EVERYMIND'S EUCLID

EUCLID'S ELEMENTS

BOOKS III AND IV

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Dedication

For my daughter, Margaret, and her husband, Jeffrey,
for providing place and opportunity to write this work,
and for my strange fate
which has provided me with more time
for intellectual pursuits
than I could possibly have deserved.

Table of Contents

Instructions

For Learners

I have to assume that you have studied Everymind's Euclid volume one and understand Euclid's Books I and II. All of Euclid builds up from that basis. And I assume you know that you must come to a real understanding of every proposition, attempt to work every problem, and seriously study every solution, whether you solve a problem or not. Again, in this volume, we have appendices of hints and solutions for every problem. In the problem diagram appendix, there are fewer problem diagrams. But I have included one for every problem for which the description of the data seemed in any way ambiguous. And my ability to make the descriptions unambiguous has improved with practice, if not otherwise.

Books I and II were all about lines, triangles, and rectangles -- the possibilities of our straightedge. Book III is *Enter the Compass* and the circle is explored not only as an entity in itself but in its relations to the lines, triangles, and rectangles of the first books. As a tool, the circle further increases the power of our mind to analyze and synthesize geometrical arguments. And after studying two books of Euclid, you know that this will further increase the general powers of your mind. Book IV is more or less Euclid's *Fun with Circles*. The regular division of a circle allows the creation of different n-gons and Euclid milks this for all it's worth, which I find less exciting than he did.

But circles, as they expand our ability to reason, make possible what was called "Modern Geometry," a bridge between Euclidean geometry and the modern study of projective geometry. In the problems, we will have in-circles and en-circles and ex-circles and orthogonal circles with their radical axes, which always remind me of those dancing guys with the hatchets in *Kung Fu Shuffle.*

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For Teachers

You will need to be on your game for this one. Any student who can make it through the first two books of Euclid will be able to recognize someone who hasn't made it through those books. Any student who has developed his or her mind by pushing against the void of one hundred and sixty-two problems will recognize the weaker mind of someone who hasn't.

As I see it, you have three choices: tyranny, hypocrisy, or camaraderie. Only the third choice is worth making. Only the third choice is worthy of respect. You need to get down in the trenches and make at least the same effort as your students. It is your job to lead them into education. The state of this uneducated world is proof that students cannot be commanded into education by hypocrites and tyrants.

If you haven't been bruising your head bloodily against the dark wall of Euclid's problems so far, it's time to start. No one who hasn't been in combat can teach combat fieldcraft. No one who hasn't actually done mathematics can teach mathematics. G. H. Hardy, the mathematician, pointed out that most people cannot do anything well. But everyone can try to do something well and as a mathematics teacher you are doing mathematics. Try to do it well. Seriously attack the problems and study the solutions when you fail as well as when you succeed. Or how can you teach what you haven't actually done?

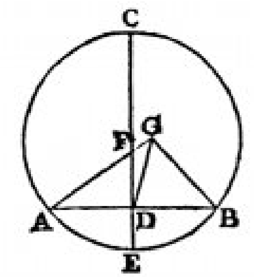
It isn't given to us all to have talent. But I haven't let that stop me. So maybe I won't have my name up there with Gauss and Riemann. But it might be there, textbook-wise, with De Morgan. Where is your name going to be? It will be in your students' minds. What are they going to hang it on? You can at least be known for making as great an effort as your students and for making it with them.

Euclid - Book III

Definitions

d.3.1 **Equal circles** (⊙) have equal radii, therefore equal diameters.
d.3.2 A line **touches** a ⊙ if it meets the ⊙ and, produced, does not cut it. This is a **tangent (tan)** with its **point of contact**.
d.3.3 ⊙s **touch** when they meet but do not cut each other. If ⊙A is in ⊙B they touch **internally**, else **externally**.
d.3.4 A line cutting a ⊙ at two points is a **secant**.
d.3.5 A **chord** is a line joining two points ∈⊙. A secant produces a chord.
d.3.6 Chords are **equidistant** (eqD) from ⊙ center when their perpendiculars (⊥) from their midpoints to ⊙ center are equal. Of two chords, the one with the greatest ⊥ is **farther** from center.
d.3.7 A **segment** of a ⊙ is a chord and what it cuts off, away from ⊙ center. Segments of circles are **similar** if their angles are equal.
d.3.8 A **segment's angle** is contained by any point ∈⊙ joined to the endpoints of its chord. This gives a segment two angles.
d.3.9 Any part of a ⊙'s circumference is an **arc**.
d.3.10 A **sector** of a ⊙ is bounded by two radii and the arc between them.
d.3.11 ⊙s with same center are **concentric**.

Propositions



Proposition 1. Problem

Given ∀⊙, find its center

Method

∀chord AB, ×/2 AB @ D (1.10)
CDE⊥AB x ⊙ @ C,E (1.11)
×/2 CE @ F (1.10) F required

Proof

Else ∀G≠F center ⊙ Join G[A,B,D]
DA=DB (con) DG∈ΔADG,BDG AD,DG = BD,DG GA=GB (d.1.15)
∴ ∠ADG = ∠BDG (1.8) = ∟ (d.1.10)
∠BDF = ∟ (con) ∴ ∠BDG = ∠BDF (a.11) ∴ less = greater ↴
∴ F center ⊙F

Corollary 1

∀⊙, ∀chord AB, if chord CD⊥AB and CD ×/2 AB then center∈CD

Problems

Problems that are more complex will be marked "*****" and problems with diagrams will be marked "**D**". A **cyclic 4-gon** is a 4-gon inside ⊙ with all vertices⊙ (on circumference of ⊙). Don't make these problems hard. All you have is d.1.15, 3.1, and its corollary 3.1.C1.

1. Problem

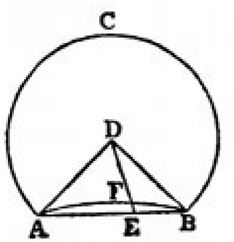
Given: ∀point A, ∀⊙B
Required: ⊙A × ⊙B @ endpoints of ∀diam(eter)CD of ⊙B

2. Theorem

∀cyclic 4-gon ∈⊙, lines ⊥ mdpt sides of 4-gon concur at a fixed point.

Recall from volume one, "∈⊙" means on the circumference and "in ⊙" means in the ⊙'s whitespace. And "Else" always means we are using "proof by contradiction." "Concur" is "intersect."

Proposition 2. Theorem



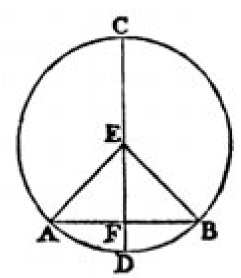
∀2 points A,B ∈⊙, then line AB in ⊙

Proof

Else line AEB outside ⊙
Find center D of ⊙ (3.1) Join D[A,B]
∀F∈arc AB, join DF.   DF(pr) × AB @ E
DA=DB (d.1.15), ∠DAB = ∠DBA, (1.5),
ΔDAE: ext∠DEB > int∠DAE (1.16)
∠DAE = ∠DAB = ∠DBA (proven)
∴ ∠DEB > ∠DBE ∴ DB > DE
DB = DF (d.1.15) ∴ DF > DE lesser > greater ⬎
∴AB !outside circle Sym. AB !∈⊙ ⬎ ∴AB in ⊙

In proofs, the symbol "**iff**" means "if and only if". If we have two propositions "If A then B" and "If B then A", we can combine them as "A iff B" to show that each implies the other. "Common chord" can mean a chord on the intersection points of two circles that cut each other or any other chord running through both of two touching or intersecting circles.

Proposition 3. Theorem



∀⊙E, ∀diamCD
∀chord AB: E∉AB
then CD ×/2 AB iff CD⊥AB

Proof

1) CD ×/2 AB

Join E[A,B] AF=FB (hyp) FE∈ΔAFE,BFE
AF,FE = BF,FE, EA=EB (d.1.15)
∴ ∠AFE = ∠BFE (1.8) = ∟ (d.1.10)
∴CD⊥AB

2) CD⊥AB

Same construction: EA = EB (d.1.15) ∠EAF = ∠EBF (1.5)
∠AFE = ∠BFE = ∟ ∴ ΔEAF = ΔEBF (1.26) ∴AF = FB
∴CD ×/2 AB

Problems

3. Theorem

If ⊙P × ⊙Q @ B,E, ∀2 ∥ lines on B,E are equal within the circles.

4. Theorem D,*

⊙A × ⊙B @ C,C', common chords DCE, FCG: D,G∈⊙A E,F∈⊙B,
If ∠FG×AB = ∠DE×AB then DE=FG

5. Problem

Given: ⊙A×⊙B @ C,D
Required: longest common chord on C

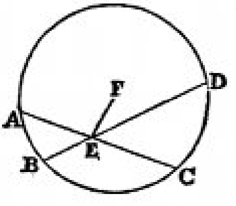
6. Theorem

⊙C,CX, diamXY, ∀chord PQ ∥ XY, ∀A∈XY, join A[P,Q]
then $AP^2 + AQ^2 = XA^2 + AY^2$

7. Theorem *

∀4-gonABCD, A',B',C',D' mdpt AB,BC,CD,DA, if ⊙A',A'A, ⊙B',B'B, ⊙C',C'C, ⊙D',D'D then common chord ∀2⊙ ∥ common chord of other 2⊙

Proposition 4. Theorem



∀⊙F, ∀chords AC,BD, if F∉AC,BD
then AC,BD !×/2 e.o.
(do not bisect each other)

Proof

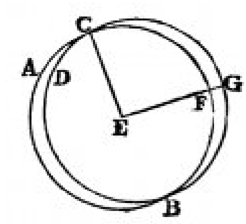
Else, clearly, if F∈AC, F∉BD then BD ! ×/2 AC
Else F∉AC and F∉BD, AC × BD @ E and AE=EC BE=ED
FE ×/2 AC (hyp) ∴ FE⊥AC (3.3) ∴ ∠FEA = ∟
FE ×/2 BD (hyp) ∴ FE⊥BD (3.3) ∴ ∠FEB = ∟
∴ ∠FEA = ∠FEB (a.11) less equal greater ⤵
∴ AE≠EC BE≠ED ∴AC,BD ! ×/2 e.o.

Let's take stock of what we've got. We can find the center of ∀⊙. The perpendicular bisector of ∀chord is on center. Only radii bisect chords at right angles. Only diameters bisect each other. Note that when you prove "iff" you first have to prove A implies B and then prove B implies A.

Proposition 5. Theorem

∀⊙A × ∀⊙B then A ≠ B

Proof



Else E center ⊙ABC,CDG

⊙ABC × ⊙CDG @ B,C

EG × ⊙ABC,CDG @ F,G

∴ ⊙ABC: EC=EF (d.1.15) ⊙CDG: EC=EG

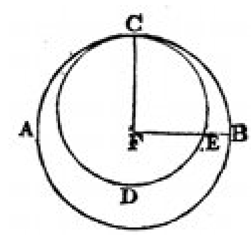
∴ EF = EG (a.1) less equal greater ↴

∴ E ! center of both ⊙ABC,CDG

Proposition 6. Theorem

∀⊙A touches ∀⊙B internally, then A≠B

Proof



Else F center ⊙ABC,CDE

⊙CDE touches ⊙ABC internally @ C

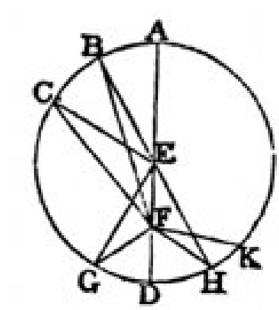
Join FC ∀FEB x ⊙ABC,CDE @ B,E

∴ FC = FB (d.1.15) FC = FE (d.1.15)

∴ FE = FB less equal greater ↴

∴ F ! center ⊙ABC,CDE

Proposition 7. Theorem



∀⊙E, ∀diamAD, ∀F∈AD ≠ E

∀line from F to circumference

1) FEA greatest, FD least,

If arc AB < arc AC then FB > FC

2) ∀FG, ∃!FH = FG (∃! "exists unique")

Proof

1) Given diagram, BE + EF > BF (1.20)

BE = AE (d.1.15) ∴ AE + EF > BF

BE = CE EF∈ΔBEF,CEF BE,EF = CE,EF ∠BEF > ∠CEF

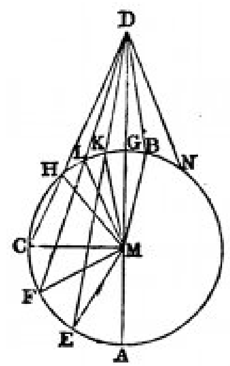
∴ FB > FC (1.24) Sym. FC > FG

GF + FE > EG (1.20) EG = ED (d.1.15) ∴ GF + FE > ED

∴ GF + EF - EF > ED - EF ∴ GF > FD

∴ FEA greatest, FD least, FB > FC > FG

2) ∠FEH = ∠FEG (1.23) Join FH
EG = EH (d.1.15) EF∈ΔGEF,HEF EG,EF = EH,EF ∠GEF = ∠HEF
∴FG = FH (1.4) ∴ ∀FG ∃! FH = FG Else ∀K, FK = FG
FH = FG ∴FH = FK (a.1) ↴ (arc AK < arc AH)



Proposition 8. Theorem
∀⊙M, ∀D outside ⊙M
1) ∀line from D to concave circumference:
DMA greatest, arcAE < arcAF then DE > DF
2) ∀line from D to convex circumference:
DG∈DMA least, arcGK < arcGL then DK < DL
3)∀DK, ∃!DN = DK

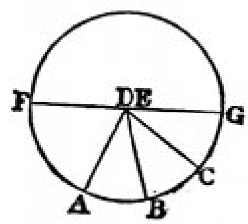
Proof
1) Given diagram, EM + MD > ED (1.20)
EM = AM (d.1.15) ∴ AM + MD = AD > ED
EM = FM MD∈ΔEMD,FMD EM,MD = FM,MD
∠EMD > ∠FMD ∴ ED > FD (1.24) Sym. FD > CD
∴ DMA greatest, DE > DF > DC
2) MK + KD > MD (1.20) MK = MG (d.1.15) ∴ GD < KD (a.3)
ΔMKD inside ΔMLD ∴ MK + KD < ML + LD (1.21) MK = ML (d.1.15)
∴KD < LD (a.3) Sym. LD < HD
∴DG least, DK < DL < DH
3) ∠DMB = ∠DMK (1.23) Join DB
MK = MB MD∈ΔKMD,BMD KM,MD = BM,MD ∠DMK = ∠DMB
∴DK = DB (1.4) ∴ ∀DK ∃!DB = DK Else ∀N, DN = DK
DB = DK ∴ DB = DN (a.1) ↴ (arcGB < arcGN)

Problems

8. Problem
Given: ∀⊙C, point A,B outside ⊙C
Required: P∈⊙C: AP,BP minimized

Proposition 9. Theorem

∀⊙, ∀P in ⊙, if more than two equal lines from P to ⊙, then P center.



Proof

From ∀D in ⊙ABC, DA = DB = DC

∴ D center ⊙ABC

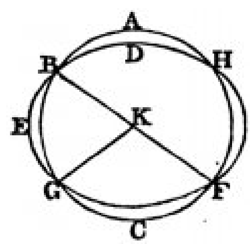
Else E center ⊙ABC Join DE

DE(pr) × ⊙ABC @ F,G ∴ FG diam ⊙ABC

∴ DC > DB > DA (3.7)⮧ ∴ D center ⊙ABC

Proposition 10. Theorem

If ∀⊙A cuts ∀⊙B there are two and only two points of intersection.



Proof

Else ⊙ABC × ⊙DEF @ B,F,G

K center ⊙ABC (3.1) Join K[B,F,G]

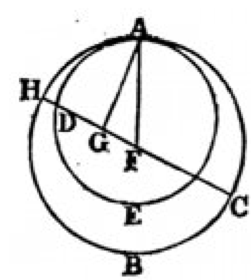
∴ KB = KF = KG (d.1.15)

∴ K center ⊙DEF (3.9) and ⊙ABC ⮧ (3.5)

Proposition 11. Theorem

∀⊙G touches ∀⊙F internally @ A

Then A∈FG(pr)



Proof

Else FG(pr) = FGDH Join A[F,G]

AG + GF > AF (1.20) AF = HF (d.1.15)

∴ AG + GF > HF ∴ AG > HG (a.3)

AG = DG (d.1.15)⮧ ∴ A∈EF(pr)

Problems

9. Theorem

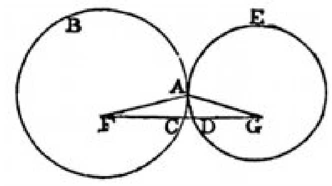
⊙B touches ⊙A internally @ C ∀2∥ diams: DE,FG∈⊙A,B then an endpoint of each diam and C are colinear

10. Theorem

∀⊙O, diamAB, ∀P∈⊙O PN⊥AB

Then bisector ∠OPN × ⊙O at one of two fixed points

Proposition 12. Theorem



∀⊙F touches ∀⊙G externally @ A
then A∈FG(pr)

Proof

Else FG(pr) = FCDG Join A[F,G]
FA = FC (d.1.15) GA = GD (d.1.15)
∴ FA + AG = FC + DG ∴ FG > FA + AG
FG < FA + AG (1.20)⤵ ∴ A∈FG(pr)

Problems

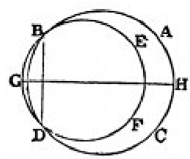
11. Theorem D,*

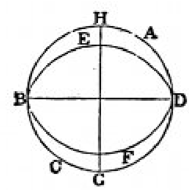
⊙O,OB, ⊙P,PA, ⊙Q,QC touch externally @ A,B,C
AB,AC × ⊙O @ D,E then 1) DE diam ⊙O 2) DE ∥ PQ

Proposition 13. Theorem

⊙A touches ⊙B 1) int. or 2) ext. at exactly one point.

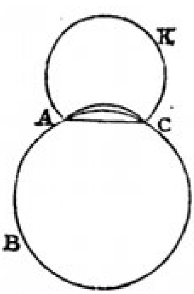
Proof





1) Else ⊙ABC touches ⊙DEF @ B,D
Join BD GH ×/2 BD GH⊥BD
∴ BD ∈⊙ABC,DEF (3.2)
∴ centers on GH (3.1.C1)
∴ B,D∈GH (3.11)⤵
∴ ⊙s touch int. at exactly one point

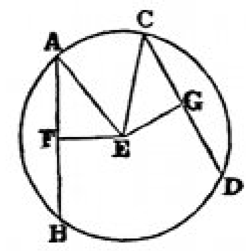
2) Else ⊙ABC touches ⊙AKC @ A,C Join AC
A,C ∈⊙AKC ∴ AC in ⊙AKC (3.2)
⊙AKC outside ⊙ABC (hyp)
∴ AC outside ⊙ABC
A,C ∈⊙ABC (hyp) ∴ AC in ⊙ABC (3.2)⤵
∴ ⊙s touch ext. at exactly one point



In 3.15, we will have "∀⊙, diam greatest chord, of others those nearer center greater than remote and conversely" which is shorthand for "∀⊙, diam greatest chord, of others those nearer center greater than remote and greater are nearer center than are lesser."

Proposition 14. Theorem

⊙E,EA, chords AB,CD with mdpt F,G,
AB = CD iff EF = EG



Proof

1) AB = CD

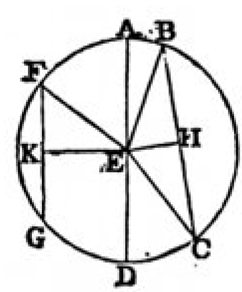
EF ≡ ⊥ ×/2 AB ∴ AF = FB ∴ AB = 2AF
Sym. CD = 2CG
AB = CD (hyp) ∴ AF = CG (a.7)
AE = CE (d.1.15) ∴ $AE^2 = CE^2$
∠AFE = ∟ ∴ $AF^2 + FE^2 = AE^2$ (1.47) Sym. $CG^2 + GE^2 = CE^2$
∴ $AF^2 + FE^2 = CG^2 + GE^2$ (a.1)
AF = CG ∴ $AF^2 = CG^2$ ∴ $FE^2 = GE^2$ (a.7) ∴ FE = GE
∴ AB,CD eqD E (d.3.6)

2) AB,CD eqD E

$AF^2 + FE^2 = CG^2 + GE^2$ (part 1)
EF = EG (hyp) ∴ $EF^2 = EG^2$ ∴ $AF^2 = GC^2$ (a.3) ∴ AF = CG
AB,CD = 2AF,2CG (part 1) ∴ AB = CD

Proposition 15. Theorem

∀ ⊙, diam greatest chord, of others those nearer center greater than remote and conversely.



Proof

1) EH < EK

Given diagram, AE = BE = CE = DE (d.1.15)
∴ AD = BE + CE (a.2)
BE + CE > BC (1.20) ∴ AD > BC
EH < EK (hyp) $EH^2 + HB^2 = EK^2 + KF^2$ (method 3.14)
EH < EK ∴ $EH^2 < EK^2$ ∴ $HB^2 > FK^2$ ∴ HB > FK ∴ BC > FG

2) BC > FG

BC > FG ∴ BH > FK
$BH^2 + HE^2 = FK^2 + KE^2$ (1.47) ∴ $BH^2 > FK^2$ ∴ BH > FK
∴ $HE^2 < KE^2$ ∴ HE < KE

Problems

12. Theorem

⊙A,diamGC touches int. ⊙C,CG @ G AH⊥CG × ⊙C @ E,F,⊙A @ H
BHD⊥EF × ⊙C @ B,D then BH = EH, DH = FH

13. Problem

Given: ∀A in ∀⊙C
Required: minimum chord on A

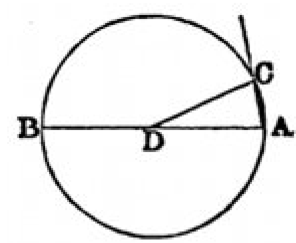
14. Problem *

Given: ⊙C, A∈AB outside ⊙C
Required: ⊙O touching ext. ⊙C, ⊙O × AB @ A

15. Problem

Given: ⊙C, AB outside ⊙C
Required: tangent ⊙C ∥ AB

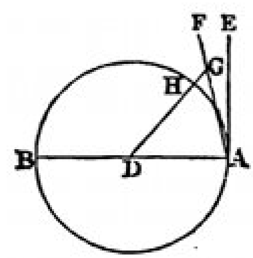
Proposition 16. Theorem



∀⊙D,DA : 1) AC⊥AB ≡ tan ⊙D @ A
and 2) AC unique tangent @ A

Proof

1) Else AC chord ⊙D Join DC
DA = DC (d.1.15) ∴ ∠DAC = ∠DCA (1.5)
∠DAC = ∟ (hyp) ∴ ∠DCA = ∟ ↴ (1.17)
∴ AC not chord ∴ AC tan @ A



2) (AC now AE)
AF •|•(⊙D,AE) DG⊥AF (1.12) × ⊙D @ H
∠DGA = ∟ (con) ∴ ∠DAG < ∟ (1.17)
∴ DA > DG (1.19)
DA = DH ∴ DH > DG less equal greater↴
∴ AC unique tangent @ A

Corollary 1

∀⊙A,AB, ∀BD⊥AB, then BD unique tangent to ⊙A @ B

Problems

16. Problem

Given: ⊙C, AB outside ⊙C

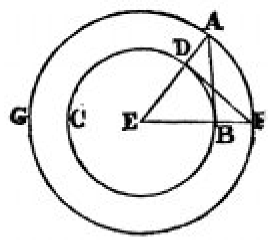
Required: tangent to ⊙C ⊥ AB

17. Problem

Given: ⊙C, diamA'A, ∀magnitude M

Required: P∈A'A(pr): tan to ⊙C from P = M

Proposition 17. Problem



Given: ⊙E,ED, A outside ⊙E

Required: tangents on A,D

Method/Proof

1) tangent on A

Join AE × ⊙E @ D Add ⊙E,EA

DF⊥AE × ⊙E,EA @ F EF × ⊙E,ED @ B

AB required

EA = EF (d.1.15) EB = ED (d.1.15)

∴AE,EB = FE,ED ∠E∈ΔAEB,FED ∴ΔAEB≡ΔFED (1.4)

∴∠ABE = ∠FDE = ∟ (con ∠FDE)

AB⊥EB ∴ AB tangent ⊙E,EB @ B

2) tangent on D

As above DF⊥ED DF required (3.16.C1)

The real goal of studying Euclid is intellectual honesty. Euclid allows you to exercise and express all of your intelligence, originality, and individuality. But you must work according to principles of geometry shared by everyone. You must learn to judge your work from this absolute standpoint and know you are exactly "this right" and exactly "this wrong." Only in this way can you close the gap between where your work is now and where it will completely express the principles of clear, correct thought. And only then will everyone who understands these principles recognize your intelligence, originality, and individuality. And this achievement of intellectual principle in geometry will permeate all of your thinking and expression, if you will let it. If you claim recognition for your abilities prematurely, those with understanding immediately recognize your dishonesty. At the risk of appearing all holy-moly, here is Paul from Galatians 6:3: "For if a man think himself to be something, when he is nothing, he deceiveth himself." But he deceives no one who understands principle.

Problems

18. Theorem

From any point outside a circle only two tangents can be drawn and they are equal.

19. Theorem

∀4-gon ABCD described on (all sides touching) ⊙PQRS:
AB,BC,CD,DA tan @ P,Q,R,S then AB + CD = BC + DA

20. Theorem

No ∥gm except a rhombus can be described on a ⊙.

21 Theorem *

∀⊙: AD,AE tan to ⊙BCF @ B,C, if DE = BD + CE
Then DE tan ⊙ (@ F)

22. Theorem D

∀4-gon described on ⊙, ∠s from center to opp sides = 2∟

23. Theorem

PQ tan ⊙C @ R, radii CA⊥CB, CA,CB × PQ @ P,Q
Then of tangents to ⊙C PM,QN: PM∥QN

24. Theorem

∀⊙O, ∀A outside ⊙O, tans AB,AC then ∠BAC = 2∠OBC

25. Theorem D

∀⊙C, diamAB, tans @ AE,BF: ∀D∈⊙ EDF tan to ⊙
Then 4-gon ABFE = ½AB•EF

26. Theorem D,*

∀4-gon ABCD described on two circles, one above the other, circles touching @ O. On top ⊙EFL × AB,BC,DA @ E,F,L
Below, ⊙GHK x BC,CD,DA @ G,H,K. MN tan both⊙ @ O.
MON × AD,BC @ M,N then AD + BC = AB + DC + 2MN

27. Problem *

Given: ⊙A,L outside ⊙B,M, L > M

Required: common tangent to ⊙A,⊙B

28. Problem

Given: ⊙ABC,⊙DEF, magnitude M

Required: tangent on ⊙ABC w/secant in ⊙DEF = M

29. Problem

Given: ⊙A,⊙B, magnitudes PQ,RS

Required: line w/secants = PQ,RS ∈ ⊙A,⊙B

A few remarks: Earlier I analyzed what we had gained in tools and results from the first several propositions. If you haven't been doing this for yourself since then, **start doing it**. Take stock now of what you have. Spell it out in a notebook. Add to, or change, your list as you progress. I'm done holding your hand.

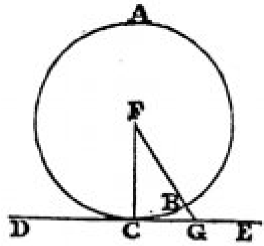
A reasonable question would be: "How hard should I work on the problems?" If you decided you would work on every problem until you solved it, you would never finish the book. Some problems are old, clever Cambridge Tripo problems. Some introduce new ideas. You can't reinvent every wheel. (Unless you can. Then kudos to you.)

A reasonable effort would be to draw the diagram, add all the labels and details, write down everything you can conclude from the diagram, and then try to see how to get from where you are to the required result. Mentally engage with the problem and **strive** to solve it. If after twenty or thirty minutes you can't get there, just sit quietly for a minute and let your eye run over what you've got. If you've still got nothing, bag it. Go find the solution, write it out, study it, and add anything you gain from it to your running list of tools.

If you do this, you will gradually build up a core of geometric understanding. At some point, that understanding will kick in, bringing you to a higher level. You get there by **striving**.

Proposition 18. Theorem

∀⊙F,FC, ∀tangent DCE, then FC⊥DE



Proof

Else ∀G, FG⊥DE × ⊙F @ B

∴ ∠FGC = ∟ (hyp) ∴ ∠FCG < ∟ (1.17)

∴ FC > FG (1.19)

FC = FB (d.1.15)

∴ FB > FG⥿ (less > greater) ∴ FG!⊥DE

Sym. no other line but FC⊥DE ∴ FC⊥DCG

Additional notation:

Distance from A to B will be denoted "D(A,B)"

Tangent PQ w/point of tangency E denoted "tangent PEQ"

Problems

30. Theorem

∀⊙C,diamMN, tans @ M,N × ∀tan PEQ @ A,B then ∠ACB = ∟

31. Theorem

Concentric ⊙O,OB in ⊙O,OA

then ∀chords∈⊙O,OA tan to ⊙O,OB are equal

32. Problem

Given: ⊙O, ∀P outside ⊙O, AB < diam ⊙O

Required: Line on P with secant in ⊙O = AB

33. Theorem D

CD tan to ⊙A,AC,⊙B,BD DC × AB @ O

OA(pr) × ⊙A,⊙B @ E',E;F,'F then CE∥DF (Sym. CE'∥DF')

34. Theorem *

∀semi⊙C,CA,base AB ∀⊙O inscribed in semi⊙:

⊙O tan to AB @ F, touches arcAB @ D

Then O eqD C and the tan to semi⊙∥AB

35. Problem *

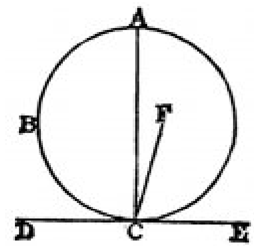
Given: AB, ⊙C, magnitude M

Required: ⊙radiusM, tangent to AB, touching ext. ⊙C

36, Theorem D,*

∀⊙C,CA touches ext. ⊙O @ B A'A tan to ⊙C @ A
Then ∃D∈⊙O: DBA one line

Proposition 19. Theorem



∀⊙ABC, ∀tan DCE then
⊙center∈AC⊥DE

Proof

Else center F∉AB Join CF
∴FC⊥DE (3.18) ∴∠FCE = ∟
∠ACE = ∟ (con)
∴ ∠FCE = ∠ACE (a.11)⌐ (less = greater) ∴ ∀F∉AC not center
∴center ⊙∈AC

Problems

37. Theorem *

∀4-gonABCD, BC∥AD, ⊙K × AB,BC,CD,DA @ E,F,G,H
LKM∥BC × AB,CD @ L,M then LM = ¼(AB+BC+CD+DA)

38. Theorem *

If a series of ⊙ all touch fixed line A'A @ A, if ∀parallel line to A'A cuts all circles then tangents at all points so touched are tangent to a fixed ⊙

39. Theorem *

∀⊙C,CA, ∀B∈radius CC'⊥CA, AB(pr) × ⊙C @ A,D
tan at D × CB(pr) @ E then ΔBDE≡isosΔ

40. Theorem *

∀⊙O,diamAB, OA(pr) to P:OA = AP,
PEC × tan on A @ E,⊙O @ C BC × tan on A @ D
then ΔCED≡eqSΔ

41 Theorem D,*

∀⊙O,OC, ∀A,B outside ⊙O then if
∀lines AC,BC × ⊙O @ C∈convex side of ⊙O
Then minimum AC+BC when tangent @ C makes eq∠ w/ AC,BC

Proposition 20. Theorem

∀⊙ABC, center E, then ∠BEC = 2∠BAC

Proof

Join AE. AE(pr) to F.

Case 1) E∈∠BAC

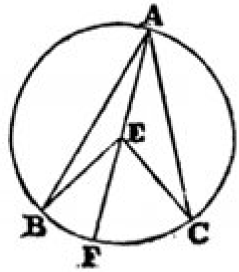
EA = EB ∴ ∠EAB = ∠EBA (1.5)

∴ ∠EAB + ∠EBA = 2∠EAB

∠BEF = ∠EAB + ∠EBA (1.32)

∴ ∠BEF = 2∠EAB Sym. ∠FEC = 2∠EAC

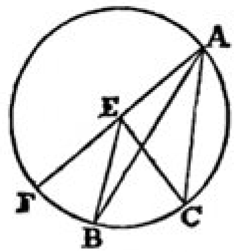
∴ ∠BEC = 2∠BAC



Case 2) E∉∠BAC

Sym. to Case 1, ∠FEC = 2∠FAC,

∠FEB = 2∠FAB ∴ ∠BEC = 2∠BAC



Proposition 21. Theorem

∀⊙F, ∀chord BD, ∀A,E∈⊙F same side BD then ∠BAD = ∠BED

Proof

Case 1) arcBAED > semi⊙

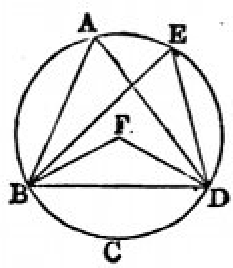
Join F[B,D]

∠BFD on center, ∠BAD ∈⊙,

both on arcBCD ∴ ∠BFD = 2∠BAD (3.20)

Sym. ∠BFD = 2∠BED

∴ ∠BAD = ∠BED (a.7)



Case 2) arcBAED < semi⊙

AF(pr) × ⊙F @ C Join CE

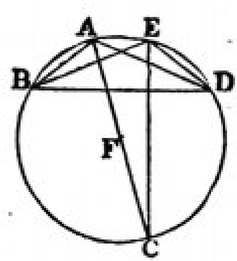
arcBAEC > semi⊙

∴ ∠BAC = ∠BEC (Case 1)

arcCAED > semi⊙

∴ ∠CAD = ∠CED (Case 1)

∴ ∠BAD = ∠BED (a.2)



Problems

42. Theorem

∀⊙, chord AB, ∀segmentARB, ∀2d segment on RB same side ARB
∀P∈RB, APD,BPC × 2d segment @ D,C   AC × BD @ Q
then ∠CQD constant.

43. Theorem D

∀cyclic 4-gon ABCD, AB × CD @ O
then angles of ΔAOC = angles of ΔBOD

44. Theorem

⊙AQP × ⊙PBR @ P,P' APB fixed common chord
then ∀common chord QPR, AQ × BR @ constant ∠

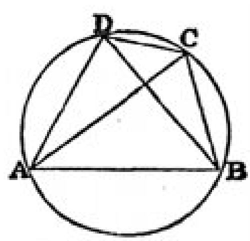
45. Theorem

∀⊙O, ∀A outside O, AB,AC tan to ⊙O, ∀D∈⊙O,∉arcBC
then ∠ABD + ∠ACD = constant

46. Theorem *

∀⊙, chord AB, ∀P,Q∈⊙ same side of AB
AR,BR ×/2 ∠PAQ,∠PBQ, AR × BR @ R
then ∠ARB constant

Proposition 22. Theorem



∀cyclic 4-gonABCD, ∑opp∠ = 2∟

Proof

Join AC,BD ∑(∠ΔCAB) = 2∟ (1.32)
∠CAB = ∠CDB on chord CB (3.21)
∠ACB = ∠ADB on chord AB (3.21)
∴ ∠CAB + ∠ACB = ∠ADC (by their shared chords)
∴ ∠CAB + ∠ACB + ∠ABC = ∠ABC + ∠ADC (a.2)
∠CAB + ∠ACB + ∠ABC = ∑∠ΔCAB = 2∟ (1.32)
∴ ∠ABC + ∠ADC = 2∟ Sym. ∠BAD + ∠BCD = 2∟

Note: In a traditional Euclid, this proposition takes about 25 lines of prose. Here we have six lines of algebra. But if you will truly realize the elements of the logic, truly grasp what is going on, you'll have about three lines of argument in your head and that's all you need.

Let's add a little notation. If a figure is inscribed in a circle, we will denote it as "figure w/en⊙" ("en" for "encircled"). In this case, the figure's vertices are on the circle. If a figure has a circle inscribed within it, it is a "figure w/in⊙". In this case, the figure's sides are tangent to the circle.

Problems

47. Theorem

If ∥gm w/en⊙ then ∥gm ≡ rect∟

48. Theorem

∀ΔABC w/en⊙DEF D,E,F ∈arcsBC,CA,AB
then ∠AFB + ∠BDC + ∠CEA = 4∟

49. Theorem D

∀cyclic 4-gon w/en⊙, ∑(ext∠on sides) = 6∟

50. Theorem

ΔAOB, ∀C,D∈BO,AO:∠ODC = ∠OBA
Then 4-gonABCD ≡ cyclic 4-gon

51. Problem

Divide a ⊙ into 2 parts: ∠ on one side of dividing chord = 2∠ on other side.

52. Problem

Divide a ⊙ into 2 parts: ∠ on one side of dividing chord = 5∠ on other side.

53. Theorem D

∀6-gonABCDEF w/en⊙: If ∀2 adj sides ∥ to their opp sides
then remaining 2 sides ∥

54. Theorem D

∀4-gonABCD AB(pr) to E ∠EBC = ∠ADC
Then ∠ADB = ∠ACB

55. Theorem D

∀⊙,∀C∈⊙, ∀DCE,ACB × ⊙ @ E,B
CP ×/2 ∠DCB × ⊙ @ P ∈ arcCB then EP = BP

56. Theorem

∀ΔABC, ∀∠M AP,BQ × BC,AC @ P,Q @ ∠M

Then ∀Δ, baseAB, apex∠C PQ ≡ constant

57. Theorem D,*

∀cyclic 4-gonABCD, AB × CD @ P, AD × BC @ Q: P,Q outside ⊙

QH ×/2 ∠AQC, PH ×/2 ∠APC, PH × QB @ E then ∠PHQ = ∟

58. Theorem D,*

∀cyclic 4-gonABCD, CD(pr) to ∀E

Bisectors of ∠ABC,∠ADE × @ F

Then AD × BF @ H ∈⊙

59. Theorem *

∀cyclic 4-gonABCD lineEFGH x AD,BC @ F,G, ⊙ @ E,H

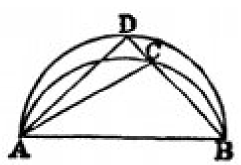
If ∠AFG = ∠BGF then EH makes equal ∠ w/AB,DC

60. Theorem *

∀4-gonEFGH w/en⊙: ∠E + ∠G = 2∟

in⊙O,OA × EF,FG,GH,HE @ A,B,C,D then BD⊥AC

Proposition 23. Theorem



∀AB, two similar segments of circles, on same side of AB, must coincide.

Proof

Else segADB ~ segACB do not coincide

⊙ACB × ⊙ADB @ A,B ∴!∃ 3d point of intersection (3.10)

∴one segment (segACB) must be inside other segment (segADB)

Add line DCB. Join A[C,D]

segADB ~ segACB ∴∠ADB = ∠ACB (d.3.17)

∴ext∠ACD = ∠CAB + ∠CBA ↴ (1.16) ∴ segments coincide

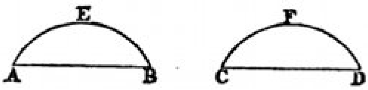
Note: This proposition serves the same purpose as 1.7. Similar segments and equal triangles on same side of same base coincide. This excludes the possibility of anything else being the case.

Proposition 24. Theorem

∀AB,CD: AB = CD,

∀segAEB,segCFD: AEB ~ CFD

Then segAEB = segCFD



Proof

AB = CD ∴ if AEB superimposed on CFD the AB,CD coincide

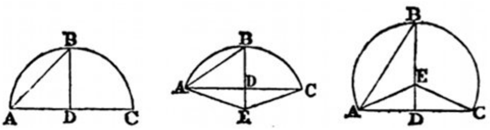
∴ AEB,CFD coincide (3.23) ∴ segAEB = segCFD

Note: Here again, Euclid is excluding other possibilities, clearing the ground for further propositions. Interestingly, both here and in 1.4, when he uses superposition to move one Δ scratched in the dirt over another Δ, he only moves rigid figures. A square can collapse. But triangles and segments cannot.

Proposition 25. Problem

Given: ∀⊙segABC

Required: implied ⊙



Method/Proof

D mdpt AB (1.10) DB⊥AC (1.11) Join AB

Case 1) ∠ABD = ∠BAD

∴DB = DA (1.6) DA = DC (con) ∴DB = DC (a.1)

∴DA = DB = DC ∴ A,B,C∈⊙D,DA (3.9) ∴⊙D,DA required

Case 2) ∠ABD ≠ ∠BAD

∠BAE = ∠ABD (1.23) Join BED or BD(pr) to E Join EC

∠BAE = ∠ABE (con) ∴EA = EB (1.6)

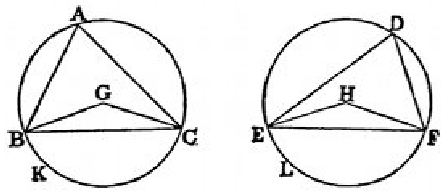
ΔADE,CDE: AD = CD (con), DE = DE, ∠ADE = ∠CDE = ∟ (con)

∴ EA = EC (1.4) EA = EB ∴EA = EB = EC

∴A,B,C∈⊙E,EA ∴⊙E,EA required

Note: In Case 1, arcABC is a semi⊙. When the center of ⊙ is outside segABC, arcABC is less than a semi⊙, if inside, greater than.

Proposition 26. Theorem



⊙G,GA = ⊙H,HD:
∠BGC = ∠EHF,
∠BAC = ∠EDF
then arcBKC = arcELF
(K,L opp side BC,EF of A,D)

Proof

Join BC,EF
⊙G = ⊙H (hyp) ∴ radii equal (d.3.1) ∴ BG,GC = EH,HF
∠G = ∠H (hyp) ∴ BC = EF (1.4)
∠A = ∠D (hyp) ∴ segBAC ~ segEDF (d.3.7) BC = EF
∴ segBAC = segEDF (3.24)
⊙G = ⊙H (hyp) ∴ segBKC = segELF (a.3) ∴ arcBKC = arcELF

Recall that "∀" ≡ "each, any, every, all" as these are logically equivalent. In 61, the first is "any", the second "all".

Problems

61. Theorem

∀ΔsCAB on same side AB
Then ∀AD(pr) ×/2 ∠C concur on common point

62. Theorem

⊙O × ⊙P @ A,B If common chords CAD,CBE × AB: C,D opp sides A, C,E opp sides B then DE constant

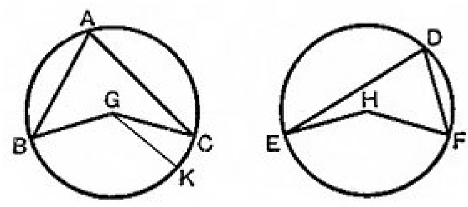
63, Problem *

Given: ⊙C,diamAB, ∀D∈⊙C
Required: chord DE on one side AB: arcAE = 3×arcBD

64. Theorem D,*

∀⊙O,diamAB, diamCD⊥AB, ∀E∈arcAC,
chordEFG × CD @ F: EF = DA
Then arcBG = 3×arcAE

Proposition 27. Theorem



⊙G,GA = ⊙H,HD:
arcBC = arcEF
then ∀A,D∈⊙G,H:
∠BGC = ∠EHF
∠BAC = ∠EDF

Proof

If ∠BGC = ∠EHF, then ∠BAC = ∠EDF (3.20, a.7)
Else ∠BGC > ∠EHF ∠BGK = ∠EHF (1.23)
∠BGK = ∠EHF ∴ arcBK = arcEF (3.26)
arcBC = arcEF (hyp) ∴ arcBK = arcBC ↴ (less = greater)
∴ ∠BGC = ∠EHF ∴ ∠A = ½∠BGC = ½∠EHF = ∠D (3.20)

Problems

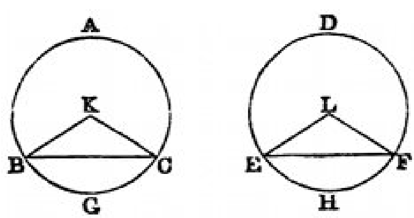
65. Theorem D

∀⊙O, chords AC,BD: AC = BD then AB∥CD

66. Theorem *

⊙K,KA × ⊙L,LA: KA < LA touch int. @ A
∀chordBDC∈⊙L tan to ⊙K @ D then ∠BAD = ∠DAC

Proposition 28. Theorem



⊙K = ⊙L, BC = EF
∀chordBC,EF∈⊙K,⊙L then
arcs on BC = arcs on EF

Proof

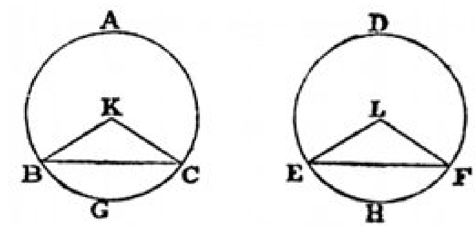
Join K[B,C], L[E,F]
A,G;D,H opp sides BC,EF
⊙K = ⊙L and BC = EF (hyp) ∴BK=KC=EL=LF ∴ ∠BKC = ∠ELF (1.8)
∴arcBGC = arcEHF (3.26) ∴arcBAC = arcEDF (a.3)

Problems

67. Theorem

⊙O,M × ⊙P,M @ A,B CAD ×⊙O,⊙P @ C,D then BC = BD

Proposition 29. Theorem



⊙ABC = ⊙DEF
arcBGC = arcEHF
then BC = EF

Proof

K,L center ⊙ABC,DEF
Join K[B,C] L[E,F]
arcBGC = arcEHF (hyp) ∴ ∠BKC = ∠ELF (3.27)
⊙K = ⊙L (hyp) ∴ KB,KC = LE,LF (d.3.1) ∴ BC = EF (1.4)

Problems

68. Theorem

∀⊙O, chords AB,CD: AB∥CD then AC = BD, AD = BC

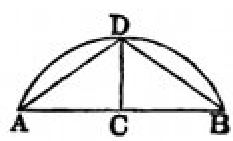
69. Theorem

∀⊙, chords OA,OB,OC: ∠AOB = ∠BOC, OA > OB
PB⊥OA ×OA @ P BQ⊥OC(pr) × OC @ Q then AP = CQ

70. Theorem

∀⊙, chord AB,AL,AM: ∠MAB = ∠LAB, AL < AM
then 1) if AB •|•(AL,AM) then AM + AL = constant
2) if AB !•|•(AL,AM) then AM - AL = constant

Proposition 30. Problem



Given: ∀arcADB Required: bisect arc

Method

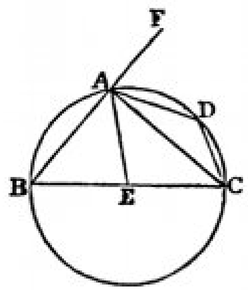
Join AB C mdpt AB CD⊥AB × arcADB @ D D required

Proof

Join D[A,B] ΔACD,BCD: AC = CB ∠ACD = ∠BCD = ∟(con)
DC=DC ∴ AD = BD (1.4) ∴ arcAD = arcBD (3.28 both on same side)

Note: Be clear about 3.28. Equal chords cut off equal arcs on both sides of the chord. That's what the geometric algebra of proposition 28 says. Euclid always says it once in Greek and once in his prosaic algebra. His 28: *In equal circles, equal lines cut off equal arcs, the greater equal to the greater, and the less equal to the less.* Our algebra is simpler but just as general.

Proposition 31. Theorem



∀⊙ABCD, center E, chord BC

1) diamBC then ∠BCA = ∟

2) arcABC > semi⊙ then ∠ABC < ∟

3) arcADC < semi⊙ then ∠ADC > ∟

Proof

Join AE BA(pr) to F

1) diamBC

EA = EB (d.1.15) ∴ ∠EAB = ∠EBA (1.5)

EA = EC (d.1.15) ∴ ∠EAC = ∠ECA (1.5) ∴ ∠BAC = ∠ABC + ∠ACB

ΔABC: ext∠FAC = ∠ABC + ∠ACB

∴ ∠BAC = ∠FAC = ∟ (d.1.10) ∴ ∠BAC = ∟

2) arcABC > semi⊙

∠ABC + ∠BAC < 2∟ (1.17) ∠BAC = ∟ ∴ ∠ABC < ∟

3) arcADC > semi⊙

4-gonABDC cyclic ∴ ∑opp∠ = 2∟ (3.22) ∴ ∠ABC + ∠ADC = 2∟

∠ABC < ∟ (proven) ∴ ∠ADC > ∟

Corollary 1

∀ΔABC: if ∀∠A = ∠B + ∠C then ∠A = ∟ (∠FAC = ∠BAC d.1.10)

Problems

71.Theorem

∀ ⊿ABC,∟C, D mdpt AB then ∀C∈⊙D,DA

72. Theorem

∀isosΔCAB: ⊙diamAC × ⊙diamBC @ mdpt AB

73. Theorem

⊙ABC × ⊙ABD @ A,B, diamsAC,AD then B∈CD

74. Theorem

∀⊙O,OA and ⊙diamOA then ⊙diamOA ×/2 ∀chordAC∈⊙O

75. Theorem D

Maximum rect∟ w/en⊙ ≡ square

76. Theorem

∀ΔABC, AD,CE⊥BC,AB Join DE then ∠ADE = ∠ACE

77. Theorem

∀⊙diamAD, ∀B,C∈⊙diamAD same side AD, DE⊥AD × BC(pr) @ E
Then $AD^2 = AB^2 + BC^2 + CD^2 + 2BC•CE$

78. Theorem

∀ ⊿ABC,∟C D mdpt AB EDF⊥AB: DE = DF = DA
Join C[E,F]: AB·|·(C,E), FE·|·(C,A) then CE,CF ×/2 ∠C,ext∠C

79. Theorem

∀AB,AC BD⊥AC DE⊥AB CF⊥AB FG⊥AC then EG∥BC

80. Theorem

∀ΔABC, ⊙O,diamAB w/diamFE∥BC: E,C same side AB
Then EB,FB ×/2 int∠B,ext∠B

81. Theorem

∀ ⊿ABC,∟B Squares on sides: ADEC, CFGB, BHKA
AE × CD @ M then MB⊥KF

82. Theorem

∀ΔABC: BE,CF alt∠B,C D mdpt BC DG⊥EF then G mdpt EF

83. Theorem

⊙diamAB × ⊙diamBC ext. @ B: AB = BC
chordBD∈⊙diamAB⊥chordBE∈⊙BC
then DE∥AC and DE = D(between ⊙ centers)

84. Theorem D,*

⊙ABC × ⊙DBE @ A,B common chords ACD,BCE: C inside ⊙DBE
⊙ABC,diamHC(pr) x DE @ K then HC⊥DE

85. Theorem D,*

∀AB, semi⊙AB ∀P∈arcAB PM⊥AB Join P[A,B]
semi⊙AM,BM × AP,BP @ Q,R
Then QR common tan to semi⊙AM,BM

86. Problem *

Given: ∀lineABC
Required: ⊙ tan to ABC @ B: tan to ⊙ on A∥tan to ⊙ on C

87. Problem *

Given: magnitude R, AB(pr both ways)
Required: ⊙C,radiusR touching AB: tan to ⊙ on A∥tan to ⊙ on B

88. Theorem D,*

∀semi⊙diamAB ∀D,E∈arcAB AD,AE × BE,BD @ F,G

Then FG(pr)⊥AB

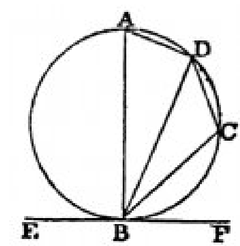
89. Problem *

Given: ⊙C,CD, CA < CD

Required: P∈⊙: CA subtends max ∠

90. Theorem *

∀⊙O, chordAB⊥chordCD × @ E then $AE^2 + EB^2 + CE^2 + ED^2 = diam^2$



Proposition 32. Theorem

∀⊙diamAB, EF tan to ⊙ @ B, ∀D,C∈arcAB

Then ∠DBF = ∠BAD, ∠DBE = ∠BCD

Proof

Join AD,DC,CB

EF tan to ⊙ @ B BA⊥EF (hyp)

∴center⊙∈BA (3.19) ∴∠ADB = ∟ (3.31)

∴∠BAD + ∠ABD = ∟ (1.32) = ∠ABF

∠BAD + ∠ABD - ∠ABD = ∠ABF - ∠ABD ∴∠BAD = ∠DBF (a.3)

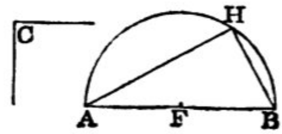
ABCD≡cyclic 4-gon ∴ ∠BAD + ∠BCD = 2∟ (3.22)

∠DBF + ∠DBE = 2∟ (1.13) ∴∠BAD + ∠BCD = ∠DBF + ∠DBE

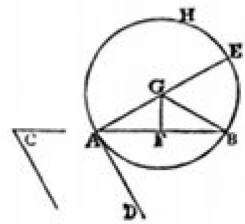
∠BAD = ∠DBF (proven) ∴ ∠BCD = ∠DBE

Euclid will make extensive use of this proposition. His expression of it is "**equal angles in alternate segments**" of a ⊙. These two pairs of equal angles **define** Euclid's idea of **alternate segments**. You can have as many pairs (2 here) as you have points (D,C) on arcAB. You have FBDA, flip DA to AD and ∠FBD = ∠BAD. You have EBDC, flip DC to CD and ∠EBD = ∠BCD. This is one way to come to an understanding of what alternate segments are. Nail this down in your mind because it appears in the next two propositions and throughout the next problem set.

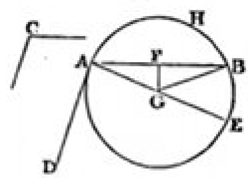
Proposition 33. Problem



Given: ∀AB, ∀∠C
Required: segmentAB = ∠C
Method/Proof
Case 1) ∠C = ∟
F mdpt AB semi⊙F,FA ∀H∈arcAB
∴ ∠AHB = ∟ (3.31)

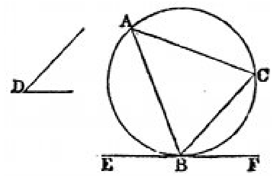


Case 2) ∠C ≠ ∟
∠BAD = ∠C (1.23) AE⊥AD (1.11)
F mdpt AB FG⊥AB × AE @ G Join GB
ΔAFG,BFG: AF = BF (con) FG = FG
∠AFG = ∠BFG (d.1.10) ∴ AG = BG (1.4)
⊙G,GA ∴ DA(pr) tan ⊙ @ A
∴ ∀H∈arcAB ∠AHB = ∠DAB (3.32) = ∠C



Note: See how ∠DAB alternates with ∠AHB? You have DABH and flip the BH to HB. We get a lot of mileage out of 3.32.

Proposition 34. Problem



Given: ∀⊙ABC ∀∠D
Required: ⊙segment = ∠D
Method
EF tan to ⊙ @ B (3.17) ∠FBC = ∠D (1.23)
segmentBAC required.
Proof
EF tan to ⊙ @ B (con) ∴ ∠FBC = ∠BAC (3.32) = ∠D (con)

Problems
91. Theorem
∀⊙ chordAB, tan AC, D mdpt arcAB
DE,DF⊥AB,AC × AB,AC @ E,F then DE = DF
92. Theorem
∀⊙C, ∀B∈⊙C, ∀P, PA tan to ⊙C × CB(pr) @ A
PD⊥CB then PB ×/2 ∠APD

93. Theorem

∀⊙ chordAB, AD tan to ⊙ @ A, ∀DPQ∥AB × ⊙ @ P,Q

Then ΔPAD eq∠ ΔQAB

94. Theorem

⊙O × ⊙P @ A,B, ∀P∈⊙O common chords PAC,PBD

Then CD ∥ tan to ⊙O @ P

95. Theorem

⊙O × ⊙P @ A,B, AC,AD tan to ⊙P,⊙O × ⊙O,⊙P @ C,D

Join B[C,D] then AB(pr) ×/2 ∠CBD

96. Theorem

⊙O × ⊙P @ B ext. ∀common chordABC (A∈⊙O)

Then segments on AB ~ (similar to) segments on BC

97. Theorem

∀⊙O chord PQ⊥chordAB × AB @ M

AN⊥tan@P × tan@P @ N then ΔNAM eq∠ΔPAQ

98. Theorem

∀⊙O diamAB⊥diamCD, ∀P∈⊙O, tan@P,AP,BP × CD(pr) @ Q,R,S

Then RQ = SQ

99. Problem

Given: baseAB, apex∠M, point D where alt apex∠ × base

Required: implied Δ

100. Problem

Given: baseAB, apex∠M, magnitude L = alt apex∠

Required: implied Δ

101. Problem

Given: baseAB, apex∠M, magnitude L = med apex∠

Required: implied Δ

102. Problem

Given: baseAB, apex∠M

Required: show area Δ max when Δ isos

103. Problem *

Given: ⊙O, A outside ⊙O

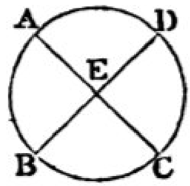
Required: ABC × ⊙O @ B,C: ΔBOC max

104. Problem *

Given: ∠M, sideAB opp ∠M, ∑(other 2 sides)

Required: implied Δ

Proposition 35. Theorem



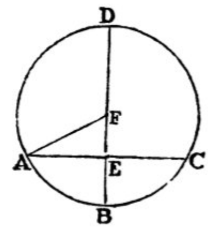
∀⊙ chordAC × chordBD @ E, AE•EC = BE•ED

Proof

Case 1) E center ⊙

E center ∴ EA = EB = EC = ED (d.1.15)

∴ AE•EC = BE•ED



Case 2) F center ⊙, AC⊥BD, F∈BD

F∈ED ∴ AE = EC (3.3)

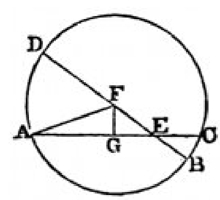
F mdpt BD ∴ BE•ED + EF^2 = FB^2 = AF^2 (2.5)

AF^2 = AE^2 + EF^2 (1.47)

BE•ED + EF^2 = AE^2 + EF^2 (a.1)

∴BE•ED = AE^2

∴BE•ED = AE•EC (EC=AE)



Case 3) F center⊙, AC!⊥BD, F∈BD

FG⊥AC × AC @ G (1.12) ∴AG = GC (3.3)

∴AE•EC + EG^2 = AG^2 (2.5)

∴AE•EC + EG^2 + GF^2 = GF^2 + AG^2

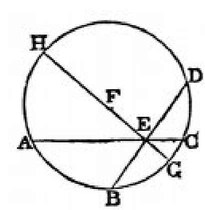
EG^2 + GF^2 = EF^2 AG^2 + GF^2 = AF^2 (1.47)

∴AE•EC + EF^2 = AF^2 = FB^2

FB^2 = BE•ED + EF^2

∴AE•EC + EF^2 = BE•ED + EF^2

∴ AE•EC = BE•ED



Case 4) F center∉AC,BD, AC!⊥BD

Add diam GEFH

∴GE•EH = AE•EC GE•EH = BE•ED (proven)

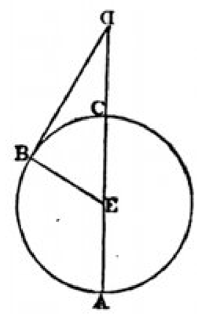
∴AE•EC = BE•ED

Proposition 36. Theorem

$\forall \odot E$, D outside $\odot E$,

$\forall$ secant DCA x $\odot E$ @ C,A,

$\forall$ tanDB x $\odot E$ @ B then $BD^2 = AD \bullet DC$



Proof

Case 1) $E \in DA$

Join BE $\therefore \angle EBD = \llcorner$ (3.18)

$AD \bullet DC + EC^2 = ED^2$ (2.6)

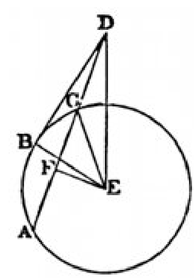
EC = EB (d.1.15)

$\therefore AD \bullet DC + EB^2 = ED^2$

$ED^2 = EB^2 + DB^2$ (1.47)

$\therefore AD \bullet DC + EB^2 = EB^2 + DB^2$

$\therefore AD \bullet DC = DB^2$



Case 2) $E \notin DA$

$EF \perp AC$ (1.12) Join E[B,C,D]

$\therefore AF = FC$ (3.3)

$AD \bullet DC + FC^2 = FD^2$ (2.6)

$AD \bullet DC + FC^2 + FE^2 = FE^2 + FD^2$

$CF^2 + FE^2 = CE^2$ $DF^2 + FE^2 = DE^2$ (1.47)

$\therefore AD \bullet DC + CE^2 = DE^2$

CE = BE (d.1.15)

$\therefore AD \bullet DC + BE^2 = DE^2$

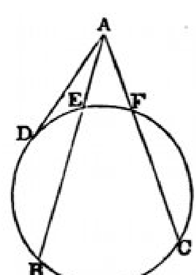
$DE^2 = DB^2 + BE^2$ (1.47)

$\therefore AD \bullet DC + BE^2 = BE^2 + DB^2$

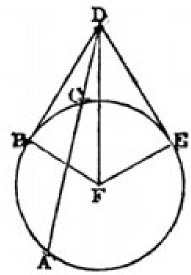
$\therefore AD \bullet DC = DB^2$

Corollary 1)

All such secants from the same point and either tangent from that point are subject to 3.36. Or $AB \bullet AE = AC \bullet AF = AD^2$ in this diagram.



Proposition 37. Theorem



∀⊙F, ∀D outside ⊙F, ∀secant DCA,
∀DB × ⊙F, if DA•DC = DB^2 then DB tan to ⊙F

Proof

Add DE tan to ⊙F (3.17) Join F[B,D,E]
∠FED = ∟ (3.18) AD•DC = DE^2 (3.36)
AD•DC = DB^2 (hyp) ∴ $DB^2 = DE^2$ ∴DB = DE
EF = BF (d.1.15) ∴DE,EF = DB,BF DF = DF
∴∠DEF = ∠DBF (1.8) = ∟ (con)
∴ BD tan to ⊙F @ B (3.16.C1)

Note: This is the converse of 3.36.

Problems

105. Theorem

⊙O × ⊙P @ A,B AB(pr) to ∀T, TP,TQ tan to ⊙O,⊙P
Then TP = TQ

106. Theorem

⊙O × ⊙O' @ A,B then AB(pr) ×/2 common tangent PQ

107. Theorem

∀ΔABC: AD,CE alt ∠A,C then BC•BD = BA•BE

108. Theorem

⊙O × ⊙P @ A,B ∀C∈AB: chordDCE,FCG∈⊙O,⊙P
Then DFEG≡cyclic 4-gon

109. Theorem *

⊙ABCD × ⊙EBCF @ B,C common tangents AE,DF × BC(pr) @ G,H
Then $GH^2 = AE^2 + BC^2 = DF^2 + BC^2$

110. Theorem *

∀⊿ABC,∟A ∀D∈BC DEF⊥BC × CA,BA(pr) @ E,F
Then DE^2 = BD•DC - AE•EC and DF^2 = BD•DC + AF•FB

111. Theorem *

A series of ⊙s intersect each other such that their tangents from a fixed point T are equal. Then the line joining any two ⊙'s points of intersection are colinear with T.

112. Problem *

Given: center ⊙A, BC(pr): C fixed, MN^2

Required: ⊙A: ⊙A × BC @ E,F: CE•CF = MN^2

113. Problem *

Given: ⊙diamAB, CB tan to ⊙ @ B, $MN^2 < AB^2$

Required: T∈BC: secantTPA: TP•PA = MN^2

Euclid - Book IV

Definitions

1. An n-gon is **inscribed** in another n-gon when every vertex of the first n-gon is on the side of the second.

2. An n-gon is **described** on another n-gon when every vertex of the second n-gon is on a side of the first.

3. An n-gon is **inscribed in a ⊙** when all of its vertices lie on the ⊙.

4. An n-gon is **described on a ⊙** when all of its sides are tangent to the ⊙.

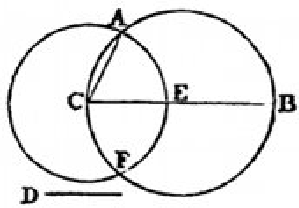
5. A ⊙ is **inscribed** in an n-gon when all sides of the n-gon are tangent to the ⊙.

6. A ⊙ is **described** on an n-gon when all vertices of the n-gon lie on the ⊙.

7. A line is **placed in a ⊙** when it is made a chord of the ⊙.

Propositions

Proposition 1. Problem



Given: ⊙diamBC, line D < BC
Required: place D in ⊙diamBC

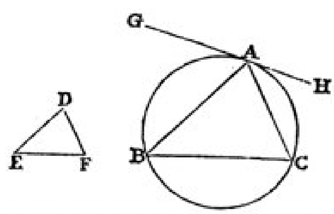
Method

If BC = D, BC required.
Else CE = D (1.3)
⊙C,CE × ⊙diamBC @ A,F
CA = CF required

Proof

CA = CE = CF (d.1.15)
CE = D (con)
∴ CA = CF = D

Proposition 2. Problem



Given: ⊙ABC, ΔDEF
Required: ΔABC eq∠ ΔDEF in ⊙ABC

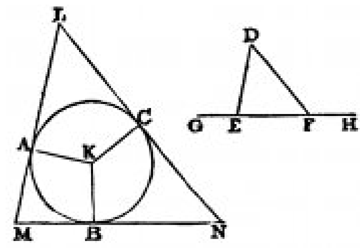
Method

GH tan to ⊙ABC @ A (3.17)
∠HAC,GAB = ∠DEF,DFE (1.23)
Join BC
ΔABC required.

Proof

GH tan to ⊙ @ ∀A
∴ ∠HAC = ∠ABC (3.32) = ∠DEF (con)
Sym. ∠ACB = ∠DFE
∴ ∠BAC = ∠EDF (1.32)

Proposition 3. Problem



Given: ⊙ABC, ΔDEF
Required: ΔLMN eq∠ ΔDEF
described on ⊙ABC

Method

EF(pr) to GEFH
K center ⊙ABC ∀radius KB
∠BKA,BKC = ∠DEG,DFH (1.23)
LAM,MBN,NCL tangents to ⊙ABC (3.17)
ΔLMN required.

Proof

LM,MN,NL⊥AK,BK,CK (3.18)
n-gonAMBK: ∑∠ = 4∟
∠KAM = ∠KBM = ∟ ∴ ∠AKB + ∠AMB = 2∟
∠DEG + ∠DEF = 2∟ (1.13) ∠AKB = ∠DEG (con) ∴ ∠AMB = ∠DEF
Sym. ∠LNM = ∠DFE ∴ ∠MLN = ∠EDF (1.32)

Problems

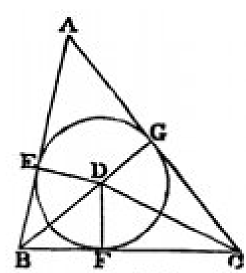
114. Theorem

In the diagram of proposition 4.3, prove tan @ A × tan @ B

115. Theorem

Using the diagram of proposition 4.3, if ∀ΔLMN w/in⊙K then the ΔABC of tangents on in⊙ of ΔLMN has only acute angles.

Proposition 4. Problem



Given: ∀ΔABC Required: in⊙

Method/Proof

BD,CD ×/2 ∠B,C x @ D (1.9)
DE,DF,DG⊥AB,BC,CA (1.12)
ΔEBD,FBD: ∠EBD = ∠FBD DB=DB
∟BED = ∟BFD ∴ DE = DF (1.26)
Sym. DF = DG ∴DE = DG
∴E,F,G∈⊙D,DE (d.1.15)
∠E,F,G = ∟ ∴AB,BC,CA tan to ⊙D @E,F,G (3.16.C1)
∴⊙D,DE≡in⊙ of ΔABC

If you are serious about your diagrams, you might want to get a plastic template with a series of circles. Mine has twenty-one circles with diameters running from 1/16th of an inch to 1½ inches. One of these will almost exactly fit as an inscribed or escribed circle. For circumcircles, you just draw the circle first and the triangle inside for most diagrams.

Problems

116. Theorem

In the diagram of proposition 4.4, prove ×/2∠B × ×/2∠C

117. Theorem

In the diagram of proposition 4.4, prove DA(pr) ×/2 ∠A

118. Theorem D

∀ΔABC: in⊙O × BC,CA,AB @ D,E,F

GHK × AB,AC @ G,K, tan ⊙O @ H

LMN × BC,BA @ L,N, tan ⊙O @ M

PQR × CA,CB @ P,R, tan ⊙O @ Q

Then ∑(perimeters ΔAGK,BLN,CPR) = perimeter ΔABC

119. Problem

Given: ⊙O, AB,AC tan to ⊙O

Required: ⊙ with tangents AB,AC, touching ⊙O ext.

120. Theorem D,*

∀ΔABC w/in⊙O, ⊙O tan to AB,AC @ D,E, AO × ⊙O @ G

Then ⊙G ≡ in⊙ of ΔADE

121. Theorem *

∀4-gonABCD: AB + CD = BC + AD ∠A,B,C,D < 2∟

Then in⊙ possible (<2∟ means no angle re-entrant)

The following proposition is not Euclid's. It proves the existence, or possibility, of ex-circles (ex⊙). These are part of Modern Geometry, the late 19th, early 20th century extension of Euclid that led to Projective Geometry. There they are called escribed circles.

Proposition 4N. Problem

Given: ∀ΔABC Required: ex⊙ on BC

Method

AB(pr),AC(pr) to ∀G,H

×/2∠GBC × ×/2∠HCB @ I

I[D-F]⊥BC,AH,AG

⊙I,ID required

Proof

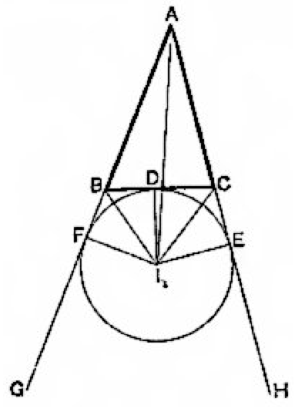
ΔIBD,IBF: ∠IBD = ∠IBF (con) IB=IB

∟IDB = ∟IFB ∴ ID = IF (1.26)

Sym. IE = IF ∴ ID = IE = IF

∴ D,E,F∈⊙I,ID (d.1.15)

∠D,E,F = ∟ ∴ AG,AH,BC tan to ⊙I @ D,E,F (3.16.C1)



Problems

122. Theorem

∀ΔABC w/ex⊙D,⊙E on BC,AC

AB,AC,BC produced to ∀K,L,M

Then D,C,E colinear

123. Theorem

∀ΔABC w/in⊙D AD(pr) × BO⊥BD @ O

Then O center ex⊙ on BC

124. Theorem D

∀ΔABC: AB>AC

ex⊙ on BC × BC,AB(pr),AC(pr) @ F,D,E

in⊙ × BC,AB,AC @ K,G,H (F·|·(B,K))

Then FK = AB - AC

125. Theorem D

∀ΔABC w/3 ex⊙ × BC,AC,AB @ D,E,F

AB(pr),AC(pr) × ex⊙GDH @ G,H

Then AE = BD BF = CE CD = AF

126. Theorem

∀ΔABC, ex⊙ on BC,CA,AB center D,E,F

Then AD×BE×CF @ O: center in⊙ΔABC

127. Theorem *

⊙HPL × ⊙KPM @ P common tangents HK,LM

Join HL,KM then in⊙ possible in 4-gonHKLM

128. Theorem *

∀ΔABC: A,∠A,perimeter constant sides vary

Then BC tangent to fixed ⊙

129. Problem *

Given: basePQ, apex∠V, radius R of in⊙

Required: implied Δ

130. Problem *

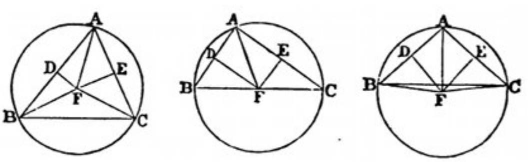
Given: points A,B, line EF

Required: ⊙: A,B∈⊙, EF tan to ⊙

Proposition 5. Problem

Given: ∀ΔABC

Required: en⊙



Method

D,E mdpt AB,AC

DF,EF⊥AB,AC

⊙F,FA required

Proof

If F∉BC, join F[B,C]

⊿AFD,BFD: AD = BD (con) DF=DF ∟FDA = ∟FDB ∴ FA = FB (1.4)

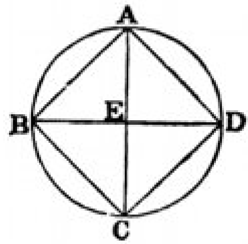
Sym. FC = FA ∴ FA = FB = FC ∴A,B,C∈⊙F,FA (d.1.15)

Corollary 1

If ⊿acute, F inside ⊿; if ⊿, F∈hypotenuse; if ⊿obtuse, F outside ⊿

Proposition 6. Problem

Given: ∀⊙ Required: inscribed square



Method

⊙E,EB: diamAC⊥diamBD (3.1,1.11)

Join AB,BC,CD,DA

squareABCD required

Proof

AC × BD @ E BE=ED ∴ E center ⊙E

ΔABE,ADE: AE=AE AE⊥BD BE=ED ∴ BA=DA (1.4)

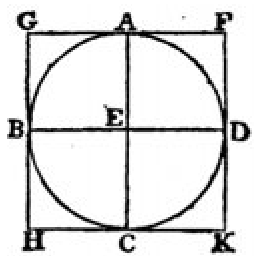
Sym. BC = DC = BA = DA ∴ ABCD eqS

diamBD ∴ arcBAD ≡ semi⊙ ∴ ∠BAD = ∟ (3.31)

Sym. ∠ABC,BCD,CDA = ∟ ∴ ABCD ≡ rect∟ ∴ ABCD ≡ square

Proposition 7. Problem

Given: ∀⊙ Required: described square



Method

⊙E,EB: diamAC⊥diamBD (3.1,1.11)

tangents FG,GH,HK,KF on A,B,C,D (3.17)

squareFGHK required

Proof

AC × BD @ E ∴ E center ⊙E

GF tan @ A ∴ ∠EAF,EAG = ∟ (3.18) Sym. ∠s @ B,C,D

∠AEB,EBG = ∟ ∴ GH∥AC (1.28) Sym. AC∥FK Sym. GF∥HK∥BD

∴ 4-gons GK,GC,CF,FB,BK ≡ ∥gm ∴ GF=HK GH=FK (1.34)

AC=BD AC=GH=FK BD=GF=HK GH=FK=GF=HK ∴ FGHK eqS

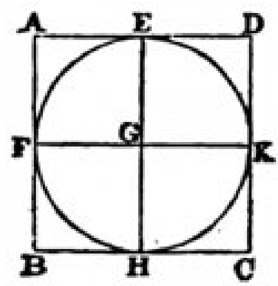
AEBG ≡ ∥gm ∠AEB = ∟ ∴ ∠AGB = ∟ Sym. ∠H,K,F = ∟

∴ FGHK ≡ rect∟ ∴ FGHK ≡ square

Note: "A rectangular, equilateral 4-gon is a square" is the lesson here.

Proposition 8. Problem

Given: ∀square Required: in⊙

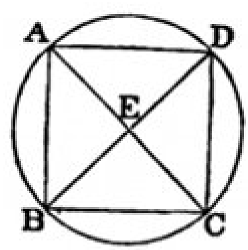


Method/Proof

squareABCD
F,E mdpt AB,AD (1.10)
EH,FK∥AB,AD (1.31)
∴ n-gonsAK,KB,AH,HD,AG,
GC,BG,GD ≡ ∥gm (1.34)
AD=AB (d.1.30) ∴ AE=½AD AF=½AB
∴ AE=AF (a.7) ∴ FG=GE (1.34)
Sym. GH,GK = FG,GE ∴ GE = GF = GH = GK
∴ E,F,H,K∈⊙G,GE
∠ @ E,F,H,K = ∟ ∴ AB,BC,CD,DA tan to ⊙E (3.16.C1) ∴ ⊙E ≡ in⊙

Proposition 9. Problem

Given: ∀square Required: en⊙



Method/Proof

squareABCD: AC × BD @ E
ΔBAC,DAC: AB=AD AC=AC ∠B = ∠D
∴ BA = DA BC=DC
∴ ∠BAC = ∠DAC (1.8) ∴ AC ×/2 ∠BAD
Sym. AC ×/2 ∠BCD BD ×/2 ∠ABC,ADC
∴ ∠DAB = ∠ABC ∠EAB = ½∠ABC
∴ ∠EAB = ∠EBA (a.7) ∴ EA=EB (1.6)
Sym. EC = ED = EA = EB ∴ A,B,C,D∈⊙E,EA ∴ ⊙E ≡ en⊙

Most of the following problems use the in⊙ and en⊙ of triangles to show more properties of circles and triangles. Propositions 4.4 and 4.5 enable the problems. But the solutions reach back into Books I and III.

Problems

131. Theorem

In diagram 4.5, if FG⊥BC × BC @ G then FG ×/2 BC

132. Theorem

∀ΔABC ∀DE∥BC then en⊙ABC,ADE have common tangent

133. Theorem

∀ΔABC: if O center in⊙ and en⊙ then ΔABC≡eqSΔ

134. Theorem

∀4-gonABCD w/en⊙: AD × BC @ E en⊙ on ΔECD
Then tanEP on en⊙ of ΔECD∥AB (P same side BC)

135. Theorem

∀ΔABC w/in⊙P,en⊙Q, if A∈PQ(pr) then ΔABC≡isosΔ (AB=AC)

136. Problem

Given: points A,B,C, magnitude M
Required: ⊙: B,C∈⊙ tan to ⊙ from A = M

137. Problem

Given: ∀ΔABC
Required: center of ⊙ cutting equal chords from AB,BC,CA

138. Theorem *

ΔABC,KLM: BC = LM ∠A = ∠K en⊙F,G of ΔABC,KLM then FB = GL

139. Theorem *

∀⊙C,CA: CA⊥CB ∀chordBP × CA @ N then BA tan to ⊙ANP

140. Theorem D,*

AB∥CD AD × BC @ E then en⊙ on ΔAEB,ECD touch ext.

141. Problem *

Given: points A,B linePQ magnitude R
Required: ⊙: A,B∈⊙, chord in ⊙ = R∈PQ

142. Problem *

Given: ∀ΔABC, magnitude P
Required: ⊙ w/center∈AB, chords∈BC(pr),CA(pr) = P

143. Theorem *

∀ΔABC, in⊙O, en⊙P AO(pr) × en⊙P @ F then FB = FO = FC

144. Theorem *

∀ΔABC: E mdpt AB CD ×/2∠C × ED⊥AB @ D
Then ∠ACB + ∠ADB = 2∟

145. Theorem *

∀AB,AC fixed in position, BC fixed length, D,E mdpt AB,AC
DF,EF⊥AB,AC then FB constant for any AB,AC

146. Theorem *

∀isosΔABC: AB=AC w/en⊙ ∀ADE × BC,en⊙ @ D,E

Then AB tangent to ⊙BDE

147. Theorem *

∀4-gonABCD: BC(pr),BA(pr) × AD(pr),CD(pr) @ P,Q

en⊙ΔPCD × en⊙QBC @ C,R then PRQ colinear

148. Problem

Given: points A,B same side line CD

Required: ⊙: A,B∈⊙, CD tan to ⊙

149. Problem *

Given: 2 lines, ⊙ w/center eqD both lines, ∠M

Required: ⊙: both given lines tangent to ⊙,

new ⊙ cuts seg⊙ = ∠M from given ⊙

150. Theorem *

∀⊙, chordAC ∀B,D∈AC ⊙BDE touching int. ⊙AC @ E

Then AB,CD subtend equal angles from E

151. Theorem *

∀semi⊙diamAB, chordAD × chordBC @ E

Then on en⊙ΔCDE tangents @ C,D⊥tangents on semi⊙ @ C,D

152. Theorem D,*

∀⊙ABC w/en⊙ tangent to en⊙ @ C × AB(pr) @ D

⊙D,DC × AB @ E then EC ×/2 ∠ACB

153. Theorem *

∀4-gonABCD AC × BD @ O en⊙ on OAB,OBC,OCD,ODA

Then centers of en⊙s: P,Q,R,S form ∥gm

154. Problem

Given: ∀rect∟ABCD

Required: en⊙

155. Theorem

No rect∟ but a square can be described on ⊙

156. Theorem

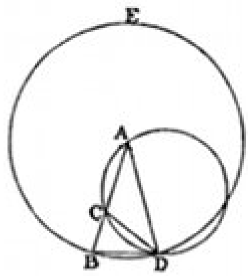
∀⊙w/inscribed squareABCD ∀P∈⊙

Then $PA^2 + PB^2 + PC^2 + PD^2 = 2diam^2$

157. Theorem *

∀⊙O ∀diamAOB,COD tangents on A-D: NAK,KDL,LBM,MCN
4-gonKLMN of tangents ≡ rhombus

Proposition 10. Problem



Create isosΔ: base∠ = 2apex∠

Method

∀AB, C∈AB: AB•BC = AC^2 (2.11)
⊙A,AB D∈⊙: BD = AC (4.1) Join AD
ΔABD: ∠B,D = 2∠A required.

Proof

Join DC ⊙ADC (4.5)
AB•BC = AC^2 AC = BD (con) ∴AB•BC = BD^2
∴BD tan,BCA secant to ⊙ADC (3.37) ∴∠BDC = ∠DAC (3.32)
∴∠BDC + ∠CDA = ∠CDA + ∠DAC ∴∠BDA = ∠CDA + ∠DAC
ext∠BCD = ∠CDA + ∠DAC ∴∠BDA = ∠BCD
∠BDA = ∠DBA (1.5) ∴∠BDA = ∠DBA = ∠BCD
∠DBC = ∠BCD ∴DB = DC (1.6)
DB = CA (con) ∴CA = CD ∴∠CAD = ∠CDA (1.5)
∴∠CAD + ∠CDA = 2∠CAD
∠BCD = ∠CDA + ∠CAD (1.32) ∴∠BCD = 2∠CAD ≡ 2∠BAD
∠BCD = ∠BDA = ∠DBA (proven) ∴∠BDA,DBA = 2∠BAD

Problems

158. Theorem

In diagram of 4.10, ∠ACD = 3∠BAD

159. Theorem

In diagram of 4.10, 1) 2 isosΔ have base∠ = 2 vertex∠
2) 1 isosΔ has base∠ = 1/3 vertex∠

160. Problem

Given: ∀KL
Required: isosΔMKL: apex∠ = 3base∠

161. Theorem *

In diagram 4.10, CD = side regular 5-gon in ⊙ACD

162. Theorem

In diagram 4.10, ⊙A,AC × ⊙ACD @ D,E then DE = DC

163. Theorem

In diagram 4.10, ⊙A,AC × ⊙ACD @ D,E AE × BD(pr) @ G

Then ΔGAB: ∠A,B = 2∠G

164. Theorem

In diagram 4.10, CA(pr),DC(pr) × ⊙A,AC @ G,H

Then ΔGCH: ∠GHC,HCG = 2∠HGC

165. Theorem

In diagram 4.10, en⊙ΔABD ≡ ⊙ACD

166. Theorem

In diagram 4.10, ⊙A,AC × ⊙ACD @ D,E Join E[A,C]

AE(pr) × BD(pr) @ G then CDGE ≡ ∥gm

167. Theorem *

In diagram 4.10, AF ≡ diam⊙ACD then DF ≡ radius en⊙O on ΔBCD

A **regular n-gon** is eqS and eq∠

Proposition 11. Problem

Given: ∀⊙ Required: inscribed regular 5-gon

Method

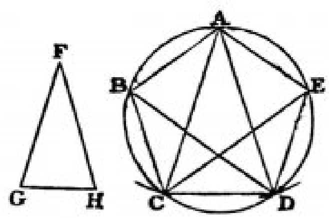
isosΔFGH: ∠G,H = 2∠F (4.10)

in ⊙, inscribe ΔACD eq∠ ΔFGH (4.2)

CE,DB ×/2 ∠C,D (1.9)

Join AB,BC,AE,ED

5-gonABCDE required.



Proof

∠ACD,ADC = 2∠CAD ∴ ∠ADB = ∠BDC = ∠CAD = ∠DCE = ∠ECA

∴arcAB = arcBC = arcCD = arcDE = arcEA (3.26)

∴AB = BC = CD = DE = EA (3.29) ∴ABCDE ≡ eqS

arcAB = arcDE ∴arcAB + arcBCD = arcBCD + arcDE

∠AED,BAE on arcABCD,BCDE ∴ ∠AED = ∠BAE (3.27)

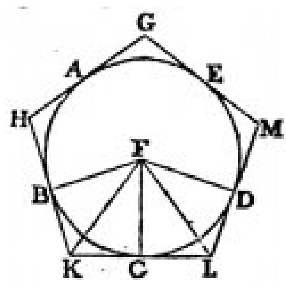
Sym. ∠ABC = ∠BCD = ∠CDE = ∠AED = ∠BAE

∴ABCDE ≡ eq∠ 5-gon ∴ ABCDE ≡ regular 5-gon

Proposition 12. Problem

Given: ∀⊙F Required: described regular 5-gon

Method



Inscribe angles of 5-gon in ⊙:
arcAB,BC,CD,DE,EA all equal (4.11)
tangentsGAH,HBK,KCL,LDM,MEG (3.17)
GHKLM required

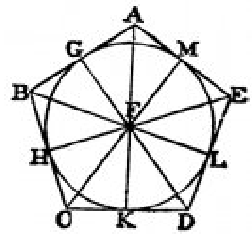
Proof

Join F[B,K,C,L,D]
KL tangent ∴ FC⊥KL (3.18)
∴ ∠C = ∟ Sym. ∠B,D = ∟
∴ $FC^2 + CK^2 = FK^2$ (1.47) = $FB^2 + BK^2$
FC = FB ∴ CK = BK
ΔBFK,CFK: FB = FC FK=FK BK=CK
∴ ∠BFK = ∠CFK (1.8) ∠BKF = ∠CKF (1.4)
∴ ∠BFC,BKC = 2∠CFK,CKF Sym. ∠CFD,CLD = 2∠CFL,CLF
arcBC = arcCD ∴ ∠BFC = CFD (3.27)
∴ ∠BFC,CFD = 2∠CFK,CFL ∴ ∠CFK = ∠CFL
∟FCK = ∟FCL ∴ΔFCK ≡ ΔFCL (1.26) ∴ CK=CL ∠FKC = ∠FLC
CK = CL ∴ LK = 2CK Sym. HK = 2BK
BK = CK ∴ HK = 2BK LK = 2CK ∴ HK = LK
Sym. GH = GM = ML = HK = LK ∴ GHKLM ≡ eqS
∠FKC = ∠FLC ∠HKL,KLM = 2∠FKC,FLC ∴ ∠HKL = ∠KLM
Sym. ∠KHG = ∠HGM = ∠GML = ∠HKL = ∠KLM
∴ GHKLM ≡ eq∠ ∴ GHKLM ≡ regular 5-gon

How to Draw a Regular N-gon

Mark ⊙ center on notepaper line. Scribe a ⊙ with 3-4 line radius. Divide 360 by n and use a small protractor to mark the angles. Use a ruler on center and marks to mark the ⊙ n times. Then connect the dots. Or do it Euclid's way and go mad.

Proposition 13. Problem



Given: regular 5-gon Required: in⊙

Method

CF,DF ×/2 ∠BCD,CDE @ F Join F[A,B,E]

ΔBCF,DCF: BC=DC (hyp) CF=CF

∠BCF = ∠DCF (con) ∴ BF=DF ∠CBF = ∠CDF

∠CDE = 2∠CDF = ∠CBA ∠CDF = ∠CBF

∴ ∠CBA = 2∠CBF ∴ ∠ABF = ∠CBF ∴ BF ×/2 ∠ABC

Sym. AF,EF ×/2 ∠BAE,AED

FG,FH,FK,FL,FM⊥AB,BC,CD,DE,EA

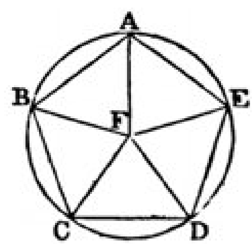
ΔFHC,FKC: ∠FCH = ∠FCK ∟FHC = ∟FKC FC=FC

∴ FH=FK HC=CK (1.26) Sym. FL = FG = FM = FH = FK

∴ G,H,K,L,M∈⊙F,FG

∠G,H,K,L,M = ∟ ∴ AB,BC,CD,DE,EA tan to ⊙F (3.16) ∴ ⊙F ≡ in⊙

Proposition 14. Problem



Given: regular 5-gonABCDE

Required: en⊙

Method

CF,DF ×/2 ∠BCD,CDE Join F[A,B,E]

∴ BF,AF,EF ×/2 ∠CBA,BAE,AED (4.13)

∠BCD = ∠CDE ∠FCD,FDC = ½∠BCD,CDE

∴ ∠FCD = ∠FDC ∴ FC=FD (1.6) Sym. FB = FA = FE = FC = FD

∴ A,B,C,D,E∈⊙F,FA ∴ ⊙F ≡ en⊙

Problems

168. Theorem

∀eqS n-gon w/en⊙ ≡ eq∠ n-gon

169. Theorem

∀reg.(regular) 5-gonABCDE AC×BE @ F then AC = AB + BF

170. Theorem

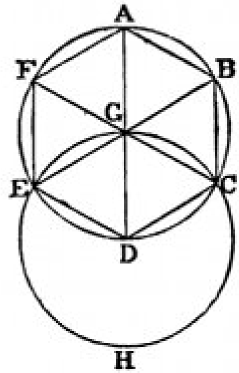
∀reg. 5-gonABCDE Join every other vertex (AC,BD,...)

AC×BD @ K BD×CE @ L ... @O then KLMNO ≡ reg. 5-gon

171. Theorem *

∀reg. 5-gonABCDE then 3ΔABC < 5-gon < 4ΔABC

Proposition 15. Problem



Given: ∀⊙G

Required: inscribed regular 6-gon

Method

diamAGD ⊙D,DG × ⊙G @ E,C

EG(pr),CG(pr) × ⊙G @B,F

6-gonABCDEF required

Proof

GE=GD DE=DG (d.1.15)

∴ GE=DE ∴ ΔEGD ≡ eqSΔ

∴ ∠EGD = ∠GDE = ∠DEG (1.5.C1)

∴ ∑ ∠EGD,GDE,DEG = 2∟ (1.32)

∴ ∠EGD = 2/3∟ Sym.∠DGC = 2/3∟

∠EGC + ∠CGB = 2∟ (1.13) ∴ ∠CGB = 2/3∟

∴ ∠EGD = ∠DGC = ∠CGB

∴ ∠BGA = ∠AGF = ∠FGE ∴ ∀ ∠@G = 2/3∟

∴ arcAB = arcBC = arcCD = arcDE = arcEF = arcFA (3.26)

∴ AB = BC = CD = DE = EF = FA (3.29) ∴ ABCDEF ≡ eqS

arcAF = arcED ∴ arcABCD + arcAF = arcED + arcABCD

∴ arcFABCD = arcABCDE ∴ ∠FED = ∠AFE (3.27)

Sym. ∀2∠ of 6-gon equal ∴ ABCDEF ≡ eq∠

∴ ABCDEF ≡ regular 6-gon

Corollary 1.

Side of 6-gon = radius of its en⊙

Also, tangents @ vertices form regular 6-gon. (Method 4.12 on ⊙G)

Problems

172. Problem

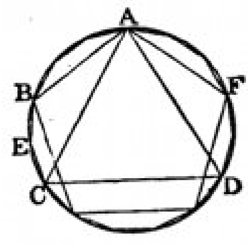
Given: ∀eqSΔABC w/en⊙O

Required: reg. 6-gon inscribed in ⊙O

173. Theorem

In #172, **1)** side 6-gon = radius ⊙O **2)** 6-gon = 2ΔABC

Proposition 16. Problem



Given: ∀ ⊙

Required: inscribed regular 15-gon

Method/Proof

AC ≡ side inscribed eqSΔ (4.2)

AB ≡ side inscribed reg. 5-gon (4.11)

arcABC = 1/3 ⊙

∴ arcABC = 5/15 ⊙ arcAB = 3/15 ⊙

∴ arcBC = 2/15 ⊙ E mdpt arcBC (3.30)

∴ arcBE = 1/15 ⊙ ∴ 15 chords = chordBE required

Corollary 1.

Using method of 4.13 and 4.14 one can create in⊙ and ex⊙ of reg. 15-gon

Problems

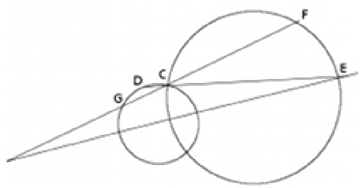
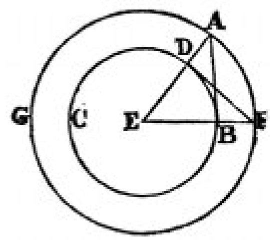
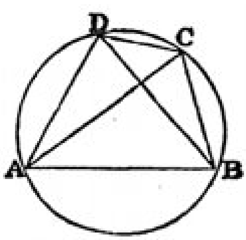
174. Problem

Given: ∀ ⊙

Required: incribed Δ: angles in relation of 2:5:8

Problem Diagrams

In the first volume, I babied you with diagrams and labelling conventions. In this volume, not so much. In the next volume, not at all.

4. 	**18.** 
49. 	
11. Draw the three touching circles and then label as indicated.	**22.** Use diagram and labelling from #19.
25. Draw tangents @ A,B. Pick ∀D ∈⊙ between them and let the tangent on D determine E,F	**27.** Draw the stacked ⊙s, then enclose them in a 4-gon, then label carefully.

33. Best if one circle much smaller than the other.	**36.** Create diagram according to description. Then produce AB to determine D ∈⊙O
41. Draw tangent @ C. Add 2 eq∠ lines from C. Put A on one line, B on other.	**43.** Draw ⊙. From O outside ⊙, draw secants OAB,ODC. Join AD,BC
53. Don't draw a regular 6-gon. Use ∀6-gon and reason it out.	**54.** Don't try to calculate ∠EBC. Draw a ⊙. Put ∀4-gon in it. Produce AB to some E outside ⊙. And **assume** ∠EBC=∠ADC
55. Draw ⊙, then lines, then juggle labels into place.	**57.** Put A,D,C on bottom left third of ⊙. Then P,Q won't run off the page.
58. Make BC < AD so H•|•(B,F) and so H in⊙	**64.** Place E near A and F out towards circumference.
65. Label both AC,BD left to right.	**75**. Draw ⊙, add rect∟ and label it clockwise ABCD
84. Draw ⊙s, label A,B, put C inside ⊙BCE, then draw chords ACD,BCE, then diam CH, then DE.	**85.** Best if AB is 5-6+ cm and P near, but not at, middle of arcAB.

88. Label arc ADEB with D,E close together at the top or F goes off the page.	**118.** Believe it or not, a rough sketch will do. Spin compass, throw three sides of Δ around ⊙, add last three lines, and label.
120. Draw ⊙, draw Δ around it, then label. ⊥radii can be used for pts of tangency.	**124.** A rouch sketch of Δ, in⊙, and upper part of ex⊙ will suffice.
125. Do a tidy rough sketch or go mad.	**134.** Draw ⊙, add EDA,ECB, join AD,BC, add PE: P same side AB
140. Make your diagram so that AD,BC are not produced in order to meet. They should intersect between the parallel lines.	**152.** Draw ⊙ then draw ΔABC with C lower than B so tan on C and AB(pr) intersect near ⊙

Problem Hints

1. diam⊙B must be chord ⊙A
2. Sides of 4-gon are chords of circle.
3. Bisect chords from centers to create rect∟
4. Use last result.
5. Simply reason from last result.
6. ΔAPQ: $AP^2 + AQ^2$ = what from Euclid Book II?
7. v1#118 (Everymind's Euclid volume I, problem 118)
8. v1#146
9. By contradiction, let CF × DE @ H
10. ‖gmize ΔOPN
11. Show PA ‖ radius OD using isosΔ
12. ‖gmize ΔAHC into square
13. This is simply 3.15. Then show ∀other chord is longer.
14. ⊙O must be ⊙O,OA and OA + radius ⊙C = OC
15. See definitions of tangent and parallel.
16. Slightly alter method of last problem.
17. 2 isosΔ with ∟ apexes (or "apices" FYI)
18. Use 3.17 on its own diagram.
19. #18
20. Reason it out from last result as a short proof.
21. By contradiction: Else DE cuts ⊙, DFG tan ⊙ @ F × AC @ G
22. 1.15.C.2
23. Show tangents make 2∟ on PQ.
24. Need an ∠ = ∠OBC. Use 1.32 and 3.1.
25. Use #18 to show 4-gon = 2 × some Δ
26. #18 combined with a bit of algebra.
27. ⊙A,(L - M)
28. Note on #27
29. Reason it out from last result as a short proof.
30. Two pairs of equivalent Δ

31. Just draw the diagram, add two chords, and think.
32. Add a ⊙ to use last result.
33. Need ∠AEC = ∠BFD so join AC,BD. IsosΔ.
34. Add EF which will be ⊥ to AB and that tangent.
35. Re #14, calculate required radius and work backward.
36. If DB,BA one line, you need ∠s to compare @ B.
37. ΔLKB to show LK = ½AB using prior results.
38. Use just one ⊙ × ∥ @ B,C, add tangents @ B,C
39. Join CD. Show ∠BDE = ∠DBE = ∠CBA
40. Add OCF:OC=CF. Use ΔPCF,PCO to get an ∠ = 2/3∟
41. ∀P∉C, join P[A,B] PA × tangent on C @ E. Join EB.
42. 3.21
43. 3.21
44. 3.21, 1.32
45. Ask yourself why some of ∠s constant and not individual ∠s.
46. ∠PAQ = ∠PAB - ∠QAB ∴ ∠RAB = ½(∠PAB + ∠QAB)
47. Skip the diagram and reason it out.
48. 3.22 and some arithmetic.
49. The 4∠s in diagram 3.22 are int∠ on sides.
50. Supplementary angles.
51. eqSΔ in ⊙
52. Oh, please ...
53. v1#36. Join AD.
54. Converse of 3.22 is true. Use it.
55. 4-gonCPBE: show ∠EBP = ∠BEP
56. Equal ∠s on base seen as equal∠s on chord.
57. Solve ∠s of ΔEHQ. sup∠A = ∠C
58. Show ∠DFB = ∠BAD
59. ∀⊙ chords AB,CD × @ E then ∠AEC = ½∠(arcAC + arcBD)
60. #59. Show BD × AC @ ½(∠AED + ∠BGC)
61. Use ⊙, equal ∠s ∴ equal chords or arcs.
62. ∠DAE = ext∠ of what Δ?
63. ⊙D,CD × AB(pr) @ F making ABF (not BAF).
64. Method of #63.

65. 1.27

66. Show arcBF = arcCF where AD(pr) × ⊙L @ F

67. Oh, please ...

68. 3.26, 3.29

69. ΔBOP,BOQ and 1.47

70. #227, constant as tied to chord and ∠s

71. v1#58

72. #71

73. CBD need to be one line. Don't make this hard.

74. Draw a clear enough diagram and you will see it.

75. Maximize ΔDAC.

76. 3.21

77. ΔABD,BCD and algebra from Book I and II

78. #77

79. Cyclic 4-gons on BC,FD

80. 1.29 for equal angles.

81. One cyclic 4-gon for very short proof.

82. Show DCEF cyclic 4-gon.

83. Join AD, CE. Build proof from ∥ lines and 3.31.

84. Show ∠EKC = some necessarily ∟ on diam.

85. C mdpt AM. PM×QR@D. DQ(pr) to ∀E. Need ∠EQC = ∟

86. For AD∥CE need ∠DAB + ∠ECB = 2∟ (1.28)

87. #30

88. Show angles of ΔAFH = angles of ΔABD.

89. AB⊥CA × ⊙ @ B. B required. Now prove it.

90. 2.4, 2.7 and squares on the sum and difference of lines.

91. ΔDAE ≡ ΔDAF

92. Chord PDF, diamBE, join PE, 3.32

93. DAPQ: what flips for equal∠s

94. Label tan@P: ∠EPA = ∠PBA

95. Prove ∠ABC = ∠ABD, result follows from 1.13.

96. TanEBF for 3.32. See def. of similar segments.

97. 3.32 and a cyclic 4-gon.

98. Show ΔQPS,QPR both isosΔ

99. Maybe use a segment of a circle?
100. Unless Δisos, two solutions fit the segment.
101. d.1.15
102. Area of Δ from Book I.
103. Use analysis and #102. Seg⊙ from 3.33 × ⊙ @ C
104. On AB: seg⊙=∠m,seg⊙=½∠M. Think about it then study soln.
105. 3.36 straight up
106. Look at your last diagram.
107. Cyclic 4-gon.
108. Any three non-colinear points define a circle.
109. ∀AB,CD (AB+CD)(AB-CD) = AB^2 - CD^2
110. ⊙diamBC × DE(pr) @ N,O and note on #109.
111. Just show this is true of any two of the circles.
112. You need a CE and CF in a diagram like 3.36.
113. Uses 3.31, 3.36, 2.3, 1.47.
114. Use an axiom.
115. #18. Prove one ∠(ABC) < ∟ and Sym. the other two.
116. #115
117. Show equal angles using 1.47.
118. #18
119. Required ⊙ is an in⊙ of Δ touching ⊙O.
120. Need 3∠bisectors × @ G. AG by #117, prove DG, Sym EG.
121. Consider #19.
122. Work with ∑ of ∠ to show colinear by 1.14
123. AO,BO,CO must bisect ∠A,ext∠B,ext∠C
124. Nothing but a bunch of equal tangents from pts outside ⊙
125. Use tans on ex⊙GDH for AE = BD. Sym. the rest.
126. #117
127. #121
128. #125
129. Locate center in⊙, tangents for sides
130. Proposition 3.37
131. Proposition 1.26
132. 3.32: show how both tangents on A are one line.

133. O[A,B,C] are what in in⊙ and en⊙?
134. 3,22 and ∑(opp∠ of 4-gon)
135. Draw a decent diagram using isos (not eqS) Δ
136. Oh, please. What equals the square on a tangent?
137. Distance from center of equal chords?
138. Draw a decent diagram. FB=GL by 1.26
139. O center ⊙ANP. If ∠OAB = ∟, what is ∠APB = ∠APN?
140. Come to a deeper understanding of 3.32.
141. AB(pr) × PQ @ C and use 3.36.
142. CD ×/2 ∠C × AB @ D. ⊙D for chords.
143. Join BO. 2 equal by AF. 2 equal by 1.6.
144. cyclic 4-gon
145. Forget the sense of movement. Show ΔFBC fixed in form.
146. Proposition 3.32.
147. Need ∠DRP + ∠DRQ = 2∟
148. Secant?
149. Reduce to last problem then reason it out.
150. 3.32 on both ⊙
151. tan on en⊙ @ C × AB @ O. Need OB = OC.
152. Need ∠ECB = ∠ECA. Use 3.32 and 1.32.
153. Use 4.5 to show PS,QR⊥OA for starters.
154. You don't need a hint for this.
155. Assume rect∟ on ⊙ then prove it is a square.
156. Proposition 1.47
157. Prove LM = LK and sym. the other three.
158. Understand the angles in diagram 4.10. Take the time.
159. Don't make this hard. Two lines. Cite the proofs.
160. Leverage 4.10 to make a short proof.
161. ⊙A,AC × ⊙ACD @ D,F. If CD side 5-gon, so are DF,CA.
162. Sound familiar?
163. Leverage your work. Three lines.
164. Use angle tools from Books I and III.
165. Use equal Δ by #162.
166. #162

167. ⊙A,AC × ⊙ACD @ D,E

168. Show ∀2 angles equal

169. AC = AF + FC ∴ need ∀side = FC

170. Outer 5-gon symmetric. Show ∠K = ∠L KL = LM

171. Create ∥gm ABCF

172. Show eqS and eq∠

173. Need BE=BO and ΔABC = ΔEBC

174. 2 + 5 + 8 = what?

Problem Solutions

1. Method

Join AB BC(pr)⊥AB × ⊙B @ C,D (1.11) ⊙A,AC required

Proof

ΔABC,ABD: AB=AB BC=BD (d.1.15) ∠ABC = ∠ABD = ∟

∴ AC=AD (1.4) ∴ D∈⊙A,AC and CD diam ⊙B

2. Proof

Each side of 4-gon is chord of ⊙ (d.3.5)

Each chord can be used to determine center ⊙ (3.1)

∴ Each ⊥ on mdpt chord × center ≡ fixed point

3. Proof

ABC ∥ DEF: A,D∈⊙P C,F∈⊙Q (1.31) PK⊥AB (1.11)

PK(pr) × DE @ M Sym. QL⊥BC × EF @ N

∴ KLNM ≡ ∥gm ∴ KM = LN (1.34)

KB = ½AB BL = ½BC (3.3) ∴ KL = ½AC Sym. MN = ½DF

KL = MN ∴ AC = DF

4. Proof

AL⊥FC AP⊥DC BM⊥CG BQ⊥CE (1.11)

BX⊥AL(pr) BY⊥AP ∥ DE

∠FG × AB = ∠DE × AB ∴ ∠BAY = ∠BAX ∴ BX = BY (1.26)

∴ BX = ½FG BY = ½DE ∴ FG = DE (#3)

5. Method

Line on either point of intersection ∥ AB

Proof

∀line on intersection !∥ AB < 2AB (#4)

∴ line on intersection ∥ AB = 2AB is maximum (#4)

6. Proof

CD⊥PQ × PQ @ D (3.3)

$AP^2 + AQ^2 = 2(AD^2 + PD^2)$ (v1#146) $= 2(AC^2 + CD^2 + PD^2)$ (1.47)

$= 2(AC^2 + CP^2)$ (1.47)

AY = CA + CY = CX + CA

AX = CX - CA

∴ $AX^2 + AY^2 = 2(AC^2 + CX^2) = 2(AC^2 + CP^2) = AP^2 + AQ^2$

Note: See v1#139

7. Proof

⊙A' × ⊙B' @ B,E

A'B' ⊥ BE and A'B' ∈ bisector of BE ∴ A'B' ×/2 AB,BC (3.3)

∴ A'B' ∥ EC,AE (v1#118) ∴ AEC one line and BE ⊥ AC

Sym. chord C'D' ∈ ⊙C',D' ⊥ AC

∴ common chords ∥

Sym. other two common chords ∥

8. Method/Proof

O mdpt AB

$AP^2 + BP^2 = 2OP^2 + 2AO^2$ (v1#146)

∴ OP must be minimized

Join OC OC × ⊙C @ P

OC minimum from O (3.8) ∴ OP minimum ∴ AP,BP minimum

9. Proof

DE diam ⊙A ∥ FG diam ⊙B Join F[D,C] ∴ DCF one line

Else let CF × DE @ H

AB × ⊙A @ C (3.11) ∴ ∠BFC = ∠AHC (1.29)

∴ ∠AHC = ∠ACH ∴ AH = AC ∴ AH = AD⤵ ∴ DFC one line

10. Proof

radiusOQ ∥ PN: P,Q opp sides AB Join PQ

∠NPQ = ∠OQP (1.29) ∠OQP = ∠OPQ (1.5) ∴ ∠NPQ = ∠OPQ

PQ ×/2 ∠OPN ∴ P,Q opp sides AB then Q∈PQ ×/2 ∠OPN

Sym. QO × ⊙O @ R P,Q same side AB then R∈RQ ×/2 ∠OPN

Note: Problem 10 is useful for diagramming as the bisector of any angle is always on the perpendicular diameter.

11. Proof

Join O[D,E] ∠PAB = ∠PBA (1.5) ∴ ∠PAB = ∠OBD = ∠ODB (1.15)
∴ PA ∥ DO (1.27) Sym. QA ∥ OE
PAQ one line (3.12) ∴ DOE one line ∴ DE ∥ PQ and DOE diam ⊙O

12. Proof

CK⊥BD ∴ ACKH ≡ square
∴ FE,BD eqD C ∴ FE = BD (3.14) ∴ AE = BK = ½FE
AH = HK ∴ EH = BH ∴ DH = FH

13. Method

Join CA chord BAD⊥CA required

Proof

Else ∀other chord EAF = BAD
CG⊥EF ∠CGA = ∟ ∴ ∠CAG < ∟ (1.32) ∴ CG < CA (1.19)
∴ EF nearer C than BD ∴ ∀EAF > BAD⫰ ∴ BD least

14. Method

AB = radius ⊙C Join BC Add radius CD: ∠CBA = ∠BCD
CD(pr) × BA @ O ∴ isosΔOBC O required

Proof

∠OCB = ∠OBC (con) ∴ OC = OB (1.6)
BA = CD (con) ∴ OD = OA
∴ ⊙O,OA × AB @ A, ⊙O tangent (touches) ⊙C

Note: If 2 ⊙s touch, they share tangent at shared point

15. Method

CA⊥AB × AB @ A,⊙C @ D DE⊥CD required

Proof

DE tan (3.16.C1) DE ∥ AB (1.28)

Note: Modernizing the statement of 3.16 makes 3.16.C1 redundant but I keep the old references to match older Euclids.

16. Method

radiusCD ∥ AB DE⊥CD required

Proof

DE tan (3.16.C1) DE⊥CD (1.29)

17, Method

AD⊥CA = M   Join CD × ⊙C @ B   BE⊥CB × CA @ E   EB required

Proof

ΔCAD,CBE: ∠C = ∠C, ∠CAD = ∠CBE = ∟, CA = CB

∴ AD = BE (1.26)   BE tan (3.16.C1)

18. Proof

In diagram 3.17:

FD(pr) × ⊙ @ H and EH × ⊙ @ K

Join AK then AK tan to ⊙ (3.17)

DH = DF (3.3) ∴ AK = AB

Note: This is a fundamental result.

19. Proof

AP = AS, BP = BQ, CQ = CR, DR = DS (#18)

∴ AB + CD = AP + PB + DR + RC = AS + DS + BQ + CQ = BC + DA

20. Proof

∥gm ABCD described on circle

∴ AB + CD = BC + DA (#19)

AB = CD BC = DA (1.34)

∴ AB = BC = CD = DA ∴ ABCD ≡ rhombus

21. Proof

Else DE cuts ⊙ then DF tan to ⊙ @ F, DF(pr) × AC @ G

∴ DF = DB GF = GC (#18)

DE = BD + CE ∴ DE + EG = EG + BD + CE

∴ DE + EG = BD + CG ∴ DE + EG = DG⤵ (2 sides Δ > 3d side 1.20)

∴ DE ! cut ⊙ Sym. DE ! outside ⊙ ∴ DE tan ⊙

Note: You must often distort the diagram for contradiction proofs.

22. Theorem

In diagram #19, ⊙ center O, join O[A-D,P-S]

ΔAOS,OAP: OA = OA, OS = OP, AS = AP (#18) ∴ ∠AOS = ∠AOP (1.8)

Sym. ∠BOP = ∠BOQ, ∠COQ = ∠COR, ∠DOR = ∠DOS

∴ ∠AOP + ∠BOP + ∠COR + ∠DOR

= ∠AOS + ∠BOQ + ∠COQ + ∠DOS

∴ ∑LHS = ∑RHS = 2L (1.15.C.2)

23. Theorem

∠MPQ = 2∠CPQ ∠NPQ = 2∠CQP (#22)

∴ ∠MPQ + ∠NQP = 2(∠CPQ + ∠CQP) = 2∟ (1.32)

∴ PM∥QN (1.28)

24. Proof

∠ABC + ∠OBC = ∟ = ∠ABC + ∠BAO (1.32)

∴ ∠OBC = ∠BAO

∠BAC = 2∠BAO ∴ ∠BAC = 2∠OBC

25. Proof

ΔEAC ≡ ΔEDC ΔFDC ≡ ΔFBC (1.5,#18)

∴ ABFE = 2ΔECF = EF•DC = ½EF•AB (DC = 1 radius, AB = 2 radii)

26. Proof

AB + CD = AL + DK + CG + FB (#18)

∴ AD + BC = AB + DC + LK + FG

LM = MK = MO NF = NG = NO

∴ AD + BC = AB + CD + 2(MN)

27. Method

⊙A,(L - M)

on B, add tangent to ⊙A,L-M (3.17) @ C

AC(pr) x ⊙A,L @ D

in ⊙B,M radius DE ∥ AD

DE required

Proof

DC = M = EB (con) ∠ACB = ∟ (3.17) ∴ ∠CBE = ∟ (1.29)

∴ CB∥DE CB = DE (1.33) ∴ DE common tangent (3.16.C.1)

Note: If you use ⊙A,L+M, tangent passes between circles

28. Method

Copy M to chord DE∈⊙DEF

⊙DEF center G (3.1)

GM⊥DE ⊙G,GM

AFH tan to ⊙ABC,⊙G × ⊙DEF @ F,H required (#27 note)

Proof

DE,FH eqD G ∴ DE = FH = M

29. Method

Copy PQ,RS to chords of ⊙A,⊙B

Describe ⊙s on A,B touching these chords

Create tangent on these inner ⊙s with secants in ⊙A,⊙B

Tangent line required

Proof

Secants equal to chords (#28)

Note: After you have established results, as in the last few problems, you can then shorten proofs by using these results in a reasoned sequence. At some point, there is no virtue in specific constructions when the relevant results are established.

30. Proof

PQ tan @ E Join C[A,B]

ΔCAE,CAM: AE = AM (#18) CM = CE (d.1.15) ∠AEC = ∠AMC = ∟

∴ ΔCAE ≡ ΔCAM (1.4) Sym. ΔBCE ≡ ΔBCN

∴ ∠ACB = ∠ACM + ∠BCN = ∟(1.13)

31. Proof

∀chords ABC,DEF, in larger, tangent to smaller @ B,E JoinO[B,E]

∠B = ∠E = ∟ (3.18) ∴ AC,DF eqD O

∴ AC = DF (3.14)

32. Method

chord AB ∈⊙O = AB radiusOC⊥chordAB ⊙O,OC

tangent from P to ⊙O,OC @ E × ⊙O,OA @ D,F PF required

Proof

AB = DF (#31)

Note: Learn to use prior results like this to shorten proofs.

33. Proof

∠OCA = ∠ODB = ∟ (3.18) ∴ ∠CAO = ∠DBO (1.32)

∠ACE = ∠AEC ∠BDF = ∠BFD (1.5)

∴ ∠AEC = ∠BFD ∴ CD∥DF (1.28)

Note: Make sure you understand the basis of every statement in every proof. Otherwise, you are doing all of this for nothing. Who cares if "you know the answer."

34. Proof

DOC one line (3.11) FO × ∥tan @ E

E∈tan∥AB ∴EF = CD (CD = radius ∥ EF)

OD = OF (d.1.15) ∴OC = OE (a.3)

35. Method

DE∥AB: D(DE,AB) = M

⊙C,(radius⊙C + M) × DE @ F,G

⊙F,M (or ⊙G,M) required

Proof

D(F,AB) = M ∴AB tan to ⊙F

D(F,C) = (M + radius C) ∴⊙F touches ext. ⊙C

Note: If AB in ⊙C, DE opp side AB from C

If AB outside ⊙C, DE •|•(AB,C)

36. Proof

Line on O ∥ CA × ⊙O @ D opp side CO from A

∴ D fixed and ABD one line

OD∥CA (con) ∠@A = ∟ (3.18)

∴ DO(pr) × AA'(pr) @ ∟

∴ DO fixed line ∴ D fixed point wrt (with respect to) A

Join B[A,D] ∠BOD = ∠BCA (1.29)

∴ ∠ODB + ∠OBD = ∠CAB + ∠CBA (1.32)

∠ODB = ∠OBD ∠CAB = ∠CBA (1.5)

∴ ∠OBD = ∠CBA ∴ DBA one line

This problem 36 should rightfully frustrate you. It's not completely clear, on the first pass, what the question even means. And even though you more or less understand the question, you are asking yourself what you even need in order to prove it. Sometimes, you have an idea of how to prove it but can't see how to say it in Euclid-speak. The point of this kind of problem is to extend your knowledge to include all these things. If it seems painful, know that any problem in life is only painful if you care. So pain is good. And if you care, mathematics continues, in this way, to be painful. Get used to it.

37. Proof

L,M mdpt AB,CD (v1.#39)

∠EBK = ∠FBK (#18)

∠FBK = ∠LKB (1.29)

∴ ∠LBK = ∠LKB LK = LB (1.6) ∴ LK = ½AB

Sym. MK = ½CD ∴ LM = ¼ ∑ sides (#19)

38. Proof

∥ line × ∀⊙ @ B,C   tan @ B × A'A @ F
AE⊥tan on B × tan on B @ E   AD⊥BC × BC @ D
∠FBA = ∠FAB (#18) ∴ ∠FBA = ∠ABD (1.29)
ΔABE,ABD: AB=AB, ∠ABE = ∠ABD, ∠AEB = ∠ADB = ∟
∴ AE = AD (1.26) ∴ tan FB ≡ tan to ⊙A,AD @ E
Sym. tan on C ≡ tan to ⊙A,AD ∴ ⊙A,AD fixed

Note: This proof works because you show that for **any** of the circles, the tangents to it are also tangent to ⊙A,AD. This is also a lesson in post-Euclid "Modern Geometry" from which projective geometry was developed.

39. Proof

Join CD. ∠CDA = ∠CAD (1.5)
∠CDA + ∠BDE = ∟ (3.18) ∠CBA + ∠CAB = ∟ (1.32)
∴ ∠BDE = ∠CBA
∠DBE = ∠CBA (1.15) ∴ ∠BDE = ∠DBE ∴ ΔBDE ≡ isosΔ (1.6)

40. Proof

OCF: OC = CF ΔPCF,PCO: PF=PO, OF=OP
∴ ΔPOF ≡ eqSΔ ∴ ∠POC = 2/3∟
∴ ∠EPA = ⅓∟ (1.32) ∴ ∠PEA = 2/3∟ (1.32)
∴ ∠CED = 2/3∟ (1.15)
∠COA = ∠OBC + ∠OCB (1.32) = 2∠OBC (1.5) ∴ ∠OBC = ⅓∟
∠BAD = ∟ ∴ ∠ADB = 2/3∟ ∴ ∠CDE = ∠CED = 2/3∟
∴ ∠ECD = 2/3∟ (1.32) ∴ ΔCED ≡ eqSΔ

Note: By this point, you can shorten equivalent Δ bits as in line 1.

41. Proof

∀P≠C∈convex side ⊙ Join P[A,B] PA × tan on C @ E Join EB
∴ PE + PB > EB (1.20) ∴ PA + PB > EA + EB
EA + EB > CA + CB (1.20) ∴ ∀P,PA + PB > CA + CB

Note: I'm rushing it a bit on the 2d (1.20). For P•|•(C,B): BD⊥tan on F:BF=FD. Join D[A,E] Now BE = DE and you can see the 1.20, if you couldn't see it before. This is a very useful result.

42. Proof

∠CQD = ∠AQB ∴ ∠AQB constant on AB (3.21)

Note: This is Todhunter's problem 205. It seems almost too trivial once you diagram it out. Maybe he just wanted you to diagram it out.

43. Proof

ΔCOA,BOD: ∠B,∠C on chord AD ∴ ∠B = ∠C (3.21)

∠O = ∠O ∴ ∠A = ∠D (1.32)

44. Proof

∠AQP,∠QRB constant on AP,PB (3.21)

∴ Δ on base QR has constant apex angle (1.32)

Note: If ∠A constant then supplement of ∠A constant.

45. Proof

4-gonABCD: ∑∠ = 4∟

∠BAC constant (con)

∠BDC constant (3.21)

∴ ∠ABD + ∠ACD = 4∟ - ∠BAC - ∠BDC = constant

46. Proof

∠RAB = ½(∠PAB + ∠QAB)

∠RBA = ½(∠PBA + ∠QBA)

∴ ∠R + ½(∠PAB + ∠QAB + ∠PBA + ∠QBA) = 2∟ (1)

½(∠Q + ∠QBA + ∠QAB) = ∟ (1.32) (2)

Sym. ½(∠P + ∠PAB + ∠PBA) = ∟ (3)

∴ (1) = (2) + (3) ∴ ∠R = ½(∠P + ∠Q) = constant (3.21)

Note: Now prove the hint.

47. Proof

∠A + ∠C = 2∟ (3.22)

∠A = ∠C (1.34)

∴ ∠A = ∠C = ∟ Sym. ∠B = ∠D = ∟ ∴ ‖gm ≡ rect∟

48. Proof

∠CAB + ∠CDB = 2∟ (3.22) Sym. ∑(∀6 ∠) = 6∟

ΔABC: ∑∠ = 2∟ (1.32)

∴ ∑ext∠ = 4∟ (a.3)

49. Proof

int∠ACB + ext∠ on AB = 2∟ (3.22)

Sym. ∑(∀4 int/ext∠) = 8∟

∠ACB + ∠CAB + ∠CBD + ∠DBA = ∑∠ΔABC = 2∟

∴ ∑ext∠ = 6∟ (a.3)

Note: Those four angles in line 3 are on the chords that make up ΔABC. If you take angles from center in this problem, then ∑(int/ext∠) = 10∟ ∑int∠ = 4∟ (= π) ∴ ∑ext∠ = 6∟.

50. Proof

∠ODC + ∠ADC = 2∟ (1.13)

∠ODC = ∠OBA = ∠ABC

∴ ∠ABC + ∠ADC = 2∟ ∴4-gonABCD cyclic

51. Method/Proof

ΔABC w/en⊙:∀∠(∠ACB) = 2/3∟

∀D∈arcAB then ∠ACB + ∠ADB = 2∟ (3.22)

2∟ - ⅓×2∟ = 2/3×2∟ (a.3) ∴ ∠ADB = 2∠ACB

52. Method/Proof

ΔABC w/en⊙:∀∠(∠ACB) = 1/6×2∟

∀D∈arcAB then ∠ACB + ∠ADB = 2∟ (3.22)

2∟ - 1/6×2∟ = 5/6×2∟ (a.3) ∴ ∠ADB = 5∠ACB

53. Proof

6-gonABCDEF: AB,BC∥DE,EF Join AD

∠FED + ∠FAD = 2∟ = ∠ABC + ∠ADC (3.22)

∠ABC = ∠FED (v1.#36)

∴ ∠FAD = ∠ADC (a.3)

∴FA∥DC (1.27)

54. Proof

∠ABC + ∠EBC = 2∟ (1.13)

∠EBC = ∠ADC (hyp)

∴ ∠ABC + ∠ADC = 2∟

∴ 4-gon cyclic (3.22 converse)

∴ ∠ADB = ∠ACB (3.21)

Note: Or: the angles from C and D, using sides and diagonals, are both on chord AB. Make sure you grasp this as an important result.

55. Proof

∠ECP + ∠EBP = 2∟ (3.22)

∠ECP + ∠ DCP = 2∟ (1.15)

∴ ∠EBP = ∠DCP = ½∠DCB

∠BEP = ∠BCP (3.21) = ½∠DCB

∴ ∠EBP = ∠BEP ∴ EP = BP (1.6)

Note: First three lines of proof are a recurring method for showing equal angles. Showing a 4-gon is cyclic is as useful as ∥gmize a Δ.

56. Proof

∠M = ∠AQB = ∠BPA ∴ chordPQ ∈ ⊙ABPQ

∴ arcABPQ constant (3.21) ∴ PQ constant

Note: Learn to see geometric elements in **all** their possible contexts. This lets you leverage the most powerful context.

57. Proof

∠P = 2∟ - (2∟ - ∠A) - ∠B (1.15,32 3.22) = ∠A - ∠B

∴ ½P = ½∠A - ½∠B

∠Q = 2∟ - ∠B - ∠A (1.32) ∴ ½∠Q = ∟ - ½∠B - ½∠A

∴ ∠E = 2∟ - (∟ - ½∠B - ½∠A) - ∠A (1.32)

∴ ∠E + ½∠Q = ∟ ∴ ∠PHQ = ∟ (1.15)

58. Proof

AD × BF @ H: H in ⊙

∠ADE = ∠ABC (3.22) ∴ ∠HDF = ∠HBA ∴ ∠DHF = ∠BHA (1.15)

∴ ∠DFH = ∠BAH (1.32) or ∠BFD = ∠BAD ∴ F∈⊙ (3.21)

Note: If AD × BF @ H outside ⊙, you can prove 4-gon ABDF is cyclic by converse of 3.22. The point of the proof, in either case, is that if ∠BAD is ∈⊙ then ∠BFD only equals ∠BAD if F∈⊙. This is an important result.

59. Proof

∠AFG = ∠BGF ∴ arcED + arcABH = arcHC + arcEAB

∴ arcHC - arcED = arcBH - arcEA

∴ HE makes equal angles with AB,DC

Note: This is a substantial result. Draw diagram. Color arcs. Figure it out. Note that if AB∥DC∥EH then ∠ = ∞. Also if EH × BC outside ⊙ then angle is ½∠(arcAC - arcBD)

60. Proof

AC × BD @ ½(arcDA + arcBC) = ½(∠DOA + ∠BOC) (#59)

∠ODE = ∠OAE = ∟ ∴ ∠DOA + ∠DEA = 2∟ (1.32)

∠DEA + ∠BGC = 2∟ (hyp) ∴ ∠DOA = ∠BGC

Sym. ∠BOC = ∠AED

∴ ∠(AC×BD) = ½(∠AED + ∠BGC) = ∟

61. Proof

AB fixed ∴ ∀ΔCAB: C∈fixed⊙ w/chordAB (3.26)

AD(pr) × ⊙ @ D'

∠ACD' = ∠BCD' ∴ AD' = BD' ∴ D' mdpt arcADB

∴ D' fixed for ∀AD(pr) ×/2 ∠C

62. Proof

∠DAE = ∠ACB + ∠AEB (1.32)

∴ ∠ACB,AEB constant (3.21) ∴ ∠DAE constant ∴ DE constant (3.26)

Note: Also true if C,D;C,E same side A,B (difference of ∠, not sum)

63. Method

⊙D,CD, × ABF @ F   FD × ⊙C @ E   arcAE = 3×arcBD required

Proof

∠ACE = ∠CEF + ∠CFE (1.32) ∴ ∠ACE = ∠CDE + ∠CFD (1.5)

∴ ∠ACE = ∠DCF + 2∠DFC (1.32) ∴ ∠ACE = 3∠DCF

∴ arcAE = 3×arcBD

Note: Recall 1.32 is ext∠ = ∑opp∠ not just ∑∠= 2∟

64. Proof

FE × OA @ H   K mdpt OF   Join EK

ΔEFK ≡ ΔEOK (1.8) ∴ ∠EKO = ∟ ∴ KE∥OH (1.28)

∴ ∠FEK = ∠EHO ∠KED = ∠EOH (1.29) ∴ ∠EHO = ∠EOH

∠GOB = ∠OGH + ∠OHG (1.32)

∴ ∠GOB = ∠OEG + ∠OHG (1.5) = ∠EOH + 2∠OHG

∴ ∠GOB = 3∠EOH ∴ BG = 3AE (3.26)

65. Proof

∠ABC = ∠BCD (3.27) ∴ AB∥CD (1.27)

66. Proof

AD(pr) × ⊙L @ F Join KD,LF

A,K,L colinear (3.1) ∠KAD = ∠KDA ∠LAD = ∠LFA (1.5)

∴ ∠KDA = ∠LFA ∴ LF∥KD (1.28)

KD⊥BC (3.18) ∴ LF⊥BC (1.29) ∴ LF ×/2 BC (3.3)

∴ arcBF = arcCF ∴ ∠BAD = ∠DAC

67. Proof

∠ACB = ∠ADB (3.27,28) ∴ BC = BD

68. Proof

AB∥CD ∴ ∠ABC = ∠BCD (1.29)

∴ arcAC = arcBD (3.26) ∴ AC=BD (3.29)

arcAC = arcBD ∴ arcAC + arcCD = arcCD + arcBD

∴ arcACD = arcBDC ∴ AD=BC (3.29)

69. Proof

ΔBOP,BOQ: OB=OB ∠BOP = ∠BOQ (hyp)
∠BPO = ∠BQO = ∟ ∴ BP=BQ (1.26)
AB=BC (3.26,29) ∴ $AB^2 = BC^2$ ∴ $AP^2 + PB^2 = CQ^2 + QB^2$ (1.47)
BP=QB (proven) ∴ $AP^2 = CQ^2$ ∴ AP=CQ
Note: Squares on lines, 1.47, and Book II propositions allow you to deal with geometry algebraically.

70. Proof

1) AB •|•(AL,AM)
BP⊥AM × AM @ P BQ⊥AL(pr) × AL @ Q
∴ PM=QL (#69) ∴ AL + AM = 2AP
AB, ∠PAB, ∠APB fixed ∴ 2AP constant
2) AB !•|•(AL,AM)
BP⊥AM × AM @ P BQ⊥LA(pr) × LA(pr) @ Q
LB=MB (#55) ∴ LQ=PM (#69) ∴ AM - AL = 2AP = constant

71. Proof

DC=DA (v1#58) ∴ C∈⊙D,DA (3.31)

72. Proof

D mdpt AB ΔADC ≡ ΔBDC (1.8) ∴ ∠ADC = ∠BDC = ∟
∴ D∈both ⊙ (#71)

73. Proof

∠ABC = ∠ABD = ∟ (3.31) ∴ BC,BD one line (1.14)

74. Proof

∀chordAC × ⊙diamOA @ B ∴ ∠OBA = ∟ (3.31)
∴ OB ×/2 AC (3.3) and B∈diamOA

75. Proof

rect∟ABCD w/en⊙ ∴ ∠ADE = ∟ ∴ AC ≡ diam⊙ABC
O center ⊙ABC rect∟ = 2ΔDAC (1.34) ΔDAC: AC fixed base
Height ΔDAC < OD unless ∠DOA = ∟
∴ max rect∟w/en⊙: AC⊥BD ∴ sides equal (1.4) ∴ square

76. Proof

∠E = ∠D = ∟ ∴ ⊙diamAC ≡ en⊙ of 4-gonACDE (#71)
∴ ∠ACE = ∠ADE same side chordAE (3.21)

77. Proof

∠ABD = ∟ (3.31) ∠BCD > ∟ (3.31)
$AD^2 = AB^2 + BD^2$ (1.47) ∴ $AD^2 = AB^2 + BC^2 + CD^2 + 2BC \bullet CE$ (2.12)

78, Proof

CD = ½AB (v1#58) ∴ B,E,A,F∈⊙D,DA
∠ECB = ½∠EDB (3.20) ∴ ∠ECB = ½∟ = ½∟ACB (3.21)
∴ EC ×/2 ∠C
Sym. ∠FCA = ½∠FDA = ½∟ ∴ FC ×/2 ∠(AC × BC(pr))

79. Proof

D,F∈⊙diamBC ∴ ∠FBC = ∠FDG
E,G∈⊙diamFD ∴ ∠FDG = ∠AEG
∴ ∠AEG = ∠FBC ∴ EG∥BC (1.28)
Note: Cyclic 4-gonFBDC: ∠FDC = sup∠FBC
∴ ∠FDG = sup∠FDC = ∠FBC
4-gon is cyclic because ∠BFC + ∠CDB = ∟ + ∟ (3.31) but these are **not** angles of the 4-gon. If they were, 4-gon ≡ rect∟ and BC not diam⊙.

80. Proof

EF∥BC ∴ ∠OEB = ∠EBC (1.29)
∠OEB = ∠OBE (1.5) ∴ ∠OBE = ∠EBC ∴ EB ×/2 ∠OBC
Sym. w/CB(pr) to D, FC ×/2 ∠OBD

81. Proof

∠M = ∠B = ∟ ∴ ∠M + ∠B = 2∟ ∴ ABCM cyclic 4-gon
∴ KF tan to ⊙ABC @ B ∴ ∠MBK = ∟ ∴ MB⊥KF
Note: KBF is one line as diag of ∥gmKF

82. Proof

DE = DF = DC (v1#58) ∴ B,E,F∈⊙D,DC ∴ D mdpt EF (3.3)

Have you noticed how often the result of v1#58 is used?

83. Proof

Join AD,CE ∠ADB = ∟ (3.31) ∠DBE = ∟ (hyp)
∴ ∠ADB = ∠DBE ∴ AD∥BE (1.27) ∴ ∠BAD = ∠CBE (1.29)
∠ADB = ∠BEC (3.31) AB = BC ∴ BE = AD
∴ DE = AB = D(between ⊙ centers) and DE∥AB (1.33)

84. Proof

∠CED = ∠CAB (3.21 both on DB)
∴ ∠CED = ∠CHB (3.21 both on CB)
∠ECK = ∠HCB (1.15) ∴ ∠EKC = ∠HBC (1.32) ∴ ∠EKC = ∟ (3.31)
Note: If you flip H and C, the same strategy will work.

85. Proof

C mdpt AM PM × QR @ D DQ(pr) to ∀E
∴ PQMR ≡ rect∟ (3.31) ∴ DQ = DP
∠EQC = ∠EQA + ∠AQC = ∠PQD + ∠AQC = ∠QPD + ∠AQC
∴ ∠EQC = ∠APM + ∠PAM = ∟ (1.32)
∴ QR tan to semi⊙AM @ Q Sym. QR tan to semi⊙BM @ R
Note: Sometimes you don't even know how geometry expresses some relation. This problem shows how we say: "QR is a common tangent."

86. Method

semi⊙AC BO⊥AC x semi⊙ @ O ⊙O,OB required
Proof
AC tan to ⊙O,OB @ B (3.16.C1) AD,CE tan to ⊙O,OB
AD = AB (#18) ∴ ΔOAB ≡ ΔOAD ∴ ∠OAB = ∠OAD
∴ ∠DAB = 2∠OAD Sym. ∠ECB = 2∠OCB
∠DAB + ∠ECB = 2(∠OAB + ∠OCB)
∠AOC = ∟ (3.31) ∴ ∠OAC + ∠OCA = ∟ (1.32)
∴ ∠DAB + ∠ECB = 2∟ ∴ AD∥CE (1.28)

87. Method/Proof (analysis)

tan from A∥tan from B ∴ ∠ACB = ∟ (#30)

D mdpt AB ∴ C∈⊙D,DA (3.31)

DE = R DE⊥AB FEG∥AB

FG × ⊙D,DA @ C

Note: Why is C∈⊙D,DA? Why is C @ FG × ⊙D,DA?

88. Proof

∠FDG = ∠FEG = ∟ (3.31) ∴ FDGE ≡ cyclic 4-gon ∴ ∠DFG = ∠DEG

∴ ∠DFG = ∠DBA (3.21)

ΔAFH,ABD: ∠A = ∠A ∠AFH = ∠ABD

∴ ∠AHF = ∠ADB (1.32) = ∟ (3.31) ∴ FH⊥AB

Note: Line 2: ∠DFG = ∠DBA. EGA one line.

∴ ∠DFG ≡ ∠DFA = ∠DBA all on chord DA.

With 3.21, always trace angles back to chords.

89. Method

AB⊥CA × ⊙ @ B B required.

Proof

A∈⊙diamCB ∀P∈⊙C,CD CP × ⊙diamCB @ Q Join AQ

∠CBA = ∠CQA (3.21) ∴ ∠CQA = ∠CPA (1.16)

∴ ∠CBA > ∠CPA for ∀P

Note: These harder problems should be thought of as solutions to study **after** you strive for your own solutions. Striving prepares the mind. Without skin in the game, you don't really care and won't really learn.

90. Proof

OG ×/2 AB @ G OH ×/2 CD @ H (3.3)

$AE^2 + EB^2 = 2(AG^2 + EG^2)$ $CE^2 + ED^2 = 2(CH^2 + HE^2)$

EG = HO EH = GO (1.34)

∴ $AE^2 + EB^2 + CE^2 + ED^2 = AG^2 + OG^2 + CH^2 + HO^2$

$= 2(OA^2 + OC^2)$ (1.47) = $diam^2$

Note: Line 2: from 2.4,2.7, AE = AG + GE, EB = AG - GE

∴ $AE^2 + EB^2 = 2(AG^2 + GE^2)$. Do the algebra...

91. Proof

Join D[A,B] ∠DAF = ∠DBA (3.32) = ∠DAB (3.27)

∴ ∠DAF = ∠DAE (DAB)

ΔDAE,DAF: ∠DAE = ∠DAF,, AD=AD ∠E = ∠F = ∟

∴ DF = DE (1.26)

92. Proof

BC(pr),PD(pr) × ⊙C @ E,F PD = DF (3.3)

ΔBPD ≡ ΔBFD (1.4) ∴ ∠BPD = ∠BFD =∠BEP (3.31)

∠BPA = ∠BEP (3.32) ∴ ∠BPA = ∠BPD ∴ PB ×/2 ∠APD

93. Proof

∠DAP = ∠PQA (3.32) = ∠QAB (1.29)

∠DPA = ∠QBA (1.13, 3.22) ∴ 3d∠s equal (1.32)

Note: line 2: ∠QBA + ∠QPA = 2∟(3.22)

∴ ∠QPA = sup∠QBA = ∠DPA (1.13)

94. Proof

PE tan to ⊙O @ P: PE,PA same side PB

∴ ∠EPA = ∠PBA (3.32)

∠PBA = ∠PCD (1.13, 3.22) ∴ EP∥CD (1.27)

95. Proof

CA(pr),DA(pr) to ∀E,F

∠CAF = ∠ABC ∠DAE = ∠ABD (3.32)

∠CAF = ∠DAE (1.15) ∴ ∠ABC = ∠ABD ∴ AB(pr) ×/2 ∠CBD (1.13)

96. Proof

Join OP(pr both ways as diams), EBF⊥OP tan both ⊙ @ B (3.16.C1)

segments on AB = ∠ABE,ABF = segments on BC (3.32)

∴ segmentsAB pairwise similar segmentsBC

Note: Compare your diagram on #96 to the diagram for 3.32. Yours doesn't have the sides of the "segment triangles" on either side of AB,BC. But segments are the same for any such sides from any point on the arcs. (3.31, 3.28, etc.)

97. Proof

∠APN = ∠AQM (3.32) ∟ANP = ∟AMQ
∴ ∠PAN = ∠QAM (1.32) ∴ ∠NAM = ∠PAQ
∟ANP = ∟AMP ∷ ANPM ≡ cyclic 4-gon
∴ ∠ANM = ∠APQ (3.21) ∴ ∠AMN = ∠AQP (1.32)

Note: Be methodical. Look at the diagram. You can see that ∠ANM=∠APQ, ∠NAM=∠PAQ, ∠PQA=∠NMA. List these out. Then list what you can see for equal ∠ using 3.32 and cyclic 4-gons. Then bring the two lists together with intermediate steps.

98. Proof

∠RPS = ∟ (3.31) ∠QPS = ∠PAB (1.15, 3.32)
∠PAB = ∠OSB (1.32) = ∠QSP ∴ ∠QPS = ∠QSP
∟RPS = ∠PSR + ∠PRS (1.32)
∠QPS = ∠QSP ∴ ∠QPR = ∠QRP
∴ QS = QP, QR = QP ∴ RQ = SQ

Note: line 3: compares ΔOAR,PSR and assumes 1.15

99. Method/Proof

On AB, seg⊙ = ∠M (3.33) DC⊥AB × arcAB @ C ∴ ΔABC required

100. Method/Proof

On AB, seg⊙ = ∠M (3.33) AD⊥AB = L DCE∥AB × seg⊙ @ C,E
ΔABC ≡ ΔABE required

101. Method/Proof

D mdpt AB On AB, seg⊙ = ∠M ⊙D,M × seg⊙ @ C,E
ΔACB ≡ ΔAEB required

102. Method/Proof

On AB, seg⊙ = ∠M D mdpt AB DC⊥AB × seg⊙ @ C
ΔCAB ≡ isosΔ center⊙ ∈ CD(pr) (3.1)
∴ line on C⊥CD is tan to ⊙ (3.16.C1)
∀P ≠ C∈arcAB: PM⊥AB × AB @ M ∴ PM < CD
∴ ΔPAB < ΔCAB for ∀P≠C (1.37)

When dealing with "area" in Euclid, go look at the propositions for "area." This is all you have and all you need. This also keeps things simple. Everything in Euclid is **defined** in Euclid.

103. Analysis

B nearer A than C ∴ OB base ΔBOC

∴ height < OC unless ∠COB = ∟ then height = OC

∴ Δ max when isosΔ (#102) ∴ ∠OBC = ∠OCB = ½∟ (1.32)

∴ On OA, seg⊙ = ½∟ seg⊙ × ⊙O @ C ∴ AC ×⊙O @ B,C

Note: Make sure you understand why ∠OCA = ½∟ and why C∈⊙O.

104. Method

On AB, seg⊙ = ∠M, seg⊙ = ½∠M

⊙A, ∑ sides × seg⊙=½∠M @ D

AD × seg⊙=∠M @ C ΔACB required.

Proof

∠ACB = ∠CDB + ∠CBD (1.32) ∠ACB = 2∠CBD (con)

∴ ∠CDB = ∠CBD ∴ CD = CB (1.6)

∴ CA + CB = CA + CD = AD = ∑ sides and AB, ∠M as required.

105. Proof

TA•TB = TP^2 = TQ^2 (3.36) ∴ TP = TQ

106. Proof

PQ × AB(pr) @ T ∴ TP = TQ (#105) ∴ AB(pr) ×/2 PQ @ T

107. Proof

∠ADC = ∠AEC ∴ AEDC cyclic 4-gon (3.21)

∴ BC•BD = BA•BE (3.36.C1)

108. Proof

AC•CB = DC•CE = FC•CG ((3.35)

D,E,F define ⊙DFE

DC•CE = FC•CG and D,E,F∈⊙ ∴ G∈⊙DFE (3.35)

Should this proof of #108 do more to show that a point on FC(pr) further or closer to C than G would not be on the circle? Does this **need** proving here?

109. Proof

K mdpt BC

$GE^2 = GB \bullet BC$ (3.36) $\therefore GE^2 = GK^2 - BK^2$ (2.5,6)

$\therefore GK^2 = GE^2 + BK^2$ $\therefore 4GK^2 = 4GE^2 + 4BK^2$

$\therefore 4GK^2 = AE^2 + BC^2$ (#106)

AE = DF (#27)

$GK^2 = GE^2 + BK^2$ $\therefore$ $HK^2 = HF^2 + CK^2$

$\therefore GK^2 = HK^2$ $\therefore$ GK = HK $\therefore 4GK^2 = GH^2 = AE^2 + BC^2$

Note: GB = GK - BK, BC = GK + BK $\therefore GB \bullet BC = GK^2 - BK^2$

110. Proof

⊙diamBC × DE(pr) @ N,O

$NE \bullet NO = DN^2 - DE^2$ (note # 109)

$DN^2 = BD \bullet DC$ (3.35)

$NE \bullet NO = AE \bullet EC$ (3.35)

$\therefore AE \bullet EC = BD \bullet DC - DE^2$ $\therefore DE^2 = BD \bullet DC - AE \bullet EC$

$FN \bullet FO = DF^2 - DN^2$ (note #109) $\therefore DF^2 = FN \bullet FO + DN^2$

$DN^2 = DC \bullet DB$ (3.35)

$FN \bullet FO = FA \bullet FB$ (3.36.C1)

$\therefore DF^2 = BD \bullet DC + FA \bullet FB$

111. Proof

∀2 ⊙s × @ A,B

Fixed T: tanTP to one ⊙ = tanTQ to other ⊙ $\therefore$ ABT one line

Else join TA TA(pr) × one⊙,other⊙ @ L,M

$TP^2 = TA \bullet TL$ $TQ^2 = TA \bullet TM$ (3.36)

But TP = TQ (hyp) $\therefore TA \bullet TL = TA \bullet TM$⤵ $\therefore$ ABT one line

Note: This problem is a lesson in stepping back and being a bit more abstract in your analysis.

112. Method

Join AC ⊙diamAC chordCD = MN∈⊙diamAC ⊙A,AD required

Proof

⊙A × BC @ E,F

∠ADC = ∟ (3.31) ∴CD tan to ⊙DEF (3.16.C1)

∴CE•CF = CD^2 = MN^2

113. Method

ChordBP = MN AP(pr) × tanCB @ T TPB required

Proof

TP•TA = TB^2 (3.36) ∴TP•PA + TP^2 = TB^2 (2.3)

∴TP•PA = TB^2 - TP^2 = PA^2 = BP^2 = MN^2 (3.31, 1.47)

114. Proof

Join AB. Add tangents at A,B

Each tan makes ∠w/AB < ∟

∴ ∠A + ∠B < 2∟ ∴tanA × tanB (a.12)

115. Proof

LA = LC ∴ ∠LAC = ∠LCA ∴ ∠LAC,LCA < ∟ (1.17)

∠LAC = ∠ABC (3.32) ∴ ∠ABC < ∟ Sym. other 2 ∠

Note: When you have to prove symmetrical things, prove the easiest and sym. the others. Just make sure they are really symmetrical.

116. Proof

∠ABC + ∠ACB < 2∟

∴ ∑(angles of bisectors) < ∠ABC + ∠ACB < 2∟

∴bisectors intersect (a.12)

Note: Todhunter, who chose these problems, has a reason for pointing out the intersection of tangents and bisectors.

117. Proof

$DA^2 = DG^2 + GA^2$ (1.47) $= DE^2 + EA^2$ ∴ $DG^2 + GA^2 = DE^2 + EA^2$

$DG^2 = DE^2$ ∴ $GA^2 = EA^2$ ∴ ∠DAG = ∠DAE ∴ DA ×/2 ∠EAG

Note: If you can't see the justification of any statement in this proof, figure out what proposition, definition, or axiom is being used.

118. Proof

GH = GF (#18) ∴ AG + GH = GF + AG

Sym. BN + NM = BF and BL + LM = BD

And so on for the entire perimeter.

In the last five proofs, you can see how the argument has been shortened in various ways. There is no virtue in a tedious proof. Your reader cannot be so stupid as to need every detail spelled out. Learn when enough of a proof is sufficient. But it must be sufficient. My approach is to make it just a little more detailed than I would need, if I were reading it.

119. Method

OA × ⊙O @ D   EDF⊥OD × AB,AC @ E,F

EG ×/2 ∠AED × AD @ G   ⊙G,GD required

Proof

Using ΔAOB,AOC we can show AG ×/2 ∠BAC

∴ ⊙G,GD ≡ in⊙ of ΔAEF (4.4) and touches ⊙O @ D

Note: Trim A[B,C] equal w/compass and ΔAOB ≡ ΔAOC (1.4)

If AO(pr) × ⊙O @ K, there is a second solution, same method.

120. Proof

OGA ×/2 ∠DAE (#117) ∴ ∠AOD = ∠AOE (1.32)

∠ADG = ∠GED (3.32) ∴ ∠ADG = ½∠DOG (3.20) = ½∠GOE

∠GDE = ½∠GOE (3.20) ∴ DG ×/2 ∠ADE Sym. EG ×/2 ∠AED

∴ ⊙G in⊙ of ΔADE (4.4)

121. Proof

BO,CO ×/2 ∠ABC,BCD   OE⊥BC × BC @ E

∴ ⊙O,OE × AB,BC,CD (4.4) ∴ ⊙O,OE in⊙

Else 1) ⊙O,OE × AD @ 2 pts

BA(pr) × CD(pr) @ S LM∥AD × SA,SD @ L,M: LM tan to ⊙O

∴ LM + BC = LB + MC (#19)

AD + BC = AB + CD (hyp) ∴ LM + BC = LA + AD + MD + BC⤵ (1.20)

2) ⊙O,OE falls short of AD

Sym. with 1)

3) 4-gonABCD ≡ ∥gm ∴ BA !× CD

Sym. with 1)

∴ AD tan to ⊙O,OE ∴ ∃ in⊙

Note: This proof is more of a lesson in adequate proofs than a solvable problem for most people at this point. If you are ambitious, prove to yourself that parts 2 and 3 are symmetrical with the first part.

122. Proof

∠ACB + ∠BCL + ∠LCM + ∠MCA = 4∟ (1.15.C1)

∠BCD,ECA = ½∠BCL,MCA (con)

∠ACB = ½(∠ACB + ∠LCM) (1.15)

∴ ∠BCD + ∠ACB + ∠ECA = 2∟

∴ D,C,E colinear (1.14)

Note: If line 4 is not obvious to you, work it out.

123. Proof

∠OBD = ∟ ∠DBC = ½∠ABC

∴ ∠OBC + ∠DBC = ∟ ∴ BO ×/2 ext∠B

∠BDO = ∠DBA + ∠DAB = ½(∠ABC + ∠BAC) (4.4)

∠OBD = ∟ (con) ∴ ∠BOD = ½∠ACB (1.32)

∴ ∠BOD = ∠BCD ∴ BDCO cyclic 4-gon ∴ ∠DCO = ∟ (3.22)

∴ CO ×/2 ext∠C ∴ ⊙O ≡ ex⊙ on BC (#122)

124. Proof

AE = AD AH = AG (#18) ∴ HE = GD

HE = HC + CE = CK + CF

GD = DB + BG = BF + BK

∴ CK + CF = BF + BK ∴ KF + 2CK = FK + 2BF ∴ BF = CK

∴ FK = BK - BF = BK - CK = BG - CH = BA - CA

125. Proof

BG = BD CH = CD AG = AH (#18) ∴ AB + BD = AC + CD

∴ AB + BD = ½ perimeter ΔABC

Sym. BA + AE = ½ perimeter ΔABC

∴ AB + BD = BA + AE ∴ AE = BD

Sym. BF = CE and CD = AF

Note: On a problem like 125, as soon as you have labelled the diagram, you can see that the solution is symmetric. So you prove one and sym. two.

126. Proof

OA x/2 ∠A (#117) Sym. DA ×/2 ∠A ∴ O∈DA

Sym. O∈EB,FC

127, Proof

SPT tan to both ⊙

BC on ⊙centers × HL,KM @ B,C ∴ ∠HBP = ∠KCP = ∟ (#27)

LT = MT = PT ∴ T mdpt LM

∴ BL + CM = 2PT (v1#89) = LM Sym. BH + CK = HK

∴ HL + KM = HK + LM ∴ in⊙ possible (#121)

Note: Lesson in usefulness of #121.

128. Proof

Produce AB,AC to D,F: ex⊙ on BC × AB,BC,AC @ D,E,F

∴ AD = AF = ½perimeterΔABC (#125)

∴ D,F constant ∴ ex⊙ on BC fixed

Note: D,F fixed on AB,AC, ∠A fixed ∴ D,F∈fixed lineDF ∴ ex⊙ fixed

A lemma is a minor proof used within another proof. When I was trying to solve this problem, the lemma seemed unfair. How are you supposed to know you need such a lemma? Obviously, the point of the problem is to learn the lemma.

129. Lemma: In diagram 4.3, we show ∠MKN = ½∠MLN + ∟
Join K[M,N] ∴ ∠KMA = ∠KMB = ½∠LMN
∴ ∠MKB + ½∠LMN + ∟ = 2∟ (1.32)
Sym. ∠NKB + ½∠LNM + ∟ = 2∟
∴ ∠MKN + ½∠LNM + ½∠LMN = 2∟
= ∟ + ½(∠L + ∠M + ∠N) (1.32)
∴ ∠MKN = ∟ + ½∠MLN
Method/Proof
DE∥PQ: D(DE,PQ) = R
On PQ, seg⊙ = ∟ + ½∠V (lemma)
∴seg⊙PQ × DE @ center in⊙
Add in⊙ Add tangents PN,QN ΔNPQ required.

130. Method/Proof
Join AB AB(pr) × EF @ C
D∈EF: CD^2 = CA•CB
⊙ABD required (3.37)

131. Proof
FB = FC ∴ ∠FBC = ∠FCB
∟FGB = ∟FGC ∴BG = CG (1.26)

132. Proof
∠(tan @ A on en⊙ΔADE × AD) = ∠AED (3.32)
∠(tan @ A on en⊙ΔABC × AB) = ∠ACB (3.32)
∠AED = ∠ACB (1.29)
∴tan @ A on en⊙ΔADE ≡ tan @ A on en⊙ΔABC

133. Proof

O center en⊙ ∴ OA = OB = OC
O center in⊙ ∴ OA,OB,OC ×/2 ∠A,B,C (4.4)
OA = OB ∴ ∠OAB = ∠OBA = ½∠BAC = ½∠ABC
∴ AC = BC (1.6) Sym. AB = AC ∴ ΔABC ≡ eqSΔ

134. Proof

∠PEC = ∠EDC (3.32)
∠EDC = ∠ABC (3.22, 1.13)
∴ ∠PEC = ∠ABC ∴ EP‖AB (1.27)
Note: ∠EDC = sup∠ADC, ∠ADC + ∠ABC = 2∟

135. Proof

PQA one line (hyp) QM,QN⊥AB,AC
ΔAQM,AQN: ∠QAM = ∠QAN (4.4) ∟AMQ = ∟ANQ AQ=AQ
∴ AM = AN (1.26)
AM = ½AB AN = ½AC (3.3) ∴ AB=AC ∴ ΔABC ≡ isosΔ
Note: Last line uses en⊙

136. Method/Proof

Join AB D∈AB(pr): AB•AD = M^2 (1.45.C1)
∴ D∈⊙ (3.36) ∴ ⊙BCD required

137. Method/Proof

Required ⊙ must be eqD from sides ∴ center in⊙ required.

138. Proof

∠A = ∠K ∴ ∠BFC = ∠LGM (3.20)
∴ ∠FBC + ∠FCB = ∠GLM + ∠GML (1.32)
∠FBC = ∠FCB ∠GLM = ∠GML (1.5)
∴ ∠FBC = ∠GLM ∠FCB = ∠GML
BC = LM (hyp) ∴ FB = GF (1.26)

139. Proof

∠APB = ½∠ACB (3.20) = ½∟ ∴ BA tan to ⊙ (3.32)
Note: Make sure you understand this short proof.

140. Proof

⊙AEB GEH tan to ⊙AEB ∴ ∠AEG = ∠ABE (3.32)
∠AEG = ∠DEH (1.15) ∠ABE = ∠ECD (1.29)
∴ ∠DEH = ∠ECD ∴ GEH tan to ⊙ECD (3.32)

141. Method

Join AB AB(pr) × PQ @ C E mdpt AB
EF⊥AB: AF > ½R ⊙F,FA
secantCKL on ⊙F: KL = R (#32)
CM∈PQ: CM = CK ⊙ABM required

Proof

⊙ABM × PQ @ M,N
∴ CM•CN = CA•CB (3.36) = CK•CL
CM = CK ∴ CN = CL
∴ MN = KL = R

142. Method

CD ×/2 ∠C × AB @ D DE⊥AC × AC @ E
M∈AC: EM = ½P ⊙D,DM required

Proof

⊙D × AC @ M,N ∴ E mdpt MN (3.3) ∴ MN = P
DF⊥CB(pr) ∴ ΔCDE ≡ ΔCDF ∴ DF = DE ∴ chord ∈CA = P (3.14)

143. Proof

Join BO ∠BAF = ∠CAF (4.4) ∴ arcBF = arcCF ∴ BF = CF (3.26,29)
∠OBF = ∠OBC + ∠FBC = ∠OBA + ∠FAC
= ∠OBA + ∠OAB = ∠BOF (1.32) ∴ BF = OF (1.6)

144. Proof

en⊙ on ΔABC ∴ CD ×/2∠C ×/2 arcAB (3.26)
ED(pr) ×/2 arcAB (3.30) ∴ D∈arcAB ∴ ACBD cyclic 4-gon
∴ ∠ACB + ∠ADB = 2∟

145. Proof

F center en⊙ on ΔABC ∴AF = BF = CF

ΔBFC: base BC fixed (hyp)

∠BFC = 2∠BAC (3.20) ∴fixed angle

∠FBC = ∠FCB ∴fixed angles (1.32)

∴ΔFBC constant ∴FB constant

Note: End of the 19th century, they tried to bring movement and the Calculus into Euclid. **Nothing** moves in Euclid. So in pure geometry, ignore movement, prove a static instance.

146. Proof

∠ABD = ∠ACD (1.5) ∠ACB = ∠AEB (3.21)

∴ ∠ABD = ∠BED ∴AB tan to ⊙BDE (3.32)

Note: This shows that the form of 3.32 forces AB to be tangent.

147. Proof

∠DRP + ∠DCP = 2∟ ∠DRQ + ∠DAQ = 2∟ (3.22)

∴ ∠DRP + ∠DCP + ∠DRQ + ∠DAQ = 4∟

∠DCP + ∠DAQ = 2∟ (1.13, 3.22)

∴ ∠DRP + ∠DRQ = 2∟ ∴PRQ colinear (1.14)

148. Method/Proof

AB(pr) × CD @ C EC: EC^2 = AC•BC ⊙ABE required (3.37)

Note: E can be either side of C, so 2 solutions. If AB∥CD,
F mdpt AB FE⊥CD x CD @ E ⊙ABE required. And, yes, this is #130; I'm repeating it for a reason.

149. Method/Proof

Centers of both⊙ on line AB≡×/2 angle of given lines.

In given ⊙, chordPQ⊥AB subtending ∠M

⊙ on P,Q,point of tangency to either given line required. (#148)

Note: (top of next page)

The difficulty in #149 is the depth of reasoning required. It must encompass the equidistance of the lines, the necessity governing the centers, the need for PQ to be ⊥ to AB. The first level of solving problems is to hammer down on each problem and study the solutions. The next level comes when the problems put enough geometry in your head that you have a sense of what is necessary.

150. Proof

TE tan to ⊙AC @ E ∠TED = ∠DBE (3.32)
∠TEC = ∠DAE (3.32) ∴ ∠DEC = ∠AEB (1.32)
Note: 1.32: ∠A + sup∠B + (∠B - ∠A) = 2∟ and diagram sym.

151. Proof

tan on en⊙ @ C × AB @ O
∴ ∠OCE = ∠CDE (3.32) = ∠CBA (3.21)
∴ OC = OB (1.6) ∴ O center semi⊙
∴ tan @ C on en⊙ ⊥ tan @ C on semi⊙
Sym. for tangents @ D

152. Proof

∠CED = ∠ECD (1.5)
∠ECD = ∠ECB + ∠BCD = ∠ECB + ∠BAC (3.32)
∠CED = ∠ECA + ∠BAC (1.32) ∴ EC ×/2 ∠ACB

153. Proof

P,S on line on mdpt OA and is ⊥OA (4.5)
∴ PS⊥AC Sym. QR⊥AC ∴ PS∥QR (1.28) Sym. PQ∥SR
Note: This is like those problems with series of circles. You have a lot of objects in the diagram. Focus on the goal. ∥gm has two pairs of ∥ sides. So a rough sketch will do to determine which points relate to what. Prove one pair; sym. one pair.

154. Proof

rect∟ ABCD AC × BD @ O ∴ O mdpt AC,BD (1.34)
OB = OC (v1#58) ∴ OA = OB = OC = OD ∴ ⊙O,OA ≡ en⊙

155. Proof

rect∟ABCD w/in⊙ AB+CD = BC+DA (#19) ∴ 2AB = 2BC
∴ AB = BC ∴ rect∟ ≡ square

156. Proof

∠B = ∟ ∴ AC ≡ diam⊙ ∴ ∠APC = ∟ (3.31) Sym. for ∠BPD
∴ $AP^2 + PC^2 = AC^2$ (1.47) Sym. $BP^2 + PD^2 = BD^2$
∴ $AP^2 + PC^2 + BP^2 + PD^2 = AC^2 + BD^2 = 2diam^2$

157. Proof

MB = MC (#18) ∴ ∠MOB = ∠MOC (1.8)
∴ MO ×/2 ∠BOC Sym. KO ×/2 ∠AOD ∴ ∠MOB = ∠KOD
ΔMOB,KOD: ∠MOB = ∠KOD OB=OD ∟MBO = ∟KDO ∴ BM=DK
LB = LD (#18) ∴ LM = LK
Sym. LK = LM = MN = NK ∴ KLMN ≡ rhombus

158. Proof

∠ACD = ∠BDC + ∠CBD (1.32) ∠BDC = ∠BAD (4.10)
∠CBD = 2∠BAD (4.10) ∴ ∠ACD = 3∠BAD

159. Proof

ΔBCD,BAD: base∠ = 2 vertex∠ (4.10)
ΔACD: base∠ = 1/3 vertex∠ (#158)

160. Method/Proof

∠MKL,MLK = ∠CAD in 4.10 ∴ ∠MKL,MLK = ∠CAD,ADC in 4.10
∴ ∠KML = ∠ACD (1.32) ∴ ∠KML = 3∠MKL,MLK (#158)

161. Proof

⊙A,AC × ⊙ACD @ D,F ∴ AF=AD ∴ ∠AFD = ∠ADB (3.32)
∠ADF = ∠AFD (1.5) ∴ ∠FAD = ∠BAD (1.32) ∴ ∠ADF = 2∠DAF
DG ×/2 ∠ADF × ⊙A,AC @ D,G
∴ ∠ADG = ∠GDF = ∠FAD = ∠DAC = ∠ADC
∴ CD side regular 5-gon in ⊙ACD

162. Proof

E ≡ F in #161 ∴ ∠EAD = ∠DAC (#161) ∴ DE = DC (3.26)

Note: This is an example of leveraging all you can from what you have already done.

163. Proof

∠BAG = 2∠BAD (#162) ∠ABD = 2∠BAD (4.10)
∴ ∠BAG,ABG = 2∠AGB

164. Proof

∠GCH = ∠BCD (1.15) = ∠ADB (4.10)
∠GHD = ∠GBD (3.21) ∴ ∠HGC = ∠BAD (1.32)
∴ ∠GHC,HCG = 2∠HGC

165. Proof

⊙A,AC × ⊙ACD @ D,E ∴ ΔBAD ≡ ΔEAD (#162)
∴ en⊙ΔACD ≡ en⊙ΔABD

166. Proof

∠DAE = ∠DAC (#162) ∴ ∠DAE = ∠ADC ∴ AE∥DC (1.27)
∠CDB = ∠CAD (4.10) = ∠ECD (#162) ∴ BD∥EC (1.27) ∴ CDGE ≡ ∥gm

167. Proof

⊙A,AC × ⊙ACD @ D,E ∴ ∠DFE + ∠DAE = 2∟ (3.22)
∠BAD = ∠DAE (#162)
∴ ∠DFE = 2∟ - ∠BAD = ∠CBD + ∠BDA = 2∠CBD
∴ ΔACD,FDE ≡ isosΔ ∴ CD=DE ∠COD = ∠DFE ∴ DF=CO (1.26)

168. Proof

eqS n-gonABCD... w/en⊙ We show ∠ABC = ∠BCD.
∠ABC subtends ⊙ - (arcAB + arcBC)
∠BCD subtends ⊙ - (arcBC + arcCD)
AB=CD (hyp) ∴ arcAB = arcCD (3.28) ∴ ∠ABC = ∠BCD (3.27)
Sym. remaining angles.

169. Proof

Add en⊙ ∴ ∠BFC = ½(arcAE + arcBC) ∠FEC = ½(arcCD + arcDE)
∴ ∠FBC = ∠BFC ∴ BC = FC = AB
∠ABF = ∠BAF ∴ AF = BF ∴ AC = AF + FC = BF + AB

170. Proof

CB=CD ∴ ∠CBD = ∠CDB (3.26) ∴ ∠CAB = ∠BCA
∠ABC = BCD ∴ ∠CBD = ∠CDB = ∠BAC = ∠BCA (1.32)
∴ outer Δs ≡ isosΔ with equal apex angles (1.6)
∴ ∠s of inner 5-gon equal (1.15)
AK = CL BL = DM ∴ KL = LM = MN = NO = OK
∴ sides inner 5-gon equal ∴ inner 5-gon regular

171. Proof

Add en⊙ Join A[C,D] CF∥AB × AD @ F
∴ AD∥BC (#65) ∴ ABCF ≡ ∥gm ∴ ΔABC ≡ ΔAFC (1.34)
∴ ΔAED ≡ ΔABC ∴ 5-gon > 3ΔABC
AC=AD AB+BC > AC (1.20) ∴ AB+BC > AD
BC=AF (1.34) ∴ AB > FD ∴ AF > FD ∴ ΔCFD < ΔCAF ∴ 5-gon < 4ΔABC
Note: 5-gon + (ΔCAF - ΔCFD) = 4ΔABC

172. Method/Proof

OD⊥BC × BC,⊙O @ D,E ∴ BE = EC Sym. remaining sides
∴ 6-gon ≡ eqS ∴ 6-gon in en⊙ ≡ eq∠ (#169) ∴ regular 6-gon

173. Proof

1) ΔBOD,COD: OD ×/2 ∠BOC ∴ ∠BOD=∠BAC (3.20) ∴ ∠BOE=2/3∟
∠OEB = ∠OBE = 2/3∟ ∴ BE = BO
2) ΔOBC ≡ ΔEBC ∴ 6-gon = 2ΔABC

174. Method/Proof

Inscribe reg. 15-gon Join vertices 1,3 3,8 8,1 Δ1,3,8 required
Note: These later solutions are minimized. Your solutions should be sufficiently detailed to defend themselves.

Notation

Labelling is done top to bottom, left to right; or clockwise from top-left apex of non-triangular figure. Labelling in propositions follows that of the original 1867 diagrams.

Operators

intersect, cut	×
bisect, bisector	×/2
trisect	×/3
at	@
parallel	∥
between	·|·
A between B and C	A·|·(B,C)
perpendicular	⊥
AB perpendicular to CD	AB⊥CD
equivalent, equal in every way	≡
equal in magnitude	=
on	∈
not on	∉
equilateral (equal sides)	eqS
equiangular	eq∠
equidistant	eqD
distance from A to B	D(A,B)
absolute difference	~
|a-b|	~(a,b) or a~b
summation	∑
A+D+C+D	∑[A-D]

Points

on or endpoints of lines	A, B, C, ...
considered in themselves	P, R, S, ..
as center of a figure	O

Lines

by endpoints	AB
creation from points	Join AB
Join AB, AC, AD	Join A[B-D]
mid-point	mdpt
P mdpt AB, Q mdpt CD	P,Q mdpt AB,CD

Angles

angle	∠
interior angle	int∠
exterior angle	ext∠
alternate angle	alt∠
opposite angle	opp∠
right angle	∟

Triangles

triangle	Δ
right triangle	⊿
∀triangle	ΔABC
equilateral triangle	eqSΔ
equiangular triangle	eq∠Δ
isosceles triangle	isosΔ
CF bisector of angle C	CF ×/2 ∠C
AD median on angle A	AD med∠A
BE altitude on angle B	BE alt∠B

Circles

circle	⊙
create by center and radius	⊙A,AB
as existing circle	⊙A
as defined by three points	⊙ABC
touching center	on center
on circumference	∈⊙, on⊙
in circle's whitespace	in ⊙
line PQ tangent @ point A	tangent PAQ
circumcircle	en⊙
inscribed circle	in⊙
escribed circle	ex⊙

Polygons

polygon	n-gon
by number of sides (4+)	4-gon
parallelogram	‖gm
rectangle	rect∟
rectangle, sides AB,CD	AB•CD
square on line AB	AB^2

Logic

therefore	∴
symmetrically	Sym.
by hypothesis	(hyp)
by construction	(con)
contradiction	↴
any, every, each, all	∀
exists, exists uniquely	∃, ∃!
not, not equivalent	! !≡
if and only if	iff

Euclid's Axioms, Postulates, and Definitions

All of the following are from Loney's last edition of Todhunter's Euclid. Their numbering differs slightly from another version of Todhunter's. And looking around, there is no conclusive numbering. All are close. Beyond that, you will find that there is a bit of back and forth between axioms and postulates from text to text as well. Corollaries date from the 17thC and can differ from text to text. **The numbering of the propositions is Euclid's and are the same in all Euclid texts.**

Euclid's Axioms
Book I
a.1 Things equal to the same thing are also equal to one another.
a.2 Things added to equals make equals.
a.3 Things taken from equals leave equals.
a.6 Things twice the same thing are equal to each other.
a.7 Things half of the same thing are equal to each other.
a.8 The whole is greater than its part.
a.9 Magnitudes which can be made to coincide are equal.
a.10 Two lines cannot enclose a space. They must have 0, 1, or all points in common.
a.11 All right angles are equal.
a.12 If a line cut two other lines such that, on one side of the first, the other two make angles summing to less than two right angles, the lines, extended on that side, must intersect.

Euclid's Postulates
p.1. A line may be drawn between any two points.
p.2. A line may be indefinitely extended.
p.3. Any point and any line from it may be used to construct a circle.

Euclid's Definitions

Book I

d.1.1 A **point** is position without magnitude.

d.1.2 A **line** is length without breadth.

d.1.3 The **extremities** and **intersections** of lines are points.

d.1.5 A **surface** is length and breadth.

d.1.6 The **boundaries** of surfaces are lines.

d.1.7 A **plane** is a surface such that, for any two points, their line lies entirely on the surface.

d.1.8 A **plane angle** is the inclination of two lines to one another which meet on the plane.

d.1.9 A **plane rectilinear angle** is the plane angle of two straight lines which meet at their **vertex**.

d.1.10 When a line meets another so that the two angles created by the former on one side of the latter are equal, these are **right angles** and the lines are **perpendicular**.

d.1.11 An **obtuse angle** is greater than a right angle.

d.1.12 An **acute angle** is less than a right angle.

d.1.13 A **plane figure** is any shape enclosed by lines, which are its perimeter.

d.1.15 A **circle** is a plane figure bounded by its **circumference** which is equidistant from its **center**.

d.1.20 A **triangle** is bounded by three straight lines. Any of its angular points can be its **apex** which is opposite its **base**.

d.1.22 A **polygon** or **n-gon** is a plane figure with n lines for sides. A figure with 4 sides is a 4-gon or "quadrilateral."

d.1.23 An **equilateral triangle** has three equal sides.

d.1.24 An **isosceles triangle** has two equal sides.

d.1.29 **Parallel lines** are coplanar lines which cannot be produced to intersect.

d.1.30 A **parallelogram** is a 4-gon of opposing parallel sides.

d.1.31 A **square** is an eqS 4-gon with one right angle.

d.1.33 A **rhombus** is an eqS 4gon with no right angles.

Book II

d.2.1 ∀rectangle ABCD is **contained** by any two adjacent sides. In our notation, this is "rectangle ABCD ≡ AB•AD".

d.2.2 In a ‖gm, there are two internal ‖gms on a diagonal and two complements. The complements combined with either internal ‖gm is a **gnomon**.

d.2.3 ∀AB produced in both directions: if we choose a point (cut) between A and B, we divide AB **internally**. If we choose a point to either side, outside of AB, we divide AB **externally**.

Book III

d.3.1 **Equal circles** (⊙) have equal radii, therefore equal diameters.

d.3.2 A line **touches** a ⊙ if it meets the ⊙ and, produced, does not cut it. This is a **tangent (tan)** with its **point of contact**.

d.3.3 ⊙s **touch** when they meet but do not cut each other. If ⊙A is in ⊙B they touch **internally**, else **externally**.

d.3.4 A line cutting a ⊙ at two points is a **secant**.

d.3.5 A **chord** is a line joining two points ∈⊙. A secant produces a chord.

d.3.6 Chords are **equidistant** (eqD) from ⊙ center when their perpendiculars (⊥) from their midpoints to ⊙ center are equal. Of two chords, the one with the greatest ⊥ is **farther** from center.

d.3.7 A **segment** of a ⊙ is a chord and what it cuts off, away from ⊙ center. Segments of circles are **similar** if their angles are equal.

d.3.8 A **segment's angle** is contained by any point ∈⊙ joined to the endpoints of its chord. This gives a segment two angles.

d.3.9 Any part of a ⊙'s circumference is an **arc**.

d.3.10 A **sector** of a ⊙ is bounded by two radii and the arc between them.

d.3.11 ⊙s with same center are **concentric**.

Book IV

1. An n-gon is **inscribed** in another n-gon when every vertex of the first n-gon is on the side of the second.
2. An n-gon is **described** on another n-gon when every vertex of the second n-gon is on a side of the first.
3. An n-gon is **inscribed in a ⊙** when all of its vertices lie on the ⊙.
4. An n-gon is **described on a ⊙** when all of its sides are tangent to the ⊙.
5. A ⊙ is **inscribed** in an n-gon when all sides of the n-gon are tangent to the ⊙.
6. A ⊙ is **described** on an n-gon when all vertices of the n-gon lie on the ⊙.
7. A line is **placed in a ⊙** when it is made a chord of the ⊙.