184	942223	425	057777	48
13	818825	240	876347	184	942478	425	057522	47
14	818969	240	876236	185	942733	425	057267	46
15	819113	240	876125	185	942988	425	057012	45
16	819257	240	876014	185	943243	425	056757	44
17	819401	240	875904	185	943498	425	056502	43
18	819545	240	875793	185	943752	425	056248	42
19	819689	239	875682	185	944007	425	055993	41
20	819832	239	875571	185	944262	425	055738	40
21	9.819976	239	9.875459	185	9.944517	425	10.055483	39
22	820120	239	875348	185	944771	424	055229	38
23	820263	239	875237	186	945026	424	054974	37
24	820406	239	875126	186	945281	424	054719	36
25	820550	239	875014	186	945535	424	054465	35
26	820693	238	874903	186	945790	424	054210	34
27	820836	238	874791	186	946045	424	053955	33
28	820979	238	874680	186	946299	424	053701	32
29	821122	238	874568	186	946554	424	053446	31
30	821265	238	874456	186	946808	424	053192	30
31	9.821407	238	9.874344	186	9.947063	424	10.052937	29
32	821550	238	874232	187	947318	424	052682	28
33	821693	237	874121	187	947572	424	052428	27
34	821835	237	874009	187	947827	424	052173	26
35	821977	237	873896	187	948081	424	051919	25
36	822120	237	873784	187	948335	424	051665	24
37	822262	237	873672	187	948590	424	051410	23
38	822404	237	873560	187	948844	424	051156	22
39	822546	237	873448	187	949099	424	050901	21
40	822688	237	873335	187	949353	424	050647	20
41	9.822830	236	9.873223	187	9.949608	424	10.050392	19
42	822972	236	873110	188	949862	424	050138	18
43	823114	236	872998	188	950116	424	049884	17
44	823255	236	872885	188	950371	424	049629	16
45	823397	236	872772	188	950625	424	049375	15
46	823539	236	872659	188	950879	424	049121	14
47	823680	236	872547	188	951133	424	048867	13
48	823821	235	872434	188	951388	424	048612	12
49	823963	235	872321	188	951642	424	048358	11
50	824104	235	872208	188	951896	424	048104	10
51	9.824245	235	9.872095	189	9.952150	424	10.047850	9
52	824386	235	871981	189	952405	424	047595	8
53	824527	235	871868	189	952659	424	047341	7
54	824668	235	871755	189	952913	424	047087	6
55	824808	234	871641	189	953167	424	046833	5
56	824949	234	871528	189	953421	424	046579	4
57	825090	234	871414	189	953675	423	046325	3
58	825230	234	871301	189	953929	423	046071	2
59	825371	234	871187	189	954183	423	045817	1
60	825511	234	871073	190	954437	423	045563	0
	Cosine.		Sine.		Cotang.		Tang.	M.

48 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.825511	234	9.871073	190	9.954437	423	10.045563	60
1	825651	234	870960	190	954691	423	045309	59
2	825791	233	870846	190	954946	423	045054	58
3	825931	233	870732	190	955200	423	044800	57
4	826071	233	870618	190	955454	423	044546	56
5	826211	233	870504	190	955708	423	044292	55
6	826351	233	870390	190	955961	423	044039	54
7	826491	233	870276	190	956215	423	043785	53
8	826631	233	870161	190	956469	423	043531	52
9	826770	233	870047	191	956723	423	043277	51
10	826910	232	869933	191	956977	423	043023	50
11	9.827049	232	9.869818	191	9.957231	423	10.042769	49
12	827189	232	869704	191	957485	423	042515	48
13	827328	232	869589	191	957739	423	042261	47
14	827467	232	869474	191	957993	423	042007	46
15	827606	232	869360	191	958247	423	041753	45
16	827745	232	869245	191	958500	423	041500	44
17	827884	231	869130	192	958754	423	041246	43
18	828023	231	869015	192	959008	423	040992	42
19	828162	231	868900	192	959262	423	040738	41
20	828301	231	868785	192	959516	423	040484	40
21	9.828439	231	9.868670	192	9.959769	423	10.040231	39
22	828578	231	868555	192	960023	423	039977	38
23	828716	231	868440	192	960277	423	039723	37
24	828855	230	868324	192	960530	423	039470	36
25	828993	230	868209	193	960784	423	039216	35
26	829131	230	868093	193	961038	423	038962	34
27	829269	230	867978	193	961292	423	038708	33
28	829407	230	867862	193	961545	423	038455	32
29	829545	230	867747	193	961799	423	038201	31
30	829683	230	867631	193	962052	423	037948	30
31	9.829821	230	9.867515	193	9.962306	423	10.037694	29
32	829959	229	867399	193	962560	423	037440	28
33	830097	229	867283	193	962813	423	037187	27
34	830234	229	867167	194	963067	423	036933	26
35	830372	229	867051	194	963320	423	036680	25
36	830509	229	866935	194	963574	423	036426	24
37	830646	229	866819	194	963828	423	036172	23
38	830784	229	866703	194	964081	423	035919	22
39	830921	228	866586	194	964335	423	035665	21
40	831058	228	866470	194	964588	422	035412	20
41	9.831195	228	9.866353	194	9.964842	422	10.035158	19
42	831332	228	866237	194	965095	422	034905	18
43	831469	228	866120	195	965349	422	034651	17
44	831606	228	866004	195	965602	422	034398	16
45	831742	228	865887	195	965855	422	034145	15
46	831879	228	865770	195	966109	422	033891	14
47	832015	227	865653	195	966362	422	033638	13
48	832152	227	865536	195	966616	422	033384	12
49	832288	227	865419	195	966869	422	033131	11
50	832425	227	865302	195	967123	422	032877	10
51	9.832561	227	9.865185	195	9.967376	422	10.032624	9
52	832697	227	865068	195	967629	422	032371	8
53	832833	227	864950	196	967883	422	032117	7
54	832969	226	864833	196	968136	422	031864	6
55	833105	226	864716	196	968389	422	031611	5
56	833241	226	864598	196	968643	422	031357	4
57	833377	226	864481	196	968896	422	031104	3
58	833512	226	864363	196	969149	422	030851	2
59	833648	226	864245	196	969403	422	030597	1
60	833783	226	864127	196	969656	422	030344	0
	Cosine.		Sine.		Cotang.		Tang.	M.

47 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.833783	226	9.864127	196	9.969656	422	10.030344	60
1	833919	226	864010	196	969909	422	030091	59
2	834054	225	863892	197	970162	422	029838	58
3	834189	225	863774	197	970416	422	029584	57
4	834325	225	863656	197	970669	422	029331	56
5	834460	225	863538	197	970922	422	029078	55
6	834595	225	863419	197	971175	422	028825	54
7	834730	225	863301	197	971429	422	028571	53
8	834865	225	863183	197	971682	422	028318	52
9	834999	225	863064	197	971935	422	028065	51
10	835134	224	862946	198	972188	422	027812	50
11	9.835269	224	9.862827	198	9.972441	422	10.027559	49
12	835403	224	862709	198	972695	422	027305	48
13	835538	224	862590	198	972948	422	027052	47
14	835672	224	862471	198	973201	422	026799	46
15	835807	224	862353	198	973454	422	026546	45
16	835941	224	862234	198	973707	422	026293	44
17	836075	223	862115	198	973960	422	026040	43
18	836209	223	861996	198	974213	422	025787	42
19	836343	223	861877	199	974466	422	025534	41
20	836477	223	861758	199	974720	422	025280	40
21	9.836611	223	9.861638	199	9.974973	422	10.025027	39
22	836745	223	861519	199	975226	422	024774	38
23	836878	223	861400	199	975479	422	024521	37
24	837012	223	861280	199	975732	422	024268	36
25	837146	222	861161	199	975985	422	024015	35
26	837279	222	861041	199	976238	422	023762	34
27	837412	222	860922	199	976491	422	023509	33
28	837546	222	860802	200	976744	422	023256	32
29	837679	222	860682	200	976997	422	023003	31
30	837812	222	860562	200	977250	422	022750	30
31	9.837945	222	9.860442	200	9.977503	422	10.022497	29
32	838078	222	860322	200	977756	422	022244	28
33	838211	221	860202	200	978009	422	021991	27
34	838344	221	860082	200	978262	422	021738	26
35	838477	221	859962	200	978515	422	021485	25
36	838610	221	859842	201	978768	422	021232	24
37	838742	221	859721	201	979021	422	020979	23
38	838875	221	859601	201	979274	422	020726	22
39	839007	221	859480	201	979527	422	020473	21
40	839140	220	859360	201	979780	422	020220	20
41	9.839272	220	9.859239	201	9.980033	422	10.019967	19
42	839404	220	859119	201	980286	422	019714	18
43	839536	220	858998	201	980538	422	019462	17
44	839668	220	858877	201	980791	422	019209	16
45	839800	220	858756	202	981044	421	018956	15
46	839932	220	858635	202	981297	421	018703	14
47	840064	220	858514	202	981550	421	018450	13
48	840196	219	858393	202	981803	421	018197	12
49	840328	219	858272	202	982056	421	017944	11
50	840459	219	858151	202	982309	421	017691	10
51	9.840591	219	9.858029	202	9.982562	421	10.017438	9
52	840722	219	857908	202	982814	421	017186	8
53	840854	219	857786	203	983067	421	016933	7
54	840985	219	857665	203	983320	421	016680	6
55	841116	219	857543	203	983573	421	016427	5
56	841247	218	857422	203	983826	421	016174	4
57	841378	218	857300	203	984079	421	015921	3
58	841509	218	857178	203	984332	421	015668	2
59	841640	218	857056	203	984584	421	015416	1
60	841771	218	856934	203	984837	421	015163	0
	Cosine.		Sine.		Cotang.		Tang.	M.

46 Degrees.

M.	Sine.	D.100″.	Cosine.	D.	Tang.	D.100″.	Cotang.	
0	9.841771	218	9.856934	208	9.984837	421	10.015163	60
1	841902	218	856812	204	985090	421	014910	59
2	842033	218	856690	204	985343	421	014657	58
3	842163	218	856568	204	985596	421	014404	57
4	842294	217	856446	204	985848	421	014152	56
5	842424	217	856323	204	986101	421	013899	55
6	842555	217	856201	204	986354	421	013646	54
7	842685	217	856078	204	986607	421	013393	53
8	842815	217	855956	204	986860	421	013140	52
9	842946	217	855833	204	987112	421	012888	51
10	843076	217	855711	205	987365	421	012635	50
11	9.843206	217	9.855588	205	9.987618	421	10.012382	49
12	843336	216	855465	205	987871	421	012129	48
13	843466	216	855342	205	988123	421	011877	47
14	843595	216	855219	205	988376	421	011624	46
15	843725	216	855096	205	988629	421	011371	45
16	843855	216	854973	205	988882	421	011118	44
17	843984	216	854850	205	989134	421	010866	43
18	844114	216	854727	206	989387	421	010613	42
19	844243	216	854603	206	989640	421	010360	41
20	844372	215	854480	206	989893	421	010107	40
21	9.844502	215	9.854356	506	9.990145	421	10.009855	39
22	844631	215	854233	206	990398	421	009602	38
23	844760	215	854109	206	990651	421	009349	37
24	844889	215	853986	206	990903	421	009097	36
25	845018	215	853862	206	991156	421	008844	35
26	845147	215	853738	206	991409	421	008591	34
27	845276	215	853614	207	991662	421	008338	33
28	845405	214	853490	207	991914	421	008086	32
29	845533	214	853366	207	992167	421	007833	31
30	845662	214	853242	207	992420	421	007580	30
31	9.845790	214	9.853118	207	9.992672	421	10.007328	29
32	845919	214	852994	207	992925	421	007075	28
33	846047	214	852869	207	993178	421	006822	27
34	846175	214	852745	207	993431	421	006569	26
35	846304	214	852620	208	993683	421	006317	25
36	846432	213	852496	208	993936	421	006064	24
37	846560	213	852371	208	994189	421	005811	23
38	846688	213	852247	208	994441	421	005559	22
39	846816	213	852122	208	994694	421	005306	21
40	846944	213	851997	208	994947	421	005053	20
41	9.847071	213	9.851872	208	9.995199	421	10.004801	19
42	847199	213	851747	208	995452	421	004548	18
43	847327	213	851622	209	995705	421	004295	17
44	847454	212	851497	209	995957	421	004043	16
45	847582	212	851372	209	996210	421	003790	15
46	847709	212	851246	209	996463	421	003537	14
47	847836	212	851121	209	996715	421	003285	13
48	847964	212	850996	209	996968	421	003032	12
49	848091	212	850870	209	997221	421	002779	11
50	848218	212	850745	209	997473	421	002527	10
51	9.848345	212	9.850619	210	9.997726	421	10.002274	9
52	848472	211	850493	210	997979	421	002021	8
53	848599	211	850368	210	998231	421	001769	7
54	848726	211	850242	210	998484	421	001516	6
55	848852	211	850116	210	998737	421	001263	5
56	848979	211	849990	210	998989	421	001011	4
57	849106	211	849864	210	999242	421	000758	3
58	849232	211	849738	210	999495	421	000505	2
59	849359	211	849611	210	999747	421	000253	1
60	849485	211	849485	210	10.000000	421	000000	0
	Cosine.		Sine.		Cotang.		Tang.	M.

45 Degrees.

APPENDIX.

THIS Appendix contains a collection of miscellaneous propositions which will be found useful to those who wish to extend their studies beyond the range which is usually prescribed in a college course, and it includes brief notices of various topics which have not hitherto been generally admitted into the Elements of Geometry. Most of these topics are embraced under the comprehensive term of Modern Geometry, since they were either entirely unknown or had attracted but little attention until recent times.

The treatise on Descriptive Geometry by Monge, published in 1794, gave a new impulse to the study of pure geometry. In 1803 Carnot published his Geometry of Position, and in 1806 an Essay on Transversals. In 1817 Brianchon extended this theory and that of harmonic pencils, applying them to the Conic Sections. In 1822 Poncelet published his treatise on the projective properties of figures, in which work he treated of Poles and Polars, Centres of Similitude, and, indeed, nearly every branch of Modern Geometry. Since that time the same subjects have been further developed by Steiner, Poinsot, Cayley, Chasles, Salmon, Cremona, and many other mathematicians.

I take pleasure in acknowledging my obligations to Prof. H. A. Newton and Prof. J. E. Clark, to each of whom I am indebted for important assistance in preparing this Appendix.

E. L.

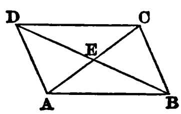
CONTENTS OF APPENDIX.

APPENDIX.

MISCELLANEOUS PROPOSITIONS.

Proposition I.

1. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



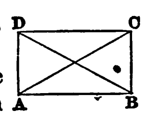
Let ABCD be a quadrilateral whose diagonals AC, BD bisect each other; then will ABCD be a parallelogram.

Because AE=EC, and EB=ED, and the angles AEB, DEC, being opposite, are equal to each other, therefore the two triangles AEB and DEC are equal to each other (B. I., Pr. 6), and the angle EAB is equal to the angle ECD. Therefore AB is parallel to CD (B. I., Pr. 22). In like manner it may be proved that AD is parallel to BC, and therefore ABCD is a parallelogram.

2. *Cor.* If the diagonals of a quadrilateral bisect each other at right angles, the figure is a rhombus.

Proposition II.

3. *The diagonals of a rectangle are equal.*



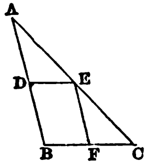
Let ABCD be a rectangle; then will its diagonals AC, BD be equal to each other.

Since the right angle ADC is equal to the right angle BCD, also the side AD is equal to BC, and CD is common to the two triangles ADC, BCD, therefore these two triangles are equal to each other, and hence AC=BD.

4. *Cor.* The diagonals of a square are equal; they bisect each other at right angles, and they also bisect the angles of the square.

Proposition III.

5. *The straight line which joins the middle points of any two sides of a triangle is parallel to the third side, and is equal to one half of that side.*



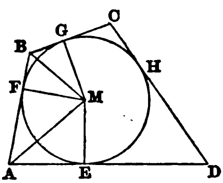
In the triangle ABC, let D be the middle point of AB, and E the middle point of AC; then will DE be parallel to BC, and will be equal to one half of BC.

Through E draw EF parallel to AB; then in the triangles ADE, EFC we have AE=EC; also AD is equal to DB, which is equal to EF (B. I., Pr. 30); and the angle DAE is equal to the angle FEC (B. I., Pr. 23); therefore the angle AED is equal to the angle ECF (B. I., Pr. 6), and DE is parallel to BC (B. I., Pr. 22). Also FC is equal to DE, which is equal to BF; and therefore DE is one half of BC.

6. *Cor.* If a straight line drawn parallel to the base of a triangle bisects one of the sides, it also bisects the other side.

Proposition IV.

7. *If a quadrilateral be described about a circle, the sum of two opposite sides is equal to the sum of the other two opposite sides.*



Let ABCD be a quadrilateral described about a circle; the sum of two opposite sides, as AB and CD, is equal to the sum of the other two sides, BC and AD.

Let E, F, G, H be the points at which the sides of the quadrilateral touch the circle. Let M be the centre of the circle, and join ME, MA, MF. The triangles MAE, MAF, being right-angled, are equal (B. I., Pr. 19); hence AE=AF. For a like reason we have

$$BF=BG;\ CG=CH;\ \text{and } DE=DH.$$

Adding the corresponding members of these four equations, we have

$$AF+BF+CH+DH=AE+DE+BG+CG;$$

that is,

$$AB+CD=AD+BC.$$

Proposition V.

8. *If the sum of two opposite sides of a quadrilateral is equal to the sum of the other two sides, a circle can be inscribed within the quadrilateral.*

Let ABCD be a quadrilateral, having the sum of the sides AB and CD equal to the sum of the sides BC and AD; a circle can be inscribed within the quadrilateral.

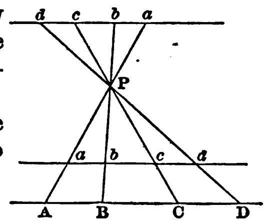
By the method employed in B. V., Pr. 15, we may describe a circle touching any three straight lines which do not pass through the same point, and which are not parallel to one another. Describe a circle touching the three sides of the quadrilateral BC, CD, and AD, in the points G, H, and E. Join M, the centre of the circle, with G, B, A, and E, and draw MF perpendicular to AB. Then will MF be equal to the radius of the circle.

Since by hypothesis, $AB+CD=BC+AD$,
and by the last Proposition, $CD=CG+DE$,
therefore $AB=BG+AE$.

Since the hypothenuse AM is common to the two right-angled triangles AME, AMF, and the hypothenuse BM is common to the two right-angled triangles BMF, BMG, if MF were less than ME or MG, BF and AF, that is, AB, would be greater than BG+AE (B. IV., Pr. 11); but if MF were greater than ME or MG, AB would be less than BG+AE. Now it has been proved that AB is equal to BG+AE; hence MF is equal to ME or MG. Therefore F is on the circumference of the circle which passes through E, G, and H; and the line AB touches the circle because the angle at F is a right angle (B. III., Pr. 9). Hence the circle touches each of the sides of the quadrilateral.

Proposition VI.

9. *If two parallel straight lines are cut by several straight lines drawn through a common point, the corresponding segments of the parallels are proportional.*



Let the two parallel lines AD, *ad* be cut by the lines PA, PB, PC, PD, drawn through the common point P; then will AD and *ad* be divided proportionally.

The triangle P*ab* is similar to the triangle PAB (B. IV., Pr. 16); also P*bc* is similar to PBC, and P*cd* to PCD. Hence we have

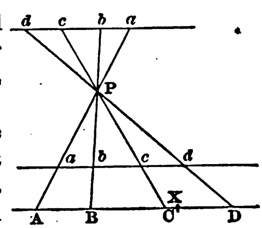
$$AB : ab :: PB : Pb :: BC : bc.$$

Also, $BC : bc :: PC : Pc :: CD : cd.$

Hence $AB : ab :: BC : bc :: CD : cd.$

Proposition VII.

10. *If three or more straight lines divide two parallel lines proportionally, they pass through a common point.*

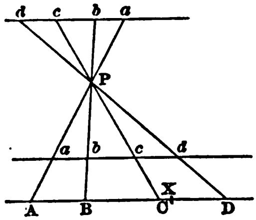


Let the two parallel lines AD, *ad* be divided proportionally by the lines A*a*, B*b*, C*c*, and D*d*; these lines, produced if necessary, meet in a common point.

Let the lines A*a* and B*b* meet in P. If the straight line C*c* does not pass through the point P, join P*c* and produce it to meet AD in X, a point supposed to be different from C. Then by the preceding Proposition we have

$$ab : AB :: bc : BX.$$

But by hypothesis we have $ab : AB :: bc : BC.$

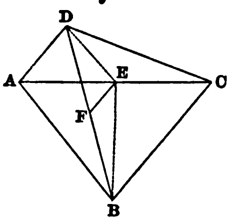


Hence BC = BX; that is, the point X coincides with C; therefore *Cc* passes through P. In the same manner it may be shown that D*d* passes through P.

11. *Scholium.* If the two lines AD, *ad* remain fixed in position while the point P recedes from them, the lines *ab* and AB will approach to equality; and when P has receded to an infinite distance, *ab* will be equal to AB, and A*a* will be parallel to B*b*. Hence every system of lines passing through a common point at an infinite distance is a system of parallel lines; and conversely every system of parallel lines may be regarded as a system of lines passing through a common point at an infinite distance.

PROPOSITION VIII.

12. *In any quadrilateral, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line which joins the middle points of the diagonals.*



Let ABCD be any quadrilateral, and let E and F be the middle points of its diagonals. Join EB, EF, and ED.

Because E is the middle point of AC, we have (B. IV., Pr. 14) $AB^2+BC^2=2AE^2+2BE^2$.

Also, $AD^2+CD^2=2AE^2+2DE^2$.

Hence, by addition,

$$AB^2+BC^2+CD^2+AD^2=4AE^2+2BE^2+2DE^2.$$

Because F is the middle point of BD, we have

$$BE^2+DE^2=2BF^2+2EF^2.$$

Hence $AB^2+BC^2+CD^2+AD^2=4AE^2+4BF^2+4EF^2$.

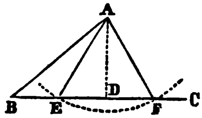
But $4AE^2=AC^2$ (B. IV., Pr. 8, Cor.); and $4BF^2=BD^2$.

Hence $AB^2+BC^2+CD^2+AD^2=AC^2+BD^2+4EF^2$.

PROPOSITION IX.

13. *If two sides and an angle opposite one of them in one triangle are equal to two sides and an angle opposite the corresponding side in another triangle, the angles opposite the other equal sides in the two triangles are either equal or one is the supplement of the other.*

This Proposition follows from Problem 11, B. V.

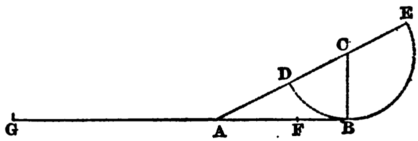


When the side opposite the given angle is less than the other given side, we may construct two triangles, ABE, ABF, having two sides and an angle of the one equal to two sides and an angle of the other; but the angle AEB is the supplement of AEF, or its equal, AFE. The angles opposite the other equal sides may therefore be supplements of each other.

But when the side opposite the given angle is not less than the other given side, the two triangles are equal and the corresponding angles are equal. In this case, the angles opposite the other equal sides are equal to each other.

PROPOSITION X.

14. *To divide a given straight line in extreme and mean ratio, either internally or externally.*



In B.V., Pr. 20, it was shown how to divide a straight line internally in extreme and mean ratio.

Produce AB to G, making AG equal to AE; then (B.IV., Pr. 29) we shall have AD : AB :: AB : AE.

By composition we have

$$AB : AD+AB :: AE : AB+AE.$$

But $$AD+AB=AD+DE=AE=AG.$$

Also, $$AB+AE=AB+AG=BG.$$

Hence $$AB : AG :: AG : BG.$$

Therefore the line AB is divided at F internally and at G externally, in extreme and mean ratio.

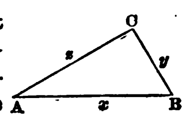
15. *Scholium.* By B. V., Pr. 20, Sch. 2, $AC=\frac{a}{2}\sqrt{5}$.

But $$AG=AE=AC+CB=AC+\frac{a}{2}.$$

Hence $$AG=\frac{a}{2}(\sqrt{5}+1).$$

PROPOSITION XI.

16. *If the three sides of a right-angled triangle are the homologous sides of similar polygons described upon them, then the polygon described on the hypothenuse is equivalent to the sum of the polygons described upon the other two sides.*



Let ABC be a right-angled triangle, having a right angle at C, and let x denote the area of a polygon described upon the hypothenuse AB, and let y and z denote the areas of similar polygons described upon the other two sides of the triangle. Then, by B. IV., Pr. 27, we have

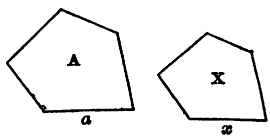
$$z : y :: AC^2 : BC^2.$$

Hence, by composition, $y+z : y :: AC^2+BC^2\ (=AB^2) : BC^2$.

Also we have $x:y::AB^2:BC^2$.
Hence $x=y+z$.

PROPOSITION XII.

17. *To construct a polygon similar to a given polygon, and such that their areas shall have a given ratio to each other.*



Let A be the given polygon, and let the given ratio be that of m to n. Let X be the required polygon similar to A. Let a be a side of the given polygon, and let x be the homologous side of the required polygon. Then we have (B. IV., Pr. 27)

$$A:X::a^2:x^2.$$

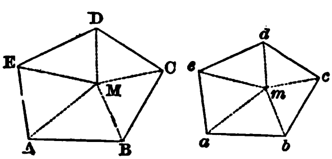
But, by hypothesis, $A:X::m:n$.
Hence $m:n::a^2:x^2$.

Find x by B. V., Pr. 30, and upon x construct the polygon X similar to P (B. V., Pr. 26); this will be the polygon required.

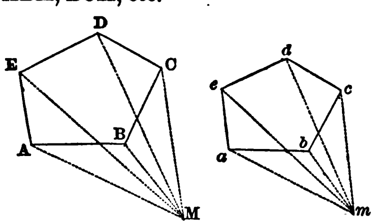
18. *Def.* In similar polygons, any points or lines similarly situated in each are called *homologous.* The ratio of a side of one polygon to its homologous side in the other is called the *ratio of similitude* of the polygons.

PROPOSITION XIII.

19. *Two similar polygons may be divided into similar triangles by lines drawn from any two homologous points.*



Let ABCDE, *abcde* be two similar polygons, and let M be any point in the plane of the first polygon. Draw MA, MB, MC, etc. In the polygon *abcde* draw *am*, making the angle *bam* equal to the angle BAM, and draw *bm*, making the angle *abm* equal to the angle ABM. The point *m* thus determined is homologous to M, and if we draw the lines *cm*, *dm*, and *em*, the triangles *abm*, *bcm*, etc., may be shown to be similar to the triangles ABM, BCM, etc.

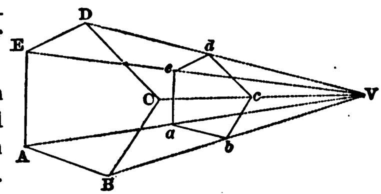


If the point M is taken without the polygon, and its homologous point *m* be found as before, the two polygons will be decomposed into triangles partly additive and partly subtractive. Thus the polygon ABCDE is equal to the sum of the three triangles MAE, MED, and MCD, diminished by the triangles MAB and MBC; and the polygon *abcde* is equal to the sum of the triangles *mae*, *med*, and *mdc*, diminished by the triangles *mab* and *mbc*.

Homologous lines in the two polygons are lines joining pairs of homologous points, such as AM and *am*, BM and *bm*, etc. Any two such homologous lines have the same ratio as any two homologous sides; that is, they have the same ratio as the *ratio of similitude* of the polygons.

PROPOSITION XIV.

20. *To construct a polygon similar to a given polygon, the ratio of similitude of the two polygons being given.*



Let ABCDE be the given polygon, and let the given ratio of similitude be as m to n.

Take any point V either within or without the given polygon, and from V draw a straight line to each of the vertices of the polygon. Upon any one of these lines, as AV, take the point a such that

$$VA : Va :: m : n.$$

Also on B take a point b, such that

$$VB : Vb :: m : n;$$

and in the same manner determine the points c, d, and e. Join ab, bc, cd, etc.; then will $abcde$ be the polygon required.

For since $\quad VA : Va :: m : n :: VB : Vb,$

ab is parallel to AB (B. IV., Pr. 16), and the angle VBA is equal to Vba. In the same manner it may be proved that bc is parallel to BC, and the angle VBC is equal to Vbc. Therefore the angle ABC is equal to abc. In like manner it may be proved that the angle BCD is equal to bcd, and so on for the other angles.

Also we have $\quad AB : ab :: BV : bV,$

and $\quad BC : bc :: BV : bV.$

Hence $\quad AB : ab :: BC : bc;$

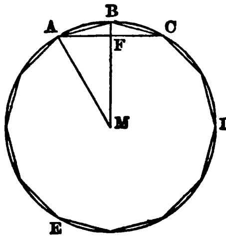
and in like manner it may be proved that the other sides about the equal angles are proportional. Hence the two polygons are similar; and the homologous sides are in the ratio of m to n.

21. The point V in the preceding figure is called the *centre of similitude* of the two polygons.

PROPOSITION XV.

22. *The area of a regular dodecagon inscribed in a circle is equal to three times the square of the radius.*

Let a regular hexagon be inscribed in a circle; by bisecting the arcs



subtended by the sides, we may construct a regular dodecagon ABCDE.

Let AC be a side of the inscribed hexagon, and AB a side of the dodecagon. Draw the radii MA, MB.

The area of $AMB = \frac{1}{2}AF.MB = \frac{1}{4}AC.MB = \frac{1}{4}MB^2$.

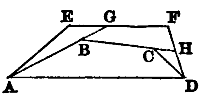
Therefore twelve times AMB equals three times MB^2.

But twelve times the triangle AMB is equal to the area of the dodecagon. Hence the dodecagon is equal to three times the square of the radius.

23. *Scholium.* Besides the polygons mentioned in Scholium, B. VI., Pr. 5, Gauss proved that a regular polygon of 257 sides may be constructed by the use of straight lines and circles only; and, in general, a polygon of any number of sides which can be expressed by 2^n+1, provided n is an integer and 2^n+1 is a prime number.

Proposition XVI.

24. *A convex polygonal line is less than any other line which envelops it and has the same extremities.*



Let ABCD be a convex polygonal line (B. I., Def. 30), that is, such that it can not be cut by a straight line in more than two points, and let it have the same extremities, A and D, as the line AEFD which envelops it; that is, it is wholly included within the space bounded by AEFD and the straight line AD; then is the polygonal line ABCD less than the polygonal line AEFD.

Produce AB and BC to meet the enveloping line in G and H. Then AG is less than AE+EG (B. I., Def. 8); hence AGFD is less than AEFD. So also BH is less than BG+GF+FH; therefore ABHD is less than AGFD, and therefore less than AEFD. Also CD is less than CH+HD; therefore ABCD is less than ABHD, and therefore less than AEFD. Hence ABCD is less than any enveloping line which has the same extremities.

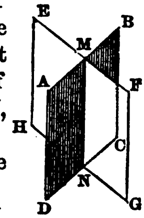
25. *Scholium.* This proposition is true when the enveloping line is a curve line of any species whatever.

26. *Cor.* The circumference of a circle is greater than the perimeter of any polygon inscribed in it.

27. *Diedral angles.* When two planes intersect, the space about the line of intersection is divided into four portions called *diedral* angles.

Thus the planes ABCD, EFGH, intersecting in the line MN, form four diedral angles. The planes MBCN, MFGN are called the *faces* of the included diedral angle, and the line MN is called the *edge* of the angle.

When several diedral angles have the same edge, each of them is designated by four letters, of which the two middle ones are on the edge of the angle, and the extremes consist of one letter on each face. Thus the four diedral angles of the annexed figure are denoted by AMNG, FMNC, BMNH, and EMND.

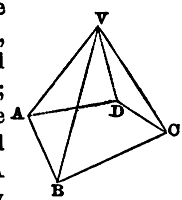


When there is but one diedral angle formed at the same edge, it may be denoted by two letters on its edge.

28. The angle AMF formed by two straight lines, AM, MF, drawn one in each face of the diedral angle perpendicular to its edge MN, from any point in that edge, is called the *plane angle* corresponding to the diedral angle.

It may be proved that any two diedral angles have the same ratio as their plane angles; hence the plane angle is taken as the measure of the diedral angle.

29. *Polyedral angles.* When three or more planes meet in a common point they form a *polyedral* angle. Thus the figure V-ABCD, formed by the planes VAB, VBC, VCD, VAD, meeting in the common point V, is a polyedral angle. The point V is called the *vertex* of the angle; the intersections of the planes are called the *edges* of the angle, and the angles formed by the edges are called the *faces*, or the plane angles of the solid angle. A polyedral angle is said to be *convex* when it lies wholly on one side of each of its faces.

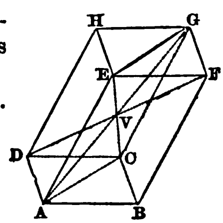


A *triedral* angle is a polyedral angle having but three faces, which is the least number of faces that can form a polyedral angle.

Proposition XVII.

30. *The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.*

Let ABGH be a parallelopiped. Draw the diagonals of the parallelopiped and also the diagonals of its opposite faces.



Because ACGE is a parallelogram, we have (B. IV., Pr. 15)

$$AG^2+CE^2=2AE^2+2AC^2;$$

and because BDHF is a parallelogram, we have

$$BH^2+DF^2=2BF^2+2BD^2.$$

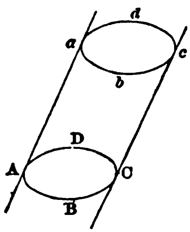
Adding, we have

$$AG^2+CE^2+BH^2+DF^2=2AE^2+2BF^2+2AC^2+2BD^2.$$

But $AC^2+BD^2=2AB^2+2AD^2$; also $AE=BF$.

Hence $AG^2+CE^2+BH^2+DF^2=4AE^2+4AB^2+4AD^2$.

31. *The cylinder.* *Def.* A *cylindrical surface* is a curved surface generated by a straight line which moves always parallel to a fixed straight line and intersects a given curve in space.



Thus if the straight line A*a* moves in such a manner as to be always parallel to a fixed straight line, and always intersects the given curve ABCD, the convex surface ABCD*abcd* is a cylindrical surface.

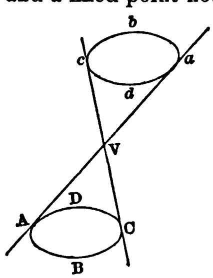
The moving line is called the *generatrix*, and the curve which it always intersects is called the *directrix*. The directrix may be any curve whatever; but in elementary geometry it is assumed to be a circle.

32. The space bounded by a cylindrical surface and two parallel planes, ABCD, *abcd*, cutting the surface is called a *cylinder*. Its plane surfaces, ABCD, *abcd*, are called its *bases*. A cylinder whose base is a circle is called a *circular cylinder*. A *right cylinder* is one whose generatrix is perpendicular to its base.

33. A right cylinder with a circular base is called a *cylinder of revolution*, because it may be generated by the revolution of a rectangle about one of its sides as an axis.

The cylinder treated of in Book X. is the cylinder of revolution.

34. *The cone.* *Def.* A *conical surface* is a curved surface generated by a straight line, which moves so as always to pass through a fixed curve and a fixed point not in the plane of the curve.



Thus if the straight line VA moves so as always to pass through a fixed point V, and a fixed curve ABCD, the surface V–ABCD is a conical surface.

The moving line is called the *generatrix;* the fixed point is called the *vertex;* and the fixed curve is the *directrix*.

If the generatrix is of indefinite extent in both directions, the whole surface generated consists of two symmetrical portions lying on opposite sides of the vertex, as V–ABCD, and V–*abcd*, which are called *nappes;* one being the *upper* and the other the *lower nappe*.

35. The space bounded by a conical surface and a plane, ABCD, cutting the surface is called a *cone*. The plane surface, ABCD, is called its *base*.

A cone whose base is a circle is called a *circular cone*. The straight line drawn from the vertex of a circular cone to the centre of its base is the *axis* of the cone.

36. A *right circular cone* is a circular cone whose axis is perpendicular

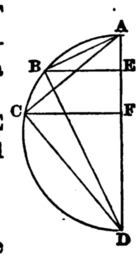
to its base. It is called a *cone of revolution*, because it may be generated by the revolution of a right-angled triangle about one of its perpendicular sides as an axis.

An *oblique circular cone* is a cone in which the directrix is a circle, but the straight line drawn from the vertex to the centre of the directrix is not perpendicular to the plane of the directrix.

The cone treated of in Book X. is the cone of revolution.

PROPOSITION XVIII.

37. ***The volume of a spherical segment with two bases is equal to the half sum of its bases multiplied by its altitude, plus the volume of a sphere whose diameter is the altitude of the segment.***



Let AD be the diameter of a circle; draw BE and CF perpendicular to AD, and let the semicircle ACD be revolved about its diameter; the figure BEFC will generate a spherical segment with two bases.

Represent CF by R, and AF by H; also BE by r and AE by h; then the volume of the spherical segment generated by the figure ABCF is (B. X., Pr. 9)

$$\tfrac{1}{6}\pi(3R^2H+H^3);$$

and the volume of the segment generated by the figure ABE is

$$\tfrac{1}{6}\pi(3r^2h+h^3).$$

Let V represent the volume of the spherical segment generated by the revolution of the figure BEFC, and we shall have

$$V=\tfrac{1}{6}\pi(3R^2H-3r^2h+H^3-h^3). \qquad (1)$$

Draw the lines AB, AC, BD, CD; then (B. IV., Pr. 23) we have

$$AD=\frac{AB^2}{AE}=\frac{AC^2}{AF};$$

that is,

$$\frac{r^2+h^2}{h}=\frac{R^2+H^2}{H}.$$

Hence $$r^2H+h^2H=R^2h+H^2h,$$

or $$3Hr^2+3Hh^2-3hR^2-3hH^2=0. \qquad (2)$$

Adding equations (1) and (2), we have

$$V=\tfrac{1}{6}\pi(3HR^2+3Hr^2-3hR^2-3hr^2+H^3-3H^2h+3Hh^2-h^3),$$

or $$V=\tfrac{1}{2}\pi(R^2+r^2)(H-h)+\tfrac{1}{6}\pi(H-h)^3.$$

If we substitute A for H$-h$, the altitude of the proposed segment, we have $$V=\tfrac{1}{2}\pi(R^2+r^2)A+\tfrac{1}{6}\pi A^3,$$

which corresponds with the proposition above enunciated, since πR^2 and

πr^2 represent the areas of the two bases of the segment; and $\frac{1}{6}\pi A^3$ represents a sphere whose diameter is A.

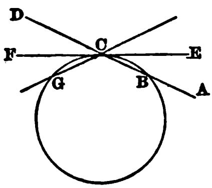
38. *Cor.* If the upper base becomes zero, the solid generated becomes a segment of one base. In this case $r=0$, and we have

$$V=\tfrac{1}{2}\pi R^2A+\tfrac{1}{6}\pi A^3,$$

which corresponds with Prop. 9, B. X.

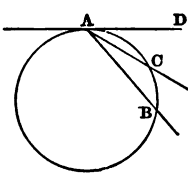
Tangents treated by the Method of Limits.

39. Let ABCD be a straight line meeting the circumference of a circle in two points, B and C. If this line be supposed to revolve about the point C, its second point of intersection, B, will move along the circumference and approach nearer and nearer to C. When the second point comes into coincidence with C, the revolving line ceases to be strictly a secant, according to Def. 9, B. III., and becomes the tangent ECF. If the line continues to revolve in the same direction, it will again meet the circumference in a second point on the other side of C, as at G.



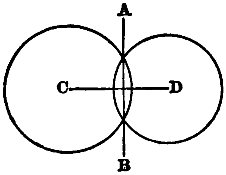
As long as the two points B and C do not coincide, the line which passes through them both is a *secant* line, according to Definition 9; but when the two points coincide, the secant becomes a *tangent*, for it meets the circumference in but one point. The tangent is therefore the *limit* toward which the secant continually approaches, as the second point of intersection approaches the first, and *the tangent may be regarded as a secant whose two points of intersection coincide.* A secant may therefore be regarded as a line drawn through *any* two points of the curve; and a tangent is the special case of such a line in which the two points coincide. By means of this principle the properties of a tangent may be deduced from those of a secant.

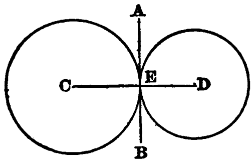
40. Ex. 1. By Prop. 15, B. III., an inscribed angle BAC is measured by half the intercepted arc BC.



Let the line AB remain fixed, while the line AC revolves about A until it arrives at the position of the tangent DA. The point of intersection C will move along the circumference, and when the line AC becomes a tangent, the point C will coincide with A, and the intercepted arc becomes ACB, half of which measures the angle BAD. Thus Prop. 16, B. III., becomes a particular case of Prop. 15.

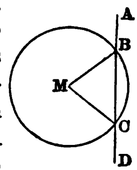
Ex. 2. In Prop. 11, B. III., it is proved that if two circumferences cut each other, the straight line joining their centres bisects their common chord at right angles. Suppose one of the circles to move so as to cause the points of intersection to approach each other; these points will ultimately meet on the line joining the centres, and unite in a single point, E, common to the two circumferences. The perpendicular to CD erected at E will then be a common tangent to the two circumferences, and take the place of the common chord. Thus Prop. 12, B. III., becomes a particular case of Prop. 11.

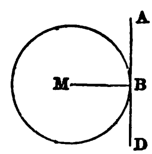




Ex. 3. At every point on the circumference of a circle the tangent is perpendicular to the radius (B. III., Pr. 9). This may be proved as follows:

Let AD be a line meeting the circumference of a circle in two points, B and C. Draw the radii MB, MC; then, since the triangle MBC is isosceles, the two external angles MBA, MCD are always equal; they are therefore equal in the particular case when B coincides with C; in which case MB coincides with MC, and the angle MBA with the angle MCA. In this case the angles MBA, MBD are equal, and therefore the radius MB is perpendicular to the tangent AD.





The principle here illustrated is applicable not only to the circle, but to any curve whatever, and often affords the most convenient method of deducing the properties of a tangent line.

PLANE LOCI.

41. A *locus*, in Plane Geometry, is a line every point of which satisfies certain conditions, which conditions no other point of the plane satisfies.

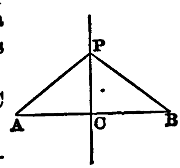
The nature and use of loci will be understood from the following examples.

Proposition I.

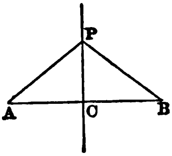
42. *Required the locus of all points which are equidistant from two given points.*

Let A and B be the two given points, and let P be a point equidistant from A and B, so that PA=PB; it is required to find the locus of P.

Join AB, bisect AB in C, and join PC. Then PC produced is the locus required.



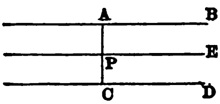
In the triangles ACP, BCP, because AC=BC by con-



struction, AP=BP by hypothesis, and PC is common to both triangles, therefore the angle ACP=the angle BCP (B. I., Pr. 15), and therefore PC is at right angles to AB (B. I., Def. 19); that is, every point equidistant from A and B lies on the line which bisects AB at right angles; or the line which bisects AB at right angles is the locus of all points equidistant from A and B.

Proposition II.

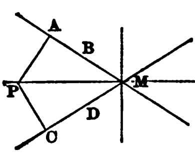
43. *Required the locus of all points which are equidistant from two given straight lines.*



Let AB, CD be the two given straight lines, and let P be a point equidistant from both of them; it is required to find the locus of P. From P draw PA perpendicular to AB, and PC perpendicular to CD; then by hypothesis AP=CP (B. I., Pr. 17, Cor. 1). If the given straight lines are parallel, then through P draw PE parallel to AB or CD, and it will be the locus required.

For AB and PE are every where equidistant from each other (B. I., Pr. 25); also CD and PE are every where equidistant from each other; and the distance of PE from AB is equal to the distance of PE from CD. Hence the required locus is a straight line which is parallel to each of the given lines, and bisects the distance between them.

If AB and CD are not parallel, let them be produced to meet in M, and join PM.



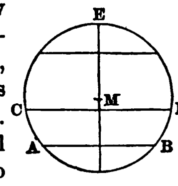
Then in the triangles PAM, PCM, because the angles PAM, PCM are by construction right angles; also by hypothesis the side PA is equal to PC, and the hypothenuse PM is common to both triangles, therefore the triangle PAM is equal to the triangle PCM (B. I., Pr. 19), and the angle AMP is equal to the angle CMP. Therefore the point P is in the straight line which bisects the angle AMC; that is, the locus of all points which are equidistant from two intersecting straight lines is the straight line which bisects the angle between the lines.

If the lines AB and CD are produced indefinitely, four angles will be formed at their point of intersection; and the line which bisects either of these angles is the locus of all points equidistant from the two given straight lines.

Proposition III.

44. *Required the locus of the middle points of parallel chords in a circle.*

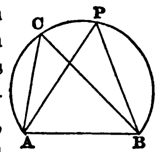
Let AB, CD be parallel chords in the circle ABE; it is required to find the locus of the middle points of all the chords parallel to AB.



Let M be the centre of the circle, and through M draw a diameter, EM, perpendicular to AB; it will also be perpendicular to CD and every chord parallel to AB (B. I., Pr. 23, Cor. 1). This diameter bisects each of the chords AB, CD, and every chord parallel to AB (B. III., Pr. 6); that is, the locus of the middle points of parallel chords in a circle is the diameter perpendicular to these chords.

Proposition IV.

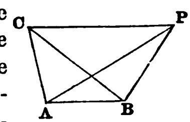
45. *Required the locus of the vertices of all triangles upon the same base and upon the same side of it, and having equal vertical angles.*



Let AB be the given base, let ACB be the given vertical angle, and let P be any point in the locus. Join PA, PB. Then because the angle APB is by hypothesis equal to the angle ACB, the point P is on the circumference of a circle passing through A, B, and C (B. III., Pr. 15, Cor. 1); that is, the arc of a circle passing through C and having AB for its chord is the locus of the vertices of all triangles upon the base AB, and upon the same side of it, and having a vertical angle equal to ACB.

Proposition V.

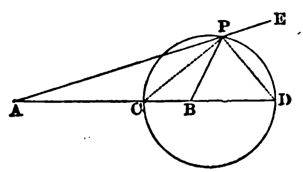
46. *Required the locus of the vertices of all equivalent triangles upon the same base and upon the same side of it.*



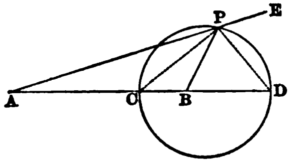
Let AB be the given base; let ABC be a triangle having the given area, and let it be situated on the given side of the base AB, and let P be a point in the locus. Join PA, PB, and PC. Then because the triangle PAB is equivalent to the triangle CAB, both triangles have the same altitude (B. IV., Pr. 6, Cor. 2), and since they are on the same base, AB, the line CP must be parallel to AB (B. I., Pr. 25); that is, the point P lies in a straight line drawn through C parallel to AB. Hence the line drawn through C parallel to AB is the locus of the vertices of all triangles situated upon the base AB, and upon the same side of it, and whose area is equal to CAB.

Proposition VI.

47. *Required the locus of all the points whose distances from two given points are in a given ratio.*



Let A and B be the given points, and let the given ratio be M : N. Suppose P is a point of the required locus. Divide AB at C so that AC : CB : : M : N (B. V., Pr. 17); and produce AB to D so that AD : BD : : M : N. Join PA, PB, PC, and PD. Then because

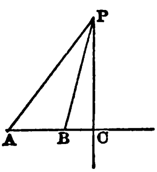


in the triangle PAB the line PC divides the base into two segments which are proportional to the adjacent sides, it bisects the vertical angle APB (B. IV., Pr. 17). Also because the line PD divides the base, AB produced, into segments which are proportional to the sides of the triangle APB, it bisects the exterior angle BPE (B. IV., Pr. 18). Therefore the angle CPD is equal to half the sum of the angles APB, BPE; that is, to the half of two right angles, or to one right angle. And because CPD is a right angle, the point P lies in the circumference of a circle described upon the diameter CD (B. III., Pr. 15, Cor. 3).

Hence we have the following construction. Divide AB at C into parts having the given ratio, and produce AB to D, so that the segments produced shall also have the given ratio. Upon CD as a diameter describe a circumference; this circumference is the required locus.

Proposition VII.

48. *Required the locus of the vertices of all triangles on a given base,* AB, *such that the square on the side terminated at A may exceed the square on the side terminated at* B *by a given square.*



Let P be a point on the required locus, and from P draw PC perpendicular to AB, or AB produced. Then the square on AP is equal to the sum of the squares on AC and PC, and the square on BP is equal to the sum of the squares on BC and PC. Therefore the square on AP exceeds the square on BP by as much as the square on AC exceeds the square on BC. Hence C is a fixed point either in AB or in AB produced through B; and the required locus is the straight line drawn through C at right angles to AB.

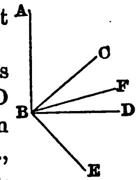
LOCI IN SPACE.

49. *Def.* In the geometry of space a geometric locus is the assemblage of all the points of space which satisfy one or more given conditions. This locus may be either a line or a surface.

Thus the line drawn through the centre of a circle perpendicular to its plane is the locus of all points which are equally distant from all points of the circumference; and the surface of a sphere is the locus of all points situated at a given distance from a given point.

PROPOSITION I.

50. *The locus of straight lines which cut a given straight line at right angles and at a given point is a plane.*

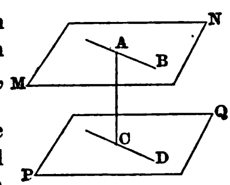


Let the lines BC, BD, BE, etc., be perpendicular to AB at the point B; they will all lie in one plane.

For, if possible, suppose that the plane which contains BC and BE does not contain BD. Through AB and BD pass a second plane, which shall intersect the plane CBE in the line BF, different from BD. Then, by Prop. 4, B. VII., AB, which is perpendicular to BC and BE, will be perpendicular to BF. But, by hypothesis, AB is perpendicular to BD. We have then two straight lines, BF, BD, perpendicular to the same line AB at the same point and in the same plane, which is impossible (B. I., Pr. 1). Hence BC, BD, and BE lie in the same plane; and in the same manner it may be proved that all lines drawn through the point B, perpendicular to AB, lie in the same plane.

PROPOSITION II.

51. *The locus of all the straight lines drawn through a given point parallel to a given plane is a plane passing through the point parallel to the given plane.*



Let A be the given point and PQ the given plane; then every straight line, AB, drawn through A parallel to the plane PQ lies in the plane MN, which passes through A and is parallel to PQ.

Through AB pass a plane intersecting the plane PQ in the straight line CD. Then AB is parallel to CD, for although produced indefinitely, it can not meet CD, since it can not meet the plane in which CD lies. But CD is parallel to the plane MN, since it is in the plane PQ, which is parallel to MN, and therefore can not meet MN.

Now a plane passing through CD and A intersects the plane MN in a line parallel to CD (B. VII., Pr. 12); and since there can not be two parallels to CD drawn through the same point A in the plane ACD, therefore AB must be this line of intersection; that is, it lies in the plane MN.

PROPOSITION III.

52. *The locus of the points in space which are equally distant from two given points is the plane that bisects at right angles the line joining the two points.* (To be proved.)

PROPOSITION IV.

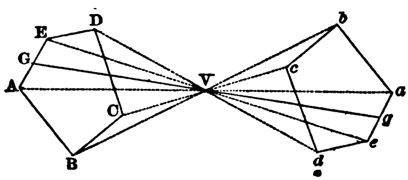
53. *Find the locus of the points which are equally distant from three given points.*

SYMMETRICAL FIGURES.

I. SYMMETRY WITH RESPECT TO A CENTRE.

54. *Def.* Two points, A and *a*, are said to be placed *symmetrically* with respect to a fixed point, called the *centre of symmetry*, when this centre bisects the straight line A*a*, which joins the two points.

Any two polygons, ABCDE, *abcde*, are said to be symmetrical with respect to a centre when every point of one polygon has its symmetrical point on the other polygon.



If in the two polygons ABCDE, *abcde* the straight lines joining the vertices A and *a*, B and *b*, C and *c*, etc., all pass through the same point V, and are bisected by it, the two polygons are said to be symmetrical with respect to the point V, and V is called their centre of symmetry.

Proposition I.

55. *If two polygons are symmetrical with respect to a centre, any two corresponding sides are equal and parallel, and point in opposite directions; also any straight line passing through the centre of symmetry and terminating in two opposite sides is bisected by this centre.*

The two triangles VAB, V*ab* have an equal angle contained by equal sides; therefore AB is equal to *ab*. Also the angle VBA is equal to the angle V*ba*, and therefore AB is parallel to *ab*. In the same manner it may be proved that BC is equal and parallel to *bc*, CD to *cd*, etc.

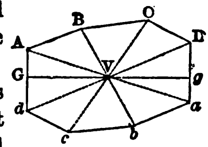
Also any straight line, G*g*, passing through V and terminating in two opposite sides, is bisected in V. For since AE is parallel to *ae*, the triangles VAG, V*ag* are equiangular; and since AV is equal to *a*V, VG must be equal to V*g*.

56. *Scholium* 1. When two polygons have their corresponding sides equal, parallel, and pointing in opposite directions, they have a centre of symmetry. For since AB is equal and parallel to *ab*, the lines A*a*, B*b* are the diagonals of a parallelogram, and therefore bisect each other (B. I., Pr. 33). So also C*c*, D*d*, etc., bisect each other; that is, all the straight lines, A*a*, B*b*, C*c*, etc., are bisected in the same point.

57. *Scholium* 2. When two polygons are symmetrical with respect to a centre, one of them may be made to coincide with the other by revolving it about the centre, through two right angles in their common plane.

58. *Def.* A polygon, ABCD*abcd*, is said to be symmetrical with respect to a centre, V, when its vertices taken two and two are symmetrical with

respect to V; that is, when the line joining A and *a* passes through V, and is bisected by it; also the line joining B and *b*, C and *c*, etc.



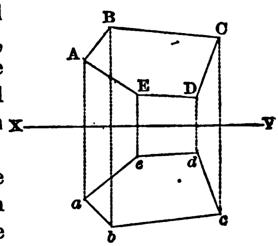
It is evident that any two opposite sides of this polygon are equal and parallel, and also any straight line, G*g*, passing through the centre of symmetry and terminating in two opposite sides, is bisected by this centre. Such a line is called a *diameter* of the polygon.

It is evident that any polygon having a centre of symmetry must have an *even* number of sides. Every parallelogram has a centre of symmetry, but no other quadrilateral figure has one. Any regular polygon having an even number of sides has a centre of symmetry, which is also the centre of the polygon.

II. SYMMETRY WITH RESPECT TO AN AXIS.

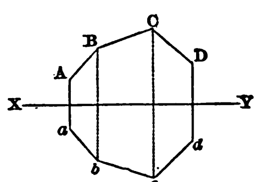
59. *Def.* Two points, A and *a*, are said to be placed *symmetrically* with respect to a straight line, XY, called the *axis of symmetry*, when this axis is perpendicular to the line which joins the two given points and divides the line into two equal parts.

Any two polygons, ABCDE, *abcde*, are said to be symmetrical with respect to an axis, XY, when this axis is perpendicular to the lines which join the vertices A and *a*, B and *b*, etc., of the two polygons, and divides each of these lines into two equal parts.



If the part of the plane which is above the line XY be turned over upon the part which is below XY, the polygon ABCDE may be superposed upon the polygon *abcde*, and the vertices of the one polygon be made to correspond with the vertices of the other polygon.

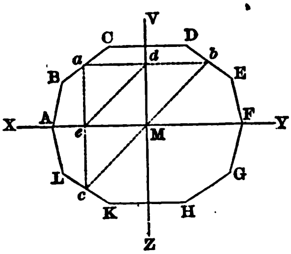
60. *Def.* Any polygon is said to be symmetrical with respect to an axis, when it is divided by that axis into two symmetrical figures. Thus the polygon ABCD*dcba* is symmetrical with respect to the axis XY when its vertices, taken two and two, are placed symmetrically with respect to that axis.



The line which bisects the vertical angle of an isosceles triangle is an axis of symmetry with respect to the two parts of the triangle; the line which bisects an angle of a regular polygon is an axis of symmetry with respect to the two halves of the polygon; the diameter of a circle is an axis of symmetry for the two halves of the circle; and the line which joins the centres of two circles is an axis of symmetry with respect to the halves of both of the circles.

Proposition II.

61. *Every figure which has two axes of symmetry perpendicular to each other is also symmetrical with respect to the point of intersection of these axes as a centre of symmetry.*



Let the figure ABDHL be symmetrical with respect to the two axes XY, VZ, which are perpendicular to each other; it will also be symmetrical with respect to their point of intersection, M, as a centre of symmetry.

Let *a* be any point in the perimeter of the figure; draw *adb* perpendicular to VZ, and *aec* perpendicular to XY. Join *c*M, *b*M, and *ed*.

Since the figure is symmetrical with respect to XY, *ae* is equal to *ec*; and since *d*M is equal and parallel to *ae*, it is equal and parallel to *ce*; therefore *de* and *c*M are also equal and parallel. In the same manner it may be proved that *de* is equal and parallel to *b*M; therefore *b*M is equal to *c*M; and since *b*M and *c*M are both parallel to the same straight line, and pass through the same point, M, they can not intersect (B. I., Ax. 12), but form one straight line. Therefore any straight line, *bc*, drawn through M is bisected at M, and hence M is the centre of symmetry of the figure (Prop. I.):

62. *Scholium.* The two rectangular axes XY, VZ divide the figure ACEHL into four equal parts. The adjacent parts may be superposed by revolving about one of the axes; the opposite parts may be superposed by revolving through two right angles in the plane of the figure.

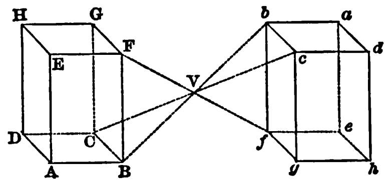
There are then two kinds of symmetrical position in a plane, one with respect to a point, the other with respect to a straight line.

SYMMETRICAL POLYEDRONS.

I. SYMMETRY WITH RESPECT TO A CENTRE.

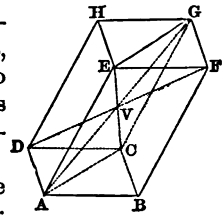
63. *Def.* Any two polyedrons are said to be symmetrical with respect to a centre, when each vertex of one polyedron has its symmetrical vertex on the other polyedron.

If in the two polyedrons AB–GH, *ab–gh* the straight lines joining the vertices A and *a*, B and *b*, C and *c*, etc., all pass through the same point



V, and are bisected by it, the two polyedrons are said to be symmetrical with respect to the point V, and V is called their centre of symmetry.

It is evident that if two polyedrons are symmetrical with respect to a centre, their homologous faces are equal each to each; their diedral angles are equal each to each; their polyedral angles are symmetrical, and the two polyedrons are equivalent.



64. *Def.* A polyedron, AB–GH, is said to be symmetrical with respect to a centre, V, when its vertices, taken two and two, are symmetrical with respect to V; that is, when the line joining A and G passes through V and is bisected by it; also the line joining C and E, D and F, etc.

It is evident that a polyedron which has a centre of symmetry must have an *even* number of edges; also its homologous edges must be equal and parallel; its homologous faces must be equal and parallel; the homologous plane angles and diedral angles must be equal; and the homologous polyedral angles must be symmetrical. Also every straight line passing through the centre of symmetry and terminating in two opposite surfaces is bisected by this centre.

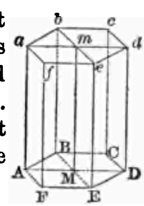
A prism whose bases are polygons which are symmetrical with respect to a point, has a centre of symmetry: viz., the middle of the straight line which joins the centres of the two bases.

A parallelopiped has a centre of symmetry: viz., the centre of figure.

II. SYMMETRY WITH RESPECT TO AN AXIS.

65. *Def.* Any two polyedrons are said to be symmetrical with respect to an axis, when this axis bisects at right angles the straight lines which join the corresponding vertices of the two polyedrons.

66. *Def.* Any polyedron is said to be symmetrical with respect to an axis, when this line bisects at right angles the lines which join its corresponding vertices taken two and two.



Thus the polyedron ACE–*ace* is symmetrical with respect to the axis M*m* when this axis is perpendicular to the lines which join the vertices A and D, *a* and *d*; B and E, *b* and *e*, etc., and divides each of these lines into two equal parts.

A right prism whose bases are symmetrical with respect to a centre has an axis of symmetry: viz., the straight line which joins the centres of symmetry of the two bases.

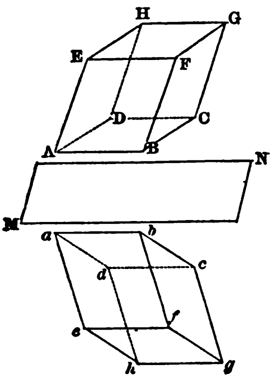
A rectangular parallelopiped has *three* axes of symmetry:

viz., the straight lines which join the centres of its opposite faces. If the base is a square, it has two other axes of symmetry: viz., the lines which join the middle points of its opposite lateral edges.

A regular pyramid which has an even number of lateral faces has an axis of symmetry: viz., the axis of the pyramid.

III. SYMMETRY WITH RESPECT TO A PLANE.

67. *Def.* Two points, A and *a*, are said to be symmetrical with respect to a plane, MN, when this plane bisects at right angles the straight line, A*a*, which joins the two points.



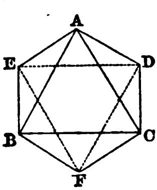
Two polyedrons are said to be symmetrical with respect to a plane when this plane bisects at right angles the straight lines which join their corresponding vertices, taken two and two.

The two polyedrons AB–GH, *ab*–*gh* are symmetrical with respect to the plane MN when this plane bisects at right angles the lines which join the vertices A and *a*, B and *b*, C and *c*, etc., of the two polyedrons.

It is evident that if two polyedrons are symmetrical with respect to a plane, their homologous faces must be equal each to each; their diedral angles must be equal each to each; their polyedral angles must be symmetrical; and the two polyedrons must be equivalent.

The symmetrical solid corresponding to any polyedron with respect to a plane is similar and equal to the solid which is symmetrical to the same polyedron with respect to a centre.

68. *Def.* A polyedron is said to be symmetrical with respect to a plane, when that plane divides it into two polyedrons which are symmetrical with respect to the plane.



Thus the regular octaedron ABCDEF is symmetrical with respect to the plane BCDE, since this plane divides the polyedron into two pyramids which are symmetrical with respect to this plane.

If a regular dodecaedron be cut by a plane passing through the centre and any one of its edges, it will be divided into two polyedrons which are symmetrical with respect to that plane.

If a polyedron has two planes of symmetry which are perpendicular to each other, their common intersection is an *axis* of symmetry; and if a polyedron has three planes of symmetry, the point common to the three planes is a *centre* of symmetry.

MAXIMA AND MINIMA OF PLANE FIGURES.

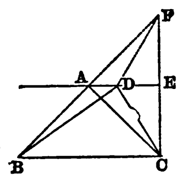
69. *Def.* A variable magnitude is said to be a *maximum* when it is the greatest of its kind, or the greatest under certain conditions; and it is called a *minimum* when it is the least of its kind, or the least under certain conditions.

Thus the diameter of a circle is a maximum among all straight lines joining two points of the circumference; and among all the lines drawn from a given point to a given straight line the perpendicular is a minimum.

When two figures have equal perimeters they are called *isoperimetric.*

PROPOSITION I.

70. *The perimeter of an isosceles triangle is less than that of any other equivalent triangle standing upon the same base.*



Let the triangles ABC, DBC have equal areas, and let them have the same base, BC, and let the triangle ABC be isosceles; the triangle ABC has a less perimeter than the triangle DBC.

The two triangles must have equal altitudes (B. IV., Pr. 6, Cor. 2), and the straight line AD joining their vertices must be parallel to BC (B. I., Pr. 25). Draw CF perpendicular to AD produced, and let it meet BA produced in F. Join DF, and let AD meet CF in E.

Since the angle EAC is equal to ACB (B. I., Pr. 23), which is equal to ABC (B. I., Pr. 10), which is equal to FAE, the two triangles AEC, AEF, having also each a right angle and one side common, are equal to each other; therefore AE is perpendicular to CF at its middle point E. Hence AF=AC and DF=DC (B. I., Pr. 18).

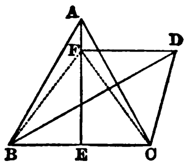
But BF<BD+DF (B. I., Pr. 8); that is, BA+AC<BD+DC. Hence, adding BC to each, the perimeter of ABC is less than the perimeter of BDC.

71. *Cor.* Of all triangles having the same area, that which is equilateral has the least perimeter. For the triangle having the least perimeter with a given area must be isosceles whichever side be taken as the base; therefore the triangle of least perimeter has each pair of its sides equal, and consequently is equilateral.

PROPOSITION II.

72. *Of all triangles having the same base and equal perimeters, the isosceles triangle has the greatest area.*

Let ABC, DBC be two triangles standing on the same base, BC, and having equal perimeters, of which ABC is isosceles and DBC is not isos-



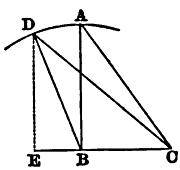
celes; then the area of ABC is greater than the area of DBC.

Through A draw AE perpendicular to BC, and through D draw DF parallel to BC. Join FB, FC. The triangles DBC, FBC, having the same base, BC, and equal altitudes, have equal areas (B. IV., Pr. 6); but the triangle FBC is isosceles (B. I., Pr. 18); therefore the triangle FBC has a less perimeter than the triangle DBC (Prop. I.), or than the triangle ABC by hypothesis. Therefore BF is less than BA, and consequently FE is less than AE (B. I., Pr. 17); that is, the altitude of the triangle DBC is less than the altitude of the triangle ABC; and since these triangles have the same base, the area of ABC is greater than that of DBC.

73. *Cor.* Of all triangles having the same perimeter, that which is equilateral has the greatest area. For the triangle having the greatest area, with a given perimeter, must be isosceles whichever side is taken as the base.

Proposition III.

74. *Of all triangles having two sides of the one equal to two sides of the other, each to each, that in which these sides are perpendicular to each other is the greatest.*

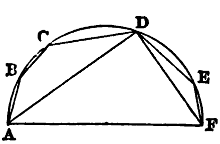


Let ABC, DBC be two triangles, having the sides AB, BC respectively equal to DB, BC; then if the angle ABC is a right angle, the area of the triangle ABC is greater than that of the triangle DBC.

For, taking BC as the common base, the two triangles ABC, DBC are as the altitudes AB, DE (B. IV., Pr. 6, Cor. 1); but the perpendicular DE is shorter than the oblique line DB (B. I., Pr. 17), or its equal AB. Therefore the triangle ABC is greater than the triangle DBC.

Proposition IV.

75. *If all the sides of a polygon except one be given, the area will be the greatest when the figure may be inscribed in a semicircumference of which the undetermined side is the diameter.*



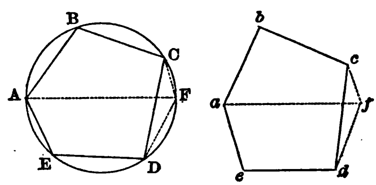
Let ABCDEF be the greatest polygon that can be contained by the sides AB, BC, CD, DE, EF given in magnitude, and another side whose length is not given. Draw the diagonals AD, DF. Then if the angle ADF were not a right angle, by making it a right angle (Prop. 3) we should enlarge the triangle ADF, without changing the parts ABCD, DEF, and consequently we should enlarge the entire polygon. But the polygon is supposed to be a *maximum*, and can not therefore be enlarged; therefore the angle ADF is a right angle. In the same manner it may be proved that each of the angles ABF, ACF, AEF is a right angle; therefore all the angles of the maximum polygon are inscribed in a semi-circumference of which the unknown side AF is the diameter.

76. *Scholium.* There is but one semicircle which will contain the maximum polygon. For suppose ABCDEF to be a semicircle containing it; the angles at its centre subtended by the chords AB, BC, etc., amount to two right angles, which would no longer be the case were the radius made either greater or less; since if the same chord AB subtends arcs described with different radii, the angle at the centre corresponding to this chord will be smallest in that circle whose radius is greatest.

PROPOSITION V.

77. *Of all polygons which have their sides equal, each to each, the greatest is that which can be inscribed in a circle.*

Let ABCDE, *abcde* be two polygons having the side AB=*ab*, BC=*bc*, etc., and let the first be inscribed in a circle, while the other is not capable of being inscribed in one; the inscribed polygon is greater than the other.

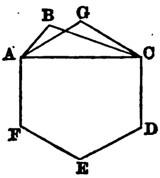


Draw the diameter AF, and join CF, DF. On *cd*, which is equal to CD, construct the triangle *cfd* equal to CFD, and join *af*. The line *af* divides the figure *abcfde* into two parts, of which *one at least*, by supposition, can not be inscribed in a semicircle. Therefore one at least of these two figures is smaller than the corresponding part of the figure ABCFDE. Therefore the whole figure ABCFDE is greater than the whole figure *abcfde;* and if from these there be taken away the respective triangles CFD, *cfd*, which are equal by construction, there will remain the polygon ABCDE greater than the polygon *abcde*.

78. *Scholium.* It is plain that the area of the inscribed polygon will be the same in whatever order the sides are arranged. For these sides are *chords* in a circle, and in whatever order they are arranged they will cut off equal segments; and the polygon is the part of the circle remaining when these segments are taken away.

PROPOSITION VI.

79. *Of all polygons having equal perimeters and the same number of sides, the regular polygon has the greatest area.*

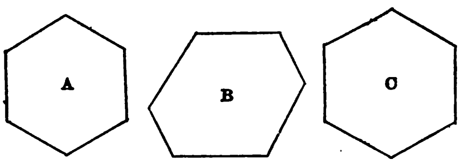


The polygon ABCDEF can not be a *maximum* among all polygons of equal perimeters and the same number of sides unless it be equilateral. For if any two of the sides, as AB, BC, are unequal, on AC describe the isosceles triangle AGC, having the sum of its sides AG, GC equal to the sum of AB, BC. The triangle AGC is greater than ABC (Prop. 2); and therefore the polygon AGCDEF is greater than the polygon ABCDEF. Hence the latter is not a maximum; and the same may be shown in a similar manner, unless the sides are all equal.

The polygon which is a maximum is therefore equilateral, and by Prop. V. it can be inscribed in a circle; it must therefore be equiangular (B. VI., Pr. 2, Sch. 2). Hence it is a regular polygon.

PROPOSITION VII.

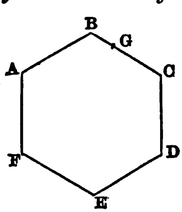
80. *A regular polygon has a less perimeter than any other polygon of equal area and the same number of sides.*



Let A and B be two polygons having the same number of sides and containing equal areas, and let A be a regular polygon, while B is an irregular polygon. The perimeter of A is less than that of B. For let C be a regular polygon having the same perimeter and the same number of sides as B. Then (Prop. VI.) C>B; but B=A by hypothesis; therefore C>A. But C and A are similar figures (B. VI., Pr. 1); consequently, perimeter of C>perimeter of A. Now by hypothesis, perimeter of C=perimeter of B; hence perimeter of B>perimeter of A.

PROPOSITION VIII.

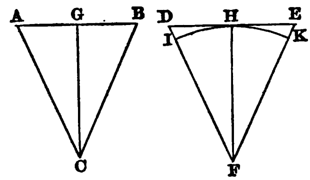
81. *Of all regular polygons having equal perimeters, that which has the greater number of sides has the greater area.*



Let ABCDEF be a regular polygon of six sides. On one of its sides, as BC, take any point G; we may regard the given polygon as an irregular polygon of seven sides, in which the sides BG, CG make an angle with each other equal to two right angles. This irregular polygon is less than a regular polygon of seven sides and having the same perimeter (Prop. VI.); that is, a regular polygon of six sides is less than a regular polygon of seven sides having the same perimeter. In the same manner it may be proved that a regular polygon of seven sides is less than a regular polygon of eight sides having the same perimeter, and so on.

PROPOSITION IX.

82. *A circle is greater than any regular polygon of the same perimeter.*



Let AG be half the side of any regular polygon, and let C be the centre of the polygon. In a circle having an equal perimeter, take the angle IFH=ACG; the arc IH will therefore be equal to the half side AG. We therefore have

$$\text{Polygon}:\text{circle}::\text{triangle } ACG:\text{sector } IFH;$$
$$::\tfrac{1}{2}AG.CG:\tfrac{1}{2}IH.HF;$$
$$::CG:FH.$$

At the point H draw the tangent HD, meeting FI produced in D. The similar triangles ACG, DFH give the proportion

$$CG:FH::AG(=IH):DH.$$

Therefore $\text{Polygon}:\text{circle}::IH:DH;$

$$::\tfrac{1}{2}IH.FH:\tfrac{1}{2}DH.FH.$$

But $\frac{1}{2}$IH.FH is the measure of the sector IFH (B. VI., Pr. 12, Cor.); and $\frac{1}{2}$DH.FH is the measure of the triangle DFH. Now the triangle is greater than the sector; hence the circle is greater than the polygon; that is, the circle is greater than any regular polygon of the same perimeter.

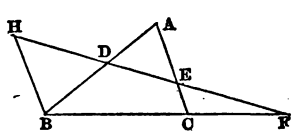
TRANSVERSALS.

83. *Def.* Any straight line cutting a system of lines is called a *transversal.*

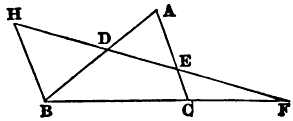
A transversal to a triangle is any straight line drawn to intersect its three sides or sides produced.

PROPOSITION I.

84. *If a straight line cut the sides of a triangle, or the sides produced, the product of three alternate segments of the sides is equal to the product of the other three segments.*



Let ABC be a triangle, and let a straight line be drawn cutting the side AB at D, the side AC at E, and the side BC produced at F. Then AD and DB are called segments of the side AB; AE and EC are



called segments of the side AC; and BF and CF are called segments of the side BC. The segments EC, AD, and BF are called alternate segments, or non-adjacent, since they have no extremity in common. So also the segments AE, BD, and CF are alternate, or non-adjacent segments.

Through B draw a straight line parallel to AC, meeting DF produced in H.

Then by similar triangles, BH : AE :: BD : AD;

or $$BH.AD = AE.BD. \qquad (1)$$

Also by similar triangles, CE : BH :: CF : BF;

or $$BF.CE = BH.CF. \qquad (2)$$

Multiplying (1) by (2), and dividing by the common factor BH, we have

$$AD.BF.CE = AE.CF.BD.$$

This theorem was known to Menelaus, A.D. 80.

PROPOSITION II. (*Converse of Prop. I.*)

85. *If three points are taken on the sides of a triangle (one of the points, or all three lying in the sides produced), so that the product of three alternate segments of the sides is equal to the product of the other three segments, the three points lie in the same straight line.*

Let the points D and E be taken upon the sides AB and AC of the triangle ABC, and let the point F be taken on the side BC produced, so that AD.BF.CE=AE.CF.BD; the points D, E, and F are in the same straight line.

Join DE; and if DE produced does not pass through F, let it meet BC produced in a point which we will call F′. Then by Prop. I. we have

$$AD.BF'.CE = AE.CF'.BD. \qquad (1)$$

But, by hypothesis, we have

$$AD.BF.CE = AE.CF.BD. \qquad (2)$$

Dividing (1) by (2), and omitting the common factors, we have

$$\frac{BF'}{BF} = \frac{CF'}{CF};$$

or $$BF' : BF :: CF' : CF.$$

Hence by division (since F is on BC produced),

$$BC : BC :: CF' : CF.$$

Therefore $$CF' = CF,$$

or the point F′ coincides with F, and therefore the three points D, E, and F are in the same straight line.

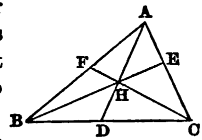
This theorem was known to Menelaus.

PROPOSITION III.

86. *If three straight lines be drawn from the angles of a triangle to the opposite sides, and meet at the same point, the product of three non-*

adjacent segments of the sides is equal to the product of the other three segments.

Let ABC be a triangle, and from the angular points to the opposite sides let the straight lines AD, BE, CF be drawn, passing through the point H; then the product AF.BD.CE will be equal to the product AE.CD.BF.



The triangle BCE is cut by the transversal AHD; hence by Prop. I. we have

$$BD.CA.EH = EA.CD.BH. \qquad (1)$$

Also the triangle BAE is cut by the transversal CHF; hence we have

$$AF.BH.EC = EH.BF.AC. \qquad (2)$$

Multiplying (1) by (2), and omitting the common factors, we obtain

$$AF.BD.CE = AE.CD.BF.$$

87. *Scholium.* We have supposed the point H to be within the triangle; if it be without the triangle, two of the points D, E, F will fall on the sides produced.

PROPOSITION IV. (*Converse of Prop. III.*)

88. *If three straight lines be drawn from the angles of a triangle to meet the opposite sides, making the product of three non-adjacent segments of the sides equal to the product of the other three segments, then the three straight lines so drawn will pass through the same point.*

Let AD, BE intersect each other in the point H, and let F be such a point in AB that AF.BD.CE=AE.CD.BF. Join CH; then CH produced passes through the point F.

If CH produced does not pass through the point F, let it pass through F′, some other point in AB. Then by Prop. III. we have

$$AF'.BD.CE = AE.CD.BF'. \qquad (1)$$

But, by hypothesis, we have

$$AF.BD.CE = AE.CD.BF. \qquad (2)$$

Dividing (1) by (2), $\dfrac{AF'}{AF} = \dfrac{BF'}{BF}$;

or $AF' : AF :: BF' : BF.$

Hence, by composition, $AB : AB :: BF' : BF.$

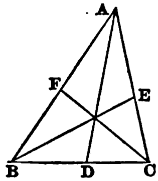
Therefore $BF' = BF$;

or the point F′ coincides with F, and therefore CHF′ coincides with CHF, and the three lines AD, BE, CF pass through the same point H.

PROPOSITION V.

89. *The three bisectors of the angles of a triangle pass through the same point.*

Let ABC be a triangle, and let AD, BE, CF be drawn bisecting the



angles A, B, and C; these three lines will intersect in the same point.

For since AD bisects the angle BAC (B. IV., Pr. 17), we have

BD : AB :: CD : AC;

or

BD.AC=AB.CD. (1)

For a similar reason

CE : BC :: AE : AB;

or

CE.AB=BC.AE. (2)

Also,

AF : AC :: BF : BC;

or

AF.BC=AC.BF. (3)

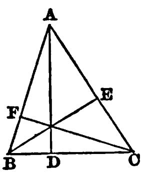
Multiplying together (1), (2), and (3), and omitting the common factors, we have

AF.BD.CE=AE.CD.BF.

Therefore, Prop. IV., the three lines AD, BE, CF pass through the same point.

Proposition VI.

90. *The three perpendiculars drawn from the vertices of a triangle to the opposite sides meet in a point.*



Let ABC be a triangle, and let AD, BE, CF be drawn from the vertices perpendicular to the opposite sides.

The triangle ABC is divided into six right-angled triangles, three of which are similar to the other three. Hence we have the proportions

AF : AE :: AC : AB;

BD : BF :: BA : BC;

CE : CD :: CB : CA.

Multiplying together the corresponding terms of these three proportions, and observing that the third and fourth terms of the resulting proportion are equal, we have

AF.BD.CE=AE.CD.BF.

Hence (Prop. IV.) the three lines AD, BE, CF meet in a point.

91. *Scholium.* The three points D, E, and F are all on the sides of the triangle, if all its angles are acute. If one angle is obtuse, two of the three points are on the sides produced, as required by Scholium of Prop. III.

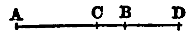
HARMONIC PROPORTION AND HARMONIC PENCILS.

92. *Def.* Three quantities are said to be in harmonic proportion when the first is to the third as the difference between the first and second is to the difference between the second and third. Thus 6, 4, 3 are in harmonic proportion, for $6:3::6-4:4-3$.

Proposition I.

93. *When a straight line is divided internally and externally in the same ratio, it is divided harmonically.*

Thus let AB be divided internally at C, and externally at D in the same ratio, so that



$$AD:BD::AC:CB;\text{ or }AD:AC::BD:CB.$$

Since BD=AD—AB, and CB=AB—AC, we have

$$AD:AC::AD-AB:AB-AC.$$

Hence AD, AB, and AC are in harmonic proportion, and the line AB is said to be divided harmonically in C and D.

94. Since the ratio of the distances of A from C and D is equal to the ratio of the distances of B from C and D, the line CD is divided harmonically at A and B.

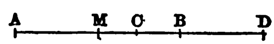
The four points A, B, C, D are called *harmonic points.* The points A and B are called *conjugate* points with respect to C and D; also the points C and D are conjugate with respect to A and B.

Since AD : BD :: AC : CB, we have AD.CB=AC.BD; that is, *when a line is divided harmonically into three parts, the rectangle under the whole line and the middle part equals the rectangle under the extreme parts;* and, conversely, *when a line is divided into three parts, such that the rectangle under the whole line and the middle part equals the rectangle under the extreme parts, the line is divided harmonically.*

95. In Art. 47 we found AC:CB::AD:BD; or AC.BD=CB.AD; hence AD is divided harmonically in C and B; that is, *the bisectors of the internal and external angles of a triangle divide the base harmonically.*

Proposition II.

96. If A, C, B, D form an harmonic system, and AB is bisected in M, then $MC.MD=MB^2$. For since AC.BD=AD.CB, we have



$$AC:CB::AD:BD.$$

By composition and division,

$$\frac{AC-CB}{2}:\frac{AC+CB}{2}::\frac{AD-BD}{2}:\frac{AD+BD}{2};$$

or

$$MC:MB::MB:MD;$$

that is,

$$MC.MD=MB^2.$$

Hence, *the half of a straight line,* AB, *is a mean proportional between the distances of the middle point* M *of this line from the two points* C *and* D *which divide it harmonically.*

97. *Conversely, if we have given* AB, *and its middle point* M, *and if* C *and* D *are so taken that* $MB^2=MC.MD$, *then* A, C, B, *and* D *are four harmonic points.*

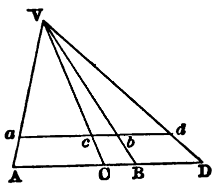
For since $MC:MB::MB:MD$,
by composition and division,

$$MB+MC:MB-MC::MD+MB:MD-MB;$$

that is, $AC:CB::AD:BD$.

98. *Scholium.* Let AB be a given segment and M its middle point, and let C move along the line AB. The conjugate point D will then also move along the same line. When C is at B, D is at the same point. As C moves toward M, the motion of D from B is at first very little greater than that of C; but as C approaches M, D moves more rapidly, and when C is indefinitely near to M, the distance of D is infinite. When C is to the left of M, D is to the left of A, and the two points approach A simultaneously. If C is in AB produced, D is on the segment AB.

99. *Def.* A system of straight lines diverging from a point is called a *pencil;* each diverging line is called a *ray*, and the point from which they diverge is called the *vertex* of the pencil.

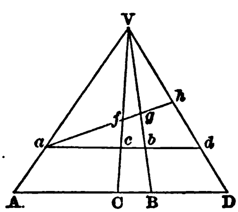


If from the point V straight lines be drawn to four harmonic points, A, C, B, D, the four lines, VA, VC, VB, VD, form an *harmonic pencil.*

It is evident that if AB is divided harmonically by the pencil, any straight line, *acbd*, drawn parallel to AD will also be divided harmonically by the pencil.

PROPOSITION III.

100. *Every straight line which cuts an harmonic pencil is divided harmonically by the pencil.*



Let the pencil V.ACBD divide the transversal ACBD harmonically, so that $AC.BD=AD.CB$; then will it divide any other transversal *afgh* harmonically.

Through the point *a* draw *ad* parallel to AD. Since the triangle *agb* is cut by the transversal VC, we have (Art. 84)

$$af.bc.gV=fg.ac.bV. \quad (1)$$

And since the same triangle is cut by the transversal V*d*, we have

$$gh.ad.bV=ah.bd.gV. \quad (2)$$

Multiplying together (1) and (2), and omitting common factors, we have

$$af.gh.bc.ad=ah.fg.ac.bd.$$

But since *ad* is parallel to AD, we have

$$ad.cb=ac.bd.$$

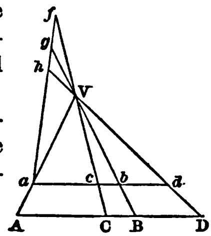
Hence $af.gh=ah.fg;$

that is, *ag* is divided harmonically.

101. Hence the rays VA, VB are called *conjugate rays* with respect to

the rays VC, VD, and are said to divide the angle AVB harmonically; also the rays VC, VD are conjugate rays with respect to the rays VA, VB, and divide the angle AVB harmonically.

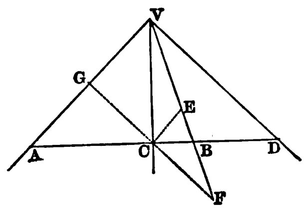
102. *Scholium.* The preceding demonstration applies when the transversal cuts one or more of the rays on opposite sides of the vertex, as in the annexed figure.



PROPOSITION IV.

103. *Three straight lines of an harmonic pencil being given, it is required to find the fourth ray of the pencil.*

Let VA, VB, VC be three rays of an harmonic pencil; it is required to find the fourth ray VD, conjugate to VC.



On VC take any point C, and draw CE parallel to VA. Take EF equal to VE; join CF, and produce it to G; and through V draw VD parallel to GF.

The line VD is conjugate to VC.

Since the triangles AGC, AVD are similar, we have

GC : VD :: AC : AD.

Also, since the triangles CFB, VBD are similar, we have

VD : CF :: BD : CB.

Multiplying together these proportions, and observing that GC=CF, since VE=EF, we have AC.BD=AD.CB.

Hence AD is divided harmonically in C and B, and the ray VD is conjugate to the ray VC.

104. *Def.* If the opposite sides of any quadrilateral be produced to meet, the figure thus formed is called a *complete quadrilateral.*

Let ABCD be any quadrilateral, and let its opposite sides be produced to meet in E and F; the line EF is called the *third diagonal* of the quadrilateral, and the whole figure formed by the four lines meeting in six points forms a complete quadrilateral. The complete quadrilateral has three diagonals: viz., two interior, AC, BD, and one exterior, EF.

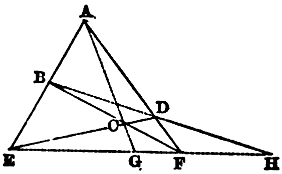
PROPOSITION V.

105. *Each diagonal of a complete quadrilateral is harmonically divided by the two other diagonals.*

Let ABCDEF be a complete quadrilateral, and let AC, BD, EF be its diagonals; any one of them, as EF, is divided harmonically by the two others in the points G and H.

The triangle AEF cut by the transversal BDH (Art. 84) gives

EH.FD.AB=FH.AD.EB. (1)



In the same triangle the straight lines AG, ED, BF, drawn from the vertices and intersecting in C (Art. 86), give the equation

FG.AD.EB=EG.FD.AB. (2)

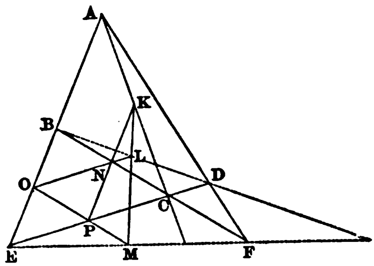
Multiplying together (1) and (2), and omitting the common factors, we have

EH.FG=FH.EG.

In a similar manner it may be proved that each of the diagonals AC, BD is harmonically divided by the two other diagonals.

PROPOSITION VI.

106. *The middle points of the three diagonals of a complete quadrilateral are in a straight line.*



Let K, L, M be the middle points of the three diagonals of the complete quadrilateral ABCDEF, and let N, O, P be the middle points of the sides of the triangle BEC. Since OP bisects BE and CE, it is parallel to BC, and therefore passes through M, the middle point of EF. For a similar reason NP passes through K, the middle point of AC, and NO passes through L, the middle point of BD.

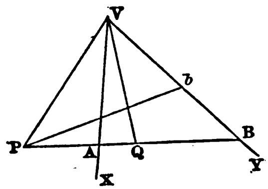
If we regard ADF as a transversal of the triangle BCE, we obtain (Art. 84) AB.DE.FC=AE.DC.FB.

Dividing each factor by 2, and observing that $\frac{1}{2}$AB=KN, $\frac{1}{2}$DE=LO, $\frac{1}{2}$FC=MP, etc., we have KN.LO.MP=KP.LN.MO.

Therefore the points K, L, M, lying in the sides of the triangle NOP, satisfy Art. 85, and are in the same straight line.

POLES AND POLARS WITH RESPECT TO AN ANGLE.

107. *Def.* If through a fixed point, P, a line, PAB, be drawn cutting the two sides of an angle, XVY, and if on PB a point Q be taken, which is the harmonic conjugate of P with respect to the points A and B, the line VQ is called the *polar* of the point P with respect to the sides of the angle XVY, and P is called the *pole* of VQ.



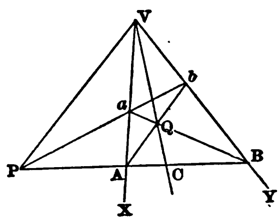
PROPOSITION I.

108. *Any straight line drawn through a pole to meet the sides of an angle is divided harmonically by these lines and the polar.*

Let P be the pole and VQ the polar with respect to the lines VX, VY, and let A and B be the points in which a straight line drawn through P meets the two lines. Since VQ is the polar of the point P, the points P, A, Q, and B form an harmonic system; and if we draw the line VP, the lines VP, VA, VQ, VB form an harmonic pencil, and any straight line Pb drawn through P will be divided harmonically by the pencil (Prop. 100).

PROPOSITION II.

109. *A point* P *and an angle* XVY *being given, if from* P *we draw two lines cutting the sides of the angle in* A, B, *a*, *b*, *and if we draw the diagonals* A*b*, B*a* *intersecting in* Q, *the line* VQ *will be the polar of the point* P *with respect to the lines* VX, VY.



If we consider the complete quadrilateral V*a*Q*b*, AB, we find the diagonal AB to be divided harmonically in P and C by the two other diagonals *ab*, VQ (Art. 105); hence the four lines VP, VX, VQ, VY form an harmonic pencil, and the straight line VQ is the polar of the point P with respect to the two lines VX, VY.

110. *Scholium.* The polar of any given point with respect to an angle may also be found by the method of Art. 103.

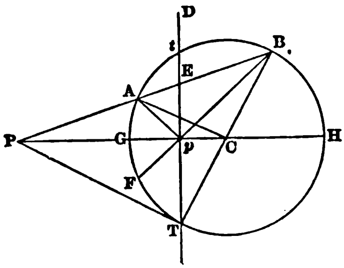
POLES AND POLARS WITH RESPECT TO A CIRCLE.

111. *Def.* If from a fixed point P a line be drawn to C, the centre of a circle, and on CP a point p be taken such that CP.Cp is equal to the square of the radius of the circle, the straight line Dp perpendicular to the line CP at the point p is called the *polar of* P *with respect to the circle*, and the point P is called the *pole of* Dp *with respect to the circle.*

PROPOSITION I.

112. *Any straight line drawn through the pole to meet a circle is divided harmonically by the circle and the polar.*

Let P be the pole and pD be the polar, and let A and B be the points in which a straight line drawn through P meets the circumference. Join CA, CB, pA, pB, and produce Bp to meet the circumference in F.



Then, since $CP.Cp=CB^2$, we have

$$CP:CB::CB:Cp;$$

and therefore the angle CBP is equal to the angle C*p*B (B. IV., Pr. 21).

Also, since $CP.Cp=CA^2$, we have

$$CP:CA::CA:Cp;$$

and therefore the angle CAP is equal to the angle C*p*A. But CBP is equal to CAB, which is the supplement of CAP; therefore C*p*B is the supplement of C*p*A. Hence C*p*B, which is equal to P*p*F, is equal to P*p*A. Also A*p*D and B*p*D are equal, being the complements of A*p*P and B*p*C. Therefore D*p* bisects the vertical angle of the triangle A*p*B, and P*p* bisects the exterior angle of the same triangle; hence the line AB is divided harmonically (Art. 95) in P and E.

113. *Cor.* 1. To find the pole of a given straight line DT, draw a diameter GH perpendicular to the given line intersecting it in *p*, and on this diameter produced, take P such that $CP.Cp=CG^2$; the point P is the pole of DT with respect to the circle.

114. *Cor.* 2. When the point P is without the circle, *its polar is the line* T*t joining the points of contact of the tangents from* P. For let T and *t* be the points where D*p* intersects the circumference, then, since $CP.Cp=CT^2$, we have

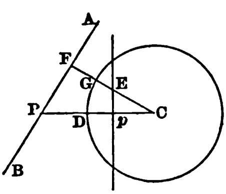
$$CP:CT::CT:Cp.$$

Hence the two triangles CPT, C*p*T are similar; therefore CTP is a right angle, and PT is a tangent to the circle at the point T. For the same reason a line drawn from P to the point *t* is a tangent to the circle.

115. *Cor.* 3. If the point P approaches the circumference of the circle, the point *p* will also approach it, and when PC becomes equal to the radius of the circle, C*p* will also be equal to the radius of the circle; therefore *the polar of any point on the circumference is the tangent at that point.*

PROPOSITION II.

116. *The polars of all the points of a straight line pass through the pole of that line.*



Let AB be any straight line, and P any point in it. Draw CP, and take *p* such that $CP.Cp=CD^2$, the straight line *p*E perpendicular to CP will contain the pole of the straight line AB.

Draw CF perpendicular to AB cutting E*p* in E. Then, since the triangles C*p*E, CFP are similar, we have

$$CF.CE=CP.Cp.$$

But

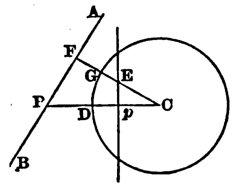
$$CP.Cp=CD^2.$$

Therefore $CF.CE = CD^2$;
and therefore the point E is the pole of AB (Art. 113).

PROPOSITION III. (*Converse of Prop. II.*)

117. *The poles of all the straight lines which pass through a given point are situated on the polar of that point.*

Let P be the given point, and AP any straight line passing through it. In order to find the pole of AP, we draw CF perpendicular to AP, and on CF take the point E such that $CF.CE = CG^2$. Join PC, and draw Ep perpendicular to CP. Then, by similar triangles,



$$CP : CF :: CE : Cp;$$

or $$CP.Cp = CF.CE = CG^2.$$

Therefore Ep is the polar of the point P, and the pole of the line AP is situated on the polar of P.

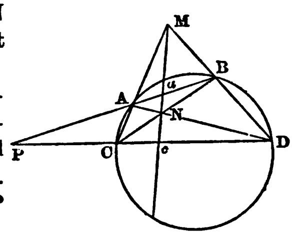
118. *Cor.* The pole of a straight line is the intersection of the polars of any two of its points.

119. *Scholium.* If P be any point within or without a given circle, and PA a line which revolves about the point P, the pole of AP will move along the polar of P; that is, the polar of a given point is the locus of the poles of all the lines passing through the given point.

PROPOSITION IV.

120. *If through a fixed point* P *in the plane of a circle any two secants,* PAB, PCD, *are drawn, also the chords* AD, BC *are drawn intersecting in* N, *and the chords* AC, BD *are produced to meet in* M, *the straight line which joins the points* M *and* N *is the polar of the fixed point* P.

Let a be the point in which the line MN intersects AB, and c the point in which it intersects CD.

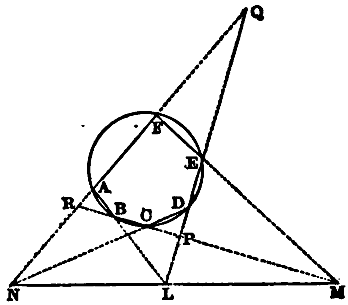


Since MANBDC is a complete quadrilateral, the diagonal AB is divided harmonically in P and a (Art. 105), and the diagonal CD is divided harmonically in P and c. Therefore ac is the polar of the point P (Art. 112).

PROPOSITION V.

121. *If a hexagon be inscribed in a circle, the points of intersection of the three pairs of opposite sides will lie in a straight line.*

Let ABCDEF be a hexagon inscribed in a circle, and let the opposite sides AB, DE be produced to meet in L; also the sides BC and EF to meet in M, and the sides CD and AF to meet in N. The points L, M, and N lie in a straight line.



The triangle PQR, formed by the alternate sides BC, DE, FA produced, is cut by the transversals ABL, DCN, and FEM. Hence (Art. 84)

PE.QF.RM=PM.RF.QE;
QA.RB.PL=QL.PB.RA;
PD.QN.RC=PC.RN.QD.

Multiplying together these equals, and observing that

PE.PD=PB.PC (B.IV., Pr.29, Cor.2);
QF.QA=QE.QD;
RB.RC=RA.RF;

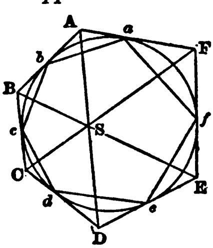
we have RM.PL.QN=PM.QL.RN.

Hence the points L, M, N are situated on a straight line (Art. 85).

This theorem is due to Pascal.

Proposition VI.

122. *If a hexagon be described about a circle, the three diagonals which join the opposite vertices intersect in the same point.*



Let ABCDEF be a hexagon described about a circle; let the side AB touch the circle in the point *b*, the side BC in the point *c*, etc., and join *ab*, *bc*, etc. The straight line *ab* is the polar of the point A (Art. 114); so also *ed* is the polar of the point D. Hence, if *ab* and *ed* be produced to meet, their point of intersection will be the pole of AD (Art. 118). So also the point of intersection of *bc* and *ef* will be the pole of BE, and the point of intersection of *cd* and *af* will be the pole of CF. But these three points of intersection are situated in a straight line (Art. 121). Hence the polars AD, BE, CF of these points intersect in a point S, which is the pole of the line joining these three points of intersection (Art. 116).

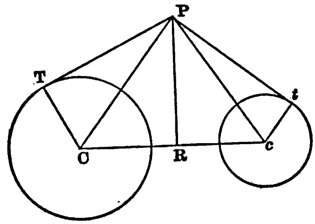
This theorem is due to Brianchon.

RADICAL AXIS OF TWO CIRCLES.

Proposition I.

123. *Required the locus of a point from which tangents drawn to two given circles are equal.*

Let C and *c* be the centres of the given circles, and P any point from which equal tangents, PT, P*t*, are drawn. Draw PR perpendicular to the line joining the centres, and join CT, ct. Then, since PT and Pt are supposed equal, we have



$$CP^2-CT^2=PT^2=cP^2-ct^2;$$

or $$CT^2-ct^2=CP^2-cP^2.$$

The first member of this equation is a given quantity; therefore the second member is a given quantity; and since Cc is also given, the required locus is the straight line PR, perpendicular to Cc, and cutting it so that $CR^2-cR^2=CT^2-ct^2$ (Art. 48).

124. *Def.* The straight line PR is called the *radical axis* of the two circles.

125. We thus see that *the radical axis of two circles is a straight line perpendicular to the line joining the centres of the circles, and dividing this line so that the difference of the squares of the two segments is equal to the difference of the squares of the radii.*

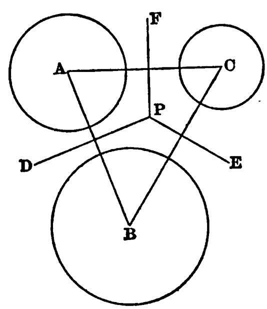
If the two circles have no point in common, the radical axis does not intersect either of them.

If the circles touch each other, either externally or internally, their common tangent at the point of contact is their radical axis.

If the circles intersect, their common chord is their radical axis.

PROPOSITION II.

126. *The radical axes of a system of three circles, taken two and two, meet in a point.*

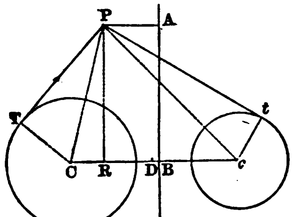


Let A, B, C be the centres of three circles. Let DP be the radical axis of A and B, and EP be the radical axis of B and C. The three centres, A, B, and C, not being in the same straight line, the axes DP, EP, which are perpendicular to the intersecting lines AB, BC (B. III., Pr. 7), will meet in a point which we designate by P. The tangents drawn from P to the circles A and B are, by definition, equal; also the tangents drawn from P to the circles B and C are equal. Hence the tangents drawn from P to the circles A and C are also equal, and therefore the radical axis of A and C passes through P.

127. If the three circles intersect, the radical axis of any two of them is their common chord; hence *the common chords of every pair of three intersecting circles meet in a point.*

PROPOSITION III.

128. *The difference of the squares of the tangents from any point to two circles is equal to twice the rectangle under the distance between the centres of the circles, and the distance of the point from their radical axis.*



Let C and c be the centres of two circles, AB their radical axis, P any point in the plane of the circles, PT, Pt the tangents drawn from P to the two circles. Draw PA perpendicular to AB, and PR perpendicular to Cc, and bisect Cc in D. Then (B. IV., Pr. 11, Cor. 1)

$$Pt^2 = Pc^2 - ct^2, \text{ and } PT^2 = PC^2 - CT^2.$$

Therefore

$$Pt^2 - PT^2 = (Pc^2 - PC^2) + (CT^2 - ct^2).$$

But $\quad Pc^2 - PC^2 = cR^2 - CR^2 = 2Cc.DR$ (B. IV., Pr. 10).

And by Art. 123, $\quad CT^2 - ct^2 = CB^2 - cB^2 = 2Cc.BD.$

Therefore $\quad Pt^2 - PT^2 = 2Cc(DR + BD);$

$$= 2Cc.BR;$$

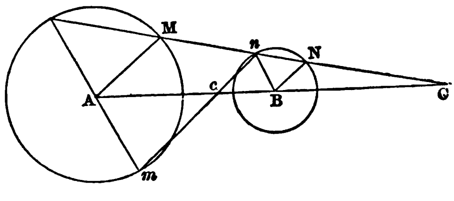
$$= 2Cc.AP.$$

CENTRES OF SIMILITUDE.

129. *Def.* If the straight line joining the centres of two circles is divided externally and internally into segments proportional to the corresponding radii, the former point of section is called the *external*, and the latter the *internal centre of similitude* of the two circles.

PROPOSITION I.

130. *If in two circles two radii are drawn parallel to each other and in the same direction, the straight line joining their extremities passes through the external centre of similitude; but if the parallel radii are drawn in opposite directions, the straight line joining their extremities passes through the internal centre of similitude.*



Let AM, BN be two parallel radii drawn in the same direction, and let the line MN intersect the line AB produced in C.

Since BN is parallel to AM, we have

$$AC : BC :: AM : BN;$$

and therefore, by the definition, C is the external centre of similitude.

If the parallel radii A*m*, B*n* are drawn in opposite directions, and the line *mn* intersects the line AB in *c*, by similar triangles we have

$$Ac : Bc :: Am : Bn;$$

and therefore *c* is the internal centre of similitude.

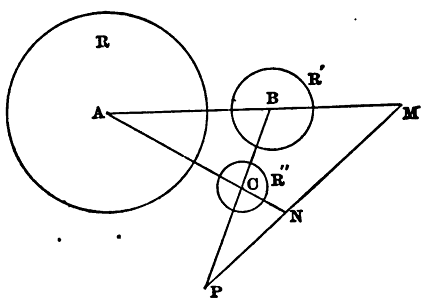
131. *Cor.* 1. If any transversal is drawn through a centre of similitude, the radii drawn to the points in which it cuts the circumferences will be parallel, two and two.

132. *Cor.* 2. If a transversal drawn through a centre of similitude is a tangent to one of the circles, it is also a tangent to the other; therefore, when one circle is wholly without the other, *the centres of similitude are the intersections of the pairs of external and internal common tangents.*

133. If the circles touch each other externally, the point of contact is their internal centre of similitude; if they touch internally, the point of contact is their external centre of similitude. Hence, *if any number of circles touch each other at the same point, and a straight line is drawn through it cutting the circles, the straight lines joining each point of intersection with the centre of the corresponding circle will be all parallel.*

PROPOSITION II.

134. *The external centres of similitude of three circles, taken two and two, are in a straight line.*



Let A, B, C be the centres of three circles; R, R′, R″ their radii; and let M be the external centre of similitude of the circles whose centres are A and B; N the external centre of similitude of the circles A and C; and P that of B and C.

By the definition, we have

$$AM : BM :: R : R';$$
$$BP : CP :: R' : R'';$$
$$CN : AN :: R'' : R.$$

Multiplying these proportions together, term by term, we have

$$AM.BP.CN : BM.CP.AN :: R.R'.R'' : R'.R''.R.$$

Hence $$AM.BP.CN = BM.CP.AN.$$

Therefore the points M, N, P are in a straight line (Art. 85).

135. *Cor.* 1. In like manner it may be shown that the straight line passing through any two internal centres of similitude of three circles, taken two and two, also passes through the external centre of similitude of the other pair of circles.

136. *Cor.* 2. Hence the six centres of similitude of three circles, taken two and two, are situated three and three on four straight lines. These lines are called *axes of similitude.*

PROJECTIONS.

137. *Projection* is the delineation on a given plane of any figure situated out of this plane.

We shall consider two modes of projection, one called *orthogonal* projection, and the other *perspective* or *conical* projection.

ORTHOGONAL PROJECTION.

138. *Def.* The *orthogonal projection of a point* upon a plane is the foot of the perpendicular let fall from that point upon the plane. The plane to which the perpendicular is drawn is called the *plane of projection.*

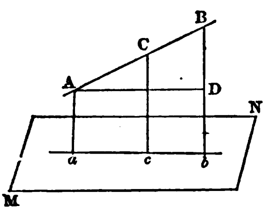
The *orthogonal projection of a line* upon a plane is the line which contains the projections of all the points of the first line.

The *orthogonal projection of a plane figure* is that surface in the plane of projection that is inclosed by the projection of the perimeter of the original figure.

In the twelve following Articles the term projection will be used as equivalent to the term orthogonal projection.

PROPOSITION I.

139. *The projection of a straight line upon a plane is a straight line.*



Let AB be the given straight line, and MN the given plane. From any point, A, in the straight line draw a perpendicular to the plane MN, meeting it in *a*. Let a plane pass through AB and A*a*, and let its intersection with the plane MN be *ab*. Then *ab* is a straight line (B. VII., Pr. 3), and it is to be proved that *ab* is the projection of AB.

In AB take any point C; and in the plane BA*a* draw C*c* parallel to A*a*, meeting *ab* at *c*. Then C*c* is perpendicular to the plane MN (B. VII., Pr. 9, Cor. 1); hence *c* is the projection of C; that is, every point in the line AB is projected into some point on the line *ab*.

140. *Scholium.* If the given straight line be perpendicular to the plane of projection, its projection is a point.

PROPOSITION II.

141. *The projection of a straight line of given length is equal to the length of the original straight line multiplied by the cosine of the angle between the straight line and its projection.*

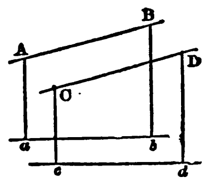
Let AB be any straight line, and ab its projection. Draw AD parallel to ab, meeting Bb at D. Then we have

$$ab = AD = AB \cos. BAD \text{ (Trig., Art. 45).}$$

142. *Scholium.* Since a straight line makes equal angles with all parallel lines which it intersects (B. I., Pr. 23), the angle between the straight line AB and its projection is understood to mean the angle between AB and any straight line, AD, parallel to its projection; that is, BAD is the angle between AB and ab.

Proposition III.

143. *The projections of parallel straight lines are themselves parallel straight lines.*

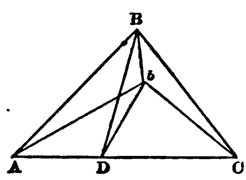


Let AB and CD be parallel straight lines; also let a be the projection of A upon a given plane, and let c be the projection of C upon the same plane. Since Aa is parallel to Cc (B. VII., Pr. 9), the plane BAa is parallel to the plane DCc (B. VII., Pr. 15); also the intersections of these planes with the plane of projection are parallel (B. VII., Pr. 12), and these intersections are the projections of AB and CD (Art. 138).

Proposition IV.

144. *If any plane figure be projected upon a plane, the area of the projected figure is equal to the area of the original figure multiplied by the cosine of the angle between the two planes.*

Case 1. Let the figure be *a triangle having one side in the plane of projection.*



Let ABC be a triangle having the side AC in the plane of projection; and let b be the projection of B. Draw BD perpendicular to AC, and join bD.

Since Bb is perpendicular to the plane ADb, it is perpendicular to the two lines Ab and Db. Hence we have (B. IV., Pr. 11)

$$AD^2 = AB^2 - BD^2 = Ab^2 + Bb^2 - (Db^2 + Bb^2) = Ab^2 - Db^2.$$

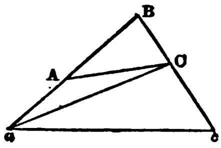
Therefore ADb is a right angle (B. IV., Pr. 11), and therefore BDb is the angle of inclination of the plane of the triangle to the plane of projection (B. VII., Def. 4).

Now the area of $\quad ABC = \frac{1}{2}AC.BD,$

and the area of $\quad AbC = \frac{1}{2}AC.bD;$

therefore, $$\frac{\text{area of } AbC}{\text{area of } ABC} = \frac{bD}{BD} = \cos. BDb.$$

Case 2. Let the figure be *any triangle.*



Let ABC be any triangle, and let its plane, if produced, meet the plane of projection in the straight line ac. Jion aC, and let x denote the angle between the plane of the triangle and the plane of projection. Then, by Case 1, the area of projection of aBc=area of aBc × cos. x; (1)

also, the area of projection of aCc=area of aCc × cos. x. (2)

Subtracting (2) from (1), we have

area of projection of aBC=area of aBC × cos. x.

Hence the proposition is true for any triangle which has one angular point in the plane of projection; therefore it is true for the triangle aAC; and therefore by subtraction it is true for the triangle ABC.

Case 3. Let the figure be *any polygon*.

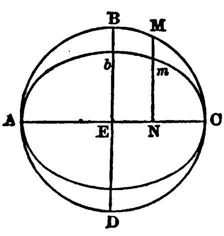
The polygon may be decomposed into triangles, and since the proposition is true for each triangle, it is true for the whole figure.

Case 4. Let the figure be bounded by *curved lines*.

We may inscribe in the figure any rectilinear polygon, and the proposition will be true of the polygon. By increasing the number of sides of the polygon, and diminishing the length of each side, the area of the polygon can be made to differ as little as we please from the area of the figure. Hence the proposition is also true for the area bounded by the curved lines.

Proposition V.

145. *The projection of a circle is an ellipse.*



First, let the plane of projection pass through the centre of the circle, and cut the circle in the diameter AC; and let BD be that diameter which is perpendicular to AC. On the circumference of the circle take any point M, and draw MN perpendicular to AC. Let b denote the projection of B, and m the projection of M, and let x denote the angle of inclination of the two planes.

Then the projection of BE will be BE cos. x; and the projection of MN will be MN cos. x.

Hence MN : BE :: MN. cos. x : BE.cos. x :: mN : bE;

or MN : mN :: BE : bE.

Therefore m is a point on an ellipse whose semi-minor axis is bE= BE cos. x, and whose semi-major axis is BE or AE (Prop. 14, Ellipse).

Second, if the plane of projection does not pass through the centre of the circle, let a plane be drawn through the centre parallel to the plane of projection. AC will be equal and parallel to its projection, and we shall still have the same proportion given above.

146. *Scholium* 1. The centre of the circle is projected into the centre of the ellipse.

Scholium 2. The cosine of the angle between the plane of the circle and the plane of the ellipse is $\frac{bE}{AE}$.

PROPOSITION VI.

147. *To find the area of an ellipse.*

Let a denote half the major axis of the ellipse, and b half the minor axis.

By Art. 145 the ellipse may be regarded as the projection of a circle whose radius is a.

By Art. 144 the area of the ellipse=area of circle×cos. x.

But the area of the circle $=\pi a^2$ (B. VI., Pr. 13, Cor. 3).

Also, $$\cos. x=\frac{b}{a} \text{ (Art. 146).}$$

Hence the area of the ellipse $=\pi a^2\times\frac{b}{a}=\pi ab$.

PROPOSITION VII.

148. *The projection of the tangent at any point of a curve is the tangent at the corresponding point of the projection of the curve.*

Let A and B be any two points on a curve, and let a and b be their projections. Then (Art. 139) the straight line drawn through a and b is the projection of the straight line through A and B. Let A move along the curve to B; then the limiting position of the secant through A and B is the tangent to the curve at B. Now as A moves to B along the curve, a moves to b along the projection of the curve, and the limiting position of the secant through a and b is the tangent at b to the projection of the curve. Hence tangents are projected into tangents.

PROPOSITION VIII.

149. *Conjugate diameters of an ellipse are the projections of diameters of a circle which are perpendicular to each other.*

If two diameters of a circle are perpendicular to each other, each diameter is parallel to the tangent at the extremity of the other diameter. Hence, by Arts. 143 and 148, the projection of each diameter is parallel to the tangent to the projection of the circle at the extremity of the other diameter. Therefore, by Def. 12, Ellipse, the projections of diameters of the circle which are perpendicular to each other are conjugate diameters of the ellipse.

150. *Cor.* Since the area bounded by two radii of a circle which are perpendicular to each other, and the intercepted arc of the circle is one

fourth of the area of the circle, therefore the area bounded by two conjugate semi-diameters of an ellipse and the intercepted arc of the ellipse is one fourth of the area of the ellipse.

PROPOSITION IX.

151. *The parallelogram formed by drawing tangents through the vertices of two conjugate diameters is equal to the rectangle of the axes.*

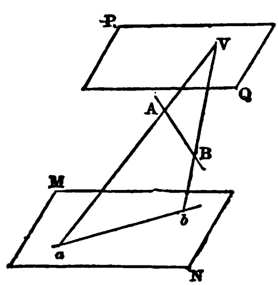
The rectangle constructed upon the two axes of an ellipse may be regarded as the projection of a square circumscribing a circle whose diameter is the major axis of an ellipse. Also, the parallelogram formed by tangents drawn through the vertices of two conjugate diameters may be regarded as the projection of another square circumscribing the same circle. Since the areas of these squares are equal, the areas of their projections upon the same plane are also equal (Art. 144).

152. *Scholium.* We thus see that certain properties of the ellipse may be deduced from corresponding properties of the circle by considering the ellipse as the projection of a circle upon a plane. Such properties are called *projective properties*, and it is evident that a large part of the properties enumerated in the Chapter on the Ellipse are of this description.

PERSPECTIVE OR CONICAL PROJECTION.

153. *Definitions.* The *perspective projection of a point* upon a plane is the intersection of the plane by a straight line drawn through the point from a fixed point in space.

The *perspective projection of a line* upon a plane is the line which contains the projections of all the points of the given line.



Let MN be a plane given in position, and V be a given point without it; then, if through any point, A, a straight line be drawn from V to meet the plane MN in a, the point a is called the perspective or conical projection of the point A upon the plane MN.

The given plane MN is called the *plane of projection;* the fixed point V is called the *vertex*, and the plane PQ, which is drawn through the point V parallel to MN, is called the *vertex plane.*

A plane figure is said to be projected upon a plane when its several lines are projected; and the figure which is contained by the projected bounding lines is called the perspective projection of that figure.

154. *Scholium.* It is obvious that the *shadow* formed on any plane by a figure when light falls on it from a point, is the perspective projection

of that figure corresponding to that point as a vertex. Also, if one point or line or surface is the projection of a second point or line or surface, the second is the projection of the first.

In the next thirty-three Articles the single term projection will be used to denote perspective projection.

Proposition I.

155. *The projection of a straight line upon a plane is a straight line.*

For suppose a point to move along the line AB (prec. fig.); its projection on MN will always be in the intersection of the plane VAB with the plane MN, and this intersection is a straight line (B. VII., Pr. 3).

156. *Scholium.* If the given straight line passes through the vertex, its projection on any plane not passing through the vertex is a point.

157. *Cor.* 1. The projection of a straight line which is parallel to the plane of projection is parallel to its original.

158. *Cor.* 2. If any number of straight lines meet in one point not in the vertex plane, their projections form a group of diverging rays all passing through one point. For, the point common to the given lines being projected into a point in the plane of projection, this second point must be common to the projections of the several given lines.

159. *Cor.* 3. If a point be situated in the vertex plane, the straight line which is drawn through it from V can not meet the plane MN, because it lies in a plane which is parallel to MN. Therefore no point in the vertex plane can be projected from V upon the plane MN. If a point is indefinitely near to the vertex plane its projection is at an indefinitely great distance. Hence we say that points in the vertex plane are *projected to infinity.*

160. *Cor.* 4. Conversely, if the projection of a point is at an infinite distance, that point must be in the vertex plane. Points on the other side of the vertex plane from MN are projected by lines produced through V.

Proposition II.

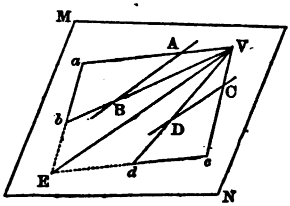
161. *If two or more straight lines intersect in the vertex plane, their projections are parallel straight lines.*

For the projections are straight lines in a plane, and can not meet because the point at which the original lines meet is projected to infinity (Art. 159).

162. *Scholium.* Since a group of lines intersecting in a point not in the vertex plane is projected into diverging rays (Art. 158), and a group of lines intersecting in a point in the vertex plane is projected into parallel lines (Art. 161), it is convenient to conceive of parallel lines as meeting at infinity, and to regard a group of parallel lines as a special case of diverging rays.

Proposition III.

163. *Two or more parallel straight lines which cut the plane of projection are projected into diverging rays.*



Let AB and CD be any two parallel straight lines, and let *ab* and *cd* be their projections on the plane MN. Through V draw VE parallel to AB and CD, meeting the plane MN in E.

The line VE must be in the plane VAB (B. VII., Pr. 2, Cor. 2), and also in the plane VCD, and hence it is the line of intersection of these two planes. Therefore *ab* and *cd* must meet in E. In the same manner it may be proved that the projection of any other straight line which is parallel to AB or CD will pass through the point E.

Or, since the given parallel lines meet in a point at infinity in their common direction, they will be projected into a group of diverging rays (Art. 158).

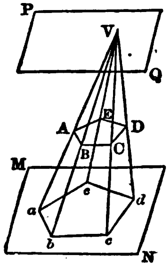
164. *Cor.* 1. If a group of parallel lines be projected into a group of diverging rays, the common point of the rays is the point E, where the line drawn from the vertex parallel to the given lines meets the plane of projection.

165. *Cor.* 2. The projections of parallel lines that are also parallel to the plane of projection are parallel straight lines. This follows from Art. 157; or otherwise, since the given lines meet in a point at infinity in the vertex plane, their projections meet in the projection of that point, that is, at infinity.

166. *Cor.* 3. Any quadrilateral may be projected into a parallelogram. For the two points of intersection of the opposite sides may be placed in the vertex plane, and the figure then projected.

Proposition IV.

167. *The projection of any plane figure on a parallel plane is similar to the original figure.*



Let ABCDE be any plane figure; let MN be a plane parallel to ABCDE; let *abcde* be the projection of ABCDE upon the plane MN, and let V be the vertex.

Since AB and *ab* are the intersections of two parallel planes by a third plane, they are parallel (B. VII., Pr. 12). Also, *bc* is parallel to BC, *cd* to CD, etc. Hence

AB : *ab* :: VB : V*b* :: BC : *bc*.

Also the angle ABC = angle *abc* (B. VII., Pr. 15).

The same may be proved of the other sides and angles of the polygons ABCDE, *abcdé*. Hence these polygons are similar.

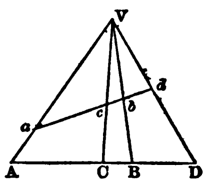
168. *Def.* *Similar curves* are such that any rectilinear figure being inscribed within the one, a similar rectilinear figure may be inscribed within the other. Similar curves may be regarded as the limits of similar polygons having an infinite number of sides which are indefinitely small.

169. *Scholium.* By the method employed in Art. 144, the demonstration of Prop. IV. may be extended to figures bounded by curve lines.

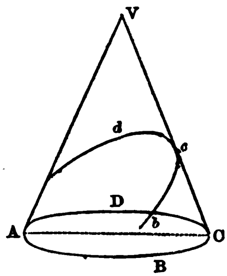
Proposition V.

170. *The projections of four harmonic points are harmonic points.*

Let A, C, B, D be four harmonic points, and let a, c, b, d be their projections. The four lines VA, VC, VB, VD form an harmonic pencil (Art. 99), and this pencil is divided harmonically by the plane of projection (Art. 100).



171. *Definition.* Let VABC be a cone with a circular base, and let its surface be cut by any plane bcd not passing through V; the intersection of the cone with the plane is called a *conic section*, or, more simply, a *conic*. Every conic may be regarded as the projection of a circle upon a plane.



172. *Scholium.* On pp. 215, 236, and 257 the intersection of a right circular cone with a plane is shown to be a parabola, an ellipse, or an hyperbola, as these terms had been previously defined. It may be easily shown that any parabola, ellipse, or hyperbola can be the section of some right cone; also, that the sections of an oblique cone of circular base with a plane, or the sections of any cone having a parabola, ellipse, or hyperbola for its base, made by a plane not passing through the vertex, will be a parabola, an ellipse, or an hyperbola. It will be a parabola if the plane through the vertex parallel to the cutting plane touches the cone; it will be an hyperbola if the plane through the vertex cuts the cone; and it will be an ellipse if the plane through the vertex does not cut the cone.

In the following propositions some properties of conics will be deduced from the above definition, without using the definitions and properties proved in pp. 215–262.

Proposition VI.

173. *A tangent to the circle is projected into a tangent to the conic.*

This may be proved in the same manner as in Art. 148.

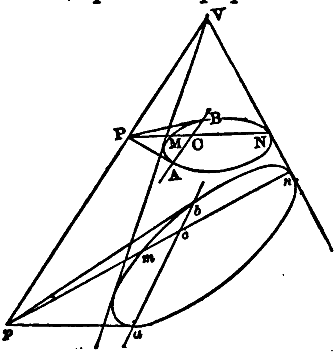
174. *Cor.* 1. A conic can be met by a straight line in only two points.

175. *Cor.* 2. From a point without a circle two tangents and only two

can be drawn to the circle; hence from a point without a conic two tangents, and only two, can be drawn to the conic.

PROPOSITION VII.

176. *A pole and polar of a circle are projected into a point and straight line which possess like properties with regard to the conic.*



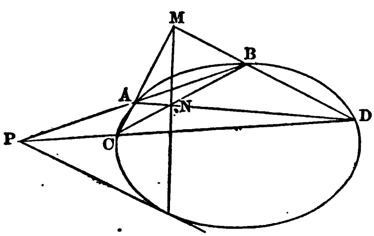
If the circle MN be projected into the conic *mn*, the pole P and its polar AB will be projected into a point *p* and a right line *ab*, in the plane of the conic, such that if any transversal be drawn through *p* cutting *ab* in *c*, and the conic in *m* and *n*, the four points *m*, *n*, *c*, and *p* will form an harmonic couple; that is,

$$cm : cn :: mp : pn.$$

For the transversal *pn* is the projection of a transversal PN in the plane of the circle, and the four points *m*, *n*, *c*, and *p* are the projections of the four harmonic points (Art. 112) M, N, C, and P, and hence they are harmonic points (Art. 170).

177. *Def.* The point *p* and the line *ab* are called a *pole* and its *polar* with respect to the conic.

178. *Cor.* 1. Props. 1, 2, and 3 (Arts. 112–117) for the circle are true for any conic, since in each case the figure in the circle is projected into the corresponding figure for the conic.



179. *Cor.* 2. Hence, a pole P being given, the polar with respect to a conic may be constructed. Through P draw the transversals PB, PD, cutting the conic in A and B, C and D; draw AD and CB to meet in N, and produce CA and DB to meet in M, and join MN; MN will be the polar of P.

180. *Cor.* 3. Hence, also, if a line be given in the plane of a conic, its pole with respect to the conic may be constructed. For if two points be taken on the line, and their polars be constructed (Cor. 2), these polars will pass through the pole of the given line (Art. 116).

181. *Cor.* 4. If the pole be without the conic, its polar cuts the conic; if the pole be within, its polar does not cut the conic.

182. *Cor.* 5. Hence, also, to draw tangents to the curve from a given point without the conic, construct the polar of the point, and from

the given point draw lines to the intersections of the polar with the conic.

183. *Cor.* 6. If the point P be projected to infinity, its polar is a diameter; that is, it bisects all the parallel chords drawn in the direction of p. For, $pn:pm::cn:mc$; hence, when p is at infinity, $cn=mc$.

[Compare Props. 11 and 12 of Parabola, and Props. 7 and 21, Cor. 1, of Ellipse and Hyperbola.]

184. *Cor.* 7. If AB be projected to infinity, p is a *centre;* that is, every chord of the conic through p is bisected at p. For, $pn:mp::cn:cm$; and when c is at infinity, $pn=mp$.

[Compare Prop. 4 of Ellipse and Hyperbola.]

185. *Cor.* 8. Let o be the middle point of the chord mn, then will $om^2=oc.op$ (Art. 96).

[Compare Props. 11 and 20 of Ellipse and of Hyperbola, as special cases of this Proposition.]

Proposition VIII.

186. *If a hexagon be inscribed in a conic, the points of intersection of the pairs of opposite sides will lie in a straight line.*

This Proposition has been proved for a circle (Art. 121); and since straight lines are projected into straight lines (Art. 155), the projections of the points L, M, and N will lie in a straight line.

187. *Cor.* Hence we can construct a conic which shall pass through five given points, no three of the points being in the same straight line. Let A, B, C, D, F be the given points. Construct a figure as in Art. 121. On BC produced take any point M, and join FM. Let DC and FA meet in N, and join NM. Let AB meet NM in L, and draw LD to meet FM in E; then will E be a point on the conic.

Proposition IX.

188. *If a hexagon be described about a conic, the three diagonals which join the opposite vertices intersect in the same point.*

This Proposition has been proved for a circle (Art. 122), and the demonstration may be extended to a conic, as in Prop. VIII.

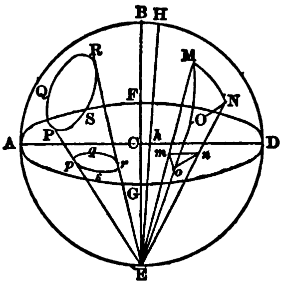
SPHERICAL PROJECTIONS.

189. *Spherical projections* are representations of portions of a spherical surface upon a plane. In order to represent a spherical surface upon a plane, various modes of projection are employed, such as the *orthographic*, *stereographic*, *globular*, etc.

190. In the *orthographic projection*, all points of a hemisphere are referred to the plane of projection by perpendiculars let fall upon the plane; so that the hemisphere is represented as it would appear to an

eye placed at an infinite distance. In this projection only the central portions are represented of their true forms, while those portions which are near the margin of the map are compressed so that their form is very much distorted. Hence the orthographic projection is of little use for representing large portions of the earth's surface.

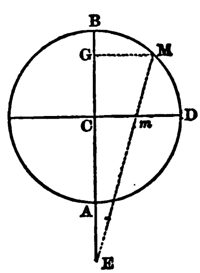
191. The *stereographic projection* is in a measure free from this defect.



Let ABD be a hemisphere, and let it be required to project its convex surface upon its base, AFDG. Draw the diameter, BCE, perpendicular to the plane AFDG, and conceive the eye to be placed at E. The point *m*, where the line EM meets the plane AFDG, is the projection of the point M. In the same manner the point N is projected into *n*, and the point O into *o*, and the figure MNO is projected into the figure *mno*.

Any minute triangle, MNO, on the surface of the sphere is represented by a similar triangle, *mno*, in the projection; and a small circle, PQRS, on the sphere is represented by a circle, *pqrs*, in the projection. Great circles passing through the vertex, B, are, however, projected into straight lines traversing the centre, C. Thus the semi-circumference ABD is projected into the straight line ACD. By this method an entire hemisphere may be projected in a single map without very great distortion of figure. There is, however, a gradual diminution in the projected dimensions of figures from the circumference, AFDG, to the centre, C. On the margin of the chart, figures are represented without any reduction in their dimensions, while at the centre the dimensions of figures upon the surface are reduced one half.

192. To avoid the inequality of the stereographic projection, the point of sight is sometimes taken without the sphere, and at a distance from it equal to the sine of 45°, $=R\sqrt{\frac{1}{2}}$. This mode of projection is called *globular*.



Bisect the quadrant BD in M; draw MG perpendicular to BC; produce BA to E, making AE equal to MG, and let E be the point of sight. Then *m* will be the projection of M, and C*m* will be equal to *m*D.

For since $EG:GM::EC:Cm$,

$$Cm=\frac{CE\times GM}{EG}. \qquad (1)$$

Also, since $GM=GC=AE=R\sqrt{\frac{1}{2}}$,

we have $CE=R+R\sqrt{\frac{1}{2}}$,

and $EG=R+2R\sqrt{\frac{1}{2}}$.

Substituting these values in (1), and reducing, we have

$$Cm=\frac{R}{2}=mD;$$

that is, the equal arcs BM, MD are projected into equal straight lines, and it may be shown that the projections of any other two equal arcs of the quadrant are very nearly equal.

In the construction of maps, various other modes of projection are frequently employed. Mercator's projection is extensively used for purposes of navigation. See Trigonometry, p. 153.

THE END.