H.D. Ikramov

LINEAR ALGEBRA

Problems Book

ИЗДАТЕЛЬСТВО • МИР •

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ЗАДАЧНИК ПО ЛИНЕЙНОЙ АЛГЕБРЕ

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Contents

Preface

An integrated course on linear algebra and analytical geometry is given to students of the computational mathematics and cybernetics department (CMC) of Moscow university (MU) during the first two terms of their study (at two lectures a week). A lecturer on the subject is presented with some complicated problems and to clarify these problems, let us make some comparisons between this course and the syllabi of similar courses in mathematics departments, and in particular the mechanics-and-mathematics department of Moscow university.

The CMC department syllabus includes the greater part of the analytical geometry course given at the mechanics-and-mathematics department of MU (excluding the affine classification of second order lines and surfaces and the projective geometry elements) and the whole linear algebra course. The latter comprises, inter alia, topics usually omitted at the mechanics-and-mathematics department like, for example, singular values of the operators, pseudo solutions of systems of linear equations, etc. The specific character of the department requires that a lecturer should draw the students' attention to the flexibility of most concepts of classical linear algebra (linear dependence, degeneracy, Jordan structure, etc.) and of its methods, as well as indicate the approaches for finding stable solutions to algebraic problems. To achieve this aim, elements of normal linear space theory are introduced into the course in such a way that the concrete metric results, such as evaluations of the perturbations of a linear system solution and the eigenvalues of a matrix, etc., can be attained later. This must all be achieved in less than the time taken by the algebra and geometry courses together at the mathematics departments, and moreover, without lowering the degree of mathematical rigor!

It is clear that without some essential restructuring of the customary course this would not be possible and it was V. Voyevodin who made such an attempt to carry out this restructuring in his book *Linear Algebra* (Mir Publishers, 1983). The book is based on the author's experience delivering a course of lectures for some years at the CMC department.

Let us indicate some of the particulars of the course devised by V. Voyevodin that helped to save his time as a lecturer.

The notion of linear spaces is easier to grasp after a study of vector algebra and so is introduced at the very beginning of the course. The customary way of repeated reintroducing linear space theory three times usually—first in analytical geometry in relation to sets of geometric vectors; then for arithmetic spaces in order to describe the structure of the solution sets of linear algebraic equation systems; and finally, in the general case—is avoided.

In the subsequent chapters, too, the development of geometry and algebra takes place simultaneously; furthermore, a new geometric notion forms the basis for an n-dimensional general case. Thus, the scalar product of geometric vectors serves to introduce Euclidean and unitary spaces and a formula for the volume of a three-dimensional parallelepiped advances the construction of n-dimensional volume theory. Thus, the theory of determinants is considered as an oriented volume of a parallelepiped in arithmetic space; straight lines and planes in three-dimensional space are a reason to introduce the notion of a plane in any linear space, and a geometric problem on intersecting hyperplanes illustrates the structure of the solution set of a system of linear equations. By contrast there are some examples when geometric results are deduced as simple corollaries of general algebraic theorems, such as the Cartesian classification of second order lines and surfaces.

Redesigning of the course of lectures also resulted in a considerable reordering of seminar classes. It turned out, moreover, that the existing problem books in linear algebra by D. Faddeev and I. Sominsky, *Problems in Higher Algebra*, Mir Publishers, 1972 and by I. Proskuryakov, *Problems in Linear Algebra*, Mir Publishers, 1978 could only be used to a very limited extent. Both of the above-mentioned books assume that when solving problems on linear and Euclidean spaces the student is already acquainted with matrix algebra and systems of linear equations. This, as shown above, is not always true in our case; besides, problems were required on the nontraditional topics of the course. All this stipulated the necessity of a new problem book to accompany V. Voyevodin's course, and this book is now offered to the reader.

The present book closely follows the structure of the book by V. Voyevodin with some insignificant deviations demanded by the particulars of the course of study. Thus, since the corresponding topic of the course of lectures is studied at the very end of the first term, seminar classes cannot keep up with the course and so the section devoted to metric spaces is included in Chapter 8.

The sequence of topics chosen in Voyevodin's course creates certain difficulties for the author of the problem book. For example, how can the computational problems of the first two chapters be solved when matrices and the greater part of the results derived by the theory of systems of linear equations cannot be used? It happens, however, that for the solution of typical computational problems on linear and Euclidean spaces, it is sufficient to combine elementary transformations of vector spaces with the method of Gauss elimination and thus a method to check consistency and definiteness and a technique for finding any solution of a system of linear equations are obtained (see the particulars in Secs. 1.0 and 2.0). Accordingly, Gaussian elimination is described in Voyevodin's book in Chapter 2: just when it is required for the seminar classes. Problems involving all the solutions of a system of linear equations are given in the problem book only from Chapter 4 on. Note that we are guided here by the same principles as A. Kurosh in *Higher Algebra*, Mir Publishers, 1980. He starts with the description of the method of successive eliminations of unknowns.

The reader will note that the first six chapters of the problem book and some sections of Chapter 7 are devoted to customary topics. But here too, because of the specific character of the CMC, the author has striven to underline the computational aspects of the topics under consideration. Consequently, a great deal of attention is paid to the considerable number of questions that arise in practical computations using the Gauss method. Therefore, in some cases computational algorithms effectively employed in practice have been formulated as a series of problems.

A number of sections in the last two chapters correspond to the new topics in Voyevodin's course and for the first time are included in a problem book on linear algebra.

It is a basic requirement that any problem book should contain a sufficient number of useful and comprehensive problems for seminar classes, home-assignments, tests and examinations. The author hopes that this requirement has been fulfilled. Moreover, he has attempted to supply the strongest students with a material for personal study, and to lead them to problems currently faced in computational algebra. Thus, he has included Wilkinson's hypothesis regarding the rate of growth of the elements in the Gauss method (Sec. 3.4), the description of the Strassen algorithm for the economical multiplication of matrices (5.4), the results obtained by Wilkinson regarding ill-conditioned eigenvalues (Sec. 8.4), and so on.

And now some notes on the use of this problem book.

The number of each problem in a section consists of three parts, the first indicating the chapter number, the second the section number, and the third the number of the problem. Formulae that may be referred to afterwards are enumerated similarly but separately.

For the reader's convenience each chapter is preceded by a "zero" section defining the concepts and, in some cases, methods used in that chapter. A number of terms are, however, also defined in the problems themselves. To make it easier for the reader to find "the origin" of some term or other there is an index at the end of the book.

The asterisks marking some of the problems should be treated as the "Attention" sign. In case of a problem requiring proof this means that it either states an important fact (irrespective of the complexity of the proof) or requires some nonstereotyped reasoning. A problem requiring computation marked by the asterisk allows for a non-stereotyped solution based, typically, on a theoretical statement. Many problems marked by asterisks are supplied with either hints or complete solutions. Anyway, the key to the solutions of all the problems may be found either in this problem book or in Voyevodin's textbook. Hints or solutions are also given for many problems having no asterisk so as to demonstrate what is, from the author's point of view, the most rational approach to such problems. In addition, problems are, as a rule, grouped logically, the asterisked one being "the leader" in a group and the others being simple corollaries to it. The position of a problem, therefore, also contains some information about it.

Whilst compiling this problem book the author resorted to many sources and it is unfortunately impossible to mention them all here. The author's task was simplified to a very great extent by the availability of a number of excellent textbooks in linear algebra by Soviet mathematicians and, in particular, by the problem books that have already been mentioned. In some cases statements borrowed from the current, specialised literature were formulated as problems.

The initiative in writing this book was taken by Prof. V. Voyevodin and Prof. I. Berezin. The author is glad to have the chance to express his profound gratitude to them. He considers it his pleasant duty to also record his gratitude to the higher algebra lecturers in the CMC department for their valuable assistance.

H. Ikramov

CHAPTER 1

Linear Spaces

1.0. Terminology and General Notes

A set V is called a *linear space over* a number *field* P if:

A. For the elements of this set, the *operation of addition* is defined so that V is a commutative (Abelian) group. This means that the following conditions are fulfilled:

(i) The operation of addition is commutative, i.e.

$$x + y = y + x.$$

(ii) The operation of addition is associative, i.e.

$$(x + y) + z = x + (y + z).$$

(iii) There exists in V the (unique) *null element* 0 satisfying, for every element x of V, the equality

$$x + 0 = x.$$

(iv) For each element x from V there is a unique *inverse element* $-x$ such that $x + (-x) = 0$.

B. The *operation of multiplication by a number* from P is defined on the elements of the set V so that for any elements x and y from V, and for any numbers α and β from P the following conditions are satisfied:

(i) $\alpha (x + y) = \alpha x + \alpha y$.

(ii) $(\alpha + \beta) x = \alpha x + \beta x$.

(iii) $(\alpha\beta) x = \alpha (\beta x)$.

(iv) $1 \cdot x = x$.

The elements of a linear space are said to be *vectors*, and the linear space itself is also called a *vector space*.

If P is the field of real or complex numbers, then the linear space over P is said to be *real* or *complex*, respectively.

In this book, with the exception of some problems in Chapter 1, only real and complex linear spaces are considered.

In a particular case, the space V may contain only one element (see Problem 1.1.1). Such a linear space is said to be the *null* (or *trivial*) space and is denoted hereafter by O. All other real or complex spaces contain infinitely many elements.

The vector $y = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k$ is said to be a *linear combination* of vectors $x_1, x_2, \ldots, x_k$ or to be *linearly expressed* in terms of these vectors. The set of all linear combinations of a fixed set of vectors $x_1, \ldots, x_k$ is called the *span* of this system and is denoted by $L(x_1, \ldots, x_k)$.

The set of vectors $x_1, \ldots, x_k$ is said to be *linearly dependent* if at least one of the vectors x_i can be linearly expressed in terms of the other vectors of the set, and *linearly independent* if otherwise. This definition is equivalent to the following: a set of vectors $x_1, \ldots, x_k$ is linearly dependent if there exist the numbers $\alpha_1, \ldots, \alpha_k$, at least one of them being different from zero, such that

$$\alpha_1 x_1 + \ldots + \alpha_k x_k = 0,$$

and linearly independent if the indicated equality holds only in the case when all α_i are zeroes.

In particular, the linear dependence of a set of two vectors x, y means that either $y = \alpha x$ or $x = \beta y$. In this case the vectors x and y are known as *collinear*.

The following *basic theorem about linear dependence* is true: if each of the vectors of a linearly independent set $y_1, \ldots, y_l$ is linearly expressed in terms of the set $x_1, \ldots, x_k$, then $l \leqslant k$.

A linearly independent set of vectors $e_1, \ldots, e_n$ in whose terms any vector of a space V can be expressed is called a *basis* for this space. The linear space is said to be *finite-dimensional* if it has a basis and *infinite-dimensional* if otherwise.

Beginning with Sec. 1.4, we shall consider only finite-dimensional linear spaces.

All bases for a finite-dimensional space V contain the same number n of vectors. The number n is called the *dimension* of the space V and is denoted by $\dim V$. Moreover, V itself is then called an *n-dimensional* space. By definition, $\dim O = 0$.

The coefficients $\alpha_1, \ldots, \alpha_n$ in the decomposition of a vector x in terms of the vectors in a basis $e_1, \ldots, e_n$, i.e. in

$$x = \alpha_1 e_1 + \ldots + \alpha_n e_n,$$

are called the *coordinates* of the vector x.

Two linear spaces over the same field are said to be *isomorphic* if there is a one-to-one correspondence between their vectors such that the image of the sum of two vectors is the sum of their images, and the image of the product of a vector by a number is the product of the image of this vector by the same number. The necessary and sufficient condition for isomorphic correspondence between two linear spaces is coincidence of their dimensions.

A subset L of a linear space V is called a *linear subspace* of this space if it is also a linear space under the operations introduced on V.

If L_1 and L_2 are linear subspaces of a space V then the set of vectors belonging both to L_1 and L_2 is called the *intersection* of the subspaces L_1 and L_2, and is denoted by $L_1 \cap L_2$. The set of all the sums $x_1 + x_2$, where $x_1 \in L_1$ and $x_2 \in L_2$, is called the *sum* of the subspaces L_1 and L_2. We denote the sum of the subspaces by $L_1 + L_2$. If for each vector x from $L = L_1 + L_2$, the representation

$$x = x_1 + x_2,$$

where $x_1 \in L_1$, $x_2 \in L_2$, is unique, then L is called the *direct sum* of the subspaces L_1 and L_2, and is denoted by $L_1 \dot{+} L_2$.

Most of the computational problems in this book are formulated for two particular linear spaces. Let us describe them in more detail.

1. The n-dimensional arithmetic space. The elements of this space are ordered sets of n numbers, whether real or complex, and are called *n-dimensional vectors*. Respectively, we shall speak of the real or complex arithmetic space, and denote them by R_n and C_n. If the n-dimensional vectors are written in the form

$$x = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad y = (\beta_1, \beta_2, \ldots, \beta_n)$$

then the operations over them are defined by the equalities

$$x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n),$$
$$\lambda x = (\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n).$$

Of the bases of the arithmetic space, one is placed in a favoured position by the very nature of this space. This basis, formed by the *unit vectors*

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0), \\ e_2 &= (0, 1, 0, \ldots, 0), \\ &\cdots\cdots\cdots \\ e_n &= (0, 0, 0, \ldots, 1), \end{aligned} \tag{1.0.1}$$

is called the *standard basis* for the arithmetic space. The "favour" is the absence of necessity to calculate the coordinates of a vector $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in this basis since they are the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

2. The space of polynomials of degree $\leqslant n$. A polynomial of degree k

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k, \quad a_k \neq 0 \tag{1.0.2}$$

is an object fully determined by the ordered set of coefficients $a_0, a_1, \ldots, a_k$. The equality of two polynomials is the equality of corresponding coefficients. Meanwhile, the coefficients of the polynomial may be real or complex numbers. In the problems, as a rule, only the first case is considered. The space of polynomials of degree $\leqslant n$ with real coefficients is denoted in this book by M_n. Real or complex numbers themselves are regarded to be polynomials of zero degree, except the number zero whose degree is not defined. This number serves as the null element in the polynomial space. The operations over polynomials are reduced to the same operations carried over their coefficients.

A polynomial (1.0.2) may be regarded as a function of a real or complex variable t. The definition of equality of two functions, however, differs from the "algebraic" definition of the equality of polynomials. Namely, functions are considered to be equal if their values are equal for all the values of the variable. Certainly, polynomials, equal in the sense of "algebraic" definition, are also equal as the functions of t, but the converse is established only at the end of Chapter 4. Therefore the notation $f(c)$ should be interpreted as a short form of writing the number $a_0 + a_1c + a_2c^2 + \ldots + a_kc^k$; the equality $f(c) = d$ as a short-hand way of writing the condition imposed on the coefficients of the polynomials considered; $f(-t)$ as a contracted designation of the polynomial $a_0 - a_1t + a_2t^2 + \ldots + (-1)^k a_kt^k$; and the equality $f(t) = f(-t)$ as the short-hand for the conditions $a_1 = 0$, $a_3 = 0, \ldots$, etc.

The following computational problems, stated for the arithmetical space, are typical of the present chapter.

1. Determine whether the given set of vectors is linearly dependent or linearly independent.
2. Find the maximum number of linearly independent vectors contained in a given set, i.e. its rank.
3. Determine whether a vector x is expressed in terms of the set of vectors $y_1, \ldots, y_k$ in which case calculate the coefficients of this decomposition, i.e.

$$x = \alpha_1y_1 + \ldots + \alpha_ky_k.$$

To solve Problems 1 and 2, "the method of elementary transformations" (see 1.2.17, 1.2.18) is developed. The idea of the method is to reduce a given set without changing the rank to a set of vectors whose linear independence or rank is self-evident.

Problem 3 reduces to the solution of a system of linear equations for which the *Gauss method* or the *method of successive elimination of unknowns* is performed. The idea of the method lies in transforming the system to its simplest form without affecting the solution set of the system. Let us describe the Gauss method in more detail bearing in mind also its numerous applications in the subsequent chapters.

Let a system of linear equations be given:

$$\begin{aligned} a_{11}^{(0)}x_1+a_{12}^{(0)}x_2+a_{13}^{(0)}x_3+\ldots+a_{1n}^{(0)}x_n&=b_1^{(0)},\\ a_{21}^{(0)}x_1+a_{22}^{(0)}x_2+a_{23}^{(0)}x_3+\ldots+a_{2n}^{(0)}x_n&=b_2^{(0)},\\ a_{31}^{(0)}x_1+a_{32}^{(0)}x_2+a_{33}^{(0)}x_3+\ldots+a_{3n}^{(0)}x_n&=b_3^{(0)},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m1}^{(0)}x_1+a_{m2}^{(0)}x_2+a_{m3}^{(0)}x_3+\ldots+a_{mn}^{(0)}x_n&=b_m^{(0)}. \end{aligned} \tag{1.0.3}$$

Assume that $a_{11}^{(0)} \neq 0$. This can always be achieved if there are some $a_{ij}^{(0)}$ among the coefficients different from zero, and reordering, if necessary, the equations of the system and/or changing the enumeration of the unknowns. Now, subtract both sides of the first equation, premultiplied by $a_{21}^{(0)}/a_{11}^{(0)}$ from the corresponding sides of the second, then subtract both sides of the first equation, premultiplied by $a_{31}^{(0)}/a_{11}^{(0)}$, from the third equation, etc. We obtain, in the end, a system in the following form:

$$\begin{aligned} a_{11}^{(1)}x_1+a_{12}^{(1)}x_2+a_{13}^{(1)}x_3+\ldots+a_{1n}^{(1)}x_n&=b_1^{(1)},\\ a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+\ldots+a_{2n}^{(1)}x_n&=b_2^{(1)},\\ a_{32}^{(1)}x_2+a_{33}^{(1)}x_3+\ldots+a_{3n}^{(1)}x_n&=b_3^{(1)},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m2}^{(1)}x_2+a_{m3}^{(1)}x_3+\ldots+a_{mn}^{(1)}x_n&=b_m^{(1)}. \end{aligned}$$

Here $a_{1j}^{(1)} = a_{1j}^{(0)}$, $j = 1, \ldots, n$; $b_1^{(1)} = b_1^{(0)}$; all the other elements are altered according to the formulae

$$a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}}\, a_{1j}^{(0)}, \qquad b_i^{(1)} = b_i^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}}\, b_1^{(0)}, \tag{1.0.4}$$

$i, j \geqslant 2$. The first stage of the Gauss method has now been completed. The coefficient $a_{11}^{(0)}$ is called the *pivot of the first stage*.

Assume now that, among the coefficients $a_{ij}^{(1)}$, $i, j \geqslant 2$, there are some different from zero and, in particular, $a_{22}^{(1)} \neq 0$. Subtract both sides of the second equation, premultiplied by

$$\frac{a_{32}^{(1)}}{a_{22}^{(1)}},\ \frac{a_{42}^{(1)}}{a_{22}^{(1)}},\ \ldots,\ \frac{a_{m2}^{(1)}}{a_{22}^{(1)}},$$

respectively, from both sides of the third and subsequent equations. We then obtain the following system:

$$\begin{aligned} a_{11}^{(2)}x_1+a_{12}^{(2)}x_2+a_{13}^{(2)}x_3+\ldots+a_{1n}^{(2)}x_n&=b_1^{(2)},\\ a_{22}^{(2)}x_2+a_{23}^{(2)}x_3+\ldots+a_{2n}^{(2)}x_n&=b_2^{(2)},\\ a_{33}^{(2)}x_3+\ldots+a_{3n}^{(2)}x_n&=b_3^{(2)},\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ a_{m3}^{(2)}x_3+\ldots+a_{mn}^{(2)}x_n&=b_m^{(2)}. \end{aligned}$$

The coefficient $a_{22}^{(1)}$ is called the *pivot of the second stage*.

Continuing the process we shall finally reduce the system of equations to the form:

$$\begin{aligned} a_{11}^{(r-1)}x_1+a_{12}^{(r-1)}x_2+\ldots+a_{1,r-1}^{(r-1)}x_{r-1}+a_{1r}^{(r-1)}x_r+\ldots+a_{1n}^{(r-1)}x_n&=b_1^{(r-1)},\\ a_{22}^{(r-1)}x_2+\ldots+a_{2,r-1}^{(r-1)}x_{r-1}+a_{2r}^{(r-1)}x_r+\ldots+a_{2n}^{(r-1)}x_n&=b_2^{(r-1)},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{r-1,r-1}^{(r-1)}x_{r-1}+a_{r-1,r}^{(r-1)}x_r+\ldots+a_{r-1,n}^{(r-1)}x_n&=b_{r-1}^{(r-1)},\\ a_{rr}^{(r-1)}x_r+\ldots+a_{rn}^{(r-1)}x_n&=b_r^{(r-1)},\\ 0\cdot x_r+\ldots+0\cdot x_n&=b_{r+1}^{(r-1)},\\ \ldots\ldots\ldots\ldots\ldots&\\ 0\cdot x_r+\ldots+0\cdot x_n&=b_m^{(r-1)}. \end{aligned} \tag{1.0.5}$$

Here $a_{11}^{(r-1)}\neq 0,\ a_{22}^{(r-1)}\neq 0,\ \ldots,\ a_{r-1,r-1}^{(r-1)}\neq 0,\ a_{rr}^{(r-1)}\neq 0$.

If some of the numbers $b_{r+1}^{(r-1)}, \ldots, b_m^{(r-1)}$ are different from zero, then system (1.0.5) evidently has no solution, and is termed *inconsistent*. Therefore the equivalent system (1.0.3) is also inconsistent. The inconsistency of the system could have been found out before if, after some stage of the elimination, or even in the original system, there was an equation

$$0\cdot x_1+0\cdot x_2+\ldots+0\cdot x_n=b,\quad b\neq 0.$$

It goes without saying that having obtained such an equation we stop the process.

If $b_{r+1}^{(r-1)}=\ldots=b_m^{(r-1)}=0$ then system (1.0.5) is *consistent*. To find its solutions it suffices to consider the first r equations. In case $r=n$ these n equations produce a system in a *triangular form*

$$\begin{aligned} a_{11}^{(n-1)}x_1+a_{12}^{(n-1)}x_2+\ldots+a_{1,n-1}^{(n-1)}x_{n-1}+a_{1n}^{(n-1)}x_n&=b_1^{(n-1)},\\ a_{22}^{(n-1)}x_2+\ldots+a_{2,n-1}^{(n-1)}x_{n-1}+a_{2n}^{(n-1)}x_n&=b_2^{(n-1)},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{n-1,n-1}^{(n-1)}x_{n-1}+a_{n-1,n}^{(n-1)}x_n&=b_{n-1}^{(n-1)},\\ a_{nn}^{(n-1)}x_n&=b_n^{(n-1)}. \end{aligned}$$

The solution of such a system is unique: from the last equation we deduce that x_n is unique, substituting its value into the preceding equation we find that x_{n-1} is unique, etc. A system of linear equations having a unique solution is said to be *determinate*. Thus, if system (1.0.3) can be reduced to a triangular form then it is determinate. This case is certain to take place when we find the decomposition of a vector in terms of the vectors in a basis for the space.

If $r < n$ then the first r equations of system (1.0.5) form a system in a *trapezoidal form* having infinitely many solutions. Given arbitrary numerical values to the *free unknowns* $x_{r+1}, \ldots, x_n$, we can find, in the way indicated above, the values of the unknowns $x_1, \ldots, x_r$. By this method all the solutions for system (1.0.5), and, therefore, of system (1.0.3), can be found. A system of linear equations having infinitely many solutions is called *indeterminate*. Thus, the trapezoidal form of the final system in the Gauss method reveals the indeterminacy of the original system (1.0.3). Such is the case, for example, when we seek the expression of a vector x in terms of a linearly dependent system $y_1, y_2, \ldots, y_k$ on condition that x belongs to $L(y_1, \ldots, y_k)$.

Note, in conclusion, that all transformations in the Gauss method may be performed with the elements of the *augmented matrix* made up of the coefficients of system (1.0.3):

$$\bar{A}_0 = \begin{Vmatrix} a_{11}^{(0)} & a_{12}^{(0)} & \ldots & a_{1n}^{(0)} & b_1^{(0)} \\ a_{21}^{(0)} & a_{22}^{(0)} & \ldots & a_{2n}^{(0)} & b_2^{(0)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}^{(0)} & a_{m2}^{(0)} & \ldots & a_{mn}^{(0)} & b_m^{(0)} \end{Vmatrix}.$$

The transfer to the subsequent matrices $\bar{A}_1, \ldots, \bar{A}_{r-1}$ is performed by the formulae of the type (1.0.4). The method of elementary transformations, suggested for the solution of Problems 1 and 2, is actually the method of Gauss elimination used for the matrix made up of the vectors of the given system.

1.1. Definition of Linear Space

In this section a number of examples of linear spaces, and also of certain sets which are not linear spaces, are given. We also touch upon (see Problems 1.1.17, 1.1.18) the axioms of a linear space.

1.1.1. A set V_0 consists of one element θ. The operations on V_0 are defined as follows:

(a) $\theta + \theta = \theta$;

(b) $\lambda\theta = \theta$ for every λ from the field P. Verify that V_0 is a linear space over the field P.

Determine, for each of the following vector sets in a plane, whether this set is a linear space under ordinary vector addition and vector multiplication by a number. In case of the negative reply, indicate which particular properties of a linear space are not fulfilled. It is assumed that the origin of each vector is at the fixed point O of the plane, being the origin of a rectangular system of coordinates.

1.1.2. All vectors whose end-points lie in the same straight line.

1.1.3. All vectors whose end-points lie: (a) in the first quadrant of the system of coordinates; (b) in the first or third quadrant; (c) in the first or second quadrant.

1.1.4. All vectors which form an angle φ, $0 \leqslant \varphi \leqslant \pi$, with a given nonzero vector a.

1.1.5. Show that (a) the set of real numbers may be considered as a rational linear space; (b) the set of complex numbers may be considered as a real linear space; (c) in general, any field P may be considered as a linear space over a subfield P_1 of the field P.

1.1.6. On the set R^+ of positive real numbers the following operations are defined: (a) "addition" $x \oplus y = xy$ (i.e. the ordinary multiplication of numbers x and y); (b) "multiplication by a real number" $\alpha \cdot x = x^{\alpha}$ (i.e. raising a number to the power of α).

Verify that the set R^+, under the indicated operations is a linear space.

1.1.7. Let $\tilde{R}_2$ be the set of all ordered pairs of real numbers $x = (\alpha_1, \alpha_2)$ under the operations: (a) if $x = (\alpha_1, \alpha_2)$ and $y = (\beta_1, \beta_2)$ then $x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$; (b) for any real number λ, $\lambda x = (\lambda\alpha_1, \alpha_2)$.

Is $\tilde{R}_2$ a real linear space?

1.1.8. The same question for the case with the following definition of the operation of multiplication by a number: if $x = (\alpha_1, \alpha_2)$ then $\lambda x = (\lambda\alpha_1, \lambda\alpha_2)$.

1.1.9. Let P_k be the set of all ordered sets of k elements from the field P: $x = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. The operations on P_k are defined as follows: (a) if $x = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $y = (\beta_1, \beta_2, \ldots, \beta_k)$, then $x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k)$; (b) for every λ from the field P, $\lambda x = (\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_k)$. Verify that P_k is a linear space over the field P.

1.1.10. Let $Z^{(2)}$ be the field of two elements 0 and 1 on which the operations are defined by the following tables:

(a) addition

	0	1
0	0	1
1	1	0

(b) multiplication

	0	1
0	0	0
1	0	1

Construct the linear space $Z_k^{(2)}$ (see Problem 1.1.9). Show that for any vector x from $Z_k^{(2)}$, $x + x = 0$. Find the number of vectors in $Z_k^{(2)}$.

1.1.11. Let s be the set of all infinite series of real numbers $x = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$. The operations on s are defined as follows: (a) if $x = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$, $y = (\beta_1, \beta_2, \ldots, \beta_n, \ldots)$, then $x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n, \ldots)$; (b) for any real λ,

$$\lambda x = (\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n, \ldots).$$

Is s a real linear space?

1.1.12. Let F be the set of all infinite series of real numbers whose elements satisfy the relation $\alpha_k = \alpha_{k-1} + \alpha_{k-2}$, $k = 3, 4, \ldots$. The operations over series are defined just like in Problem 1.1.11. Is F a linear space?

Verify, for each of the following sets of polynomials in one variable and with real coefficients, whether this set is a linear space under the ordinary operations of addition of polynomials and multiplication of a polynomial by a number.

1.1.13. The set of polynomials of any degree, with zero adjoined.

1.1.14. The set of polynomials of degrees $\leqslant n$, with zero adjoined.

1.1.15. The set of all polynomials of a given degree n.

1.1.16. The set of all polynomials $f(t)$ satisfying the conditions: (a) $f(0) = 1$; (b) $f(0) = 0$; (c) $2f(0) - 3f(1) = 0$; (d) $f(1) + f(2) + \ldots + f(k) = 0$.

1.1.17.* Give an example of a set M in which all the axioms of a linear space are fulfilled except the axiom: $1 \cdot x = x$ for every x from M. How important is this axiom for the definition of a linear space?

1.1.18.* Prove that the commutative law follows from the other axioms of a linear space.

1.2. Linear Dependence

Besides the problems involving the notion of linear dependence, we provide, in the present section, the computational means for the solution of a problem of linear dependence, or independence, of a concrete set of vectors in arithmetic space, viz., the elementary transformations for the set.

1.2.1. Prove that a set of vectors containing the null vector is linearly dependent.

1.2.2. Prove that a set of vectors two of whose vectors differ only by a scalar multiplier, is linearly dependent.

1.2.3. Prove that if, in a set of vectors, some subset is linearly dependent, then the whole set is linearly dependent.

1.2.4. Prove that in a linearly independent set of vectors, any subset is also linearly independent.

1.2.5. Let a set of vectors $x_1, \ldots, x_m$ be linearly independent and let the set $x_1, x_2, \ldots, x_m, y$ be linearly dependent. Prove that the vector y is linearly expressed in terms of $x_1, \ldots, x_m$.

1.2.6. Show that the decomposition of the vector y in terms of $x_1, \ldots, x_m$ (see the previous problem) is unique.

1.2.7. Conversely, let the decomposition of the vector y in some set $x_1, \ldots, x_m$ be unique. Prove that the set $x_1, \ldots, x_m$ is linearly independent.

1.2.8. Let a vector y be linearly expressed in terms of a linearly dependent set $x_1, \ldots, x_m$. Show that y has infinitely many decompositions in terms of the set.

1.2.9. Let x, y, z be a linearly independent set of vectors. Are the following sets of vectors linearly independent:

$$\text{(a)}\ x,\ x+y,\ x+y+z;\quad \text{(b)}\ x+y,\ y+z,\ z+x;$$
$$\text{(c)}\ x-y,\ y-z,\ z-x?$$

1.2.10. Show that for any vectors x, y, z, and any numbers α, β, γ, the set of vectors $\alpha x - \beta y$, $\gamma y - \alpha z$, $\beta z - \gamma x$ is linearly dependent.

1.2.11. Let r, s, v be different real numbers. Is the following set of polynomials linearly dependent:

$$(t-r)(t-s),\ (t-r)(t-v),\ (t-s)(t-v)?$$

1.2.12. Find the linear combination $3x_1 - 2x_2 + 7x_3$ of the vectors of the arithmetic space R_4:

$$x_1 = (3,\ 1,\ -7,\ 4),$$
$$x_2 = (1,\ 5,\ 0,\ 6),$$
$$x_3 = (-1,\ 1,\ 3,\ 0).$$

Discuss the result obtained. What can be said of the set of vectors x_1, x_2, x_3?

1.2.13. Given a set of polynomials $f_1(t) = 1 - t^2$, $f_2(t) = 1 + t^3$, $f_3(t) = t - t^3$, $f_4(t) = 1 + t + t^2 + t^3$, find the following linear combinations of the polynomials of this set: (a) $5f_1 + f_2 - 4f_3$; (b) $f_1 + 9f_2 - 4f_4$. Discuss the obtained results. What can be said of the given set of polynomials?

1.2.14. Find other decompositions of the polynomial obtained in Problem 1.2.13 in terms of the set $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$.

1.2.15. Prove that the following "trapezoidal" set of vectors of the space P_k (see Problem 1.1.9) is linearly independent:

$$\begin{array}{l}
y_1 = (\alpha_{11}, \ldots, \alpha_{1p}, \alpha_{1,p+1}, \ldots, \alpha_{1q}, \alpha_{1,q+1}, \ldots, \alpha_{1t}, \alpha_{1,t+1}, \ldots, \alpha_{1k}),\\
y_2 = (0, \ldots, 0, \quad \alpha_{2,p+1}, \ldots, \alpha_{2q}, \alpha_{2,q+1}, \ldots, \alpha_{2t}, \alpha_{2,t+1}, \ldots, \alpha_{2k}),\\
y_3 = (0, \ldots, 0, \quad 0, \quad \ldots, 0, \quad \alpha_{3,q+1}, \ldots, \alpha_{3t}, \alpha_{3,t+1}, \ldots, \alpha_{3k}),\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_r = (0, \ldots, 0, \quad 0, \quad \ldots, 0, \quad 0, \quad \ldots, 0, \quad \alpha_{r,t+1}, \ldots, \alpha_{rk}).
\end{array} \qquad (1.2.1)$$

Here $\alpha_{2,p+1}, \alpha_{3,q+1}, \ldots, \alpha_{r,t+1}$ are elements of the field P and differ from zero. At least one of the elements $\alpha_{11}, \ldots, \alpha_{1p}$ is also nonzero.

1.2.16. Prove that in polynomial space, each finite set consisting of polynomials of various degrees and not containing zero is linearly independent.

1.2.17. Prove that linear dependence or linear independence of a set of vectors is not affected during the following transformations of the set, called the *elementary transformations*: (a) interchanging two vectors of the set; (b) multiplication of a vector of the set by a nonzero number; (c) addition of one vector of the set to another multiplied by an arbitrary number.

1.2.18. Prove that an arbitrary set of vectors of the arithmetic space can be reduced, by the elementary transformations, to a set of vectors of the type indicated in Problem 1.2.15, probably, with some zero vectors adjoined. How can it be determined whether the original set was linearly dependent?

Determine whether the following vector sets of arithmetic spaces are linearly dependent:

1.2.19. $x_1 = (-3, 1, 5)$, $x_2 = (6, -2, 15)$.

1.2.20. $x_1 = (4, -12, 28)$, $x_2 = (-7, 21, -49)$.

1.2.21. $x_1 = (1, 2, 3, 0)$, $x_2 = (2, 4, 6, 1)$.

1.2.22. $x_1 = (1, i, 2 - i, 3 + i)$, $x_2 = (1 - i, 1 + i, 1 - 3i, 4 - 2i)$.

1.2.23. $x_1 = (1, 2, 3)$, $x_2 = (2, 5, 7)$, $x_3 = (3, 7, 10)$.

1.2.24. $x_1 = (1, 2, 3)$, $x_2 = (2, 5, 7)$, $x_3 = (3, 7, 11)$.

1.2.25. $x_1 = (1, 2, 3)$, $x_2 = (2, 5, 7)$, $x_3 = (3, 7, 10 + \varepsilon)$.

Here ε is however small number other than zero.

1.2.26. $x_1 = (1, 1, 1, 1)$, $x_2 = (1, -1, -1, 1)$, $x_3 = (1, -1, 1, -1)$, $x_4 = (1, 1, -1, -1)$.

1.2.27. $x_1 = (5, -3, 2, 1, 10)$, $x_2 = (-1, 8, 1, -4, 7)$, $x_3 = (2, 1, 9, -3, 6)$, $x_4 = (1, 3, -5, 9, 11)$.

1.2.28.* Let a set of vectors of an arithmetic space be given

$$x_1 = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}),$$
$$x_2 = (\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}),$$
$$\cdots\cdots\cdots$$
$$x_s = (\alpha_{s1}, \alpha_{s2}, \ldots, \alpha_{sn}),$$

where $s \leqslant n$. Prove that if $|\alpha_{jj}| > \sum_{i=1, i \neq j} |\alpha_{ij}|$, $j = 1, \ldots, s$, then the given set of vectors is linearly independent.

1.3. Spans. Rank of Vector Sets

In this section we offer problems involving the definitions of the span; the equivalence of sets of vectors; the base and rank of a vector set; and also a number of computational problems for finding the rank, and constructing the base of a vector set in arithmetic space. The method of elementary transformations developed in the previous section is the way to solve the latter problems.

Describe the spans of the following vector sets in the space R_5:

1.3.1. $x_1 = (1, 0, 0\ 0, 0)$,
$x_2 = (0, 0, 1, 0, 0)$,
$x_3 = (0, 0, 0, 0, 1)$.

1.3.2. $x_1 = (1, 0, 0, 0, 1)$,
$x_2 = (0, 1, 0, 1, 0)$,
$x_3 = (0, 0, 1, 0, 0)$.

1.3.3. $x_1 = (1, 0, 0, 0, -1)$,
$x_2 = (0, 1, 0, 0, -1)$,
$x_3 = (0, 0, 1, 0, -1)$,
$x_4 = (0, 0, 0, 1, -1)$.

Find the spans of the following sets of polynomials:

1.3.4. $1, t, t^2$.

1.3.5. $1 + t^2, t + t^2, 1 + t + t^2$.

1.3.6. $1 - t^2, t - t^2, 2 - t - t^2$.

1.3.7. $1 - t^2, t - t^2$.

1.3.8.* Consider the span generated by numbers 1 and $\sqrt{3}$ in the set of real numbers and treated as a rational linear space. Does $\sqrt[4]{3}$ belong to this span?

1.3.9. If every vector of a set $y_1, y_2, \ldots, y_n$ is a linear combination of the vectors $x_1, \ldots, x_m$, then the *set* $y_1, \ldots, y_n$ is said to be *linearly expressed in terms of the set* $x_1, \ldots, x_m$. Prove the transitive law for this concept, i.e. if the set $y_1, \ldots, y_n$ is linearly expressed in terms of the set $x_1, \ldots, x_m$ and the set $z_1, \ldots, z_p$ is linearly expressed in terms of $y_1, \ldots, y_n$, then the set $z_1, \ldots, z_p$ is linearly expressed in terms of $x_1, \ldots, x_m$.

1.3.10. Show that if a set $y_1, \ldots, y_n$ is linearly expressed in terms of $x_1, \ldots, x_m$, then the span of the first set is contained in the span of the second.

1.3.11. The set of vectors z_1, z_2 is linearly expressed in terms of the set y_1, y_2, y_3, y_4:

$$z_1 = 2y_1 + y_2 + 3y_4,$$
$$z_2 = y_1 - 5y_2 + 4y_3 - 2y_4.$$

Also the set y_1, y_2, y_3, y_4 is linearly expressed in terms of the set x_1, x_2, x_3:

$$y_1 = x_1 + x_2 + x_3,$$
$$y_2 = x_1 + x_2 - x_3,$$
$$y_3 = x_1 - x_2 + x_3,$$
$$y_4 = -x_1 + x_2 + x_3.$$

Find the expressions of the vectors z_1, z_2 in terms of the vectors x_1, x_2, x_3.

1.3.12. Two sets of vectors $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are said to be *equivalent* if each of these sets is linearly expressed in terms of the other. Prove that the equivalence relation is reflexive, symmetric and transitive.

1.3.13. Show that two sets of vectors are equivalent if, and only if, their spans coincide.

Are the following sets of vectors equivalent?

1.3.14. $x_1 = (1, 0, 0), \quad y_1 = (0, 0, 1),$

$x_2 = (0, 1, 0), \quad y_2 = (0, 1, 1),$

$x_3 = (0, 0, 1); \quad y_3 = (1, 1, 1).$

1.3.15. $x_1 = (1, 0, 0), \quad y_1 = (1, 0, 0),$

$x_2 = (0, 1, 0), \quad y_2 = (0, 1, 1),$

$x_3 = (0, 0, 1); \quad y_3 = (1, 1, 1).$

1.3.16.* Prove that two equivalent linearly independent sets contain the same number of vectors.

1.3.17. In a set of vectors $x_1, \ldots, x_m, y_1, \ldots, y_n$, the vectors $y_1, \ldots, y_n$ are linearly dependent on the vectors $x_1, \ldots, x_m$. Show that the set $x_1, \ldots, x_m, y_1, \ldots, y_n$ is equivalent to the set $x_1, \ldots, x_m$.

1.3.18.* Prove that in each set of vectors $x_1, \ldots, x_m$ containing at least one nonzero vector, an equivalent linearly independent subset may be chosen. (Any such set is called the *base* of the given set of vectors.)

1.3.19. Prove that all the bases of a given set $x_1, \ldots, x_m$ consist of the same number of vectors. (This number is called the *rank* of the given set. If all vectors of the set are zero then its rank is zero by definition).

1.3.20. Let the rank of a set $x_1, \ldots, x_m$ be equal to r. Prove that (a) any of its subsets containing more than r vectors is linearly dependent; (b) any linearly independent subset containing r vectors is a base of the given set. Note that it follows from (a) that the rank of a set of vectors equals the maximal number of its linearly independent vectors.

1.3.21. Prove that (a) any nonzero vector of a given set can be included into a certain base of this set; (b) any linearly independent subset of the given set of vectors can be extended to form the base of this set.

1.3.22. Prove that if a set $y_1, \ldots, y_n$ is linearly expressed in terms of a set $x_1, \ldots, x_m$, then the rank of the first set is not greater than the rank of the second.

1.3.23. Prove that if a set $y_1, \ldots, y_n$ is linearly expressed in terms of a set $x_1, \ldots, x_m$, then the rank of the set $x_1, \ldots, x_m, y_1, \ldots, y_n$ equals the rank of the set $x_1, \ldots, x_m$.

1.3.24. Prove that equivalent vector sets have the same rank. Determine whether the converse is true, viz. whether any two sets of the same rank are equivalent.

1.3.25.* Prove that if two vector sets have the same rank and one of these sets is linearly expressed in terms of the other, then these sets are equivalent.

1.3.26. Prove that elementary transformations of a vector set do not alter the rank of this set.

1.3.27. Apply the method of "reduction to the trapezoidal form" worked out in Problem 1.2.18 to the solution of the following problem: find the rank of a given vector set of the arithmetic space.

Find the rank of the following vector sets:

1.3.28. $x_1 = (1, 2, 3)$,
$x_2 = (4, 5, 6)$,
$x_3 = (7, 8, 9)$,
$x_4 = (10, 11, 12)$.

1.3.29. $x_1 = (1, 4, 7, 10)$,
$x_2 = (2, 5, 8, 11)$,
$x_3 = (3, 6, 9, 12)$.

1.3.30. $x_1 = (1, -1, 0, 0)$,
$x_2 = (0, 1, -1, 0)$,
$x_3 = (0, 0, 1, -1)$,
$x_4 = (0, 0, 0, 1)$,
$x_5 = (7, -3, -4, 5)$.

1.3.31. $x_1 = (1, -1, 0, 0)$,
$x_2 = (0, 1, -1, 0)$,
$x_3 = (0, 0, 1, -1)$,
$x_4 = (-1, 0, 0, 1)$.

1.3.32. $x_1 = (1, 10, 0, 0)$,
$x_2 = (0, 1, 10, 0)$,
$x_3 = (0, 0, 1, 10)$,
$x_4 = (10, 0, 0, 1)$.

1.3.33. $x_1 = (1, 1, 1, 1, 1)$,
$x_2 = (1, i, -1, -i, 1)$,
$x_3 = (1, -1, 1, -1, 1)$,
$x_4 = (1, -i, -1, i, 1)$.

1.3.34.* Use the method of Problem 1.3.27 to find a base for a given vector set in the arithmetic space.

Find a base for each of the following vector sets:

1.3.35. $x_1 = (-1, 4, -3, -2)$,
$x_2 = (3, -7, 5, 3)$,
$x_3 = (3, -2, 1, 0)$,
$x_4 = (-4, 1, 0, 1)$.

1.3.36. $x_1 = (0, 2, -1)$,
$x_2 = (3, 7, 1)$,
$x_3 = (2, 0, 3)$,
$x_4 = (5, 1, 8)$.

1.3.37.* $x_1 = (14, -27, -49, 113)$,
$x_2 = (43, -82, -145, 340)$,
$x_3 = (-29, 55, 96, -227)$,
$x_4 = (85, -163, -293, 677)$.

1.3.38. $x_1 = (3 - i,\ 1 - 2i,\ -7 + 5i,\ 4 + 3i)$,

$x_2 = (1 + 3i,\ 1 + i,\ -6 - 7i,\ 4i)$,

$x_3 = (0,\ 1,\ 1,\ -3)$.

1.3.39.* The vectors $x_{i_1}, \ldots, x_{i_r}$ of a set $x_1, \ldots, x_m$ form a base and the vector x_j, the nonzero vector, is not in this base. Prove that, among the vectors of the base, there is a vector x_{i_l} such that on replacing it with x_j in the subset $x_{i_1}, \ldots, x_{i_r}$, a new base of the given set $x_1, \ldots, x_m$ is obtained. Is this vector, x_{i_l}, unique?

1.3.40.* What can be said about a vector set of rank r if it has (a) a unique base; (b) precisely two bases; (c) precisely three bases? NB. Two bases, differing only in the order of vectors, are treated as the same.

Find all the bases of the following vectors:

1.3.41. $x_1 = (4,\ -2,\ 12,\ 8)$,

$x_2 = (-6,\ 12,\ 9,\ -3)$,

$x_3 = (-10,\ 5,\ -30,\ -20)$,

$x_4 = (-14,\ 28,\ 21,\ -7)$.

1.3.42. $x_1 = (1,\ 2,\ 3,\ 0,\ -1)$,

$x_2 = (0,\ 1,\ 1,\ 1,\ 0)$,

$x_3 = (1,\ 3,\ 4,\ 1,\ -1)$.

1.3.43. $x_1 = (1 + i,\ 1 - i,\ 2 + 3i)$,

$x_2 = (i,\ 1,\ 2)$,

$x_3 = (1 - i,\ -1 - i,\ 3 - 2i)$,

$x_4 = (4,\ -4i,\ 10 + 2i)$.

1.3.44.* Apply the method of Problem 1.3.27 to the solution of the following problem: determine whether a given vector set $y_1, \ldots, y_n$ is expressed in terms of a vector set $x_1, \ldots, x_m$, both being in the arithmetic space.

1.3.45. Given two vector sets,

$x_1 = (1,\ 1,\ 1)$, $\quad y_1 = (1,\ 2,\ 3)$,

$x_2 = (1,\ 0,\ -1)$, $\quad y_2 = (0,\ 1,\ 2)$,

$x_3 = (1,\ 3,\ 5)$; $\quad y_3 = (3,\ 4,\ 5)$,

$y_4 = (4,\ 6,\ 8)$

determine whether the set y_1, y_2, y_3, y_4 is linearly expressed in terms of the set x_1, x_2, x_3.

1.3.46. Are the sets indicated in the previous problem equivalent?

1.4. Basis and Dimension of Space

We begin this section with examples of finite-dimensional and infinite-dimensional linear spaces so as to consider, hereafter, only the finite-dimensional spaces. Furthermore, we discuss the notion of a basis. If, in a linear space, a basis is fixed, then the problems involving the elements of this space are reduced to similar problems involving the vectors of arithmetic space. Some of these problems (finding the rank of a vector set, the dimension and basis of the span, etc.) are solved by the method of elementary transformations, others (e.g. the decomposition in terms of a basis) are reduced to the solution of certain systems (known beforehand) of linear equations, the rational way of their solution being the Gauss method. The section is concluded by problems involving linear subspaces.

Determine, for each of the linear spaces indicated below, whether they are finite-dimensional. In case of the positive reply find the dimension and construct a basis for the space.

1.4.1. The space R^+ (see Problem 1.1.5).

1.4.2. The space P_k whose elements are ordered sets of k elements of the field P (see Problem 1.1.9).

1.4.3. The space s of all infinite real sequences (see Problem 1.1.11).

1.4.4. The space F of infinite real sequences whose elements satisfy the relationship $\alpha_k = \alpha_{k-1} + \alpha_{k-2}$, $k = 3, 4, \ldots$ (see Problem 1.1.12).

1.4.5. The space M of polynomials of all degrees (see Problem 1.1.13).

1.4.6. The space M_n of polynomials whose degree does not exceed a given nonnegative number n (see Problem 1.1.14).

1.4.7. Find the dimension of the field of complex numbers considered as (a) the complex linear space; (b) the real linear space.

1.4.8. Let C_n be the set of all ordered sets of n complex numbers under the customarily defined operations on these sets (see Problem 1.1.9). Find the dimension of C_n as (a) a complex space; (b) a real space.

Show that the following vector spaces are the bases for the space R_n:

1.4.9.
$$\begin{aligned} x_1 &= (1, 2, 3, \ldots, n), \\ x_2 &= (0, 2, 3, \ldots, n), \\ x_3 &= (0, 0, 3, \ldots, n), \\ &\ldots \\ x_n &= (0, 0, 0, \ldots, n). \end{aligned}$$

1.4.10.
$$\begin{aligned} x_1 &= (1, 1, \ldots, 1, 1, 1), \\ x_2 &= (1, 1, \ldots, 1, 1, 0), \\ x_3 &= (1, 1, \ldots, 1, 0, 0), \\ &\ldots \\ x_n &= (1, 0, \ldots, 0, 0, 0). \end{aligned}$$

1.4.11.
$$\begin{aligned} x_1 &= (1, 1, 1, 1, \ldots, 1), \\ x_2 &= (0, 1, 0, 0, \ldots, 0), \\ x_3 &= (0, 1, 1, 0, \ldots, 0), \\ x_4 &= (0, 1, 1, 1, \ldots, 0), \\ &\ldots \\ x_n &= (0, 1, 1, 1, \ldots, 1). \end{aligned}$$

1.4.12. Prove that in the space M_n of polynomials of degree $\leqslant n$, the basis is any set of nonzero polynomials containing one polynomial of each degree k, $k = 0, 1, 2, \ldots, n$.

1.4.13. Determine which of the following two vector sets is a basis for the space R_4:

(a) $x_1 = (1, 2, -1, -2)$, $x_2 = (2, 3, 0, -1)$, $x_3 = (1, 2, 1, 3)$, $x_4 = (1, 3, -1, 0)$;

(b) $x_1 = (1, 2, -1, -2)$, $x_2 = (2, 3, 0, -1)$, $x_3 = (1, 2, 1, 4)$, $x_4 = (1, 3, -1, 0)$.

Henceforward, only the finite-dimensional spaces will be considered.

1.4.14. Prove that (a) any nonzero vector of a space may be included in a certain basis for this space; (b) any linearly independent vector set can be extended to form a basis for the space.

1.4.15. Find two different bases for the space R_4 having the vectors $e_1 = (1, 1, 0, 0)$, and $e_2 = (0, 0, 1, 1)$ in common.

1.4.16. Extend the set of polynomials $t^5 + t^4$, $t^5 - 3t^3$, $t^5 + 2t^2$, $t^5 - t$ to form a basis for the space M_5.

1.4.17. Prove that the decomposition of a vector in terms of vectors of any basis is unique.

1.4.18. Let every vector of a space V be linearly expressed in terms of the vectors in a set $e_1, \ldots, e_n$ and let the decomposition of a certain vector x in terms of this set be unique. Prove that the vectors $e_1, \ldots, e_n$ form a basis for the space V.

1.4.19. Let $e_1, \ldots, e_n$ be an arbitrary basis for a space V. Prove that (a) the coordinates of the vector $x + y$ in terms of the basis $e_1, \ldots, e_n$ are equal to the sums of the corresponding coordinates of the vectors x and y in the same basis; (b) the coordinates of a vector λx in the basis $e_1, \ldots, e_n$ equal the corresponding coordinates of the vector x multiplied by the number λ.

1.4.20. In a space V some basis $e_1, \ldots, e_n$ is fixed. Each vector x is matched with the row of its coordinates in this basis, i.e.

$$x \to x_e = (\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Prove that (a) the linear dependence (or linear independence) of a vector set $x, y, \ldots, z$ induces the linear dependence (linear independence) of the set of rows $x_e, y_e, \ldots, z_e$ considered as the elements of the corresponding arithmetic space; (b) the rank of a vector set $x, y, \ldots, z$ equals the rank of the row set $x_e, y_e, \ldots, z_e$; (c) if a vector u is linearly dependent on the vectors of a set $x, y, \ldots, z$, i.e. $u = \lambda x + \mu y + \ldots + \nu z$, then this is true for the rows $u_e, x_e, y_e, \ldots, z_e$ and, that besides, $u_e = \lambda x_e + \mu y_e + \ldots + \nu z_e$.

Determine the rank and find a base for each of the following sets of polynomials:

1.4.21. $3t^2 + 2t + 1$, $4t^2 + 3t + 2$, $3t^2 + 2t + 3$, $t^2 + t + 1$, $4t^2 + 3t + 4$.

1.4.22. $t^3 + 2t^2 + 3t + 4$, $2t^3 + 3t^2 + 4t + 5$, $3t^3 + 4t^2 + 5t + 6$, $4t^3 + 5t^2 + 6t + 7$.

Verify that the vectors $e_1, \ldots, e_n$ form a basis for the space R_n; find the coordinates of the vector x in this basis:

1.4.23. $e_1 = (2, 2, -1)$, $e_2 = (2, -1, 2)$, $e_3 = (-1, 2, 2)$; $x = (1, 1, 1)$.

1.4.24. $e_1 = (1, 5, 3)$, $e_2 = (2, 7, 3)$, $e_3 = (3, 9, 4)$; $x = (2, 1, 1)$.

1.4.25. $e_1 = (1, 2, -1, -2)$, $e_2 = (2, 3, 0, -1)$, $e_3 = (1, 2, 1, 4)$, $e_4 = (1, 3, -1, 0)$; $x = (7, 14, -1, 2)$.

1.4.26. $e_1 = (1, 2, 1, 1)$, $e_2 = (2, 3, 1, 0)$, $e_3 = (3, 1, 1, -2)$, $e_4 = (4, 2, -1, -6)$; $x = (0, 0, 2, 7)$.

1.4.27. Find the coordinates of the polynomial $t^5 - t^4 + t^3 - t^2 - t + 1$ in each of the following bases for the space M_5:

(a) $1, t, t^2, t^3, t^4, t^5$;

(b) $1, t + 1, t^2 + 1, t^3 + 1, t^4 + 1, t^5 + 1$;

(c) $1 + t^3, t + t^3, t^2 + t^3, t^3, t^4 + t^3, t^5 + t^3$.

1.4.28. Verify that the sequences

$$e_1 = (2, 3, 5, 8, 13, \ldots),$$
$$e_2 = (1, 2, 3, 5, 8, \ldots)$$

form the basis for the space F (see Problem 1.1.12); express the sequence

$$e = (1, 1, 2, 3, 5, 8, \ldots)$$

in terms of the elements of this basis.

1.4.29. Prove that the span of an arbitrary finite vector set of a linear space V is its linear subspace.

1.4.30. Let V be an n-dimensional linear space. Prove that any linear subspace of the space V is finite-dimensional, its dimension not exceeding n.

1.4.31. Prove that if L is a linear subspace of a space V and the dimension of L equals the dimension of V, then L coincides with V.

1.4.32. Prove that any subspace of an n-dimensional space V may be considered as the span of a certain vector set. Besides, the set may be chosen to contain not more than n vectors.

1.4.33. Prove that in an n-dimensional space V, a linear subspace of any dimension k, $0 \leqslant k \leqslant n$, may be found.

1.4.34. Given that a linear subspace L is the span of a vector set $x_1, \ldots, x_k$, prove that the dimension of L equals the rank of the set $x_1, \ldots, x_k$, and that any basis for this set may serve as its basis.

Determine the dimension of and find a basis for the linear subspaces spanned by the following sets of vectors of the arithmetic space:

1.4.35. $x_1 = (1, 2, 2, -1), x_2 = (2, 3, 2, 5), x_3 = (-1, 4, 3, -1)$, $x_4 = (2, 9, 3, 5)$.

1.4.36. $x_1 = (-3, 1, 5, 3, 2), x_2 = (2, 3, 0, 1, 0), x_3 = (1, 2, 3, 2, 1), x_4 = (3, -5, -1, -3, -1), x_5 = (3, 0, 1, 0, 0)$.

1.4.37. Find a basis and the dimension of a linear subspace of the space R_n if L is determined by the equation

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0.$$

1.4.38. In the space M_n of polynomials with real coefficients of degree $\leqslant n$ the subsets of polynomials satisfying the following conditions are considered: (a) $f(0) = 0$; (b) $f(1) = 0$; (c) $f(a) = 0$, where a is any real number; (d) $f(0) = f(1) = 0$. Prove that each of the indicated subsets is a linear subspace of the space M_n; find the dimensions for these subspaces.

1.4.39. Find the dimension and a basis for the span generated by the set of polynomials: $t^6 + t^4, t^6 + 3t^4 - t, t^6 - 2t^4 + t, t^6 - 4t^4 + 2t$.

1.4.40. Let L be an m-dimensional subspace of an n-dimensional space V. Prove that a basis $e_1, \ldots, e_n$ for the space V may be found such that its first m vectors $e_1, \ldots, e_m$ are in the subspace L.

1.4.41.* Prove that for whichever m-dimensional subspace L of an n-dimensional space V, where $m < n$, there is a basis for V such that (a) it contains no vectors from L; (b) it contains precisely k vectors from L, $k < m$.

1.4.42. Construct a basis for the space M_5 of polynomials of the fifth degree.

1.4.43. Conversely, can a basis for the space M_5, containing no polynomials of the fifth degree, be found?

1.5. Sum and Intersection of Subspaces

In this section we seek:

To present the computational methods used to find the basis for the sum and intersection of two linear subspaces.

To indicate various criteria for the "directness" of the subspace sum.

To stress that, in the general case, the decomposition of a vector in terms of subspaces is not unique. It is unique only in the case of a direct sum. The subspaces that produce, in their sum, the whole linear space serve as the generalized basis for it.

To illustrate the existence of a complementary subspace (which is not unique) of any subspace.

1.5.1. Prove that the sum and intersection of two linear subspaces of a space V are also linear subspaces of this space.

1.5.2. Consider the set of all linear subspaces of a given space V under the operation of subspace addition.

Verify that (a) the operation is associative; (b) there is a zero element. Is this set a group?

1.5.3. Consider the set of all the linear subspaces of a given space V under the operation of intersection of subspaces. Show that (a) the operation is associative; (b) there is an identity element. Is this set a group?

1.5.4. Prove for any subspaces L_1 and L_2, the validity of the formula

$$\dim L_1 + \dim L_2 = \dim (L_1 + L_2) + \dim (L_1 \cap L_2).$$

Here and henceforth, $\dim L$ means the dimension of the linear space L.

1.5.5. Prove that, for any P,

$$\dim (L_1 + \ldots + L_p) \leqslant \dim L_1 + \ldots + \dim L_p.$$

1.5.6. Let L_1 be the span of the set of vectors $x_1, \ldots, x_k$ and L_2 the span of vectors $y_1, \ldots, y_l$. Prove that any base of the set $x_1, \ldots, x_k, y_1, \ldots, y_l$ serves as a basis for the sum $L_1 + L_2$. In particular, the basis $L_1 + L_2$ may be obtained by extending a basis for L_1 (L_2).

Find a basis and the dimension of the sum of the two subspaces, viz. L_1 spanned by the vectors $x_1, \ldots, x_k$ and L_2 by the vectors $y_1, \ldots, y_l$. Determine the dimension of the intersection of these subspaces.

1.5.7. $x_1 = (0, 1, 1, 1)$, $x_2 = (1, 1, 1, 2)$, $x_3 = (-2, 0, 1, 1)$; $y_1 = (-1, 3, 2, -1)$, $y_2 = (1, 1, 0, -1)$.

1.5.8. $x_1 = (2, -5, 3, 4)$, $x_2 = (1, 2, 0, -7)$, $x_3 = (3, -6, 2, 5)$; $y_1 = (2, 0, -4, 6)$, $y_2 = (1, 1, 1, 1)$, $y_3 = (3, 3, 1, 5)$.

1.5.9.* Let $x_1, \ldots, x_k$ be a basis for a subspace L_1 and $y_1, \ldots, y_l$ a basis for a space L_2. Further, let $x_1, \ldots, x_k, y_1, \ldots, y_s$ be a base of the set $x_1, \ldots, x_k, y_1, \ldots, y_l$ and the vectors $y_{s+1}, \ldots, y_l$ not in this base, have the following decompositions in terms of this base:

$$y_i = \alpha_{i1}x_1 + \ldots + \alpha_{ik}x_k + \beta_{i1}y_1 + \ldots + \beta_{is}y_s, \quad i = s + 1, \ldots, l.$$

Prove that the set of vectors $z_1, \ldots, z_{l-s}$ where

$$z_{i-s} = -\beta_{i1}y_1 - \ldots - \beta_{is}y_s + y_i, \quad i = s + 1, \ldots, l,$$

or, written in other way,

$$z_{i-s} = \alpha_{i1}x_1 + \ldots + \alpha_{ik}x_k, \quad i = s + 1, \ldots, l,$$

forms a basis for the intersection $L_1 \cap L_2$.

Find the bases for the sum and intersection of the linear subspaces spanned by the sets $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$, respectively:

1.5.10. $x_1 = (2, 1, 0)$, $x_2 = (1, 2, 3)$, $x_3 = (-5, -2, 1)$; $y_1 = (1, 1, 2)$, $y_2 = (-1, 3, 0)$, $y_3 = (2, 0, 3)$.

1.5.11. $x_1 = (1, 1, 1, 1)$, $x_2 = (1, 1, -1, -1)$, $x_3 = (1, -1, 1, -1)$; $y_1 = (1. -1, -1, 1)$. $y_2 = (2, -2, 0, 0)$, $y_3 = (3, -1, 1, 1)$.

1.5.12. $x_1 = (1, 2, 1, 1)$, $x_2 = (2, 3, 1, 0)$, $x_3 = (3, 1, 1, -2)$; $y_1 = (0, 4, 1, 3)$, $y_2 = (1, 0, -2, -6)$, $y_3 = (1, 0, 3, 5)$.

1.5.13. Find two different decompositions of the vector $x = (1, 0, 1)$ in terms of the subspaces L_1 and L_2 (see Problem 1.5.10).

1.5.14. Prove that the sum L of subspaces $L_1, \ldots, L_p$ is their direct sum if and only if the union of the bases for these subspaces produces a basis for L.

1.5.15. Prove that the condition stated in Problem 1.5.14 is equivalent to the following condition:

$$\dim (L_1 + \ldots + L_p) = \dim L_1 + \ldots + \dim L_p.$$

1.5.16. Prove that a subspace $L = L_1 + \ldots + L_p$ is the direct sum of the subspaces $L_1, \ldots, L_p$ if and only if the intersection of each of the subspaces L_i, $1 \leqslant i \leqslant p$, and the sum of the remaining subspaces consists of the null vector only.

1.5.17. Let a set of subspaces $L_1, \ldots, L_p$ be ordered. Verify that the necessary and sufficient condition stated in Problem 1.5.16 may be weakened, viz. the intersection of each of the subspaces L_i, $2 \leqslant i \leqslant p$, and the sum of the previous subspaces should consist of the null vector only.

1.5.18. Prove that the sum of subspaces $L_1, \ldots, L_p$ is their direct sum if and only if any set of nonzero vectors $x_1, \ldots, x_p$, all chosen from different subspaces L_j, $j = 1, \ldots, p$, is linearly independent.

1.5.19. Prove the associative law for the direct sum of subspaces, viz. if $L = L_1 \dot{+} \tilde{L}$ and $\tilde{L} = L_2 \dot{+} L_3$, then $L = L_1 \dot{+} L_2 \dot{+} L_3$.

1.5.20. Verify that the direct sum of the linear subspaces L_1 and L_2 spanned by the sets of vectors $x_1 = (2, 3, 11, 5)$, $x_2 = (1, 1, 5, 2)$, $x_3 = (0, 1, 1, 1)$ and $y_1 = (2, 1, 3, 2)$, $y_2 = (1, 1, 3, 4)$, $y_3 = (5, 2, 6, 2)$, respectively, produces the whole space R_4; find the decomposition of the vector $x = (2, 0, 0, 3)$ in terms of these subspaces.

1.5.21. Prove that in the space M_n of polynomials of degree $\leqslant n$ (a) the set L_1 of the even polynomials $f(t)$ (i.e. $f(-t) = f(t)$) and the set L_2 of the odd polynomials (i.e. $f(-t) = -f(t)$) are linear subspaces; (b) the following equality is valid

$$M_n = L_1 \dot{+} L_2.$$

1.5.22. Prove that, for any subspace L_1 of a linear space V, there is a *complementary* subspace, i.e. a subspace L_2 such that

$$V = L_1 \dot{+} L_2.$$

Is the complementary subspace of a given subspace L_1 unique?

1.5.23. Find two different complementary subspaces of the subspace L generated by vectors $x_1 = (1, 3, 0, -1)$, $x_2 = (2, 5, 1, 2)$, $x_3 = (1, 2, 1, 3)$.

1.5.24. In the space M_n of polynomials of degree $\leqslant n$ find a complementary subspace to the space L of the polynomials satisfying the condition $f(1) = 0$.

1.5.25. A space V is decomposed into the direct sum of subspaces $L_1, \ldots, L_p$. Prove that (a) if a vector x is decomposed as follows:

$$x = x_1 + \ldots + x_p, \quad x_i \in L_i,$$

then the decomposition of the vector λx, in terms of the subspaces $L_1, \ldots, L_p$, is of the form

$$\lambda x = \lambda x_1 + \ldots + \lambda x_p;$$

(b) if y is a vector with the decomposition $y_1 + \ldots + y_p$, $y_i \in L_i$, then the decomposition of the vector $x + y$, in terms of the subspaces $L_1, \ldots, L_p$, is of the form

$$x + y = (x_1 + y_1) + \ldots + (x_p + y_p).$$

CHAPTER 2

Euclidean and Unitary Spaces

2.0. Terminology and General Notes

A real linear space E is said to be a *Euclidean space* if for each pair of vectors x, y from E there is a corresponding real number designated by the symbol (x, y) and called the *scalar product* of the vectors x and y; at that the following conditions are fulfilled:

(1) $(x, y) = (y, x)$.

(2) $(x + y, z) = (x, z) + (y, z)$.

(3) $(\alpha x, y) = \alpha (x, y)$.

(4) $(x, x) > 0$ if $x \neq 0$.

Here x, y, z are arbitrary vectors from E and α is an arbitrary real number.

A (nonnegative) number is called the *length* of a vector x if

$$|x| = \sqrt{(x, x)}.$$

A vector whose length equals unity is said to be *normalized*.

For any two vectors x and y, the *Cauchy-Buniakowsky inequality* holds:

$$|(x, y)| \leqslant |x| \cdot |y|.$$

Vectors x and y are called *orthogonal* if their scalar product equals zero. A set of vectors is called *orthogonal* if each pair of the vectors in this set is orthogonal.

Given a linearly independent set of vectors $x_1, x_2, \ldots, x_k$, let us describe an *orthogonalization procedure* that will permit this set to be transformed to an orthogonal set of nonzero vectors $y_1, y_2, \ldots, y_k$.

Set $y_1 = x_1$; the subsequent vectors $y_2, \ldots, y_k$ are then constructed by the following formulae:

$$y_l = x_l - \sum_{i=1}^{l-1} \alpha_i^{(l)} y_i, \quad l = 2, \ldots, k,$$

$$\alpha_i^{(l)} = \frac{(x_l, y_i)}{(y_i, y_i)} \qquad i = 1, \ldots, l-1.$$

A basis for a Euclidean space is called *orthogonal* if it is an orthogonal set. If the vectors of this set are normalized then this basis is called *orthonormal*. Thus, an orthonormal basis $e_1, \ldots, e_n$ is specified by the relations

$$(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For two nonzero vectors x and y of a Euclidean space, the concept of an angle may also be considered, its cosine being determined by the formula

$$\cos(\widehat{x, y}) = \frac{(x, y)}{|x|\,|y|}.$$

A complex linear space U is called *unitary space* if, for each pair of vectors x, y from U, there is a corresponding complex number denoted by (x, y), called the *scalar product* of the vectors x and y, provided that the following conditions are true:

(1) $(x, y) = \overline{(y, x)}$.

(2) $(x + y, z) = (x, z) + (y, z)$.

(3) $(\alpha x, y) = \alpha (x, y)$.

(4) $(x, x) > 0$ if $x \neq 0$.

In a unitary space an angle between vectors is not defined. However, all the above-mentioned definitions and results regarding a Euclidean space also remain valid for a unitary space.

A typical example of a Euclidean space is the arithmetical space R_n in which the scalar product of vectors $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$ is determined by the rule

$$(x, y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n. \qquad (2.0.1)$$

Similarly, a typical example of a unitary space is the space C_n in which, for vectors x and y,

$$(x, y) = \alpha_1\overline{\beta}_1 + \alpha_2\overline{\beta}_2 + \ldots + \alpha_n\overline{\beta}_n. \qquad (2.0.2)$$

In both cases the standard basis for the arithmetical space turns out to be orthonormal.

Let us make some other notes concerning the computational problems of the present chapter.

Suppose it is required to extend an orthogonal set $a_1, \ldots, a_k$ of nonzero vectors of an arithmetic space to form an orthogonal basis for this space. We shall look for a vector a_{k+1} using the conditions for the orthogonality

$$(a_{k+1}, a_1) = 0,$$
$$\cdots\cdots\cdots$$
$$(a_{k+1}, a_k) = 0.$$

Defined by relations (2.0.1) and (2.0.2), these conditions determine a system of linear equations involving the components of the vector a_{k+1}. An arbitrary nonzero solution of this system may be chosen to yield a_{k+1}. Now, the vector a_{k+2} may be determined by the relations

$$\begin{aligned}(a_{k+2}, a_1) &= 0,\\ &\cdots\cdots\\ (a_{k+2}, a_k) &= 0,\\ (a_{k+2}, a_{k+1}) &= 0,\end{aligned}$$

and so on. At each stage of this extension procedure the results of the previous computations can be used and the solution of the linear equation systems found by the Gauss elimination method.

The problem of constructing a basis for an orthogonal complement (see Problem 2.3.2) to the span of a given vector set in an arithmetic space is solved in a similar way. Gaussian elimination can also be applied both to the problems on computation of the vector projections on a given span and to those concerning the construction of bases biorthogonal to the given (see Problems 2.3.10 and 2.3.15).

2.1. Definition of Euclidean Space

In the present section we have set ourselves the following principal goals:
To draw the simplest corollaries from the axioms of the scalar product.
To show that the scalar product may be defined for any real linear space, and in infinitely many ways. Speaking of arithmetic spaces R_n, we demonstrate the concrete techniques that transform the spaces into Euclidean spaces.
To draw the reader's attention to the fact that not only is any subspace of a Euclidean space itself, Euclidean, but, conversely, the scalar product, defined for an arbitrary subspace of a linear space is "extensible" to the whole space.
And, finally, we intend to illustrate the significance of the axiom concerning the positiveness of the scalar product.

2.1.1. Prove that it follows from the axioms for the scalar product that (a) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ for any vectors of a Euclidean space; (b) $(x, \alpha y) = \alpha (x, y)$ for any vectors x, y of a Euclidean space and real number α;

(c) $(x_1 - x_2, y) = (x_1, y) - (x_2, y)$;

(d) $(0, x) = 0$;

(e) $\left(\sum_{i=1}^{k} \alpha_i x_i, \sum_{j=1}^{l} \beta_j x_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_i \beta_j (x_i, y_j)$.

2.1.2. Prove that the scalar product may be defined for any real linear space.

2.1.3. Define the scalar product for the n-dimensional arithmetic space R_n.

2.1.4. Define the scalar product for the space M_n of polynomials of degree $\leqslant n$ with real coefficients.

2.1.5. Let V be a Euclidean space with the scalar product (x, y). Show that if we set

$$\langle x, y\rangle = \lambda (x, y),$$

where λ is a fixed positive number, then, for $\langle x, y\rangle$, all the axioms of the scalar product are fulfilled. What is the geometric meaning of the transfer from (x, y) to $\langle x, y\rangle$ in the three-dimensional space of geometric vectors?

2.1.6. Prove that if $(x, y)_1$ and $(x, y)_2$ are two different scalar products on the same linear space V, then the following are also the scalar products

(a) $(x, y) = (x, y)_1 + (x, y)_2$;

(b) $(x, y) = \lambda (x, y)_1 + \mu (x, y)_2$,

where λ and μ are arbitrary nonnegative numbers that are not simultaneously equal to zero.

2.1.7. Let $x = (\alpha_1, \alpha_2)$ and $y = (\beta_1, \beta_2)$ be arbitrary vectors of the arithmetic space R_2. Show that the scalar product on R_2 may be defined in the following ways:

(a) $(x, y) = \alpha_1\beta_1 + \alpha_2\beta_2$;

(b) $(x, y) = 2\alpha_1\beta_1 + 5\alpha_2\beta_2$;

(c) $(x, y) = \alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1 + 2\alpha_2\beta_2$.

Evaluate the scalar product of the vectors $x = (1, 1)$ and $y = (-3, 2)$ by each of the above rules.

2.1.8.* Prove that the scalar product on R_2 may be defined by the formula

$$(x, y) = a\alpha_1\beta_1 + b\alpha_1\beta_2 + b\alpha_2\beta_1 + c\alpha_2\beta_2$$

if and only if $a > 0$ and $ac > b^2$ simultaneously.

2.1.9.* Prove that the scalar product on R_3 may be defined in the following way: if $x = (\alpha_1, \alpha_2, \alpha_3)$ and $y = (\beta_1, \beta_2, \beta_3)$, then

$$(x, y) = 10\alpha_1\beta_1 + 3\alpha_1\beta_2 + 3\alpha_2\beta_1 + 2\alpha_2\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_3\beta_3.$$

2.1.10.* Prove that the scalar product on R_n may be defined by the formula

$$\begin{aligned}(x, y) = {} & a_{11}\alpha_1\beta_1 + a_{12}\alpha_1\beta_2 + \ldots + a_{1n}\alpha_1\beta_n \\ & + a_{21}\alpha_2\beta_1 + a_{22}\alpha_2\beta_2 + \ldots + a_{2n}\alpha_2\beta_n \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + a_{n1}\alpha_n\beta_1 + a_{n2}\alpha_n\beta_2 + \ldots + a_{nn}\alpha_n\beta_n\end{aligned}$$

on condition that

(a) $a_{ij} = a_{ji}$ if $i \neq j$;

(b) $a_{ii} > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$

2.1.11. Let a be a fixed vector of a Euclidean space V and α a fixed real number. Is the set of all vectors x, for which $(x, a) = \alpha$, a linear subspace of the space V?

2.1.12. Prove that each subspace of a Euclidean space V is also Euclidean in the sense of the scalar product defined for V.

2.1.13. A linear space V is resolved into the direct sum of subspaces $L_1, \ldots, L_p$. For each of the subspaces L_i the scalar product is defined. Prove that the scalar product may be defined for the whole space V, assuming that: if x and y are two arbitrary vectors from V, with decompositions in terms of the subspaces $L_1, \ldots, L_p$ respectively $x = x_1 + \ldots + x_p$ and $y = y_1 + \ldots + y_p$, then

$$(x, y) = (x_1, y_1)_1 + \ldots + (x_p, y_p)_p,$$

where the scalar product $(x_i, y_i)_i$ is found by the rule given for L_i.

2.1.14. In the arithmetic space R_4 the scalar product of two vectors of the form $\tilde{x} = (\alpha_1, \alpha_2, 0, 0)$ and $\tilde{y} = (\beta_1, \beta_2, 0, 0)$ is defined as follows:

$$(\tilde{x}, \tilde{y})_1 = \alpha_1\beta_1 + 2\alpha_2\beta_2,$$

and that of the vectors $\tilde{\tilde{x}}$ and $\tilde{\tilde{y}}$ of the form

$$\tilde{\tilde{x}} = (0, 0, \alpha_3, \alpha_4) \quad \text{and} \quad \tilde{\tilde{y}} = (0, 0, \beta_3, \beta_4)$$

is specified by another rule

$$(\tilde{\tilde{x}}, \tilde{\tilde{y}})_2 = \alpha_3\beta_3 + \alpha_3\beta_4 + \alpha_4\beta_3 + 2\alpha_4\beta_4.$$

Define the scalar product for the whole space R_4 (by the method indicated in Problem 2.1.13). Compute the scalar product of the vectors $x = (1, 2, 3, 4)$ and $y = (-3, 1, -3, 2)$ by the rule obtained.

2.1.15.* The scalar product (x, y) is defined for a subspace L of a linear space V. Prove that the scalar product may be defined for the whole space V so as to be identical with the original scalar product (x, y) of the vectors x and y from L.

2.1.16.* Prove that in the Cauchy-Buniakowski inequality for vectors x and y of Euclidean space, viz.

$$(x, y)^2 \leqslant (x, x)(y, y),$$

the equality sign is upheld if and only if the vectors x and y are linearly dependent.

2.1.17. Prove the following by the Cauchy-Buniakowski inequality:

(a) $\left(\sum_{i=1}^{n} \alpha_i\beta_i\right)^2 \leqslant \left(\sum_{i=1}^{n} \alpha_i^2\right)\left(\sum_{i=1}^{n} \beta_i^2\right);$

(b) $\left(\sum_{i=1}^{n} \alpha_i\beta_i\right)^2 \leqslant \left(\sum_{i=1}^{n} \lambda_i\alpha_i^2\right)\left(\sum_{i=1}^{n} \frac{1}{\lambda_i}\beta_i^2\right),$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are arbitrary real numbers and $\lambda_1, \ldots, \lambda_n$ are positive numbers.

2.1.18.* Given a space V for which the "scalar product" is defined with the fourth axiom replaced by a weaker requirement, viz. $(x, x) \geqslant 0$ for any vector x, prove that (a) the Cauchy-Buniakowski inequality is valid; (b) the set M of vectors, such that $(x, x) = 0$, forms a subspace; (c) for any vector x from M and any vector y from V the scalar product equals zero; (d) if N is an arbitrary, complementary subspace of M, and

$$x = x_M + x_N, \quad y = y_M + y_N$$

are the decompositions of the vectors x and y in terms of the subspaces M and N, then the equality sign in the Cauchy-Buniakowski relationship for the vectors x and y is valid if and only if x_N and y_N are linearly dependent.

2.1.19. Will the Cauchy-Buniakowski inequality be upheld if the fourth axiom in the definition of the scalar product is discarded?

2.2. Orthogonality, Orthonormal Basis, Orthogonalization Procedure

The problems of the present section concern the following two principal topics:

The orthogonalization procedure, its applications to the construction of an orthogonal basis for a space and to the determination of the linear dependence of a given vector set.

The orthonormal bases of a Euclidean space and their significance in evaluating the scalar product. We also intend to show the dependence of the orthonormality property of a basis on the method of defining the scalar product for a given linear space.

2.2.1. Prove that in a Euclidean space E (a) the null vector is the only one possessing the property of orthogonality for all vectors of the space; (b) if the equality $(a, x) = (b, x)$ is valid for any vector x from E, then $a = b$.

2.2.2. Prove that if $x, y, \ldots, z$ is an orthogonal set of vectors, then, for any numbers $\lambda, \mu, \ldots, \nu$, the set of vectors $\lambda x, \mu y, \ldots, \nu z$ is also orthogonal.

2.2.3. Prove that if a vector x is orthogonal to each of the vectors $y_1, \ldots, y_l$, then it is also orthogonal to any linear combination of these vectors.

2.2.4. Prove that an orthogonal set of nonzero vectors is linearly dependent.

We shall assume, hereafter, that the scalar product of the vectors $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $y = (\beta_1, \beta_2, \ldots, \beta_n)$ *belonging to an arithmetic space* R_n *is determined by the formula*

$$(x, y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n. \qquad (2.2.1)$$

Apply the procedure of orthogonalization to the following sets of vectors in the space R_n:

2.2.5. $x_1 = (1, -2, 2)$,
$x_2 = (-1, 0, -1)$,
$x_3 = (5, -3, -7)$.

2.2.6. $x_1 = (1, 1, 1, 1)$,
$x_2 = (3, 3, -1, -1)$,
$x_3 = (-2, 0, 6, 8)$.

2.2.7.* Prove that the orthogonalization procedure applied to a linearly independent set of vectors $x_1, \ldots, x_k$ leads to an orthogonal set of nonzero vectors $y_1, \ldots, y_k$.

2.2.8. Prove that in any Euclidean space there exists (a) an orthogonal basis; (b) an orthonormal basis.

2.2.9. Prove that (a) any nonzero vector may be included in some orthogonal basis for a Euclidean space; (b) any orthogonal set of nonzero vectors may be extended to form an orthogonal basis for the space.

Verify that the following sets of vectors are orthogonal. Extend them to form orthogonal bases.

2.2.10. $x_1 = (1, -2, 1, 3)$,
$x_2 = (2, 1, -3, 1)$.

2.2.11. $x_1 = (1, -1, 1, -3)$,
$x_2 = (-4, 1, 5, 0)$.

Extend the following sets of vectors to form orthonormal bases:

2.2.12. $x_1 = \left(-\frac{11}{15}, -\frac{2}{15}, \frac{2}{3}\right)$,

$x_2 = \left(-\frac{2}{15}, -\frac{14}{15}, -\frac{1}{3}\right)$.

2.2.13. $x_1 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$,

$x_2 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$.

2.2.14. Prove that the scalar product of any two vectors x and y of a Euclidean space is expressed in terms of their coordinates in certain bases by the formula

$$(x, y) = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n,$$

if and only if these bases are orthonormal.

2.2.15. Prove that the coordinates $\alpha_1, \ldots, \alpha_n$ of a vector x in an orthonormal basis $e_1, \ldots, e_n$ are found by the formulae

$$\alpha_i = (x, e_i), \quad i = 1, \ldots, n.$$

2.2.16. Find the dimension of the subspace formed by all vectors x such that $(a, x) = 0$. Here a is a fixed nonzero vector of a Euclidean space.

2.2.17.* Let $e_1, \ldots, e_n$ be an orthonormal basis for a Euclidean space. Find an expression for the scalar product of two arbitrary vectors x and y in terms of their coordinates in: (a) the basis $\lambda_1 e_1, \lambda_2 e_2, \ldots, \lambda_n e_n$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are nonzero numbers; (b) the basis $e_1 + e_2, e_2, e_3, \ldots, e_n$.

2.2.18. Let the procedure of orthogonalization be applied to an arbitrary set of vectors $x_1, \ldots, x_k$. Prove that (a) if the set $x_1, \ldots, x_k$ is linearly dependent, then at some stage of the orthogonalization procedure a zero vector is obtained; (b) if the vectors $y_1, \ldots, y_{l-1}$ ($l \leqslant k$) obtained in the orthogonalization process are nonzero vectors and $y_l = 0$, then, in the original set of vectors $x_1, \ldots, x_k$, the subset $x_1, \ldots, x_{l-1}$ is linearly independent and the vector x_l is linearly dependent on this subset.

Applying the orthogonalization procedure, construct orthogonal bases for the subspaces spanned by the given sets of vectors:

2.2.19.
$$x_1 = (2, 3, -4, -6),\quad x_2 = (1, 8, -2, -16),\quad x_3 = (12, 5, -14, 5),\quad x_4 = (3, 11, 4, -7).$$

2.2.20.
$$x_1 = (1, 1, -1, -2),\quad x_2 = (-2, 1, 5, 11),\quad x_3 = (0, 3, 3, 7),\quad x_4 = (3, -3, -3, -9).$$

2.2.21. Prove that if a set of vectors of the arithmetic space R_n

$$\begin{aligned} x_1 &= (\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots, \alpha_{1n}),\\ x_2 &= (\ 0,\ \alpha_{22}, \alpha_{23}, \ldots, \alpha_{2n}),\\ x_3 &= (\ 0,\ \ 0,\ \alpha_{33}, \ldots, \alpha_{3n}),\\ &\ldots\ldots\ldots\ldots\\ x_n &= (\ 0,\ \ 0,\ \ 0, \ldots, \alpha_{nn}) \end{aligned}$$

forms an orthogonal basis for this space, then (a) $\alpha_{ii} \neq 0$, $i = 1, \ldots, n$; (b) $\alpha_{ij} = 0$ if $i \neq j$.

2.2.22.* In the space R_n ($n > 1$) there is an orthogonal basis $e_1, \ldots, e_n$ such that all the components of each of the vectors e_i are either 1 or -1. Prove that the dimension of the space R_n is either 2 or a multiple of 4.

2.2.23.* Given a linearly independent set of vectors $x_1, \ldots, x_k$ and two orthogonal sets of nonzero vectors $y_1, \ldots, y_k$ and $z_1, \ldots, z_k$ such that the vectors y_i and z_i are linearly expressed in terms of $x_1, \ldots, x_i$ ($i = 1, \ldots, k$), prove that $y_i = \alpha_i z_i$ ($i = 1, \ldots, k$) where $\alpha_i \neq 0$.

2.2.24. A scalar product is defined arbitrarily for the space M_n of polynomials with real coefficients of degree $\leqslant n$. Prove that in the Euclidean space so formed (a) an orthogonal basis exists containing one polynomial of each degree k, $0 \leqslant k \leqslant n$; (b) if $f_0(t), f_1(t), \ldots, f_n(t)$ and $g_0(t), g_1(t), \ldots, g_n(t)$ are the two orthogonal bases possessing the above property, then the polynomials enumerated in the same way making up these bases are different only in scalar multipliers, i.e. $g_i(t) = \alpha_i f_i(t)$, $i = 0, 1, \ldots, n$.

2.2.25. Let $e_1, \ldots, e_n$ be an arbitrary basis for a real linear space V. Prove that the scalar product may be defined for the space V so that the set of vectors $e_1, \ldots, e_n$ may be an orthonormal basis for the obtained Euclidean space.

2.2.26. Define the scalar product for the space M_n of polynomials of degree $\leqslant n$ so that the basis

$$1, t, \frac{t^2}{2!}, \ldots, \frac{t^n}{n!}$$

becomes orthonormal.

2.3. Orthogonal Complement, Orthogonal Sums of Subspaces

The principal goals of this section:

To show the various properties of what will become a very important notion, that of an orthogonal complement to a subspace.

To provide computational problems on how to find orthogonal complements and, in particular, to underline the relation of these problems to the solution of systems of linear equations. The problem on the perpendicular (see Problems 2.3.10 to 2.3.14) is also included into this category.

To indicate the important corollary of the theorems about orthogonal complements that is the existence, for any basis for a Euclidean space, of a biorthogonal basis.

To note the similarity between the theorems about the direct sums of subspaces of a linear space and the theorems concerning the orthogonal sums in a Euclidean space. In particular, the decomposition of a Euclidean space into the orthogonal sum of subspaces is the analogue of a decomposition in terms of an orthonormal basis and in the sense the subspaces that yield a given linear space when directly summed, play the role of a generalized basis for the whole space.

2.3.1. Let L be a k-dimensional subspace of a Euclidean space E, $k < n$. Prove that there is in E a nonzero vector orthogonal to all vectors of L (or, in other words, a vector orthogonal to the subspace L).

2.3.2. Prove that the set $L^\perp$ of all vectors orthogonal to a linear subspace L is also a linear subspace. $L^\perp$ is called the *orthogonal complement of the subspace L*.

2.3.3. Let L be an arbitrary subspace of a Euclidean space E. Prove that E is the direct sum of the subspaces L and $L^\perp$. Note the relation between the dimensions of the subspaces L and $L^\perp$ that follows from the statement.

2.3.4. Prove that the orthogonal complement of a linear subspace of a Euclidean space E possesses the following properties:

(a) $(L^\perp)^\perp = L$;

(b) if $L_1 \subset L_2$, then $L_2^\perp \subset L_1^\perp$;

(c) $(L_1 + L_2)^{\perp} = L_1^{\perp} \cap L_2^{\perp}$;
(d) $(L_1 \cap L_2)^{\perp} = L_1^{\perp} + L_2^{\perp}$;
(e) $E^{\perp} = O$, $O^{\perp} = E$.

Here O is the null subspace containing the zero vector only.

2.3.5. The direct sum of subspaces L_1 and L_2 produces a Euclidean space E. Prove that the same holds for their orthogonal complements, i.e. $E = L_1^{\perp} \dot{+} L_2^{\perp}$.

2.3.6. Find a basis for the orthogonal complement $L^{\perp}$ of the span L of the following set of vectors R_4:

$$x_1 = (1, 3, 0, 2),\ x_2 = (3, 7, -1, 2),\ x_3 = (2, 4, -1, 0).$$

2.3.7. In a Euclidean space E, an orthonormal basis $e_1, \ldots, e_n$ is fixed. Prove the following:

(a) If

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1n}\alpha_n &= 0, \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{2n}\alpha_n &= 0, \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \ldots + a_{mn}\alpha_n &= 0 \end{aligned}$$

is an arbitrary set of linear equations in n unknowns, then the set of vectors z, whose coordinates with respect to the basis $e_1, \ldots, e_n$ satisfy this system, is a linear subspace of the space E. The dimension of this subspace equals $n - r$ where r is the rank of the following set of vectors of the arithmetic space:

$$\begin{aligned} n_1 &= (a_{11}, a_{12}, \ldots, a_{1n}), \\ n_2 &= (a_{21}, a_{22}, \ldots, a_{2n}), \\ \ldots\ldots\ldots \\ n_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}); \end{aligned}$$

(b) any subspace L of the space E may be described by a particular system of linear equations. This means that a vector z belongs to the subspace L if and only if its coordinates in the basis $e_1, \ldots, e_n$ satisfy the given system. If r is the dimension of the subspace L, then any system describing this subspace consists of not less than $n - r$ equations; in addition, there exists a system consisting of precisely $n - r$ equations;

(c) systems of linear equations describing the subspace L and its orthogonal complement $L^{\perp}$ in a given basis are related to each other as follows: the coefficients of the system describing one of these subspaces act as the coordinates of the vectors spanning the other subspace.

2.3.8. In the space M_n of polynomials with real coefficients of degree $\leqslant n$ a scalar product for the polynomials $f(t) = a_0 + a_1 t +$

$+ \ldots + a_n t^n$ and $g(t) = b_0 + b_1 t + \ldots + b_n t^n$ (in which higher-order coefficients of the polynomials may be equal to zero) is defined by the formula

$$(f, g) = a_0 b_0 + a_1 b_1 + \ldots + a_n b_n. \quad (2.3.1)$$

Find the orthogonal complement of (a) the subspace of all polynomials satisfying the condition $f(1) = 0$; (b) the subspace of all even polynomials of the space M_n.

2.3.9. Find the systems of linear equations that describe the subspace L defined in Problem 2.3.6 and that describe its orthogonal complement $L^\perp$.

2.3.10. Let L be a linear subspace of a Euclidean space E. Prove that any vector x from E can be represented, in a unique way, as $x = y + z$ where y belongs to L and z is orthogonal to L. The vector y is called the *orthogonal projection* of the vector x on the subspace L, and z is called the *perpendicular* drawn from x to L. Find, for the given subspace L and vector x, a method of evaluating y and z.

2.3.11.* Let $x_1, x_2, \ldots, x_k$ be an arbitrary set of vectors in a Euclidean space E. Prove that for any vector x from E the system of linear equations

$$(x_1, x_1)\,\alpha_1 + (x_2, x_1)\,\alpha_2 + \ldots + (x_k, x_1)\,\alpha_k = (x, x_1),$$
$$(x_1, x_2)\,\alpha_1 + (x_2, x_2)\,\alpha_2 + \ldots + (x_k, x_2)\,\alpha_k = (x, x_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(x_1, x_k)\,\alpha_1 + (x_2, x_k)\,\alpha_2 + \ldots + (x_k, x_k)\,\alpha_k = (x, x_k)$$

has at least one solution. In which case is the solution unique?

Find the orthogonal projection and perpendicular drawn from the vector x to the subspace L.

2.3.12. $x = (14, -3, -6, -7)$. L is spanned by the vector $y_1 = (-3, 0, 7, 6)$, $y_2 = (1, 4, 3, 2)$, $y_3 = (2, 2, -2, -2)$.

2.3.13. $x = (2, -5, 3, 4)$. L is spanned by the vectors $y_1 = (1, 3, 3, 5)$, $y_2 = (1, 3, -5, -3)$, $y_3 = (1, -5, 3, -3)$.

2.3.14. $x = (-3, 0, -5, 9)$. L is determined by the system of equations:

$$3\alpha_1 + 2\alpha_2 + \alpha_3 - 2\alpha_4 = 0,$$
$$5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = 0,$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 10\alpha_4 = 0.$$

2.3.15. Two sets of vectors $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in a Euclidean space are called *biorthogonal* if

$$(x_i, y_j) = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Prove that each of the two biorthogonal sets of vectors is linearly independent.

2.3.16. Prove that a unique biorthogonal basis exists for any basis of a Euclidean space.

2.3.17. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be a pair of biorthogonal bases for a Euclidean space. Prove that for any k, $1 \leqslant k < n$ the orthogonal complement to the subspace spanned by the vectors $e_1, \ldots, e_k$ coincides with the span of the vectors $f_{k+1}, \ldots, f_n$.

Find biorthogonal bases to the following bases of the space R_4:

2.3.18. $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 2, 0, 0)$, $e_3 = (0, 0, 3, 0)$, $e_4 = (0, 0, 0, 4)$.

2.3.19. $e_1 = (1, 0, 1, 0)$, $e_2 = (0, 1, 2, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 3, 1)$.

2.3.20. $e_1 = (1, 1, 1, 1)$, $e_2 = (0, 1, 1, 1)$, $e_3 = (0, 0, 1, 1)$, $e_4 = (0, 0, 0, 1)$.

2.3.21. $e_1 = (1, 1, 1, 1)$, $e_2 = (1, 1, -1, -1)$, $e_3 = (1, -1, 1, -1)$, $e_4 = (1, -1, -1, 1)$.

2.3.22. In a Euclidean space E biorthogonal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are fixed. Prove that

(a) if x is an arbitrary vector from E, then in its decomposition in terms of the basis $e_1, \ldots, e_n$ viz. $x = \alpha_1 e_1 + \ldots + \alpha_n e_n$, the coefficients α_i are determined by the formulae $\alpha_i = (x, f_i)$, $i = 1, \ldots, n$;

(b) the scalar product of arbitrary vectors x and y is determined by the formula

$$(x, y) = \sum_{i=1}^{n} (x, f_i)(y, e_i) = \sum_{i=1}^{n} \alpha_i \beta_i$$

where $\beta_1, \ldots, \beta_n$ are the coefficients of the decomposition of the vector y in terms of the basis $f_1, \ldots, f_n$.

2.3.23. The linear subspaces $L_1, \ldots, L_p$ of a Euclidean space E, are mutually *orthogonal* (this means that, for each subspace L_i, any vector of that subspace is orthogonal to all the other subspaces). Prove that the sum of the subspaces $L_1, \ldots, L_p$ is their direct sum. (The sum of mutually orthogonal subspaces is called their *orthogonal sum* and denoted by $L_1 \oplus \ldots \oplus L_p$.)

2.3.24. Prove that the sum L of subspaces $L_1, \ldots, L_p$ is their orthogonal sum if and only if the union of the orthogonal bases for these subspaces yields the orthogonal basis for L.

2.3.25. Prove the associative law for the orthogonal sum of subspaces, i.e. if $L = L_1 \oplus \tilde{L}$ and $\tilde{L} = L_2 \oplus L_3$, then

$$L = L_1 \oplus L_2 \oplus L_3.$$

2.3.26. The direct sum of the subspaces $L_1, \ldots, L_p$ yields a Euclidean space E. Prove that this sum is orthogonal if and only if for any vectors x and y from E, decomposed in terms of the subspaces $L_1, \ldots, L_p$, respectively: $x = x_1 + \ldots + x_p$ and $y = y_1 + \ldots + y_p$, their scalar product satisfies the equality

$$(x, y) = (x_1, y_1) + \ldots + (x_p, y_p).$$

2.3.27. A linear space V is arbitrarily decomposed into the direct sum of subspaces, i.e. $V = L_1 \dot{+} \ldots \dot{+} L_p$. Prove that a scalar product for V may be defined so that every pair of the subspaces L_i is orthogonal.

2.4. Lengths, Angles, Distances

We intend in this section:

To provide a number of simple problems regarding the definitions of the length, angle, and distance, and corroborating the validity of elementary Euclidean geometry theorems in an arbitrary Euclidean space.

To interpret the problem of decomposing a vector in terms of the orthogonal complementary subspace as the problem of determining the least distance from the vector to the subspace.

To determine the angle between a vector and a subspace and show that this definition generalizes the notion of an angle between a vector and a plane in three-dimensional Euclidean space.

2.4.1. Prove that the lengths of the vectors x and $y = \alpha x$ satisfy the equality

$$|y| = |\alpha|\,|x|.$$

2.4.2. How is the angle between nonzero vectors x and y altered if: (a) the vector x is multiplied by a positive number; (b) the vector x is multiplied by a negative number; (c) both vectors x and y are multiplied by negative numbers?

In the subsequent problems the ordered set of three vectors x, y and $x - y$ in an arbitrary Euclidean space is called, just as it is in three-dimensional Euclidean space, a *triangle* "generated by or drawn on the vectors x and y". Accordingly, the parallelogram generated by the vectors x and y is considered to have the vectors $x + y$ and $x - y$ as its diagonals.

2.4.3. Prove that the triangles generated by vectors x, y, and αx, αy respectively, where α is an arbitrary nonzero number, have equal corresponding angles.

2.4.4. Find the lengths of the sides of the triangle generated by the vectors of the space R_4 $x = (2, -1, 3, -2)$ and $y = (3, 1, 5, 1)$. Find the angles between the sides of the triangle, i.e. vectors x, y and $x - y$. Which of these angles is it natural to consider as interior and exterior angles of the triangle?

2.4.5. Formulate and prove the cosine law for a triangle generated by vectors x and y in an arbitrary Euclidean space.

2.4.6. Determine whether the triangle generated by the polynomials $t^2 + 3t$ and $2t^2 + 2t - 1$ is acute-angled or obtuse-angled if the scalar product of the polynomials $f(t) = a_0 + a_1 t + a_2 t^2$ and $g(t) = b_0 + b_1 t + b_2 t^2$ is given by the following formulae (a) $(f, g) = a_0 b_0 + a_1 b_1 + a_2 b_2$, (b) $(f, g) = a_0 b_0 + 2a_1 b_1 + a_2 b_2$.

2.4.7. Prove Pythagoras theorem and its converse, viz. that two vectors x and y of a Euclidean space are orthogonal if, and only if, $|x - y|^2 = |x|^2 + |y|^2$.

2.4.8. Prove that for an arbitrary triangle in a Euclidean space (a) the length of each side does not exceed the sum of the lengths of the other two sides; (b) the length of each side is not less than the absolute value of the difference of the other two sides.

2.4.9. Prove that in a parallelogram generated by vectors x and y, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.

2.4.10. Prove that $|x| = |y|$ if, and only if, the vectors $x + y$ and $x - y$ are orthogonal. Specify the geometric sense of the statement.

2.4.11. Let $e_1, \ldots, e_n$ be an orthonormal basis for a Euclidean space, and x be an arbitrary normalized vector. Prove that the coordinates of the vector x in the basis $e_1, \ldots, e_n$ are equal to the cosines of the angles $\alpha_1, \ldots, \alpha_n$ formed by x and the basis vectors. Hence deduce the relation

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \ldots + \cos^2 \alpha_n = 1.$$

2.4.12. The number

$$\rho(x, y) = |x - y|$$

is called the *distance between vectors* x and y of a Euclidean space. Show that the distance thus defined satisfies the triangle inequality

$$\rho(x, z) \leqslant \rho(x, y) + \rho(y, z),$$

for any three vectors x, y, z.

2.4.13. Prove that, in the triangle inequality for vectors x, y and z, the equality sign appears if, and only if, $(x - y) = \alpha (y - z)$, $\alpha \geqslant 0$.

2.4.14. In the space M_n of polynomials of degree $\leqslant n$ a scalar product for polynomials $f(t) = a_0 + a_1 t + \ldots + a_n t^n$ and $g(t) = b_0 + b_1 t + \ldots + b_n t^n$ is defined by formula (2.3.1). Given polynomials

$$f_1(t) = 3t^2 + 2t + 1, \quad f_2(t) = -t^2 + 2t + 1,$$
$$f_3(t) = 3t^2 + 2t + 5, \quad f_4(t) = 3t^2 + 5t + 2,$$

(a) find a polynomial $f_0(t)$ of degree $\leqslant 2$ equidistant from $f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$;

(b) find the distance between $f_0(t)$ and each of the polynomials $f_1(t), f_2(t), f_3(t), f_4(t)$;

(c) prove that any polynomial of the form

$$f_0(t) + m_3t^3 + \ldots + m_nt^n$$

is also equidistant from $f_1(t), f_2(t), f_3(t), f_4(t)$, and find the distance between these polynomials.

2.4.15.* Given a subspace L and an arbitrary vector x of a Euclidean space, the number

$$\rho(x, L) = \inf_{y \in L} \rho(x, y)$$

is called the *distance between the vector x and subspace L*. Prove that (a) the distance $\rho(x, L)$ is equal to the length of the perpendicular drawn from x to L; (b) the nearest vector of the subspace L to the vector x is the orthogonal projection of x on L; (c) for any y from L,

$$\rho(x + y, L) = \rho(x, L).$$

2.4.16.* A subspace L is the orthogonal direct sum of subspaces L_1 and L_2. If a vector x is orthogonal to the subspace L_1, prove that

$$\rho(x, L) = \rho(x, L_2).$$

2.4.17.* Let a be a fixed vector in a Euclidean space, and let L be the subspace of all vectors orthogonal to a. Prove that the distance between an arbitrary vector x and the subspace L may be found by the formula

$$\rho(x, L) = \frac{|(x, a)|}{|a|}.$$

2.4.18. A scalar product for the space M_n of polynomials of degree $\leqslant n$ is computed in terms of the coefficients of the polynomials by Formula (2.3.1). Find the distance between the subspace M_{n-1} of all polynomials of degree $\leqslant n - 1$, and (a) the polynomial t^n; (b) the polynomial $t^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0$; (c) the polynomial $at^n + a_{n-1}t^{n-1} + \ldots + a_1t + a_0$.

2.4.19. In the space M_n with the same scalar product as defined in (2.3.1), consider the subspace L of all polynomials fulfilling the condition $f(1) = 0$. Prove that the distance between an arbitrary polynomial $g(t)$ and the subspace L equals

$$\rho(g, L) = \frac{g(1)}{\sqrt{n+1}}.$$

2.4.20.* Given a subspace L and vector x of a Euclidean space, the least angle formed by x with any vector from L is called the *angle between the vector x and subspace L*. Prove that the angle between x and L is equal to the angle between x and its orthogonal projection y on L. Show that vectors of the subspace L form the same angle with the vector x if, and only if, they are of the form αy, $\alpha > 0$.

2.4.21. Prove that the sum of the angle formed by a vector x with an arbitrary subspace L, and of the angle formed by x and the orthogonal complement $L^{\perp}$ equals $\pi/2$.

2.4.22. A Euclidean space E is resolved into the orthogonal sum of subspaces $L_1, \ldots, L_p$. Prove that the angles $\alpha_1, \ldots, \alpha_p$ formed by an arbitrary vector x with the subspaces $L_1, \ldots, L_p$ satisfy the relation

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \ldots + \cos^2 \alpha_p = 1.$$

Compare this formula with the formula obtained in Problem 2.4.11.

2.4.23. A subspace L is the orthogonal sum of subspaces L_1 and L_2. A vector x is orthogonal to the subspace L_1. Prove that the angle between x and L equals the angle between x and L_2.

Find the angle between the vector x and linear subspace L generated by the vectors y_1, y_2, y_3:

2.4.24. $x = (-3, 15, 1, -5)$; $y_1 = (2, 3, -4, -6)$, $y_2 = (1, 8, -2, -16)$, $y_3 = (1, -5, -2, 10)$.

2.4.25. $x = (3, 1, \sqrt{2}, -2)$; $y_1 = (2, -1, 2, 1)$, $y_2 = (-1, 2, -2, 1)$, $y_3 = (-1, 1, -1, 0)$.

2.5. Unitary Spaces

Most of the problems in the present section are similar to the problems on Euclidean spaces given earlier. We intend to show, by this similarity, that the basic results proved for the case of a Euclidean space remain valid for arbitrary unitary spaces as well. At the same time, we have also attempted to illustrate the theoretical differences between the real and complex cases, particularly for geometric theorems. We describe, in conclusion, a technique for transferring from a Euclidean to unitary space (the so-called "complexification" of a unitary space), and the inverse transfer (the "decomplexification").

2.5.1. Prove that it follows from the axioms of the scalar product on a unitary space that (a) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$, for any vectors in a unitary space; (b) for any vectors x and y in a unitary space and any complex number α, $(x, \alpha y) = \bar{\alpha}(x, y)$; (c) $(0, x) = (x, 0) = 0$;

(d) $\left(\sum_{i=1}^{k} \alpha_i x_i, \sum_{j=1}^{l} \beta_j y_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_i \bar{\beta}_j (x_i, y_j)$.

2.5.2. Prove that the scalar product may be defined for any complex linear space.

2.5.3. Define the scalar product for the n-dimensional complex arithmetic space C_n.

2.5.4. Define the scalar product for the space of polynomials with complex coefficients of degree $\leqslant n$ regarded as a complex linear space under the usual operations for the addition of polynomials and the multiplication of a polynomial by a complex number.

2.5.5. Prove that the scalar product of the vectors $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $y = (\beta_1, \beta_2, \ldots, \beta_n)$ can be defined for the space C_n by the formula

$$\begin{aligned}(x, y) = & a_{11}\alpha_1\bar{\beta}_1 + a_{12}\alpha_1\bar{\beta}_2 + \ldots + a_{1n}\alpha_1\bar{\beta}_n \\ & + a_{21}\alpha_2\bar{\beta}_1 + a_{22}\alpha_2\bar{\beta}_2 + \ldots + a_{2n}\alpha_2\bar{\beta}_n \\ & \ldots\ldots\ldots\ldots\ldots\ldots \\ & + a_{n1}\alpha_n\bar{\beta}_1 + a_{n2}\alpha_n\bar{\beta}_2 + \ldots + a_{nn}\alpha_n\bar{\beta}_n\end{aligned}$$

on condition that

(a) $a_{ij} = \bar{a}_{ji}$, for all i, j;

(b) $a_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$

2.5.6. Prove that the scalar product of two vectors x and y is expressed in terms of their coordinates by the formula

$$(x, y) = \alpha_1\bar{\beta}_1 + \alpha_2\bar{\beta}_2 + \ldots + \alpha_n\bar{\beta}_n$$

if, and only if, the corresponding bases of the unitary space are orthonormal.

2.5.7. Prove the Cauchy-Buniakowski inequality for the case of a unitary space

$$|(x, y)|^2 \leqslant (x, x)(y, y).$$

Hence deduce the relation

$$\left|\sum_{i=1}^{n} \alpha_i\bar{\beta}_i\right|^2 \leqslant \left(\sum_{i=1}^{n} |\alpha_i|^2\right)\left(\sum_{i=1}^{n} |\beta_i|^2\right)$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are arbitrary complex numbers.

2.5.8. Prove that the Pythagoras theorem remains valid in an arbitrary unitary space, viz., if vectors x and y are orthogonal then

$$|x - y|^2 = |x|^2 + |y|^2.$$

Show, however, that the converse theorem is incorrect.

2.5.9. Prove that vectors x and y of a unitary space are orthogonal if, and only if, for any numbers α and β

$$|\alpha x + \beta y|^2 = |\alpha x|^2 + |\beta y|^2.$$

2.5.10. Prove that the statement of Problem 2.4.10 does not hold in the case of a unitary space. Which statement, precisely, of the two listed in the problem becomes invalid?

2.5.11. Prove, however, that the statement of Problem 2.4.9

$$|x + y|^2 + |x - y|^2 = 2|x|^2 + 2|y|^2$$

is also valid for a unitary space.

2.5.12. Prove the equality

$$4(x, y) = |x + y|^2 - |x - y|^2 + i|x + iy|^2 - i|x - iy|^2. \tag{2.5.1}$$

2.5.13.* Let R be a real space and C the set made up of all formal sums $x + iy$ where $x \in R$, $y \in R$. Prove that

(a) the set C is a complex linear space if the linear operations are defined for it by the formulae

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$
$$\lambda(x + iy) = (\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x)$$

where $\lambda = \alpha + i\beta$ is an arbitrary complex number;

(b) a set of vectors $x_1, \ldots, x_k$ of the space R is linearly dependent, or independent, when the set of vectors $x_1 + i0, \ldots, x_k + i0$ of the space C is linearly dependent, or independent;

(c) the dimension of the space C equals the dimension of the space R.

The technique just described for constructing a complex space from a given real linear space R, and with the same dimension is called the *complexification* of the space R.

2.5.14. Let R be a Euclidean space with scalar product (x, y), and C the complex space obtained from R by the complexification. Prove that

(a) the space C can be converted into a unitary space if the scalar product is determined by the formula

$$\langle x_1 + iy_1, x_2 + iy_2\rangle = [(x_1, x_2) + (y_1, y_2)] + i[(y_1, x_2) - (x_1, y_2)];$$

(b) if $e_1, \ldots, e_k$ is an orthogonal set of vectors from R, then the set of vectors $e_1 + i0, \ldots, e_k + i0$ from the space C with the scalar product just given is also orthogonal;

(c) if $e_1, \ldots, e_n$ is an orthonormal basis for R, then $e_1 + i0, \ldots, e_n + i0$ is an orthonormal basis for C.

2.5.15. Complexify the n-dimensional real arithmetic space R_n (with the customary scalar product). What sort of complex space is obtained?

2.5.16. Let C be an arbitrary complex space. Prove that the set of vectors forming C can, at the same time, be also considered as a real linear space R in which (a) the operation of addition coincides with that on C; (b) for any real number α and any vector z,

$$\alpha z = (\alpha + i0)z,$$

where the right-hand side of the equality is the product of the vector z by the number $\alpha + i0$, and is defined in C. The transfer from the complex space C to the real space R is called the *decomplexification* of the space C.

2.5.17*. Let C be an n-dimensional complex space obtained from R by the decomplexification. Prove that

(a) if $z_1, \ldots, z_k$ is a linearly independent (or linearly dependent) set of vectors of the space C, then $z_1, iz_1, \ldots, z_k, iz_k$ is a linearly independent (or linearly dependent) set of vectors in the space R (the product iz_j is defined by the same rule as for C, and is an element of this space and, therefore, an element of the space R);

(b) the dimension of the space R equals $2n$; in addition, to any basis $e_1, \ldots, e_n$ in the space C there is a corresponding basis $e_1, ie_1, \ldots, e_n, ie_n$ for the space R.

2.5.18*. Let C be an n-dimensional unitary space with the scalar product (x, y) and R the real space obtained from C by decomplexification. Prove that

(a) the space R can be converted into a Euclidean space by defining a scalar product for it by the formula

$$\langle z_1, z_2 \rangle = \operatorname{Re}(z_1, z_2);$$

(b) for any vector z from C, the vectors z and iz, considered as elements of the obtained Euclidean space, are orthogonal;

(c) if $e_1, \ldots, e_k$ is an orthogonal set of vectors from C, then the set of vectors $e_1, ie_1, \ldots, e_k, ie_k$ is orthogonal in R;

(d) if $e_1, \ldots, e_n$ is an orthonormal basis for C, then $e_1, ie_1, \ldots, e_n, ie_n$ is an orthonormal basis for R.

2.5.19. Prove that the decomplexification of the n-dimensional complex arithmetic space C_n can be performed by matching each vector $z = (\alpha_1 + i\beta_1, \ldots, \alpha_n + i\beta_n)$ from C_n with the vector $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ from the real arithmetic space R_{2n}. Which vector in R_{2n} corresponds to the vector iz? Which scalar product is induced in R_{2n} if, in C_n, the customary scalar product of $z = (\lambda_1, \ldots, \lambda_n)$ and $w = (\mu_1, \ldots, \mu_n)$ is defined as follows: $(z, w) = \lambda_1\bar{\mu}_1 + \ldots + \lambda_n\bar{\mu}_n$?

CHAPTER 3

Determinants

3.0. Terminology and General Notes

Let $x_1, x_2, \ldots, x_n$ be an arbitrary set of vectors of an n-dimensional Euclidean or unitary space, and let

$$L_0 = O, \ L_k = L(x_1, \ldots, x_k).$$

Denote the perpendicular drawn from x_k to the subspace L_{k-1} by y_k.
The number

$$V(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} |y_k| \tag{3.0.1}$$

is known as the *volume of the parallelepiped* drawn on the set of vectors $x_1, x_2, \ldots, x_n$. It is evident that the volume of such a parallelepiped equals zero if, and only if, the set $x_1, x_2, \ldots, x_n$ is linearly dependent. Since

$$|y_k| \leqslant |x_k|, \quad k = 1, \ldots, n,$$

the volume of a parallelepiped satisfies *Hadamard's inequality*

$$V(x_1, x_2, \ldots, x_n) \leqslant \prod_{k=1}^{n} |x_k| \tag{3.0.2}$$

and the equality is upheld here if, and only if, either there is at least one nonzero vector among the vectors $x_1, x_2, \ldots, x_n$ or each pair of these vectors is orthogonal.

Following V. Voyevodin we will define axiomatically an *oriented volume* $V^{\pm}(x_1, x_2, \ldots, x_n)$ *of the parallelepiped* drawn on the set of vectors $x_1, \ldots, x_n$. Viz., we shall require that the following conditions should be fulfilled:

(1) $V^{\pm}(x_1, x_2, \ldots, x_n)$ is a linear function for each of its vector arguments;

(2) $V^{\pm}(x_1, x_2, \ldots, x_n) = 0$ if the set $x_1, x_2, \ldots, x_n$ is linearly dependent;

(3) $V^{\pm}(e_1, e_2, \ldots, e_n) = 1$, for a certain orthonormal basis.

It can be shown (see V. Voyevodin, *Linear Algebra*, Mir Publishers, 1983, Chapter 4) that an oriented volume of a parallelepiped exists, and its modulus equals the volume of this parallelepiped. In particular, the condition

$$V^{\pm}(x_1, x_2, \ldots, x_n) = 0$$

turns out to be necessary and sufficient for the set of vectors $x_1, x_2, \ldots$ $\ldots, x_n$ to be linearly dependent.

Since an oriented volume is not uniquely defined, then to single out a concrete oriented volume it is necessary to give an orthonormal basis $e_1, e_2, \ldots, e_n$ with respect to which it will assume unit value.

A square number table with *n rows* and *n columns*

$$A = \left\| \begin{matrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{matrix} \right\|$$

is called a *square matrix of order n*. The *elements* a_{ij} of a matrix A can be real or complex numbers. Accordingly, we will speak of *real* and *complex* matrices.

The elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to constitute the *principal diagonal* of a matrix A, all the other elements a_{ij}, $i \neq j$, being called the *off-diagonal* elements. A matrix all of whose off-diagonal elements are zero is called a *diagonal matrix*. A diagonal matrix is called the *unit matrix* if all elements on the principal diagonal of this matrix are equal to unity. Another term is the *secondary diagonal* of a matrix A, its elements being $a_{1n}, a_{2,n-1}, \ldots, a_{n1}$.

The algebraic sum of $n!$ terms is called the *determinant of a matrix A* if these terms are all the possible products of n of the elements of the matrix taken one in each row and in each column. The term $a_{1\alpha_1}$, $a_{2\alpha_2} \ldots a_{n\alpha_n}$ has a plus sign if the permutation $\alpha_1, \alpha_2, \ldots, \alpha_n$ contains an even number of inversions, and a minus sign otherwise. An *inversion* of the numbers α_i and α_j exists when $\alpha_i > \alpha_j$ but α_i precedes α_j in the permutation $\alpha_1, \ldots, \alpha_n$. We will denote, hereafter, the determinant of a matrix A by $|A|$ or by det A.

If all the rows of a matrix A are regarded as vectors of an n-dimensional arithmetic space, then the determinant det A is nothing but an orientation volume of a parallelepiped in this space, the corresponding orthonormal basis being the standard basis (1.0.1). Hence it follows that

(i) det A is a linear function of the matrix A;

(ii) det $A = 0$ if and only if the rows of the matrix A are linearly dependent. The matrix is called *degenerate* if its determinant is equal to zero, and *nondegenerate* otherwise;

(iii) the value of the determinant of a matrix is unaltered by adding to a row a linear combination of the other rows;

(iv) the determinant changes sign when two rows are interchanged.

The *transposition* of a matrix A is that transformation of a matrix where the matrix rows are interchanged with the columns that have the same index number T. A^T is the transpose of matrix A

$$A^T = \begin{Vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{Vmatrix}. \tag{3.0.3}$$

Transposition leaves the determinant of a matrix unaltered: $\det A = \det A^T$. Hence the above properties, valid for the rows of a matrix, are also valid for its columns.

Choose any k rows with indices $i_1, i_2, \dots, i_k$ and k columns with indices $j_1, j_2, \dots, j_k$ from a matrix A. A matrix of order k, whose determinant is called the *minor of order* k of the matrix A (or its determinant) emerges from the intersection of these rows and columns. In particular, minors of order 1 are the elements a_{ij}. To indicate the position of the minor, being considered, in a matrix A, the following designation is used

$$M = A\begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}. \tag{3.0.4}$$

If, further, the indices of the rows coincide with the indices of the columns, then we will use the shorter notation: $A\,(i_1 i_2 \dots i_k)$.

If the rows with indices $i_1, \dots, i_k$ and columns with indices $j_1, \dots, j_k$ are deleted from the matrix A, then the remaining minor of order $n-k$ is called the *complementary minor* of the deleted minor (3.0.4). The cofactor of minor (3.0.4) is defined to be its complementary minor multiplied by $(-1)^{s_M}$ where

$$s_M = i_1 + i_2 + \dots + i_k + j_1 + j_2 + \dots + j_k.$$

The cofactor of the element a_{ij} is denoted by A_{ij}.

There is held the following

Laplace theorem. Let k rows (or k columns) of a matrix A be chosen arbitrarily $1 \leqslant k \leqslant n-1$. Then the determinant $\det A$ is equal to the sum of the products of all the minors of order k, and their cofactors in any row or columns of A.

In a particular case, the Laplace theorem yields the following formulae for *expanding a determinant by a row*:

$$a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in} = \det A, \tag{3.0.5}$$
$$a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn} = 0, \quad i \neq j.$$

It is evident that similar formulae for *expanding a determinant by a column* are also valid.

Let us make some other notes concerning the methods of evaluating the determinants used in the present chapter.

It can be seen from Sec. 1.0 that Gaussian elimination reduces a square matrix to a triangular form. Due to Properties 3 and 4 of determinants above, the transformations used in doing so can only change the sign of the determinant. The determinant of a matrix of triangular form equals (see 3.1.8) the product of the entries on the principal diagonal and the application of Gaussian elimination relies on just this fact. Different aspects of the method are discussed in detail in 3.4.

Let us now describe the *method of iterative formulae* which can be used for evaluating *tridiagonal* determinants of the following form:

$$D_n = \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ c & a & b & \dots & 0 & 0 \\ 0 & c & a & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & \dots & c & a \end{vmatrix}. \tag{3.0.6}$$

Consider then set s of infinite numerical sequences

$$x = (\alpha_1, \alpha_2, \dots, \alpha_n, \dots). \tag{3.0.7}$$

Assume, for example, that they are complex. Define linear operations on these sequences by the formulae:

(i) $\lambda x = (\lambda\alpha_1, \lambda\alpha_2, \dots, \lambda\alpha_n, \dots)$;

(ii) if $y = (\beta_1, \beta_2, \dots, \beta_n, \dots)$, then

$$x + y = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n, \dots)$$

and it is obvious that s becomes a linear space.

The set F of all sequences (3.0.7), for whose elements the *iteration formula* or the *difference equation of the second order* holds

$$\alpha_n = p\alpha_{n-1} + q\alpha_{n-2}, \quad n = 3, 4, \dots, \tag{3.0.8}$$

(where p and q are fixed numbers and $q \neq 0$), is a subspace of s. It is easy to see that the dimension of the subspace F equals 2. Let us show how to construct a basis for this space.

Form the algebraic equation (called the *characteristic equation*) from the coefficients of the difference relation (3.0.8)

$$\lambda^2 - p\lambda - q = 0.$$

Two cases must be considered here:

(i) The roots λ_1 and λ_2 of the characteristic equation are different. In this case the basis for the subspace F is made up of the sequences

$$e_1 = (\lambda_1, \lambda_1^2, \lambda_1^3, \dots, \lambda_1^n, \dots),$$
$$e_2 = (\lambda_2, \lambda_2^2, \lambda_2^3, \dots, \lambda_2^n, \dots).$$

(ii) The characteristic equation has a root of multiplicity 2. In which case the basis for F consists of the sequences

$$e_1 = (\lambda, \lambda^2, \lambda^3, \ldots, \lambda^n, \ldots),$$
$$e_2 = (1, 2\lambda, 3\lambda^2, \ldots, n\lambda^{n-1}, \ldots).$$

Any sequence (3.0.7) belonging to F can be decomposed in terms of the basis e_1, e_2 thus:

$$x = c_1 e_1 + c_2 e_2.$$

In terms of the components this relation is translated into

$$\alpha_n = c_1\lambda_1^n + c_2\lambda_2^n$$

for case 1, and

$$\alpha_n = c_1\lambda^n + c_2 n\lambda^{n-1} = \lambda^{n-1}(c_1\lambda + c_2 n)$$

for case 2.

The coordinates c_1 and c_2 of the sequence x may be determined by its first two components only. These are the solutions of the system of linear equations

$$\lambda_1 c_1 + \lambda_2 c_2 = \alpha_1,$$
$$\lambda_1^2 c_1 + \lambda_2^2 c_2 = \alpha_2,$$

or

$$\lambda c_1 + c_2 = \alpha_1,$$
$$\lambda^2 c_1 + 2\lambda c_2 = \alpha_2$$

according as which case is considered.

Returning to determinant (3.0.6) and expanding it along the last row, we obtain

$$D_n = aD_{n-1} - bcD_{n-2}, \quad n = 3, 4, \ldots,$$

i.e. an iterative relation of the second order. Here

$$D_1 = a,$$
$$D_2 = a^2 - bc,$$

and the above construction may be performed.

3.1. Evaluation and the Simplest Properties of Determinants

A customary set of problems on the evaluation and the simplest properties of determinants is presented in this section. We mean by this the linear properties, invariance on transposition, changing sign on the interchange of rows (or columns), the existence of a zero determinant when its rows (or columns) are linearly dependent.

Can a determinant of the seventh order have any of the following products of its elements as one of its terms? If so, what are their respective signs?

3.1.1. $a_{45}\,a_{71}\,a_{23}\,a_{67}\,a_{34}\,a_{12}\,a_{56}$.

3.1.2. $a_{23}\,a_{52}\,a_{77}\,a_{34}\,a_{61}\,a_{12}\,a_{45}$.

3.1.3. $a_{71}\,a_{17}\,a_{26}\,a_{62}\,a_{53}\,a_{35}\,a_{44}$.

3.1.4. $a_{26}\,a_{35}\,a_{44}\,a_{17}\,a_{53}\,a_{62}\,a_{31}$.

3.1.5. Extend the product of the elements $a_{13}\,a_{24}\,a_{35}\,a_{46}\,a_{57}$ of a determinant of the seventh order to obtain a term of the determinant (a) with a plus sign; (b) with a minus sign.

3.1.6. Find the relationship between the indices of the elements placed (a) on the principal diagonal; (b) above the principal diagonal; (c) below the principal diagonal.

3.1.7. What is the sign of the product of elements on the principal diagonal?

3.1.8. Using only its definition, evaluate the determinant

$$\begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

3.1.9. What is the relationship between the indices of the elements of a determinant of order n placed (a) on the secondary diagonal; (b) above the secondary diagonal; (c) below the secondary diagonal.

3.1.10. What sign has the product of the elements of the secondary diagonal when considered as a term of a determinant of order n.

3.1.11. Evaluate the determinant using its definition only

$$\begin{vmatrix} 0 & \dots & 0 & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2,\,n-1} & a_{2n} \\ 0 & \dots & a_{3,\,n-2} & a_{3,\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,\,n-2} & a_{n,\,n-1} & a_{nn} \end{vmatrix}.$$

Using the definition only evaluate the following determinants

3.1.12. $\begin{vmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{vmatrix}$.

3.1.13. $\begin{vmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{vmatrix}$.

3.1.14.*

$$\begin{vmatrix} 1 & 0 & 0 & \dots & 0 & m_{1j} & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & m_{2j} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & m_{3j} & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & m_{j-1,\,j} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & m_{j+1,\,j} & 1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & m_{n-2,j} & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & m_{n-1,j} & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & m_{nj} & 0 & \dots & 0 & 0 & 1 \end{vmatrix}.$$

3.1.15.

$$\begin{vmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\varphi & \dots & -\sin\varphi & & \\ & & \vdots & \ddots & \vdots & & \\ & & \sin\varphi & \dots & \cos\varphi & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{vmatrix} \begin{matrix} \\ \\ i\text{-th row} \\ \\ j\text{-th row} \\ \\ \\ \end{matrix}$$

i-th column $\quad$ j-th column

The elements of the principal diagonal missed out here are equal to unity, and all the other elements are zeroes.

3.1.16. Show that if, in a determinant of order n, more than $n^2 - n$ elements are zeroes, then the determinant equals zero.

Only with the help of the definition, evaluate the determinants:

3.1.17. $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}$ **3.1.18.** $\begin{vmatrix} 0 & 0 & 5 & 1 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 7 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}.$

3.1.19. $\begin{vmatrix} 0 & 0 & 5 & 1 \\ 0 & 0 & 6 & 2 \\ 5 & 6 & 7 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}.$

3.1.20. Prove that if, in a determinant of order n the elements placed at the intersections of k rows and l columns are zeroes and that $k + l > n$, then the determinant is equal to zero.

Find the maximum possible number of nonzero terms in a determinant of order n in the following form:

3.1.21. $$\begin{vmatrix} a_1 & 0 & 0 & \dots & 0 & b_1 \\ c_1 & a_2 & 0 & \dots & 0 & b_2 \\ 0 & c_2 & a_3 & \dots & 0 & b_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & c_{n-1} & a_n \end{vmatrix}.$$

3.1.22*. $$\begin{vmatrix} a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & \dots & c_n & a_n \end{vmatrix}.$$

3.1.23*. $$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,\,n-2} & a_{1,\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,\,n-2} & a_{2,\,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3,\,n-2} & a_{3,\,n-1} & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_{n-1,\,n-2} & a_{n-1,\,n-1} & a_{n-1,\,n} \\ 0 & 0 & 0 & \dots & 0 & a_{n,\,n-1} & a_{nn} \end{vmatrix}.$$

Represent the determinants of order n with entries expressed in terms of t, as polynomials with powers of t in descending order:

3.1.24. $$\begin{vmatrix} -t & 0 & 0 & \dots & 0 & a_1 \\ a_2 & -t & 0 & \dots & 0 & 0 \\ 0 & a_3 & -t & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -t & 0 \\ 0 & 0 & 0 & \dots & a_n & -t \end{vmatrix}.$$

3.1.25*. $$\begin{vmatrix} t & -1 & 0 & \dots & 0 & 0 \\ 0 & t & -1 & \dots & 0 & 0 \\ 0 & 0 & t & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & t & -1 \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n+t \end{vmatrix}.$$

What is the degree of the polynomials in t represented by the following determinants of order n:

3.1.26. $$\begin{vmatrix} a_{11}+t & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}+t & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn}+t \end{vmatrix}.$$

3.1.27. $$\begin{vmatrix} a_{11}+t & a_{12}+t & \dots & a_{1n}+t \\ a_{21} & a_{22}+t & \dots & a_{2n}+t \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn}+t \end{vmatrix}.$$

3.1.28.* Is it always true that a determinant of the following form:

$$\begin{vmatrix} a_{11}+b_{11}t & a_{12}+b_{12}t & \dots & a_{1n}+b_{1n}t \\ a_{21}+b_{21}t & a_{22}+b_{22}t & \dots & b_{2n}+b_{2n}t \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}+b_{n1}t & a_{n2}+b_{n2}t & \dots & a_{nn}+b_{nn}t \end{vmatrix}$$

has degree n if represented by a polynomial in the unknown t?

3.1.29*. Find the necessary and sufficient condition for the above determinant to be of a degree less than n if it is represented by a polynomial in t.

3.1.30. Find the free term of the polynomial indicated in Problem 3.1.28.

3.1.31. How will a determinant with complex entries be altered if all the elements are replaced by their respective conjugates?

3.1.32. How will changing the sign of each of its elements alter an n-order determinant?

3.1.33. If each element of an n-order determinant is multiplied by α, how is the determinant altered?

3.1.34.* How will a determinant be altered if each of its elements a_{ik} is multiplied by α^{i-k}, where the number α is nonzero?

3.1.35. The position of an element a_{ik} in a determinant is called *even* or *odd* according to whether the sum $i + k$ is even or odd. Prove that a determinant is not altered by changing sign of all its odd-placed elements; if, however, all even-placed elements have their sign changed, then an even-ordered determinant remains unaltered, but an odd-ordered determinant's sign is changed.

3.1.36*. A determinant is called *skew-symmetric* if its entries, symmetric about the principal diagonal, differ in sign, i.e. $a_{ij} = -a_{ji}$, for all i, j.

Prove that a skew-symmetric determinant of odd order equals zero.

3.1.37.* Prove that the value of a determinant is real if all entries, symmetric about the principal axis are complex conjugates (i.e. $a_{ij} = \bar{a}_{ji}$ for all i, j).

3.1.38. How will an n-order determinant be altered if each row is written in reverse order? Which element of the original determinant occupies the i, j entry of the new one?

3.1.39. Find the element of an n-order determinant symmetric to a_{ij} with respect to the "centre" of the determinant.

3.1.40. How will a determinant be altered if each of its elements is replaced by the one symmetric to it with respect to the "centre" of the determinant?

3.1.41. Find the element of an n-order determinant symmetric to a_{ij} about the principal diagonal.

3.1.42. How will a determinant be altered if each element is replaced by the element, symmetric about the secondary diagonal?

3.1.43.* How will rotating the matrix of an n-order determinant through 90° about its centre alter the determinant?

Solve the following equations whose left-hand side is represented in determinant form:

3.1.44. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5-t^2 & 2 & 3 & 4 \\ 2 & 3 & 5-t^2 & 1 \\ 2 & 3 & 4 & 1 \end{vmatrix} = 0.$ **3.1.45.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ t+1 & 2 & t+3 & 4 \\ 1 & 3+t & 4+t & 5+t \\ 1 & -3 & -4 & -5 \end{vmatrix} = 0.$

Evaluate the following determinants:

3.1.46. $\begin{vmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \dots & \dots & \dots & \dots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{vmatrix}.$

3.1.47. $\begin{vmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \dots & \dots & \dots & \dots \\ n(n-1)+1 & n(n-1)+2 & \dots & n^2 \end{vmatrix}.$

3.1.48. Let $f_1(t), \dots, f_n(t)$ be polynomials of degree not greater than $n-2$. Prove that for arbitrary numbers $a_1, a_2, \dots, a_n$, the determinant

$$\begin{vmatrix} f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & \dots & f_2(a_n) \\ \dots & \dots & \dots & \dots \\ f_n(a_1) & f_n(a_2) & \dots & f_n(a_n) \end{vmatrix}$$

equals zero.

3.1.49. How will a determinant be altered if (a) from each row (except the first) the previous row is subtracted; (b) from each row (beginning with the second) the previous row is subtracted, and at the same time the last of the original rows is subtracted from the first row?

3.1.50. Prove that any determinant equals half the sum of the following two determinants: one obtained by adding a number b to all elements of the i-th row of the original determinant, and the other by adding $-b$ to them.

Evaluate the following determinants representing them as a sum of determinants:

3.1.51. $\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \dots & \dots & \dots & \dots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}.$

3.1.52.
$$\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \dots & \cos(\alpha_1-\beta_n) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \dots & \cos(\alpha_2-\beta_n) \\ \dots & \dots & \dots & \dots \\ \cos(\alpha_n-\beta_1) & \cos(\alpha_n-\beta_2) & \dots & \cos(\alpha_n-\beta_n) \end{vmatrix}.$$

3.1.53*.
$$\begin{vmatrix} 1+x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & 1+x_2y_2 & \dots & x_2y_n \\ \dots & \dots & \dots & \dots \\ x_ny_1 & x_ny_2 & \dots & 1+x_ny_n \end{vmatrix}.$$

3.1.54.
$$\begin{vmatrix} 1-2w_1^2 & -2w_1w_2 & \dots & -2w_1w_n \\ -2w_2w_1 & 1-2w_2^2 & \dots & -2w_2w_n \\ \dots & \dots & \dots & \dots \\ -2w_nw_1 & -2w_nw_2 & \dots & 1-2w_n^2 \end{vmatrix},$$

where $w_1^2 + w_2^2 + \dots + w_n^2 = 1$.

3.1.55. The numbers 20604, 53227, 25755, 20927, and 78421 are all divisible by 17. Prove that the determinant

$$\begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{vmatrix}$$

is also divisible by 17.

3.1.56. All elements of a determinant Δ are differentiable functions in one variable t. Prove that for the derivative of this determinant considered as a function in t, the following formula is valid

$$\Delta'(t) = \begin{vmatrix} a'_{11}(t) & a'_{12}(t) & \dots & a'_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a'_{21}(t) & a'_{22}(t) & \dots & a'_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots + \begin{vmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \dots & \dots & \dots & \dots \\ a'_{n1}(t) & a'_{n2}(t) & \dots & a'_{nn}(t) \end{vmatrix}.$$

3.1.57. Omit selecting sign of the entries from the definition of a determinant, i.e. consider the following function against the entries of the matrix A:

$$p(A) = \sum_{j_1, j_2, \dots, j_n} a_{1j_1}a_{2j_2} \dots a_{nj_n},$$

where the subscripts $j_1, j_2, \dots, j_n$ run through the whole set of permutations of numbers $1, 2, \dots, n$. This function is called the *permanent*. Prove that for both the determinant and permanent, the following properties hold: (a) if all the entries of an arbitrary row of the matrix A are multiplied by a number α, then the permanent is also multiplied by this number; (b) if all the entries of the i-th row of the matrix A are the sums

$$a_{ij} = b_j + c_j, \quad j = 1, \ldots, n,$$

then the permanent of the matrix A equals the sum of the permanents of the two matrices that differ from A only in the i-th row, viz., the first having all its entries in this row equal to the numbers b_j, and the second to the numbers c_j; (c) when the matrix is transposed the permanent remains unaltered.

In contrast to the determinant, however, (d) interchanging the rows (or columns) of the matrix leaves the permanent unaltered.

Construct some examples demonstrating that the permanent may be nonzero even if the rows of its matrix are linearly dependent, and may equal zero in the case of a matrix with linearly independent rows.

3.2. Minors, Cofactors and the Laplace Theorem

The contents of this section are:

Problems about finding a minor, a complementary minor and a cofactor. The adjoint and associated determinants and certain their properties are also considered here.

Examples of the use of the Laplace theorem and some computational problems.

Exercises to use the method of recurrent relations, described in the introduction to the chapter, to evaluate three-diagonal determinants.

3.2.1. Find, for a determinant of order n: (a) the number of minors of order k contained in k fixed rows; (b) the number of all minors of order k.

3.2.2. Let M be an arbitrary minor of a determinant of order n; M' be the complementary minor; and let $(-1)^{s_M}M'$ be the corresponding cofactor in M (here s_M is the sum of the numbers of those rows and columns of the determinant which form the minor M). Show that the cofactor corresponding to the minor M' equals $(-1)^{s_M}M$.

3.2.3. The minor placed at the intersection of the k-th row and the k-th column of a determinant with the same numbers is called the *principal minor* of order k. Find how many principal minors of order k there are in a determinant of order n.

3.2.4*. Find expressions for the coefficients of the polynomial $f(t)$ given by the determinant

$$\begin{vmatrix} a_{11}+t & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}+t & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn}+t \end{vmatrix},$$

in terms of the minors of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

3.2.5. Find the maximum possible number of nonzero minors of order k in the first k columns of this *almost triangular* determinant of order n

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,\,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3,\,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n,\,n-1} & a_{nn} \end{vmatrix}.$$

3.2.6. Let D be a determinant of order n ($n > 1$). The determinant D' obtained from D by replacing each element a_{ij} with its cofactor A_{ij} is said to be *adjoint* of D. The determinant D'' obtained from D by replacing each element a_{ij} with its complementary minor M_{ij} is said to be *associated* with D. Prove that $D' = D''$.

3.2.7. Prove that if a determinant D is *symmetric* (i.e. each element of the determinant D is equal to the one symmetric to it about the principal diagonal), then the adjoint determinant D' is also symmetric. A similar statement is valid for the associated determinant D''.

3.2.8. Is the following statement valid: If a determinant D is skew-symmetric, then the adjoint determinant D' is also skew-symmetric?

3.2.9.* Prove that the determinant, adjoint to the triangular determinant of Problem 3.1.8, is of the form

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ 0 & A_{22} & A_{23} & \dots & A_{2n} \\ 0 & 0 & A_{33} & \dots & A_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_{nn} \end{vmatrix}.$$

3.2.10.* Find the relation between the value of a triangular de terminant of order n and the value of its adjoint.

3.2.11.* How will the adjoint determinant D' be altered if for a given determinant D of order n (a) all elements of the i-th row are multiplied by a number α; (b) the i-th and j-th rows are interchanged; (c) the j-th row is added to i-th multiplied by an arbitrary number α; (d) the determinant D is transposed.

3.2.12. Show that the Laplace expansion of a determinant of order n by any of its k rows (or columns) coincides with its decomposition by the remaining $n - k$ rows (or columns).

3.2.13. Prove that if, for a determinant of order n all the minors of order k ($k < n$) are equal to zero, then all minors of an order higher than k are also equal to zero.

3.2.14. Prove that among the minors of order k, made up of the first k columns of this *quasi-triangular* determinant

$$\begin{vmatrix} a_{11} & \dots & a_{1k} & a_{1,k+1} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k1} & \dots & a_{kk} & a_{k,k+1} & \dots & a_{kn} \\ 0 & \dots & 0 & a_{k+1,k+1} & \dots & a_{k+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{n,k+1} & \dots & a_{nn} \end{vmatrix},$$

only the principal minor can be nonzero. Find the Laplace expansion of this determinant over the first k columns.

3.2.15.* Given that the principal minor of order k, formed by the first k columns of a determinant d of order n, is nonzero, and all the other minors of order k equal zero, prove that d is of the form indicated in Problem 3.2.14.

Using the Laplace theorem evaluate the following determinants:

3.2.16. $\begin{vmatrix} 1 & 0 & 0 & -1 \\ 2 & 3 & 4 & 7 \\ -3 & 4 & 5 & 9 \\ -4 & -5 & 6 & 1 \end{vmatrix}.$ **3.2.17.** $\begin{vmatrix} 3 & -1 & 5 & 2 \\ 2 & 0 & 7 & 0 \\ -3 & 1 & 2 & 0 \\ 5 & -4 & 1 & 2 \end{vmatrix}.$

3.2.18. $\begin{vmatrix} 5 & 62 & -79 & 4 \\ 0 & 2 & 3 & 0 \\ 6 & 183 & 201 & 5 \\ 0 & 3 & 4 & 0 \end{vmatrix}.$

3.2.19. $\begin{vmatrix} 9 & 7 & 6 & 8 & 5 \\ 3 & 0 & 0 & 2 & 0 \\ 5 & 3 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 7 & 5 & 4 & 6 & 0 \end{vmatrix}.$ **3.2.20.** $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{vmatrix}.$

3.2.21. $\begin{vmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 4 \\ 0 & 3 & 0 & 4 & 0 \\ 3 & 0 & 4 & 0 & 3 \end{vmatrix}.$ **3.2.22.** $\begin{vmatrix} 7 & -3 & 9 & 5 & -4 \\ 4 & 0 & 0 & 0 & 3 \\ -6 & 0 & 1 & 0 & 8 \\ 5 & 0 & 0 & 0 & 4 \\ 1 & 8 & -2 & -9 & 3 \end{vmatrix}.$

3.2.23. $\begin{vmatrix} 2 & 3 & 1 & 2 & 9 & 8 \\ 3 & 4 & 2 & 7 & 5 & 3 \\ 0 & 0 & 5 & 3 & 3 & 1 \\ 0 & 0 & 8 & 5 & 7 & 5 \\ 0 & 0 & 0 & 0 & 9 & 7 \\ 0 & 0 & 0 & 0 & 4 & 3 \end{vmatrix}$.

3.2.24. $\begin{vmatrix} 2 & -3 & 7 & 1 & 9 & 11 \\ 1 & 0 & 3 & 0 & -4 & 0 \\ 7 & 4 & 9 & -1 & 11 & -5 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 9 & -4 & 11 & 1 & 13 & 2 \\ 4 & 0 & 1 & 0 & -1 & 0 \end{vmatrix}$.

3.2.25. $\begin{vmatrix} 1 & 30 & 94 & 46 & 14 & 2 \\ 0 & 7 & 6 & 9 & 4 & 0 \\ 0 & 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 5 & 1 & 4 & 3 & 0 \\ 2 & 7 & 47 & 23 & 15 & 1 \end{vmatrix}$.

3.2.26. $\begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{vmatrix}$.

3.2.27. Prove that

(a)
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} & a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & 0 & 0 & a_{2,n+2} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n,2n} \\ a_{n+1,1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & a_{n+1,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,n-1} & 0 & 0 & a_{2n-1,n+2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,n-1} & a_{2n,n} & a_{2n,n+1} & a_{2n,n+2} & \cdots & a_{2n,2n-1} & a_{2n,2n} \end{vmatrix}$$
$$= \begin{vmatrix} a_{1n} & a_{1,n+1} \\ a_{2n,n} & a_{2n,n+1} \end{vmatrix} \cdot \begin{vmatrix} a_{2,n-1} & a_{2,n+2} \\ a_{2n-1,n-1} & a_{2n-1,n+2} \end{vmatrix} \cdots \begin{vmatrix} a_{n1} & a_{n,2n} \\ a_{n+1,1} & a_{n+1,2n} \end{vmatrix};$$

(b)
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} & a_{1,n+1} & a_{1,n+2} & \cdots & a_{1,2n-1} & a_{1,2n} \\ 0 & a_{22} & \cdots & a_{2,n-1} & a_{2n} & 0 & a_{2,n+2} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & a_{nn} & 0 & 0 & \cdots & 0 & a_{n,2n} \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n-1} & a_{n+1,n} & a_{n+1,n+1} & a_{n+1,n+2} & \cdots & a_{n+1,2n-1} & a_{n+1,2n} \\ 0 & a_{n+2,2} & \cdots & a_{n+2,n-1} & a_{n+2,n} & 0 & a_{n+2,n+2} & \cdots & a_{n+2,2n-1} & a_{n+2,2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & a_{2n,n} & 0 & 0 & \cdots & 0 & a_{2n,2n} \end{vmatrix}$$
$$= \begin{vmatrix} a_{11} & a_{1,n+1} \\ a_{n+1,1} & a_{n+1,n+1} \end{vmatrix} \cdot \begin{vmatrix} a_{22} & a_{2,n+2} \\ a_{n+2,2} & a_{n+2,n+2} \end{vmatrix} \cdots \begin{vmatrix} a_{nn} & a_{n,2n} \\ a_{2n,n} & a_{2n,2n} \end{vmatrix};$$

(c)
$$\begin{vmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & a_{1n} & 0 \\ 0 & b_{11} & 0 & b_{12} & \cdots & 0 & b_{1n} \\ a_{21} & 0 & a_{22} & 0 & \cdots & a_{2n} & 0 \\ 0 & b_{21} & 0 & b_{22} & \cdots & 0 & b_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & 0 & a_{n2} & 0 & \cdots & a_{nn} & 0 \\ 0 & b_{n1} & 0 & b_{n2} & \cdots & 0 & b_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{vmatrix}.$$

Using the Laplace theorem, and having first transformed them evaluate the following determinants:

3.2.28. $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -8 & 5 & 9 & 5 \\ -11 & 7 & 7 & 4 \end{vmatrix}.$

3.2.29. $\begin{vmatrix} 9 & 7 & 9 & 7 \\ 8 & 6 & 8 & 6 \\ -9 & -7 & 9 & 7 \\ -8 & -6 & 8 & 6 \end{vmatrix}.$

3.2.30. $\begin{vmatrix} 6 & 8 & -9 & -12 \\ 4 & 6 & -6 & -9 \\ -3 & -4 & 6 & 8 \\ -2 & -3 & 4 & 6 \end{vmatrix}.$

3.2.31. $\begin{vmatrix} 213 & 186 & 162 & 137 \\ 344 & 157 & 295 & 106 \\ 419 & 418 & 419 & 418 \\ 417 & 416 & 417 & 416 \end{vmatrix}.$

3.2.32. $\begin{vmatrix} 8 & 10 & 3 & 1 & 4 \\ 7 & 9 & 4 & 1 & 6 \\ 1 & -2 & 2 & 1 & 3 \\ 2 & 5 & -4 & -2 & -6 \\ -1 & 2 & 6 & 3 & 9 \end{vmatrix}.$

3.2.33*. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 7 & 6 & 7 & 8 \\ 2 & 5 & 9 & 10 & 11 \\ 5 & 9 & 1 & 1 & 1 \\ 9 & 1 & 2 & 3 & 4 \end{vmatrix}.$

3.2.34. $\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 & 3 & 6 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 \end{vmatrix}.$

3.2.35. $\begin{vmatrix} 2 & 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 3 & 1 & 1 \\ 1 & 1 & 3 & 4 & 1 & 1 \\ 1 & 3 & 1 & 1 & 4 & 1 \\ 3 & 1 & 1 & 1 & 1 & 4 \end{vmatrix}.$

3.2.36. Prove that for the permanents (see Problem 3.1.57), a theorem similar to the Laplace is valid, namely if, in a square matrix A of order n, k rows (or columns) $1 \leqslant k \leqslant n - 1$ are fixed, then the permanent of the matrix A equals the sum of the products of the permanents of all the submatrices of order k placed in these fixed rows (or columns) and the permanents of their complementary submatrices (of order $n - k$).

Using iterative relations evaluate the following determinants of order n:

3.2.37. $\begin{vmatrix} 5 & 2 & 0 & \dots & 0 \\ 2 & 5 & 2 & \dots & 0 \\ 0 & 2 & 5 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 5 \end{vmatrix}.$

3.2.38. $\begin{vmatrix} 7 & 6 & 0 & \dots & 0 \\ 2 & 7 & 6 & \dots & 0 \\ 0 & 2 & 7 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 7 \end{vmatrix}.$

3.2.39.
$$\begin{vmatrix} 3 & 2 & 0 & 0 & \dots & \cdot & \cdot & \cdot & \dots \\ 1 & 3 & 1 & 0 & \dots & \cdot & \cdot & \cdot & \dots \\ 0 & 2 & 3 & 2 & \dots & \cdot & \cdot & \cdot & \dots \\ 0 & 0 & 1 & 3 & \dots & \cdot & \cdot & \cdot & \dots \\ & & & & \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & 3 & 2 & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & 1 & 3 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & 2 & 3 & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots \end{vmatrix}.$$

3.2.40.
$$\begin{vmatrix} 5 & 3 & 0 & 0 & \dots & 0 & 0 \\ -10 & -1 & -4 & 0 & \dots & 0 & 0 \\ 0 & 5 & 1 & -4 & \dots & 0 & 0 \\ 0 & 0 & 5 & 1 & \cdot & 0 & 0 \\ & & & & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & -4 \\ 0 & 0 & 0 & 0 & \dots & 5 & 1 \end{vmatrix}.$$

3.2.41.
$$\begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{vmatrix}.$$

3.2.42.
$$\begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ & & & & \cdot & & \\ & & & & \cdot & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \end{vmatrix}.$$

3.2.43.
$$\begin{vmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ & & & \cdot & \\ & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \\ 0 & 0 & 0 & \dots & 2 \end{vmatrix}.$$

3.2.44.
$$\begin{vmatrix} 12 & 9 & 0 & \dots & 0 \\ 4 & 12 & 9 & \dots & 0 \\ 0 & 4 & 12 & \dots & 0 \\ & & \cdot & & \\ & & & \cdot & \\ & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 12 \end{vmatrix}.$$

3.2.45. Prove the equality:

$$\begin{vmatrix} \cos\alpha & 1 & 0 & \dots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & \dots & 0 & 0 \\ 0 & 1 & 2\cos\alpha & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & \\ & & & & \cdot & \\ 0 & 0 & 0 & \dots & 1 & 2\cos\alpha \end{vmatrix} = \cos n\alpha.$$

3.2.46. Find the iteration relations between the polynomials of the sequence $f_0(\lambda), f_1(\lambda), f_2(\lambda), \dots, f_n(\lambda)$, where $f_0(\lambda) \equiv 1$, and the polynomial $f_i(\lambda)$ $(1 \leqslant i \leqslant n)$ is the principal minor of order i placed in the upper left-hand corner of the determinant

$$\begin{vmatrix} \lambda-a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & \lambda-a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & \lambda-a_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & & \\ & & & \cdot & & \\ 0 & 0 & 0 & & \lambda-a_{n-1} & b_n \\ 0 & 0 & 0 & & c_n & \lambda-a_n \end{vmatrix}.$$

3.3. Determinants and the Volume of a Parallelepiped in a Euclidean Space

In this section some of the properties of determinants and the volumes of parallelepipeds in an n-dimensional Euclidean or unitary space are established using the natural relationships between them. Thus, the determinant of a square matrix of order n is an orientation volume of a parallelepiped generated by an ordered set of rows (or columns) for this matrix; the rows (columns) are considered here to be vectors of the corresponding arithmetic space, and the modulus of the determinant coincides with the volume of the parallelepiped (see V. Voyevodin, *Linear Algebra*, Chapter 4). In particular, this relationship makes it possible to extend Hadamard's inequality to determinants and to obtain the related approximations to the values of determinants, and consequently to its volume. We also consider the Gram determinants and find their relation to the volumes. Finally, we provide some problems to illustrate the stability of an orthogonal determinant and instability of a determinant of general form.

3.3.1. Let $a_1, a_2, \dots, a_n$ be an ordered set of rows of a determinant d of order n, these rows being considered as vectors in an n-dimensional arithmetic space; and let $b_1, b_2, \dots, b_n$ be the set obtained from $a_1, a_2, \dots, a_n$ by the orthogonalization procedure. Prove that the determinant d', whose rows are the vectors $b_1, b_2, \dots, b_n$, equals the determinant d.

3.3.2*. Prove that a determinant equals zero if and only if its rows (or columns) are linearly dependent.

3.3.3. Let d be a determinant of order n

$$d=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Using the relationship between the modulus of a determinant and the volume of the parallelepiped in an arithmetic space, prove Hadamard's inequality:

$$|d|\leqslant\Big(\sum_{j=1}^{n}|a_{1j}|^2\Big)^{1/2}\Big(\sum_{j=1}^{n}|a_{2j}|^2\Big)^{1/2}\dots\Big(\sum_{j=1}^{n}|a_{nj}|^2\Big)^{1/2}.$$

3.3.4. Prove that the equality sign in Hadamard's inequality holds if and only if either each pair of the rows of the determinant are orthogonal, or all the elements of at least one row equal zero. A similar statement holds for the columns of the determinant.

3.3.5*. Prove that if the modulus of all elements a_{ij} of a determinant of order n is bounded by a number M, $|a_{ij}|\leqslant M$, then (a) the modulus of the determinant does not exceed $M^n n^{n/2}$; (b) this approximation is achieved for determinants with complex entries for any n; (c) for determinants with real entries, this approximation is achieved if n is a number of the form $n=2^m$.

3.3.6*. Prove that the maximum f_n of the moduli of determinants of order n all of whose elements are real numbers from the line segment $[-1, 1]$ coincides with the maximum g_n of the moduli of determinants whose elements only assume the values 1 and -1.

3.3.7*. Let h_n be the maximum of the moduli of determinants of order n compiled from units and zeroes, and let g_n be determined as in Problem 3.3.6. Prove that for the numbers g_n and h_n, the following relations are valid

$$h_{n-1}\leqslant h_n\leqslant g_{n-1}\leqslant g_n\leqslant 2^{n-1}h_{n-1}.$$

Note, in particular, that g_n is divisible by 2^{n-1}.

3.3.8. Using Hadamard's inequality and the inequalities proved in Problem 3.3.7, prove, for the case of determinants of order 3, that (a) $h_3=2$; (b) $g_3=4$. Note that it follows from (b) that the approximation of the value of the determinant, indicated in Problem 3.3.5 (a), is not achieved for a determinant with real coefficients of order 3.

3.3.9*. Strengthen the approximation indicated in Problem 3.3.7 by proving that

$$g_n\geqslant 2g_{n-1}.$$

3.3.10*. Find the number g_5 and a determinant with entries 1 and -1, equal to g_5. Note that the approximation of Problem 3.3.5(a) is not achieved for determinants with real entries of order 5.

3.3.11*. Prove that if in the conditions of Problem 3.3.5 all elements a_{ij} of the determinant d are real and nonnegative, then for the modulus of d the approximation is valid:

$$|d| \leqslant M^n 2^{-n} (n+1)^{(n+1)/2}.$$

3.3.12. Reformulate Problems 3.3.5-3.3.11 for the volumes of the corresponding parallelepipeds.

3.3.13. *The Gram determinant of a set of vectors* $x_1, x_2, \ldots, x_k$ of a Euclidean (or unitary) space is a determinant of the form:

$$G(x_1, \ldots, x_k) = \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \ldots & (x_1, x_k) \\ (x_2, x_1) & (x_2, x_2) & \ldots & (x_2, x_k) \\ \cdot & \cdot & \cdot & \cdot \\ (x_k, x_1) & (x_k, x_2) & \ldots & (x_k, x_k) \end{vmatrix}.$$

The matrix of this determinant is called the *Gram matrix of the set of vectors* $x_1, x_2, \ldots, x_k$.

What are the form and value of the Gram determinant if (a) the set $x_1, \ldots, x_k$ is orthogonal; (b) the span of the vectors $x_1, \ldots, x_l$ $(1 \leqslant l < k)$ is orthogonal to the span of the vectors $x_{l+1}, \ldots, x_k$.

3.3.14. How is the Gram determinant of a set of vectors $x_1, \ldots$ $\ldots, x_k$ altered if (a) two vectors, x_i and x_j, are interchanged; (b) a vector of the set is multiplied by a number α; (c) the vector x_i is added to the vector x_j premultiplied by the number β.

Hence deduce that the property of the Gram determinant being equal or unequal to zero is maintained during these elementary transformations of the set of vectors $x_1, \ldots, x_k$.

3.3.15*. Prove that a set of vectors $x_1, \ldots, x_k$ of a Euclidean (or unitary) space is linearly dependent if and only if the Gram determinant of this set is equal to zero.

3.3.16*. A certain principal minor M of order m, $m < k$, in the Gram determinant $G(x_1, \ldots, x_k)$ equals zero. Prove that in this case any principal minor enclosing the minor M is also equal to zero. (A minor M_2 is said to enclose a minor M_1 if the matrix of the minor M_2 contains the matrix of the minor M_1 as a submatrix.) In particular, the determinant $G(x_1, \ldots, x_k)$ is also zero.

3.3.17. Prove that the Gram determinant of a set of vectors $x_1, \ldots, x_k$ is unaltered if a vector of this set is replaced by the perpendicular drawn from this vector to the span of the other vectors in the set.

3.3.18*. Let $x_1, \ldots, x_k$ be an arbitrary set of vectors of a Euclidean (or unitary) space; and let $y_1, \ldots, y_k$ be the orthogonal set obtained from the vectors $x_1, \ldots, x_k$ by the orthogonalization process. Prove that

$$G(x_1, \ldots, x_k) = G(y_1, \ldots, y_k) = |y_1|^2 |y_2|^2 \ldots |y_k|^2.$$

Using this result, establish the relationship between the Gram determinant of the vector set $x_1, \ldots, x_k$, and the volume of the parallelepiped generated by this set.

3.3.19. Prove that the Gram determinant $G(x_1, \ldots, x_k)$ equals zero if the set of vectors $x_1, \ldots, x_k$ is linearly dependent and is positive if this system is linearly independent.

3.3.20. Let A be an arbitrary square matrix of order n, either real or complex; $a_1, \ldots, a_n$ be the rows of this matrix, regarded as vectors of the corresponding arithmetic space; and $G(a_1, \ldots, a_n)$ be the Gram determinant of this set (we assume, as usual, that the scalar product of the vectors $x = (\alpha_1, \ldots, \alpha_n)$ and $y = (\beta_1, \ldots, \beta_n)$ is determined by formula (2.2.1) in the space R_n, and by the formula

$$(x, y) = \alpha_1\overline{\beta}_1 + \ldots + \alpha_n\overline{\beta}_n \tag{3.3.1}$$

in the space C_n). Prove that

$$|\det A|^2 = G(a_1, \ldots, a_n).$$

3.3.21*. Verify that the proof of the properties of the Gram determinant stated in Problems 3.3.13-3.3.19 can be given without the use of the Cauchy-Buniakowski inequality, i.e. only with the help of the theorems on vector orthogonality. Deduce this inequality from the nonnegativeness of the Gram determinant.

3.3.22. Prove that the element of the Gram determinant with the maximum modulus lies on the principal diagonal of this determinant (and if there are several elements of this kind, then at least one of them lies on the principal diagonal).

3.3.23. Prove that the distance from a vector x in a Euclidean (or unitary) space to a linear subspace L, spanned by the linearly independent set of vectors $x_1, \ldots, x_k$, can be computed by the formula

$$\rho(x, L) = \left[\frac{G(x, x_1, \ldots, x_k)}{G(x_1, \ldots, x_k)}\right]^{1/2}.$$

3.3.24. Prove Hadamard's inequality for the Gram determinants

$$G(x_1, \ldots, x_k) \leqslant |x_1|^2 \ldots |x_k|^2.$$

Show that the equality sign holds here if and only if either each pair of the vectors $x_1, \ldots, x_k$ is orthogonal, or at least one of these vectors equals zero.

3.3.25*. Prove the following generalization of Hadamard's inequality for the volumes of parallelepipeds

$$V(x_1, \ldots, x_l, x_{l+1}, \ldots, x_k) \leqslant V(x_1, \ldots, x_l) \cdot V(x_{l+1}, \ldots, x_k),$$

where $V(\ldots)$ denotes the volume of the parallelepiped generated by the corresponding vector set.

Show that the equality sign is valid here if and only if either

$$(x_i, x_j) = 0,\ i = 1, \ldots, l,\ j = l + 1, \ldots, k,$$

or at least one of the subsets $x_1, \ldots, x_l$ and $x_{l+1}, \ldots, x_k$ is linearly dependent. Enunciate the corresponding property for the Gram determinants.

3.3.26. Let $x_1, \ldots, x_{k-1}, x_k$ be a linearly independent set of vectors of a Euclidean (or unitary) space. Prove that for any vector z, the following relation between the volumes of parallelepipeds is valid

$$\frac{V(x_1, \ldots, x_k, z)}{V(x_1, \ldots, x_k)} \leqslant \frac{V(x_1, \ldots, x_{k-1}, z)}{V(x_1, \ldots, x_{k-1})}.$$

Hence deduce the corresponding property for the principal minors of the Gram determinant.

3.3.27*. Let $x_1, \ldots, x_k$ be an arbitrary set of vectors. Prove the inequality

$$V^{k-1}(x_1, \ldots, x_k) \leqslant \prod_{j=1}^{n} V(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k).$$

Explain the geometric sense of this inequality. Formulate a similar property for the principal minors of the Gram determinant.

3.3.28*. Let $x_1, \ldots, x_k$ be an orthonormal set of vectors. Prove that for any vector ε whose length is less than unity, the following inequalities are valid

$$1 - |\varepsilon| \leqslant V(x_1, \ldots, x_{i-1}, \tilde{x}_i, x_{i+1}, \ldots, x_k) \leqslant 1 + |\varepsilon|;\quad \tilde{x}_i = x_i + \varepsilon.$$

3.3.29. A determinant is said to be *orthogonal* if its rows, regarded as vectors in an arithmetic space form an orthonormal set. Reformulate the statement of Problem 3.3.28 for orthogonal determinants.

3.3.30*. Interpreting the modulus of a determinant of order n as the volume in an n-dimensional arithmetic space, explain the geometric sense of the moduli of the minors of order k ($k < n$).

3.3.31. Prove that the modulus of the minors of any order of an orthogonal determinant does not exceed unity using (a) Hadamard's inequality for determinants (see Problem 3.3.3.); (b) a geometric interpretation of the moduli of minors (see Problem 3.3.30).

Is a similar statement valid for arbitrary determinants whose modulus equals unity and which are not orthogonal?

3.3.32. Show that the following determinant of order n

$$\begin{vmatrix} 1 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 2 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & 2 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix} \qquad (3.3.2)$$

can be made equal to zero by a perturbation of a certain element whose modulus equals $2^{-(n-1)}$. Find this perturbation. Relate this result to the question in the previous problem and give its geometric interpretation.

3.4. Computing the Determinants by the Elimination Method

In the present section we consider various topics related to Gauss elimination applied to the evaluation of determinants. We provide four groups of problems on the following:

The relations between the elements of determinants, obtained at various stages of the reduction to triangular form, and the minors of the original determinant.

Problems for practising the Gauss method.

The computational aspects of the method, i.e. the number of arithmetic operations, the necessity to control the growth of elements during the reduction process, and hence, the use of various tactics whilst interchanging them.

The application of the Gauss method to the proof of a useful theorem on the Kronecker product of determinants and some of its corollaries.

3.4.1. A determinant with a matrix A was evaluated by the Gauss method without interchanging any rows or columns, i.e. the pivots at the various stages were the elements in the $(1, 1)$, $(2, 2)$, ..., $(n-1, n-1)$ positions, respectively. Prove that after the $(p-1)$th stage of this reduction all the minors of order p contained in the first p rows of the matrix were unaltered. Show also that these minors are unaltered in the subsequent stages of the reduction.

3.4.2. Let A be a square matrix of order n. The principal minor of order r at the intersection of the rows and columns numbered $1, 2, \ldots, r$ is called the *leading principal minor* of order r of the matrix A. Prove that if all leading principal minors of the matrix A of orders $1, 2, \ldots, n-1$ are nonzero, then all the pivots $a^{(p)}_{p+1,p+1}$ employed in the elimination method for the matrix are also nonzero. Find an expression for the pivots in terms of the principal minors of the matrix A.

3.4.3*. Prove that if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n$$

then the condition indicated in the previous problem for the matrix A of order n is fulfilled. Moreover, the determinant of the matrix A, in this case, is not equal to zero either.

3.4.4*. Prove that for the Gram matrix of a linearly independent set of vectors $x_1, \ldots, x_k$ of a Euclidean (or unitary) space, the condition indicated in Problem 3.4.2 is fulfilled. Moreover, all the pivots of this matrix involved in the Gauss elimination are positive and do not exceed the maximum entry of the original matrix.

3.4.5. Prove that if the determinant of a matrix A is nonzero then, by interchanging the rows and columns of this matrix, all the leading principal minors can be made to be nonzero.

3.4.6*. During the process of Gauss elimination for a matrix A, no interchanging was necessary. Find expressions for the nonzero elements of the p-th row of the matrix $A^{(p-1)}$, obtained after the $(p-1)$th stage, in terms of the original matrix.

3.4.7. Using the result of Problem 3.4.6 show that if, in a matrix A of order n, all the minors of order $r+1$ enclosing the nonzero leading principal minor of order r, $1 \leqslant r \leqslant n-1$ (the definition of an enclosing minor is given in Problem 3.3.16), are equal to zero, then the determinant of the matrix A equals zero.

3.4.8*. Prove that if in a matrix A of order n there is a nonzero minor M of order r, $1 \leqslant r \leqslant n-1$, such that all its enclosing minors of order $r+1$ equal zero, then the determinant equals zero. Note that for the above statement to be valid, it is only necessary that the enclosing minors that are situated in the $r+1$ rows of the matrix A (r of these rows coinciding with the rows forming the minor M) are equal to zero.

3.4.9*. Using the Gauss method prove that the relation between a determinant d of order n and its adjoint d',

$$d' = d^{n-1},$$

established in Problem 3.2.10 for triangular determinants, is valid for any determinant d.

Using the Gauss method, evaluate the following determinants:

3.4.10. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 1 & 4 \end{vmatrix}.$ **3.4.11.** $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{vmatrix}.$

3.4.12. $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix}.$ **3.4.13.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 8 & 11 \\ 7 & 13 & 20 & 26 \\ 31 & 23 & 55 & 42 \end{vmatrix}.$

3.4.14. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 31 & 23 & 55 & 42 \end{vmatrix}.$ **3.4.15.** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 9 \\ 31 & 23 & 55 & 42 \end{vmatrix}.$

3.4.16*. $\begin{vmatrix} 30 & 20 & 15 & 12 \\ 20 & 15 & 12 & 10 \\ 105 & 84 & 70 & 60 \\ 168 & 140 & 120 & 105 \end{vmatrix}.$ **3.4.17*.** $\begin{vmatrix} 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \\ 1/5 & 1/6 & 1/7 & 1/8 \end{vmatrix}.$

3.4.18. $\begin{vmatrix} 2 & 1000 & 4 & 0.08 \\ 1 & 3000 & -6 & 0.02 \\ 3 & -2000 & 2 & -0.02 \\ 2 & -1000 & 2 & 0 \end{vmatrix}$.

3.4.19. $\begin{vmatrix} 128 & 256 & 384 & 512 \\ 1/4 & 3/8 & 1/8 & 1/4 \\ 1/64 & 1/64 & 1/64 & -1/64 \\ 2 & 0 & -4 & -12 \end{vmatrix}$.

3.4.20. $\begin{vmatrix} 1001 & 1002 & 1003 & 1004 \\ 1002 & 1003 & 1001 & 1002 \\ 1001 & 1001 & 1001 & 999 \\ 1001 & 1000 & 998 & 999 \end{vmatrix}$.

3.4.21. $\begin{vmatrix} 1 & 2 & -1 & -0.002 \\ 3 & 8 & 0 & -0.004 \\ 2 & 2 & -4 & -0.003 \\ 3000 & 8000 & -1000 & -6 \end{vmatrix}$.

3.4.22. $\begin{vmatrix} 0 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 \end{vmatrix}$.

3.4.23. $\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 6 & 4 & 1 & 0 \\ 1 & 10 & 10 & 5 & 1 \\ 1 & 15 & 20 & 15 & 6 \end{vmatrix}$.

3.4.24. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$.

3.4.25. $\begin{vmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix}$.

3.4.26. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{vmatrix}$.

3.4.27. $\begin{vmatrix} 1 & 10 & 100 & 1000 & 10000 & 100000 \\ 0.1 & 2 & 30 & 400 & 5000 & 60000 \\ 0 & 0.1 & 3 & 60 & 1000 & 15000 \\ 0 & 0 & 0.1 & 4 & 100 & 2000 \\ 0 & 0 & 0 & 0.1 & 5 & 150 \\ 0 & 0 & 0 & 0 & 0.1 & 6 \end{vmatrix}$.

3.4.28. Evaluate the polynomial $f(t)$ given by the determinant

$$\begin{vmatrix} 3-t & 1 & 0 & 0 & 0 & 0 \\ -1 & 3-t & 1 & 0 & 0 & 0 \\ 0 & -1 & 3-t & 1 & 0 & 0 \\ 0 & 0 & -1 & 3-t & 1 & 0 \\ 0 & 0 & 0 & -1 & 3-t & 1 \\ 0 & 0 & 0 & 0 & -1 & 3-t \end{vmatrix}$$

when $t = 2$.

3.4.29. Find the number of multiplication and division operations necessary for evaluating a determinant of order n by the Gauss method. Compare this number with the number of multiplication operations when evaluating a determinant by the definition only.

3.4.30. Assuming that in Gaussian elimination no interchange occurred, find the number of multiplications and divisions necessary to evaluate (a) an almost triangular determinant of order n

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2,n-2} & a_{2,n-1} & a_{2n} \\ 0 & a_{32} & a_{33} & \dots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & a_{n,n-1} & a_{nn} \end{vmatrix};$$

(b) a tridiagonal determinant of order n

$$\begin{vmatrix} a_1 & b_2 & 0 & \dots & 0 & 0 \\ c_2 & a_2 & b_3 & \dots & 0 & 0 \\ 0 & c_3 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & \dots & c_n & a_n \end{vmatrix}.$$

3.4.31. Let it be required to evaluate an n-order determinant d_n known to be nonzero, and its enclosing determinant d_{n+1} of order $n+1$, i.e.

$$d_{n+1} = \begin{vmatrix} & & & a_1 \\ & & & a_2 \\ & d_n & & \vdots \\ & & & a_n \\ b_1 & b_2 & \dots\ b_n & c \end{vmatrix}.$$

Organize the sequence of computations using Gaussian elimination so that for evaluating both determinants d_n and d_{n+1}, there may be required the same number of multiplications and divisions as for evaluating only the determinant of order $n+1$.

3.4.32. It is required to evaluate k determinants of order n which differ from each other only in the last column. Given that all the determinants are nonzero, what is the sequence of computations using the Gauss method in which the evaluation of all k determinants requires only $O(kn^2)$ more multiplications than the number of operations necessary to evaluate one determinant of order n.

3.4.33*. Find a method to evaluate a determinant of order n of the form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \dots & a_{2,\,n-1} & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \dots & a_{3,\,n-1} & a_{3n} \\ a_{41} & 0 & 0 & a_{44} & \dots & a_{4,\,n-1} & a_{4n} \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ a_{n-1,\,1} & 0 & 0 & 0 & & a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \dots & a_{n,\,n-1} & a_{nn} \end{vmatrix}$$

(where the diagonal elements $a_{22}, \ldots, a_{n-1,n-1}$ are nonzero) such that the number of operations is a second degree polynomial in n (and not third degree as would be the case for the Gauss method when applied to determinants in a general form).

3.4.34. Consider the set D_n of determinants of order n that fulfil the following conditions: (a) the modulus of all the elements of the determinants is bounded by unity; (b) there is an element whose modulus equals unity; (c) all the leading principal minors are nonzero.

The last condition makes it possible to use the Gauss method to evaluate a determinant from the set D_n without any interchange (see Problem 3.4.2). Prove, however, that if k is any integer from the set $1, 2, \ldots, n-1$, and N is any positive number, then there is a determinant from the set D_n such that in the matrix obtained after k stages of the Gauss method without any interchange, there may be found an element $a_{ij}^{(k)}$ whose modulus is greater than the number N. Thus, whatever the word size used by a computer is, there exists a determinant from the set D_n whose evaluation by Gaussian elimination will lead to overflow.

3.4.35*. Consider the following modification of the Gauss method intended to avoid the overflow indicated in Problem 3.4.34. After completing k stages of the reduction to triangular form, the element with the greatest modulus out of the elements $a_{k+1,k+1}^{(k)}$ $a_{k+2,k+1}^{(k)}, \ldots, a_{n,k+1}^{(k)}$ is chosen as the pivot of the $(k+1)$th stage. Let it be an element $a_{j,k+1}^{(k)}$, $j \geqslant k+1$. Then the rows $k+1$ and j are interchanged so that this element is positioned at $(k+1, k+1)$. Then the customary Gaussian transformations are performed for the $(k+1)$th stage. This modification is called the elimination method with the choice of a pivot in a column (or partial pivoting). Prove that during this procedure (a) if all elements $a_{k+1,k+1}^{(k)}, a_{k+2,k+1}^{(k)}, \ldots, a_{n,k+1}^{(k)}$ of the pivotal column are zeroes, then the original determinant is also equal to zero; (b) for any position (i, j)

$$|a_{ij}^{(k+1)}| \leqslant 2 \max_{r, s} |a_{rs}^{(k)}|;$$

(c) for a determinant of order n, the modulus of all the elements obtained in the process of the reduction to triangular form is not more than 2^{n-1} times as great as the maximum element in the original determinant.

3.4.36*. Construct an example to support the probability that the maximum modulus of elements will grow to the estimated value indicated in Problem 3.4.35 (c), during the reduction to the triangular form using the Gauss method with partial pivoting.

3.4.37. Prove that whilst applying the Gauss method with partial pivoting (a) the maximum modulus of the elements of an almost triangular determinant of order n will not increase more than n times; (b) the growth of the maximum modulus of the elements of a tridiagonal determinant of order n, in the process of the reduction, is not more than twofold, i.e.

$$\max_{r, s} |a_{rs}^{(k)}| \leqslant 2 \max_{r, s} |a_{rs}|, \quad 1 \leqslant k \leqslant n-1.$$

3.4.38. It follows from Problem 3.4.36 that even faster growth of the maximum modulus of the elements is possible using the Gauss method with partial pivoting. As a consequence, a computer may, once again, overflow during the computation of a determinant of a sufficiently high order. Thus, another modification of the Gauss method can be applied: the pivot of the $(k + 1)$th stage is chosen to be the element with the greatest modulus out of the submatrix of order $n - k$ situated in the lower right-hand corner of the matrix $A^{(k)}$ after the previous k stages. The rows and columns having indices greater than k are interchanged so that the element with the maximum modulus is positioned at $(k + 1, k + 1)$. The $(k + 1)$th stage of the Gauss method is then performed as usual. This modification is termed the Gauss method with complete pivoting. Prove that for the Gauss method with complete pivoting, the modulus of the pivot of the $(k + 1)$th stage is not more than twice as great as the modulus of the pivot of the k-th stage. Is a similar statement valid for the Gauss method with the partial pivoting?

3.4.39*. There exists a hypothesis that for the Gauss method with the complete pivoting, the growth of the maximum modulus of the elements of a determinant of order n does not exceed n. Using the result of Problem 3.3.8, prove this hypothesis for $n = 3$.

3.4.40. Deduce from the result of the previous problem that whilst applying the Gauss method with complete pivoting (a) the growth of the maximum modulus does not exceed 6 in the case of real matrices of order 4; (b) the growth of the maximum modulus does not exceed 9 in the case of real matrices of order 5.

3.4.41*. The following determinant of order mn

$$D=\begin{vmatrix} a_{11}b_{11} & \dots & a_{1n}b_{11} & a_{11}b_{12} & \dots & a_{1n}b_{12} & \dots & a_{11}b_{1m} & \dots & a_{1n}b_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}b_{11} & \dots & a_{nn}b_{11} & a_{n1}b_{12} & \dots & a_{nn}b_{12} & \dots & a_{n1}b_{1m} & \dots & a_{nn}b_{1m} \\ a_{11}b_{21} & \dots & a_{1n}b_{21} & a_{11}b_{22} & \dots & a_{1n}b_{22} & \dots & a_{11}b_{2m} & \dots & a_{1n}b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}b_{21} & \dots & a_{nn}b_{21} & a_{n1}b_{22} & \dots & a_{nn}b_{22} & \dots & a_{n1}b_{2m} & \dots & a_{nn}b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{11}b_{m1} & \dots & a_{1n}b_{m1} & a_{11}b_{m2} & \dots & a_{1n}b_{m2} & \dots & a_{11}b_{mm} & \dots & a_{1n}b_{mm} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}b_{m1} & \dots & a_{nn}b_{m1} & a_{n1}b_{m2} & \dots & a_{nn}b_{m2} & \dots & a_{n1}b_{mm} & \dots & a_{nn}b_{mm} \end{vmatrix}.$$

is called the *Kronecker product of the determinant*

$$\det A=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{vmatrix}$$

and the determinant

$$\det B=\begin{vmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \dots & b_{mm} \end{vmatrix}$$

of order n and m, respectively. Thus, the matrix of the determinant D consists of m^2 blocks of order n. These blocks are obtained from the matrix A by multiplying all its elements by $b_{11}, b_{12}, \dots, b_{1m}, b_{21}, b_{22}, \dots, b_{2m}, \dots, b_{m1}, b_{m2}, \dots, b_{mm}$, respectively. By means of the Gauss method, prove that

$$D = (\det A)^m (\det B)^n.$$

3.4.42. Prove that the Kronecker product of two orthogonal determinants, d of order n and d' of order m, is an orthogonal determinant of order mn.

3.4.43. Find the relation between the determinant of a matrix of order n and determinants of matrices of order $2n$ arranged as follows:

(a) $\begin{pmatrix} A & -A \\ A & A \end{pmatrix}$. (b) $\begin{pmatrix} A & -A \\ -A & A \end{pmatrix}$;

(c) $\begin{pmatrix} 2A & 3A \\ A & 2A \end{pmatrix}$; (d) $\begin{pmatrix} A & 3A \\ 2A & 5A \end{pmatrix}$.

CHAPTER 4

Systems of Linear Equations

4.0. Terminology and General Notes

A rectangular number table consisting of m rows and n columns,

$$A = \left\| \begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right\|,$$

is called a *rectangular matrix of order* $m \times n$ (or an $m \times n$ matrix). It is said to be *real* or *complex* depending on whether the *elements* a_{ij} of this matrix are real or complex.

A minor of order k is defined for a rectangular matrix, as in the particular case of a square matrix (see Sec. 3.0), assuming that $k \leqslant \min(m, n)$. The minor is designated in the same way, i.e. as

$$A \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

The highest order r of the minors of a matrix A that are nonzero is called the *rank* of this matrix, and any nonzero minor of order r is termed the *basis minor* of the matrix A. If all elements of a matrix are zeroes (the *zero matrix*), then its rank is defined to be zero.

If the rows of a matrix A are considered as n-dimensional vectors and its columns as m-dimensional vectors, then the rank of the set of rows and the rank of the set of columns are both equal to the rank of the matrix A. Hence, the rank of the *transposed* matrix A^T, order $n \times m$ (see Sec. 3.0), coincides with the rank of A.

Let x_0 be a fixed vector and L a subspace in an n-dimensional linear space V. The set P of all vectors of the form

$$x = x_0 + y, \quad y \in L,$$

is called the *plane obtained by translation of the subspace L by the vector x_0* and denoted by $x_0 + L$, x_0 being termed the *translation vector*, and L the *directional subspace* of the plane P.

If the subspace L is represented as the span $L(q_1, q_2, \dots, q_k)$ then a *parametric equation* of the plane P can be formed

$$x = x_0 + t_1 q_1 + t_2 q_2 + \dots + t_k q_k,$$

where the parameters $t_1, t_2, \dots, t_k$ assume arbitrary values.

It can be shown (see 4.2.1) that for a given plane, the directional subspace is defined uniquely. Thus, any plane can have a *dimension* equal to that of its directional subspace. Thus, a plane of dimension 1 is called a *straight line* and a plane of dimension $(n - 1)$ is termed a *hyperplane*.

The planes $P_1 = x_1 + L_1$ and $P_2 = x_2 + L_2$ are called *parallel* if either $L_1 \subset L_2$ or $L_2 \subset L_1$.

Let us list some other definitions and results regarding systems of linear equations (the systems are also considered in Sec. 1.0).

A system of linear equations is called *homogeneous* if the right-hand sides of all equations of this system equal zero and *nonhomogeneous* otherwise.

A matrix A, made up of the coefficients of the unknowns, is called the *coefficient matrix of the* given *system of equations*. If the column of the right-hand sides of this system is ascribed to A then the so-called *augmented matrix* $\bar{A}$ *of the system* is obtained.

Let the number of the unknowns in a system of equations be equal to the number of the equations. Then the coefficient matrix A of the system is a square matrix and the condition $\det A \neq 0$ stipulates whether the system is consistent or determinate. A unique solution $x_1, \ldots, x_n$ can be found by *Cramer's formulae*

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, \ldots, n,$$

where A_i is the matrix obtained from A by replacing the i-th column with the column of the right-hand sides.

Two remarks about the problems of this chapter. A number of computational problems are given in Sec. 4.2 concerning determination of the mutual disposition of planes in a linear space. These problems, including those on finding the rank of a given system of vectors, intersection of two spans, etc., can be solved by the methods of Chapter 1.

As it is shown in Sec. 4.5, the solution set of a nonhomogeneous system of linear equations can be considered to be a plane in an arithmetic space. If the scalar product can be defined in a space, then among the vectors of any plane there exists a unique vector, orthogonal to the directional subspace of this plane (see Problem 4.3.11). It is called the *normal vector*, and the corresponding solution of the system of linear equations is called the *normal solution*. This concept is used, for example, in Problem 4.5.36.

4.1. The Rank of a Matrix

In this section we provide a number of problems illustrating various definitions of the rank of a matrix and their applications to finding the rank of concrete matrices.

4.1.1. Prove that in any r linearly independent rows (or columns) of a matrix, there is a nonzero minor of order r.

4.1.2*. In a rectangular $m \times n$ matrix A $(m \geqslant n)$ there is a nonzero minor of order $n-1$, but all its *enclosing* minors of order n are zero. Prove that *all* the minors of order n of the matrix A equal zero, and hence the rank of A equals $n-1$.

4.1.3*. In a matrix A there is a nonzero minor M of order r but all the enclosing M minors are zero. Prove that the rank of A equals r.

4.1.4. What can be said of an $m \times n$ matrix $(m > n)$ of rank n if it has only one basic minor?

4.1.5*. What can be said of an arbitrary $m \times n$ matrix if it has only one basic minor?

4.1.6*. Prove that the minor of order r situated at the intersection of any r linearly independent rows and r linearly independent columns is nonzero.

4.1.7. A square matrix A is said to be *symmetric* if $a_{ij} = a_{ji}$ for any i, j. Prove that the rank of a symmetric matrix equals the highest order of the nonzero principal minors of this matrix.

4.1.8. Show that the statement of Problem 4.1.7 is also valid for a complex *Hermitian* matrix A, i.e. the matrix in which $a_{ij} = \overline{a}_{ji}$ for any i, j.

4.1.9*. Prove that the rank of an arbitrary set of vectors in a Euclidean (or unitary) space equals the rank of the Gram matrix of this system.

4.1.10. A square matrix A is said to be *skew-symmetric* if $a_{ij} = -a_{ji}$ for any i, j. Prove that the rank of a skew-symmetric matrix equals the highest order of the nonzero principal minors of this matrix.

4.1.11. Prove that the rank of a skew-symmetric matrix is an even integer.

4.1.12. The determinant of a square matrix of order n is nonzero. Prove that for any r, $1 \leqslant r \leqslant n-1$, the leading principal minor of order r of this matrix may be made nonzero by interchanging only its rows.

4.1.13. Prove that all the leading principal minors of a square matrix with a nonzero determinant may be made nonzero by interchanging only its rows.

4.1.14. The rank of an $m \times n$ matrix A equals 1. Prove that there are numbers $b_1, \ldots, b_m$ and $c_1, \ldots, c_n$ such that

$$a_{ij} = b_i c_j$$

for any i, j. Are these numbers uniquely determined?

4.1.15. The rows of an $m \times n$ matrix A, considered as vectors in an n-dimensional arithmetic space, are orthogonal. Moreover, in each row there is at least one nonzero element. Prove that $n \geqslant m$.

4.1.16. Prove that the rank of a matrix A of the form

$$A=\begin{Vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{Vmatrix}$$

equals the sum of the ranks of the submatrices A_{11} and A_{22} (0 designates zero submatrices of the corresponding order).

4.1.17. Does the following statement hold "the rank of a matrix A of the form

$$A=\begin{Vmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{Vmatrix}$$

is always equal to the sum of the ranks of the submatrices A_{11} and A_{22}"?

4.1.18. How can the rank of matrix alter if the value of one of its elements is altered?

4.1.19. How can the rank of a matrix alter if the elements of only one row are altered? of k rows?

4.1.20*. Prove that in an $n \times n$ matrix of order r, there are k elements such that however small a change in their absolute value is, it increases the rank of the matrix to $r + k$, $1 \leqslant k \leqslant n - r$.

4.1.21. Indicate the possible values of the rank of a matrix of the form

$$\begin{Vmatrix} 0 & 0 & \dots & 0 & a_{1n} \\ 0 & 0 & \dots & 0 & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & a_{m-1,\,n} \\ a_{m1} & a_{m2} & \dots & a_{m,\,n-1} & a_{mn} \end{Vmatrix}.$$

4.1.22. Prove that in a square $n \times n$ matrix with a nonzero determinant, the rank of any square submatrix of order $n - 1$ is not less than $n - 2$.

4.1.23. Prove that the two systems of vectors in the arithmetic space

$$\begin{aligned} x_1 &= (a_{11}, \dots, a_{1i}, \dots, a_{1j}, \dots, a_{1n}), \\ x_2 &= (a_{21}, \dots, a_{2i}, \dots, a_{2j}, \dots, a_{2n}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_k &= (a_{k1}, \dots, a_{ki}, \dots, a_{kj}, \dots, a_{kn}) \end{aligned}$$

and

$$\begin{aligned} y_1 &= (a_{11}, \dots, a_{1j}, \dots, a_{1i}, \dots, a_{1n}), \\ y_2 &= (a_{21}, \dots, a_{2j}, \dots, a_{2i}, \dots, a_{2n}), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_k &= (a_{k1}, \dots, a_{kj}, \dots, a_{ki}, \dots, a_{kn}) \end{aligned}$$

have the same rank.

4.1.24. Prove that the dimension of the span generated by a set of vectors $x_1, \ldots, x_k$ equals the rank of the matrix made up from their coordinates in any basis for the space.

4.1.25. Prove that a vector b belongs to the span of $x_1, \ldots, x_k$ if and only if the rank of the matrix, made up from the coordinates of the vectors $x_1, \ldots, x_k$ in some basis for the space, equals the rank of the augmented matrix made up from the coordinates in the same basis of the vectors $x_1, \ldots, x_k, b$.

4.1.26. Prove that the elementary row (or column) transformations (see Problem 1.2.17) of a matrix do not alter its rank.

4.1.27. Prove that any $m \times n$ matrix can be reduced, by elementary transformations of its rows and columns, to the form

$$\left\| \begin{matrix} a_{11} & a_{12} & \ldots & a_{1r} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2r} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & a_{rr} & \ldots & a_{rn} \\ 0 & 0 & \ldots & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & \ldots & 0 \end{matrix} \right\|,$$

where $a_{11}, a_{22}, \ldots, a_{rr}$ are nonzero, and r equals the rank of the original matrix. Compare this statement with that of Problem 1.2.18.

Evaluate the rank of the following matrices:

4.1.28. $\left\| \begin{matrix} 37 & 259 & 481 & 407 \\ 19 & 133 & 247 & 209 \\ 25 & 175 & 325 & 275 \end{matrix} \right\|.$ **4.1.29.** $\left\| \begin{matrix} 2 & 5 & -1 & 4 & 3 \\ -3 & 1 & 2 & 0 & 1 \\ 4 & 1 & 6 & -1 & -1 \\ -2 & 3 & 0 & 4 & -9 \end{matrix} \right\|.$

4.1.30*. $\left\| \begin{matrix} 1241 & 381 & 273 & -165 \\ 134 & -987 & 562 & 213 \\ 702 & 225 & -1111 & 49 \end{matrix} \right\|.$

4.1.31. $\left\| \begin{matrix} -3 & 2 & 0 & 1 & 4 \\ -1 & 5 & 2 & 3 & 5 \\ 6 & -12 & 3 & -7 & -8 \\ -3 & 7 & 9 & 4 & 15 \end{matrix} \right\|.$

4.1.32. $\left\| \begin{matrix} -5 & 3 & -4 & 0 & 3 & -4 \\ 4 & 0 & 0 & -1 & 3 & 0 \\ -3 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & -3 & 0 & 4 \\ -1 & 0 & 2 & 4 & 0 & 0 \end{matrix} \right\|.$

4.1.33. $\left\| \begin{matrix} 9 & -12 & 3 & -4 & 12 & -16 \\ -15 & 21 & -5 & 7 & -20 & 28 \\ 18 & -24 & 6 & -8 & 15 & -20 \\ -30 & 42 & -10 & 14 & -25 & 35 \end{matrix} \right\|.$

4.1.34. Find the dimension of the span generated by the set of vectors $x_1 = (73, -51, 13, 42, 15)$, $x_2 = (44, -32, 5, 25, 3)$, $x_3 = (76, -52, 16, 44, 18)$, $x_4 = (-37, 27, -4, -21, -2)$.

4.1.35. A linear space L is spanned by the vectors $x_1 = (2, 4, 8, -4, 7)$, $x_2 = (4, -2, -1, 3, 1)$, $x_3 = (3, 5, 2, -2, 4)$, $x_4 = (-5, 1, 7, -6, 2)$.

Do the vectors

(a) $b_1 = (6, 18, 1, -9, 8)$;

(b) $b_2 = (6, 18, 1, -9 + \varepsilon, 8)$;

(c) $b_3 = (6, 18, 1, -9, 8 + \varepsilon)$

belong to this subspace? Here ε is any nonzero number.

4.1.36*. Prove that the rank of a $k \times n$ matrix A of the form

$$\begin{Vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_k & a_k^2 & \dots & a_k^{n-1} \end{Vmatrix}$$

is equal to k, where $k \leqslant n$ and $a_1, a_2, \dots, a_k$ are different numbers.

4.2. Planes in a Linear Space

Most of the problems in the present section concern the following two topics: the determination of a plane and its dimension in a linear space; and the mutual disposition of planes.

At the end of the section we state some of the relationships between planes of an arbitrary dimension and hyperplanes.

4.2.1. Prove that the two planes $P_1 = x_1 + L_1$ and $P_2 = x_2 + L_2$ coincide if and only if $L_1 = L_2$ and $x_1 - x_2 \in L_1$. Hence the direction subspace of any plane is uniquely determined.

4.2.2. Deduce from the result of the previous problem that for any plane any of its vectors can be chosen to be the translation vector.

4.2.3. Prove that if vectors x_1 and x_2 belong to the plane $P = x_0 + L$, then $x_1 - x_2 \in L$. Conversely, if $x_1 \in P$ and $x_1 - x_2 \in L$ then $x_2 \in P$.

4.2.4. Prove that the plane $P = x_0 + L$ is a subspace if and only if $x_0 \in L$.

4.2.5. Prove that for the plane $P = x_0 + L$ to be a subspace, it is necessary and sufficient that the sum of any vectors x_1 and x_2 should belong to L.

4.2.6. Prove that the intersection of the plane $P = x_0 + L$ with any subspace, complementary to L, consists of only one vector.

4.2.7. What represents a plane of zero dimension?

4.2.8. What represents a plane of dimension n in an n-dimensional linear space V?

4.2.9. Prove that in the space of polynomials of degree $\leqslant n$, the set of polynomials $f(t)$ satisfying the condition $f(a) = b$, where a and b are fixed numbers, is a plane. Find the dimension of this plane.

4.2.10. Prove that in a plane of dimension k, not a subspace, a linearly independent set consisting of $k+1$ vectors can be found.

4.2.11. Prove that in a plane of dimension k, any set consisting of $k + 2$ vectors is linearly dependent.

4.2.12. Prove that for any $k + 1$ linearly independent vectors, there exists, and is unique, a plane of dimension k containing these vectors.

4.2.13. Prove that the plane of dimension k containing the linearly independent vectors $x_0, x_1, \ldots, x_k$ can be described by the set of all the linear combinations $\alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_k x_k$ satisfying the condition $\alpha_0 + \alpha_1 + \ldots + \alpha_k = 1$.

4.2.14. Prove that if the intersection of two planes $P_1 = x_1 + L_1$ and $P_2 = x_2 + L_2$ is nonempty, then it is a plane with the direction subspace $L_1 \cap L_2$.

4.2.15. We define the *sum* $P_1 + P_2$ *of the planes* $P_1 = x_1 + L_1$ and $P_2 = x_2 + L_2$ to be the set of all vectors of the form $z_1 + z_2$ where $z_1 \in P_1$, $z_2 \in P_2$. Prove that the sum of the planes P_1 and P_2 is also a plane. Find its directional subspace.

4.2.16. We define the *product* λP *of the plane* $P = x_0 + L$ *by a number* λ to be the set of all vectors of the form λz where $z \in P$. Prove that the product of the plane P by the number λ is also a plane. Find its direction subspace.

4.2.17. A subspace L is fixed in a linear space V. Is the set M of all planes in the space under the operations of addition and multiplication by a number defined in Problems 4.2.15 and 4.2.16 a linear space?

4.2.18. Alter the definition for the multiplication of a plane by a number so that the set M of Problem 4.2.17 may become a linear space. Indicate the zero element of this space. (The obtained space M is called a *factor-space of the space* V *by the space* L).

4.2.19. Let L be a k-dimensional subspace of an n-dimensional space V in the context of Problem 4.2.18. What is the dimension of the space M?

4.2.20. Given a plane $x = x_0 + t_1 p_1 + t_2 p_2$ where $x_0 = (2, 3, -1, 1, 1)$, $p_1 = (3, -1, 1, -1, 1)$, $p_2 = (-1, 1, 1, 1, -1)$, determine whether the vectors $z = (1, 6, 4, 4, -2)$ and $v = (1, 6, 5, 4, -2)$ belong to this plane.

4.2.21. Prove that if a straight line has two vectors in common with a plane then it is contained in this plane.

4.2.22. State the mutual disposition of the plane $P = x_0 + L$, where $x_0 = (1, 0, 0, 1)$, and L is spanned by the vectors $y_1 = (5, 2,$

$-3, 1)$, $y_2 = (4, 1, -1, 0)$, $y_3 = (-1, 2, -5, 3)$ and the straight lines

(a) $x = x_1 + tq_1$, $x_1 = (3, 1, -4, 1)$, $q_1 = (-1, 1, 2, 1)$;

(b) $x = x_2 + tq_2$, $x_2 = (3, 0, -4, 1)$, $q_2 = (-1, 1, 2, 1)$;

(c) $x = x_3 + tq_3$, $x_3 = (-2, 0, -1, 2)$, $q_3 = (1, 1, -2, 1)$.

4.2.23. Prove that the straight lines $x = x_1 + tq_1$ and $x = x_2 + tq_2$, where $x_1 = (9, 3, 6, 15, -3)$, $q_1 = (7, -4, 11, 13, -5)$, $x_2 = (-7, 2, -6, -5, 3)$, $q_2 = (2, 9, -10, -6, 4)$, intersect. Find their intersection. Indicate the plane of dimension 2 in which these straight lines lie.

4.2.24*. Prove that the straight lines $x = x_1 + tq_1$ and $x = x_2 + tq_2$ where $x_1 = (8, 2, 5, 15, -3)$, $q_1 = (7, -4, 11, 13, -5)$, $x_2 = (-7, 2, -6, -5, 3)$, $q_2 = (2, 9, -10, -6, 4)$, do not meet. Construct a plane of dimension 3 that contains them both.

Determine the mutual disposition of the planes $P_1 = x_0 + t_1p_1 + t_2p_2$ and $P_2 = y_0 + t_1q_1 + t_2q_2$:

4.2.25. $x_0 = (3, 1, 2, 0, 1)$, $p_1 = (2, -6, 3, 1, -6)$,
$y_0 = (1, 0, 1, 1, 0)$, $q_1 = (-1, 1, -1, 0, 1)$,
$p_2 = (0, 5, -2, -1, 6)$,
$q_2 = (-1, 3, -1, -1, 2)$.

4.2.26. $x_0 = (7, -4, 0, 3, 2)$, $p_1 = (-1, 1, 1, 1, 1)$,
$y_0 = (6, -5, -1, 2, 3)$, $q_1 = (1, 1, -1, 1, 1)$,
$p_2 = (1, -1, 1, 1, 1)$,
$q_2 = (1, 1, 1, -1, 1)$.

4.2.27. $x_0 = (2, -3, 1, 5, 0)$, $p_1 = (3, -2, 1, 0, 1)$,
$y_0 = (0, -1, 0, 4, 1)$, $q_1 = (1, 2, 4, 0, -2)$,
$p_2 = (-1, 5, -2, 0, 3)$,
$q_2 = (6, 3, 4, 0, 3)$.

4.2.28. $x_0 = (-3, -2, 1, -1, 2)$, $p_1 = (1, -1, 1, 1, 3)$,
$y_0 = (-1, 0, 3, 3, 8)$, $q_1 = (1, 1, -3, -3, 1)$,
$p_2 = (-1, 2, 1, 2, -2)$,
$q_2 = (0, 1, 2, 3, 1)$.

4.2.29. $x_0 = (1, 2, 0, 2, 1)$, $p_1 = (5, -2, 6, 1, -4)$,
$y_0 = (1, 2, 1, 2, 1)$, $q_1 = (1, -4, 0, 1, -6)$,
$p_2 = (2, 1, 3, 0, 1)$,
$q_2 = (-3, 3, -3, -1, 5)$.

4.2.30. $x_0 = (4, 1, 10, -3, 5)$, $p_1 = (2, 1, 3, 0, 1)$,

$y_0 = (-3, 2, 1, -4, 8)$, $q_1 = (3, -3, 3, 1, -5)$,

$p_2 = (1, -4, 0, 1, -6)$,

$q_2 = (5, -2, 6, 1, -4)$.

4.2.31. Prove that if a straight line $x = x_0 + tq$ and a hyperplane $\pi = y_0 + L$ do not meet, then $q \in L$.

4.2.32. Prove that if the hyperplanes $\pi_1 = x_0 + L_1$ and $\pi_2 = y_0 + L_2$ do not intersect then $L_1 = L_2$.

4.2.33*. Prove that if the intersection of hyperplanes $\pi_1, \ldots, \pi_k$ of an n-dimensional space is nonempty, then it is a plane whose dimension is not less than $n - k$.

4.2.34*. Prove that any k-dimensional plane can be defined, in an n-dimensional space, as the intersection of $n - k$ hyperplanes.

4.3. Planes in a Euclidean Space

Various techniques for representing hyperplanes in a Euclidean space are discussed, and a one-to-one correspondence between planes and systems of linear equations is established. Then we introduce the notion of a normal vector to a plane and consider certain geometric problems related to the determination of distance. In conclusion, we consider it important to note that the description of planes by systems of linear equations obtained here for the orthonormal bases of a Euclidean space, actually holds for any basis of a linear space.

4.3.1*. Prove that the set of all vectors of a Euclidean (unitary) space E satisfying the condition $(n, x) = b$ where n is a fixed nonzero vector and b is a given number, is a hyperplane of this space. In what case will this hyperplane be a subspace?

4.3.2. Show that the hyperplane described by the condition $(n, x) = b$ can be also described by the condition $(n, x - x_0) = 0$, where x_0 is an arbitrary vector of this hyperplane.

4.3.3*. Prove that any hyperplane of a Euclidean space can be determined by a condition of the form $(n, x) = b$.

4.3.4. Prove that if the conditions $(n_1, x) = b_1$ and $(n_2, x) = b_2$ determine one and the same hyperplane, then for a certain nonzero number α, $n_1 = \alpha n_2$ and $b_1 = \alpha b_2$.

4.3.5. The scalar product for the space of polynomials of degree $\leqslant n$ is defined by formula (2.3.1). Describe the hyperplane, given by the condition $f(c) = d$, by a relation of the form $(n, f) = b$. Indicate the corresponding polynomial $n(t)$.

4.3.6. Can an arbitrary hyperplane of the space of polynomials (see the previous problem) be described by a condition of the form $f(c) = d$?

4.3.7. Prove that any hyperplane can be described, with respect to any orthonormal basis, by an equation of the first degree

$$A_1\alpha_1 + A_2\alpha_2 + \ldots + A_n\alpha_n = b$$

in the coordinates $\alpha_1, \alpha_2, \ldots, \alpha_n$ of vectors of this hyperplane.

4.3.8*. Prove that if the intersection of hyperplanes in an n-dimensional space

$$\begin{aligned} (n_1, x) &= b_1, \\ (n_2, x) &= b_2, \\ &\ldots\ldots \\ (n_k, x) &= b_k \end{aligned}$$

is nonempty, then it represents a plane whose dimension equals $n - r$ where r is the rank of the set of vectors $n_1, \ldots, n_k$.

4.3.9. An orthonormal basis $e_1, \ldots, e_n$ is fixed for a Euclidean (unitary) space. Prove that (a) if

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\alpha_2 + \ldots + a_{1n}\alpha_n &= b_1, \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \ldots + a_{2n}\alpha_n &= b_2, \\ &\ldots\ldots \\ a_{m1}\alpha_1 + a_{m2}\alpha_2 + \ldots + a_{mn}\alpha_n &= b_m \end{aligned}$$

is an arbitrary consistent system of linear equations in n unknowns, then the set of vectors z, whose coordinates in the basis $e_1, \ldots, e_n$ satisfy this system, is a plane of the space E. The dimension of this plane equals $n - r$ where r is the rank of the matrix

$$\begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{Vmatrix}.$$

(b) any plane P in the space E can be described by a certain system of linear equations. This means that a vector z belongs to the plane P if and only if its coordinates with respect to the basis $e_1, \ldots, e_n$ satisfy the given system. If r is the dimension of the plane P, then any system describing this plane consists of at least $n - r$ equations, and there exists a system containing precisely $n - r$ equations.

4.3.10. Find a system of linear equations describing the plane $P = x_0 + L$ where $x_0 = (1, 1, 1, 1)$ and L is spanned by the vectors $y_1 = (1, 3, 0, 2)$, $y_2 = (3, 7, -1, 2)$, $y_3 = (2, 4, -1, 0)$.

4.3.11. Prove that among the vectors in any plane P there is a unique vector z_0, orthogonal to the directional subspace of this plane. The vector z_0 is called the *normal vector* of the plane P.

4.3.12. Show that of all vectors in a plane P the normal vector z_0 has the least length.

4.3.13. Show that the normal vector z_0 of a plane P equals the perpendicular drawn from an arbitrary vector of this plane to the directional subspace.

4.3.14. Find the normal vector z_0 of the hyperplane given by the condition $(n, x) = b$.

4.3.15. Let z_0 be the normal vector of a plane P (not coincident with the whole space). Prove that the plane P is contained in the hyperplane $(z_0, x) = (z_0, z_0)$.

4.3.16. Find for the space of polynomials of degree $\leqslant n$ with the scalar product (2.3.1) the normal vector of a plane defined by the conditions $f(0) = 1$, $f(1) = 1$.

4.3.17. We define the *distance between a vector x and the plane $P = x_0 + L$* to be the number

$$\rho(x, P) = \inf_{u \in P} \rho(x, u).$$

Prove that the distance $\rho(x, P)$ is equal to the length of the perpendicular drawn from the vector $x - x_0$ to the subspace L.

4.3.18. A subspace L is generated by a linearly independent set of vectors $y_1, \ldots, y_k$. With the use of the result of Problem 4.3.17 and the Gram determinant properties, prove that the distance from the vector x to the plane $P = x_0 + L$ is equal to

$$\rho(x, P) = \left(\frac{G(y_1, \ldots, y_k, x - x_0)}{G(y_1, \ldots, y_k)} \right)^{1/2}.$$

4.3.19. Find the distance from the vector $x = (5, 3, -1, -1)$ to the plane $P = x_0 + L$ where $x_0 = (0, 0, -3, 6)$ and L is spanned by the set of vectors $y_1 = (1, 0, 2, -2)$, $y_2 = (0, 1, 2, 0)$, $y_3 = (2, 1, 6, -4)$.

4.3.20. A number

$$\rho(P_1, P_2) = \inf_{u_1 \in P_1,\ u_2 \in P_2} \rho(u_1, u_2)$$

is called the *distance between two planes* $P_1 = x_1 + L_1$ and $P_2 = x_2 + L_2$. Prove that the distance $\rho(P_1, P_2)$ is equal to the length of the perpendicular drawn from the vector $x_1 - x_2$ to the subspace $L = L_1 + L_2$.

4.3.21. Prove that the square of the distance between the straight lines $l_1 = x_1 + tq_1$ and $l_2 = x_2 + tq_2$ equals

(a) $\rho^2(l_1, l_2) = \dfrac{G(q_1, q_2, x_1 - x_2)}{G(q_1, q_2)}$

if the straight lines l_1 and l_2 are not parallel;

(b) $\rho^2(l_1, l_2) = \dfrac{G(q_1, x_1 - x_2)}{(q_1, q_1)}$

if the straight lines l_1 and l_2 are parallel.

Find the distance between the straight lines $l_1 = x_1 + tq_1$ and $l_2 = x_2 + tq_2$:

4.3.22. $x_1 = (5, 2, 0, 3)$, $q_1 = (1, 2, -4, 1)$; $x_2 = (3, -1, 3, 1)$, $q_2 = (1, 0, -1, 0)$.

4.3.23. $x_1 = (5, 4, 3, 2)$, $q_1 = (1, 1, -1, -1)$; $x_2 = (2, 1, 4, 3)$, $q_2 = (-3, -3, 3, 3)$.

Find the distance between the planes $P_1 = x_0 + t_1p_1 + t_2p_2$ and $P_2 = y_0 + t_1q_1 + t_2q_2$:

4.3.24. $x_0 = (89, 37, 111, 13, 54)$, $p_1 = (1, 1, 0, -1, -1)$,
$y_0 = (42, -16, -39, 71, 3)$, $q_1 = (1, 1, 0, 1, 1)$,
$p_2 = (1, -1, 0, -1, 1)$,
$q_2 = (1, -1, 0, 1, -1)$.

4.3.25. $x_0 = (5, 0, -1, 9, 3)$, $p_1 = (1, 1, 0, -1, -1)$,
$y_0 = (3, 2, -4, 7, 5)$, $q_1 = (1, 1, 0, 1, 1)$,
$p_2 = (1, -1, 0, -1, 1)$,
$q_2 = (0, 3, 0, 1, -2)$.

4.3.26. $x_0 = (4, 2, 2, 2, 0)$, $p_1 = (1, 2, 2, -1, 1)$,
$y_0 = (-1, 1, -1, 0, 2)$, $q_1 = (8, 7, -2, 1, -1)$,
$p_2 = (2, 1, -2, 1, -1)$,
$q_2 = (-5, -4, 2, -1, 1)$.

4.3.27. Prove that with respect to any basis for a linear space, any hyperplane can be described by a first degree equation in the vector coordinates of the hyperplane (cf. Problem 4.3.7).

4.3.28. Prove that with respect to any basis of a linear space, any plane of dimension r can be described by a system of $n - r$ linear equations in the vector coordinates of the plane.

4.3.29. Let P be a certain plane in a linear space, not being a subspace, and x an arbitrary vector in this plane. Show that the scalar product can be defined for the space so that x is the normal vector of the plane P.

4.4. Homogeneous Systems of Linear Equations

We considered it appropriate to group the problems referring to homogeneous systems of linear equations in a separate section. In contrast to the nonhomogeneous case, the question of consistency does not arise here; moreover, the algebraic structure of the solution set is also different, being a subspace for a homogeneous system and a plane for a nonhomogeneous system.

Great attention has been paid to the two traditional tasks, viz. finding the complete solution and constructing the fundamental system of solutions. It was our intentions to emphasize the relationship of these two methods that describe the solution subspace of a homogeneous system and Problem 4.4.13 shows that the formulae for the complete solution coincide, in fact, with the description of this subspace in terms of a special fundamental system.

At the end of the section we have indicated some of the applications of homogeneous systems of linear equations to the problems concerning linear spaces, i.e. finding the basis and dimension of a subspace, testing the equivalence of two vector sets, etc.

4.4.1. Show that the solution set of an arbitrary homogeneous system of linear equations is a subspace, whereas the solutions may be considered as vectors of the corresponding arithmetic space.

4.4.2. Two homogeneous systems of linear equations

$$a_{11}x_1 + \dots + a_{1n}x_n = 0,$$
$$\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = 0$$

and

$$b_{11}x_1 + \dots + b_{1n}x_n = 0,$$
$$\dots\dots\dots\dots\dots$$
$$b_{l1}x_1 + \dots + b_{ln}x_n = 0$$

are said to be *equivalent* if they have the same solution set. Prove that the indicated systems are equivalent if and only if the vector sets

$$u_1 = (a_{11}, \dots, a_{1n}),$$
$$\dots\dots$$
$$u_m = (a_{m1}, \dots, a_{mn})$$

and

$$v_1 = (b_{11}, \dots, b_{1n}),$$
$$\dots\dots$$
$$v_l = (b_{l1}, \dots, b_{ln})$$

are equivalent.

4.4.3. A homogeneous system of m equations in n unknowns has a coefficient matrix with rank r. Prove that the dimension of the solution subspace of this system equals $n - r$.

4.4.4. Indicate all the values of the parameter λ for which the system of equations

$$(8 - \lambda) x_1 + 2x_2 + 3x_3 + \lambda x_4 = 0,$$
$$x_1 + (9 - \lambda) x_2 + 4x_3 + \lambda x_4 = 0,$$
$$x_1 + 2x_2 + (10 - \lambda) x_3 + \lambda x_4 = 0,$$
$$x_1 + 2x_2 + 3x_3 + \lambda x_4 = 0$$

is indeterminate.

Find the dimension of the solution subspaces of the following systems in relation to the value of the parameter λ:

4.4.5. $(1-\lambda)x_1+\lambda x_2+2\lambda x_3+2\lambda x_4=0,$
$(-1+\lambda)x_1+(2-2\lambda)x_2-2\lambda x_3-2\lambda x_4=0,$
$(1-\lambda)x_1+\lambda x_2+(2+\lambda)x_3+(1+2\lambda)x_4=0,$
$(-1+\lambda)x_1-\lambda x_2-2\lambda x_3+(2-3\lambda)x_4=0.$

4.4.6. $-4x_1+(2+2\lambda)x_2+2\lambda x_3+2\lambda x_4=0,$
$\lambda x_1+(1+\lambda)x_2+\lambda x_3+\lambda x_4=0,$
$\lambda x_1+(1+\lambda)x_2-2x_3+\lambda x_4=0,$
$-\lambda x_1-(1+\lambda)x_2-\lambda x_3-(2+2\lambda)x_4=0.$

4.4.7. Let a homogeneous system of equations be of rank r. Prove that the minors of orders $1, 2, \ldots, r$ placed in the upper left-hand corner can be made different from zero by interchanging the equations and renumbering the unknowns.

In Problems 4.4.8-4.4.15 the following homogeneous system of equations is considered

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=0,\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n&=0. \end{aligned} \tag{4.4.1}$$

The rank of this system is assumed to be equal to r, and the corner minors of orders $1, 2, \ldots, r$ of the matrix of the system are taken to be nonzero (which can be always achieved, according to Problem 4.4.7, by changing the order of the equations and unknowns).

4.4.8. Prove that, when applying the elimination method to system (4.4.1), $a_{11}, a_{22}^{(1)}, \ldots, a_{rr}^{(r-1)}$ can be chosen as the pivot at the separate stages while the resultant system of equations is of the form

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1r}x_r+a_{1,\,r+1}x_{r+1}+\ldots+a_{1n}x_n&=0,\\ a_{22}^{(1)}x_2+\ldots+a_{2r}^{(1)}x_r+a_{2,\,r+1}^{(1)}x_{r+1}+\ldots+a_{2n}^{(1)}x_n&=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&\\ a_{rr}^{(r-1)}x_r+a_{r,\,r+1}^{(r-1)}x_{r+1}+\ldots+a_{rn}^{(r-1)}x_n&=0 \end{aligned} \tag{4.4.2}$$

(those equations whose coefficients are all zero are omitted).

4.4.9. Show that system (4.4.2) can be solved for $x_1, \ldots, x_r$ in terms of the free unknowns $x_{r+1}, \ldots, x_n$, in the following form:

$$\begin{aligned} x_1&=c_{11}x_{r+1}+c_{12}x_{r+2}+\ldots+c_{1,\,n-r}x_n,\\ x_2&=c_{21}x_{r+1}+c_{22}x_{r+2}+\ldots+c_{2,\,n-r}x_n,\\ \ldots&\ldots\ldots\ldots\ldots\ldots\\ x_r&=c_{r1}x_{r+1}+c_{r2}x_{r+2}+\ldots+c_{r,\,n-r}x_n. \end{aligned} \tag{4.4.3}$$

These formulae are termed the *complete solution* of system (4.4.1). This means that one possible solution of the first r components of system (4.4.1) may be found by giving arbitrary values to the free unknowns by formulae (4.4.3), and conversely, any solution of this system can be obtained in this way for the corresponding values of the free unknowns.

4.4.10*. Prove that the solutions of system (4.4.1)

$$y_1 = (x_{11}, \ldots, x_{1r}, x_{1,\, r+1}, \ldots, x_{1n}),$$
$$y_2 = (x_{21}, \ldots, x_{2r}, x_{2,\, r+1}, \ldots, x_{2n}),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$y_k = (x_{k1}, \ldots, x_{kr}, x_{k,\, r+1}, \ldots, x_{kn})$$

are linearly dependent if and only if the $(n - r)$-dimensional vectors

$$z_1 = (x_{1,\, r+1}, \ldots, x_{1n}),$$
$$z_2 = (x_{2,\, r+1}, \ldots, x_{2n}),$$
$$\cdots\cdots\cdots\cdots$$
$$z_k = (x_{k,\, r+1}, \ldots, x_{kn})$$

are linearly dependent.

4.4.11. Prove that a basis for the solution subspace of system (4.4.1) can be obtained as follows: fix some nonzero determinant of order $n - r$, and letting the values of the free unknowns $x_{r+1}, \ldots, x_n$ fill the matrix rows in turn, find the corresponding values of the unknowns $x_1, \ldots, x_r$ by formulae (4.4.3). The constructed $n - r$ solutions form the basis. Any basis for the solution subspace of a homogeneous system of equations is called the *fundamental solution set* of this system of equations.

4.4.12. Show that any fundamental solution set of system (4.4.1) can be constructed by the method indicated in Problem 4.4.11 by a correct choice of the determinant.

4.4.13. Show that the vectors

$$y_1 = (c_{11}, \quad c_{21}, \ldots, c_{r1}, \quad 1, \; 0, \ldots, 0),$$
$$y_2 = (c_{12}, \quad c_{22}, \ldots, c_{r2}, \quad 0, \; 1, \ldots, 0),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_{n-r} = (c_{1,\, n-r}, c_{2,\, n-r}, \ldots, c_{r,\, n-r}, \quad 0, \; 0, \ldots, 1)$$

form the fundamental solution set of system (4.4.1), where c_{ij} are the coefficients of formulae (4.4.3).

Adjoining the identities

$$x_{r+1} = x_{r+1},$$
$$x_{r+2} = \quad x_{r+2},$$
$$\cdots\cdots\cdots$$
$$x_n = \quad x_n$$

to formulae (4.4.3), interpret the complete solution as the representation of any solution of system (4.4.1) by a linear combination of the solutions $y_1, y_2, \ldots, y_r$ whose coefficients are the values of the free unknowns.

4.4.14*. Prove that the rank of the $r \times (n-r)$ matrix C, made up of the coefficients of formulae (4.4.3), is equal to the rank of the submatrix

$$\begin{Vmatrix} a_{1,\,r+1} & a_{1,\,r+2} & \cdots & a_{1n} \\ a_{2,\,r+1} & a_{2,\,r+2} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m,\,r+1} & a_{m,\,r+2} & \cdots & a_{mn} \end{Vmatrix}$$

of the coefficient matrix of system (4.4.1).

4.4.15. Prove that all the coefficients of a free unknown x_k $(r < k \leqslant n)$ in formulae (4.4.3) will equal zero if and only if all the coefficients of this unknown in the original system (4.4.1) equal zero.

Find the general solution and the fundamental solution set of the following systems of equations:

4.4.16. $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0.$

4.4.17.
$$\begin{aligned} 9x_1 + 21x_2 - 15x_3 + 5x_4 &= 0, \\ 12x_1 + 28x_2 - 20x_3 + 7x_4 &= 0. \end{aligned}$$

4.4.18.
$$\begin{aligned} 14x_1 + 35x_2 - 7x_3 - 63x_4 &= 0, \\ -10x_1 - 25x_2 + 5x_3 + 45x_4 &= 0, \\ 26x_1 + 65x_2 - 13x_3 - 117x_4 &= 0. \end{aligned}$$

4.4.19.
$$\begin{aligned} 2x_1 - 5x_2 + 4x_3 + 3x_4 &= 0, \\ 3x_1 - 4x_2 + 7x_3 + 5x_4 &= 0, \\ 4x_1 - 9x_2 + 8x_3 + 5x_4 &= 0, \\ -3x_1 + 2x_2 - 5x_3 + 3x_4 &= 0. \end{aligned}$$

4.4.20.
$$\begin{aligned} 2x_1 + x_2 + 4x_3 + x_4 &= 0, \\ 3x_1 + 2x_2 - x_3 - 6x_4 &= 0, \\ 7x_1 + 4x_2 + 6x_3 - 5x_4 &= 0, \\ x_1 \qquad + 8x_3 + 7x_4 &= 0. \end{aligned}$$

4.4.21.
$$\begin{aligned} x_1 + 4x_2 + 2x_3 \qquad - 3x_5 &= 0, \\ 2x_1 + 9x_2 + 5x_3 - 2x_4 + x_5 &= 0, \\ x_1 + 3x_2 + x_3 - 2x_4 - 9x_5 &= 0. \end{aligned}$$

4.4.22.
$$
\begin{aligned}
2x_1 - 2x_2 + 3x_3 + 6x_4 + 5x_5 &= 0,\\
-4x_1 + 5x_2 - 7x_3 - 3x_4 + 8x_5 &= 0,\\
8x_1 - 9x_2 + 13x_3 + 15x_4 + 2x_5 &= 0,\\
10x_1 - 12x_2 + 17x_3 + 12x_4 - 11x_5 &= 0,\\
-6x_1 + 7x_2 - 10x_3 - 9x_4 + 3x_5 &= 0,\\
-14x_1 + 17x_2 - 24x_3 - 15x_4 + 19x_5 &= 0.
\end{aligned}
$$

4.4.23.
$$
\begin{aligned}
2x_1 - x_2 - x_3 - x_4 - x_5 &= 0,\\
-x_1 + 2x_2 - x_3 - x_4 - x_5 &= 0,\\
4x_1 + x_2 - 5x_3 - 5x_4 - 5x_5 &= 0,\\
x_1 + x_2 + 2x_3 + x_4 + x_5 &= 0,\\
x_1 + x_2 + x_3 + 2x_4 + x_5 &= 0.
\end{aligned}
$$

4.4.24.
$$
\begin{aligned}
3x_1 + 6x_2 + 10x_3 + 4x_4 - 2x_5 &= 0,\\
6x_1 + 10x_2 + 17x_3 + 7x_4 - 3x_5 &= 0,\\
9x_1 \qquad\quad + 3x_3 + 2x_4 + 3x_5 &= 0,\\
12x_1 - 2x_2 + x_3 + 8x_4 + 5x_5 &= 0.
\end{aligned}
$$

4.4.25.
$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 2x_4 - 6x_5 &= 0,\\
2x_1 + 3x_2 + 7x_3 + 6x_4 - 18x_5 &= 0,\\
3x_1 + 5x_2 + 11x_3 + 9x_4 - 27x_5 &= 0,\\
2x_1 - 7x_2 + 7x_3 + 16x_4 - 48x_5 &= 0,\\
x_1 + 4x_2 + 5x_3 + 2x_4 - 6x_5 &= 0.
\end{aligned}
$$

4.4.26. Verify that the system

$$
\begin{aligned}
2x_1 + 4x_2 + 6x_3 + 5x_4 + 3x_5 &= 0,\\
5x_1 + 6x_2 + 7x_3 + 9x_4 + 6x_5 &= 0,\\
4x_1 + 6x_2 + 8x_3 + 7x_4 + 5x_5 &= 0,\\
5x_1 + 5x_2 + 5x_3 + 8x_4 + 6x_5 &= 0,\\
3x_1 + 4x_2 + 5x_3 + 6x_4 + 4x_5 &= 0.
\end{aligned}
$$

has infinitely many solutions, while in each of its solutions $x_4 = x_5 = 0$. Explain these facts in terms of linear dependence or linear independence of the columns of the matrix of the system.

4.4.27. Indicate all the sets of unknowns that can be free unknowns of the system

$$7x_1 - 4x_2 + 9x_3 + 2x_4 + 2x_5 = 0,$$
$$5x_1 + 8x_2 + 7x_3 - 4x_4 + 2x_5 = 0,$$
$$3x_1 - 8x_2 + 5x_3 + 4x_4 + 2x_5 = 0,$$
$$7x_1 - 2x_2 + 2x_3 + x_4 - 5x_5 = 0.$$

4.4.28. In the space of polynomials of degree $\leqslant n$, determine the dimension of the subspace of polynomials $f(t)$ satisfying the conditions $f(a_1) = f(a_2) = \ldots = f(a_k) = 0$ where $a_1, \ldots, a_k$ are different numbers.

4.4.29. In the space of polynomials of degree $\leqslant 5$, find the basis for a linear subspace of polynomials $f(t)$ fulfilling the conditions $f(0) = f(1) = f(2) = f(3) = 0$.

4.4.30. Find a homogeneous system of linear equations consisting of (a) two equations; (b) three equations; (c) four equations, and for which the vector set

$$y_1 = (1, 4, -2, 2, -1),$$
$$y_2 = (3, 13, -1, 2, 1),$$
$$y_3 = (2, 7, -8, 4, -5)$$

is a fundamental solution set.

4.4.31. Can a system of linear equations be found for which the vector sets

$$y_1 = (2, 3, 1, 2),$$
$$y_2 = (1, 1, -2, -2),$$
$$y_3 = (3, 4, 2, 1)$$

and

$$z_1 = (1, 0, 2, -5),$$
$$z_2 = (0, 1, 8, 7),$$
$$z_3 = (4, 5, -2, 0)$$

are two fundamental solution sets?

4.4.32*. The rank of a homogeneous system of linear equations consisting of $n-1$ equations with n unknowns equals $n-1$. Prove that a nonzero solution of this system can be constructed by the formulae

$$x_i = (-1)^i A_i, \quad i = 1, \ldots, n$$

where A_i is a minor derived from the coefficient matrix of the system by deleting the i-th column. In addition, show that any other solution of the system and the indicated one are collinear.

4.4.33. With the aid of the result of Problem 4.4.32, find a vector orthogonal to the set of vectors

$$z_1 = (2, -1, 3, 1),$$
$$z_2 = (1, 0, 2, -3),$$
$$z_3 = (2, 3, 1, 4).$$

4.4.34*. Prove the following theorem: For two linearly independent vector sets $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_{n-1}$ in an n-dimensional linear space to be equivalent, it is necessary and sufficient that all n minors of order $n-1$, made up of the coordinates of the vectors $x_1, \ldots, x_{n-1}$ in any basis for this space, should be proportional to the corresponding minors made up of the coordinates of the vectors $y_1, \ldots, y_{n-1}$.

4.4.35. By means of the result of Problem 4.4.34, determine whether the vector sets of Problem 4.4.31 are equivalent.

4.5. Nonhomogeneous Systems of Linear Equations

The topics touched upon in this section are, for the most part, as follows: Criteria for the consistency of nonhomogeneous systems of equations and investigation into the consistency of individual systems.

Finding the general solution of a system. In addition to purely computational problems, we also provide problems concerning the determination of the complete solution. As with the homogeneous case, we stress the fact that the formulae for complete solution yield, actually, the parametric equations for the solution plane of the given system of linear equations (see Problem 4.5.9). Note that the computational problems also point out certain techniques that are generally employed in practical computations, viz. a change of numeration and scaling of the equations and unknowns, use of individual features of a system, e.g. its property to decompose into several systems of the lesser order, or the tridiagonal form of the matrix of coefficients.

In conclusion we list a number of problems referring to particular systems of linear equations. We indicate, *inter alia*, some applications of Cramer's formulae.

4.5.1. Prove that a nonhomogeneous system of m linear equations is consistent if the rank of the coefficient matrix of the unknowns is m.

4.5.2. Prove that for the system

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \quad \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array} \tag{4.5.1}$$

to be consistent, it is necessary and sufficient that the vector

$$b = (b_1, b_2, \ldots, b_m)$$

should belong to the span of the vectors

$$
\begin{aligned}
a_1 &= (a_{11}, a_{21}, \ldots, a_{m1}), \\
a_2 &= (a_{12}, a_{22}, \ldots, a_{m2}), \\
&\cdots\cdots\cdots\cdots \\
a_n &= (a_{1n}, a_{2n}, \ldots, a_{mn}).
\end{aligned}
$$

4.5.3. Prove the *Fredholm theorem*: for a homogeneous system (4.5.1) to be consistent, it is necessary and sufficient that the m-dimensional vector $b = (b_1, b_2, \ldots, b_m)$ should be orthogonal to all the solutions of the conjugate homogeneous system

$$
\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m &= 0, \\
a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m &= 0.
\end{aligned}
$$

4.5.4. Given that a nonhomogeneous system of m equations in n unknowns is consistent and the rank of the coefficient matrix of the unknowns equals r, prove that the solution set for this system is a plane in an n-dimensional arithmetic space with dimension $n - r$, and whose directional subspace is the solution set of the corresponding homogeneous system with the same coefficient matrix of the unknowns.

4.5.5. Two nonhomogeneous systems of linear equations

$$
\begin{aligned}
a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\
\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

and

$$
\begin{aligned}
c_{11}x_1 + \ldots + c_{1n}x_n &= d_1, \\
\cdots\cdots\cdots\cdots \\
c_{l1}x_1 + \ldots + c_{ln}x_n &= d_l
\end{aligned}
$$

are called *equivalent* if either they are both inconsistent or both consistent, and have the same solution set. Prove that the indicated systems, if consistent, are equivalent if and only if the vector sets are equivalent

$$
\begin{aligned}
u_1 &= (a_{11}, \ldots, a_{1n}, b_1), \\
&\cdots\cdots\cdots\cdots \\
u_m &= (a_{m1}, \ldots, a_{mn}, b_m)
\end{aligned}
$$

and

$$
\begin{aligned}
v_1 &= (c_{11}, \ldots, c_{1n}, d_1), \\
&\cdots\cdots\cdots\cdots \\
v_l &= (c_{l1}, \ldots, c_{ln}, d_l).
\end{aligned}
$$

In the following verify whether the system is consistent and determine the dimension of the solution plane in relation to the value of the parameter λ.

4.5.6.

$$\begin{aligned} (5-\lambda)x_1 - 2x_2 - x_3 &= 1, \\ -2x_1 + (2-\lambda)x_2 - 2x_3 &= 2, \\ -x_1 - 2x_2 + (5-\lambda)x_3 &= 1. \end{aligned}$$

4.5.7.

$$\begin{aligned} -x_1 + (1+\lambda)x_2 + (2-\lambda)x_3 + \lambda x_4 &= 3, \\ \lambda x_1 - x_2 \qquad + (2-\lambda)x_3 + \lambda x_4 &= 2, \\ \lambda x_1 + \lambda x_2 \qquad + (2-\lambda)x_3 + \lambda x_4 &= 2, \\ \lambda x_1 + \lambda x_2 \qquad + (2-\lambda)x_3 - x_4 &= 2. \end{aligned}$$

Problems 4.5.8-4.5.11 concern consistent nonhomogeneous systems of equations of rank r, similar to (4.5.1). The equations are assumed to have a numeration that stipulates that the minors of orders $1, 2, \ldots, r$, positioned in the upper left-hand corner, are nonzero.

4.5.8. Applying the method of elimination of the unknowns, show that system (4.5.1) can be solved for $x_1, \ldots, x_r$ so that their expressions in terms of the free unknowns $x_{r+1}, \ldots, x_n$ may be of the following form:

$$\begin{aligned} x_1 &= c_{10} + c_{11}x_{r+1} + c_{12}x_{r+2} + \ldots + c_{1,\,n-r}x_n, \\ x_2 &= c_{20} + c_{21}x_{r+1} + c_{22}x_{r+2} + \ldots + c_{2,\,n-r}x_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_r &= c_{r0} + c_{r1}x_{r+1} + c_{r2}x_{r+2} + \ldots + c_{r,\,n-r}x_n. \end{aligned} \qquad (4.5.2)$$

These formulae are known as the *complete solution* of system (4.5.1). Formulae (4.4.3) are a particular case of formulae (4.5.2).

4.5.9. Show that the vector

$$x_0 = (c_{10}, c_{20}, \ldots, c_{r0}, 0, 0, \ldots, 0)$$

is a solution of system (4.5.1) and the vectors

$$\begin{aligned} y_1 &= (c_{11}, c_{21}, \ldots, c_{r1}, 1, 0, \ldots, 0), \\ y_2 &= (c_{12}, c_{22}, \ldots, c_{r2}, 0, 1, \ldots, 0), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n-r} &= (c_{1,\,n-r}, c_{2,\,n-r}, \ldots, c_{r,\,n-r}, 0, 0, \ldots, 1) \end{aligned}$$

form a fundamental solution set of the corresponding homogeneous system.

By adding the identities

$$\begin{aligned} x_{r+1} &= x_{r+1}, \\ x_{r+2} &= \qquad x_{r+2}, \\ &\ldots\ldots \\ x_n &= \qquad x_n \end{aligned}$$

to formulae (4.5.2), interpret the obtained relations as parametric equations of the solution plane of system (4.5.1).

4.5.10. Prove that the rank of the matrix,

$$\left\| \begin{matrix} c_{10} & c_{11} & c_{12} & \dots & c_{1,\, n-r} \\ c_{20} & c_{21} & c_{22} & \dots & c_{2,\, n-r} \\ \dots & \dots & \dots & \dots & \dots \\ c_{r0} & c_{r1} & c_{r2} & \dots & c_{r,\, n-r} \end{matrix} \right\|,$$

made up of the coefficients of formulae (4.5.2), equals the rank of the submatrix

$$\left\| \begin{matrix} a_{1,\, r+1} & \dots & a_{1n} b_1 \\ a_{2,\, r+1} & \dots & a_{2n} b_2 \\ \dots & \dots & \dots \\ a_{m,\, r+1} & \dots & a_{mn} b_m \end{matrix} \right\|$$

of the augmented matrix of system (4.5.1).

4.5.11. Prove that the vector

$$z_i = (c_{1i}, c_{2i}, \dots, c_{ri}), \quad i = 1, \dots, n-r$$

is a solution of the system of equations

$$\begin{aligned} a_{11}x_1 + \dots + a_{1r}x_r &= -a_{1i}, \\ \dots \dots \dots \dots \\ a_{r1}x_1 + \dots + a_{rr}x_r &= -a_{ri}, \end{aligned}$$

and the vector $z_0 = (c_{10}, c_{20}, \dots, c_{r0})$ is a solution of the system of equations

$$\begin{aligned} a_{11}x_1 + \dots + a_{1r}x_r &= b_1, \\ \dots \dots \dots \dots \\ a_{r1}x_1 + \dots + a_{rr}x_r &= b_r. \end{aligned}$$

Investigate the consistency and find the general solution of the following systems of equations

4.5.12.
$$\begin{aligned} 38x_1 - 74x_2 + 46x_3 + 84x_4 &= 90, \\ -95x_1 + 185x_2 - 115x_3 - 210x_4 &= -225, \\ 57x_1 - 111x_2 + 69x_3 + 126x_4 &= 135. \end{aligned}$$

4.5.13.
$$\begin{aligned} 105x_1 - 175x_2 - 315x_3 + 245x_4 &= 84, \\ 90x_1 - 150x_2 - 270x_3 + 210x_4 &= 72, \\ 75x_1 - 125x_2 - 225x_3 + 175x_4 &= 59. \end{aligned}$$

4.5.14.
$$\begin{aligned} 7x_1 - 5x_2 - 2x_3 - 4x_4 &= 8, \\ -3x_1 + 2x_2 + x_3 + 2x_4 &= -3, \\ 2x_1 - x_2 - x_3 - 2x_4 &= 1, \\ -x_1 \qquad + x_3 + 2x_4 &= 1, \\ -x_2 + x_3 + 2x_4 &= 3. \end{aligned}$$

4.5.15.
$$\begin{aligned} x_1+2x_2+3x_3+4x_4&=0,\\ 7x_1+14x_2+20x_3+27x_4&=0,\\ 5x_1+10x_2+16x_3+19x_4&=-2,\\ 3x_1+5x_2+6x_3+13x_4&=5, \end{aligned}$$

4.5.16.
$$\begin{aligned} x_1+x_2&=1,\\ x_1+x_2+x_3&=4,\\ x_2+x_3+x_4&=-3,\\ x_3+x_4+x_5&=2,\\ x_4+x_5&=-1. \end{aligned}$$

4.5.17.
$$\begin{aligned} 12x_1-18x_2+102x_3-174x_4-216x_5&=132,\\ 14x_1-21x_2+119x_3-203x_4-252x_5&=154,\\ x_3+2x_4+3x_5&=-1,\\ 4x_3+5x_4+6x_5&=-2,\\ 7x_3+8x_4+9x_5&=-3. \end{aligned}$$

4.5.18.
$$\begin{aligned} 24x_1+9x_2+33x_3-15x_4&=21,\\ 8x_1+3x_2+11x_3-5x_4&=7,\\ 40x_1+15x_2+55x_3-25x_4+213x_5&=35,\\ 56x_1+21x_2+77x_3-35x_4+197x_5&=49. \end{aligned}$$

4.5.19*.
$$\begin{aligned} 2000x_1+0.003x_2-0.3x_3+40x_4&=5,\\ 3000x_1+0.005x_2-0.4x_3+90x_4&=8,\\ 500x_1+0.0007x_2-0.08x_3+8x_4&=1.5,\\ 60000x_1+0.09x_2-9x_3+1300x_4&=190. \end{aligned}$$

4.5.20.
$$\begin{aligned} x_1+2x_2-5x_3+4x_4+x_5&=4,\\ 3x_1+7x_2-x_3-3x_4+2x_5&=10,\\ -x_2-13x_3-2x_4+x_5&=-14,\\ x_3-16x_4+2x_5&=-11,\\ 2x_4+5x_5&=12. \end{aligned}$$

4.5.21.
$$\begin{aligned} 8x_1+12x_2&=20,\\ 14x_1+21x_2&=35,\\ 9x_3+11x_4&=0,\\ 16x_3+20x_4&=0,\\ 10x_5+12x_6&=22,\\ 15x_5+18x_6&=33, \end{aligned}$$

4.5.22.
$$\begin{aligned} x_1-5x_3+2x_6&=6,\\ 2x_2+x_4+3x_5&=6,\\ 2x_1-7x_3+3x_6&=4,\\ 3x_2+2x_4+4x_5&=7,\\ 2x_1-x_3+x_6&=-12,\\ 4x_2+3x_4+5x_5&=9. \end{aligned}$$

Investigate the following systems and find their general solutions in relation to the value of the parameter λ:

4.5.23.
$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= -1, \\ 7x_1 + 6x_2 + 5x_3 &= \lambda, \\ 5x_1 + 4x_2 + 3x_3 &= 2. \end{aligned}$$

4.5.24.
$$\begin{aligned} \lambda x_1 + x_2 + x_3 &= 0, \\ 5x_1 + x_2 - 2x_3 &= 2, \\ -2x_1 - 2x_2 + x_3 &= -3. \end{aligned}$$

4.5.25.
$$\begin{aligned} 24x_1 - 38x_2 + 46x_3 &= 26, \\ 60x_1 + \lambda x_2 + 115x_3 &= 65, \\ 84x_1 - 133x_2 + 161x_3 &= 91. \end{aligned}$$

4.5.26.
$$\begin{aligned} x_1 + x_2 + \lambda x_3 &= 1, \\ x_1 + \lambda x_2 + x_3 &= 1, \\ \lambda x_1 + x_2 + x_3 &= 1. \end{aligned}$$

4.5.27.
$$\begin{aligned} x_1 + x_2 + \lambda x_3 &= 2, \\ x_1 + \lambda x_2 + x_3 &= -1, \\ \lambda x_1 + x_2 + x_3 &= -1. \end{aligned}$$

4.5.28.
$$\begin{aligned} x_1 + x_2 + \lambda x_3 &= 3, \\ x_1 + \lambda x_2 + x_3 &= 0, \\ \lambda x_1 + x_2 + x_3 &= 0. \end{aligned}$$

4.5.29.
$$\begin{aligned} (3-2\lambda) x_1 + (2-\lambda) x_2 + x_3 &= \lambda, \\ (2-\lambda) x_1 + (2-\lambda) x_2 + x_3 &= 1, \\ x_1 + x_2 + (2-\lambda) x_3 &= 1. \end{aligned}$$

4.5.30.
$$\begin{aligned} (3+2\lambda) x_1 + (1+3\lambda) x_2 + \lambda x_3 + (\lambda-1) x_4 &= 3, \\ 3\lambda x_1 + (3+2\lambda) x_2 + \lambda x_3 + (\lambda-1) x_4 &= 1, \\ 3\lambda x_1 + 3\lambda x_2 + 3x_3 + (\lambda-1) x_4 &= 1, \\ 3\lambda x_1 + 3\lambda x_2 + \lambda x_3 + (\lambda-1) x_4 &= 1. \end{aligned}$$

4.5.31. Verify that in all solutions of the system of equations
$$\begin{aligned} 2x_1 + 3x_2 + x_3 \qquad + x_5 &= 6, \\ x_1 + 2x_2 + x_3 \quad x_4 \qquad &= 5, \\ -x_1 + x_2 + 3x_3 + 5x_4 + x_5 &= 8, \\ 2x_1 - x_2 + x_3 - 8x_4 + 2x_5 &= -6, \end{aligned}$$

the values of the unknowns x_3 and x_5 are constant and equal to 1 and 0, respectively. Account for this fact in terms of the linear dependence and linear independence of the columns in the augmented matrix of the system.

4.5.32. Can the general solution of the same system of linear equations in 8 unknowns be described by the formulae

$$\begin{aligned} x_1 &= x_5 + 2x_6 + 3x_7 + 4x_8, \\ x_2 &= 2x_5 + 3x_6 + x_7 + 2x_8, \\ x_3 &= x_5 + x_6 + x_7 - x_8, \\ x_4 &= x_5 \qquad - 2x_7 - 6x_8 \end{aligned}$$

and

$$\begin{aligned} x_5 &= 21x_1 - 6x_2 - 26x_3 + 17x_4, \\ x_6 &= -17x_1 + 5x_2 + 20x_3 - 13x_4, \\ x_7 &= -x_1 \qquad + 2x_3 - x_4, \\ x_8 &= 4x_1 - x_2 - 5x_3 + 3x_4? \end{aligned} \tag{4.5.3}$$

4.5.33. Replace the first relation in formulae (4.5.3) of Problem 4.5.32 by

$$x_5 = 22x_1 - 6x_2 - 26x_3 + 17x_4$$

and answer the problem question again.

4.5.34. Prove that the set of polynomials $f(t)$ of degree $\leqslant n$ satisfying the conditions $f(a_1) = b_1, f(a_2) = b_2, \ldots, f(a_k) = b_k$ (where $k \leqslant n + 1$ and $a_1, \ldots, a_k, b_1, \ldots, b_k$ are arbitrary numbers, whereas all a_i, $1 \leqslant i \leqslant k$, are different) is nonempty and produces a plane. Find the dimension of this plane.

4.5.35. Find three linear independent polynomials $f(t)$ of degree $\leqslant 5$ fulfilling the conditions $f(0) = 1$, $f(1) = 0$, $f(2) = -5$, $f(3) = -20$.

4.5.36*. Verify that the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 - 2x_4 &= -2, \\ 8x_1 + 7x_2 + 7x_3 - 9x_4 &= 3, \\ 6x_1 + 5x_2 + 5x_3 - 5x_4 &= 7 \end{aligned}$$

is consistent and find a normal solution to this system.

4.5.37. Prove that for a nonhomogeneous system of linear equations, with the number of the equations equal to the number of the unknowns, to be consistent it is necessary and sufficient that the reduced homogeneous system should have a unique solution.

4.5.38. The columns $q_1, q_2, \ldots, q_n$ of the coefficient matrix of a system of n linear equations in n unknowns form an orthonormal set. Prove that this system is fully defined, and that its solution can be evaluated by the formulae $x_i = (b, q_i)$, $i = 1, \ldots, n$. Here b is an n-dimensional vector made up of the right-hand sides of the system, and the scalar product is defined by the usual rule for an arithmetic space.

4.5.39. Prove that the statement of Problem 4.5.38 also holds for a consistent system in which the number of equations does not equal the number of the unknowns (the same condition of the orthonormality of the columns, however, being maintained).

4.5.40. Using the result of Problem 4.5.38, solve the following system of equations:

$$\begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= p, \\ -bx_1 + ax_2 + dx_3 - cx_4 &= q, \\ -cx_1 - dx_2 + ax_3 + bx_4 &= r, \\ -dx_1 + cx_2 - bx_3 + ax_4 &= s \end{aligned}$$

assuming that $A = a^2 + b^2 + c^2 + d^2 \neq 0$.

4.5.41. Deduce from the result of Problem 4.5.34 that if the values of two polynomials $f(t)$ and $g(t)$ of degree $\leqslant n$ coincide for more than n different values of the argument, then these polynomials are equal (i.e. the corresponding coefficients of the polynomials coincide). Hence deduce that the given definition of equality for two polynomials is equivalent to their equality as functions (i.e. to coincidence of their values for all values of the unknown).

4.5.42. Find a polynomial $f(t)$ of the third degree for which $f(1) = -2$, $f(2) = -4$, $f(3) = -2$, $f(4) = 10$.

4.5.43. Find a polynomial $f(t)$ of degree $\leqslant 4$ for which $f(-2) = 10$, $f(1) = 4$, $f(-3) = 60$, $f(2) = -10$, $f(-1) = -4$.

4.5.44*. Prove that a polynomial $f(t)$ of degree $\leqslant 2k$, satisfying the conditions $f(a_i) = f(-a_i)$, $i = 1, \ldots, k$, where $a_1^2, \ldots, a_k^2$ are different nonzero numbers, is necessarily even, i.e. the equality $f(-t) = f(t)$ holds true.

4.5.45. Prove that a polynomial $f(t)$ of degree $\leqslant 2k - 1$ fulfilling the conditions $f(a_i) = -f(-a_i)$, $i = 1, \ldots, k$, where $a_1^2, \ldots, a_k^2$ are different nonzero numbers, is necessarily odd, i.e. the equality $f(-t) = -f(t)$ is valid.

4.5.46. Prove that whatever the numbers $a, b_0, b_1, \ldots, b_n$ are, there exists, and is unique, a polynomial $f(t)$ of degree $\leqslant n$ such that $f(a) = b_0$, $f'(a) = b_1, \ldots, f^{(n)}(a) = b_n$.

4.5.47. Find a polynomial $f(t)$ of degree $\leqslant 4$ such that $f(2) = 5$, $f'(2) = 19$, $f^{(2)}(2) = 40$, $f^{(3)}(2) = 48$, $f^{(4)}(2) = 24$.

4.5.48*. Prove that whatever the numbers $a_1, a_2, b_0, b_1, \ldots, b_{n-1}, c_0$ $(a_1 \neq a_2)$ are, there exists, and is unique, a polynomial $f(t)$ of degree $\leqslant n$ such that $f(a_1) = b_0$, $f'(a_1) = b_1, \ldots, f^{(n-1)}(a_1) = b_{n-1}$, $f(a_2) = c_0$.

4.5.49. Find a polynomial $f(t)$ of degree $\leqslant 4$ such that $f(1) = -3$, $f'(1) = -3$, $f^{(2)}(1) = 12$, $f^{(3)}(1) = 42$, $f(-1) = 3$.

4.5.50*. Prove that whatever the numbers $a_1, a_2, b_0, b_1, \ldots, b_k, c_0, c_1, \ldots, c_l$ ($a_1 \neq a_2$; $k + l = n - 1$) are, there exists, and is unique, a polynomial of degree $\leqslant n$ such that the conditions $f(a_1) = b_0$, $f'(a_1) = b_1, \ldots, f^{(k)}(a_1) = b_k$, $f(a_2) = c_0$, $f'(a_2) = c_1, \ldots, f^{(l)}(a_2) = c_l$ are met.

4.5.51. Find a polynomial $f(t)$ of degree $\leqslant 5$ such that $f(1) = -2$, $f'(1) = -7$, $f^{(2)}(1) = -14$, $f^{(3)}(1) = 24$, $f(2) = -4$, $f'(2) = 25$.

4.5.52. The right-hand sides b_i of a certain system of n linear equations in n unknowns are differentiable functions of a variable t; the coefficients a_{ij} of the unknowns are constant numbers. Prove that the components $x_1, \ldots, x_n$ are also differentiable functions of t, and

$$x_i'(t) = \frac{\begin{vmatrix} a_{11} & \ldots & b_1'(t) & \ldots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \ldots & b_n'(t) & \ldots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & \ldots & a_{1i} & \ldots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \ldots & a_{ni} & \ldots & a_{nn} \end{vmatrix}}, \quad i = 1, \ldots, n.$$

4.5.53*. By means of Cramer's formulae, deduce, for the n-th derivative of a function

$$f(t) = \frac{g(t)}{h(t)},$$

the following relation

$$f^{(n)}(t) = \frac{1}{h^{(n+1)}(t)} \begin{vmatrix} h(t) & 0 & 0 & \ldots & g(t) \\ h'(t) & h(t) & 0 & \ldots & g'(t) \\ h^{(2)}(t) & 2h'(t) & h(t) & \ldots & g^{(2)}(t) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ h^{(n)}(t) & {}^nC_1 h^{(n-1)}(t) & {}^nC_2 h^{(n-2)}(t) & \ldots & g^{(n)}(t) \end{vmatrix}.$$

4.5.54. Evaluate the 5-th derivative of the function

$$f(t) = \frac{(t-1)^5}{37t^6 - 61t^5 + 13t^2 - 74t + 25}$$

when $t = 1$.

4.5.55. Prove that the solutions $x_1, \ldots, x_k$ of certain systems of linear equations with the same coefficient matrix (and with right-hand sides $b_1, \ldots, b_k$) are linearly dependent if and only if the right-hand sides are linearly dependent.

CHAPTER 5

Linear Operators and Matrices

5.0. Terminology and General Notes

Given two linear spaces X and Y both real or both complex. A relation between the elements of these spaces that matches each vector $x \in X$ with one particular vector $y \in Y$ is called a *linear operator* A from X to Y. The vector y is called the image of the vector x and is denoted by Ax, moreover

$$A\ (\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2$$

for any vectors x_1 and x_2 and any numbers α and β. Since we will only be considering linear operators from now on, the word "linear" is sometimes omitted.

The set of all vectors $Ax, x \in X$, is called the *range* or *image* of the operator A and is designated by T_A. The set of all vectors x, for which $Ax = 0$, is called the *kernel* of the operator A and designated by N_A. The image and kernel of a linear operator are linear spaces (see Sec. 5.1). The dimension of the subspace T_A is denoted by r_A and is called the *rank* of the operator A and the dimension of the subspace N_A is denoted by n_A and is called the *defect* of the operator A.

Let the set of all linear operators from X to Y be ω_{XY}. The structure of a linear space can be defined for the set ω_{XY}, that is we can put

(i) $(A + B)\ x = Ax + Bx$;

(ii) $(\lambda A)\ x = \lambda\ (Ax)$,

where x is an arbitrary vector from X. The operators defined by these relations, i.e. $A + B$ and λA, are called the *sum of the operators* A and B, and the *product of the operator* A *by a number* λ, respectively. The zero element of the linear space ω_{XY} is the *zero operator* from X to Y, i.e. the operator matching each vector from X with the zero element of the space Y.

Now, let $A \in \omega_{XY}$, $B \in \omega_{YZ}$. An operator $C = BA$ from X to Z and defined by the relation

$$Cx = B\ (Ax),$$

is called the *product of the operator* B *by the operator* A. For the product BA to have any sense it is a necessary and sufficient condition that the image of the operator A should be contained in the domain of the operator B. This condition is inevitably fulfilled in the case of operators from ω_{XX}. We will say about each of such operators that it is on the space X.

For an operator A from ω_{XX}, a natural power A^k may be defined as the product of k operators equal to A. By definition, for any operator A we put

$$A^0 = E,$$

where E is the *identity* or *unit operator* (i.e. the operator matching each $x \in X$ with the same vector x). If

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k$$

is an arbitrary polynomial, then the operator

$$f(A) = a_0 E + a_1 A + a_2 A^2 + \ldots + a_k A^k$$

is called a polynomial $f(A)$ in the operator A.

The operator A on an n-dimensional space X is called *nondegenerate* if the defect of this operator equals zero, or in other words, if the rank equals n. For a nonhomogeneous operator A there exists, and is unique, a linear operator B such that

$$AB = BA = E.$$

The operator B is called the *inverse* of the operator A and is denoted by A^{-1}.

With the aid of the inverse operator, the whole negative powers of a nondegenerate operator A can be defined. Namely, if k is a natural number, we put

$$A^{-k} = (A^{-1})^k$$

or, equivalently

$$A^{-k} = (A^k)^{-1}.$$

A matrix $C = A + B$ of order $m \times n$ is called the *sum of the matrices A and B of order $m \times n$* if

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

An $m \times n$ matrix $D = \lambda A$ such that

$$d_{ij} = \lambda a_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

is called the *product of the matrix A by the number λ*.

The *unit matrix* (cf. Sec. 3.0), just like the identity operator, is denoted by E. If the order n of a unit matrix should be explicit, the notation E_n is employed. Matrices of the form λE are said to be *scalar*.

A matrix C of order $p \times n$ such that

$$c_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, n$$

is called the *product* BA of a $p \times m$ matrix B and an $m \times n$ matrix A. For the product BA to have sense it is a necessary and sufficient condition that the number of columns of the matrix B should equal the number of rows of the matrix A. This condition is necessarily fulfilled if both matrices are square matrices of the same order.

For a *nondegenerate* matrix A (i.e. a square matrix with a nonzero determinant, cf. Sec. 3.0) there exists an inverse matrix A^{-1} satisfying the equalities

$$AA^{-1} = A^{-1}A = E.$$

If we put $B = A^{-1}$ then the elements b_{ij} of the matrix B can be computed by the formulae

$$b_{ij} = \frac{A_{ji}}{\det A}, \tag{5.0.1}$$

A_{ji} being the cofactor of the element a_{ji}.

If a square $n \times n$ matrix C is the product AB of two rectangular matrices A and B of dimensions $n \times m$ and $m \times n$, respectively, m being $\geqslant n$, then for the determinant of the matrix C the *Binet-Cauchy formula* is valid:

$$\det C = \sum_{1 \leqslant k_1 < k_2 < \ldots < k_n \leqslant m} A\begin{pmatrix} 1 & 2 & \ldots & n \\ k_1 & k_2 & \ldots & k_n \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \ldots & k_n \\ 1 & 2 & \ldots & n \end{pmatrix}. \tag{5.0.2}$$

In particular, if A and B are also square matrices then

$$\det AB = \det A \cdot \det B. \tag{5.0.3}$$

Let A be an operator from ω_{XY} and let $e_1, \ldots, e_n$ and $q_1, \ldots, q_m$ be fixed bases of the spaces X and Y, respectively. Resolve the vectors $Ae_1, \ldots, Ae_n$ in terms of the basis vectors $q_1, \ldots, q_m$:

$$\begin{aligned} Ae_1 &= a_{11}q_1 + a_{21}q_2 + \ldots + a_{m1}q_m, \\ Ae_2 &= a_{12}q_1 + a_{22}q_2 + \ldots + a_{m2}q_m, \\ &\cdots\cdots\cdots\cdots\cdots \\ Ae_n &= a_{1n}q_1 + a_{2n}q_2 + \ldots + a_{mn}q_m. \end{aligned}$$

Construct an $m \times n$ matrix of the coefficients in these decompositions

$$A_{qe} = \begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{Vmatrix}. \tag{5.0.4}$$

A_{qe} is said to be the *matrix of the operator* A *in the pair of bases* $e_1, \ldots, e_n$ and $q_1, \ldots, q_m$, or to *define the operator* A *in* this *pair of bases*.

An $m \times 1$ matrix is called an m-dimensional *column vector*, an $1 \times n$ matrix is called an n-dimensinnal *row vector*. Let each vector

$x \in X$ correspond to an n-dimensional column vector x_e made up of the coordinates of this vector in a basis $e_1, \ldots, e_n$. Similarly, each vector $y \in Y$ can be matched with an m-dimensional column vector y_q of the coordinates of this vector in a basis $q_1, \ldots, q_m$. Then the relation between the coordinates of the vector x and vector $y = Ax$ can be determined by a matrix equality

$$y_q = A_{qe}x_e. \qquad (5.0.5)$$

To describe an operator A from ω_{XX} it suffices to fix one basis $e_1, \ldots, e_n$, with the vectors $Ae_1, \ldots, Ae_n$ being decomposed in terms of this basis. The matrix made up of the coefficients in the decompositions is denoted by A_e and is called the *matrix of the operator A in the basis $e_1, \ldots, e_n$*. A_e is also said to *define the operator A in this basis.* Formula (5.0.5) is converted into

$$y_e = A_e x_e. \qquad (5.0.6)$$

Let in a space X two bases be fixed: $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$. Decompose the vectors $f_1, \ldots, f_n$ in terms of the basis $e_1, \ldots, e_n$

$$\begin{aligned} f_1 &= p_{11}e_1 + p_{21}e_2 + \ldots + p_{n1}e_n, \\ f_2 &= p_{12}e_1 + p_{22}e_2 + \ldots + p_{n2}e_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ f_n &= p_{1n}e_1 + p_{2n}e_2 + \ldots + p_{nn}e_n. \end{aligned}$$

A matrix can be constructed from the coefficients in these decompositions

$$P = \begin{Vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{Vmatrix}. \qquad (5.0.7)$$

This matrix P is called the *matrix of the transfer from the basis $e_1, \ldots, e_n$ to the basis $f_1, \ldots, f_n$*. If x_e and x_f are columns of the coordinates of a vector x in these two bases, then the relation between them is given by the equality

$$x_e = Px_f. \qquad (5.0.8)$$

It is now possible also using the transfer matrix to write down the relation between the two matrices A_e and A_f, that determine the operator A on the space X; thus:

$$A_f = P^{-1}A_eP. \qquad (5.0.9)$$

If $A \in \omega_{XY}$ and if in each of the spaces X and Y two bases are fixed, i.e. $e_1, \ldots, e_n$; $f_1, \ldots, f_n$ and $q_1, \ldots, q_m$; $t_1, \ldots, t_m$, respectively, then the matrices A_{qe} and A_{tf} are connected by a relation

$$A_{tf} = Q^{-1}A_{qe}P, \tag{5.0.10}$$

where P is the matrix of the transfer from $e_1, \ldots, e_n$ to $f_1, \ldots, f_n$, and Q is the matrix of the transfer from $q_1, \ldots, q_m$ to $t_1, \ldots, t_m$.

If the elements of an arithmetic space are written in the vector-column form, then formula (5.0.5) makes it possible to identify operators in R_n with those in R_m, or operators in C_n with those in C_m, using $m \times n$ matrices which are real or complex, respectively (for further details see Problem 5.6.7). With this remark in mind, we will speak, hereafter, of the image of a matrix, its kernel, etc.

5.1. The Definition of a Linear Operator, the Image and Kernel of an Operator

In addition to examples of operators in concrete linear spaces, we provide a number of problems related to the definition of a linear operator. Further, the effect produced by a linear operator on the principal relations of a linear space (such as linear dependence, equivalence of vector sets, the sum of subspaces, etc.) is given prominence.

At the end of the section the important concepts of a kernel and image are discussed.

Determine, for each of the following operators on the three-dimensional Euclidean space of geometric vectors, whether the operator is linear. All the operators are described by their effect on an arbitrary vector x. Further, a and b signify fixed vectors of the space, and α is a fixed number.

5.1.1. $Ax = a$. **5.1.2.** $Ax = x + a$. **5.1.3.** $Ax = \alpha x$. **5.1.4.** $Ax = (x, a)\, a$. **5.1.5.** $Ax = (a, x)\, b$. **5.1.6.** $Ax = (a, x)\, x$. **5.1.7.** $Ax = [x, a]$. **5.1.8.** $Ax = [a, [x, b]]$.

Verify which of the following mappings of the three-dimensional Euclidean space of geometric vectors into the set of real numbers are linear operators. All the mappings are described by their effect on an arbitrary vector x, a and b being fixed vectors of the space and α a fixed number.

5.1.9. $f(x) = \alpha$. **5.1.10.** $f(x) = (x, a)$.

5.1.11. $f(x) = \cos(x, a)$. **5.1.12.** $f(x) = (x, x)$.

5.1.13. $f(x) = ([a, x], b)$. **5.1.14.** $f(x) = (x, [a, x])$.

Determine which of the following transformations of the three-dimensional arithmetic space are linear. Each transformation is described by its effect on an arbitrary vector x, while the components of the image vector are given by functions of the components of the vector x.

5.1.15. $Ax = (x_1, x_2, x_3^2)$. **5.1.16.** $Ax = (x_3, x_1, x_2)$.

5.1.17. $Ax = (x_3, x_1, x_2 - 1)$.

5.1.18. $Ax = (x_1 + 2x_2 - 3x_3, 3x_1 - x_2 + 3x_3, 2x_1 + 3x_2 + 2x_3)$.

Find the linear operators on the space M_n of polynomials of degree $\leqslant n$ in the real variable t among the transformations given below. Each transformation is described by its effect on an arbitrary polynomial $f(t)$.

5.1.19. $Af(t) = f(-t)$. **5.1.20.** $Af(t) = f(t+1)$.

5.1.21. $Af(t) = f(at+b)$, where a and b are fixed numbers, while $a \neq 0$.

5.1.22. $Af(t) = f'(t)$. This operator is called the *differential operator*.

5.1.23. $Af(t) = f^{(k)}(t)$. This operator is called the *differential operator of multiplicity k*.

5.1.24. $Af(t) = f(t+1) - f(t)$.

5.1.25. $Af(t) = f(t+1) - g(t)$, where $g(t)$ is a fixed nonzero polynomial.

5.1.26. $Af(t) = tf(t)$. **5.1.27.** $Af(t) = f(t^2)$.

5.1.28. Show that (a) the transformation indicated in Problem 5.1.22 can be treated as a linear operator from M_n into M_{n-1}; (b) the transformation indicated in Problem 5.1.26 is a linear operator from M_n into M_{n+1}; (c) the transformation indicated in Problem 5.1.27 is a linear operator from M_n into M_{2n}.

5.1.29. Given that a linear space X is the direct sum of subspaces L_1 and L_2, show that the operator P that assigns to each vector x from the space X with the decomposition

$$x = x_1 + x_2$$

where $x_1 \in L_1$, $x_2 \in L_2$, the vector x_1 of this decomposition, is linear. The operator P is known as a *projection operator of the space X on L_1 parallel to L_2*.

5.1.30. A linear space X is the direct sum of subspaces L_1 and L_2. Prove that the operator R matching each vector x from the space X with the decomposition

$$x = x_1 + x_2,$$

where $x_1 \in L_1$ and $x_2 \in L_2$, with the vector $y = x_1 - x_2$, is linear. The operator R is called the *reflection of the space X in L_1 parallel to L_2*.

5.1.31. State the geometric sense of the orthogonal reflection of a three-dimensional Euclidean space in a two-dimensional subspace L.

5.1.32. Given that in a linear space X a basis $e_1, \ldots, e_n$ is fixed, prove that the mapping that matches each vector x of the space with its i-th coordinate in this basis, is a linear operator from X into the space of real or complex numbers. The linear operator mapping the space X into the corresponding number field is called a *linear functional on X*.

5.1.33. Prove that each linear operator on a one-dimensional space can be reduced to a multiplication of all the vectors of the space by a number fixed for the given operator.

5.1.34. Describe all linear operators of the space R^+ (see Problem 1.1.6).

5.1.35. Prove that any linear operator transforms a linearly dependent set of vectors into a linearly dependent vector set.

5.1.36. Is the following statement valid: a linearly independent vector set is transformed by any linear operator into a linearly independent set?

5.1.37. Is the statement true: if vector sets $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ are equivalent, then for any linear operator A the vector sets $Ax_1, \ldots, Ax_k$ and $Ay_1, \ldots, Ay_l$ are also equivalent?

5.1.38. Let $A \in \omega_{XY}$ and L be an arbitrary subspace of a space X. The set of vectors Ax, where $x \in L$, is called the *image of the subspace* L and denoted by AL. Prove that AL is a subspace of the space Y.

5.1.39. Prove that the dimension of a subspace AL does not exceed the dimension of the subspace L.

5.1.40. Let L be the sum of subspaces L_1 and L_2, and L_0 their intersection. Is it true that for any linear operator A (a) $AL = AL_1 + AL_2$; (b) $AL_0 = AL_1 \cap AL_2$?

5.1.41. Give an example of a linear operator for which the formula (b) of Problem 5.1.40 does not hold.

5.1.42. Show that a linear operator A has a unique effect on any vector from a space X given that the images $Ae_1, \ldots, Ae_n$ of the vectors $e_1, \ldots, e_n$, which form a basis for the space X, are known.

5.1.43. Let $e_1, \ldots, e_n$ be a basis for a space X, $y_1, \ldots, y_n$ an arbitrary vector set of a space Y. Prove that there exists, and is unique, an operator A from ω_{XY} such that $Ae_i = y_i$, $i = 1, \ldots, n$.

5.1.44. Let $x_1, \ldots, x_k$ be an arbitrary vector set of a space X, $y_1, \ldots, y_k$ an arbitrary vector set of a space Y. Is the following statement true: there exists a linear operator A from ω_{XY} that transforms the vectors x_i into the vectors y_i, $i = 1, \ldots, k$?

5.1.45. In addition to the data of Problem 5.1.44, assume that the vector set $x_1, \ldots, x_k$ is linearly independent. Will the statement of the problem still remain valid?

5.1.46. Given that a basis $e_1, \ldots, e_n$ for a space X is fixed, show that the operation of a linear functional f on an arbitrary vector x can be determined by the formula

$$f(x) = c_1\alpha_1 + \ldots + c_n\alpha_n, \tag{5.1.1}$$

where $\alpha_1, \ldots, \alpha_n$ are the coordinates of the vector x, and $c_1, \ldots, c_n$ are the images of the basis vectors. Conversely, formula (5.1.1) determines a linear functional on X for any numbers $c_1, \ldots, c_n$.

5.1.47. Show that the formula

$$\varphi f(t) = f(a_0)$$

defines a linear functional φ on the space M_n of polynomials of degree $\leqslant n$. Here f is an arbitrary polynomial from M_n and a_0 is a fixed number. Is the converse statement valid: any linear functional φ on M_n can be defined thus, given a convenient choice of the number a_0?

5.1.48. Let L be a subspace of a space X and A an arbitrary operator from ω_{XY}. Show that the effect produced by the operator A on the subspace L can be considered as (a) the operation of a linear operator from L into Y; (b) that of a linear operator from L into AL.

5.1.49. Let L be a subspace of a space X and A a linear operator from L into a certain space Y. Show that there is a linear operator from X into Y whose effect on the subspace L coincides with that of the operator A.

5.1.50. Construct two different linear operators on the space M_n of polynomials of degree $\leqslant n$ that coincide with the differential operator on the subspace M_{n-1}.

5.1.51. Let a space X be the direct sum of subspaces $L_1, \ldots, L_k$. Show that the effect of a linear operator A on any vector of the space is uniquely determined, if the effect of this operator on each of the subspaces $L_1, \ldots, L_k$ is known.

5.1.52. Let A be a linear operator on a real linear space R, and C be a complex space obtained from R by complexification (see Problem 2.5.13). Define an operator $\hat{A}$ on C as follows: for any vector $z = x + iy$ from C where $x, y \in R$, we put

$$\hat{A}z = Ax + iAy.$$

Show that the operator $\hat{A}$ is linear.

Can any linear operator of the space C be obtained in this way?

5.1.53. Can a linear functional on a complex linear space assume only real values?

5.1.54. Show that the kernel N_A of an arbitrary linear operator A from ω_{XY} is a linear subspace of the space X.

5.1.55. Is it true that any subspace of a space X is the kernel of a certain linear operator from X to Y?

5.1.56. According to Problem 5.1.38, the image T_A of an arbitrary linear operator A from ω_{XY} is a subspace of the space Y. Is it true that any subspace of a space Y is the image of a certain linear operator from X to Y?

5.1.57. Prove that the set of all preimages of a vector y from T_A is a plane from the space X with the directional subspace N_A.

5.1.58*. Construct, for an operator A from ω_{XY}, a one-to-one correspondence between T_A and the planes of the space X of the form $P = x_0 + N_A$.

5.1.59. The set M of all planes of the space X of the form $P = x_0 + N_A$ is, according to Problem 4.2.18, a linear space.

Prove that the correspondence between planes from M and vectors from T_A constructed in Problem 5.1.58 is a linear operator (from M to T_A). Find the kernel and defect of this operator.

5.1.60*. Prove that for any operator A from ω_{XY} the sum of the rank and defect equals the dimension of the space X.

5.1.61. Give an example of a linear operator from ω_{XX} such that the space X is not the direct sum of the image and kernel of this operator.

5.1.62. Let M be any subspace complementary to the kernel N_A of an operator A. Prove that (a) any linearly independent vector set from M is transformed by the operator A into a linearly independent set (cf. Problem 5.1.36); (b) the subspace M is mapped by the operator A onto its image T_A by a one-to-one mapping.

5.1.63. Prove that for any two subspaces, e.g. N of an n-dimensional space X, and T of a space Y, such that $\dim N + \dim T = n$, there is a linear operator A from ω_{XY} whose kernel coincides with N and whose image coincides with T.

5.1.64. Construct two different linear operators on M_n having the same image and kernel.

5.1.65. Let A be an operator from X to Y, and L be a subspace satisfying the inclusion $L \subset T_A$. Prove that the set of vectors x of the space X whose images belong to L (called the *complete preimage of the subspace* L) is also a subspace, and its dimension equals $\dim L + n_A$.

5.1.66. Find the defect of a linear functional f on an n-dimensional space X.

5.1.67. Find the kernel of each of the linear functionals on a three-dimensional Euclidean space $f_1(x) = (x, a)$ and $f_2(x) = ([a, x], b)$.

5.1.68. Find the image and kernel of the linear operator on a three-dimensional Euclidean space defined by the formula $Ax = [x, a]$.

5.1.69*. Do the above for the operator $Ax = [a, [x, b]]$.

Determine the defect and rank of the following transformations of a three-dimensional arithmetic space and construct the bases for their kernels and images. Each transformation is described by its effect on an arbitrary vector x, while the components of the vector Ax are given as functions of components of the vector x.

5.1.70. $Ax = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$.

5.1.71. $Ax = (2x_1 - x_2 - x_3, x_1 - 2x_2 + x_3, x_1 + x_2 - 2x_3)$.

5.1.72. $Ax = (-x_1 + x_2 + x_3, x_1 - x_2 + x_3, x_1 + x_2 - x_3)$.

5.1.73. Describe the image and kernel of the differential operator on the space M_n.

5.1.74. Consider the *difference operator* A_h on the same space M_n

$$A_h f(t) = \frac{f(t+h) - f(t)}{h},$$

where h is a fixed nonzero number. Find its image and kernel.

5.1.75. Consider the following mapping of the space M_n into an arithmetic space:

$$f(t) \to (f(a_1), \ldots, f(a_k)),$$

where $a_1, \ldots, a_k$ are different numbers. Find the defect of this operator.

5.1.76. Find the image and kernel of the projection operator (see Problem 5.1.29).

5.1.77. Prove that in complexifying a real space R, the rank and defect of an operator A from ω_{XY} are preserved during the transfer to the operator $\hat{A}$ (see Problem 5.1.52).

5.2. Linear Operations over Operators

The set ω_{XY} of all linear operators from X to Y is considered, in the present section, as a linear space. Particular attention is drawn to the following topics:

(i) The dimension of the space ω_{XY}.

(ii) Some of the classes of subspaces of ω_{XY}. Here we examine the details of the relation between the properties of linear dependence of operators from ω_{XY} and the mutual disposition of the images of these operators.

(iii) The rank of the sum of operators, and the conditions stipulating its equality to the sum of the ranks of the addends.

5.2.1. Prove that the set ω_{XY} of all linear operators from a space X into a space Y is a linear space under the operations of addition of operators and multiplication of an operator by a number.

5.2.2. Prove that the space of all linear operators on a one-dimensional linear space is also one-dimensional.

5.2.3. The linear space X^* of all functionals on a space X is said to be *conjugate to the space* X. Prove that the conjugate linear space X^* is isomorphic to the space X.

5.2.4. Show that for any subspace L of a space X, the following relations hold: (a) $(\lambda A) L = AL$ if $\lambda \neq 0$; (b) $(A + B) L \subset AL + BL$, where A and B are operators from ω_{XY}. Show that, generally speaking, the equality sign does not hold in the relation (b).

5.2.5. Prove that nonzero operators A and B from ω_{XY}, whose images are different, are linearly independent.

5.2.6. Let $q_1, \ldots, q_m$ be a basis for a space Y, and x a nonzero vector of a space X. Prove that operators $B_1, \ldots, B_m$ such that

$$B_j x = q_j, \quad j = 1, \ldots, m \qquad (5.2.1)$$

are linearly independent.

5.2.7*. Prove that for any operator A from ω_{XY} there are operators $B_1, \ldots, B_m$ such that $A = B_1 + \ldots + B_m$ whereas (a) the rank of each of the operators B_i does not exceed unity; (b) the image of a nonzero operator B_i is the vector q_i, where $q_1, \ldots, q_m$ is a fixed basis for the space Y.

5.2.8. Let $e_1, \ldots, e_n$ be a basis for a space X, and y be a nonzero vector of a space Y. Prove that the operators $A_1, \ldots, A_n$ such that

$$A_je_k = \begin{cases} y, & k=j \\ 0, & k \neq j \end{cases}$$

are linearly independent.

5.2.9. Prove that any operator of rank 1 whose image contains a vector y is a linear combination of the operators $A_1, \ldots, A_n$ (see the previous problem).

5.2.10*. Let bases $e_1, \ldots, e_n$ and $q_1, \ldots, q_m$ for spaces X and Y, respectively, be fixed. Using the results of Problems 5.2.7 and 5.2.9, show that each operator from ω_{XY} is a linear combination of the operators $A_{11}, \ldots, A_{mn}$ satisfying the conditions

$$A_{ij}e_k = \begin{cases} q_i, & k=j, \\ 0, & k \neq j \end{cases}, \quad i=1, \ldots, m, \ j=1, \ldots, n. \quad (5.2.2)$$

5.2.11. By means of the results of Problems 5.2.6 and 5.2.8 show that a set of operators defined by relations (5.2.2) is linearly independent. Hence deduce the dimension of the space ω_{XY} (use also the results of Problem 5.2.10).

5.2.12. Is the set of linear operators having (a) the same image T; (b) the same kernel N, a linear subspace of the space ω_{XY}?

5.2.13. Show that if T is a subspace of a space Y, then the set ω_{XT} of all linear operators mapping the space X into T is a subspace of the space ω_{XY}. Find the dimension of this subspace if $\dim X = n$, $\dim T = k$.

5.2.14. Show that if N is a subspace of a space X, then the set K_N of all linear operators from ω_{XY} whose kernel contains the subspace N is a subspace of the space ω_{XY}. Find the dimension of this subspace if $\dim X = n$, $\dim N = l$, $\dim Y = m$.

5.2.15*. Let L_1 and L_2 be arbitrary subspaces of a space Y, $L = L_1 + L_2$, $L_0 = L_1 \cap L_2$. Prove the following relations:

(a) $\omega_{XL} = \omega_{XL_1} + \omega_{XL_2}$;

(b) $\omega_{XL_0} = \omega_{XL_1} \cap \omega_{XL_2}$.

5.2.16. Let a space Y be decomposed into the direct sum of subspaces $L_1, \ldots, L_k$. Prove that

$$\omega_{XY} = \omega_{XL_1} \dot{+} \omega_{XL_2} \dot{+} \ldots \dot{+} \omega_{XL_k}.$$

5.2.17. Prove that the rank of the sum of operators A and B from ω_{XY} does not exceed the sum of the ranks of these operators.

5.2.18. Let operators A and B from ω_{XX} satisfy the equality

$$X = T_A \dot{+} T_B = N_A \dot{+} N_B.$$

Prove that the rank of the operator $A + B$ equals the sum of the ranks of the operators A and B.

5.2.19. Deduce the following inequality from Problem 5.2.17

$$r_{A+B} \geqslant | r_A - r_B |.$$

5.2.20*. Prove that any operator A from ω_{XY} with rank r can be represented as the sum of r operators of rank 1 but cannot be represented as the sum of less than r such operators.

5.2.21*. Find the necessary and sufficient condition for the sum of two operators of rank 1 to be of rank $\leqslant 1$.

5.2.22*. Given that a space X has the dimension n ($n > 1$), prove that in ω_{XX} any subspace L of dimension $n + 1$ contains at least one operator > 1.

5.2.23. Let operators A and B from ω_{XY} be such that for any vector x from X, vectors Ax and Bx are collinear. Does this imply that the operators A and B are themselves collinear?

5.2.24*. The condition that n ($n = \dim X$) must equal the rank of the operator B is added to the data in Problem 5.2.23. Are the operators A and B collinear in this case?

5.2.25. Prove that operators A and B of rank 1, having the same image T and kernel N, are collinear.

5.2.26. Prove that for any projection operator P, the operator $E - P$ is also a projection operator. Find the relation between the kernel and image of the operator $E - P$ and the kernel and image of P.

5.2.27. Prove that for operators P and R carrying out the projection and reflection of a space X into L_1 parallel to L_2, respectively, the following relation is valid: $E + R = 2P$.

5.2.28. Show that when a real space R is transformed into a complex one: (a) an operator $A + B$ corresponds to the operator $\hat{A} + \hat{B}$ (see 5.1.52); (b) an operator αA corresponds to the operator $\alpha \hat{A}$, α being a real number.

5.3. Multiplication of Operators

In the present section the following topics related to the multiplication of operators are scrutinized:

(i) The image and kernel of the product of operators.
(ii) Polynomials in operators.
(iii) Commutativity of operators.
(iv) Nondegenerate operators.

We assume in the following that the products of operators which may be on different spaces make sense.

5.3.1. Prove that the product BA of operators A and B satisfies the inequalities:

(a) $r_{BA} \leqslant \min(r_A, r_B)$;

(b) $n_{BA} \geqslant n_A$.

If the operators A and B are defined on the same space, then

(c) $n_{BA} \geqslant n_B$.

5.3.2. Prove that the product BA of operators A and B satisfies the relations: (a) $r_{BA} = r_A - \dim(T_A \cap N_B)$; (b) $n_{BA} = n_A + \dim(T_A \cap N_B)$.

Note that from (b) an inequality follows:

$$n_{BA} \leqslant n_A + n_B.$$

5.3.3*. Prove the *Frobenius inequality*:

$$r_{BA} + r_{AC} \leqslant r_A + r_{BAC}.$$

5.3.4. Let A and B be operators from ω_{XX} whereas $BA = 0$. Does it follow from here that $AB = 0$?

5.3.5. Give an example of two operators A and B such that $AB = BA = 0$.

5.3.6. Prove that the set of all linear operators B from ω_{XX} satisfying, for a fixed operator A, the condition $AB = 0$, is a subspace of the space ω_{XX}. Find the dimension of this subspace if $\dim X = n$ and the rank of the operator A equals r.

5.3.7. The same question for the set of operators C from ω_{XX} satisfying the condition $CA = 0$ for a fixed operator A of rank r.

5.3.8. Let X be an n-dimensional space and A an operator of rank r from ω_{XX}. Using the operator A, construct a transformation of the space ω_{XX} that matches any operator B with the operator AB. Prove that this transformation is linear. Find its rank and defect.

5.3.9. Let A be an arbitrary operator from ω_{XX}, and let N_i and T_i be the kernel and image of the operator A^i, respectively. Prove that

(a) $N_1 \subset N_2 \subset N_3 \subset \ldots$;

(b) $T_1 \supset T_2 \supset T_3 \supset \ldots$.

5.3.10*. Prove that if in the sequence of subspaces $N_1, N_2, N_3, \ldots$ (see Problem 5.3.9) for some q for the first time $N_q = N_{q+1}$, then $N_q = N_{q+k}$ for any $k \geqslant 1$.

5.3.11. An operator A from ω_{XX} is said to be *nilpotent* if there exists a natural number q such that $A^q = 0$. The least such number q is called the *nilpotence index* of the operator A. Prove that the index of any nilpotent operator on an n-dimensional space does not exceed n.

5.3.12. Show that the differential operator on polynomials of the space M_n is nilpotent. Find its index of nilpotence.

5.3.13. Let A be a nilpotent operator of index q, and a vector x satisfy the inequality $A^{q-1}x \neq 0$. Prove that the vector set $x, Ax, A^2x, \ldots, A^{q-1}x$ is linearly independent.

5.3.14*. Prove that for any operator A from ω_{XX} and with rank 1, there is a number α such that $A^2 = \alpha A$.

5.3.15. Show that any operator of reflection R satisfies the relation $R^2 = E$.

5.3.16. Show that any projection operator P satisfies the equality $P^2 = P$.

5.3.17*. Conversely, prove that any operator P satisfying the condition $P^2 = P$ is a projection operator.

5.3.18. Show that it follows from the conditions $P_1 + P_2 = E$, $P_1P_2 = 0$ that

(a) P_1, P_2 are projection operators;

(b) $P_2P_1 = 0$.

5.3.19. Prove that an operator A on the space M_n which assigns the polynomial $g(t) = f(t + 1)$ to any polynomial $f(t)$ is a polynomial in the differential operator.

5.3.20. Given an operator A, a polynomial $f(t)$ ($f(t) \neq 0$) is called an *A-annihilator* if $f(A) = 0$. Prove that for any linear operator A on an n-dimensional space, there exists an A-annihilator of degree $\leqslant n^2$.

5.3.21. Let $m(t)$ be the polynomial of the least degree out of all the A-annihilators. Prove that $m(t)$ is a divisor of all the other A-annihilators.

5.3.22. Prove that the polynomial $m(t)$ of Problem 5.3.21 is uniquely determined by the operator A depending only on a nonzero multiplier. Normalized so that the higher-order coefficient equals unity, the polynomial $m(t)$ is called the *minimal polynomial* of the operator A.

5.3.23*. Find the minimal polynomial (a) for a projection operator; (b) for a reflection operator; (c) for a nilpotent operator of index q.

5.3.24. Show that for an operator of rank 1, the minimal polynomial is of the second degree.

5.3.25. Operators A and B from ω_{XX} are said to be *commuting* if $AB = BA$. Let A commute with B, and B commute with C. Does it follow that A commutes with C?

5.3.26. Show that any two polynomials in the same operator A are commuting.

5.3.27. Show that if operators A and B are commuting then any polynomials $f(A)$ and $f(B)$ in these operators are also commuting.

5.3.28. Prove that for commuting operators A and B

$$(A+B)^n = A^n + nA^{n-1}B + \frac{n(n-1)}{2}A^{n-2}B^2 + \ldots + B^n.$$

5.3.29. Prove that operators of rank 1 having the same kernel and the same image are commuting.

5.3.30. Given two commuting operators, prove that $BN_A \subset N_A$.

5.3.31*. Prove that if the projection operators P_1 and P_2 are commuting, then their product is also a projection operator. Moreover

(a) $T_{P_1P_2} = T_{P_1} \cap T_{P_2}$;

(b) $N_{P_1P_2} = N_{P_1} + N_{P_2}$.

5.3.32*. Prove that the sum of the projection operators P_1 and P_2 is a projection operator if and only if $P_1P_2 = P_2P_1 = 0$. In addition,

(a) $T_{P_1+P_2} = T_{P_1} \dot{+} T_{P_2}$;

(b) $N_{P_1+P_2} = N_{P_1} \cap N_{P_2}$.

5.3.33*. Prove that if an operator A commutes with each operator from ω_{XX}, then for any subspace L from X, $AL \subset L$. In particular, for any vector x from X, the vectors x and Ax are collinear.

5.3.34. Using the result of Problem 5.3.33, prove the *Schur lemma*: if an operator A commutes with each operator from ω_{XX} then it is *scalar*, i.e. $A = \alpha E$ for a certain number α.

5.3.35. Show that if A is a nondegenerate operator, then for any subspace L, the equality $\dim L = \dim AL$ holds.

5.3.36. Given that a space X is the direct sum of subspaces $L_1, \ldots, L_k$, and A_i is a nondegenerate operator defined on the subspace L_i, $i = 1, \ldots, k$, show that an operator A from ω_{XX} coinciding, on each of the subspaces L_i, with the corresponding operator A_i is nondegenerate.

5.3.37. Verify that the differential operator (a) is degenerate on the space M_n of polynomials of degree $\leqslant n$; (b) is nondegenerate on the two-dimensional linear space generated by the functions $f_1 = \cos t$ and $f_2 = \sin t$ (under the operations of function addition and multiplication of a function by a number, both defined in the usual way).

5.3.38. Find the inverse operator of the differential operator defined in Problem 5.3.37(b).

5.3.39. Find the inverse operator of a reflection operator R.

5.3.40. Show that for a nondegenerate operator A and any nonzero number α,

$$(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}.$$

5.3.41*. Prove that if an operator A is of rank 1, then at least one of the operators $E + A$ and $E - A$ is nondegenerate.

5.3.42. Prove that if an operator A is nondegenerate, then for any operator B,

$$r_{AB} = r_{BA} = r_B.$$

5.3.43. Prove that the product of operators A and B is a nondegenerate operator if and only if each of the operators A and B is nondegenerate. In this case:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

5.3.44. Prove that for a nondegenerate operator A and an arbitrary operator B, the identity is valid

$$(A + B) A^{-1} (A - B) = (A - B) A^{-1} (A + B).$$

5.3.45. Let A be a nilpotent operator of index q. Prove that the operator $E - A$ is nondegenerate and that

$$(E - A)^{-1} = E + A + A^2 + \dots + A^{q-1}.$$

5.3.46. Given that operators A and B are connected by a relation $AB + A + E = 0$, prove that A is a nondegenerate operator while $A^{-1} = -E - B$.

5.3.47. Prove that if an A-annihilator has a nonzero free term, then the operator A is nondegenerate.

5.3.48. Prove that the absolute term of the minimal polynomial $m(t)$ annihilating a nondegenerate operator is nonzero.

5.3.49. Prove that for a nondegenerate operator A on an n-dimensional space, the inverse operator A^{-1} is represented as a polynomial in A of a degree not greater than $n^2 - 1$.

5.3.50. Show that any two polynomials $f(A)$ and $g(A^{-1})$, where A is a nondegenerate operator, commute.

5.3.51. Let A be an operator from ω_{XY} and let there exists an operator B from ω_{YX} such that $BA = E_X$ (the identity operator of the space X). Does it follow from this that $AB = E_Y$?

5.3.52. Let X be the span of polynomials $t, t^2, \dots, t^n$; and let Y be the space of polynomials of degree $\leqslant n - 1$. Consider the differentiation of polynomials as an operator A from X into Y and integration (i.e. the transformation matching each polynomial with its antiderivative) as an operator B from Y into X. Show that

$$BA = E_X, \quad AB = E_Y.$$

5.3.53. Let, in addition to the data of Problem 5.3.51, $\dim Y > > \dim X$. Prove that the operator AB is a projection operator on Y.

5.3.54. Show that, when complexifying the real space R: (a) to operator AB there corresponds the operator $\hat{A}\hat{B}$; (b) to a nondegenerate operator A there corresponds the nondegenerate operator $\hat{A}$; (c) if A is nondegenerate then to the inverse operator A^{-1} there corresponds the inverse operator $\hat{A}^{-1}$.

5.4. Operations over Matrices

We consider here various properties of the operations defined on matrices, and first of all, the operation of multiplication. Amongst the topics treated, the greatest weight is given to the following:

(i) The formal properties of the operation of multiplication, viz. the dimensions of factors and the product; the number of fundamental arithmetic operations; etc.

(ii) Matrices of elementary transformations (or elementary matrices).

(iii) Commuting matrices.

(iv) Classes of matrices, closed under multiplication.

(v) The rank of the product of matrices.

(vi) Operations with matrices partitioned into blocks, i.e. partitioned matrices.

(vii) The Kronecker product of matrices.

Find the products AB and BA where

5.4.1. $A=(2\ \ -3\ \ 0),\quad B=\begin{Vmatrix}4\\3\\1\end{Vmatrix}.$

5.4.2. $A=\begin{Vmatrix}-2&3&0&1\\1&1&2&-1\end{Vmatrix},\quad B=\begin{Vmatrix}2&0\\1&-1\\-1&2\\1&3\end{Vmatrix}.$

Find the product AB where

5.4.3. $A=\begin{Vmatrix}83&-29&-52&46\\-15&97&78&-112\\38&-4&69&85\end{Vmatrix},\quad B=\begin{Vmatrix}0\\0\\1\\0\end{Vmatrix}.$

5.4.4. $A=\|0\ 1\ 0\|,\quad B=\begin{Vmatrix}83&-29&-52&46\\-15&97&78&-112\\38&-4&69&85\end{Vmatrix}.$

5.4.5. $A=\begin{Vmatrix}5&2&-3&-3\\-7&-2&4&2\\-1&2&1&1\\2&-2&-3&4\end{Vmatrix},\quad B=\begin{Vmatrix}1\\1\\1\\1\end{Vmatrix}.$

5.4.6. $A=\|1\ 1\ 1\ 1\|,\quad B=\begin{Vmatrix}5&2&-3&-3\\-7&-2&4&2\\-1&2&1&1\\2&-2&-3&4\end{Vmatrix}.$

5.4.7. $A=\begin{Vmatrix} 923 & 2115 & 0 & 0 \\ 1097 & 518 & 0 & 0 \\ 652 & 769 & 0 & 0 \\ 841 & 134 & 0 & 0 \end{Vmatrix}, \quad B=\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 476 & 372 & 1505 & 882 \\ 549 & 795 & 999 & 400 \end{Vmatrix}.$

5.4.8*. Evaluate the product ABC of matrices

$$A=\begin{Vmatrix} 991 & 992 & 993 \\ 994 & 995 & 996 \\ 997 & 998 & 999 \\ 1000 & 1001 & 1002 \end{Vmatrix}, \quad B=\begin{Vmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 24 & -12 & -4 \end{Vmatrix}, \quad C=\begin{Vmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 0 \end{Vmatrix}.$$

5.4.9*. Evaluate the product $ABCD$ of matrices

$$A=\begin{Vmatrix} 3 \\ 5 \\ 7 \end{Vmatrix}, \quad B=\| 213 \;\; 510 \;\; 128 \|. \quad C=\begin{Vmatrix} 3 \\ -1 \\ -1 \end{Vmatrix}, \quad D=\| 1 \;\; -2 \;\; 1 \|.$$

5.4.10. Show that a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be rewritten in the form of a matrix equation $Ax = b$ by introducing matrices

$$A=\begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{Vmatrix}, \quad b=\begin{Vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{Vmatrix}, \quad x=\begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix}.$$

5.4.11. Conversely, show that the solution of a matrix equation $AX = B$, where A and B are $m \times n$ and $m \times p$ matrices, respectively, is reduced to the solution of p systems of linear equations with the same coefficient matrix A but different right-hand sides.

Solve the following matrix equations:

5.4.12. $\begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix} X = \begin{Vmatrix} 3 & 0 \\ 7 & 2 \end{Vmatrix}.$

5.4.13. $X\begin{Vmatrix} 3 & 2 \\ -2 & -1 \end{Vmatrix} = \begin{Vmatrix} -1 & 2 \\ -1 & 1 \end{Vmatrix}$

5.4.14. $\begin{Vmatrix} 2 & 1 \\ 1 & 2 \end{Vmatrix} X - X\begin{Vmatrix} 1 & -1 \\ 1 & 1 \end{Vmatrix} = \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix}.$

5.4.15. $X\begin{Vmatrix} 1 & 2 \\ 2 & 3 \end{Vmatrix} X \begin{Vmatrix} 2 & -1 \\ -1 & 1 \end{Vmatrix} = \begin{Vmatrix} 1 & -1 \\ -1 & 0 \end{Vmatrix}.$

5.4.16. $X\begin{Vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{Vmatrix} = \begin{Vmatrix} 6 & 9 & 8 \\ 0 & 1 & 6 \end{Vmatrix}$.

5.4.17. $\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{Vmatrix} X = \begin{Vmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{Vmatrix}$.

5.4.18. Show that if both the products AB and BA have sense and A is an $m \times n$ matrix, then B is an $n \times m$ matrix.

5.4.19. Evaluate the number of multiplication and addition operations in multiplying an $m \times n$ matrix A and an $n \times p$ matrix B together.

5.4.20. Let A, B and C be matrices of orders $m \times n$, $n \times p$, $p \times q$, respectively. Evaluate the number of multiplications required to compute the product ABC. Note that this number of operations depends on the place of brackets in the product ABC.

5.4.21. Verify that for square matrices A and B of order 2, the procedure for computing the matrix $C = AB$ indicated below requires 7 multiplication operations whereas employing the usual algorithm to construct AB requires 8 multiplications:

$$\begin{aligned}
\alpha_1 &= (a_{11} + a_{22})(b_{11} + b_{22}), \\
\alpha_2 &= (a_{21} + a_{22})\, b_{11}, \\
\alpha_3 &= a_{11}(b_{12} - b_{22}), \\
\alpha_4 &= a_{22}(b_{21} - b_{11}), \\
\alpha_5 &= (a_{11} + a_{12})\, b_{22}, \\
\alpha_6 &= (a_{21} - a_{11})(b_{11} + b_{12}), \\
\alpha_7 &= (a_{12} - a_{22})(b_{21} + b_{22}), \\
c_{11} &= \alpha_1 + \alpha_4 - \alpha_5 + \alpha_7, \\
c_{12} &= \alpha_3 + \alpha_5, \\
c_{21} &= \alpha_2 + \alpha_4, \\
c_{22} &= \alpha_1 + \alpha_3 - \alpha_2 + \alpha_6.
\end{aligned}$$

This algorithm was suggested by Strassen.

5.4.22. The sum of the elements on the principal diagonal of a square matrix is called its *trace*. The trace of a matrix A is denoted by $\operatorname{tr} A$.

Prove that the following properties are fulfilled

(a) $\operatorname{tr}(A + B) = \operatorname{tr} A + \operatorname{tr} B$;

(b) $\operatorname{tr}(\alpha A) = \alpha \operatorname{tr} A$;

(c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

The last equality is also valid for rectangular matrix A and B if both the products AB and BA are defined.

5.4.23. Matrices A and B of orders $m \times n$ and $n \times p$, respectively, possess the property of having the same sums of the elements in any row, equal to r in the matrix A, and to s in the matrix B. Prove that the same property is possessed by the product AB, with the corresponding sums equal to rs. Enunciate and prove a similar statement for the columns.

5.4.24. Show that the elementary row transformations of a matrix A (see Problem 4.1.26) are equivalent to a premultiplication of this matrix by special matrices, called the *matrices of elementary transformations*, that is (a) interchanging the i-th and j-th rows corresponds to a multiplication by a matrix P_{ij}

$$P_{ij}=\begin{Vmatrix} 1 & & & & \\ & \ddots & & & \\ & & \begin{matrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{matrix} & & \\ & & & \ddots & \\ & & & & 1 \end{Vmatrix}$$

(the elements of the principal diagonal that are not indicated are equal to unity; all the other elements, except elements (i, j) and (j, i), are equal to zero); (b) multiplication of the i-th row by a number α corresponds to a multiplication by a diagonal matrix D_i

$$D_i=\begin{Vmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{Vmatrix};$$

(c) addition of the j-th row multiplied by a number α to the i-th row is equivalent to a multiplication by a matrix L_{ij}:

$$L_{ij}=\begin{Vmatrix} 1 & & & & \\ & \ddots & & & \\ & & \begin{matrix} 1 & \dots & \alpha \\ & \ddots & \vdots \\ & & 1 \end{matrix} & & \\ & & & \ddots & \\ & & & & 1 \end{Vmatrix}$$

(all off-diagonal elements of this matrix, except the element (i, j), are equal to zero).

Formulate and prove similar statements for the elementary transformations of the columns of the matrix A.

5.4.25. Determine how a matrix A is altered by post-multiplication by (a) the matrix N_i

$$N_i = \begin{Vmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \alpha_{i+1,\,i} & \ddots & \\ & & \vdots & & \\ & & \alpha_{ni} & & 1 \end{Vmatrix};$$

(b) the matrix S_i

$$S_i = \begin{Vmatrix} 1 & & \alpha_{1i} & & \\ & \ddots & \vdots & & \\ & & 1 \;\; \alpha_{i-1,\,i} & & \\ & & 1 & & \\ & & \alpha_{i+1,\,i} & & \\ & & \vdots & \ddots & \\ & & \alpha_{ni} & & 1 \end{Vmatrix}.$$

The off-diagonal elements of both matrices that are not indicated are equal to zero.

A similar problem for premultiplying the matrix A by the matrices S_i and N_i.

5.4.26. Prove that (a) the matrix N_i (see the previous problem) is the product of matrices L_{ki}, $k = i + 1, \ldots, n$ (see Problem 5.4.24 (c)); (b) the matrix S_i is the product of matrices L_{ki}, $k = 1, \ldots, i - 1, i + 1, \ldots, n$; (c) the nontrivial elements of the factors L_{ki} coincide with the corresponding nontrivial elements of the matrix N_i (S_i); (d) the order of the factors in the product in both cases is arbitrary.

5.4.27. Prove that the product N_iN_j of the matrices N_i and N_j is (when $i < j$) of the following form:

$$N_iN_j = \begin{Vmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \alpha_{i+1,\,i} & \ddots & & \\ & & \vdots & & & \\ & & \alpha_{j,\,i} & 1 & & \\ & & \alpha_{j+1,\,i} & \alpha_{j+1,\,j} & & \\ & & & & 1 & \\ & & \vdots & \vdots & & \ddots \\ & & \alpha_{ni} & \alpha_{nj} & & 1 \end{Vmatrix}$$

(the off-diagonal elements which are not indicated are equal to zero).

5.4.28*. A square matrix P in which each row and each column have only one nonzero element equal to unity, is called a *permutation matrix*. Prove that any permutation matrix is the product of the matrices P_{ij} (see Problem 5.4.24 (a)).

Evaluate the following expressions (if the order of a matrix is not explicit, it is equal to n):

5.4.29. $\begin{Vmatrix} 1 & 1 \\ 0 & 1 \end{Vmatrix}^k$. 5.4.30. $\begin{Vmatrix} 1 & 1 \\ 1 & 0 \end{Vmatrix}^k$.

5.4.31. $\begin{Vmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{Vmatrix}^k$ (all the off-diagonal elements are zero).

5.4.32. $\begin{Vmatrix} 0 & & & \lambda_1 \\ & & \lambda_2 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \lambda_n & & & 0 \end{Vmatrix}^k$. 5.4.33. $\begin{Vmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{Vmatrix}^k$

(all the elements, except the elements positioned at $(i, i+1)$, $i = 1, \ldots, n-1$, are zero).

5.4.34. $\begin{Vmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & \ldots & & 0 \end{Vmatrix}^k$

(all the elements, except the elements positioned at (1, 2), (2, 3), (3, 4), . . ., $(n-1, n)$, $(n, 1)$, are zero).

5.4.35*. Prove that for an $n \times n$ matrix

$$J_\lambda = \begin{Vmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{Vmatrix}$$

the matrix J_λ^k is of the form ($k \geqslant n$):

$$J_\lambda^k = \left\| \begin{matrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} & \dots & {}^{n-1}C_k\lambda^{k-n+1} \\ & \lambda^k & k\lambda^{k-1} & \dots & {}^{n-2}C_k\lambda^{k-n+2} \\ & & \lambda^k & \dots & {}^{n-3}C_k\lambda^{k-n+3} \\ & 0 & & \ddots & \vdots \\ & & & & \lambda^k \end{matrix} \right\| .$$

The matrix J_λ is called a *Jordan block corresponding to the number* λ.

5.4.36. Let D be a diagonal matrix of order n with all the diagonal elements different. Prove that (a) any polynomial in the matrix D will be diagonal matrix; (b) any diagonal matrix can be represented as a polynomial $f(D)$ in the matrix D; (c) $f(t)$ can be chosen so that its degree does not exceed $n-1$.

5.4.37. Prove that for any diagonal matrix of order n, the minimal polynomial has a degree not exceeding n. The definition of the minimal polynomial of a matrix is similar to the definition of the minimal polynomial of an operator. The latter is given in Problem 5.3.20.

5.4.38. Show that the minimal polynomial of a diagonal matrix of order n with all its diagonal elements different is of degree n.

5.4.39. Prove that a matrix, commuting with a diagonal matrix which has all its diagonal elements different, is also diagonal.

5.4.40*. A square matrix A is called *scalar* if it is diagonal and all its diagonal elements are equal. Using the result of Problem 5.4.39 prove the Schur lemma: if a square matrix A commutes with all square matrices of the same order, then it is scalar (cf. Problem 5.3.34).

5.4.41. Show that for any matrix A, the set of matrices that commute with A is (a) a subspace; (b) a ring.

Find the general form of matrices that commute with the following matrix:

5.4.42. $\left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{matrix} \right\|$.

5.4.43*. $\left\| \begin{matrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & \ddots & 1 \\ & & & & & 0 \end{matrix} \right\|$

(the matrix is of order n).

5.4.44. Prove that any matrix, that commutes with a matrix A, will also commute with the matrix $A - \lambda E$ for any number λ.

Hence deduce that the set of matrices, symmetric to a Jordan block J_λ, is the same for all λ and therefore coincides with the set obtained in Problem 5.4.43. According to Problem 5.4.41, this set is a subspace, determine its dimension.

5.4.45. A square matrix A is called *upper* (or *right-hand*) *triangular* if $a_{ij} = 0$ for $i > j$. Similarly, a square matrix A in which $a_{ij} = 0$ for $i < j$, is called *lower* (or *left-hand*) *triangular*. Prove that the product of upper (lower) triangular matrices of the same order is an upper (lower) triangular matrix.

5.4.46. Find the number of multiplications necessary for the evaluation of the product of two triangular n-order matrices of the same form (i.e. both the matrices are either upper triangular or lower triangular).

5.4.47. A square matrix A is called *strictly upper* (*lower*) *triangular* if $a_{ij} = 0$ for $i \geqslant j$ $(i \leqslant j)$. Prove that for the product B of two strictly triangular matrices A_1 and A_2 of the same form, $b_{ij} = 0$ when $i \geqslant j - 1$ $(i \leqslant j + 1)$.

5.4.48. Prove that for a strictly triangular n-order matrix A the power with index n is equal to the zero matrix.

5.4.49. A square matrix A of order $n + 1$ is called a *greenhouse matrix* if it has the following structure

$$A = \begin{Vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_{-1} & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{-2} & a_{-1} & a_0 & \dots & a_{n-3} & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \dots & a_0 & a_1 \\ a_{-n} & a_{-n+1} & a_{-n+2} & \dots & a_{-1} & a_0 \end{Vmatrix}.$$

Such a matrix is fully determined, therefore, by $2n + 1$ numbers.

Prove that an upper triangular matrix A is a greenhouse matrix if and only if it is a polynomial in the Jordan block J_0.

5.4.50. Deduce from the result of Problem 5.4.49 that (a) the product of upper triangular greenhouse matrices is also a matrix of the same form; (b) any two matrices of this class commute.

5.4.51. Prove that the product of two permutation matrices is also a permutation matrix.

5.4.52. A square matrix A of order $n + 1$ is called a *circulant* if it has the following structure

$$\begin{Vmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_0 & \dots & a_{n-3} & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_0 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_n & a_0 \end{Vmatrix}.$$

Thus, a matrix of this class is fully determined by $n + 1$ numbers.

Prove that a matrix C is a circulant if and only if it is a polynomial in the permutation matrix P of Problem 5.4.34.

5.4.53. Deduce from the result of Problem 5.4.52 that (a) the product of circulants is also a circulant; (b) any two circulants commute.

5.4.54. How many multiplication operations are sufficient to evaluate the product of two circulants of order n?

5.4.55. An n-order square matrix A is called a *band matrix* if for a certain number m ($< n$), all its elements a_{ij}, such that $|i - j| > m$, equal zero. The number $2m + 1$ is called the *bandwidth*.

Prove that the product of strip matrices is also a strip matrix. Determine the minimum strip width of the product if the width of the factors equals $2m_1 + 1$ and $2m_2 + 1$, respectively.

5.4.56. A square product A with nonnegative elements is said to be *stochastic* if the sum of the elements in each row of this matrix equals 1. Moreover, if the sum of the elements in each column equals unity, then the matrix is said to be *doubly stochastic*. Prove that (a) the product of stochastic matrices is a stochastic matrix; (b) the product of doubly stochastic matrices is a doubly stochastic matrix.

5.4.57. Using the matrix multiplication rule prove that the rank of the product AB does not exceed the rank of each of the factors A and B.

5.4.58. Given that an $n \times n$ matrix C is the product of two rectangular matrices A and B of orders $n \times m$ and $m \times n$, respectively, $m < n$, prove that the determinant of the matrix C equals zero.

5.4.59. Prove that an $m \times n$ matrix A with rank 1 can be represented as the product $A = xy$ where x is an $m \times 1$ matrix and y is a $1 \times n$ matrix. Is such a representation unique?

5.4.60. Let $A = xy$ be an $n \times n$ matrix of rank 1. Prove that there is a number α such that $A^2 = \alpha A$. Find an expression of this number in terms of the elements of the matrices x and y.

5.4.61. Given the representations $A = xy$ and $B = uv$ of two matrices with rank 1, find the number of multiplications necessary to evaluate their product.

5.4.62*. Prove that an $m \times n$ matrix A with rank r can be represented as the product $A = BC$ where B and C are $m \times r$ and $r \times n$ matrices, respectively. Is such a representation unique?

The representation of a matrix A derived in Problem 5.4.62 is called the *skeletal decomposition* of this matrix. Find the skeletal decomposition of the following matrices:

5.4.63. $\left\|\begin{matrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{matrix}\right\|$. **5.4.64.** $\left\|\begin{matrix} 1 & -1 & 2 & 0 \\ -1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 1 \end{matrix}\right\|$.

5.4.65. A rectangular matrix A divided by horisontal and vertical lines into submatrices is called a *partitioned matrix*. These submatrices are called *blocks* and denoted by A_{ij}. For example, if the matrix A is partitioned into three "block rows" and two "block columns", then it is written in the form

$$A=\begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{Vmatrix}.$$

Show that (a) the multiplication of a partitioned matrix by a number is equivalent to the multiplication of each of its blocks by this number; (b) the addition of two rectangular matrices of the same order and partitioned in the same way is reduced to the addition of the corresponding blocks; (c) if A and B are two rectangular partitioned matrices of orders $m \times n$ and $n \times p$, respectively, whereas

$$A=\begin{Vmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ A_{21} & A_{22} & \ldots & A_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ A_{r1} & A_{r2} & \ldots & A_{rs} \end{Vmatrix}, \quad B=\begin{Vmatrix} B_{11} & B_{12} & \ldots & B_{1t} \\ B_{21} & B_{22} & \ldots & B_{2t} \\ \cdot & \cdot & \cdot & \cdot \\ B_{s1} & B_{s2} & \ldots & B_{st} \end{Vmatrix}$$

and the number of columns in each block A_{ij} is equal to the number of rows in the block B_{jk}, then the matrix $C = AB$ can also be represented in a partitioned form

$$C=\begin{Vmatrix} C_{11} & C_{12} & \ldots & C_{1t} \\ C_{21} & C_{22} & \ldots & C_{2t} \\ \cdot & \cdot & \cdot & \cdot \\ C_{r1} & C_{r2} & \ldots & C_{rt} \end{Vmatrix},$$

where

$$C_{ik}=\sum_{j=1}^{s} A_{ij}B_{jk}, \quad i=1, \ldots, r; \quad k=1, \ldots, t.$$

This condition can be reformulated thus: the number of the columns of A included in each of its block columns equals the number of the rows of B included in the corresponding block row; (d) if A and B are square matrices of the same order and are similarly partitioned into blocks, with the diagonal blocks A_{ii} and B_{ii}, $i = 1, \ldots, r$, being square, then the matrix $C = AB$ can be represented in the same partitioned form, and

$$C_{ik}=\sum_{j=1}^{r} A_{ij}B_{jk}, \quad i, k=1, \ldots, r.$$

5.4.66. A square matrix D partitioned into blocks is said to be *quasi-diagonal* if its diagonal blocks are square, and its off-diagonal blocks are zero submatrices. Show that operations over quasi-diagonal matrices of the same block structure result in quasi-diagonal matrices of the same structure. Note that when quasi-diagonal matrices A and B are multiplied together, the diagonal blocks of the matrix $C = AB$ equal the products $A_{ii}B_{ii}$ of the corresponding diagonal blocks of the factors. Hence deduce that quasi-diagonal matrices A and B of the same structure commute if and only if the corresponding diagonal blocks are symmetric.

5.4.67*. Find the general form of matrices that commute with this quasi-diagonal matrix

$$\begin{Vmatrix} \lambda_1 E_{k_1} & & & 0 \\ & \lambda_2 E_{k_2} & & \\ & & \ddots & \\ 0 & & & \lambda_r E_{k_r} \end{Vmatrix}$$

($\lambda_i \neq \lambda_j$ when $i \neq j$).

5.4.68. A square partitioned matrix A is said to be *quasi-triangular* if its diagonal blocks are square, and off-diagonal blocks A_{ij}, $i > j$ ($i < j$) are zero submatrices. Show that operations over quasi-triangular matrices of the same block structure, either upper or lower, result in quasi-triangular matrices of the same structure. Note that when upper (lower) quasi-triangular matrices A and B are multiplied together, diagonal blocks of the matrix $C = AB$ equal the products $A_{ii}B_{ii}$ of the corresponding diagonal blocks of the factors.

5.4.69*. Using the Strassen algorithm (see Problem 5.4.21), indicate a method of evaluating the product $C = AB$ of square matrices A and B of order 4 requiring only 49 multiplication operations (compared with 64 operations in the customary method).

5.4.70. Let A be a complex n-order matrix. Represent A as $A = B + iC$ where B and C are real matrices, and assign to it a real matrix D of order $2n$, viz.

$$D = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Show that if A_1 and A_2 are complex $n \times n$ matrices, and D_1 and D_2 are real double-order matrices made up in the indicated way, then the product A_1A_2 corresponds to the product D_1D_2. Note that in the particular case where $n = 1$ the correspondence between the complex numbers $z = x + iy$ and the real matrices of order 2 of the form

$$\begin{Vmatrix} x & -y \\ y & x \end{Vmatrix}$$

is obtained.

5.4.71. Let a complex column vector z_0 of order n be a solution to a system of linear equations $Az = b$ where A is a complex $m \times n$ matrix and b is a complex column vector of order m. Represent A, b and z_0 as $A = B + iC$, $b = f + ig$, $z_0 = x_0 + iy_0$, where B and C are real matrices; f, g, x_0, y_0 are real column vectors. Show that the real column vector

$$u_0 = \begin{Vmatrix} x_0 \\ y_0 \end{Vmatrix}$$

of order $2n$ is a solution to the system of $2m$ equations with real coefficients $Du = d$, where

$$D = \begin{Vmatrix} B & -C \\ C & B \end{Vmatrix}, \quad d = \begin{Vmatrix} f \\ g \end{Vmatrix}.$$

5.4.72. Show that the transposition operation is related to the other operations on matrices by the following properties:

(a) $(\alpha A)^T = \alpha A^T$;

(b) $(A + B)^T = A^T + B^T$;

(c) $(AB)^T = B^T A^T$.

5.4.73*. Let A and B be rectangular matrices of orders $m \times n$ and $p \times q$, respectively. A matrix C of order $mp \times nq$ that can be represented in a block form as

$$C = \begin{Vmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{Vmatrix}$$

is called the *Kronecker product* $A \times B$ of the matrices A and B.

Prove that for the Kronecker product of matrices the following is valid:

(a) $(\alpha A) \times B = A \times (\alpha B) = \alpha (A \times B)$;

(b) $(A + B) \times C = A \times C + B \times C$;

(c) $A \times (B + C) = A \times B + A \times C$;

(d) if the products AB and CD are defined, then

$$(AB) \times (CD) = (A \times C)(B \times D);$$

(e) the matrix $A \times B$ can be reduced to the matrix $B \times A$ by interchanging its rows and columns; moreover, if A and B are square, then the rows and columns undergo a similar interchange.

5.4.74. Show that the representation of a matrix A of rank 1 as the product $A = xy$ (see Problem 5.4.59) can be interpreted as a representation of A as the Kronecker product

$$A = y \times x.$$

5.4.75*. Let $e_1, \ldots, e_m$ be a basis for the space of column vectors of order m (i.e. $m \times 1$ matrices), and $f_1, \ldots, f_n$ a basis for the space of row vectors of order n (i.e. $1 \times n$ matrices). Prove that the Kronecker products $f_j \times e_i$ produce a basis for the space of $m \times n$ matrices.

5.4.76. Prove that the Kronecker product of square matrices A and B, perhaps, of different orders, is (a) a diagonal matrix if A and B are diagonal; (b) an upper (lower) triangular matrix if A and B are upper (lower) triangular; (c) a stochastic (doubly stochastic) matrix if A and B are stochastic (doubly stochastic).

5.4.77*. Let A and B be square matrices of orders m and n, respectively. Prove that

(a) $\operatorname{tr}(A \times B) = (\operatorname{tr} A)(\operatorname{tr} B)$;

(b) $\det(A \times B) = (\det A)^n (\det B)^m$

5.4.78. Let A, B, and C be rectangular matrices of orders $m \times n$, $p \times q$ and $m \times q$, respectively. Consider the matrix equation $A \times B = C$, where X is an $n \times p$ matrix, as a system of mq linear equations in the np unknown coefficients of this matrix, numbered as follows:

$$x_{11}, x_{12}, \ldots, x_{1p}, x_{21}, x_{22}, \ldots, x_{2p}, \ldots, x_{n1}, x_{n2}, \ldots, x_{np}.$$

The equations of the system are numbered in accordance with the familiar "by row" numeration of the coefficients in the matrix C:

$$c_{11}, c_{12}, \ldots, c_{1q}, c_{21}, c_{22}, \ldots, c_{2q}, \ldots, c_{m1}, c_{m2}, \ldots, c_{mq}.$$

Prove that this system of linear equations has $A \times B^T$ as its matrix. If, however, the coefficients of the matrices X and C are numbered by column, i.e.

$$x_{11}, x_{21}, \ldots, x_{n1}, x_{12}, x_{22}, \ldots, x_{n2}, \ldots, x_{1p}, x_{2p}, \ldots, x_{np};$$

$$c_{11}, c_{21}, \ldots, c_{m1}, c_{12}, c_{22}, \ldots, c_{m2}, \ldots, c_{1q}, c_{2q}, \ldots, c_{mq},$$

then the system has $B^T \times A$ as its matrix.

5.4.79. Show that if a matrix equation

$$AX + XB = C,$$

where A, B and C are $m \times m$, $n \times n$ and $m \times n$ matrices, respectively, is considered as a system of linear equations in the coefficients of the $m \times n$ matrix X, then the matrix of this system is given by the following: (a) $A \times E_n + E_m \times B^T$ if the coefficients of the matrices X and C are numbered by row; (b) $E_n \times A + B^T \times E_m$ if the coefficients of the matrices X and C are numbered by columns.

5.4.80. Given that the elements of an $m \times n$ matrix A are real differentiable functions of a real variable t, an $m \times n$ matrix dA/dT

$$\frac{dA}{dt}=\begin{Vmatrix} \frac{da_{11}}{dt} & \frac{da_{12}}{dt} & \dots & \frac{da_{1n}}{dt} \\ \frac{da_{21}}{dt} & \frac{da_{22}}{dt} & \dots & \frac{da_{2n}}{dt} \\ \dots & \dots & \dots & \dots \\ \frac{da_{m1}}{dt} & \frac{da_{m2}}{dt} & \dots & \frac{da_{mn}}{dt} \end{Vmatrix}$$

is called the *derivative* of the matrix A.

Prove that for the differentiation of matrices so defined the following relations are valid:

(a) $\frac{d}{dt}(\alpha A)=\alpha\frac{dA}{dt}$;

(b) $\frac{d}{dt}(A+B)=\frac{dA}{dt}+\frac{dB}{dt}$;

(c) $\frac{d}{dt}(AB)=\frac{dA}{dt}B+A\frac{dB}{dt}$;

(d) $\frac{d}{dt}(A^T)=\left(\frac{dA}{dt}\right)^T$.

5.5. The Inverse of a Matrix

In this section various techniques to evaluate the inverse matrix and the forms of the inverse matrices in the cases of some frequent classes of matrices are indicated. Just like in Sec. 5.4, great attention is paid to the matrices of the elementary transformations and to partitioned matrices. At the end of the section we provide problems on the use of the Binet-Cauchy formula.

Using explicit expressions of the elements of A^{-1} in terms of elements of A, evaluate the inverse matrices of the following:

5.5.1. $\begin{Vmatrix} 5 & -4 \\ -8 & 6 \end{Vmatrix}$.

5.5.2. $\begin{Vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{Vmatrix}$.

5.5.3. $\begin{Vmatrix} a & -b \\ b & a \end{Vmatrix}$, $a^2+b^2\neq 0$.

5.5.4. $\begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$, $ad-bc\neq 0$.

5.5.5. $\begin{Vmatrix} -2 & 3 & 1 \\ 3 & & 2 \\ 1 & 2 & 1 \end{Vmatrix}$.

5.5.6. $\begin{Vmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{Vmatrix}$.

5.5.7. $\begin{Vmatrix} 1 & -3 & -1 \\ -2 & 7 & 2 \\ 3 & 2 & -4 \end{Vmatrix}$.

5.5.8. $\begin{Vmatrix} 2 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & 0 & 1 \end{Vmatrix}$.

5.5.9. $\begin{Vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{Vmatrix}$.

5.5.10. $\begin{Vmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{Vmatrix}$.

5.5.11. $\begin{Vmatrix} 3 & 2 & 1 & 2 \\ 7 & 5 & 2 & 5 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 11 & 5 \end{Vmatrix}$.

5.5.12*. $\begin{Vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{Vmatrix}$, $a^2+b^2+c^2+d^2 \neq 0$.

5.5.13. Prove that the set of matrices of the form

$$\begin{Vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{Vmatrix},$$

where α is any real number, forms a commuting group under multiplication.

5.5.14. Prove that the set of real matrices of the form

$$\begin{Vmatrix} a & -b \\ b & a \end{Vmatrix}$$

has the structure of a field with respect to the usual operations of addition and multiplication of matrices. Show that the correspondence between such matrices and the complex numbers

$$\begin{Vmatrix} a & -b \\ b & a \end{Vmatrix} \to z = a + ib$$

is one-to-one and preserves the operations.

5.5.15*. Prove that the set of real matrices of the form

$$\begin{Vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{Vmatrix}$$

has the structure of a ring with respect to the usual operations of addition and multiplication of matrices.

Prove that nonzero matrices of the indicated form is a group (noncommuting) under multiplication.

5.5.16*. Can a set of matrices in which (a) all matrices are degenerate; (b) there are both degenerate and nondegenerate matrices, be a group under multiplication?

5.5.17*. Prove that the matrix, inverse to an upper (lower) triangular matrix, is also upper (lower) triangular. Hence, using the result of Problem 5.4.45, deduce a corollary: the set of nondegenerate triangular matrices of the same form is a group under multiplication.

5.5.18*. Prove that the matrix, inverse to a greenhouse triangular matrix, is also a greenhouse triangular matrix of the same form.

Hence, with the aid of the result of Problem 5.4.50, deduce a corollary: the set of nondegenerate greenhouse triangular matrices of the same form is a group under multiplication.

5.5.19*. Prove that the matrix, inverse to a circulant, is also a circulant. Bearing in mind the result of Problem 5.4.53, deduce the following corollary: the set of nondegenerate circulants is a group under multiplication.

5.5.20*. In a nondegenerate matrix A the sum of all row elements is the same for all the rows. Prove that the inverse matrix A^{-1} possesses the same property. Moreover, if the row sum equals $r \neq 0$ for the matrix A, then they are equal to $1/r$ for A^{-1}.

Enunciate and prove a similar statement for the columns.

5.5.21. Prove that (a) the set of nondegenerate stochastic matrices, (b) the set of nondegenerate doubly stochastic matrices, are groups under multiplication.

Find the inverse matrices of the following matrices of order n:

5.5.22. $\left\| \begin{matrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{matrix} \right\|$. **5.5.23.** $\left\| \begin{matrix} 0 & & & \lambda_1 \\ & & \lambda_2 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \lambda_n & & & 0 \end{matrix} \right\|$

(all λ_i are different from zero).

5.5.24. $\left\| \begin{matrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{matrix} \right\|$. **5.5.25.** $\left\| \begin{matrix} 1 & -2 & 0 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{matrix} \right\|$.

5.5.26. $\left\| \begin{matrix} 1 & 0 & 0 & \dots & 0 & 0 \\ a & 1 & 0 & \dots & 0 & 0 \\ 0 & a & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a & 1 \end{matrix} \right\|$. **5.5.27*.** $\left\| \begin{matrix} a & 1 & 0 & \dots & 0 \\ 0 & a & 1 & \dots & 0 \\ 0 & 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a \end{matrix} \right\|$.

5.5.28. $\left\| \begin{matrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{matrix} \right\|$.

5.5.29. Find the inverse matrices to the matrices of elementary transformations P_{ij}, D_i and L_{ij} (see Problem 5.4.24).

5.5.30. How is the inverse matrix A^{-1} altered if in the matrix A (a) the i-th and j-th rows are interchanged; (b) the i-th row is multiplied by a nonzero number α; (c) the j-th row premultiplied by a number α is added to the i-th row?

Answer the similar questions for the columns of A.

5.5.31. Find the inverse matrices of the matrices N_i and S_i (see Problem 5.4.25).

5.5.32. Prove that for a nondegenerate matrix A of the form

$$A = \begin{Vmatrix} 0 & 0 & \dots & 0 & a_{1n} \\ 0 & 0 & \dots & a_{2,\,n-1} & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n-1,\,2} & \dots & a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n1} & a_{n2} & \dots & a_{n,\,n-1} & a_{nn} \end{Vmatrix}$$

the inverse matrix $B = A^{-1}$ is of the form

$$B = \begin{Vmatrix} b_{11} & b_{12} & \dots & b_{1,\,n-1} & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2,\,n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-1,\,1} & b_{n-1,\,2} & \dots & 0 & 0 \\ b_{n1} & 0 & \dots & 0 & 0 \end{Vmatrix}.$$

5.5.33. Prove that the matrix, inverse to a permutation matrix, is also a permutation matrix. Show that the set of permutation matrices of a given order n is a group under multiplication. Find the number of elements in this group.

5.5.34. Show that the evaluation of the matrix, inverse to an $n \times n$ matrix A, can be reduced to the solution of n systems of linear equations, each of which consists of n equations in n unknowns and has the matrix A as its coefficient matrix for the unknowns. Compare the number of arithmetic operations needed in solving such systems by the Gauss method with that in finding the inverse matrix using the explicit expressions for its elements in terms of the elements of A.

Find the inverse matrices of the following by the method indicated in Problem 5.5.34:

5.5.35. $\begin{Vmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{Vmatrix}$. **5.5.36.** $\begin{Vmatrix} 1 & -2 & 2 & -4 \\ -2 & 3 & -4 & 6 \\ 3 & -6 & 5 & -10 \\ -6 & 9 & -10 & 15 \end{Vmatrix}$.

5.5.37*. All the leading principal minors of an $n \times n$ matrix A are nonzero. Prove that using the Gauss method the matrix A can be represented as the product of a lower triangular matrix L by an upper triangular matrix R, i.e. $A = LR$. The diagonal elements of one of these matrices can be set equal to unity.

5.5.38. Prove that the representation of a matrix A as the product $A = LR$, obtained in Problem 5.5.37, is unique if the diagonal elements of the matrix L are chosen to be equal to unity.

5.5.39. Prove that any nondegenerate matrix A can be represented as the product $A = PLR$, where P is a permutation matrix, L is a lower triangular, and R is an upper triangular matrix.

5.5.40*. Prove that any nondegenerate matrix A can be reduced to the unit matrix by elementary transformations of its rows and columns.

5.5.41. Show that the statement of Problem 5.5.40 is valid even if only elementary transformations of the rows (columns) are permitted.

5.5.42. Using the result of Problem 5.5.40, prove that any nondegenerate matrix can be represented as the product of matrices of elementary transformations.

5.5.43. Show that if elementary row transformations by which a given matrix A is reduced to the unit matrix are applied in the same sequence to the rows of the unit matrix, then the resulting matrix is the inverse A^{-1}.

Find the inverse matrices of the following by the method indicated in Problem 5.5.43:

5.5.44. $\begin{Vmatrix} 2 & 3 & 2 & 2 \\ -1 & -1 & 0 & -1 \\ -2 & -2 & -2 & -1 \\ 3 & 2 & 2 & 2 \end{Vmatrix}$. **5.5.45.** $\begin{Vmatrix} 2 & 3 & 4 & 5 \\ 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 \end{Vmatrix}$.

5.5.46. Let J_n be a matrix of order n all of whose elements are equal to unity. Prove that

$$(E-J_n)^{-1}=E-\frac{1}{n-1}J_n.$$

5.5.47. Let B be a matrix of rank 1. According to Problem 5.4.60, $B^2 = \alpha B$ for some number α. Assuming that $\alpha \neq -1$, prove that

$$(E + B)^{-1} = E - \beta B,$$

where $\beta = \frac{1}{1+\alpha}$. Show that Problem 5.5.46 is a particular case of this statement.

5.5.48. Show that if a matrix A is nondegenerate, then the matrices $A + B$ and $E + A^{-1}B$ are either both degenerate or both nondegenerate.

5.5.49*. Let A be a nondegenerate matrix whose inverse A^{-1} is known; further, let $B = xy$ be a matrix of rank 1. Prove that if the matrix $A + B$ is nondegenerate, then its inverse can be found by the formula

$$(A + B)^{-1} = A^{-1} - \beta A^{-1}BA^{-1}$$

where $\beta = \frac{1}{1+\alpha}$, $\alpha = yA^{-1}x$. Thus, if a matrix of rank 1 is added to the matrix A, then a matrix of rank 1 is also added to the inverse matrix.

5.5.50. Calculate the number of multiplications and divisions necessary to transform A^{-1} to $(A + B)^{-1}$ in Problem 5.5.49, assuming that the matrices x and y that make up the matrix B are known.

5.5.51. A number γ is added to an element a_{ij} of a nondegenerate matrix A yielding a matrix $\tilde{A}$ which is also nondegenerate. Find an expression for $\tilde{A}^{-1}$ in terms of γ and the elements of the matrix A^{-1}.

5.5.52. In a nondegenerate matrix A of order n, the elements $\gamma_1, \ldots, \gamma_n$ are added to the last row in such a way that the nondegeneracy of the matrix is preserved. Find an expression for the inverse of the new matrix $\tilde{A}$, in terms of the elements of A^{-1} and the numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$.

5.5.53. A number a is added to each of the elements of a nondegenerate matrix A. The obtained matrix $\tilde{A}$ will still be nondegenerate. Find an expression for $\tilde{A}^{-1}$ in terms of the elements of A^{-1} and the number a.

Find the inverse matrices to the following matrices of order n:

5.5.54. $\left\| \begin{matrix} a & b & b & \ldots & b \\ b & a & b & \ldots & b \\ b & b & a & \ldots & b \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \ldots & a \end{matrix} \right\|, \quad \begin{matrix} a \neq b, \\ a \neq b(1-n). \end{matrix}$

5.5.55. $\left\| \begin{matrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 0 \end{matrix} \right\|.$

5.5.56. $\left\| \begin{matrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 0 \end{matrix} \right\|.$

5.5.57. $\left\| \begin{matrix} 1+a_1 & 1 & 1 & \ldots & 1 \\ 1 & 1+a_2 & 1 & \ldots & 1 \\ 1 & 1 & 1+a_3 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 1+a_n \end{matrix} \right\|$

(all a_i are nonzero).

5.5.58. Prove that the inverse of a nondegenerate quasi-diagonal matrix D is also quasi-diagonal and has the same block structure as D. Note that the diagonal blocks of D^{-1} are the inverse matrices of the corresponding diagonal blocks of D.

5.5.59. Prove that the inverse matrix of a nondegenerate upper (lower) quasi-triangular matrix A is also upper (lower) quasi-triangular and has the same block structure as A. Note that the diagonal blocks of A^{-1} are the inverse matrices of the corresponding diagonal blocks of A.

5.5.60. Find the inverse to the matrix A of order $k + l$

$$A = \begin{Vmatrix} E_k & B \\ 0 & E_l \end{Vmatrix}.$$

5.5.61*. Let the submatrix A of the following square partitioned matrix

$$M = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$$

be square and nondegenerate. Prove that the determinant of the matrix M satisfies the relation

$$| M | = | A | \, | D - CA^{-1}B |.$$

5.5.62*. The inverse matrix A_{n-1}^{-1} of a matrix A_{n-1} of order $n - 1$ is known. Find the inverse to the *enclosing* matrix A_n of order n

$$A_n = \begin{Vmatrix} A_{n-1} & u_{n-1} \\ v_{n-1} & a \end{Vmatrix},$$

assuming it to be nondegenerate.

5.5.63. Calculate the number of multiplications and divisions necessary to employ the formulae for A_n^{-1} derived in Problem 5.5.62.

5.5.64. Verify that the inverse matrix M^{-1} of the square partitioned matrix M of order $k + l$

$$M = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix},$$

where A and D are square blocks of orders k and l, respectively, is also partitioned, viz.,

$$M^{-1} = \begin{Vmatrix} P & Q \\ R & S \end{Vmatrix},$$

where $P = (A - BD^{-1}C)^{-1}$, $Q = -PBD^{-1}$, $R = -D^{-1}CP$, $S = D^{-1} - D^{-1}CQ$ or

$$S = (D - CA^{-1}B)^{-1}, \quad R = -SCA^{-1},$$
$$P = A^{-1} - A^{-1}BR, \quad Q = -A^{-1}BS.$$

The inverse matrices indicated here are assumed to be defined. These so-called *Frobenius formulae* make it possible to reduce the evaluation of the inverse to a matrix of order $k + l$ to the computation of one matrix of order k and one matrix of order l.

5.5.65. Let A and B be square nondegenerate matrices of orders m and n, respectively. Prove that the Kronecker product of these matrices is also nondegenerate and that

$$(A \times B)^{-1} = A^{-1} \times B^{-1}.$$

Find the inverse matrices to the following

5.5.66. $\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 83 & -47 & 1 & 0 & 0 \\ -55 & 94 & 0 & 1 & 0 \\ 62 & -71 & 0 & 0 & 1 \end{Vmatrix}.$

5.5.67. $\begin{Vmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 4 & 1 & 5 \end{Vmatrix}.$

5.5.68. $\begin{Vmatrix} 1 & 0 & 3 & 9 \\ 0 & 1 & 7 & 21 \\ -3 & -12 & 4 & 0 \\ 1 & 4 & 0 & 1 \end{Vmatrix}.$ 5.5.69.* $\begin{Vmatrix} 24 & 32 & 9 & 12 \\ 40 & 56 & 15 & 21 \\ 15 & 20 & 6 & 8 \\ 25 & 35 & 10 & 14 \end{Vmatrix}.$

5.5.70. Let $A = B + iC$ be a complex matrix of order n, and $A^{-1} = F + iG$ be the inverse of A. Prove that the real matrices of order $2n$, i.e.

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

and

$$\begin{pmatrix} F & -G \\ G & F \end{pmatrix}$$

are inverse to one another.

5.5.71. Prove that the operations of transposing and finding the inverse are commuting, i.e. $(A^T)^{-1} = (A^{-1})^T$.

5.5.72. The elements of a square matrix A are differentiable functions of a real variable t. Assuming that the matrix A is nondegenerate for a given value of t, prove the formula:

$$\frac{d}{dt}(A^{-1}) = -A^{-1}\frac{dA}{dt}A^{-1}.$$

5.5.73. Show that the solution of a system of linear equations $Ax = b$ with a nondegenerate square coefficient matrix A is $x = A^{-1}b$. Hence deduce Cramer's formulae.

5.5.74. Let the coefficients of the matrix A and column vector b (see Problem 5.5.73) be differentiable functions of a real variable t. Prove the formula

$$\frac{dx}{dt} = -A^{-1}\frac{dA}{dt}x + A^{-1}\frac{db}{dt}.$$

5.5.75. Let A and B be rectangular matrices of orders $m \times n$ and $n \times p$, respectively. Prove that the minors of the matrix $C = AB$

satisfy the relations:

$$C\begin{pmatrix} i_1 & i_2 & \dots & i_q \\ j_1 & j_2 & \dots & j_q \end{pmatrix} = \sum_{\substack{1\leqslant k_1<k_2<\dots \\ \dots<k_q\leqslant n}} A\begin{pmatrix} i_1 & i_2 & \dots & i_q \\ k_1 & k_2 & \dots & k_q \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \dots & k_q \\ j_1 & j_2 & \dots & j_q \end{pmatrix}$$

$$(1\leqslant i_1<i_2<\dots<i_q\leqslant m;\ 1\leqslant j_1<j_2<\dots<j_q\leqslant p).$$

5.5.76. Using the Binet-Cauchy formula, prove that the rank of each of the matrices AA^T and A^TA equals the rank of the matrix A. A is assumed to be a real matrix.

5.5.77. Prove that the sum of all the principal minors of a given order k ($1 \leqslant k \leqslant \min(n, m)$) of the matrices AB and BA, where A and B are rectangular matrices of orders $m \times n$ and $n \times m$, respectively, is the same.

5.5.78. A square matrix A is said to be *totally nonnegative* (*totally positive*) if all minors of each order are nonnegative (positive). Prove that the product of totally nonnegative (totally positive) matrices is also a totally nonnegative (totally positive) matrix.

5.5.79*. Let A be a square matrix of order n. Given a natural number p, $1 \leqslant p \leqslant n$, list in lexicographic order all the $N = {}^nC_p$ combinations of n numbers $1, 2, \dots, n$ taken p numbers $k_1 < k_2 < \dots < k_p$ at a time. Lexicographic order means that the combination $k_1 < k_2 < \dots < k_p$ precedes the combination $k_1' < k_2' < \dots < k_p'$ if $k_1 = k_1', \dots, k_{l-1}' = k_{l-1}'$, but $k_l < k_l'$, $1 \leqslant l \leqslant p$. Construct the square matrix $A_p = (a_{ij,p})$ of order N as follows:

$$a_{ij,\,p} = A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix},$$

if the number of the combination $i_1 < i_2 < \dots < i_p$ equals i, and the number of the combination $j_1 < j_2 < \dots < j_p$ equals j. The obtained matrix A_p is termed the *p-th associated* with A. In particular $A_1 = A$, $A_n = |A|$.

Prove that

(a) $(E_n)_p = E_N$,

(b) an associated matrix with a diagonal matrix D is also diagonal; (c) an associated matrix with an upper (lower) triangular matrix A is also upper (lower) triangular;

(d) $(AB)_p = A_pB_p$,

(e) if A is a nondegenerate matrix then $(A^{-1})_p = (A_p)^{-1}$.

5.5.80*. Let A be a nondegenerate matrix of order n. Prove that the minors of any order of the inverse matrix $B = A^{-1}$ are related to the minors of the matrix A by the relations

$$B\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} = \frac{(-1)^{\sum_{s=1}^{p}(i_s+k_s)} A\begin{pmatrix} k_1' & k_2' & \dots & k_{n-p}' \\ i_1' & i_2' & \dots & i_{n-p}' \end{pmatrix}}{|A|},$$

where $i_1 < i_2 < \ldots < i_p$ along with $i'_1 < i'_2 < \ldots < i'_{n-p}$, and $k_1 < k_2 < \ldots < k_p$ along with $k'_1 < k'_2 < \ldots < k'_{n-p}$ make the complete system of indices 1, 2, ..., n.

5.6. The Matrix of a Linear Operator, Transfer to Another Basis, Equivalent and Similar Matrices

These problems are in three groups corresponding to the topics in the section heading.

5.6.1. The Euclidean plane E_2 is assumed to have a right-hand orientation (i.e. positive angles are those measured counterclockwise). Let Oe_1e_2 be a dextral Cartesian system of coordinates on the plane E_2. Construct the matrix of a linear transformation consisting in the rotation of E_2 through an angle α about the origin for the basis e_1, e_2.

5.6.2. Let e_1, e_2, e_3 be a dextral orthonormal basis for the three-dimensional Euclidean space E_3 of geometric vectors. Consider the following linear operator A of the space E_3

$$Ax = [x, a].$$

Here a is a fixed vector whose coordinates with respect to the basis e_1, e_2, e_3 are equal to α, β, γ. Find the matrix of the operator A in this basis.

5.6.3. Write the matrices of: (a) the differential operator; (b) the difference operator A_1; in the space M_n of polynomials of degree $\leqslant n$ with respect to the basis $1, t, t^2, \ldots, t^n$.

5.6.4. If the differential operator is an operator from M_n to M_{n-1}, write its matrix with respect to the two bases $1, t, t^2, \ldots, t^n$ and $1, t, t^2, \ldots, t^{n-1}$. Find the matrix of the integration operator with respect to the two bases as if it were an operator from M_{n-1} to M_n.

5.6.5. Find the matrix of the differential operator on the two-dimensional linear space drawn on the basis functions

(a) $f_1(t) = \cos t$, $f_2(t) = \sin t$;

(b) $g_1(t) = e^{at} \cos bt$, $g_2(t) = e^{at} \sin bt$.

5.6.6. A space X is the direct sum of subspaces L_1 and L_2. A basis $e_1, \ldots, e_n$ is selected so that the vectors $e_1, \ldots, e_r$ form a basis for the subspace L_1 and $e_{r+1}, \ldots, e_n$ form a basis for L_2. Using the basis $e_1, \ldots, e_n$, construct (a) the matrix of the operator that projects onto L_1, parallel to L_2; (b) the matrix of the operator that projects onto L_2, parallel to L_1; (c) the matrix of the operator that reflects in L_1 parallel to L_2.

5.6.7. Consider the n-dimensional arithmetic space X (either real or complex) and the corresponding m-dimensional arithmetic space Y where the "natural" (standard) bases, made up of the unit vectors of these spaces, are used. We can match each $m \times n$ matrix A with an operator $\tilde{A}$ from X to Y when the operator is defined as follows:

$$x \xrightarrow{\tilde{A}} y = Ax,$$

i.e. each column vector x from X is multiplied by the matrix A. Prove that (a) this correspondence between the $m \times n$ matrices and the operators from X to Y is one-to-one; (b) the matrix of the operator $\tilde{A}$ with respect to the two standard bases coincides with matrix of A. Thus, the operators on arithmetic spaces can be identified with rectangular matrices of the corresponding orders.

5.6.8. An operator A on a three-dimensional arithmetic space converts linearly independent vectors a_1, a_2, a_3 into vectors b_1, b_2, b_3, where

$$a_1 = \begin{Vmatrix} 5 \\ 3 \\ 1 \end{Vmatrix}, \quad a_2 = \begin{Vmatrix} 1 \\ -3 \\ -2 \end{Vmatrix}, \quad a_3 = \begin{Vmatrix} 1 \\ 2 \\ 1 \end{Vmatrix}, \quad b_1 = \begin{Vmatrix} -2 \\ 1 \\ 0 \end{Vmatrix}, \quad b_2 = \begin{Vmatrix} -1 \\ 3 \\ 0 \end{Vmatrix}, \quad b_3 = \begin{Vmatrix} -2 \\ -3 \\ 0 \end{Vmatrix}.$$

Find the matrix of this operator (a) with respect to the basis a_1, a_2, a_3;

(b) with respect to the standard basis e_1, e_2, e_3.

5.6.9. In the space of square matrices of order 2 a basis consisting of matrices (in the order indicated)

$$\begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}$$

is fixed. Write with respect to this basis (a) the matrix of the transposition operator, i.e. the operator that assigns to each matrix X its transpose; (b) the matrix of the operator G_{AB} that assigns to each matrix X the matrix AXB where A and B are given matrices; (c) the matrix of an operator F_{AB} defined by the relation

$$X \to AX + XB.$$

How are these matrices altered if in the basis the matrices

$$\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}$$

are interchanged?

5.6.10. Let in the space of $m \times n$ matrices a basis $E_{11}, E_{12}, \ldots, E_{1n}, E_{21}, E_{22}, \ldots, E_{2n}, \ldots, E_{m1}, E_{m2}, \ldots, E_{mn}$ (in the order indicated) be fixed, E_{ij} being an $m \times n$ matrix in which the only nonzero element is placed at (i, j) and is equal to 1. Further, let A and B be given square matrices of orders m and n, respectively.

Consider the operators G_{AB} and F_{AB}, defined by the relations

$$X \xrightarrow{G_{AB}} AXB,$$

$$X \xrightarrow{F_{AB}} AX + XB.$$

Prove that with respect to the indicated basis (a) the matrix of the operator G_{AB} is the Kronecker product $A \times B^T$; (b) the matrix of the operator F_{AB} is $A \times E_n + E_m \times B^T$.

Find matrices of the same operators with respect to the basis $E_{11}, E_{21}, \ldots, E_{m1}, E_{12}, E_{22}, \ldots, E_{m2}, \ldots, E_{1n}, E_{2n}, \ldots, E_{mn}$.

5.6.11. Let A be an operator from ω_{XY}. Prove that all the matrices defining the operator A with respect to various pairs of bases for the spaces X and Y have the same rank equal to the rank of A.

5.6.12. Find the rank of an operator F_{AB}

$$X \to \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} X + X \begin{pmatrix} -3 & 1 \\ -4 & 1 \end{pmatrix}.$$

5.6.13. Prove that the operator F_{AB} (see Problem 5.6.12) is nilpotent and find the nilpotence index of this operator.

5.6.14. What can be said about the matrix of an operator A of rank r if, in the basis $e_1, \ldots, e_n$ of the space X, the vectors $e_{r+1} \ldots \ldots, e_n$ belong to the kernel of this operator?

5.6.15*. An operator A from ω_{XY} has rank r. Prove that in the spaces X and Y, the respective bases $e_1, \ldots, e_n$ and $q_1, \ldots, q_m$ can be chosen such that the matrix A_{qe} of the operator A is of the form

$$\left\| \begin{matrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{matrix} \right\|. \qquad (5.6.1)$$

The number of nonzero columns in the matrix A_{qe} equals the rank r of the operator.

5.6.16. Show that any real or complex nondegenerate matrix of order n can be regarded as the matrix that defines in the n-dimensional space X, respectively real or complex, the transfer from one basis $e_1, \ldots, e_n$ to another, $f_1, \ldots, f_n$; moreover, one of the bases can be chosen arbitrarily.

5.6.17. Let a matrix A define the transfer from a basis $e_1, \ldots, e_n$ to a basis $f_1, \ldots, f_n$, and a matrix B from $f_1, \ldots, f_n$ to $g_1, \ldots, g_n$. Show that (a) the transfer matrix from $f_1, \ldots, f_n$ to $e_1, \ldots, e_n$ is A^{-1}; (b) the transfer matrix from $e_1, \ldots, e_n$ to $g_1, \ldots, g_n$ is $C = AB$.

5.6.18. How is the transfer matrix from $e_1, \ldots, e_n$ to $f_1, \ldots, f_n$ altered if (a) the vectors e_i and e_j are interchanged?
(b) the vectors f_k and f_l are interchanged?

5.6.19. An operator A is defined with respect to the basis 1, t, t^2 in the space M_2, by the matrix

$$\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$

Find the matrix of this operator when the basis comprises the polynomials $3t^2 + 2t$, $5t^2 + 3t + 1$, $7t^2 + 5t + 3$.

5.6.20. Two operators are defined on the space M_3. The operator A transforms any polynomial $a_0 + a_1t + a_2t^2 + a_3t^3$ into the polynomial $a_0 + a_1t + a_2t^2$. The operator B transforms the polynomials $t^3 + t^2$, $t^3 + t$, $t^3 + 1$, $t^3 + t^2 + t + 1$ into $t^3 + t$, $t^3 + 1$, $t^3 + t^2 + t + 1$ and the zero polynomial, respectively. Construct the matrices of the operators AB and BA with respect to the basis 1, t, t^2, t^3.

5.6.21. Let P and Q be nondegenerate matrices of orders m and n, respectively. Show that matrices $F_{11}, F_{12}, \ldots, F_{1n}, F_{21}, F_{22}, \ldots, F_{mn}$ (where $F_{ij} = PE_{ij}Q$, and E_{ij} are the matrices defined in Problem 5.6.10) form a basis for the space of $m \times n$ matrices. Find the transfer matrix from the basis made up of the matrices E_{ij} to this basis, and also the matrix of the inverse transfer.

5.6.22. Find the matrices of the operators G_{AB} and F_{AB} of Problem 5.6.10 with respect to the basis $F_{11}, F_{12}, \ldots, F_{mn}$ (see Problem 5.6.21).

5.6.23. Let $\tilde{A}_1$ be an operator defined by a square $n \times n$ matrix A with respect to a basis $e_1, \ldots, e_n$ of a space X, $\tilde{A}_2$ the operator defined by the same matrix with respect to a basis $f_1, \ldots, f_n$. Prove that

$$\tilde{A}_2 = P\tilde{A}_1P^{-1},$$

where P is an operator transforming the vectors $e_1, \ldots, e_n$ into $f_1, \ldots, f_n$.

5.6.24. Rectangular matrices A and B are said to be *equivalent* if there exist nondegenerate matrices R and S such that $B = RAS$. Show that the equivalence relation on the set of rectangular matrices of fixed order $m \times n$ is reflexive, symmetric, and transitive.

5.6.25. Square matrices A and B are said to be *similar* if there exists a nondegenerate matrix P such that $B = P^{-1}AP$. In addition, the matrix P is said to *transform* A to B. Show that the similarity relation on a set of square matrices of a given order n is reflexive, symmetric, and transitive.

5.6.26. Prove that any two equivalent (similar) matrices have the same rank.

5.6.27. Let X and Y be an n-dimensional and m-dimensional space, respectively. Prove that any two equivalent $m \times n$ matrices A and B can be regarded as matrices defining the same operator from ω_{XY} with respect to certain pairs of bases $e_1, \ldots, e_n, q_1, \ldots, q_m$, and $f_1, \ldots, f_n, t_1, \ldots, t_m$ of these spaces. One of the pairs of bases can be chosen arbitrarily.

5.6.28. Prove that any two similar matrices of order n are matrices that define the same operator of an n-dimensional space X with respect to two bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ for this space. The choice of one of the bases is arbitrary.

5.6.29*. Prove that any matrix A is equivalent to a matrix of form (5.6.1).

5.6.30. Prove the statement, converse to that in Problem 5.6.26, viz.. two $m \times n$ matrices A and B having the same rank are equivalent.

5.6.31. Let matrices A and B be similar, i.e. $B = P^{-1}AP$. Is the transforming matrix P unique?

5.6.32*. Show that a scalar matrix αE is similar only to itself. Prove that this property is intrinsic only to scalar matrices.

5.6.33. Let A be a fixed square matrix. Prove that the set of all matrices P transforming A into A is a group under multiplication.

5.6.34. Let A and B be similar matrices. Prove that if P_0 is some matrix that transforms A into B, then the whole set of the transforming matrices is obtained from the set of the matrices transforming A into A by multiplying the latter matrices on the right by the matrix P_0.

5.6.35. Show that a matrix A is transformed into a similar matrix by the following procedure: (a) the i-th row is multiplied by a nonzero number α and then the i-th column is multiplied by the number $1/\alpha$; (b) the j-th row is multiplied by a number α and added to the i-th row; then the i-th column premultiplied by α is subtracted from the j-th column; (c) the i-th and j-th rows, and then the i-th and j-th columns are interchanged.

5.6.36*. Show that the mirror reflection in the centre of a square matrix is a similar transformation of this matrix.

5.6.37. Prove that similar matrices A and B have the same trace and determinant.

5.6.38. Prove that if at least one of two square matrices A and B of the same rank is nondegenerate, then the matrices AB and BA are similar. Give an example of degenerate matrices A and B for which AB and BA are not similar.

5.6.39. Show that if matrices A and B are similar, then (a) the matrices A^2 and B^2 are similar; (b) the matrices A^k and B^k, where k

is any natural number, are similar; (c) for any polynomial $f(t)$, the matrices $f(A)$ and $f(B)$ are similar.

5.6.40. Does the equivalence of matrices A and B of order $n \times n$ mean that the matrices A^2 and B^2 are equivalent (cf. Problem 5.6.39 (a))?

5.6.41. Show that similar matrices A and B have the same minimal polynomial.

5.6.42. Matrices A and B of orders m and n, respectively, are similar to matrices C and D. Prove that (a) the matrix $A \times B$ is similar to the matrix $C \times D$; (b) the matrix $A \times E_n + E_m \times B^T$ is similar to the matrix $C \times E_n + E_m \times D^T$.

5.6.43. Prove that if matrices A and B are similar, then their associated matrices A_p and B_p are similar.

5.6.44. Show that if the complex matrices $A_1 = B_1 + iC_1$ and $A_2 = B_2 + iC_2$ are similar, then the real matrices D_1 and D_2 are also similar:

$$D_i = \begin{Vmatrix} B_i & -C_i \\ C_i & B_i \end{Vmatrix}, \quad i = 1, 2.$$

CHAPTER 6

Linear Operator Structure

6.0. Terminology and General Notes

Let A be an operator from ω_{XX}. A number λ is called an *eigenvalue* of the operator A if a nonzero vector x exists such that

$$Ax = \lambda x. \tag{6.0.1}$$

Any vector $x \neq 0$ satisfying (6.0.1) is called an *eigenvector* of the operator A associated with the eigenvalue λ.

If A_e is the matrix of an operator A with respect to an arbitrary basis $e_1, \ldots, e_n$ for the space X then the polynomial $\det(\lambda E - A)$ does not depend on the selection of the basis and is called the *characteristic polynomial* of the operator A.

The roots (in the given field) of the characteristic polynomial, and only the roots, are the eigenvalues of an operator.

According to the *fundamental theorem of algebra*, any polynomial of degree n ($n \geqslant 1$) with complex coefficients has precisely n roots in the field of complex numbers (if each is counted as many times as its multiplicity). If the *algebraic multiplicity of an eigenvalue* is defined to be equal to its multiplicity as a root of the characteristic polynomial, then

in a complex linear space of dimension n, each operator has n eigenvalues (taking their multiplicity into account). In addition, *there exists at least one eigenvector.*

A subspace L is said to be *invariant with respect to an operator* A if from $x \in L$ it follows that $Ax \in L$. An operator A, considered only for vectors from an invariant space L, is called an *induced operator* and denoted by A/L.

If a space X is the direct sum of subspaces L_1 and L_2, invariant with respect to an operator A, then for any vector x with the decomposition

$$x = x_1 + x_2, \quad x_1 \in L_1, \quad x_2 \in L_2,$$

we obtain

$$Ax = Ax_1 + Ax_2 = (A/L_1)\,x_1 + (A/L_2)\,x_2,$$

whereupon the operator A is said to be the *direct sum* of the induced operators A/L_1 and A/L_2. This is equivalent to saying that the operator A *is reduced* by the pair of subspaces L_1 and L_2.

For any operator A on a complex space, there exists a basis for this space, called a *Jordan canonical basis*, in which the matrix of this operator is of quasi-diagonal form

$$J = \begin{Vmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{Vmatrix},$$

where each of the diagonal blocks J_i is a Jordan block corresponding to one of the eigenvalues of the operator A. The matrix J is called the *Jordan form* of the operator A.

Terms such as "an eigenvector of a matrix", "an invariant subspace of a matrix", etc. are used in the present chapter in the same sense attributed to them at the end of Sec. 5.0. For example, an eigenvector of an $n \times n$ matrix is considered as an n-dimensional column vector, etc.

6.1. Eigenvalues and Eigenvectors

The present section includes problems, pertaining to the eigenvalues and eigenvectors of an operator, that can be solved without the use of the characteristic polynomial. These problems mostly concern the following topics:

(i) Definition of eigenvalues and eigenvectors.

(ii) A theorem about linear independence of eigenvectors associated with different eigenvalues, and corollaries to it.

(iii) Operators and matrices of simple structures.

6.1.1. Prove that it is a necessary and sufficient condition for nondegeneracy of an operator A, that it should not have an eigenvalue equal to zero.

6.1.2. Show that (a) the eigenvectors of an operator A associated with a zero eigenvalue, and no others, belong to the kernel of this operator; (b) the eigenvectors associated with nonzero eigenvalues belong to the image of the operator.

6.1.3. Prove that if an operator A is nondegenerate, then both A and A^{-1} have the same eigenvectors. Find the relation between the eigenvalues of these operators.

6.1.4. Show that when an operator is multiplied by a nonzero number, the eigenvectors are unaltered and the eigenvalues are also multiplied by this number.

6.1.5. Show that the operator $A - \lambda_0 E$ has, for any number λ_0, the same eigenvectors as the operator A. Find the relation between the eigenvalues of these operators.

6.1.6. Prove that if x is an eigenvector of an operator A, associated with an eigenvalue λ, then x is also an eigenvector of the operator (a) A^2; (b) A^k for any natural k; and (c) $f(A)$ where $f(t)$ is any polynomial. Find the corresponding eigenvalues.

6.1.7. Is the following statement valid: if x is an eigenvector of a certain polynomial $f(A)$ in the operator A, then x is also an eigenvector of the operator A itself?

6.1.8. Prove that a nilpotent operator has no eigenvalues other than zero.

6.1.9. Prove that the operator which rotates the Euclidean plane through an angle α, not a multiple of π has no eigenvectors.

6.1.10. Find the eigenvalues and eigenvectors of the operator A of the three-dimensional Euclidean space such that $Ax = [x, a]$, where a is a fixed vector.

6.1.11. Find the eigenvalues and eigenvectors of the differential operator on the space of polynomials M_n.

6.1.12. Find the eigenvectors of the differential operator on the space generated by $f_1(t) = \cos t$, $f_2(t) = \sin t$.

6.1.13. Prove that the eigenvalues of a diagonal matrix coincide with its diagonal elements.

6.1.14. Prove that a stochastic matrix has an eigenvalue equal to unity. Find the corresponding eigenvector.

6.1.15*. Find the eigenvalues of a matrix $A = xy$ having unit rank.

6.1.16. Find the eigenvalues and eigenvectors of the $n \times n$ matrix J_n

$$J_n = \begin{Vmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 \end{Vmatrix}.$$

6.1.17. Find the eigenvalues and eigenvectors of the $n \times n$ matrix A:

$$A = \begin{Vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \dots & a \end{Vmatrix}.$$

6.1.18. Prove that if the matrices A and B are similar, then every eigenvalue of A is also an eigenvalue of B, and vice versa. Find the relation between the eigenvectors of the matrices A and B.

6.1.19*. Prove that an operator's eigenvectors which are associated with different eigenvalues, are linearly independent.

6.1.20. Using the result of Problem 6.1.19, deduce that an operator A on an n-dimensional space X cannot have more than n different eigenvalues. If there are precisely n different eigenvalues then a basis for the space X exists that consists of the eigenvectors of the operator A.

6.1.21. Prove that the set of all eigenvectors of an operator A associated with a given eigenvalue λ_0, together with the zero vector is a subspace called an *eigensubspace* of the operator A associated with the eigenvalue λ_0.

6.1.22. A space X is the direct sum of subspaces L_1 and L_2. Find the eigenvalues and eigensubspaces of (a) the projection operator on L_1 parallel to L_2; (b) the reflection operator in L_1 parallel to L_2.

6.1.23. The dimension of the eigensubspace of an operator A associated with eigenvalue λ_0 is called the *geometric multiplicity* of the eigenvalue λ_0. Show that the geometric multiplicity of λ_0 is equal to the defect of the operator $A - \lambda_0 E$.

6.1.24. Prove that the sum of the eigensubspaces of an operator A is the direct sum.

6.1.25. Prove that all nonzero vectors of a space are the eigenvectors of an operator A if and only if A is a scalar operator.

6.1.26*. Prove that the sum of the geometric multiplicities of all the eigenvalues of an operator A from ω_{XX} does not exceed the dimension of the space X. Moreover, it is a necessary and sufficient condition that the indicated sum equal the dimension of the space X for a basis, made up of the eigenvectors of the operator A, to exist in the space X.

6.1.27. An operator A is called an *operator with a simple structure* when there exists a basis for the space consisting of the eigenvectors of this operator. What is the geometric meaning of such an operator? What is the form of the matrix of the operator A with respect to the basis of eigenvectors?

6.1.28. A square matrix is called a *matrix of simple structure* if it is similar to some diagonal matrix. Prove that an operator A from ω_{XX} is an operator of simple structure if and only if its matrix with respect to an arbitrary basis for the space is a matrix of simple structure.

6.1.29. Prove that an operator of simple structure possesses the following properties: (a) the image is the span of the eigenvectors associated with the nonzero eigenvalues; (b) the intersection of the kernel and image consists of the zero vector only.

6.1.30. Show that the projection and reflection operators are of simple structure.

6.1.31. Prove that among nilpotent operators only the zero operator is of simple structure.

6.1.32. Prove that any polynomial $f(A)$ in an operator of simple structure is also of simple structure. In particular, if A is nondegenerate, then A^{-1} is of simple structure.

6.1.33. Prove that if an operator A of an n-dimensional space is of simple structure, then the minimal polynomial of this operator has a degree not exceeding n.

6.1.34. An operator A on an n-dimensional space X has n different eigenvalues. Prove that any operator B that commutes with A is an operator of simple structure.

6.1.35. Show that the operator B (see Problem 6.1.34) can be represented by a polynomial of the operator A.

6.1.36. Let A be an operator on a real space R, and let $\hat{A}$ be the operator obtained from A by the complexification of the space R. Show that if x is an eigenvector of the operator A, associated with an eigenvalue λ, then the vector $x + i0$ is an eigenvector of the operator $\hat{A}$ associated with the same eigenvalue.

6.1.37. Show that the operator $\hat{A}$ (see Problem 6.1.36) is of simple structure if A is an operator of simple structure.

6.1.38. According to the definition of a matrix A of simple structure, there exists a nondegenerate matrix P such that $P^{-1}AP = \Lambda$ is a diagonal matrix. Prove that the diagonal elements of the matrix Λ are the eigenvalues, and the columns of the matrix P the eigenvectors of the matrix A. Conversely, a nondegenerate matrix P whose columns are the eigenvectors of a matrix A reduces this matrix to a diagonal matrix.

6.1.39. Prove that if a matrix A is of simple structure, then the same is valid for the transpose of A, i.e. for A^T.

6.1.40. Let λ be an eigenvalue and x the associated eigenvector of an $m \times m$ matrix A, and let μ be an eigenvalue and y the associated eigenvector of an $n \times n$ matrix B. Prove that the Kronecker product $x \times y$ is: (a) an eigenvector of the matrix $A \times B$; (b) an eigenvector of the matrix $A \times E_n + E_m \times B$. Find the associated eigenvalues.

6.1.41. Prove that if the matrices A and B (see Problem 6.1.40) are of simple structure, then the same holds true for the matrices $A \times B$ and $A \times E_n + E_m \times B$.

6.1.42. Deduce a corollary from Problem 6.1.41: if matrices A and B are of simple structure, then the operators G_{AB} and F_{AB} (see Problem 5.6.10) are of simple structure.

6.1.43. Prove that if A is a matrix of simple structure, then so are all the associated matrices A_p.

6.2. The Characteristic Polynomial

We intended in this section to illustrate the following topics related to the characteristic polynomial:

(i) The definition of characteristic polynomial, the expression of its coefficients in terms of minors of the matrix, and the relation of the coefficients to the eigenvalues.

(ii) The characteristic polynomial as a means of computing the eigenvalues.

(iii) The companion matrix of this polynomial.

(iv) The characteristic polynomials of special classes of operators and matrices.

As in the previous section, a great consideration is given to operators and matrices of simple structure. The test, established in Problem 6.1.26, reveals its significance only here, i.e. when a method of computing the eigenvalues is available.

6.2.1. Write explicit expressions for the characteristic polynomials of matrices of order (a) 1; (b) 2; (c) 3.

6.2.2*. Prove that in the expression of the characteristic polynomial $|\lambda E - A|$ of a matrix A in terms of powers of λ

$$|\lambda E - A| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0,$$

the coefficient a_k equals the sum of all principal minors of order $n-k$ of the matrix A multiplied by $(-1)^{n-k}$.

Set up the characteristic polynomials of the matrices

6.2.3. $\begin{Vmatrix} x_1y_1 & x_1y_2 & \ldots & x_1y_n \\ x_2y_1 & x_2y_2 & \ldots & x_2y_n \\ \cdot & \cdot & \cdot & \cdot \\ x_ny_1 & x_ny_2 & \ldots & x_ny_n \end{Vmatrix}.$ **6.2.4.** $\begin{Vmatrix} 0 & 0 & \ldots & 0 & b_1 \\ 0 & 0 & \ldots & 0 & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & b_{n-1} \\ c_1 & c_2 & \ldots & c_{n-1} & a \end{Vmatrix}.$

6.2.5. Prove that the characteristic polynomial of the transpose A^T of a matrix A coincides with the characteristic polynomial of the matrix A.

6.2.6. Prove that if each coefficient of a complex matrix A is replaced by its conjugate, then the coefficients of the characteristic polynomial are also replaced by their conjugates.

6.2.7*. Given that A and B are square matrices of the same order, prove that the matrices AB and BA have the same characteristic polynomial.

6.2.8. Prove that the characteristic polynomial $f(\lambda)$ of a matrix A and that $g(\lambda)$ of the matrix $A - \lambda_0 E$ are related by the formula

$$g(\lambda) = f(\lambda + \lambda_0).$$

6.2.9. Let an $n \times n$ matrix A be nondegenerate. Prove that the characteristic polynomial $f(\lambda)$ of the matrix A is related to the characteristic polynomial $h(\lambda)$ of the matrix A^{-1} by the formula

$$h(\lambda) = (-\lambda)^n \frac{1}{|A|} \cdot f\left(\frac{1}{\lambda}\right).$$

Hence deduce the relationship between the sums of all principal minors of a given order of the matrices A and A^{-1}. (Another method of stating this relationship is given in Problem 5.5.80.)

6.2.10. Prove that similar matrices possess the same characteristic polynomial. Give an example demonstrating that the converse statement, viz., matrices having the same characteristic polynomial are similar, does not hold.

6.2.11*. Prove that the following function in the elements of a matrix A

$$m(A) = \sum_{i,j=1}^{n} a_{ij}a_{ji}$$

is unaltered when the similarity transformation is applied to the matrix.

6.2.12. Assuming that the matrix A (see Problem 6.2.11) is complex, write an expression for the function $m(A)$ in terms of the eigenvalues of this matrix.

6.2.13. Generalizing the statement of Problem 6.2.11, prove that the function

$$m_l(A)=\sum_{i=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\cdots\sum_{k_l=1}^{n} a_{ik_1}a_{k_1k_2}\cdots a_{k_{l-1}k_l}a_{k_li}$$

is unaltered when the similarity transformation is applied to the matrix A.

6.2.14. If n eigenvalues $\lambda_1, \ldots, \lambda_n$ of a matrix A of order $n+1$ are given, how can another eigenvalue λ_{n+1} be found?

6.2.15. Find the characteristic polynomial and eigenvalues of the triangular matrix

$$\begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & a_{nn} \end{Vmatrix}.$$

6.2.16. Prove that the characteristic polynomial of the matrix

$$C(f(\lambda))=\begin{Vmatrix} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 & 0 \end{Vmatrix}$$

is equal to $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$. The matrix $C(f(\lambda))$ is called the *companion of the polynomial* $f(\lambda)$ (or the *Frobenius matrix*).

6.2.17. Use the result of Problem 6.2.16 to prove that any n-degree polynomial with the higher-order coefficient equal to unity can be the characteristic polynomial of a certain square n-order matrix.

6.2.18. Find the characteristic polynomial for the operator that rotates the Euclidean plane through an angle α.

6.2.19. Find the characteristic polynomial for the operator A of the three-dimensional Euclidean space such that $Ax = [x, a]$, where a is a fixed vector.

6.2.20. Find the characteristic polynomial of the differential operator on the space M_n.

6.2.21. Find the characteristic polynomial of an arbitrary nilpotent operator on the n-dimensional complex space.

6.2.22. Prove that the rank of a projection operator equals its trace.

6.2.23. Let an operator R reflect an n-dimensional space X in a subspace L. Prove that the dimension of L is related to the trace of the operator R by the following:

$$\operatorname{tr} R = 2 \dim L - n.$$

Evaluate the eigenvalues and eigenvectors of the following matrices:

6.2.24. $\begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix}$.

6.2.25. $\begin{Vmatrix} 3+i & -i \\ 2i & 1-i \end{Vmatrix}$.

6.2.26. $\begin{Vmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{Vmatrix}$.

6.2.27. $\begin{Vmatrix} 4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{Vmatrix}$.

6.2.28. $\begin{Vmatrix} 2 & -5 & -3 \\ -1 & -2 & -3 \\ 3 & 15 & 12 \end{Vmatrix}$.

6.2.29. $\begin{Vmatrix} 4 & -4 & 2 \\ 2 & -2 & 1 \\ -4 & 4 & -2 \end{Vmatrix}$.

6.2.30. $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$.

6.2.31. $\begin{Vmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{Vmatrix}$.

6.2.32. $\begin{Vmatrix} 1 & 2 & 0 & 3 \\ -1 & -2 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \end{Vmatrix}$.

6.2.33. $\begin{Vmatrix} 3 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 \end{Vmatrix}$.

6.2.34. Prove that any operator of a real space of dimension $n = 2k + 1$ has at least one eigenvector.

Find the eigenvalues of the following matrices (a) in the field of real numbers; (b) in the field of complex numbers.

6.2.35. $\begin{Vmatrix} 1 & 2 \\ -2 & 1 \end{Vmatrix}$.

6.2.36. $\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{Vmatrix}$.

6.2.37. $\begin{Vmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 0 \\ 0 & -1 & 0 & 3 \end{Vmatrix}$.

6.2.38. $\begin{Vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{Vmatrix}$.

6.2.39. Show that the characteristic polynomial of a quasi-triangular (quasi-diagonal) matrix equals the product of the characteristic polynomials of the diagonal blocks.

6.2.40. Using the results of Problems 6.2.8 and 6.2.9, show that the algebraic multiplicities of corresponding eigenvalues of the operators A and $A - \lambda_0 E$ are equal; the same is true for the corresponding eigenvalues of the operators A and A^{-1}.

6.2.41*. Prove that the geometric multiplicity of any eigenvalue λ of an arbitrary operator A does not exceed its algebraic multiplicity.

6.2.42. Prove that an operator A on a complex space is of simple structure if and only if the geometric multiplicity of each eigenvalue of this operator coincides with the algebraic multiplicity. Is the similar statement valid for a real space?

Determine if each of the following matrices is of simple structure. If so, find a matrix that reduces the given one to diagonal form and give the diagonal matrix.

6.2.43. $\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{Vmatrix}.$ **6.2.44.** $\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{Vmatrix}.$

6.2.45. $\begin{Vmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 2 \end{Vmatrix}.$ **6.2.46.** $\begin{Vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{Vmatrix}.$

6.2.47. $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 1 & 7 & -1 \end{Vmatrix}.$ **6.2.48.** $\begin{Vmatrix} 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{Vmatrix}.$

6.2.49*. Can the companion matrix of a polynomial $f(\lambda)$ be of simple structure if this polynomial has at least one multiple root?

6.2.50. Prove that matrices A and B of simple structure are similar if and only if they have the same characteristic polynomial.

6.2.51. Prove that a complex matrix with different eigenvalues is similar to the companion matrix of its characteristic polynomial.

6.2.52. Find the characteristic polynomial of the n-order matrix P

$$P = \begin{Vmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{Vmatrix}.$$

6.2.53*. Find the eigenvalues of the matrix P (see the previous problem) in the field of complex numbers, and the associated eigenvectors.

6.2.54. Using the result of Problem 6.2.53, show that any circulant over the field of complex numbers is a matrix of simple structure. Find expressions for the eigenvalues of a circulant in terms of its elements.

6.2.55*. Let $\lambda_1, \ldots, \lambda_m$ be all the different roots of a polynomial $f(\lambda)$. Find the eigenvectors of the companion matrix to this polynomial.

6.2.56*. Deduce the result of Problem 6.2.53 from Problem 6.2.55.

6.2.57*. Let a matrix A be of simple structure. Prove that for any number λ_0, the rank of the matrix $A - \lambda_0 E$ is equal to the highest order of the nonzero principal minors in this matrix.

6.2.58. Prove that any operator of simple structure is annihilated by its characteristic polynomial.

6.2.59. Let A be an operator of simple structure on an n-dimensional space, and let $\lambda_1, \ldots, \lambda_m$ be all the different eigenvalues of the operator A. Find the minimal polynomial of this operator.

6.2.60*. Let A and B be rectangular matrices of orders $m \times n$ and $n \times m$, respectively. Prove that the characteristic polynomials of the matrices AB and BA satisfy the equality:

$$\lambda^n \mid \lambda E_m - AB \mid = \lambda^m \mid \lambda E_n - BA \mid.$$

In particular, when $m=n$ we obtain the result of Problem 6.2.7.

6.2.61*. Prove that the characteristic polynomial of the matrix M

$$M = \begin{Vmatrix} A & B \\ B & A \end{Vmatrix},$$

where A and B are square matrices of the same order, equals the product of the characteristic polynomials of the matrices $A + B$ and $A - B$.

6.2.62. Prove that on complexifying the real linear space, an operator A is transformed into the operator $\hat{A}$ with the same characteristic polynomial.

6.2.63. Show that the result of Problem 6.2.21 also holds for a nilpotent operator on the n-dimensional real space.

6.2.64. Prove that the characteristic polynomial of a real $2n$-order matrix D

$$D = \begin{Vmatrix} B & -C \\ C & B \end{Vmatrix}$$

equals the product of the characteristic polynomials of $n \times n$ complex matrices $A = B + iC$ and $\bar{A} = B - iC$.

6.3. Invariant Subspaces

The first half of this section is devoted to problems in invariant subspaces and induced operators. In the second half, we consider a theorem and its corollaries concerning the possibility of reducing the matrix of an operator to triangular form.

6.3.1. Prove that the sum and intersection of A-invariant subspaces L_1 and L_2 are also invariant with respect to the operator A.

6.3.2. Show that the kernel and image of an operator A from ω_{XX} are A-invariant.

6.3.3. Prove that if an operator A is degenerate, then any subspace containing its image is A-invariant.

6.3.4. State the geometric meaning of the one-dimensional invariant subspaces of an operator and show that in a complex space any operator has at least one one-dimensional invariant subspace.

6.3.5. What can be said about an operator A from ω_{XX} such that any subspace of the space X is A-invariant?

6.3.6*. Prove that if any subspace of dimension k (where k is a fixed natural number, $1 \leqslant k < n$) of an n-dimensional space X is A-invariant, then A is a scalar operator.

6.3.7. Prove that the span of any set of the eigenvectors of an operator A is A-invariant. In particular, eigensubspaces of the operator A are A-invariant.

6.3.8. Prove that operators A and $A - \lambda E$, where λ is any number, possess the same invariant subspaces.

6.3.9*. Show that any operator on an n-dimensional complex space has an invariant subspace of dimension $n - 1$.

6.3.10. Prove that if an operator A is nondegenerate, then both A and A^{-1} possess the same invariant subspaces.

6.3.11. Show that any A-invariant subspace is also invariant with respect to any polynomial of this operator. Is the converse statement true?

6.3.12. Prove that both the kernel and image of any polynomial $f(A)$ in an operator A are A-invariant.

6.3.13. Let operators A and B commute. Prove that the kernel and image of the operator B are A-invariant.

6.3.14. Prove that any eigensubspace of an operator A is invariant with respect to any operator commuting with A.

6.3.15. Prove that if an operator A on an n-dimensional space has n different eigenvalues, then any operator B, commuting with A, is of simple structure. Further, all the eigenvectors of the operator A are also eigenvectors of the operator B.

6.3.16. Find all A-invariant subspaces of the three-dimensional Euclidean space, where $Ax = [x, a]$ and a is a fixed vector. Determine the induced operator A/L for each invariant subspace L.

6.3.17*. Find all invariant subspaces of the differential operator on the space of polynomials M_n.

6.3.18. A space X of dimension n is the direct sum of a subspace L_1 of dimension k (> 0) and subspace L_2 of dimension $n - k$. Suppose a basis $e_1, \ldots, e_n$ for the space is selected so that the vectors $e_1, \ldots, e_k$ belong to L_1, and the vectors $e_{k+1}, \ldots, e_n$ to the subspace L_2. Represent the matrix of an operator A with respect to the basis $e_1, \ldots, e_n$ in partitioned form

$$A_e = \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix},$$

where A_{11} and A_{22} are square submatrices of orders k and $n - k$, respectively. Prove that (a) $A_{21} = 0$ if and only if L_1 is A-invariant; (b) $A_{21} = 0$ and $A_{12} = 0$ if and only if both the subspaces L_1 and L_2 are A-invariant.

6.3.19. Show that any complex n-order square matrix A is similar to a matrix B of the form

$$B = \begin{Vmatrix} b_{11} & B_{12} \\ 0 & B_{22} \end{Vmatrix},$$

where B_{22} is a submatrix of order $n - 1$. Give a method of constructing a transforming matrix P such that $B = P^{-1}AP$.

6.3.20. If an operator A is on a complex space, then prove that any A-invariant subspace contains at least one eigenvector of this operator.

6.3.21. Let L be an A-invariant subspace. Prove that (a) the characteristic polynomial of the induced operator A/L is a divisor of the characteristic polynomial of the operator A; (b) the minimal polynomial of the induced operator A/L is a divisor of the minimal polynomial of the operator A.

6.3.22. Subspaces L_1 and L_2 are invariant with respect to an operator A with $L_1 \subset L_2$. Prove that the characteristic polynomial of the operator A/L_1 is a divisor of the characteristic polynomial of the operator A/L_2. A similar statement is valid for the minimal polynomials.

6.3.23. Subspaces L_1 and L_2 are invariant with respect to an operator A. Prove that the characteristic polynomial of the operator $A/(L_1 + L_2)$ is a common multiple of, and that of the operator $A/(L_1 \cap L_2)$ is a common divisor of, the characteristic polynomials of the operators A/L_1 and A/L_2. A similar statement holds for the minimal polynomials.

6.3.24*. Prove that an operator A of simple structure induces an operator of simple structure A/L on each of its invariant subspaces L.

6.3.25. Deduce the following corollary to Problem 6.3.24: any nontrivial invariant subspace of an operator A of simple structure is spanned on a certain set of the eigenvectors of this operator.

6.3.26. Prove that for commuting operators of simple structure A and B, there exists a basis for the space consisting of the common eigenvectors of these operators.

6.3.27. Prove that any two commuting operators on a complex space have a common eigenvector.

6.3.28. Prove that for any (even infinite) set G consisting of mutually commuting operators on a complex space, there is a common eigenvector for all the operators from G.

6.3.29. An operator A is reducible by two subspaces L_1 and L_2. Prove that (a) the rank of the operator A is equal to the sum of the ranks of the operators A/L_1 and A/L_2; (b) the characteristic polynomial of the operator A equals the product of the characteristic polynomials of the operators A/L_1 and A/L_2; (c) the minimal polynomial A is the least common multiple of the minimal polynomials A/L_1 and A/L_2; (d) the operator A^k is, for any whole number, the direct sum of the operators $(A/L_1)^k$ and $(A/L_2)^k$; (e) for any polynomial $f(t)$, the operator $f(A)$ is the direct sum of the operators $f(A/L_1)$ and $f(A/L_2)$.

6.3.30. Prove that the differential operator on the space M_n cannot be reduced by any pair of subspaces.

6.3.31. Let R be a real linear space and let C be a complex space obtained from R by complexification. Let L be a subspace of R which is invariant with respect to an operator A, and let $e_1, \ldots, e_k$ be a basis for L. Show that the span of the vectors $e_1 + i0, \ldots, e_k + i0$ of the space C is $\hat{A}$-invariant, $\hat{A}$ being the operator corresponding to A.

6.3.32*. Using complexification, prove that any operator on a real linear space has an invariant subspace of dimension 1 or 2.

6.3.33. Find a two-dimensional invariant subspace of the matrix

$$\begin{Vmatrix} 4 & -6 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix},$$

considered as an operator of the real arithmetic space R_3.

6.3.34. Let n-dimensional column vectors $z_1, \ldots, z_k$, $z_j = x_j + iy_j$ form a basis for a k-dimensional subspace of the complex matrix $A = B + iC$. Prove that the $2n$-dimensional real column vectors $u_1, \ldots, u_k, v_1, \ldots, v_k$ where

$$u_j = \begin{Vmatrix} x_j \\ y_j \end{Vmatrix}, \quad v_j = \begin{Vmatrix} -y_j \\ x_j \end{Vmatrix},$$

form a basis for a $2k$-dimensional invariant subspace of the real matrix

$$D = \begin{Vmatrix} B & -C \\ C & B \end{Vmatrix}.$$

6.3.35. Vectors $e_1, \ldots, e_k$ form a basis for a k-dimensional invariant subspace of an $m \times m$ matrix A and vectors $f_1, \ldots, f_l$ form a basis for an l-dimensional invariant subspace of an $n \times n$ matrix B. Prove that in the following matrices the span of the Kronecker products $e_i \times f_j$, $i = 1, \ldots, k$, $j = 1, \ldots, l$, is a kl-dimensional invariant subspace: (a) of the matrix $A \times B$; (b) of the matrix $A \times E_n + E_m \times B$.

6.3.36*. Prove that for any operator A on an n-dimensional complex space X, there is a sequence of invariant subspaces L_1, L_2, . . ., L_{n-1}, L_n such that the dimension of the subspace L_k equals k and

$$L_1 \subset L_2 \subset \ldots \subset L_{n-1} \subset L_n = X.$$

Show that the matrix of the operator A, with respect to a basis for the space having $e_i \in L_i$, is upper triangular. However, if the order of the basis vectors is reversed to e_n, . . ., e_1, then the matrix of the operator assumes lower triangular form. State the meaning of the diagonal elements of these matrices.

6.3.37. Deduce the following corollary to Problem 6.3.36: any complex square matrix is similar to an upper (lower) triangular matrix.

6.3.38*. Prove the statement of the previous problem without using the result of Problem 6.3.36.

6.3.39. Prove that triangular form of a given complex matrix A is not unique, viz., each order of the eigenvalues λ_1, . . ., λ_n of the matrix A has a corresponding upper (lower) triangular matrix similar to A in which the elements λ_1, . . ., λ_n on the principal diagonal are arranged in the required order.

Reduce the following matrices to triangular form by a similarity transformation (indicate the obtained triangular forms and the transforming matrices):

6.3.40. $\left\| \begin{matrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{matrix} \right\|$. **6.3.41.** $\left\| \begin{matrix} 5 & 2 & 1 \\ -8 & -3 & -2 \\ 7 & 4 & 3 \end{matrix} \right\|$.

6.3.42*. Let λ_1, . . ., λ_m be all the different eigenvalues for a complex $n \times n$ matrix A, and k_1, . . ., k_m the algebraic multiplicities of these eigenvalues. Prove that the matrix A is of simple structure if and only if it is similar to a matrix B of the following block structure

$$B = \left\| \begin{matrix} \lambda_1 E_{k_1} & B_{12} & B_{13} & \ldots & B_{1m} \\ 0 & \lambda_2 E_{k_2} & B_{23} & \ldots & B_{2m} \\ 0 & 0 & \lambda_3 E_{k_3} & \ldots & B_{3m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \lambda_m E_{k_m} \end{matrix} \right\|.$$

6.3.43. Prove that an operator on a complex space is nilpotent if and only if all its eigenvalues are equal to zero.

6.3.44*. Prove that the community matrices A and B can be reduced to triangular form by the same similarity transformation.

6.3.45. What does the statement of Problem 6.3.44 mean for the commuting operators A and B?

6.3.46*. Prove that any real square matrix is similar to an upper (lower) quasitriangular matrix whose diagonal blocks are of order 1 or 2.

6.3.47. Deduce the following corollary to the result of Problem 6.3.46, viz., that any operator on an n-dimensional space has an invariant subspace of dimension $n-1$ or $n-2$.

6.3.48*. $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of a complex $m \times m$ matrix A, and $\mu_1, \ldots, \mu_n$ the eigenvalues of a complex $n \times n$ matrix B (each sequence may contain equal terms). Prove that (a) the mn products $\lambda_i\mu_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ represent collectively all the eigenvalues of the matrix $A \times B$ and the operator G_{AB} (see Problem 5.6.10); (b) the mn sums $\lambda_i + \mu_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ represent collectively all the eigenvalues of the matrix $A \times E_n + E_m \times B$ and operator F_{AB}.

6.3.49. Prove that the matrix equation

$$AX + XB = C$$

(where A, B and C are given complex matrices of orders $m \times m$, $n \times n$ and $m \times n$, respectively) has a unique solution if there are no eigenvalues λ_i, of the matrix A, and μ_j, of the matrix B, such that $\lambda_i + \mu_j = 0$.

6.3.50*. An operator A on a complex space is reducible by two subspaces L_1 and L_2, and the induced operators A/L_1 and A/L_2 have no equal eigenvalues. Prove that any operator B that commutes with A is reducible by the same two subspaces L_1 and L_2. Extend this statement to the case of any finite number of subspaces.

6.3.51. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a complex $n \times n$ matrix A (some of the numbers λ_i may be equal). Prove that all the possible products of p numbers from $\lambda_1, \ldots, \lambda_n$ represent all the eigenvalues of the p-th associated matrix A_p.

6.4. Root Subspaces and the Jordan Form

The problems in this section are grouped in the following sequence:

Root subspaces. The basic tools here are: a theorem on the decomposition of a complex space into the direct sum of the root subspaces of an operator A, and the characterization of the root subspace K_{λ_i} that corresponds to an eigenvalue λ_i, as the kernel of some power of the operator $A - \lambda_i E$. Corollaries to this theorem as well as computational problems on the construction of root subspaces are given and the concept of the height of a root vector is discussed.

The structure of a root subspace. The material is expounded by degrees, beginning with the simplest case, i.e. when the maximum height of a root vector coincides with the dimension of the root subspace. The situation then is gradually becoming more complex until the most general case is considered. At each stage we illustrate the structure of a canonical basis, and provide some computational examples. Having got an insight into the structure of an individual root subspace, we proceed to

The construction of the Jordan form of an arbitrary operator. In addition to computational problems, we also offer here a number of theoretical problems about the use of the Jordan form. In particular, the formulae which enable the Jordan form to be computed without resorting to the construction of a canonical basis are derived. The section is concluded by problems on

The relation between the similarity of matrices and the Jordan form.

Throughout this section we will consider only operators on a complex space and complex matrices unless otherwise stated.

6.4.1. Using Problems 5.3.9 and 5.3.10, prove that for any operator A on an n-dimensional, real or complex, space X, the space X can be decomposed into the direct sum of subspaces

$$X = N \dot{+} T, \tag{6.4.1}$$

where N is the kernel, and T the image of the operator A^q for some natural number q. Moreover, for the least possible q, this inequality is valid: $q \leqslant n$.

Show that the operator A induces a nilpotent operator on the subspace N, and a nondegenerate operator on the subspace T. Thus, the statement of the problem may be reformulated as follows: any operator A is the direct sum of a nilpotent and a nondegenerate operator.

6.4.2*. Prove that the decomposition of an operator A into the direct sum of nilpotent and nondegenerate operators is unique.

6.4.3. Prove that the dimension of a subspace N in the decomposition (6.4.1) equals the algebraic multiplicity of the zero eigenvalue of an operator A.

6.4.4*. Prove that for any operator A, a space X can be decomposed into the direct sum of subspaces $K_{\lambda_1}, \ldots, K_{\lambda_m}$

$$X = K_{\lambda_1} \dot{+} K_{\lambda_2} \dot{+} \ldots \dot{+} K_{\lambda_m} \tag{6.4.2}$$

(where $\lambda_1, \ldots, \lambda_m$ are all the different eigenvalues of the operator A having algebraic multiplicities $k_1, \ldots, k_m$, respectively) such that each of the subspaces K_{λ_i} is A-invariant, and the induced operator A/K_{λ_i} has the characteristic polynomial $(\lambda - \lambda_i)^{k_i}$.

6.4.5. Prove that the decomposition in (6.4.2) is unique if the operator A satisfies the conditions listed in Problem 6.4.4.

6.4.6. A subspace K_{λ_i} in the decomposition (6.4.2) is called a *root subspace* associated with an eigenvalue λ_i. Show that it follows from Problems 6.4.1-6.4.5 that (a) the subspace K_{λ_i} can be described as a set of all vectors x such that $(A - \lambda_i E)^s x = 0$, where s is any natural number; (b) the subspace K_{λ_i} can be described as the kernel of the operator $(A - \lambda_i E)^{q_i}$, where q_i is a certain natural number not exceeding k_l; (c) the eigensubspace L_{λ_l} associated with an eigenvalue λ_l, is contained in the root subspace K_{λ_l}.

6.4.7. Show that for an operator A to be of simple structure it is necessary and sufficient that each eigensubspace L_{λ_l} of this operator associated with the eigenvalue λ_l should coincide with the root subspace K_{λ_l}.

6.4.8. Prove that if K_{λ_l} is a root subspace of an operator A associated with an eigenvalue λ_l, then (a) K_{λ_l} is the root subspace of the operator $A - \lambda_0 E$ associated with the eigenvalue $\lambda_l - \lambda_0$; (b) K_{λ_l} is the root subspace of the operator A^{-1} associated with the eigenvalue $1/\lambda_l$.

6.4.9*. Prove that any root subspace of an operator A is invariant with respect to any operator B which commutes with A.

6.4.10*. Prove the *Cayley-Hamilton theorem*, viz., that any operator A is annihilated by its characteristic polynomial.

6.4.11. Prove that if an operator A of an n-dimensional space is nondegenerate, then the inverse operator A^{-1} can be represented as a polynomial of degree $n - 1$ in A.

Construct the root subspaces of the following matrices:

6.4.12. $\begin{Vmatrix} -1 & 1 & 1 \\ -3 & 2 & 2 \\ -1 & 1 & 1 \end{Vmatrix}$.

6.4.13. $\begin{Vmatrix} 1 & 1 & 0 \\ -4 & -2 & 1 \\ 4 & 1 & -2 \end{Vmatrix}$

6.4.14. $\begin{Vmatrix} 2 & 3 & 0 & 3 \\ -1 & 0 & 1 & 2 \\ 0 & 3 & 2 & 3 \\ 1 & 2 & -1 & 0 \end{Vmatrix}$.

6.4.15. $\begin{Vmatrix} 2 & -3 & 4 & -6 \\ 1 & -2 & 2 & -4 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -2 \end{Vmatrix}$.

6.4.16. Any vector of a root subspace K_{λ_l} of an operator A is called a *root vector* of this operator associated with the eigenvalue λ_l. A natural number h such that $(A - \lambda_l E)^h x = 0$, but $(A - \lambda_l E)^{h-1} x \neq 0$ is called the *height* of a root vector x from K_{λ_l}. By definition, the height of the null vector is zero.

Show that (a) the height of each vector from K_{λ_l} does not exceed the algebraic multiplicity k_l of the eigenvalue λ_l; (b) the height of an eigenvector equals 1; (c) the set H_k of all vectors from $K_{\lambda l}$ whose height does not exceed a given natural k is a subspace.

6.4.17*. Let x be a root vector of an operator A associated with an eigenvalue λ_l and with a height h (> 0). Prove that (a) the vector $(A - \lambda_l E)\, x$ has the height $h - 1$; (b) the vector $(A - \lambda_j E)\, x$, where λ_j is an eigenvalue of the operator A other than λ_l, has the height h; (c) if λ_l is a root of a polynomial $f(t)$ with a multiplicity of l, where $l \leqslant h$, then the vector $f(A)\, x$ has height $h - l$; (d) the vector

$A^{-1}x$ has height h; (e) if B is an operator that commutes with A, then the height of the vector Bx does not exceed h.

6.4.18. Show that a root vector x of an operator A is also a root vector with the same height of (a) the operator $A - \lambda_0 E$; (b) the operator A^{-1}.

6.4.19. Prove that a set of nonzero vectors from K_{λ_i}, all of whose heights are different, is linearly independent.

6.4.20. Let x be a vector from K_{λ_i} with a height h.

Show that (a) the vector set $(A - \lambda_i E)^{h-1}x$, $(A - \lambda_i E)^{h-2}x$, ..., $(A - \lambda_i E)\,x$, x is linearly independent; (b) the span of the set is A-invariant.

In Problems 6.4.21-6.4.62 the only operators of an n-dimensional space and matrices of order n that will be considered are those which have only one eigenvalue λ_0 with an algebraic multiplicity n. This condition will not be made explicit henceforward. It should be clear that all the results obtained will also be valid for an arbitrary operator that is considered only on a root subspace.

6.4.21. An operator A on an n-dimensional space X is said to be a *one-block* operator if the maximum possible height of the root vector coincides with the dimension n of the space. Prove that (a) any basis for the space X contains at least one vector of height n; (b) if x is a vector of height n, then the vector set $(A - \lambda_0 E)^{n-1}\,x$, $(A - \lambda_0 E)^{n-2}x$, . . ., $(A - \lambda_0 E)\,x$, x is a basis for the space X; (c) the matrix of the operator A with respect to this basis is a Jordan block of order n corresponding to the number λ_0. The last statement accounts for the term "one-block operator".

Thus, in the case of a one-block operator the canonical basis is the set $(A - \lambda_0 E)^{n-1}x$, . . ., $(A - \lambda_0 E)\,x$, x called the *series* constructed from the vector x, and the Jordan form consists of one block of order n.

6.4.22. Find the matrix of the operator A (see Problem 6.4.21, (b)) with respect to the basis x $(A - \lambda_0 E)\,x$, . . ., $(A - \lambda_0 E)^{n-1}x$.

Construct the canonical basis and find the Jordan form of the following matrices:

6.4.23. $\begin{Vmatrix} 11 & 4 \\ -4 & 3 \end{Vmatrix}$.

6.4.24. $\begin{Vmatrix} 5 & -9 & -4 \\ 6 & -11 & -5 \\ -7 & 13 & 6 \end{Vmatrix}$.

6.4.25. $\begin{Vmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & -1 & -1 & 1 \end{Vmatrix}$.

6.4.26. $\begin{Vmatrix} 5 & -10 & 10 & -5 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}$.

Find the Jordan form of the following matrices of order n:

6.4.27. $\begin{Vmatrix} -1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & -1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{Vmatrix}.$

6.4.28. $\begin{Vmatrix} 1 & \alpha & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \alpha & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \alpha & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & \alpha \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{Vmatrix}, \quad \alpha \neq 0.$

6.4.29. $\begin{Vmatrix} 9 & \alpha_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 9 & \alpha_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 9 & \alpha_3 & \dots & 0 & 0 \\ \cdot & \cdot & \ddots & \ddots & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 9 & \alpha_{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 9 \end{Vmatrix},$

$$\alpha_1\alpha_2 \dots \alpha_{n-1} \neq 0.$$

6.4.30. $\begin{Vmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 0 & 1 & 2 & 3 & \dots & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 \end{Vmatrix}.$ **6.4.31.** $\begin{Vmatrix} 2 & 3 & 4 & 5 & \dots & n+1 \\ 0 & 2 & 3 & 4 & \dots & n \\ 0 & 0 & 2 & 3 & \dots & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 2 \end{Vmatrix}.$

6.4.32. $\begin{Vmatrix} \alpha & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \alpha & a_{23} & \dots & a_{2n} \\ 0 & 0 & \alpha & \dots & a_{3n} \\ \cdot & \cdot & & \ddots & \cdot \\ 0 & 0 & 0 & \dots & \alpha \end{Vmatrix},$

$$a_{12}a_{23} \dots a_{n-1,\,n} \neq 0.$$

6.4.33. Find the canonical basis and Jordan form of the differential operator on the space of polynomials M.

6.4.34. Prove that if A is a one-block operator associated with an eigenvalue $\lambda_0 \neq 0$, then (a) the operator A^2; (b) the operator A^l where l is any natural number; (c) the operator A^{-1}, are also one-block.

6.4.35. Show that if A is a one-block operator associated with the zero eigenvalue, then A^2 is not a one-block operator (the dimension of the space is assumed to be greater than 1).

6.4.36. Given a one-block operator A, prove that the subspace H_k, which is the kernel of the operator $(A - \lambda_0 E)^k$, has the dimension k, $0 < k \leqslant n$.

6.4.37. Prove that a one-block operator A has no nontrivial invariant subspaces that are different from the subspaces H_k (see Problem 6.4.36).

6.4.38. Let A and B be commutative one-block operators. Prove that A-invariant and B-invariant subspaces coincide.

6.4.39. Prove that the minimal polynomial of a one-block operator A coincides with its characteristic polynomial.

6.4.40*. Let the maximum height of a vector in a space X be equal to t. Given that the vectors $x_1, \ldots, x_p$ are linearly independent and have a height t, and that the intersection of the span of the vectors $x_1, \ldots, x_p$ and the subspace H_{t-1} consists of the null vector only, prove that for any natural k, $0 < k < t$, the vectors $(A - \lambda_0 E)^k x_1, \ldots, (A - \lambda_0 E)^k x_p$ are linearly independent, and that the intersection of the span of these vectors and the subspace H_{t-k-1} consists of the null vector only (remember that the subspace H_{t-k-1} is the kernel of the operator $(A - \lambda_0 E)^k$).

6.4.41. Denote the defect of the operator $(A - \lambda_0 E)^k$ by m_k. Deduce the following inequalities from the result of Problem 6.4.40: $n - m_{t-1} = m_t - m_{t-1} \leqslant m_k - m_{k-1}$, where $0 < k < t$, $m_0 = 0$.

6.4.42*. Prove that the series, constructed in Problem 6.4.40 on the vectors $x_1, \ldots, x_p$, form a linearly independent set.

6.4.43. Show that if, in addition to the data of Problem 6.4.40, the relation $n = (n - m_{t-1})\, t$ (where n is the dimension of the space X) is valid, then (a) the series $(A - \lambda_0 E)^{t-1} x_1, \ldots, \ldots, (A - \lambda_0 E) x_1, x_1, \ldots, (A - \lambda_0 E)^{t-1} x_p, \ldots, (A - \lambda_0 E) x_p, x_p$ form a basis for the space X (we put $p = n - m_{t-1}$); (b) the matrix of the operator A with respect to this basis is of the following quasi-diagonal form

$$\begin{Vmatrix} J_1 & & 0 \\ & J_2 & \\ & \ddots & \\ 0 & & J_p \end{Vmatrix},$$

where each of the matrices $J_1, J_2, \ldots, J_p$ is a Jordan block of order t, corresponding to the number λ_0.

Thus, in the above case, the canonical basis for the operator A consists of a number of series that have the maximum possible length, and the Jordan form consists of a number of Jordan blocks of the same order.

Construct the canonical basis, and find the Jordan form of the following matrices:

6.4.44. $\begin{Vmatrix} 1 & 1 & -2 & 0 \\ 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{Vmatrix}$. **6.4.45.** $\begin{Vmatrix} 99 & 0 & 0 & 101 \\ 0 & 99 & 0 & 0 \\ 0 & 101 & 99 & 0 \\ 0 & 0 & 0 & 99 \end{Vmatrix}$.

6.4.46. $\begin{Vmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{Vmatrix}$. **6.4.47.** $\begin{Vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -5 & 0 & 0 & 2 \end{Vmatrix}$.

6.4.48. Find the canonical basis and Jordan form of the double differentiation operator on the space of polynomials M_n, assuming that $n = 2k - 1$, where k is a whole number.

6.4.49. The maximum height of a vector in a space X equals t. Linearly independent vectors $x_1, \ldots, x_p$ all have height t, and the space X is the direct sum of the subspace H_{t-1} and the span drawn on this vector set. Prove that if the following inequality holds for the numbers m_k (see Problem 6.4.41),

$$m_t - m_{t-1} < m_{t-1} - m_{t-2},$$

then (a) the series constructed on the vectors $x_1, \ldots, x_p$ do not form a basis for the space X; (b) the series constructed on the vectors $(A-\lambda_0 E)\,x_1, \ldots, (A-\lambda_0 E)\,x_{p_1}$ do not form a basis for the subspace H_{t-1}; (c) if linearly independent vectors $x_{p_1+1}, \ldots, x_{p_2}$ having height $t-1$ are such that the direct sum of the span, drawn on the vector set $(A-\lambda_0 E)\,x_1, \ldots, (A-\lambda_0 E)\,x_{p_1},\ x_{p_1+1}, \ldots, x_{p_2}$ and the subspace H_{t-2}, is the subspace H_{t-1}, then the series constructed on the vectors $x_1, \ldots, x_{p_1}, x_{p_1+1}, \ldots, x_p$ form a linearly independent set; (d) the numbers m_k satisfy the relations

$$m_{t-1} - m_{t-2} \leqslant m_k - m_{k-1},$$

where $0 < k < t-1$, $m_0 = 0$.

6.4.50. Find a relationship connecting the dimension n of the space X, the maximum height t of the vectors and the numbers m_k, which will imply that the series, constructed in Problem 6.4.49 (c), form a basis for X. Construct the Jordan form of the operator A for this case.

Construct the canonical basis and find the Jordan form of the following matrices:

6.4.51. $\left\|\begin{matrix} 4 & 1 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 4 \end{matrix}\right\|$. 6.4.52. $\left\|\begin{matrix} 5 & 1 & -1 & -1 \\ 1 & 5 & -1 & -1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{matrix}\right\|$.

6.4.53. $\left\|\begin{matrix} -2 & 0 & 3 & 4 & 5 \\ 0 & -2 & 0 & 6 & 7 \\ 0 & 0 & -2 & 0 & 8 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{matrix}\right\|$.

6.4.54. $\left\|\begin{matrix} 1 & -1 & 0 & -1 & 0 \\ 2 & -2 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -2 & 0 \\ 2 & -1 & 0 & -1 & -1 \end{matrix}\right\|$.

6.4.55. Find the canonical basis and Jordan form of the double differentiation operator on the space of polynomials M_n, assuming that $n = 2k$, where k is a whole number.

6.4.56. Show that generally a basis for a space can be made up of p_1 series with a maximum length t, $p_2 - p_1$ series of length $t - 1$, and for any $0 < k < t$, $p_{t-k+1} - p_{t-k}$ series of length k. Here

$$p_k = m_{t-k+1} - m_{t-k}.$$

Find the Jordan form of the operator for this case.

6.4.57. Deduce the following corollary to the result of Problem 6.4.56: the numbers m_k satisfy the inequalities

$$m_{r+1} - m_r \leqslant m_{s+1} - m_s$$

if $r > s$.

6.4.58. Could there be a nilpotent operator A on an 8-dimensional space such that the ranks r_k of the operators A^k form the sequence 6, 4, 3, 1, 0?

Construct the canonical basis, and find the Jordan form of the following matrices:

6.4.59. $\left\|\begin{matrix} 1 & 1 & 1 & 0 \\ -1 & 3 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{matrix}\right\|$. 6.4.60. $\left\|\begin{matrix} -3 & 1 & -3 & -2 & -2 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{matrix}\right\|$.

6.4.61. $\begin{Vmatrix} 3 & 6 & 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & -2 & -3 \end{Vmatrix}.$

6.4.62*. $\begin{Vmatrix} 2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 & 0 & -9 \\ 2 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{Vmatrix}.$

Using the procedure for constructing a canonical basis for a root subspace, described in the previous section, and also the decomposition of the space into a direct sum of root subspaces, find the canonical basis and the Jordan form of the following matrices:

6.4.63. $\begin{Vmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{Vmatrix}.$ 6.4.64. $\begin{Vmatrix} 3 & -1 & 1 \\ -2 & 4 & -2 \\ -2 & 2 & 0 \end{Vmatrix}.$

6.4.65. $\begin{Vmatrix} -4 & 4 & 2 \\ -1 & 1 & 1 \\ -5 & 4 & 3 \end{Vmatrix}.$ 6.4.66. $\begin{Vmatrix} 3 & 0 & -1 \\ -2 & 1 & 1 \\ 3 & -1 & -1 \end{Vmatrix}.$

6.4.67. $\begin{Vmatrix} 0 & 0 & -5 & 3 \\ 0 & 0 & -3 & 1 \\ -5 & 3 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{Vmatrix}.$ 6.4.68. $\begin{Vmatrix} -3 & 4 & 3 & 15 \\ -1 & 1 & 0 & 5 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 2 & 2 \end{Vmatrix}.$

6.4.69. $\begin{Vmatrix} -2 & 4 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -2 & 4 & -1 & 0 \\ 3 & -6 & 0 & -1 \end{Vmatrix}.$ 6.4.70. $\begin{Vmatrix} 4 & 1 & 1 & 1 \\ -1 & 2 & -1 & -1 \\ 6 & 1 & -1 & 1 \\ -6 & -1 & 4 & 2 \end{Vmatrix}.$

6.4.71. The vectors of the canonical basis for an operator A have been numerated in reverse order. How is the matrix of the operator altered?

6.4.72. Given the Jordan form of an operator A, find the Jordan form of the operator (a) $A - \lambda_0 E$; (b) A^{-1}.

6.4.73. Show that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues (some of which may be equal) of an operator A of an n-dimensional space, then the numbers $f(\lambda_1), \ldots, f(\lambda_n)$ are the eigenvalues of the polynomial $f(A)$.

6.4.74. Prove that any operator on a complex space is the direct sum of one-block operators.

6.4.75*. Given the Jordan form of an operator A, find the Jordan form of the operator A^2.

6.4.76. Prove that any operator on a complex space can be represented as the sum of an operator of simple structure and a nilpotent operator.

6.4.77*. Prove that a non-scalar operator A, fulfilling the condition $A^2 = E$, is a reflection operator.

6.4.78. Prove that an operator A, fulfilling the condition $A^k = E$ for a certain natural number k, is of simple structure.

6.4.79*. Prove that in any Jordan form of an operator A, the number of the Jordan blocks, corresponding to an eigenvalue λ_0, equals the defect m_1 of the operator $A - \lambda_0 E$.

6.4.80*. Prove that in any Jordan form of an operator A, the number of the Jordan blocks, corresponding to an eigenvalue λ_0 and having an order greater than or equl to k, is determined by the formula

$$S_{\geqslant k} = m_k - m_{k-1},$$

where $m_0 = 0$, and m_k is the defect of the operator $(A - \lambda_0 E)^k$.

6.4.81. Deduce from the result of Problem 6.4.80 that

$$S_k = 2m_k - m_{k+1} - m_{k-1},$$

where S_k is the number of the Jordan blocks corresponding to the eigenvalue λ_0 and having the order k.

Thus, the Jordan form of any operator is uniquely determined by the position of the Jordan blocks on the diagonal.

Without computing the canonical basis, find the Jordan form of the following matrices:

6.4.82. $\begin{Vmatrix} 3 & 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{Vmatrix}.$

6.4.83. $\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 2 & 1 \end{Vmatrix}.$

6.4.84. $\begin{Vmatrix} 5 & 0 & 6 & 7 & 9 & 14 \\ 0 & 5 & 0 & 8 & 10 & 15 \\ 0 & 0 & 5 & 0 & 11 & 16 \\ 0 & 0 & 0 & 5 & 12 & 17 \\ 0 & 0 & 0 & 0 & 13 & 18 \\ 0 & 0 & 0 & 0 & 0 & 19 \end{Vmatrix}.$

6.4.85. $\begin{Vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & -4 & 0 & 0 & 0 \\ 3 & 2 & 1 & -4 & 0 & 0 \\ -2 & 2 & 5 & 7 & -4 & 0 \\ 4 & 3 & 8 & 6 & 0 & -4 \end{Vmatrix}.$

6.4.86*. $\begin{Vmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{Vmatrix}.$

6.4.87*. $\begin{Vmatrix} 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -3 & 1 & -1 & -3 \\ 0 & 1 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{Vmatrix}.$

6.4.88. Find the Jordan form of the difference operator A_1 on the space of polynomials M_n.

6.4.89. Find the Jordan form of (a) the triple differentiation operator; (b) the operator A_1^3, where A_1 is a difference operator, on the space of polynomials M_8.

6.4.90. Show that in each class of similar matrices there is a unique Jordan form for each positioning of the diagonal Jordan blocks.

Determine whether the following matrices A, B and C are similar:

6.4.91.
$$A^* = \begin{Vmatrix} -3 & 2 & 5 \\ -12 & 8 & 20 \\ 3 & -2 & -5 \end{Vmatrix}, \quad B = \begin{Vmatrix} 59 & -63 & 52 \\ -147 & 159 & -132 \\ -244 & 263 & -219 \end{Vmatrix},$$
$$C = \begin{Vmatrix} 59 & -63 & 52 \\ -147 & 159 & -132 \\ -244 & 263 & -218 \end{Vmatrix}.$$

6.4.92.
$$A = \begin{Vmatrix} 3 & 1 & -1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{Vmatrix}, \quad B = \begin{Vmatrix} 5 & 5 & -2 \\ -2 & -1 & 1 \\ -1 & -1 & 2 \end{Vmatrix},$$
$$C = \begin{Vmatrix} 6 & 0 & 8 \\ 3 & 2 & 6 \\ -2 & 0 & -2 \end{Vmatrix}.$$

6.4.93.
$$A=\begin{Vmatrix} -1 & -1 & 2 \\ 3 & -5 & 6 \\ 2 & -2 & 2 \end{Vmatrix}, \quad B=\begin{Vmatrix} -8 & 12 & -6 \\ -10 & 18 & -10 \\ -12 & 24 & -14 \end{Vmatrix},$$
$$C=\begin{Vmatrix} 0 & 6 & 6 \\ -2 & 16 & 12 \\ 4 & -28 & -20 \end{Vmatrix}.$$

6.4.94. Prove that any complex matrix A is similar to its transpose A^T.

6.4.95. What can be said about the Jordan form of a matrix A similar to its inverse A^{-1}?

6.4.96. Prove that a Jordan block is similar to the companion matrix of its characteristic polynomial, i.e., to the Frobenius matrix.

6.4.97. Prove that any complex matrix is similar to a quasi-diagonal matrix all of whose diagonal blocks are the Frobenius matrices.

6.4.98*. Find the necessary and sufficient condition for the minimal polynomial of a complex matrix to coincide with its characteristic polynomial.

6.4.99. Let $\lambda_1, \ldots, \lambda_m$ be eigenvalues, all different, of an $n \times n$ complex matrix A. Prove that the matrix A is of simple structure if and only if the polynomial $(\lambda-\lambda_1)\ldots(\lambda-\lambda_m)$ is the minimal polynomial of A.

6.4.100*. Given an m-order matrix A of simple structure and the Jordan form J of an $n \times n$ matrix B. Find the Jordan form of the matrix (a) $A \times B$; (b) $A \times E_n + E_m \times B$.

Apply the derived results to the operators G_{AB} and F_{AB} (see Problem 5.6.10).

6.4.101. Find the Jordan form of the n-order matrix
$$A=\begin{Vmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \\ & & & & 1 \\ \varepsilon & & & & 1 \end{Vmatrix},$$
where ε is positive and positioned in $(n, 1)$, and the off-diagonal elements which are not indicated are zeroes.

6.4.102*. Replace the unit off-diagonal elements, if any, of the Jordan form of a matrix A by an arbitrary number $\varepsilon \neq 0$. Prove that the matrix obtained is similar to the matrix A.

CHAPTER 7

Unitary Space Operators

7.0. Terminology and General Notes

Assume that X and Y are two spaces that are either both Euclidean or both unitary, and consider a linear operator A from ω_{XY}. A linear operator A^* from ω_{XY} is said to be the *conjugate of the operator A* if for any two vectors $x \in X$ and $y \in Y$,

$$(Ax, y) = (x, A^*y). \tag{7.0.1}$$

Every operator A has a conjugate operator A^* which is unique.

Given a complex $m \times n$ matrix A, an $n \times m$ matrix A^* is said to be *the conjugate of the matrix A* if

$$a^*_{ij} = \bar{a}_{ji}$$

for all i, j.

The conjugate operator has a corresponding conjugate matrix and vice versa with respect to every pair of orthonormal bases for the unitary spaces X and Y. In the case of the Euclidean spaces X and Y, it can be shown that a similar relationship exists between the conjugate operators and the transposed matrices.

Consider now the operators on a unitary space X for which the following theorem is true.

The Schur theorem. *For each operator A there is an orthonormal basis for the space X with respect to which the matrix of the operator is triangular.*

A number of important classes of operators on a unitary space X can be identified, using the notion of a conjugate operator.

An operator A is said to be *normal* if

$$A^*A = AA^*. \tag{7.0.2}$$

An operator U is said to be *unitary* if

$$U^*U = UU^* = E. \tag{7.0.3}$$

An operator H is said to be *Hermitian* if

$$H^* = H. \tag{7.0.4}$$

An operator K is said to be *skew Hermitian* if

$$K^* = -K. \tag{7.0.5}$$

A Hermitian operator H is said to be *positive-semidefinite* (*positive-definite*) if for each non-zero vector x

$$(Hx, x) \geqslant 0 \ (> 0). \qquad (7.0.6)$$

We define *normal, unitary, Hermitian, skew Hermitian, positive-semidefinite*, and *positive-definite matrices* in exactly the same way and in the last two cases matrices can be identified, as usual, with operators on the arithmetic space.

The following results hold for all the above classes of operators:

An operator A is normal if and only if there exists an orthonormal basis for it containing the eigenvectors.

A normal operator A is unitary if and only if the moduli of all the eigenvalues are equal to unity.

A normal operator A is Hermitian if and only if all its eigenvalues are real.

A Hermitian operator H is positive-semidefinite (positive-definite) if and only if all its eigenvalues are nonnegative (positive).

Any operator A from ω_{XX} may be represented as

$$A = H_1 + iH_2, \qquad (7.0.7)$$

where H_1 and H_2 are Hermitian operators. This is called the *Hermitian decomposition* of operator A. Moreover,

$$H_1 = \frac{1}{2}(A + A^*), \quad H_2 = \frac{1}{2i}(A - A^*).$$

In the Euclidean space X the relations (7.0.2)-(7.0.6) also identify classes of operators which are called, respectively, *normal, orthogonal, symmetric, skew-symmetric, positive-semidefinite, positive-definite*. Matrices with the same names are defined in similar manner.

The following definitions and results are true for both unitary and Euclidean spaces.

If A is an operator with rank r from X to Y, then the nonzero eigenvalues of operators A^*A and AA^* coincide (taking their multiplicity into account) and are positive.

If n and m are the dimensions of spaces X and Y, respectively, then the multiplicity of the eigenvalue zero is equal to $n - r$ for the operator A^*A, and to $m - r$ for the operator AA^*.

Let $s = \min(n, m)$ and denote the common eigenvalues of the operators A^*A and AA^* by $\alpha_1^2, \ldots, \alpha_s^2$ ($\alpha_i \geqslant 0$). The numbers $\alpha_1, \ldots, \alpha_s$ are then called the *singular values of the operator* A. The *singular values of a matrix* are defined similarly.

In all cases orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ (if $A \in \omega_{XX}$, then $m = n$) exist for an operator A such that: (1) the vectors $e_1, \ldots, e_n$ are the eigenvectors of the operator A^*A; (2) the vectors

$f_1, \ldots, f_m$ are the eigenvectors of the operator AA^*; (3) if $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$ are associated with nonzero numbers $\alpha_1^2, \ldots, \alpha_r^2$, then

$$f_i = \frac{1}{\alpha_i} Ae_i, \quad i = 1, \ldots, r.$$

A pair of bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ that possess these properties is said to be a pair of *singular bases* for the operator A.

It is possible to represent any operator A on a space X as the product of a positive-semidefinite and unitary (orthogonal) operator:

$$A = HU. \tag{7.0.8}$$

This is called the *polar representation* of operator A.

Assume that A is an operator from ω_{XY} and b is a fixed vector in space Y. If the equality

$$Ax = b \tag{7.0.9}$$

is considered for finding vectors x from X, then the equation is consistent if and only if $b \in T_A$. Thus, the solutions of (7.0.9) are all pre-images of the vector b. If $b \notin T_A$, then it is sensible also to find the vectors x such that the vector

$$y = b - Ax$$

has the least possible length. These vectors x are called *pseudosolutions* of equation (7.0.9). The pseudosolution which has the least length is said to be the *normal pseudosolution* of the equation (7.0.9). It always exists and is unique.

By considering equation (7.0.9) for all vectors b from Y, we can match the normal pseudosolution of the corresponding equation to each vector b and thus obtain a linear operator from Y to X. This operator is called *pseudoinverse* of the operator A and is denoted by A^+.

A *quadratic form* F in n real variables $x_1, \ldots, x_n$ is a function of the form

$$F = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \tag{7.0.10}$$

where a_{ij} are real numbers; we assume that $a_{ij} = a_{ji}$.

If a symmetric matrix A of the coefficients a_{ij} (called the *matrix of a quadratic form*) and a column vector x having the variables $x_1, \ldots, x_n$ are constructed, then the definition of a quadratic form may be rewritten as

$$F = (Ax, x). \tag{7.0.11}$$

The scalar product is defined here by the familiar rule (7.1.4) and the *rank* of a quadratic form F is the rank of the matrix A.

When the variables are changed by

$$x = Py \tag{7.0.12}$$

the form F is transformed into a new quadratic form in the new variables $y_1, \ldots, y_n$ and the matrix B, defined by this form, is related to the matrix A by the relation

$$B = P^T AP. \tag{7.0.13}$$

This change of variables (7.0.12) is said to be *nondegenerate* if the matrix P is nondegenerate. The rank of a quadratic form remains unaltered by a nondegenerate transformation of its variables.

Every quadratic form F of rank r may be reduced by a nondegenerate transformation of variables to the form

$$F = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_r y_r^2, \tag{7.0.14}$$

which is called the *canonical form* of F. Here $\lambda_1, \lambda_2, \ldots, \lambda_r$ are all nonzero.

Generally speaking, the canonical form of a given quadratic form is not uniquely defined. In particular, it is always possible to make nonzero coefficients equal to 1 or -1. Such a canonical form is called the *normal form* of the quadratic form and despite the ambiguity of the canonical form the following statement remains valid.

The law of inertia of quadratic forms. The number of positive (and negative) coefficients among $\lambda_1, \ldots, \lambda_r$ is the same in each canonical form to which a given quadratic form may be reduced by a nondegenerate transformation of variables.

The above-mentioned numbers are called, respectively, the *positive and negative indices of inertia*, and their difference is said to be the *signature* of the quadratic form.

Note that each quadratic form F may be reduced to the canonical form by an *orthogonal* transformation of variables (a transformation defined by an orthogonal coefficient matrix). For this it is sufficient to substitute an orthogonal matrix, whose columns are the eigenvectors of the matrix A, for P in (7.0.12). The coefficients of the canonical form then obtained are the eigenvalues of A.

A quadratic form (7.0.11) is called *positive-definite* if

$$(Ax, x) > 0$$

and when $x \neq 0$. A positive-definite form F becomes normal when

$$F = y_1^2 + y_2^2 + \ldots + y_n^2. \tag{7.0.15}$$

Two quadratic forms F and G in the same variables may be reduced to a canonical form by one transformation if at least one of the forms (F, for example) is positive-definite. If in this case the transformation $x = Py$, which reduces the form F to a normal form

(7.0.15), is carried out first, then the form G is turned into some form in the variables $y_1, \ldots, y_n$. Next the orthogonal transformation $y = Qz$, which reduces G to a canonical form, is carried out; the form F still retains its normal form and is changed into $F = z_1^2 + z_2^2 + \ldots + z_n^2$.

Note in conclusion that the symbols $e^{i\varphi}$ are used in this chapter as a contracted way of writing the complex number $z = \cos\psi + i\sin\psi$.

7.1. Conjugate Operator. Conjugate Matrix

In this section the following topics are considered:

The definition and algebraic properties of conjugate operators and conjugate matrices.

Examples of conjugate operators.

The relation between conjugate operators and conjugate matrices with respect to orthonormal bases for the space.

Relationships between the geometric characteristics of an operator A and the conjugate operator A^* such as the kernel, image, eigenvalues, etc.

It is stressed throughout that this property (two operators being conjugate) depends on the definition of the scalar product for a given linear space.

7.1.1. Deduce the following properties from the definition of a conjugate operator:

(i) $(A^*)^* = A$;

(ii) $(A + B)^* = A^* + B^*$;

(iii) $0^* = 0$;

(iv) $(\alpha A)^* = \bar{\alpha} A^*$;

(v) $(AB^*) = B^*A^*$;

(vi) $E^* = E$;

(vii) if an operator A is nondegenerate, then $(A^{-1})^* = (A^*)^{-1}$;

(viii) $(A^m)^* = (A^*)^m$ for any whole nonnegative m;

(ix) if an operator A is nondegenerate, then the previous property is true for any whole number m;

(x) if $f(t) = a_0 + a_1 t + \ldots + a_m t^m$ is an arbitrary polynomial, then

$$[f(A)]^* = \bar{f}(A^*),$$

where $\bar{f}(t) = \bar{a}_0 + \bar{a}_1 t + \ldots + \bar{a}_m t^m$.

7.1.2. Prove that the properties listed in the previous problem also hold for conjugate matrices.

7.1.3. Show that for a nilpotent operator A with the nilpotence index q, the conjugate operator A^* is also nilpotent and has the same nilpotence index.

7.1.4. Show that if operators A and B commute, then the conjugate operators A^* and B^* also commute.

7.1.5. Two bases $e_1, \ldots, e_n$ and $q_1, \ldots, q_m$ for unitary (Euclidean) spaces X and Y, respectively, are held fixed. Assume that the following relations are valid for linear operators A and B

$$(Ae_i, q_j) = (e_i, Bq_j), \ i = 1, \ldots, n; \ j = 1, \ldots, m.$$

Prove that in this case $A^* = B$.

7.1.6. Let $e_1, \ldots, e_n$ be an orthogonal (but not orthonormal!) basis for a space X. Find the relationship between the matrices defined by an operator A from ω_{XX} with respect to this basis and the matrices of the conjugate operator A^*.

7.1.7*. Let an operator A be defined by a matrix A_e with respect to a certain basis $e_1, \ldots, e_n$ for a unitary (Euclidean) space X. Prove that with respect to the basis $f_1, \ldots, f_n$, which is biorthogonal to the basis $e_1, \ldots, e_n$, the conjugate operator A^* is defined by the conjugate matrix $(A_e)^*$.

7.1.8. If there is an operator A on the one-dimensional unitary (Euclidean) space, what does the transformation A^*, which is conjugate of A, constitute?

7.1.9. Find the conjugate operator of the rotation of the Euclidean plane through an angle α.

7.1.10*. Find the conjugate operator for the operator of the Euclidean three-dimensional space $Ax = [x, a]$, where a is a fixed vector.

7.1.11. The scalar product is given on the space of polynomials M_2 by the formula:

$$(f, g) = a_0b_0 + a_1b_1 + a_2b_2, \qquad (7.1.1)$$

where $f(t) = a_0 + a_1t + a_2t^2$, $g(t) = b_0 + b_1t + b_2t^2$. Find all the matrices of the differential operator A and the conjugate operator A^* with respect to the basis: (a) $1, t, t^2$; (b) $\frac{1}{2}t^2 - \frac{1}{2}t, t^2 - 1, \frac{1}{2}t^2 + \frac{1}{2}t$; (c) $1, t, \frac{3}{2}t^2 - \frac{1}{2}$.

7.1.12. On the space M_2 the scalar product is defined by:

$$(f, g) = f(-1)\,g(-1) + f(0)\,g(0) + f(1)\,g(1). \qquad (7.1.2)$$

Find the matrix defined by the operator, which is conjugate to the differential operator, with respect to each of the bases listed in Problem 7.1.11. Compare the matrices obtained with the corresponding matrices of Problem 7.1.11.

7.1.13. The scalar product on the space M_2 is given by the formula:

$$(f, g) = \int_{-1}^{1} f(t)\,g(t)\,dt. \qquad (7.1.3)$$

Find the matrix of the operator which is conjugate to the differential operator with respect to each of the bases listed in Problem 7.1.11.

7.1.14. On the n-dimensional arithmetic space, whose elements are the column vectors, the natural scalar product is defined by:

$$(x, y) = \alpha_1\overline{\beta}_1 + \ldots + \alpha_n\overline{\beta}_n. \qquad (7.1.4)$$

Here

$$x = \begin{Vmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{Vmatrix}, \quad y = \begin{Vmatrix} \beta_1 \\ \vdots \\ \beta_n \end{Vmatrix}$$

(the sign of complex conjugation is omitted in the real case).

Show that if $n \times n$ matrices are identified with operators on this space as in Problem 5.6.7, then the conjugate operator of the matrix A is: (a) the transpose of the matrix A^T in the case of the real space R_n; (b) the conjugate matrix A^* in the case of the complex space C_n.

7.1.15. Show that in the case of the Kronecker product $A \times B$, the conjugate matrix is of the form $A^* \times B^*$.

7.1.16. Prove that if A is a square matrix, then for associated matrices the following relationship holds true

$$(A^*)_p = (A_p)^*.$$

7.1.17. Denote the spaces of real and of complex $n \times n$ matrices by $R_{n\times n}$ and $C_{n\times n}$, respectively, for which the scalar product is given by the formula

$$(A, B) = \sum_{i, j=1}^{n} a_{ij}\overline{b}_{ij}. \qquad (7.1.5)$$

(In the real case, the sign of the complex conjugation is omitted).

Show that

$$(A, B) = \mathrm{tr}\,(B^*A) = \mathrm{tr}\,(AB^*). \qquad (7.1.6)$$

7.1.18. Show that on the spaces $R_{n\times n}$ and $C_{n\times n}$ the conjugates to the operators $G_{A\,B}$ and $F_{A\,B}$ defined in Problem 5.6.10 are the operators $G_{A^*\,B^*}$ and $F_{A^*\,B^*}$.

7.1.19. Let $A_1, \ldots, A_n$ be fixed real $n \times n$ matrices. Consider the following operator A from R_n to $R_{n\times n}$:

$$x = \begin{Vmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{Vmatrix} \xrightarrow{A} Ax = \alpha_1 A_1 + \ldots + \alpha_n A_n.$$

The scalar product is defined according to (7.1.4). Show that the conjugate of an operator A is the operator

$$B \to y = \begin{Vmatrix} \beta_1 \\ \vdots \\ \beta_n \end{Vmatrix}, \quad \beta_i = \operatorname{tr}(B^T A_i) = \operatorname{tr}(A_i^T B), \quad i = 1, \ldots, n.$$

Extend this result to the complex case.

7.1.20. Show that each linear functional $f(x)$ on a unitary (Euclidean) space X may be defined as the scalar product

$$f(x) = (x, f),$$

where f is a certain vector of the space being held constant for the given functional.

7.1.21. Show that the conjugate to a projection operator is also a projection operator.

7.1.22. Show that the conjugate to a reflection operator is also a reflection operator.

7.1.23. Show that the rank of a conjugate operator A^* equals the rank of the operator A.

7.1.24. Prove that the kernel of an operator A^* coincides with the orthogonal complement of the image of the operator A.

7.1.25. In the three-dimensional Euclidean space, a Cartesian system of coordinates $Oxyz$ is fixed. Let A be the projection operator on the coordinate plane and parallel to the straight line determined by the equations $x = y = z$. Find the conjugate operator A^*.

7.1.26. Find the kernel and image of the operator in the M_2-space which is conjugate to the differential operator, if the scalar product for M_2 is given by the formula: (a) (7.1.1); (b) (7.1.2); (c) (7.1.3).

7.1.27. Prove the *Fredholm theorem*: a non-homogeneous system of linear equations $Ax = b$ is consistent if and only if the column vector b is orthogonal to all the solutions of the conjugate homogeneous system $A^*y = 0$ (cf. 4.5.3).

7.1.28. Prove the following *Fredholm alternative*: either a system of equations $Ax = b$ is consistent, whatever the right-hand side b is, or the conjugate homogeneous system $A^*y = 0$ has nonzero solutions.

7.1.29. Prove that the kernel of an operator A^*A coincides with the kernel of the operator A.

7.1.30. Prove that the image of an operator A^*A coincides with the image of the operator A^*.

7.1.31. Let operators A and B satisfy the equality $B^*A = 0$. Prove that the images of these operators are orthogonal subspaces.

7.1.32*. Prove that if $AB^* = 0$ and $B^*A = 0$, then the rank of the operator $A + B$ equals the sum of the ranks of the operators A and B. Moreover, the kernel of the operator $A + B$ is the intersection of the kernels of the operators A and B.

7.1.33. Prove that if a subspace L of a unitary (Euclidean) space is A-invariant, then its orthogonal complement $L^{\perp}$ is invariant under the conjugate operator A^*.

7.1.34*. On the space M_n of polynomials of degree $\leqslant n$, the scalar product is given by the formula

$$(f, g) = \sum_{i=1}^{n} a_i b_i, \qquad (7.1.7)$$

where $f(t) = a_0 + a_1 t + \ldots + a_n t^n$, $g(t) = b_0 + b_1 t + \ldots + b_n t^n$. Describe all the invariant subspaces of the operator conjugate to the differential operator.

7.1.35. The scalar product for M_n is determined by the formula:

$$(f, g) = \sum_{k=0}^{n} f(k)\, g(k). \qquad (7.1.8)$$

Find the n-dimensional invariant subspace of the operator which is conjugate to the differential operator.

7.1.36. Solve a similar problem for the case when the scalar product for M_n is defined by the formula

$$(f, g) = \int_{-1}^{1} f(t)\, g(t)\, dt. \qquad (7.1.9)$$

7.1.37. Prove that in the unitary space of dimension n, each operator has: (a) an invariant subspace of dimension $n - 1$; (b) an invariant subspace of dimension k, $0 < k < n$ (cf. 6.3.9 and 6.3.36).

7.1.38. Prove the following Schur theorem: for each operator A on a unitary space there exists an orthonormal basis with respect to which the matrix of the operator A is triangular (cf. 6.3.36).

7.1.39. Find the Schur basis for the differential operator in the space M_2 if the scalar product of M_2 is determined by the formula: (a) (7.1.1); (b) (7.1.2); (c) (7.1.3).

7.1.40*. Prove that commuting operators A and B on a unitary space have a common Schur basis with respect to which the matrices, defined by these operators, are triangular and have the same form.

7.1.41. Find the relation between the eigenvalues of an operator A on a unitary space and the conjugate operator A^*.

7.1.42. Let x be an eigenvector common to the conjugate operators A and A^*. Prove that the eigenvalues λ and μ of the operators A and A^*, and associated with the vector x, are conjugate numbers.

7.1.43. Let x be the eigenvector of an operator A, associated with an eigenvalue λ; y is the eigenvector of the operator A^*, associated with an eigenvalue μ, with $\mu \neq \bar{\lambda}$. Prove that the vectors x and y are orthogonal.

7.1.44*. Let K_λ and K_μ^* be root subspaces of operators A and A^*, associated with eigenvalues λ and μ, respectively, with $\mu \neq \bar{\lambda}$. Prove that the subspaces K_λ and K_μ^* are orthogonal.

7.1.45. How are the Jordan forms of the conjugate operators A and A^* related?

7.1.46. Find the Jordan canonical bases for the differential operator and its conjugate in the polynomial space M_2, with the scalar product introduced as in (7.1.1).

7.1.47*. Prove that the Schur basis for an operator A is defined ambiguously. Namely, for each sequence of the operator's eigenvalues $\lambda_1, \ldots, \lambda_n$, there is an orthonormal basis for the unitary space with respect to which the matrix determining this operator not only is upper (lower) triangular, but also has eigenvalues λ_i positioned on the main diagonal in the original sequence.

7.2. Normal Operators and Matrices

We discuss here various properties of normal operators and normal matrices. The most important of these is, certainly, the existence of an orthonormal basis for these operators and matrices that is made up of the eigenvectors. The greater part of the problems are devoted to just this fact. Further, we wished to illustrate the following important statement: of all operators of simple structure, the normal operators are peculiar with respect to the scalar product defined on the space, since the basis for these, being made up of the eigenvectors, is orthogonal, and not merely arbitrary. If, however, the scalar product on this space is defined in another way then those linear operators that were normal would cease to be such, generally speaking, and conversely, another subset of the operators of simple structure would become the class of normal operators.

7.2.1. Show that any scalar operator of a unitary (Euclidean) space is normal.

7.2.2. Show that if A is a normal operator, then the following operators are also normal: (a) αA, where α is any number; (b) A^k, where k is any natural number; (c) $f(A)$, where $f(t)$ is an arbitrary polynomial; (d) A^{-1}, if A is nondegenerate; (e) A^*.

7.2.3. Give examples demonstrating that the sum $A + B$ and product AB of normal operators A and B are not, generally speaking, normal operators either.

7.2.4. Show that the matrix of a normal operator with respect to any orthonormal basis is also normal. Conversely, any normal matrix defines a normal operator with respect to that basis.

7.2.5. Give examples demonstrating that the matrix of a normal operator with respect to a non-orthogonal basis (a) may prove not to be normal; (b) may be normal.

7.2.6. Show that any linear operator in a one-dimensional unitary (Euclidean) space is a normal operator.

7.2.7. Show that rotation operator on the Euclidean plane is a normal operator.

7.2.8. Show that an operator on the three-dimensional Euclidean space, such that $Ax = [x, a]$, is normal.

7.2.9. Show that the following operators on the space M_n of polynomials with the scalar product (7.1.7) are normal:

(a) $f(t) \to f(-t)$;

(b) $f(t) \to t^n f(1/t)$.

7.2.10. Prove that any circulant is a normal matrix.

7.2.11. Let $A = B + iC$ be a normal complex matrix of order n. Prove that the real matrix D of order $2n$

$$D = \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \tag{7.2.1}$$

is also normal.

7.2.12. Prove that if the rows and columns of a normal matrix are considered as vectors of the arithmetic space with the natural scalar product (7.1.4), then (a) the length of the i-th row equals the length of the i-th column; (b) the scalar product of the i-th and j-th rows equals the scalar product of the j-th and i-th columns (in the indicated order).

7.2.13. Prove that a quasi-triangular normal matrix is necessarily quasi-diagonal.

7.2.14. Prove that if A is a normal matrix, then the associated matrix A_p is also normal.

7.2.15. Prove that the sum of the squares of the moduli of all minors of order k, selected from the rows $i_1, \ldots, i_k$ of a normal matrix A, equals a similar sum of the minors selected from the columns with the same indices.

7.2.16. Prove that the Kronecker product of normal matrices A and B (perhaps, of different orders) is also a normal matrix.

7.2.17. Let A and B be normal matrices of order $n \times n$. Prove that the operators G_{AB} and F_{AB} (see Problem 5.6.10) are normal operators on the space $C_{n\times n}$ ($R_{n\times n}$).

7.2.18. Prove that if A is a normal operator, then for any vector x, the following equality is true

$$|Ax| = |A^*x|. \tag{7.2.2}$$

7.2.19. Prove that the kernel of a normal operator is the orthogonal complement to its image.

7.2.20*. Prove the following statement: for an operator A on a unitary space to be normal, it is necessary and sufficient that the image and kernel of the operator $A - \lambda E$, where λ is any number, should be orthogonal. Is a similar statement valid for a Euclidean space?

7.2.21. Prove that a projection operator P is normal if and only if the image and kernel of this operator are orthogonal. If this is the case, the operator P is called an *operator of orthogonal projection.*

7.2.22. Let A and B be normal operators, and $AB = 0$. Do these conditions imply the equality $BA = 0$?

7.2.23. Prove that any eigenvector of a normal operator A is also an eigenvector of the conjugate operator A^*.

7.2.24*. Prove the statement converse to that in Problem 7.2.23: if each eigenvector of an operator A on a unitary space is also an eigenvector of the conjugate operator A^* the operator A is normal.

7.2.25*. Prove that each invariant subspace of a normal operator A is also invariant with respect to A^*.

7.2.26. Prove that an operator, induced on an arbitrary invariant subspace by a normal operator, is also normal.

7.2.27. Show that the eigensubspaces of a normal operator are orthogonal to one another.

7.2.28. Prove that the operator R of reflection in L_1 parallel to L_2 is normal if and only if the subspaces L_1 and L_2 are orthogonal. In this case R is called an *orthogonal reflection* operator.

7.2.29. Can a normal operator have a nonorthogonal basis made up of the eigenvectors?

Verify that the matrices, indicated below, are normal and find, for each of them, an orthonormal (in the sense of (7.1.4)) basis of eigenvectors:

7.2.30. $\left\|\begin{matrix} 1 & i \\ i & 1 \end{matrix}\right\|$. 7.2.31. $\left\|\begin{matrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{matrix}\right\|$.

7.2.32*. $\left\|\begin{matrix} 2-i & -1 & 0 \\ -1 & 1-i & 1 \\ 0 & 1 & 2-i \end{matrix}\right\|$. 7.2.33. $\left\|\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{matrix}\right\|$.

7.2.34. Can the scalar product be defined on the space M_n $(n \geqslant 1)$ so that the differential operator is normal?

7.2.35. An operator on the space of polynomials M_n $(n \geqslant 1)$, $f(t) \to f(t+a)$ (where a is a certain fixed number), is considered. Can the scalar product on M_n be defined so that this operator is normal?

7.2.36. Let X be an arbitrary linear space. Prove that for any operator A of simple structure on X the scalar product on X may be defined so that A is normal.

7.2.37*. An operator A on the arithmetic space R_3 has the matrix

$$\left\|\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{matrix}\right\|$$

with respect to the standard basis. Define the scalar product on R_3 so that the operator A is a normal operator.

7.2.38*. Prove that an operator A is a normal operator if and only if the conjugate operator A^* can be represented by a polynomial of A.

7.2.39. Let A be a normal operator, and let A commute with some operator B. Prove that (a) A^* commutes with B; (b) A commutes with B^*.

7.2.40. Prove that the commuting normal operators A and B have a common orthonormal basis composed of their eigenvectors.

Verify that the matrices A and B, indicated below, are normal and commuting and construct a common orthonormal basis of their eigenvectors for them:

7.2.41. $$A=\begin{Vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{Vmatrix}, \quad B=\begin{Vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{Vmatrix}.$$

7.2.42. $$A=\begin{Vmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{Vmatrix}, \quad B=\begin{Vmatrix} \frac{1+5i}{6} & \frac{-1+i}{3} & \frac{1-i}{6} \\ \frac{-1+i}{3} & \frac{2+i}{3} & \frac{-1+i}{3} \\ \frac{1-i}{6} & \frac{-1+i}{3} & \frac{1-5i}{6} \end{Vmatrix}.$$

7.2.43. Prove that the operators $A + B$, AB, and BA (see Problem 7.2.40) are normal as well as the operators A and B.

7.2.44*. Prove the following statement which contrasts slightly to that of Problem 7.2.43: if A, B, and AB are normal operators, and at least one of the operators A or B has not only simple but different in modulus eigenvalues then A and B are commuting.

7.2.45*. Prove the following strong version of the statement of Problem 7.2.44, viz. that if A, B, and AB are normal operators, and at least one of the operators A or B has no different eigenvalues with equal moduli, then A and B are commuting.

7.2.46. Give an example of normal operators A and B for which the operators AB and BA are normal and different.

7.2.47. The maximum value of the moduli of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of an operator A

$$\rho(A) = \max_i |\lambda_i|$$

is called its *spectral radius*. Prove the following estimate of the extreme value for the spectral radius of a normal operator A:

$$\rho(A) = \max_{x \neq 0} \frac{|(Ax, x)|}{(x, x)}.$$

What can be said about vectors for which such a maximum occurs?

7.2.48. Prove that the following estimate of the spectral radius of a normal $n \times n$ matrix A is valid

$$\rho(A) \geqslant \frac{1}{n} \left| \sum_{i,\, j=1}^{n} a_{ij} \right|.$$

7.2.49. Prove that for the spectral radius of a normal operator A the formula is valid

$$\rho(A) = \max_{x \neq 0} \frac{|Ax|}{|x|}.$$

Is it correct to say that each vector x, for which the indicated equality occurs, is an eigenvector of the operator A?

7.2.50*. Let R be a Euclidean space, C a unitary space obtained from R by complexification (see Problem 2.5.14). Show that the correspondence between the operators A of the space R and the operators $\hat{A}$ of the space C (see Problem 5.1.52): (a) assigns the conjugate operator $\hat{A}^*$ to the conjugate operator A^*; (b) assigns the normal operator $\hat{A}$ to a normal operator A.

Using (b), show that if λ is an eigenvalue for a normal operator A, then its geometric and algebraic multiplicities coincide.

7.3. Unitary Operators and Matrices

The first part of the section is devoted to unitary operators. We shall be especially concerned with the following two of their properties: the spectral characteristic (unitary operators are normal operators all of whose eigenvalues have unit moduli), and the preservation of the scalar product.

In the second part of the section we consider unitary matrices. Having discussed their formal properties, we introduce the notion of unitarily similar matrices and formulate the matrix analogues of a number of propositions that have already been proved for the operators. Finally, we demonstrate some important computational applications of certain unitary matrices of special form.

7.3.1. Show that the set of all unitary operators from ω_{XX} forms a group under multiplication.

7.3.2. Show that the sum of unitary operators is not, generally speaking, a unitary operator.

7.3.3. Show that the product of a unitary operator by a number α is a unitary operator if and only if $|\alpha| = 1$.

7.3.4. Describe all the unitary operators on a one-dimensional space.

7.3.5. Show that an operator that rotates the Euclidean plane is an orthogonal operator.

7.3.6. Is the operator $Ax = [x, a]$ on a three-dimensional Euclidean space orthogonal?

7.3.7. Show that the operators of Problem 7.2.9 are orthogonal.

7.3.8. Let the scalar product on the space M_n $(n \geqslant 1)$ be defined by formula (7.1.9). Are the operators mentioned in Problem 7.2.9 orthogonal on such a Euclidean space?

7.3.9. Let A be a normal operator on a three-dimensional unitary space. Prove that if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of this operator considered as points in the complex plane are not in the same straight line, then the operator A can be represented in the form

$$A = aE + \rho U,$$

where U is a unitary operator and a is a complex number, $\rho > 0$.

7.3.10. Can a projection operator be unitary?

7.3.11. Show that an orthogonal reflection operator is a unitary operator.

7.3.12. Show that the operators mentioned in Problem 7.2.9 are orthogonal reflection operators. Find the eigensubspaces of each of them.

7.3.13*. An operator A on the space M_2 has the matrix

$$\left\| \begin{matrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{matrix} \right\|$$

with respect to the basis $1, t, t^2$. Show that A is a reflection operator. Define the scalar product on M_2 so that A becomes an orthogonal operator.

7.3.14. Prove that a normal operator A, fulfilling the condition $A^k = E$ (where k is a certain whole number $k \neq 0$), is a unitary operator.

7.3.15. Prove that the modulus of the determinant of a unitary operator equals unity.

7.3.16*. An orthogonal operator Q on the space of polynomials M_2 with the scalar product (7.1.1) transforms the polynomials $1 + t + t^2$ and $1 - t^2$ into $-1 - t + t^2$ and $1 - t$, respectively. The determinant of this operator equals -1. Find its matrix with respect to the basis $1, t, t^2$.

7.3.17. Prove that if U is a unitary operator, then for any vectors x and y

$$(Ux, Uy) = (x, y),$$

i.e. the unitary operator preserves the scalar product. Conversely, if a certain linear operator U preserves the scalar product of any two vectors, then U is a unitary operator.

7.3.18. An operator on the arithmetic space R_4 with the scalar product (7.1.4) transforms the vectors $x_1 = (2, 2, 2, 2,)$, $x_2 = (2, 0, 2, 2)$, $x_3 = (2, 2, 0, 2)$, $x_4 = (2, 2, 2, 0)$, respectively, into the vectors $y_1 = (4, 0, 0, 0)$, $y_2 = (3, -1, 1, 1)$, $y_3 = (3, 1, -1, 1)$, $y_4 = (3, 1, 1, -1)$. Is this operator orthogonal?

7.3.19. Prove that for a linear operator on a space X to be unitary, it is necessary and sufficient that it should preserve the scalar products of the vectors of a certain basis for the space X. In particular, an operator is unitary if it transforms an orthonormal basis into another orthonormal basis.

7.3.20*. Prove that for a linear operator U on a space X to be unitary, it is sufficient that U should preserve the lengths of all vectors from X.

7.3.21*. Prove that a linear operator preserving the orthogonality of any two vectors differs from a certain unitary operator only by a numerical multiplier.

7.3.22. Prove that the requirement of a matrix U to be unitary is equivalent to the requirement that the columns (or rows) of U, considered as vectors of the arithmetic space with the scalar product (7.1.4), form an orthonormal basis for this space.

7.3.23. Prove that any permutation matrix is a unitary matrix.

7.3.24. Prove that the modulus of each element of a unitary matrix equals its complementary minor.

7.3.25. Let $U = P + iQ$ be a complex unitary matrix of order n. Prove that the real matrix of order $2n$

$$D = \begin{Vmatrix} P & -Q \\ Q & P \end{Vmatrix}$$

is orthogonal.

7.3.26. Prove that if U is a unitary matrix, then the associated matrix U_p is also unitary.

7.3.27. Prove that the sum of the squares of the moduli of all minors of order k, selected from arbitrary k rows (or columns) of a unitary matrix, equals unity.

7.3.28. Let the modulus of the leading principal minor of order k of a unitary matrix U equal unity. Prove that in this case U is of quasi-diagonal form

$$U = \begin{Vmatrix} U_{11} & 0 \\ 0 & U_{22} \end{Vmatrix},$$

where U_{11} is a block of order k.

7.3.29. Prove that the Kronecker product of unitary matrices U and V, being, perhaps, of different orders, is also a unitary matrix.

7.3.30. Let U and V be unitary matrices of order $n \times n$. Show that (a) the operator G_{UV} (see Problem 5.6.10) is unitary; (b) the operator F_{UV} is not, generally speaking, unitary.

7.3.31. Prove that the transfer matrix, from an orthonormal basis into another orthonormal basis for a unitary space, is a unitary matrix.

7.3.32. Matrices A and B are said to be *unitarily similar* if there exists such a unitary matrix U that $B = U^{-1}AU$. Show that the relation of unitary similarity on the set of square matrices of a given order n is reflexive, symmetric, and transitive.

7.3.33. Prove that any complex matrix is unitarily similar to a triangular matrix.

7.3.34. Prove that an upper triangular matrix is unitarily similar to some lower triangular matrix.

7.3.35. Show that a unitarily similar transformation transforms a normal matrix into another normal matrix.

7.3.36. Show that a complex normal matrix is unitarily similar to a diagonal matrix.

7.3.37. Find a condition for a matrix of the form

$$\begin{array}{l} \\ \\ i\text{-th row} \\ \\ j\text{-th row} \\ \\ \\ \end{array} \left\| \begin{array}{ccccc} 1 & & & & \\ & \ddots & & & \\ & & \cos\varphi\cdot e^{i\psi_1} & \dots & -\sin\varphi\cdot e^{i\psi_2} \\ & & \vdots & \ddots & \vdots \\ & & \sin\varphi\cdot e^{i\psi_3} & \dots & \cos\varphi\cdot e^{i\psi_4} \\ & & & & \ddots \\ & & & & & 1 \end{array} \right\| \qquad (7.3.1)$$

to be unitary (the off-diagonal elements, that are not indicated, are equal to zero). The unitary matrix obtained is called an *elementary unitary matrix* and is further denoted by T_{ij}.

7.3.38. Let A be a square matrix of order n ($n \geqslant 2$). Select an elementary unitary matrix T_{ij} so that the (j, i) element of the matrix $B = T_{ij}A$ is equal to zero. In this way we may put $\psi_1 = \psi_4 = 0$ (see Problem 7.3.37).

7.3.39. Given an n-order matrix A, how should such a sequence of elementary unitary matrices $T^{(1)}$, $T^{(2)}$, . . ., be chosen so that all the elements of the first column below the diagonal of the product . . . $T^{(2)}T^{(1)}A$ are equal to zero?

7.3.40*. Using Problems 7.3.38 and 7.3.39, indicate a method to decompose a square matrix into the product of a unitary and upper triangular matrices.

7.3.41. Prove that any unitary matrix can be decomposed into the product of elementary unitary matrices and, perhaps, by a diagonal unitary matrix.

7.3.42. Let $A = U_1R_1$ and $A = U_2R_2$ be two decompositions of a nondegenerate matrix A into the product of a unitary and upper triangular matrix. Prove that

$$U_2 = U_1Q, \quad R_1 = QR_2,$$

where Q is a certain diagonal unitary matrix.

7.3.43. How should the method derived in Problem 7.3.40 be applied to the solution of a system of linear equations $Ax = b$ with a square nondegenerate coefficient matrix?

7.3.44. Find a condition for a column vector w so that a matrix of the form

$$H = E - 2ww^* \tag{7.3.2}$$

is unitary.

7.3.45. Let w be a normalized column vector. Prove that the corresponding matrix (7.3.2), treated as an operator on the artithmetic space, defines an orthogonal reflection in it. Such a matrix H is called a *reflection matrix*.

7.3.46. Find the eigenvalues and eigenvectors of a reflection matrix.

7.3.47. Find the determinant of a reflection matrix.

7.3.48. Show that any unitary matrix all of whose eigenvalues are $+1$ and -1, -1 being an eigenvalue of multiplicity unit, can be represented in the form (7.3.2).

7.3.49. Show that the matrix

$$T = \begin{Vmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{Vmatrix}$$

is a reflection matrix. Find the corresponding vector w.

7.3.50. Let H be a reflection matrix whose vector w is known. How should the product of the matrix H by a column vector x be computed so that this operation requires performing only $(2n + 1)$ multiplication operations?

7.3.51. Select the vector w so that the reflection matrix, defined by it, transforms a given vector x into a vector collinear to the unit column vector e_1 (we assume that the vector x itself is not collinear to e_1).

7.3.52*. Use the result of Problem 7.3.51 to construct an algorithm that decomposes a square matrix into the product of a unitary and an upper triangular matrix.

7.3.53. Let $Ax = b$ be a system of linear equations with a nondegenerate square matrix A. Describe a method to solve this system based on the procedure derived in Problem 7.3.52.

7.3.54*. Let A be a square matrix of order n $(n > 2)$. How should such a reflection matrix H be selected so that the matrix $B = HAH^*$ has zeroes for all the elements of the first column from the third element down?

7.3.55. A square matrix B is said to be *upper* (*lower*) *almost triangular* if $b_{ij} = 0$ for $i > j + 1$ $(j > i + 1)$. Using the result of Problem 7.3.54, show that any square matrix is unitarily similar to an upper (lower) almost triangular matrix. Restate this statement in the language of operators.

7.4. Hermitian Operators and Matrices

In the first half of the section, we discuss the simplest properties of Hermitian operators and matrices. The sequence of the problems is determined by the same principles as in the previous sections. The second half comprises the problems concerning the eigenvalues of Hermitian operators. A special consideration is given to their remarkable extreme properties. The application of these properties enables one of the most effective computational techniques for finding the eigenvalues of Hermitian matrices to be used, i.e. the method of bisection which is described in Problems 7.4.43-7.4.48.

7.4.1. Show that the set of all Hermitian operators from ω_{XX} forms a group under addition.

7.4.2. Show that the set of all symmetric operators from the linear space ω_{XX} of all linear operators on a Euclidean space X is a linear subspace. A similar statement is valid for the set of all skew-symmetric operators from ω_{XX}.

7.4.3. Show that the product of a nonzero Hermitian operator and a number α is a Hermitian operator if and only if α is a real number.

7.4.4. Show that an operator K is skew Hermitian if and only if the operator iK is Hermitian.

7.4.5. Show that the product of Hermitian operators H_1 and H_2 is a Hermitian operator if and only if H_1 and H_2 commute.

7.4.6. Show that the inverse operator of a nondegenerate Hermitian operator is also Hermitian.

7.4.7. Describe all the Hermitian operators that act on a one-dimensional space.

7.4.8. A linear operator A is defined on a two-dimensional Euclidean space and for two particular noncollinear operators x and y

$$(Ax, y) = (x, Ay).$$

Prove that A is a symmetric operator.

7.4.9. Show that the operator $Ax = [x, a]$ on a three-dimensional Euclidean space is skew-symmetric.

7.4.10*. Prove that any skew-symmetric operator K of a three-dimensional Euclidean space can be represented in the form $Kx = [x, a]$ having selected a convenient vector a.

7.4.11. An operator on the arithmetic space R_4 with the scalar product (7.1.4) converts the vectors $x_1 = (0, 1, 1, 1)$, $x_2 = (-1, 0, 1, 1)$, $x_3 = (-1, -1, 0, 1)$, $x_4 = (-1, -1, -1, 0)$ into the vectors $y_1 = (3, -1, -1, -1)$, $y_2 = (1, -3, -1, -1)$, $y_3 = (-1, -3, -1, 1)$, $y_4 = (-3, -1, -1, 1)$, respectively. Is this operator symmetric?

7.4.12. Show that the operators of Problem 7.2.9 are symmetric.

7.4.13. Show that any orthogonal reflection operator is Hermitian. In particular, the reflection matrix (7.3.2) is Hermitian.

7.4.14. Show that an operator, both unitary and Hermitian, is either equal to $\pm E$ or is an orthogonal reflection operator.

7.4.15*. A symmetric operator S on the space M_2 of polynomials with the scalar product (7.1.1) transforms the polynomials

$2 + 2t - t^2$ and $2 - t + 2t^2$ into $5 - t - t^2$ and $3 + 3t + 3t^2$,

respectively. The trace of this operator is equal to 3. Find the matrix with respect to the basis 1, t, t^2.

7.4.16. Let H_1 and H_2 be complex Hermitian matrices of the same order. Prove that the trace of the matrix H_1H_2 is a real number.

7.4.17. Let a Hermitian matrix H be represented in the form $H = S + iK$, where S and K are real matrices. Show that S is a symmetric, and K a skew-symmetric matrix.

7.4.18. Prove that the real matrix (see Problem 7.4.17)

$$D = \begin{Vmatrix} S & -K \\ K & S \end{Vmatrix}$$

is symmetric.

7.4.19. Prove that if H is a Hermitian matrix, then the associated matrix H_p is also Hermitian.

7.4.20. Prove that the Kronecker product of Hermitian matrices H_1 and H_2 of different orders, perhaps, is also a Hermitian matrix.

7.4.21. Let H_1 and H_2 be Hermitian matrices of order $n \times n$. Show that the operators $G_{H_1H_2}$ and $F_{H_1H_2}$ (see Problem 5.6.10) are Hermitian.

7.4.22. Prove that if an operator H is Hermitian, then for an arbitrary vector x the scalar product (Hx, x) is a real number.

7.4.23. Let K be a skew-symmetric operator on a Euclidean space X. Prove that $(Kx, x) = 0$ for any vector x from X.

7.4.24. What can be said about a Hermitian operator H if $(Hx, x) = 0$ for all vectors x?

7.4.25. Show that if the equality $(H_1x, x) = (H_2x, x)$ is valid for Hermitian operators H_1 and H_2 and any vector x, then $H_1 = H_2$.

7.4.26. Prove the statement, converse to that in Problem 7.2.18, viz., that if the equality (7.2.2) holds for any vector x, and A is a linear operator, then A is a normal operator.

7.4.27. The eigenvalues of a normal operator A on a unitary space belong to the same straight line of the complex plane. Prove that the operator A can be represented in the form

$$A = aE + \alpha H,$$

where H is a Hermitian operator, a and α are complex numbers, $|\alpha| = 1$.

7.4.28. Show that the eigenvalues of a skew-Hermitian operator are pure imaginary numbers.

7.4.29. Show that an orthogonal projection operator is a Hermitian operator.

It is assumed in Problems 7.4.30-7.4.37 that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a Hermitian operator (or a Hermitian matrix) H are numerated so that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \tag{7.4.1}$$

If, besides the eigenvalues, an orthonormal basis $e_1, \ldots, e_n$ consisting of the eigenvectors of the operator H is considered, then the enumeration of its vectors will be assumed to correspond to the same ordering (7.4.1).

7.4.30. Prove the validity of the following representations of the maximum and minimum values of a Hermitian operator H:

$$\lambda_1 = \max_{x \neq 0} \frac{(Hx, x)}{(x, x)}, \quad \lambda_n = \min_{x \neq 0} \frac{(Hx, x)}{(x, x)}. \tag{7.4.2}$$

Show that the vectors for which the indicated extreme values occur are eigenvectors of the operator H.

7.4.31. Show that the extreme eigenvalues of a Hermitian matrix H satisfy the relations:

$$\lambda_1 \geqslant \max_i h_{ii}, \quad \lambda_n \leqslant \min_i h_{ii}.$$

7.4.32. Assume that the equality $\lambda_1 = h_{ii}$ holds for a Hermitian matrix H. Prove that all the off-diagonal elements of the i-th row and i-th column of the matrix H are zeroes.

7.4.33. Prove that for the linear subspace L drawn on the eigenvectors $e_{i_1}, \ldots, e_{i_k}$ $(i_1 < \ldots < i_k)$ of a Hermitian operator H, the following relations are valid:

$$\lambda_{i_1} = \max_{\substack{x \neq 0 \\ x \in L}} \frac{(Hx, x)}{(x, x)}, \quad \lambda_{i_k} = \min_{\substack{x \neq 0 \\ x \in L}} \frac{(Hx, x)}{(x, x)}. \tag{7.4.3}$$

7.4.34*. Prove the following *Courant-Fischer theorem*: an eigenvalue λ_k of a Hermitian operator H on an n-dimensional space X satisfies the conditions

$$\lambda_k = \max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{(Hx, x)}{(x, x)} \tag{7.4.4}$$

$$\lambda_k = \min_{L_{n-k+1}} \max_{\substack{x \neq 0 \\ x \in L_{n-k+1}}} \frac{(Hx, x)}{(x, x)}. \tag{7.4.5}$$

The maxima must be found for all k-dimensional subspaces L_k of the space X for use in the equality (7.4.4); similarly, L_{n-k+1} in (7.4.5) means an arbitrary subspace of order $n - k + 1$.

7.4.35*. Let H_{n-1} be an arbitrary principal submatrix of an n-order Hermitian matrix H. Using the Courant-Fischer theorem, prove that the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ of the matrix H_{n-1} enumerated in descending order *separate* the eigenvalues of the matrix H.

This means that

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \ldots \geqslant \lambda_{n-1} \geqslant \mu_{n-1} \geqslant \lambda_n.$$

7.4.36. Without computing the eigenvalues for the n-order matrix H

$$H = \begin{Vmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & n-1 \\ 1 & 2 & \ldots & n-1 & n \end{Vmatrix},$$

indicate the number of nonzero eigenvalues and their signs.

7.4.37. Let the rank of a Hermitian matrix H be two greater than the rank of the principal submatrix H_{n-1}. Prove that the matrix H has one positive and one negative eigenvalue more than H_{n-1}.

7.4.38. Let the eigenvalues of Hermitian operators H_1, H_2 and $H_1 + H_2$ be enumerated in descending order

$$\begin{gathered} H_1 - \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n, \\ H_2 - \beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_n, \\ H_1 + H_2 - \gamma_1 \geqslant \gamma_2 \geqslant \ldots \geqslant \gamma_n. \end{gathered} \tag{7.4.6}$$

With the aid of the Courant-Fischer theorem, prove that the following inequalities are true ($k = 1, 2, \ldots, n$):

$$\gamma_k \leqslant \alpha_1 + \beta_k, \quad \gamma_k \leqslant \alpha_k + \beta_1,$$
$$\gamma_k \geqslant \alpha_n + \beta_k, \quad \gamma_k \geqslant \alpha_k + \beta_n.$$

7.4.39. Show that by a unitary similar transformation, a Hermitian matrix is reduced to another Hermitian matrix.

7.4.40. A band matrix is said to be *tridiagonal* if the band width equals 3. Deduce the following corollary to Problem 7.3.55: any Hermitian matrix is unitarily similar to a tridiagonal matrix. Restate this proposition in the language of operators.

7.4.41. We call a tridiagonal matrix C *irreducible* if $c_{ij} \neq 0$ when $|i - j| = 1$. Prove that if a tridiagonal Hermitian matrix has an eigenvalue λ of multiplicity r, then it is quasi-diagonal and, moreover, that there are at least r irreducible submatrices of lesser order on the diagonal.

The following Problems 7.4.42-7.4.49 concern a given tridiagonal irreducible Hermitian matrix C of order n, for which a sequence of polynomials $f_0(\lambda), f_1(\lambda), \ldots, f_n(\lambda)$ is considered, where $f_0(\lambda) \equiv 1$, and $f_i(\lambda)$ is the characteristic polynomial of the leading principal submatrix C_i of the matrix C (so that the polynomial $f_i(\lambda)$ is of degree i). The iteration formulae, connecting the polynomials of this set, were derived in Problem 3.2.46 (in the Hermitian case under consideration, $c_i = \overline{b}_i$), and are used from now on without further reference. The roots of the polynomial $f_i(\lambda)$, i.e. the eigenvalues of the submatrix C_i are denoted by $\lambda_1^{(i)}, \ldots, \lambda_i^{(i)}$ and enumerated in descending order so that $\lambda_1^{(i)} \geqslant \lambda_2^{(i)} \geqslant \ldots \geqslant \lambda_i^{(i)}$ (however, see Problem 7.4.43, (b)), whereas $\lambda_i^{(n)} = \lambda_i$, $i = 1, \ldots, n$.

7.4.42. Construct a sequence of polynomials $f_0(\lambda), f_1(\lambda), \ldots, f_5(\lambda)$ for the matrix

$$\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}. \tag{7.4.7}$$

7.4.43. Prove that in the set of polynomials $f_0(\lambda), f_1(\lambda), \ldots, f_n(\lambda)$: (a) adjacent polynomials have no common roots; (b) the roots of a polynomial $f_i(\lambda)$, $1 \leqslant i \leqslant n-1$, *strictly* separate the roots of the polynomial $f_{i+1}(\lambda)$:

$$\lambda_1^{(i+1)} > \lambda_1^{(i)} > \lambda_2^{(i+1)} > \lambda_2^{(i)} > \ldots > \lambda_i^{(i+1)} > \lambda_i^{(i)} > \lambda_{i+1}^{(i+1)};$$

(c) if $\lambda_k^{(i)}$, $i < n$, is a root of a polynomial $f_i(\lambda)$, then the numbers $f_{i-1}(\lambda_k^{(i)})$ and $f_{i+1}(\lambda_k^{(i)})$ have different signs.

7.4.44*. A real number μ is not a root of any polynomial $f_i(\lambda)$. Prove that the number of changes of sign in the number sequence

$$f_0(\mu), f_1(\mu), \ldots, f_n(\mu) \tag{7.4.8}$$

equals the number of the eigenvalues of the matrix C (i.e. the roots of the polynomial $f_n(\lambda)$) which are strictly greater than μ.

7.4.45*. Now let the number μ be a root of the polynomials in the set $f_0(\lambda), f_1(\lambda), \ldots, f_n(\lambda)$. As before, count the number of changes of sign in the sequence (7.4.8), ascribing the sign of the number $f_{i-1}(\mu)$ to each zero value $f_i(\mu)$. Prove that the statement of Problem 7.4.44 remains valid in this case also.

7.4.46*. Given that an eigenvalue λ_k of matrix C lies in an interval (a, b). In this case λ_k is said to be *localized* in (a, b). How, using the results of Problems 7.4.44 and 7.4.45, can λ_k be localized in an interval of half the length?

7.4.47. Let all eigenvalues of a matrix C lie in an interval (m, M). Using the result of Problem 7.4.46, indicate a method of finding the numbers λ_i to the accuracy of a given member ε.

7.4.48. Show that the sequence (7.4.8) can be computed by performing only $2(n-1)$ operations of multiplication (assuming that the $|b_i|^2$, in the iteration formulae connecting the polynomials $f_i(\lambda)$, are already evaluated).

7.4.49. The method of evaluating the eigenvalues of a tridiagonal Hermitian matrix derived in Problem 7.4.47 is termed the *bisection method*. Perform the bisection method for computing the greatest eigenvalue of the matrix (7.4.7) to the accuracy of $\varepsilon = 1/16$.

7.4.50*. Using the results of Problems 7.4.40, 7.4.41, 7.4.47, describe a method for approximating the eigenvalues of an arbitrary Hermitian matrix.

7.4.51*. A tridiagonal irreducible matrix A is said to be *Jacobian* if $a_{i,\,i+1}a_{i+1,\,i} > 0$ for all i. Show that for Jacobian matrices with real diagonal entries the results of Problems 7.4.43-7.4.47 are valid.

7.4.52. Using the correspondence between operators on a Euclidean space R and a unitary space C, obtained from R by complexification, prove that (a) a Hermitian operator $\hat{S}$ on the space C corresponds to a symmetric operator S on the space R; (b) for any symmetric operator on the space R, there is an orthonormal basis for R such that the matrix of this operator, with respect to this basis, is diagonal.

Reformulate statement (b) for matrices.

7.4.53. Let $z_1, \ldots, z_n$, $z_j = x_j + iy_j$ be an orthonormal basis of the eigenvectors of a Hermitian matrix of order $n \times n$ $H = S + iK$, and $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Prove that the $2n$-dimensional column vectors $u_1, v_1, \ldots, u_n, v_n$, where

$$u_j = \begin{Vmatrix} x_j \\ y_j \end{Vmatrix}, \quad v_j = \begin{Vmatrix} -y_j \\ x_j \end{Vmatrix},$$

form an orthonormal basis of the eigenvectors of a real matrix

$$D = \begin{Vmatrix} S & -K \\ K & S \end{Vmatrix},$$

whose corresponding eigenvalues are $\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n$.

7.5. Positive-Semidefinite and Positive-Definite Operators and Matrices

This section mostly deals with:

The formal properties of positive-semidefinite and positive-definite operators which follow from the definition directly.

Positive-definite matrices and Gram matrices. In this part of the section, we show that positive-definite matrices are, in a sense, a universal means of defining a scalar product on a given linear space.

The nonnegativeness (positiveness) of the eigenvalues of a positive-semidefinite (positive-definite) operator (matrix).

Various criteria for the positive definiteness of Hermitian matrices, and, in particular, the diagonal dominance (see Problem 7.5.24), the Sylvester criterion, etc. We also provide computational problems on their use.
The relation of a partial ordering on the set of Hermitian operators.
The square root of a positive-semidefinite numerical operator and some examples on evaluating the square root.
Finally, applications of an important theorem about the eigenvalues of an operator HS, where H and S are Hermitian operators, and S is positive-definite.

7.5.1. Can a positive-definite operator H convert a nonzero vector x into a vector y, orthogonal to x?

7.5.2. Deduce from the definition that a positive-definite operator is nondegenerate.

7.5.3. Let H be a positive-definite operator on a Euclidean space X. Show that for any nonzero vector x from X, its image makes an acute angle with x.

7.5.4. Let H and S be positive-semidefinite operators. Show that for any nonnegative numbers α and β, the operator $\alpha H + \beta S$ is positive-semidefinite.

7.5.5. Let H and S be positive-semidefinite operators, and assume that for certain real numbers α_0 and β_0 the operator $\alpha_0 H + \beta_0 S$ is positive-definite. Show that in this case, all operators $\alpha H + \beta S$ (where α and β are arbitrary positive numbers) are positive-definite.

7.5.6. Prove that the inverse operator of a positive-definite operator is also positive-definite.

7.5.7. Show that any orthogonal projection operator is a positive-semidefinite operator.

7.5.8. Let H be a complex positive-definite matrix. Prove that the transpose of H, i.e. H^T, is also positive-definite.

7.5.9. Prove that any principal submatrix of a positive-semidefinite (positive definite) matrix is also positive-semidefinite (positive-definite).

7.5.10*. Let $x_1, \ldots, x_k$ be an arbitrary vector set of a unitary (Euclidean) space X. Prove that the Gram matrix of the set $x_1, \ldots, x_k$ is a positive-semidefinite matrix. This matrix is positive-definite if the set $x_1, \ldots, x_k$ is linearly independent.

7.5.11. Let $e_1, \ldots, e_n$ be an arbitrary basis for a unitary (Euclidean) space X. Prove that the scalar product of any two vectors x and y from X can be computed by the formula

$$(x, y) = (GX_e, Y_e). \tag{7.5.1}$$

G^T denotes the Gram matrix of the set $e_1, \ldots, e_n$; X_e, Y_e are n-dimensional vector columns constituted from the coordinates of the vectors x and y with respect to the basis $e_1, \ldots, e_n$, and the scalar product on the right-hand side of (7.5.1) is defined in the usual way (as in 7.1.4).

7.5.12. Let $e_1, \ldots, e_n$ be an arbitrary basis for a linear space X, and let G be an arbitrary positive-definite matrix. Show that the formula (7.5.1) defines a scalar product on X. Moreover, the matrix G^T is the Gram matrix of the set $e_1, \ldots, e_n$ with respect to the derived scalar product.

Thus, the formula (7.5.1) (just like the method of Problem 2.1.2) describes all possible methods of defining a scalar product on a given linear space X.

7.5.13. Let $(x, y)_1$ and $(x, y)_2$ be two different scalar products on an arithmetic space. Prove that (a) there is a nondegenerate matrix A such that

$$(x, y)_2 = (Ax, y)_1;$$

(b) it follows from (a) that

$$(x, y)_1 = (A^{-1}x, y)_2.$$

7.5.14. Let A be an arbitrary linear operator from a unitary (Euclidean) space X to a unitary (Euclidean) space Y. Show that the product A^*A is a positive-semidefinite operator on X, and the product AA^* is a positive-semidefinite operator of the space Y. Accordingly, for any rectangular matrix A, the matrices A^*A and AA^* are positive-semidefinite.

7.5.15. Let H be a complex positive-definite matrix. Prove that in the representation of the matrix H

$$H = S + iK$$

(where S and K are real matrices) the matrix S is positive-definite.

7.5.16. Let H be a positive-semidefinite operator, and $(Hx, x) = 0$ for some vector x. Prove that (a) x belongs to the kernel N_H of the operator H; (b) the operator H/T_H, induced on the image T_H of the operator H, is positive-definite.

7.5.17. Show that a positive-definite operator can be defined as a nondegenerate positive-semidefinite operator.

7.5.18. Show that a Hermitian operator H is positive-semidefinite (positive-definite) if and only if for any positive (nonnegative) number ε, the operator $H + \varepsilon E$ is nondegenerate.

7.5.19. A Hermitian operator H is said to be *negative-semidefinite* (*negative-definite*) if for any nonzero vector x the scalar product (Hx, x) is nonpositive (negative). *Negative-semidefinite* and *negative-definite matrices* are similarly defined.

Prove that any Hermitian operator can be represented as the sum of positive-semidefinite and negative-semidefinite operators.

7.5.20*. A complex square matrix A is said to be *stable* if for any eigenvalue λ for this matrix, the condition Re $\lambda < 0$ is fulfilled.

Prove that if the *Lyapunov matrix equation* for an $n \times n$ matrix A

$$A^*X + XA = C$$

(where C is a certain negative-definite matrix) has a positive-definite solution B, then A is a stable matrix. Hence, deduce that B is the unique solution of the indicated equation.

7.5.21. What can be said about a negative-semidefinite operator H if its trace equals zero?

7.5.22. Show that the determinant of a positive-definite operator is positive. Hence, deduce that all the principal minors of a positive-definite matrix are positive.

7.5.23. Show that the element with maximum modulus in a positive-definite matrix is on the principal diagonal.

7.5.24*. Prove that a Hermitian $n \times n$ matrix H is positive-definite if

$$h_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |h_{ij}|, \quad i = 1, \ldots, n. \tag{7.5.2}$$

7.5.25. Let $H = S + iK$ be a complex positive-definite matrix. Prove that the real matrix

$$D = \left\| \begin{matrix} S & -K \\ K & S \end{matrix} \right\|$$

is also positive.

7.5.26*. Let H be a positive-definite matrix. Prove that the associated matrix H_p is also positive definite.

7.5.27. Prove that of all the k-order minors of a positive-definite matrix H, the one with the maximum modulus is one of the principal minors.

7.5.28. Prove that the Kronecker product of positive-definite matrices H_1 and H_2 (perhaps, of different orders) is also a positive-definite matrix.

7.5.29*. Let A and B both be n-order square matrices. The *Schur product* of the matrices A and B is a matrix C of order $n \times n$ such that for all i, j

$$c_{ij} = a_{ij} b_{ij}.$$

Prove that the Schur product of positive-definite matrices H_1 and H_2 is also a positive-definite matrix.

7.5.30. Let H be a positive-definite n-order matrix. Prove that an $n \times n$ matrix S such that $s_{ij} = |h_{ij}|^2$ for all i, j is also positive-definite.

7.5.31. Let H and S be Hermitian operators, and let the difference $H - S$ be a positive-semidefinite (positive-definite) operator. We will write in this case that $H \geqslant S$ ($H > S$). Show that for the relation $\geqslant$ the following properties are valid:

(a) $H \geqslant S,\ S \geqslant T \Rightarrow H \geqslant T$;

(b) $H_1 \geqslant S_1,\ H_2 \geqslant S_2 \Rightarrow \alpha H_1 + \beta H_2 \geqslant \alpha S_1 + \beta S_2$, where α and β are any nonnegative numbers,

(c) $H \geqslant S \Rightarrow A^*HA \geqslant A^*SA$ for any operator A.

7.5.32. Let H and S be Hermitian operators, $H \geqslant S$. Prove that the eigenvalues of the operator S placed in descending order do not exceed the corresponding (also in the same order) eigenvalues of the operator H.

7.5.33. A positive-definite operator H satisfies the inequality $H \geqslant E$. Prove that $H^{-1} \leqslant E$.

7.5.34. Matrices H and S are positive-definite, $H \geqslant S$. Prove that

(a) $\max\limits_{i,j} |h_{ij}| \geqslant \max\limits_{i,j} |s_{ij}|$;

(b) the principal minors of the matrix S do not exceed the corresponding minors of the matrix H; and in particular, (c) $\det H \geqslant \det S$.

7.5.35. A diagonal element h_{ii} of a positive-definite matrix H has been increased. Prove that the determinant of the obtained matrix $\tilde{H}$ is greater than the determinant of the matrix H.

7.5.36*. Prove the following *Sylvester criterion* for positive definiteness: for a Hermitian matrix H to be a positive-definite, it is necessary and sufficient that all its leading principal minors be positive.

7.5.37. A leading principal k-order minor of a nonnegative matrix H is equal to zero. Prove that all the leading principal minors with orders higher than k are equal to zero.

7.5.38. Prove that all the principal odd-order minors of a negative-definite matrix H are negative, whereas all the principal even-order minors are positive.

Determine for each of the following tridiagonal n-order matrices whether the matrix is positive-definite or positive-semidefinite.

7.5.39. $$\begin{Vmatrix} n-1 & 1 & & & & \\ 1 & n-2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 2 & 1 & \\ & & & 1 & 1 & 1 \\ & & & & 1 & 0 \end{Vmatrix}.$$

7.5.40. $$\begin{Vmatrix} n-1 & 1 & & & & \\ 1 & n-2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 2 & 1 & \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{Vmatrix}.$$

7.5.41. $\begin{Vmatrix} n+1 & 1 & & & & \\ 1 & n & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & 4 & 1 & \\ & & & 1 & 3 & 1 \\ & & & & 1 & 2 \end{Vmatrix}.$

7.5.42*. $\begin{Vmatrix} n & 1 & & & & \\ 1 & n-1 & \cdot & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & 3 & 1 \\ & & & 1 & 2 & 1 \\ & & & & 1 & 1 \end{Vmatrix}.$

7.5.43. $\begin{Vmatrix} n^2 & 1 & & & \\ 1 & (n-1)^2 & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & \cdot & 4 & 1 \\ & & & 1 & 1 \end{Vmatrix}.$

7.5.44. $\begin{Vmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & 2 & 1 \\ & & & & 1 & 1 \end{Vmatrix}.$

7.5.45*. Prove that for any positive semidefinite (positive-definite) operator H, there is a unique positive-semidefinite (positive-definite) operator K such that $K^2 = H$. The operator K is called (the principal value of) the *square root of the operator H* and denoted by $H^{1/2}$.

Find the square roots of the following matrices:

7.5.46. $\begin{Vmatrix} 5 & -3 \\ -3 & 5 \end{Vmatrix}.$ 7.5.47. $\begin{Vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{Vmatrix}.$

7.5.48. $\begin{Vmatrix} 24 & 6 & -12 \\ 6 & 33 & 6 \\ -12 & 6 & 24 \end{Vmatrix}.$ 7.5.49. $\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{Vmatrix}.$

7.5.50*. Using the existence of a square root, prove that the determinant of a positive-definite n-order matrix H satisfies the inequality

$$\det H \leqslant h_{11}h_{22} \ldots h_{nn}.$$

The equality occurs if and only if H is a diagonal matrix.

7.5.51*. A positive definite matrix H is represented in a partitioned form thus:

$$H = \begin{Vmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{Vmatrix},$$

where H_{11} and H_{22} are square submatrices. Prove that

$$\det H \leqslant \det H_{11} \cdot \det H_{22},$$

and that the equality occurs if and only if $H_{12} = 0$.

7.5.52. Let H and S be Hermitian operators, and let S be positive-semidefinite. Prove that if H and S commute, then H^{-1} and $S^{1/2}$ also commute.

7.5.53*. Operators H and S are positive-definite and $H \geqslant S$. Prove that $H^{-1} \leqslant S^{-1}$.

7.5.54. Show that the product HS of commuting positive-semidefinite operators H and S is also a positive-semidefinite operator.

7.5.55. Let $H \geqslant S$, and let T be a positive-semidefinite operator that commutes with H and S. Prove that $HT \geqslant ST$.

7.5.56*. Let H and S be Hermitian operators, and let S be positive-definite. Prove that the eigenvalues of the operator HS are real numbers and that the operator is of simple structure.

7.5.57. The operator H is positive-semidefinite (see Problem 7.5.56). Show that all the eigenvalues of the operator HS are nonnegative.

7.5.58. Show that the statement, converse to that in 7.5.57, is true: if operators H and S are Hermitian, S is positive-definite, and all the eigenvalues of the operator HS are nonnegative, then H is a positive-semidefinite operator.

7.5.59*. Let H and S be Hermitian n-order matrices, and let S be positive-definite. Prove that (a) the left-hand side of the equation

$$\det (\lambda S - H) = 0 \qquad (7.5.3)$$

is a polynomial in λ of degree n with the higher-order coefficient equal to the determinant of the matrix S; (b) the equation (7.5.3) has n real roots if each root is counted as many times as its multiplicity is.

7.5.60. Let H and S be positive-definite operators whose greatest eigenvalues are equal to α_1 and β_1, respectively. Prove that the greatest eigenvalue γ_1 of the operator HS satisfies the inequality $\gamma_1 \leqslant \alpha_1\beta_1$.

7.5.61*. Prove that the following statements are valid: (a) the eigenvalues of the matrix $iS^{-1}K$ (see Problem 7.5.15) are real and have absolute values less than unity; (b) $\det S \geqslant \det H$, and the equality occurs if and only if $H = S$; (c) $\det S > \det K$.

7.5.62*. Let A be an operator of rank r from an n-dimensional space X to an m-dimensional space Y, and let $e_1, \ldots, e_n$ be an orthonormal basis containing the eigenvectors of a positive-semidefinite operator A^*A, the vectors $e_1, \ldots, e_r$ corresponding to the nonzero eigenvalues $\alpha_1^2, \ldots, \alpha_r^2$ ($\alpha_i > 0$, $i = 1, 2, \ldots, r$). Prove that (a) the vectors $e_{r+1}, \ldots, e_n$ constitute a basis for the kernel N_A of the operator A; (b) the vectors $e_1, \ldots, e_r$ constitute a basis for the image T_{A^*} of the conjugate operator A^*; (c) the vectors $Ae_1, \ldots, Ae_r$ are orthogonal and form a basis for the image T_A of the operator A; (d) the length of the vector Ae_i equals α_i, $i = 1, \ldots, r$; (e) each of the vectors Ae_i is an eigenvector of the operator AA^* and corresponds to the eigenvalue α_i^2; (f) if we put

$$f_i = \frac{1}{\alpha_i} Ae_i, \quad i = 1, \ldots, r,$$

then

$$A^* f_i = \alpha_i e_i.$$

7.6. Singular Values and the Polar Representation

When discussing singular values, we shall mostly be interested in the various methods that in concrete cases facilitate computation and estimation in practical cases. The principal applications of singular values are related to metric problems and are discussed in Sec. 7.8 and later sections. Here, however, we only provide some of the inequalities that connect singular values to the eigenvalues of an operator. The singular-value decomposition of an arbitrary rectangular matrix, and the polar representation of operators from ω_{XX} and square matrices, are discussed in detail.

In all the problems, singular values $\alpha_1, \ldots, \alpha_s$ are assumed to be enumerated in descending order

$$\alpha_1 \geqslant \ldots \geqslant \alpha_s.$$

7.6.1. Given the singular values of an operator A, find the singular values of (a) the operator A^*, (b) the operator αA, where α is an arbitrary complex number.

7.6.2. Prove that the singular values of an operator are unaltered when the operator is multiplied by unitary operators.

7.6.3. Let an operator A be defined on a space X. Show that A is nondegenerate if and only if all the singular values of this operator are nonzero.

7.6.4. Show that the modulus of an operator's determinant equals the product of its singular values.

7.6.5. Assuming that an operator A is nondegenerate, find the relation between the singular values of the operators A and A^{-1}.

7.6.6. Prove that the singular values of a normal operator coincide with the moduli of its eigenvalues.

7.6.7. Prove that an operator A on a unitary space is unitary if and only if all the singular values of this operator equal unity.

7.6.8*. Find the singular values of the differential operator on the space of polynomials M_n with the scalar product (7.1.7).

7.6.9*. Find the singular values of the differential operator on the space M_2 of polynomials if the scalar product is defined by the formula (7.1.2). Contrast this result with that of Problem 7.6.8.

7.6.10*. Let A be a rectangular $m \times n$ matrix of rank r, either real or complex. Prove that the matrix A can be represented in the form

$$A = U\Lambda V, \tag{7.6.1}$$

where U and V are orthogonal (unitary) matrices of orders m and n, respectively; Λ is an $m \times n$ matrix such that $\lambda_{11} \geqslant \lambda_{22} \geqslant \ldots \ldots \geqslant \lambda_{rr} > 0$, and all the other elements are zeroes. This representation (7.6.1) is called the *singular-value decomposition of the matrix* A.

7.6.11. Show that the matrix Λ in the decomposition (7.6.1) is uniquely determined by the matrix A itself, viz. the numbers $\lambda_{11}, \ldots, \lambda_{rr}$ are the nonzero eigenvalues of the matrix $(A^*A)^{1/2}$ (as they are of the matrix $(AA^*)^{1/2}$).

7.6.12. Determine the meaning of the matrices U and V in the singular-value decomposition of a matrix A.

7.6.13. Rectangular $m \times n$ matrices A and B are said to be *unitarily equivalent* if there exist such unitary matrices U and V that $B = UAV$. Prove that the relation of unitary equivalence on the set of $m \times n$ matrices is reflexive, symmetric and transitive.

7.6.14. Prove that $m \times n$ matrices A and B are unitarily equivalent if and only if they have the same singular values.

7.6.15. Show that matrices A and B are unitarily equivalent if and only if the matrices A^*A and B^*B are similar.

7.6.16. Given the singular-value decomposition $A = U\Lambda V$ of a matrix A, find the singular-value decompositions and singular values of the matrices: (a) A^T, (b) A^*, (c) A^{-1} if A is a square, nondegenerate matrix.

7.6.17. Show that for any $m \times n$ matrix A, there is a unitary m-order matrix W such that the rows of the matrix WA are orthogonal. Similarly, a unitary n-order matrix Z exists such that the columns of the matrix AZ are orthogonal.

7.6.18. The rows of a matrix are orthogonal. Prove that the singular values of this matrix equal the lengths of its rows.

7.6.19. Find the singular values of an $m \times n$ matrix A with unity rank.

7.6.20. Let A be a partitioned matrix of the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are not necessarily square matrices. Prove that the nonzero singular values of the blocks A_1 and A_2 produce, collectively, all the nonzero singular values of the matrix A. The same statement is also valid for a partitioned matrix of the form

$$\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}.$$

7.6.21. Deduce the following corollary to the statement of Problem 7.6.20: if a pair of orthogonal subspaces L and M reduce an operator A, then the singular values of the operators A/L and A/M produce, collectively, all the singular values of the operator A.

7.6.22. Prove that the singular value decomposition (7.6.1) of a matrix A can be rewritten in the form

$$A = \tilde{U}\Lambda_r\tilde{V}, \tag{7.6.2}$$

where $\tilde{U}$ is an $m \times r$ matrix with orthonormal columns, $\tilde{V}$ is an $r \times n$ matrix with orthonormal rows, and Λ_r is a diagonal matrix with positive diagonal elements. The representation (7.6.2) is also called the singular decomposition of the matrix A.

7.6.23. Prove that for the singular values $\alpha_1, \ldots, \alpha_n$ of an operator A, the following version of the Courant-Fischer theorem is valid

$$\alpha_k = \max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{|Ax|}{|x|},$$

$$\alpha_k = \min_{L_{n-k+1}} \max_{\substack{x \neq 0 \\ x \in L_{n-k+1}}} \frac{|Ax|}{|x|}.$$

Here, as in 7.4.34, L_k and L_{n-k+1} are arbitrary subspaces of dimensions k and $n-k+1$, respectively, of the n-dimensional space X. In particular, the following relations hold

$$\alpha_1 = \max_{x \neq 0} \frac{|Ax|}{|x|}, \quad \alpha_n = \min_{x \neq 0} \frac{|Ax|}{|x|}.$$

7.6.24. Prove that the spectral radius of an operator does not exceed its greatest singular value.

7.6.25. Prove that the eigenvalue λ_n with the minimal modulus and the minimal singular value α_n of an operator A satisfy the relation

$$|\lambda_n| \geqslant \alpha_n.$$

7.6.26. Let $\alpha_1, \ldots, \alpha_n$ be the singular values of an $n \times n$ matrix A. Prove that the singular values of the associated matrix A_p are all the possible products of p numbers from $\alpha_1, \ldots, \alpha_n$.

7.6.27. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of an $n \times n$ matrix A are ordered so that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. Prove that the following *Weyl inequalities* are valid

$$|\lambda_1| \ldots |\lambda_k| \leqslant \alpha_1 \ldots \alpha_k,$$

$$|\lambda_k| |\lambda_{k+1}| \ldots |\lambda_n| \geqslant \alpha_k \alpha_{k+1} \ldots \alpha_n, \quad 1 \leqslant k \leqslant n.$$

They generalize 7.6.24 and 7.6.25.

7.6.28. Prove that the greatest and least singular values of an $n \times n$ matrix A satisfy the estimates

$$\alpha_1 \geqslant \max \{\max_i (\sum_{j=1}^{n} |a_{ij}|^2)^{1/2}, \quad \max_j (\sum_{i=1}^{n} |a_{ij}|^2)^{1/2}\},$$

$$\alpha_n \leqslant \min \{\min_i (\sum_{j=1}^{n} |a_{ij}|^2)^{1/2}, \quad \min_j (\sum_{i=1}^{n} |a_{ij}|^2)^{1/2}\}.$$

7.6.29*. For an operator A the equality $|\lambda_1| = \alpha_1$ is valid. Here λ_1 is the eigenvalue of A with the greatest modulus. Prove that the operators A and A^* have a common eigenvector corresponding to the eigenvalue λ_1 ($\bar{\lambda}_1$).

7.6.30*. Prove the statement, converse to 7.6.6, viz., if the singular values of an operator A coincide with the moduli of the eigenvalues, then A is a normal operator.

7.6.31*. Let A be a rectangular $m \times n$ matrix, and let $\tilde{A}$ be an arbitrary submatrix of the matrix A. Prove that the singular values of the matrix $\tilde{A}$ do not exceed the corresponding singular values of A.

7.6.32. Let $\tilde{A}$ be an arbitrary square submatrix of a normal matrix A. Prove that the spectral radius of $\tilde{A}$ does not exceed the spectral radius of A.

7.6.33. Prove that the singular values α_k, β_k, γ_k of operators A, B and $A + B$ satisfy the inequalities:

$$\gamma_k \leqslant \alpha_1 + \beta_k, \quad \gamma_k \leqslant \alpha_k + \beta_1,$$
$$\gamma_k \geqslant -\alpha_1 + \beta_k, \quad \gamma_k \geqslant \alpha_k - \beta_1, \quad 1 \leqslant k \leqslant n.$$

7.6.34*. Operators A and B are defined on an n-dimensional space X. Prove that the singular values α_k, β_k, δ_k of the operators A, B and AB satisfy the relations:

$$\delta_k \leqslant \alpha_1\beta_k, \quad \delta_k \leqslant \alpha_k\beta_1,$$
$$\delta_k \geqslant \alpha_n\beta_k, \quad \delta_k \geqslant \alpha_k\beta_n, \quad 1 \leqslant k \leqslant n.$$

7.6.35. Let A and B be positive-definite operators. Prove that the eigenvalues of the operator AB are equal to the squares of the singular values of the operator $A^{1/2}B^{1/2}$.

7.6.36. Given the singular values $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ of matrices A and B of orders n and m, respectively. Find the singular values of the Kronecker product $A \times B$.

Find the singular values of the following matrices:

7.6.37. $\begin{Vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{Vmatrix}$. 7.6.38. $\begin{Vmatrix} 0 & 2 & 0 \\ 0 & 0 & -3 \\ 1 & 0 & 0 \end{Vmatrix}$.

7.6.39. $\begin{Vmatrix} 4 & -2 & 2 \\ 4 & 4 & -1 \\ -2 & 4 & 2 \end{Vmatrix}$.

7.6.40. $\begin{Vmatrix} 4 & -2 & 4 \\ 2 & -1 & 2 \\ -4 & 2 & -4 \end{Vmatrix}$.

7.6.41. $\begin{Vmatrix} 4 & -3i & 0 \\ -3i & 4 & 0 \\ 0 & 0 & -3 \end{Vmatrix}$.

7.6.42. $\begin{Vmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \end{Vmatrix}$.

7.6.43. $\begin{Vmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{Vmatrix}$.

7.6.44. $\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{Vmatrix}$.

7.6.45. $\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{Vmatrix}$.

7.6.46*. $\begin{Vmatrix} 2 & -1 & 4 & -2 \\ ? & 1 & 4 & 2 \\ -4 & 2 & 2 & -1 \\ -4 & -2 & 2 & 1 \end{Vmatrix}$.

7.6.47. What becomes of the polar representation of a matrix of order n when $n = 1$?

7.6.48. Show that in a polar representation $A = HU$ of an operator A, the positive-semidefinite operator H is uniquely determined.

7.6.49*. Let $A = HU$ be an *arbitrary* polar representation of an operator A. Show that the operator U transforms the orthonormal basis, containing the eigenvectors of the operator A^*A, into a similar basis for the operator AA^*.

7.6.50. Show that for whatever polar representation $A = HU$ of an operator A, the unitary operator U transforms the subspace T_{A^*} into T_A, and the subspace N_A into N_{A^*}.

7.6.51*. Let $A = HU$ be an arbitrary polar representation of an operator A. Prove that effect of the unitary operator U on the subspace T_{A^*} is uniquely determined by the operator A.

7.6.52. Prove that a nondegenerate operator possesses a unique polar representation.

7.6.53. Prove that any operator A on a unitary (Euclidean) space can be represented in the form

$$A = U_1H_1,$$

where U_1 is a unitary (orthogonal), and H_1 a positive-semidefinite operator. Show that the operator H_1 is, in this representation, uniquely determined.

7.6.54*. Prove that an operator A is normal if and only if the operators H and U in its polar representation $A = HU$ are commuting.

7.6.55. Let A be a nondegenerate normal operator on a unitary space, and let its eigenvalues $\lambda_1, \ldots, \lambda_n$ be given in trigonometric form

$$\lambda_1 = \rho_1 (\cos \varphi_1 + i \sin \varphi_1), \ldots, \lambda_n = \rho_n (\cos \varphi_n + i \sin \varphi_n).$$

Prove that the operators H and U in a polar representation of the operator A have the eigenvalues $\rho_1, \ldots, \rho_n$ and $\cos \varphi_1 + i \sin \varphi_1, \ldots, \cos \varphi_n + i \sin \varphi_n$, respectively.

7.6.56. An operator S is positive-semidefinite. Find its polar representation.

7.6.57. Find a polar representation of the differential operator on the space M_n of polynomials with the scalar product (7.1.7).

7.6.58. Given a polar representation $A = HU$ of a matrix A, find a polar representation of the associated matrix A_p.

7.6.59. Given square matrices A and B, perhaps, of different orders and letting $A = HU$ and $B = KV$ be their polar representations, find a polar representation of the Kronecker product $A \times B$.

Find polar representations of the following matrices:

7.6.60. $\left\| \begin{matrix} -1 & -7 \\ 1 & 7 \end{matrix} \right\|$. **7.6.61.** $\left\| \begin{matrix} 0 & 3 & -1 \\ 0 & 4 & 2 \\ -5i & 0 & 0 \end{matrix} \right\|$.

7.6.62*. $\left\| \begin{matrix} 2 & -1 & 4 & -2 \\ 2 & 1 & 4 & 2 \\ -4 & 2 & 2 & -1 \\ -4 & -2 & 2 & 1 \end{matrix} \right\|$.

7.6.63*. Using the polar representation, prove the converse of 7.5.56, viz., that if an $n \times n$ matrix A, whose eigenvalues $\lambda_1, \ldots, \lambda_n$ are real numbers, is of simple structure, then A can be represented in the form $A = HS$, where H and S are Hermitian operators and S is positive-definite. If the matrix A is real, then the factors H and S can be chosen to be real.

7.6.64*. Prove that the sum of the singular values $\alpha_1, \ldots, \alpha_n$ of an $n \times n$ matrix A satisfies the representations

$$\alpha_1 + \ldots + \alpha_n = \max_W |\operatorname{tr}(AW)| = \max_W \operatorname{Re} \operatorname{tr}(AW),$$

where W ranges over the whole set of unitary n-order matrices.

7.7. Hermitian Decomposition

The purpose of this section is to illustrate that, despite its simplicity, Hermitian decomposition is a useful instrument. In many cases a problem, posed in terms of arbitrary operators, can be transferred using the Hermitian decomposition to an analogous task posed in Hermitian operators, the solution of which proves to be much simpler to obtain. At the end of the section we demonstrate an analogue of the Hermitian decomposition of operators on a Euclidean space (see Problem 7.7.23).

7.7.1. What does the Hermitian decomposition of an n-order matrix, when $n = 1$, turn into?

7.7.2. What can be said about a linear operator A if $(Ax, x) = 0$ for any vector x?

7.7.3. What can be said about linear operators A and B if for any vector x:

(a) $(Ax, x) = (Bx, x)$?
(b) $(Ax, x) = (x, Bx)$?

7.7.4. Prove the converse to 7.4.22, viz., that if the scalar product (Ax, x) is a real number for any operator A, then for any operator x, A is a Hermitian operator.

7.7.5. Show that in the definition of a positive-definite operator on a unitary space the requirement that it should be Hermitian is extra.

7.7.6. Let H and S be Hermitian operators. Show that the scalar product (Hx, Sx) is real for any vector x if and only if H and S are commuting.

7.7.7. What can be said about an $n \times n$ matrix A if it is orthogonal to any Hermitian matrix with the scalar product defined as in (7.1.5)?

7.7.8. Let the trace of the product AH of an $n \times n$ matrix A and any Hermitian matrix H be a real number. Prove that the matrix A is Hermitian.

7.7.9. Let $A = H_1 + iH_2$ be the Hermitian decomposition of an operator A. Find the Hermitian decomposition of the conjugate operator A^*.

7.7.10. Prove that an operator A is normal if and only if the operators H_1 and H_2 in its Hermitian decomposition are commuting.

7.7.11. Show that the eigenvalues of the operators H_1 and H_2 from the Hermitian decomposition of a normal operator A coincide with the real and the imaginary parts, respectively, of the eigenvalues of the operator A.

7.7.12. Show that any orthonormal basis, containing the eigenvectors of a normal operator A is also a basis made up of the eigenvectors of the operators H_1 and H_2 of its Hermitian decomposition.

7.7.13. Let A and B be commuting normal operators, and let $A = H_1 + iH_2$, $B = S_1 + iS_2$ be their Hermitian decompositions. Prove that all the operators H_1, H_2, S_1, S_2 are commuting.

7.7.14. Let A be an operator on an n-dimensional space with the Hermitian decomposition $A = H_1 + iH_2$. Prove that the set of values for the ratio

$$\frac{(Ax, x)}{(x, x)}$$

(where x is an arbitrary nonzero vector) is bounded by a rectangle in the complex plane with vertices (α_1, β_1), (α_1, β_n), (α_n, β_n),

(α_n, β_1), α_1, α_n and β_1, β_n being the greatest and least of the eigenvalues of the matrices H_1 and H_2, respectively.

7.7.15. Deduce the following *Bendixson theorem* from 7.7.14: the real (imaginary) parts of the eigenvalues of an operator A are confined between the greatest and least eigenvalues of the operator H_1 (H_2) of its Hermitian decomposition.

7.7.16. The operator H_1 from the Hermitian decomposition of an operator A is positive-definite. Prove that the operator A is nondegenerate.

7.7.17*. The matrix H_1 in the Hermitian decomposition of a matrix A is negative-definite. Prove that (a) the matrix A is stable (see 7.5.20); (b) the product of the matrix A by any positive-definite matrix H is also a stable matrix.

7.7.18*. The diagonal elements a_{ii} of a complex tridiagonal matrix A are real numbers and the off-diagonal elements satisfy the inequalities $a_{i,\, i+1}a_{i+1,\, i} < 0$, $i = 1, 2, \ldots, n-1$. Prove that the eigenvalues of the matrix A are bounded by the strip in the complex plane:

$$\min_i a_{ii} \leqslant \operatorname{Re} z \leqslant \max_i a_{ii}.$$

7.7.19*. A square real matrix A is called a *tournament* matrix if all the diagonal elements a_{ii} are zeroes and the off-diagonal elements satisfy the condition $a_{ij} + a_{ji} = 1$ for all i, j $(i \neq j)$. Prove that the eigenvalues of a tournament matrix A, considered in the field of complex numbers, lie in the strip of the complex plane

$$-\frac{1}{2} \leqslant \operatorname{Re} z \leqslant \frac{1}{2}(n-1),$$

where n is the order of A.

7.7.20*. Prove that in the context of Problem 7.7.16

$$|\det A| \geqslant \det H_1.$$

When does the equality occur in this relation?

7.7.21*. By means of the Schur theorem, prove that the following relationship between the eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\alpha_1, \ldots, \alpha_n$ of the operators A and H_1, respectively, is true (see Problem 7.7.16)

$$\operatorname{Re} \lambda_1 \operatorname{Re} \lambda_2 \ldots \operatorname{Re} \lambda_n \geqslant \alpha_1\alpha_2 \ldots \alpha_n.$$

The equality occurs if and only if $\operatorname{Re} \lambda_i = \alpha_i$, $i = 1, \ldots, n$, with the appropriate ordering of the eigenvalues.

7.7.22. Show that the greatest singular value α_1 of an operator A satisfies the inequality

$$\alpha_1 \leqslant \rho(H_1) + \rho(H_2).$$

Here $\rho(H_1)$ and $\rho(H_2)$ are the spectral radii of the operators H_1 and H_2 of the Hermitian decomposition.

7.7.23. Show that any linear operator A on a Euclidean space can be uniquely represented in the form

$$A = S \dotplus K,$$

where S is a symmetric, and K a skew-symmetric operator

7.7.24. Prove that the space $R_{n\times n}$ (see 7.1.17) is the orthogonal sum of the subspace of symmetric matrices and the subspace of skew-symmetric matrices.

7.7.25. What can be said about a linear operator A on a Euclidean space if $(Ax, x) = 0$ for any vector x? Contrast this with the result of 7.7.2.

7.8. Pseudosolutions and Pseudoinverse Operators

The first half of the section is devoted to the properties of pseudosolutions and a normal pseudosolution of the equation $Ax = b$, where A is generally an operator from ω_{XY}, and b is a fixed vector from the space Y. Several methods of evaluating pseudosolutions are demonstrated, using predetermined orthonormal bases that were obtained from the eigenvectors of the operators A^*A or AA^*. Remember, by the way, that these bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ are called the singular bases of an operator A (and of A^*) if the vectors $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$, corresponding to nonzero eigenvalues $\alpha_1^2, \ldots, \alpha_r^2$, are connected by the relations

$$f_i = \frac{1}{\alpha_i} Ae_i, \quad i = 1, \ldots, r.$$

Singular bases play a principal role in proving various properties about the pseudoinverse operator to which the second part of the section is devoted. We treat this subject more extensively than is required for immediate academic purposes, taking into account the scarcity of the material on the pseudoinverse operator in general text-books on linear algebra. In specialized literature interpretations of this notion can be found that seem totally different to each other at first glance, but which are certainly equivalent. We present a number of such definitions to prove their equivalence. We demonstrate also that a number of classes of operators on a unitary space (normal, Hermitian, positive-semidefinite) is closed under the operation of pseudoinverse determination.

7.8.1. Let b_T be a projection of a vector b onto the image T_A of an operator A. Prove that any pseudosolution of the equation $Ax = b$ is a pre-image of the vector b_T.

7.8.2. Show that the set of all pseudosolutions of the equation $Ax = b$ is a plane whose directional subspace is the kernel N_A of the operator A. This plane is a subspace if and only if b belongs to the kernel N_{A^*} of the conjugate operator A^*.

7.8.3. Show that a normal pseudosolution of the equation $Ax = b$ can be specified as a pseudosolution of this equation, orthogonal to the kernel of the operator A, or in other words, as a pseudosolution belonging to the image of the conjugate operator A^*.

7.8.4. Let A be a differential operator on the space M_n of polynomials with the scalar product defined as in (7.1.7), and let $g(t)$ be a given polynomial from M_n. Find all the pseudosolutions and a normal pseudosolution of the equation $Af = g$.

7.8.5. How are pseudosolutions and normal pseudosolutions of the equation $Ax = b$ and the equations (a) $\alpha Ax = b$, (b) $Ax = \alpha b$, (c) $\alpha Ax = \alpha b$ related where α is a nonzero number?

7.8.6. How are normal pseudosolutions of the equation $Ax = b$ and the equations (a) $UAx = Ub$, (b) $AVx = b$ related? Here U and V are unitary operators.

7.8.7. Let A be a normal operator, and let an orthonormal basis, $e_1, \ldots, e_n$, containing the eigenvectors of this operator, be given. How are the pseudosolutions and a normal pseudosolution of the equation $Ax = b$ found?

7.8.8*. Let A be an operator of rank r from an n-dimensional space X into an m-dimensional space Y. Given an orthonormal basis $e_1, \ldots, e_n$, containing the eigenvectors of the operator A^*A and the corresponding eigenvalues $\alpha_1^2, \ldots, \alpha_n^2$ ($\alpha_i > 0$, $i = 1, \ldots, r$), prove that (a) the pseudosolutions of the equation $Ax = b$ are described by the formula

$$x = \beta_1 e_1 + \ldots + \beta_r e_r + \gamma_{r+1} e_{r+1} + \ldots + \gamma_n e_n,$$

where

$$\beta_i = \frac{(b, Ae_i)}{(Ae_i, Ae_i)} = \frac{(A^*b, e_i)}{\alpha_i^2}, \quad i = 1, \ldots, r,$$

and $\gamma_{r+1}, \ldots, \gamma_n$ are arbitrary numbers; (b) the normal pseudosolution is the vector

$$x_0 = \beta_1 e_1 + \ldots + \beta_r e_r.$$

7.8.9. Given an orthonormal basis $f_1, \ldots, f_m$, containing the eigenvectors of the operator AA^* (while $\alpha_i > 0$, $i = 1, \ldots, r$), prove that the normal pseudosolution of the equation $Ax = b$ can be found by the formula

$$x_0 = \xi_1 A^* f_1 + \ldots + \xi_r A^* f_r,$$

where

$$\xi_i = \frac{(b, f_i)}{\alpha_i^2}, \quad i = 1, \ldots, r.$$

Find the normal pseudosolutions of the following systems of linear equations, assuming that the scalar products on the corresponding arithmetic spaces are defined by (7.1.4).

7.8.10.
$$279x_1 + 362x_2 - 408x_3 = 0,$$
$$515x_1 - 187x_2 + 734x_3 = 0$$

7.8.11*.
$$\begin{aligned} 27x_1-55x_2&=1,\\ -13x_1+27x_2&=1,\\ -14x_1+28x_2&=1.\end{aligned}$$

7.8.12.
$$\begin{aligned} x_1+x_2+x_3+x_4&=2,\\ x_1+x_2+x_3+x_4&=3,\\ x_1+x_2+x_3+x_4&=4.\end{aligned}$$

7.8.13.
$$\begin{aligned} x_1+x_2&=2,\\ x_1-x_2&=0,\\ 2x_1+x_2&=2.\end{aligned}$$

7.8.14.
$$\begin{aligned} -x_1-2x_2&=1,\\ 2x_1+4x_2&=0,\\ x_1+2x_2&=0,\\ 3x_1+6x_2&=0.\end{aligned}$$

7.8.15.
$$\begin{aligned} 2x_1-x_2\quad &=1,\\ -x_1+x_2+x_3&=0,\\ x_2+2x_3&=1.\end{aligned}$$

7.8.16.
$$\begin{aligned} 2x_1-x_2\qquad\qquad &=1,\\ -x_1+(1+\varepsilon)\,x_2+x_3&=0,\quad (\varepsilon\neq 0),\\ x_2+2x_3&=1.\end{aligned}$$

7.8.17.
$$\begin{aligned} 2x_1-x_2\qquad\qquad &=1,\\ -x_1+x_2+x_3\qquad &=0,\\ x_2+(2+\varepsilon)\,x_3&=1,\quad (\varepsilon\neq 0).\end{aligned}$$

7.8.18*.
$$\begin{aligned} 5x_1-3x_4\qquad &=2,\\ 4x_2+2x_3+2x_5&=3,\\ 2x_2+2x_3\qquad &=0,\\ -3x_1+x_4\qquad &=-2,\\ 2x_2+2x_5\qquad &=3.\end{aligned}$$

7.8.19. Find pseudoinverse operator of the null operator from X into Y.

7.8.20. Prove that the pseudoinverse operator of a nondegenerate operator coincides with its inverse.

7.8.21. Find the pseudoinverse operator of the differential operator on the space M_n of polynomials with the scalar product defined as in (7.1.7). Compare the obtained operator with the conjugate (see 7.1.34).

7.8.22. Prove that for any operator A and a nonzero number α

$$(\alpha A)^+=\frac{1}{\alpha}A^+.$$

7.8.23. Prove that for any unitary operators U and V:

(a) $(UA)^+ = A^+U^*$,

(b) $(AV)^+ = V^*A^+$.

7.8.24. Show that the image and kernel of the pseudoinverse operator A^+ coincide with the image and kernel, respectively, of the conjugate operator A^*.

7.8.25. Consider an operator A as an operator from T_{A^*} into T_A, and the pseudoinverse operator A^+ as an operator from T_A into T_{A^*}. Show that the operators A and A^+ are inverse to each other on this pair of subspaces. This means that for any vector x from T_{A^*} and any vector y from T_A,

$$A^+Ax = x, \quad AA^+y = y.$$

7.8.26. Show that stating the properties of the pseudoinverse operator listed in 7.8.24 and 7.8.25, together with that of linearity, is equivalent to the definition of the pseudoinverse operator.

7.8.27. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_m$ be singular bases for an operator A. Find the matrix of the pseudoinverse operator A^+ with respect to this pair of bases.

7.8.28. Show that singular bases for an operator A are also singular for the pseudoinverse operator A^+. Meanwhile, the nonzero singular values of the operators are reciprocal.

7.8.29. Show that $(A^+)^+ = A$.

7.8.30. Show that $(A^*)^+ = (A^+)^*$.

7.8.31. Show that the pseudoinverse operator of a Hermitian operator is also Hermitian.

7.8.32. Show that the pseudoinverse operator A^+ of a normal operator A is also normal. Find the relationship between the eigenvalues of the operators A and A^+.

7.8.33. Prove that a normal operator A satisfies for any k the relation $(A^k)^+ = (A^+)^k$.

7.8.34. Prove that the pseudoinverse operator of a positive-semidefinite operator is also positive-semidefinite.

7.8.35. Let $A = HU$, and let $A = U_1H_1$ be polar representations of an operator A. Find polar representations of the operator A^+.

7.8.36. Prove that for an operator A to coincide with its pseudoinverse operator, it is necessary and sufficient that (a) the image T_A and kernel N_A should be orthogonal; (b) the induced operator A/T_A should satisfy the equality

$$(A/T_A)^{-1} = A/T_A.$$

In particular, these conditions are fulfilled for an operator of orthogonal projection.

7.8.37*. Let operators A and B satisfy the relations $A^*B = 0$ and $BA^* = 0$. Prove that $(A + B)^+ = A^+ + B^+$.

7.8.38*. Operators A and B are such that $T_A = T_{B^*}$. Prove that $(BA)^+ = A^+B^+$.

7.8.39. Prove the equality

$$AA^+A = A.$$

7.8.40. Show that the geometric meaning of the equation

$$AXA = A \tag{7.8.1}$$

in a linear operator X is that the operators A and X are inverse to one another on the pair of subspaces XT_A and T_A in the sense defined in 7.8.25.

7.8.41. Prove that a pseudoinverse operator A^+ can be defined as a linear operator satisfying equation (7.8.1) and having the same image and kernel as the conjugate operator A^*.

7.8.42*. Prove that each of the definitions indicated below is equivalent to the definition of a pseudoinverse operator:

(a) an operator X satisfying equation (7.8.1) and such that

$$X = A^*B = CA^*$$

for certain linear operators B and C; (b) an operator X satisfying equation (7.8.1) and such that

$$X = A^*DA^*$$

for a certain linear operator D; (c) an operator X satisfying the equation $A^*AX = A^*$ and such that

$$X = A^*AF$$

for a certain linear operator F.

7.8.43. Prove that the rank of the operator $(A^+)^2$ equals the rank of the operator A^2.

7.8.44. Given an operator A from a space X to a space Y, prove that the operator A^+A is Hermitian and projects the space X orthogonally onto the subspace T_{A^*}.

7.8.45. Describe the geometric meaning of the requirements for an operator X to be specified by the system of equations

$$AXA = A,$$
$$(XA)^* = XA. \tag{7.8.2}$$

7.8.46. Prove the equality $A^+AA^+ = A^+$.

7.8.47. An operator X satisfies system (7.8.2). What does a new requirement for this operator specified by the equation

$$XAX = X$$

mean?

7.8.48. Prove that the operator A (see Problem 7.8.44) stipulates that the operator AA^+ is Hermitian and projects the space Y orthogonally onto the subspace T_A.

7.8.49. Prove that the conditions

$$AXA = A, \qquad XAX = X,$$
$$(XA)^* = XA, \quad (AX)^* = AX$$

determine the pseudoinverse operator uniquely. These conditions are called the *Penrose equations* after a British mathematician who was one of the first to introduce the notion of a pseudoinverse operator (a pseudoinverse matrix, actually).

7.9. Quadratic Forms

In this section we shall focus our attention mostly on:

The reduction of a quadratic form to its canonical form by an orthogonal transformation of the unknowns.

The law of inertia, relation of congruence, and the use of principal minors for finding the indices of inertia.

Simultaneous reduction of a pair of quadratic forms.

The Lagrange method of the reduction to canonical form, considered only with regard to positive-definite forms. Hence, the possibility that a positive-definite matrix may be reduced into the product of two triangular matrices, each the transpose of the other. Such a reduction forms the basis for one of the most efficient techniques for the solution of systems of linear equations with matrices of this class. We have paid particular attention to this method and its computational aspects.

Note that all the matrices considered in the present section are assumed to be real.

For each of the quadratic forms below, find an orthogonal transformation of the unknowns that makes the form canonical, and state the canonical form obtained.

7.9.1. $2x_1^2 + 5x_2^2 + 2x_3^2 - 4x_1x_2 - 2x_1x_3 + 4x_2x_3$.

7.9.2. $-3x_2^2 + 4x_1x_2 + 10x_1x_3 - 4x_2x_3$.

7.9.3. $-x_1^2 + x_2^2 - 5x_3^2 + 6x_1x_3 + 4x_2x_3$.

7.9.4. $2x_1x_4 + 6x_2x_3$.

7.9.5. $x_1^2 + 4x_2^2 + x_3^2 + 4x_4^2 + 4x_1x_2 + 2x_1x_3 + 4x_1x_4 + 4x_2x_3 + 8x_2x_4 + 4x_3x_4$.

7.9.6*. Suppose a quadratic form $F(x_1, \ldots, x_n)$ is reduced by some (even degenerate) transformation to the form

$$F = y_1^2 + \ldots + y_k^2 - y_{k+1}^2 - \ldots - y_{k+l}^2.$$

Prove that the positive index of inertia for the form F does not exceed k, and that the negative index of inertia does not exceed l.

7.9.7. Prove that for separating a quadratic form into the product of two linear forms, it is necessary and sufficient that the rank of the form should not exceed two, and that the signature should be equal to zero if the rank equals two.

7.9.8. Show that the rank and signature of a quadratic form are either both odd or both even.

7.9.9. Real $n \times n$ matrices A and B are said to be *congruent* if there exists a nondegenerate matrix P such that $B = P^T A P$. Show that the congruence relation on the set of square matrices of a given order is reflexive, symmetric and transitive.

7.9.10. Prove that a matrix A is congruent to a diagonal matrix if and only if it is symmetric.

7.9.11. Prove that symmetric matrices A and B are congruent if and only if they have the same number of positive and negative eigenvalues.

7.9.12*. Using the properties of the eigenvalues and principal submatrices of symmetric matrices (see 7.4.35), prove that if a matrix A is the matrix of a quadratic form F in n unknowns and if all the leading principal minors of the matrix A are different from zero, then the positive (negative) index of inertia of the form F equals the number of repetitions (changes) of sign in the number sequence

$$1, D_1, D_2, \ldots, D_n,$$

where D_i is the leading principal minor of order i. This *rule* for finding the indices of inertia was introduced by *Jacobi*.

7.9.13*. The minor D_k, $k < n$ (see Problem 7.9.12) is zero, but the minors D_{k-1} and D_{k+1} are nonzero. Prove that $D_{k-1}D_{k+1} < 0$.

7.9.14*. Assume that the determinant $D_n \neq 0$ in the sequence $1, D_1, \ldots, D_n$ but if $k < n$, then the minor D_k may be zero. In which case, assume, additionally, that both D_{k-1} and D_{k+1} are nonzero. By giving arbitrary signs to the zero values of D_k, show that the Jacobi rule for finding the indices of inertia is still valid for this case. This modification to the Jacobi rule is due to *Gundelfinger*.

7.9.15. Deduce statements 7.4.44 and 7.4.45 from 7.9.12 and 7.9.14.

Compute the indices of inertia for the following quadratic forms.

7.9.16. $x_1x_2 + x_2x_3 + x_3x_4$.

7.9.17. $x_1x_2 + 2x_1x_3 + 3x_1x_4 + x_2x_3 + 2x_2x_4 + x_3x_4$.

7.9.18. $x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 4x_2x_3 + 4x_2x_4 + 6x_3x_4$.

7.9.19. Let the coefficient a_{11} in a quadratic form $F(x_1, \ldots, x_n)$ be greater than zero. What will be the result of the following transformation of the unknowns

$$y_1 = \frac{1}{\sqrt{a_{11}}}(a_{11}x_1 + \ldots + a_{1n}x_n),$$

$$y_i = x_i, \quad i = 2, \ldots, n?$$

7.9.20*. Prove that a positive-definite quadratic form can be reduced to the normal one by a triangular transformation of the unknowns, i.e. by a transformation of the form

$$\begin{aligned} y_1 &= s_{11}x_1 + s_{12}x_2 + \dots + s_{1n}x_n, \\ y_2 &= \qquad\quad s_{22}x_2 + \dots + s_{2n}x_n, \\ &\dots\dots\dots\dots\dots\dots \\ y_n &= \qquad\qquad\qquad\quad s_{nn}x_n, \end{aligned} \tag{7.9.1}$$

where $s_{11}, s_{22}, \dots, s_{nn}$ are different from zero.

Reduce the following quadratic forms to normal ones by a triangular transformation:

7.9.21. $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 + 4x_2x_3$.

7.9.22. $x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$.

7.9.23. $x_1^2 + 4x_2^2 + 11x_3^2 + 24x_4^2 - 2x_1x_3 - 4x_1x_4 + 4x_2x_3 + 16x_3x_4$.

7.9.24. Prove that for any positive-definite matrix A, there is a so-called *triangular decomposition*, i.e.

$$A = S^T S, \tag{7.9.2}$$

where S is an upper triangular matrix.

7.9.25. Show that the diagonal elements of the matrix S in the triangular decomposition (7.9.2) and the leading principal minors D_i of the matrix A are connected by the relations

$$s_{ii}^2 = \frac{D_i}{D_{i-1}}, \quad i = 1, \dots, n; \ D_0 = 1.$$

The same is true for the diagonal elements of formulae (7.9.1).

7.9.26. Prove that a triangular decomposition (7.9.2) of a positive-definite matrix A, where all the diagonal elements s_{ii} are positive, is unique.

7.9.27. Show that the elements of the matrix S in decomposition (7.9.2) can be calculated consecutively in the order $s_{11}, s_{12}, \dots, s_{1n}, s_{22}, s_{23}, \dots, s_{nn}$ by the formulae

$$s_{11} = \sqrt{a_{11}}, \quad s_{1j} = \frac{a_{1j}}{s_{11}}, \quad j = 2, \dots, n,$$

$$s_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} s_{ki}^2} \quad (i > 1), \tag{7.9.3}$$

$$s_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} s_{ki}s_{kj}}{s_{ii}} \quad (j > i).$$

Using formulae (7.9.3), find triangular decompositions of the following matrices:

7.9.28. $\begin{Vmatrix} 4 & 2 & -2 \\ 2 & 5 & 1 \\ -2 & 1 & 3 \end{Vmatrix}$. **7.9.29.** $\begin{Vmatrix} 9 & -3 & 0 \\ -3 & 5 & -4 \\ 0 & -4 & 5 \end{Vmatrix}$.

7.9.30. $\begin{Vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{Vmatrix}$.

7.9.31. A positive-definite matrix A is of the band structure, i.e. $a_{ij} = 0$ when $|i - j| > d > 0$. Using formulae (7.9.3), show that $s_{ij} = 0$ when $j - i > d$.

7.9.32. Find the triangular decomposition of the following tri-diagonal matrix of order n

$$A = \begin{Vmatrix} 1 & \sqrt{2} & & & & \\ \sqrt{2} & 3 & \sqrt{2} & & & \\ & \sqrt{2} & 3 & \sqrt{2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 3 & \sqrt{2} \\ & & & & \sqrt{2} & 3 \end{Vmatrix}.$$

7.9.33. Show that if a matrix S represents the triangular decomposition of a matrix A, then its principal submatrix S_k represents the triangular decomposition of the submatrix A_k of the matrix A.

7.9.34. Using the result of Problem 7.9.30, find the triangular decomposition of the matrix

$$\begin{Vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 8 & 14 & 20 & 26 \\ 4 & 11 & 20 & 30 & 40 \\ 5 & 14 & 26 & 40 & 55 \end{Vmatrix}.$$

7.9.35. Prove that for the elements of the matrix S of the triangular decomposition in (7.9.2), the following inequality holds:

$$\max_{i,j} |s_{ij}| \leqslant \max_i \sqrt{a_{ii}}.$$

Hence, deduce that if for the matrix A, $\max\limits_{i,j} |a_{ij}| = 1$, then $\max\limits_{i,j} |s_{ij}| \leqslant 1$.

In these circumstances when the triangular decomposition of a positive-definite matrix is being computed, growth of elements (in the indicated sense) will not occur.

7.9.36. Find the number of multiplication, division, and square root operations that are needed to obtain the triangular decomposition matrix using formulae (7.9.3).

7.9.37. Given the triangular decomposition of a positive-definite matrix A, find a method to solve the system of linear equations $Ax = b$.

7.9.38. Find the total number of multiplications and divisions necessary to solve the system of linear equations $Ax = b$ (where A is a positive-definite matrix) when a triangular decomposition using formulae (7.9.3) followed by an application of triangular matrices to the subsequent systems (see 7.9.37) is used. Compare this with the number of multiplication and division operations necessary for Gaussian elimination.

The method indicated for the solution of a system of linear equations with a positive-definite matrix is called the *square root method.*

7.9.39. Prove that a positive-definite matrix A can be also represented as the product

$$A = S_1 S_1^T, \tag{7.9.4}$$

where S_1 is an upper triangular matrix.

7.9.40. Let A be a positive-definite matrix, and $\tilde{A}$ the matrix obtained when the elements of A are reflected through the centre of A, $\tilde{A} = \tilde{S}^T\tilde{S}$ is then the triangular decomposition of the matrix $\tilde{A}$. Prove that this representation (7.9.4) of A can be obtained by reflecting each of the matrices $\tilde{S}^T$ and $\tilde{S}$ through their centres.

7.9.41. Prove that two quadratic forms F and G both in the same unknowns can be reduced to canonical form by the same nondegenerate linear transformation if at least one of the forms F and G is positive-definite.

7.9.42. Given quadratic forms F and G both in the same unknowns, the form G being nondegenerate. Prove that if a nondegenerate linear transformation exists that can reduce both F and G to canonical form:

$$F = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2,$$
$$G = \mu_1 y_1^2 + \ldots + \mu_n y_n^2,$$

then the set of ratios

$$\frac{\lambda_1}{\mu_1}, \frac{\lambda_2}{\mu_2}, \ldots, \frac{\lambda_n}{\mu_n}$$

is the same for any such transformation. These ratios are the roots of the so-called *z-equation of the pair of forms F* and *G*, i.e. $|A - zB| = 0$, where A and B are the matrices for F and G, respectively.

7.9.43. Quadratic forms F and G are positive-definite. Consider two nondegenerate linear transformations, the first reducing F to the canonical form $\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2$ and G to a normal form, and the other reducing F to a normal form and G to the canonical form $\mu_1 z_1^2 + \ldots + \mu_n z_n^2$. How are the coefficients $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ related?

7.9.44. Prove that the forms F and G can both be reduced to canonical form by the same nondegenerate linear transformation if the matrices of these forms are commuting.

For each of the following pairs of quadratic forms, find a nondegenerate linear transformation that reduces both to canonical form and state the obtained forms:

7.9.45. $F = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3,$

$G = 2x_1^2 + 8x_2^2 + 3x_3^2 + 8x_1x_2 + 2x_1x_3 + 4x_2x_3.$

7.9.46. $F = x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3,$

$G = x_1^2 - 2x_2^2 + x_3^2 + 4x_1x_2 - 10x_1x_3 + 4x_2x_3.$

7.9.47. $F = -x_1^2 - 5x_2^2 - 14x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3,$

$G = -x_1^2 - 14x_2^2 - 4x_3^2 + 8x_1x_2 - 2x_1x_3 + 4x_2x_3.$

7.9.48. $F = x_1^2 + 3x_2^2 + x_3^2 - x_4^2 - 2x_1x_2 - 4x_2x_3 + 2x_3x_4,$

$G = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4.$

7.9.49. $F = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 4x_2x_4 + 2x_3x_4,$

$G = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4 - 2x_2x_3 + 2x_2x_4 - 2x_3x_4.$

7.9.50. Assume that F and G are quadratic forms both in the same unknowns $x_1, \ldots, x_n$, and G is positive-definite, and enumerate the roots of the z-equation in descending order $z_1 \geqslant z_2 \geqslant \ldots \geqslant z_n$. Prove that for the biggest root z_1 and smallest root z_n, the following representations are true

$$z_1 = \max_{x_1^2+\ldots+x_n^2 \neq 0} \frac{F(x_1, \ldots, x_n)}{G(x_1, \ldots, x_n)},$$

$$z_n = \min_{x_1^2+\ldots+x_n^2 \neq 0} \frac{F(x_1, \ldots, x_n)}{G(x_1, \ldots, x_n)}.$$

7.9.51. Formulate and prove the analogue of the Courant-Fischer theorem for the pair of the forms in the preceding problem.

Metric Problems in Linear Space

8.0. Terminology and General Notes

A set X is called a *metric space* if to each pair of its elements x and y, there is assigned a nonnegative number $\rho(x, y)$ called the *distance between x and y*, and the following conditions are fulfilled:
(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$;
(iii) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

If M_1 is a subset of a metric space X, then the set of all elements $x \in X$, not belonging to M_1, is called the *complement* of the set M_1. If $M_1, M_2, \ldots$ are the subsets of X, then the set of all elements, each of which belongs to at least one of the sets $M_1, M_2, \ldots$, is called the *union* of $M_1, M_2, \ldots$. The set of all elements, which are elements of each of the sets $M_1, M_2, \ldots$, is called the *intersection* of $M_1, M_2, \ldots$.

The set of all elements x from X fulfilling the condition

$$\rho(a, x) < r$$

is called *ι sphere* $S(a, r)$. The element a is called the *centre* of the sphere, and the positive number r the *radius* of the sphere.

The *neighbourhood* of an element x is any sphere, centre x. A set M in a metric space X is said to be *open* if it contains for every element x some neighbourhood of the element.

An element $x \in X$ is called a *boundary point* of a set M if any neighbourhood of this element contains at least one element from M which does not coincide with x. The set obtained from M by adding all its boundary points, is called the *closure* of the set M and denoted by $\overline{M}$. A set M is *closed* if $M = \overline{M}$.

The set $\overline{S}(a, r)$ of all elements x from X fulfilling the condition

$$\rho(x, a) \leqslant r$$

is called a *closed sphere*, centre a and radius r.

An element x_0 from a metric space X is called the *limit of a sequence* $\{x_n\}$ of elements $x_1, x_2, \ldots, x_n, \ldots$ from X if $\rho(x_0, x_n) \to 0$ as $n \to \infty$. We write this as

$$x_n \to x$$

or

$$\lim_{n\to\infty} x_n = x_0.$$

A sequence $\{x_n\}$ which has a limit (x_0) is said to be *convergent*.

A sequence $\{x_n\}$ of elements from a metric space is said to be *fundamental* if for any number $\varepsilon > 0$ there is another number $N(\varepsilon)$ such that $\rho(x_n, x_m) < \varepsilon$, when $n, m \geqslant N(\varepsilon)$.

If any fundamental sequence in a metric space X converges to a limit, then the space is said to be *complete*.

A real or complex linear space X is called a linear *normed space* if each vector $x \in X$ has an associated real number $\| x \|$ called the *norm* of the vector x, and which fulfils the following conditions:

(i) $\| x \| \geqslant 0$, moreover $\| x \| = 0$ only if $x = 0$;
(ii) $\| x + y \| \leqslant \| x \| + \| y \|$ (the triangle inequality); (8.0.1)
(iii) $\| \lambda x \| = | \lambda | \, \| x \|$.

A normed space can be treated as a metric space if we put

$$\rho(x, y) = \| x - y \|.$$

The convergence of a sequence with respect to the distance thus defined is called the *convergence with respect to the norm*.

A set M in a linear normed space X is said to be *bounded* if there is a positive number C such that $\| x \| \leqslant C$ for all x from M.

The *unit sphere* of a normed space X is the set of all vectors x for which $\| x \| \leqslant 1$ $(\| x \| = 1)$.

A set M in a normed space is said to be *convex* if, in addition to any two of its vectors x and y, it also contains the whole segment $\lambda x + (1 - \lambda) y$, $0 \leqslant \lambda \leqslant 1$.

Any finite-dimensional linear normed space X is a complete metric space. Moreover, describing a set M from X as bounded is equivalent to describing the coordinates of all vectors x from M with respect to any basis for the space X as bounded. Similarly, the convergence of a sequence $\{x_k\}$ to a vector x_0 is equivalent to the convergence of the coordinates of the vectors x_k to the corresponding coordinates of the vector x_0 with respect to any basis for the space X.

An example of a normed space is the n-dimensional arithmetic space in which the norm of a vector $x = (\alpha_1 \alpha_2, \ldots, \alpha_n)^T$ is defined by the equality

$$\| x \|_p = (| \alpha_1 |^p + | \alpha_2 |^p + \ldots + | \alpha_n |^p)^{1/p}, \quad p \geqslant 1. \qquad (8.0.2)$$

The triangle inequality for this norm is called the *Minkowski inequality*. Its proof is based on the following *Holder inequality*

$$\sum_{k=1}^{n} |\alpha_k \beta_k| \leqslant \Big(\sum_{k=1}^{n} |\alpha_k|^p \Big)^{1/p} \Big(\sum_{k=1}^{n} |\beta_k|^q \Big)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1. \qquad (8.0.3)$$

Let X and Y be normed spaces with the norms $\| x \|_X$ and $\| y \|_Y$, respectively. The norm $\| A \|$ on the space of the operators ω_{XY} is said to be *consistent* with the vector norms on spaces X and Y if

$$\| Ax \|_Y \leqslant \| A \| \| x \|_X \tag{8.0.4}$$

for all $x \in X$ and any operator $A \in \omega_{XY}$.

If X is a normed space with the norm $\| x \|$, then the norm on the space ω_{XX}, defined by the equality

$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}, \tag{8.0.5}$$

is said to be *subordinate* to the vector norm $\| x \|$. Besides the usual conditions (8.0.1), the secondary norm also possesses the following special property with respect to operator multiplication:

$$\| AB \| \leqslant \| A \| \| B \|. \tag{8.0.6}$$

The definitions of a consistent and subordinate norm can be extended immediately to spaces of matrices considered as operators on arithmetic spaces. In particular, if the norm $\| x \|_p$ (see (8.0.2)) is defined on an arithmetic space, then the corresponding subordinate norm is designated by $\| A \|_p$. The norms $\| A \|_1$, $\| A_2 \|$, $\| A \|_\infty$ are considered most often.

Even if the matrix norm under consideration is not subordinate, we shall assume that equality (8.0.6) is valid for it.

If a matrix A has the form $A = E + B$, where $\| B \| < 1$ for some matrix norm, then A is nondegenerate and the norm of the inverse matrix can be estimated by:

$$\| A^{-1} \| \leqslant \frac{\| E \|}{1 - \| B \|}. \tag{8.0.7}$$

Consider the system of linear equations

$$Ax = b$$

with a square nondegenerate matrix A and a *perturbed* system

$$(A + \varepsilon_A) \tilde{x} = b + \varepsilon_b.$$

The matrix ε_A is assumed to satisfy the inequality

$$\| \varepsilon_A \| < \| A^{-1} \|^{-1}.$$

This condition is sufficient for the matrix $A + \varepsilon_A$ to be nondegenerate. If we put

$$\delta x = \frac{\| x - \tilde{x} \|}{\| x \|}, \quad \delta A = \frac{\| \varepsilon_A \|}{\| A \|}, \quad \delta b = \frac{\| \varepsilon_b \|}{\| b \|},$$

then the following estimate is true

$$\delta x \leqslant \frac{\| A \| \| A^{-1} \|}{1 - \| A \| \| A^{-1} \| \delta A} (\delta A - \delta b). \tag{8.0.8}$$

Here, the matrix norm $\| A \|$ is assumed to be subordinate to the vector norm $\| x \|$.

The product $\| A \| \| A \|^{-1}$ is called the *condition number* of the matrix A and denoted by cond (A). If it is necessary to state explicitly to which matrix norm a condition number refers, then we shall write $\text{cond}_1 (A)$, $\text{cond}_2 (A)$ or $\text{cond}_\infty (A)$.

As can be seen from estimate (8.0.8), a condition number characterizes the sensitivity of a system of linear equations $Ax = b$ to perturbations of its coefficients. Matrices with large condition numbers are said to be *ill-conditioned*.

Suppose an $n \times n$ matrix A with the eigenvalues $\lambda_1, \ldots, \lambda_n$ is of simple structure, and X is a nondegenerate matrix whose columns are the eigenvectors of the matrix A. Then all the eigenvalues of the matrix $A + \varepsilon_A$ are in a region of the complex plane which is the union of n circles

$$| z - \lambda_i | \leqslant \text{cond}(X) \| \varepsilon_A \|, \quad i = 1, \ldots, n. \tag{8.0.9}$$

Here, the matrix norm is understood to be one of the norms $\| A \|_1$, $\| A \|_2$, $\| A \|_\infty$.

8.1. Normed Linear Space

In addition to the basic metric notions, another two topics are considered in this section: the equivalence of the norms on a finite-dimensional linear space, and a duality relation between the norms and the scalar product. The theory of dual norms will make it possible to introduce in the next section a relation ordering the set of operator norms.

8.1.1. Show that the length of a vector in a Euclidean (unitary) space fulfils the conditions of a norm.

8.1.2. Given a fixed basis $e_1, \ldots, e_n$ for an n-dimensional space X and an arbitrary vector x from X whose decomposition with respect to this basis is

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n.$$

Show that a norm can be defined for X by any of the following equalities

(a) $\| x \|_1 = | \alpha_1 | + | \alpha_2 | + \ldots + | \alpha_n |$;
(b) $\| x \|_2 = (| \alpha_1 |^2 + | \alpha_2 |^2 + \ldots + | \alpha_n |^2)^{1/2}$;
(c) $\| x \|_\infty = \max\limits_i | \alpha_i |$;
(d) generally, for any positive number p, $p > 1$,

$$\| x \|_p = (| \alpha_1 |^p + | \alpha_2 |^p + \ldots + | \alpha_n |^p)^{1/p}.$$

8.1.3. Let $m(x)$ and $n(x)$ be two norms for a linear space X. Show that

(a) $p(x) = \max(m(x), n(x))$;

(b) $q(x) = \alpha m(x) + \beta n(x)$, where α and β are fixed nonnegative numbers and never both zero;

(c) $r(x) = (m^2(x) + n^2(x))^{1/2}$

are also norms for this space.

8.1.4. Let P be a linear nondegenerate operator on a normed linear space X with respect to the norm $\| x \|$. Prove that $m(x)$, where

$$m(x) = \| Px \|, \tag{8.1.1}$$

is also a norm for the space X.

8.1.5. A linear space X is the direct sum of subspaces L_1 and L_2 and, in addition, norms $m(x)$ and $n(x)$ are defined on L_1 and L_2, respectively. Let x be an arbitrary vector from X, and let $x = x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$. Show that a norm on the space X can be defined thus:

$$\| x \| = m(x_1) + n(x_2).$$

8.1.6. If the requirement that the norm in the definition of a norm should be equal to zero only in the case of the null vector is omitted then the vector function thus obtained is called the *seminorm*. Hence, the seminorm $\| x \|$ is specified by the conditions:

(a) $\| x \| \geqslant 0$;

(b) $\| \alpha x \| = | \alpha | \, \| x \|$;

(c) $\| x + y \| \leqslant \| x \| + \| y \|$.

Prove that if a seminorm $\| x \|$ is defined on a linear space X, then: (a) the set of vectors, for which the seminorm equals zero, is a linear subspace L of the space X; (b) all the vectors in the plane $x_0 + L$ have the same seminorm; (c) by matching each plane $x_0 + L$ with the common value of the seminorm of its vectors, a norm on the factor-space of the space X is obtained with respect to the subspace L.

8.1.7. Prove that for any four vectors x, y, z, u of a normed space, the following inequality holds

$$| \, \| x - y \| - \| z - u \| \, | \leqslant \| x - z \| + \| y - u \|.$$

8.1.8. Prove that the sphere $\| x - x_0 \| < r$ is an open set.

8.1.9. Prove that the union of any number of open sets is an open set.

8.1.10. Show that any sphere is a bounded set.

8.1.11. Show that any plane of positive dimension is not a bounded set.

8.1.12. Show that any sphere is a convex set.

8.1.13. Show that any plane of positive dimension is a convex set.

8.1.14. Prove that the sphere $\| x - x_0 \| \leqslant r$ is a closed set.

8.1.15. Prove that the complement of an open set is a closed set.

8.1.16. Prove that the complement of a closed set is open.

8.1.17. Show that the intersection of any number of closed sets is a closed set.

8.1.18. Show that the union of any finite number of closed sets is a closed set. Set up an example demonstrating that the union of infinitely many closed sets may not be a closed set.

8.1.19. Prove that if $x_k \to x_0$ and $y_k \to y_0$, then: (a) $\| x_k \| \to \| x_0 \|$; (b) $\| x_k - a \| \to \| x_0 + a \|$ for any vector a; (c) $\alpha x_k + \beta y_k \to \alpha x_0 + \beta y_0$ for any numbers α and β; (d) if a sequence of numbers λ_k converges to a number λ_0, then $\lambda_k x_k \to \lambda_0 x_0$.

8.1.20. Prove that if any nontrivial subsequence of a sequence $\{x_k\}$ converges, then the sequence $\{x_k\}$ itself also converges. A subsequence is trivial if it coincides with the original sequence from some term onwards.

8.1.21. Prove that if x_0 is a boundary point of a set M, then there is a sequence $\{x_k\}$, $x_k \in M$, convergent to x_0.

8.1.22. Prove that the closure of a convex set is also a convex set.

8.1.23. Prove that a convergent subsequence can be singled out from any bounded sequence of vectors of a normed space.

8.1.24. Prove that any infinite bounded set has boundary points.

8.1.25. The quantity

$$\rho(x, M) = \inf_{y \in M} \| x - y \|$$

is called the *distance from a vector x to a set M*. Show that if M is a closed set, then there is $y_0 \in M$ such that $\rho(x, M) = \| x - y_0 \|$.

8.1.26. The quantity

$$\rho(M_1, M_2) = \inf_{x \in M_1, \quad y \in M_2} \| x - y \|$$

is called the *distance between the sets* M_1 and M_2. Prove that if the sets M_1 and M_2 are closed and bounded, then there are $x_0 \in M_1$ and $y_0 \in M_2$ such that $\rho(M_1, M_2) = \| x_0 - y_0 \|$.

8.1.27. Show that the result of Problem 8.1.26 remains valid if the requirement for the boundedness of one of the sets M_1 and M_2 is omitted. Give an example to demonstrate that this statement becomes invalid if neither set (M_1 and M_2) is bounded.

8.1.28. Are the vectors x_0 and y_0 in Problems 8.1.25, 8.1.26, 8.1.27 unique?

8.1.29*. Assume M to be a convex set of a Euclidean (unitary) space, and consider the length of a vector as its norm. Prove that the vector y_0 (see Problem 8.1.25) is determined uniquely in these circumstances.

8.1.30. Let M_1 and M_2 be closed bounded sets. Prove that the set N, made up of all vectors having the form $x + y$, where $x \in M_1$, $y \in M_2$, is closed and bounded.

8.1.31. The sets M_1 and M_2 are closed, moreover the set M_1 is bounded. Prove that the statement of Problem 8.1.30 about the boundedness of the set N is also valid in this case. Give an example demonstrating that when the sets M_1 and M_2 are closed and not bounded, then the set N is not closed.

8.1.32*. Given that X is a real (complex) linear space. A *functional* on X is a mapping from the space X into a set of real (complex) numbers. For a normed space X, a functional $F(x)$ is *continuous at a point* x_0 if $F(x_k) \to F(x_0)$ as $x_k \to x_0$. A functional $F(x)$ is *continuous on a set* M if it is continuous at every x_0 in M and a continuous functional is *continuous* at every x from X.

Prove that (a) any linear functional on the space X is continuous; (b) if $\| x \|$ is a norm defined on X, then any other norm $m(x)$ on the space X is a continuous functional with respect to $\| x \|$.

8.1.33*. Let M be a closed bounded set, and let a functional $F(x)$ be continuous on the set M. Prove that there is a positive number c such that $| F(x) | \leqslant c$ for all x from M.

8.1.34*. Prove that in the set M (see the previous problem), there is a vector x_0 such that $| F(x_0) | = \max\limits_{x \in M} | F(x) |$.

8.1.35*. Prove that for any two norms $m(x)$ and $n(x)$ on a linear space X, there are two positive numbers c_1 and c_2 such that

$$c_1 n(x) \leqslant m(x) \leqslant c_2 n(x). \tag{8.1.2}$$

How can the largest possible number c_1 and smallest possible number c_2 be selected?

8.1.36. For each pair of the three norms $\| x \|_1$, $\| x \|_2$, $\| x \|_\infty$ (see 8.1.2), find the best possible c_1 and c_2 for inequalities (8.1.2).

8.1.37*. Consider for n-dimensional arithmetic space, the norms

$$\| x \|_2 = (| \alpha_1 |^2 + \ldots + | \alpha_n |^2)^{1/2}$$

and

$$m(x) = \| Px \|_2,$$

where P is a nondegenerate $n \times n$ matrix. How can the best possible constants c_1 and c_2 in inequalities (8.1.2) be computed?

8.1.38. Prove that a set M, contained by a space X, and open with respect to a norm $m(x)$ on this space, is also open with respect to any other norm.

8.1.39. Prove that a set M, closed with respect to a norm on a space X, is also closed with respect to any other norm on this space.

8.1.40. Prove that any plane in a normed space X is a closed and not an open set (with the exception of the set X itself).

8.1.41*. A set X is the direct sum of the subspaces L_1 and L_2. A closed set M_1 is contained in L_1, and a closed set M_2 in L_2. Prove that the set N made up of all sums $x + y$, where $x \in M_1$, $y \in M_2$, is closed. Note that in contrast to 8.1.31, no condition on the bounds of the sets M_1 and M_2 is required here.

8.1.42. A norm $m(x)$ is considered other than the length of the vector on a Euclidean (unitary) space X. Show that for any y from X, the expression

$$m^*(y) = \sup_{x \neq 0} \frac{|(x, y)|}{m(x)} \tag{8.1.3}$$

is always finite and satisfies all the conditions of a norm. This norm $m^*(y)$ is said to be *dual to the norm* $m(x)$ with respect to the scalar product (x, y).

8.1.43. Show that the definition of a dual norm is equivalent to each of the following expressions:

(a) $m^*(y) = \sup_{m(x)=1} |(x, y)|$; (d) $m^*(y) = \max_{x \neq 0} \frac{\operatorname{Re}(x, y)}{m(x)}$;

(b) $m^*(y) = \max_{x \neq 0} \frac{|(x, y)|}{m(x)}$; (e) $m^*(y) = \max_{m(x)=1} \operatorname{Re}(y, x)$.

(c) $m^*(y) = \max_{m(x)=1} |(x, y)|$;

8.1.44. Show that for any two vectors x and y (see Problem 8.1.42), the following inequality is valid

$$|(x, y)| \leqslant m(x)\, m^*(y). \tag{8.1.4}$$

Moreover, for any y there is a vector x_0 such that

$$(x_0, y) = m(x_0)\, m^*(y).$$

8.1.45. Find the dual norm for the length of vectors.

8.1.46. Find the dual norm for the norm $\| x \|_\infty = \max_i |\alpha_i|$ on the n-dimensional arithmetic space with the scalar product defined as (7.1.4).

8.1.47*. Generalizing 8.1.46, prove that the norm

$$\| x \|_q = (|\alpha_1|^q + \ldots + |\alpha_n|^q)^{1/q}$$

is dual with respect to the norm

$$\| x \|_p = (|\alpha_1|^p + \ldots + |\alpha_n|^p)^{1/p}, \quad p > 1, \; \frac{1}{p} + \frac{1}{q} = 1.$$

What happens to inequality (8.1.4) for this pair of norms?

8.1.48. Given two norms $m(x)$ and $n(x)$ on a Euclidean (unitary) space X and any vector x for which the inequality $m(x) \geqslant n(x)$ obtains, show that for the dual norms $m^*(y)$ and $n^*(y)$, the reverse relation is valid: $m^*(y) \leqslant n^*(y)$ for any vector y.

8.1.49*. Prove that for any vector x, there is a vector y such that inequality (8.1.4) turns into an equality.

8.1.50. Show that the norm $m^{**}(x)$, dual to the dual norm $m^{*}(y)$, coincides with the original norm $m(x)$.

8.2. Norms of Operators and Matrices

We give consideration almost exclusively here to matrix space norms, having in mind the applications mentioned in subsequent sections. Obviously, all these statements can be reformulated for operators. It should be stressed that a matrix norm fulfils a further condition related to the operation of matrix multiplication in addition to the three usual axioms, namely:

$$\| AB \| \leqslant \| A \| \| B \|.$$

Various classes of matrix norms and, in particular, the properties of the spectral and Euclidean norms are considered. In the latter case we have listed a number of interesting metric relations, similar to those valid for the complex plane. In conclusion, properties of subordinate norms and the consistency between the vector and matrix norms are analyzed. This analysis leads to a relation of partial ordering on the norm set.

8.2.1. Prove that any linear operator transforms a bounded set into another bounded set.

8.2.2. Is it correct to say that an open set is transformed by a linear operator into another open set?

8.2.3. Is it true that a closed set is transformed by a linear transformation to a closed set?

8.2.4. Prove that a closed and bounded set is transformed to a closed set by an arbitrary linear operator.

8.2.5*. If M is a closed set and A a linear operator, prove that the *complete pre-image of the set* M (i.e. the set of all x for which $Ax \in M$) is also a closed set.

8.2.6*. Let $\{A_k\}$ be a sequence of linear operators on a normed space X, and assume that for any x from X the sequence $\{A_k x\}$ is convergent. If

$$Ax = \lim_{k \to \infty} A_k x,$$

show that (a) an operator A, defined by this equality, is linear; (b) $A_k \to A$ for any norm on the space of operators.

8.2.7. Show that the sequence of matrices $A_k = (a_{ij}^{(k)})$ converges (under any norm) to the matrix $A = (a_{ij})$ if and only if $a_{ij}^{(k)} \to a_{ij}$ for all i, j.

8.2.8. Show that the limit of a sequence of normal matrices can only be a normal matrix. Similarly, show that a sequence of unitary matrices can only converge to a unitary matrix, a sequence of Hermitian matrices to a Hermitian matrix, and a sequence of positive-definite matrices to a positive-definite matrix.

8.2.9. Show that for any norm on a matrix space, the norm of the unit matrix is not less than unity.

8.2.10. Let $\| A \|$ be a norm on the space of $n \times n$ matrices. Show that the following are also matrix norms:

(a) $M(A) = \alpha \| A \|$, $\alpha > 1$;

(b) $L(A) = \| A^* \|$;

(c) $N(A) = \| P^{-1}AP \|$, where P is a nondegenerate n-order matrix.

8.2.11. Show that if $M(A)$ and $L(A)$ are matrix norms, then $N(A) = \max\{M(A), L(A)\}$ is also a matrix norm.

8.2.12. Prove that the following function of an $n \times n$ matrix

$$K(A) = \sum_{i,j=1}^{n} |a_{ij}| \tag{8.2.1}$$

is a matrix norm.

8.2.13. Let E_{ij} be an n-order matrix, in which the only nonzero element is at (i, j) and equals unity. Show that if a matrix norm $\| A \|$ satisfies the inequality

$$\| E_{ij} \| \leqslant 1$$

for all i, j then

$$\| A \| \leqslant K(A),$$

where $K(A)$ is a norm defined by the formula (8.2.1).

8.2.14. The natural scalar product (7.1.4) is defined on the n-dimensional arithmetic space. A matrix norm, subordinate to vector length in this space, is called the *spectral norm* and denoted by $\| A \|_2$. Prove that the spectral norm of a matrix equals its greatest singular value.

8.2.15. How can the spectral norm be calculated for (a) a diagonal matrix, (b) a quasidiagonal matrix?

8.2.16. Define the scalar product on the space of $n \times n$ matrices as in (7.1.5). The length of a matrix in the Euclidean (unitary) space thus obtained is expressed by the formula

$$\| A \|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

and called the *Euclidean norm of the matrix*. Show that for any matrices A and B

$$\| AB \|_E \leqslant \| A \|_E \| B \|_E.$$

8.2.17. Find the Euclidean norm of a unitary n-order matrix.

8.2.18*. Derive an expression for the Euclidean norm of an $n \times n$ matrix A in terms of its singular values $\alpha_1, \ldots, \alpha_n$.

8.2.19. Prove that the spectral norm of a matrix A equals its Euclidean norm if and only if A is a matrix of unit rank.

8.2.20. Prove that for any unitary matrices U and V

$$\| UAV \|_2 = \| A \|_2, \quad \| UAV \|_E = \| A \|_E.$$

8.2.21*. Prove the following inequalities:

(a) $\| A \|_E \leqslant \sqrt{n} \| A \|_2$;

(b) $\| AB \|_E \leqslant \| A \|_2 \| B \|_E$;

(c) $\| AB \|_E \leqslant \| A \|_E \| B \|_2$.

8.2.22. Let a matrix A have a Hermitian decomposition $A = H_1 + iH_2$. Prove that

(a) $\| H_1 \|_2 \leqslant \| A \|_2, \quad \| H_2 \|_2 \leqslant \| A \|_2$;

(b) $\| H_1 \|_E^2 + \| H_2 \|_E^2 = \| A \|_E^2$.

8.2.23. Prove that for any Hermitian matrix H

$$\| A - H \|_E \geqslant \| A - H_1 \|_E.$$

In this case, the matrix H_1 from the Hermitian decomposition of A is the Hermitian matrix closest in the sense of Euclidean distance to matrix A and, similarly, the matrix iH_2 is the closest skew Hermitian matrix. Indicate the analogue of this property on the complex plane.

8.2.24. Let $A = HU$ be a polar representation of a matrix A. Show that

$$\| H \|_E^2 = \| H_1 \|_E^2 + \| H_2 \|_E^2.$$

Which property of complex numbers does this equality correspond to?

8.2.25*. Prove that for any positive-definite matrix H, the closest (in the Euclidean distance sense) unitary matrix is the unit matrix E, and the farthest is the matrix $-E$. What happens if H is a positive-semidefinite matrix?

8.2.26. Let $A = HU$ be an arbitrary polar representation of a matrix A. Prove that for any unitary matrix V, the following inequalities are valid

$$\| A - U \|_E \leqslant \| A - V \|_E \leqslant \| A + U \|_E.$$

What is the corresponding property of complex numbers?

8.2.27*. Let A be an $n \times n$ matrix with the singular values $\alpha_1, \ldots, \alpha_n$. Assuming

$$S(A) = \alpha_1 + \ldots + \alpha_n, \tag{8.2.2}$$

prove that $S(A)$ is a matrix norm.

8.2.28. Prove that for any positive-semidefinite matrices A and B and any nonnegative numbers α and β

$$S(\alpha A + \beta B) = \alpha S(A) + \beta S(B).$$

The norm $S(A)$ is defined by (8.2.2).

8.2.29*. Show that in the definition of a secondary norm

$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|},$$

sup may be replaced by max.

8.2.30. Find subordinate matrix norms for the following norms on the n-dimensional arithmetic space:

(a) $\| x_1 \| = |\alpha_1| + \ldots + |\alpha_n|$,

(b) $\| x \|_\infty = \max |\alpha_i|$.

What are the values of these norms for a diagonal matrix D.

8.2.31. Prove that for any $n \times n$ matrix A, the following equality holds

$$\max_{i,j} |a_{ij}| = \max_{x \neq 0} \frac{\| Ax \|_\infty}{\| x \|_1}.$$

8.2.32. Norms $m(x)$ and $n(x)$ on an arithmetic space such that for any vector x $m(x) = cn(x)$ where c is a constant number. Show that the corresponding subordinate norms are identical.

8.2.33. Suppose $M(A)$ is a matrix norm, subordinate to a vector norm $m(x)$. Find a matrix norm, subordinate to the norm $n(x) = m(Px)$, where P is a constant nondegenerate matrix.

8.2.34. Let A be a matrix of rank 1 which can be represented as the product $A = xy^*$, where x and y are n-dimensional column vectors. Given any norm $m(x)$ on the arithmetic space and its corresponding subordinate matrix norm $M(A)$, prove the equality

$$M(A) = m(x)\, m^*(y), \tag{8.2.3}$$

where $m^*(y)$ is the norm dual to $m(x)$ with respect to the scalar product (7.1.4).

8.2.35. Find the value of the norm $\| A \|_\infty$ for a matrix having rank 1 given that $A = xy^*$.

8.2.36. If $M(A)$ is a subordinate matrix norm, prove that $M(A)$ can be represented as

$$M(A) = \max_{B \neq 0} \frac{M(AB)}{M(B)}. \tag{8.2.4}$$

8.2.37. Prove that the representation (8.2.4) remains valid if, instead of all nonzero matrices B, only unit rank matrices are considered.

8.2.38*. Prove that the subordinate matrix norm $M(A)$ may be represented as follows:

$$M(A) = \max_{r_B=1} \frac{|\operatorname{tr}(AB)|}{M(B)}. \qquad (8.2.5)$$

Here B ranges over the set of matrices of rank 1.

8.2.39. Given that $M(A)$ and $N(A)$ are subordinate norms, and that $M(A) \geqslant N(A)$ for all A, prove that $M(A) \equiv N(A)$.

8.2.40*. Given that $m(x)$ and $m^*(x)$ are dual norms on an arithmetic space, and that $M(A)$ and $M^*(A)$ are their subordinate matrix norms. Prove that for any matrix A

$$M(A) = M^*(A^*).$$

8.2.41*. Prove that any matrix norm is consistent with a certain norm on the arithmetic space.

8.2.42. Show that if a matrix norm $\| A \|$ is consistent with a vector norm $m(x)$ and $M(A)$ is subordinate to $m(x)$, then $\| A \| \geqslant M(A)$ for all matrices A. Thus, the subordinate norm $M(A)$ is the least of all norms, consistent with the vector norm $m(x)$.

8.2.43*. Prove that any subordinate matrix norm is consistent with a unique (dependent on a numerical multiplier) vector norm.

8.2.44. Show that any subordinate matrix norm $M(A)$ is *minimal*, i.e. another matrix norm $L(A)$ does not exist, for which

$$L(A) \leqslant M(A)$$

for any matrix A.

8.2.45*. Let a matrix norm $\| A \|$ be consistent with a vector norm $m(x)$ for which $M(A)$ is subordinate. Moreover, $\| A \|$ coincides with $M(A)$ for the set of matrices of rank 1. Prove that $m(x)$ is a vector norm (unique for a given numerical multiplier) consistent with $\| A \|$.

8.2.46. Show that the Euclidean matrix norm and the norm $S(A)$ (see (8.2.2)) are consistent only with the norm $\| x \|_2 = (|\alpha_1|^2 + \ldots + |\alpha_n|^2)^{1/2}$ (depending on a given numerical multiplier).

8.2.47. A matrix norm $M(A)$ is subordinate to the unit matrix E. Does this mean that $M(A)$ is a subordinate norm?

8.3. Matrix Norms and Systems of Linear Equations

Here the application of matrix norms to the solution of certain systems of linear equations is discussed (indeterminate and inconsistent systems have been considered in Sec. 7.8). The basic topics are the following:

Criteria of nondegeneracy of matrices.

Estimates of norms of inverse matrices.

The conditioning of a system of linear equations, properties of condition numbers.

The estimation of a perturbation in the solution of a system for a given perturbation of its coefficients.
Approximate solution of a system and estimation of the accuracy of the derived solution.

8.3.1. Prove that a matrix $A + B$, where A is nondegenerate and $\| A^{-1}B \| < 1$, is also nondegenerate.
8.3.2. Prove that if a matrix A is nondegenerate and the matrix $A - B$ is degenerate, then the condition number of the matrix A satisfies the inequality

$$\operatorname{cond}(A) \geqslant \frac{\| A \|}{\| B \|}.$$

8.3.3. Find the estimate from below for the condition number $\operatorname{cond}_\infty (A)$ of the matrix

$$A = \begin{Vmatrix} 1 & -1 & 1 \\ -1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{Vmatrix}, \quad \varepsilon \neq 0.$$

8.3.4. Prove that a matrix $U + B$ is nondegenerate given that U is a unitary matrix and the spectral norm of the matrix B is less than unity.
8.3.5*. Let α_n be the smallest singular value of an $n \times n$ matrix A. Prove that the distance (in the sense of the spectral norm) from the matrix A to the set M of degenerate matrices equals

$$\rho_2 (A, M) = \alpha_n.$$

8.3.6*. Prove that the smallest singular value of the matrix of the determinant (3.3.1) does not exceed $2^{-(n-1)}$.
8.3.7*. An n-order matrix A has the singular values $\alpha_1 \geqslant \dots \geqslant \alpha_n$. Prove that the distance (in the sense of the spectral norm) from the matrix A to the set M_r of matrices, whose rank is less than r, is equal to

$$\rho_2 (A, M_r) = \alpha_r, \quad r = 1, 2, \dots, n.$$

8.3.8. An n-order matrix A is said to be *diagonally dominant* (*with respect to its rows*) matrix if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \dots, n.$$

Prove that a diagonally dominant matrix is nondegenerate. Formulate a similar criterion for dominance with respect to its columns.
8.3.9*. Let A be a partitioned matrix of the form

$$A = \begin{Vmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \dots & \dots & \dots & \dots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{Vmatrix},$$

where all the blocks A_{ij} are square and have the same order m, and the diagonal blocks A_{ii} are nondegenerate. Moreover, for all i, $1 \leqslant i \leqslant k$, the following inequalities hold true:

$$\| A_{ii}^{-1} \| (\| A_{i1} \| + \ldots + \| A_{i,\, i-1} \| + \| A_{i,\, i+1} \| + \ldots + \| A_{ik} \|) < 1.$$

Prove that the matrix A is nondegenerate. What happens when $m = 1$?

8.3.10. Is the matrix

$$A = \begin{Vmatrix} 0 & 1 & 0.1 & -0.2 \\ 1 & 0 & -0.1 & 0.1 \\ 0.4 & 0.5 & 2 & 1 \\ -0.5 & 0.4 & 1 & 1 \end{Vmatrix}$$

nondegenerate?

8.3.11*. Assume that A is a diagonally dominant n-order matrix and for a certain positive number $\alpha < 1$

$$\alpha\, |a_{ii}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$$

Prove that for the norm of the inverse matrix A^{-1} the estimates are true:

$$\frac{1}{\min\limits_i |a_{ii}|} \cdot \frac{1}{1+\alpha} \leqslant \| A^{-1} \|_\infty \leqslant \frac{1}{\min\limits_i |a_{ii}|} \cdot \frac{1}{1-\alpha}. \qquad (8.3.1)$$

8.3.12. Estimate from below and from above the condition number $\mathrm{cond}_\infty (A)$ (see Problem 8.3.11) in terms of the diagonal elements of the matrix A and the number α.

8.3.13. Estimate from below and from above the condition number $\mathrm{cond}_\infty (A)$ of the $n \times n$ matrix

$$\begin{Vmatrix} 1 & 10^{-1} & 10^{-2} & \ldots & 10^{-(n-1)} \\ 10^{-1} & 2 & 10^{-2} & \ldots & 10^{-(n-1)} \\ 10^{-2} & 10^{-2} & 3 & \ldots & 10^{-(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 10^{-(n-1)} & 10^{-(n-1)} & 10^{-(n-1)} & \ldots & n \end{Vmatrix}.$$

8.3.14. Let R be a triangular N-order matrix for which

(a) $|r_{ij}| \leqslant 1$ for all i, j;

(b) $r_{ii} = 1$ for all i.

Find the maximum possible value of the condition number $\mathrm{cond}_\infty(R)$.

8.3.15. Given a sequence of matrices A_k of a fixed order n, with $\| A_k \| = 1$ and $\mathrm{cond}\,(A_k) \to \infty$ as $k \to \infty$. Prove that $\det A_k \to 0$ as $k \to \infty$.

Thus, for a fixed order of a matrix an increase in the condition number is related to a decrease in determinant size. However, as is shown in 8.3.14, for a sufficiently large n, the condition number of a matrix may be very large even if its determinant equals 1.

8.3.16. Show that the condition number of any matrix has a lower bound of 1.

8.3.17. Show that the condition number cond (A) is unaltered when the matrix A is multiplied by a nonzero number.

8.3.18. Find the expression of the spectral condition number of a nondegenerate normal matrix A in terms of its eigenvalues $\lambda_1, \ldots, \lambda_n$.

8.3.19. Find the expression of the spectral condition number of a nondegenerate $n \times n$ matrix A in terms of its singular values $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$.

8.3.20. Show that the equality $\text{cond}_2(A) = 1$ occurs if and only if $A = \alpha U$, where U is a unitary matrix and α is a nonzero number.

8.3.21. Show that the condition numbers $\text{cond}_{1,\,2,\,\infty,\,E}(A)$ are unaltered when the rows and columns of the matrix A are interchanged.

8.3.22. Show that the spectral and Euclidean condition numbers of a matrix A are unaltered when it is left-multiplied and right-multiplied by arbitrary unitary matrices U and V.

8.3.23. Prove the inequalities

$$\max\left\{\frac{\text{cond}(A)}{\text{cond}(B)}, \frac{\text{cond}(B)}{\text{cond}(A)}\right\} \leqslant \text{cond}(AB) \leqslant \text{cond}(A)\,\text{cond}(B).$$

8.3.24. For a nondegenerate 2×2 matrix A, give an explicit expression of the Euclidean condition number $\text{cond}_E(A)$ in terms of the elements of this matrix.

8.3.25. Show that the matrix

$$\left\|\begin{matrix} 100 & 99 \\ 99 & 98 \end{matrix}\right\|$$

has the greatest Euclidean condition number among all nondegenerate 2×2 matrices whose elements are nonnegative integers not exceeding 100.

8.3.26. The solution of a system of two linear equations in two unknowns:

$$a_{11}x + a_{12}y = a_1, \quad a_{21}x + a_{22}y = a_2$$

with a real and nondegenerate matrix A is equivalent to the geometric problem of finding the point of intersection of two straight lines determined by the equations of the system. Prove that the angle α between these straight lines satisfies the inequality

$$|\cot \alpha| \leqslant \frac{1}{2}\,\text{cond}_E(A).$$

8.3.27. If A is a positive-definite matrix, prove that the spectral condition number of the matrix $A + \alpha E$ is a steadily decreasing function of α when $\alpha > 0$.

8.3.28. Suppose A is a positive-definite matrix and A_k is an arbitrary principal submatrix of the matrix A. Prove that

$$\operatorname{cond}_2(A_k) \leqslant \operatorname{cond}_2(A).$$

8.3.29. Let $A = S^T S$ be a triangular decomposition of a real positive-definite matrix A. How are the spectral condition numbers of the matrices A and S related?

8.3.30. Estimate from below the spectral condition number of the matrix of the system of linear equations

$$\begin{aligned} 10x_1 + 10x_2 + 30x_3 &= -5, \\ 0.1x_1 + 0.5x_2 + 0.1x_3 &= 0.55, \\ 0.03x_1 + 0.01x_2 + 0.01x_3 &= 0.045. \end{aligned}$$

Indicate a method to decrease the condition number so that in the obtained system $\hat{A}x = \hat{b}$ $\operatorname{cond}_2(\hat{A}) = 3$. Find the solution of this system.

8.3.31*. Estimate from below the spectral condition number of the matrix of the system

$$\begin{aligned} x_1 + 20x_2 - 400x_3 &= 1, \\ 0.2x_1 - 2x_2 - 20x_3 &= 0.2, \\ -0.04x_1 - 0.2x_2 + x_3 &= 0.05. \end{aligned}$$

Indicate a method for decreasing the condition number so that in the obtained system $\hat{A}y = \hat{b}$, $\operatorname{cond}_2(\hat{A}) = 2$. Find the solution of this system.

8.3.32. Let $\| x \|$ be a norm on an arithmetic space, and let $\| A \|$ be its subordinate matrix norm. Show that when the right-hand side of a system of linear equations $Ax = b$ is replaced by a vector with the norm $\varepsilon > 0$, the solution of the system can be changed to a vector with the norm $\varepsilon \| A^{-1} \|$.

8.3.33. Estimate the possible perturbation of the system

$$\begin{aligned} x - 2y &= -1, \\ -2x + 4.01y &= 2 \end{aligned}$$

when the components of the right-hand side are changed by 0.01. Find the solution of this system and of the system with the same matrix and the right-hand side

$$\tilde{b} = \begin{Vmatrix} -1 \\ 2.01 \end{Vmatrix}.$$

8.3.34. Find the condition number $\operatorname{cond}_\infty(A)$ of the matrix of the system

$$\begin{aligned} 5x - 3.31y &= 1.69, \\ 6x - 3.97y &= 2.03. \end{aligned}$$

Indicate the change of the solution of this system in the transfer to the system with the same matrix but with the right-hand side being

$$\tilde{b}=\begin{Vmatrix}1.7\\2\end{Vmatrix}.$$

8.3.35. Find an approximate solution of the system

$$\begin{aligned}2.503x_1+0.002x_2-0.004x_3+0.001x_4&=5,\\0.006x_1-3.002x_2+0.001x_3-0.001x_4&=3,\\-0.002x_1+0.002x_2+4.998x_3+0.004x_4&=10,\\0.005x_1-0.001x_2\qquad\qquad+3.997x_4&=4\end{aligned}$$

such that the error in each component may not exceed 0.01.

8.3.36. Find an approximate solution of the system

$$\begin{aligned}0.501x_1-0.499x_2+0.001x_3\qquad\qquad&=0.5,\\0.498x_1+0.502x_2\qquad\qquad-0.001x_4&=0.5,\\0.006x_1+0.007x_2+3.008x_3-1.991x_4&=0,\\-0.001x_1\qquad\qquad-2.001x_3+1.000x_4&=0\end{aligned}$$

such that the error in each component may not exceed 0.06.

8.3.37. Prove the inequality

$$\frac{\|B^{-1}-A^{-1}\|}{\|B^{-1}\|}\leqslant \operatorname{cond}(A)\frac{\|B-A\|}{\|A\|}.$$

8.4. Matrix Norms and Eigenvalues

In this section we intended to demonstrate some of numerous applications of matrix norms to problems involving the eigenvalues of complex matrices.

Some inequalities connecting the eigenvalues and the matrix norms are considered at first. These inequalities can be used to specify a region on the complex plane containing all the eigenvalues of a matrix. The Gershgorin theorem (see Sec. 8.4.20) and a theorem on eigenvalue perturbations (see Sec. 8.0) can also be applied for the same purpose.

Using the properties of the eigenvalues of Hermitian matrices, the perturbation theorem can be modified so as to derive an estimate of each individual eigenvalue (see Problems 8.4.25-8.4.32).

Given an approximation to a well-separated eigenvalue λ_1 and the corresponding approximate eigenvector $\tilde{x}$ of a normal matrix, the Rayleigh ratio for the vector $\tilde{x}$ gives an approximation to λ_1 with considerably higher accuracy. We discuss this point in Problems 8.4.33-8.4.39.

Finally the relation between close eigenvalues of a matrix and ill-conditioning of the eigenvector matrix is investigated. It is easy to demonstrate that in a small neighbourhood of a matrix with close or multiple eigenvalues, there is a matrix with Jordan structure. The latter can be considered as the limiting case of a matrix with ill-conditioned eigenvectors. As it was established by Wilkinson, the converse relation is also true: if for a matrix A (even with well-separated eigenvalues) the matrix of the eigenvectors is ill-conditioned, then in a small neighbourhood of A there is a matrix with a multiple root.

8.4.1*. Prove that for the spectral radius of a matrix A, the following inequality holds

$$\rho(A) \leqslant \| A \| \qquad (8.4.1)$$

for any matrix norm $\| A \|$.

8.4.2. Indicate a circle on the complex plane containing all the eigenvalues of the matrix

$$\begin{Vmatrix} -1 & 0 & 1+2i \\ 0 & 2 & 1+i \\ 1+2i & 1+i & 0 \end{Vmatrix}.$$

8.4.3. Prove that all the eigenvalues of the matrix

$$\begin{Vmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & -1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 1 & 2 & -1 \end{Vmatrix}$$

lie within the circle of the complex plane $|z| \leqslant 6$.

8.4.4. Show that the greatest eigenvalue λ_1 and the least eigenvalue λ_4 of the symmetric matrix

$$\begin{Vmatrix} 6 & 2 & -3 & 0 \\ 2 & 9 & 5 & 1 \\ -3 & 5 & 13 & -2 \\ 0 & 1 & -2 & 20 \end{Vmatrix}$$

satisfy the inequalities

$$20 \leqslant \lambda_1 \leqslant 23, \quad 0 \leqslant \lambda_4 \leqslant 6.$$

8.4.5. Prove that the moduli of all the eigenvalues of a stochastic matrix do not exceed unity.

8.4.6. Prove that the eigenvalues of a tridiagonal matrix

$$C = \begin{Vmatrix} a_1 & b_2 & & & & \\ c_2 & a_2 & b_3 & & & \\ & c_3 & a_3 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & a_{n-1} & b_n \\ & & & & c_n & a_n \end{Vmatrix}$$

satisfy the inequality

$$|\lambda| \leqslant \max_i \{ |a_i| + |b_{i+1}| + |c_i| \}, \quad c_1 = b_{n+1} = 0.$$

How can this result be used to evaluate the eigenvalues of a Hermitian matrix by the bisection method?

8.4.7. Prove that all roots of a polynomial $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$, $a_n \neq 0$, are contained in each of the following circles on the complex plane

(a) $|z| \leqslant \max \left\{ 1, \left| \frac{a_{n-1}}{a_n} \right| + \ldots + \left| \frac{a_1}{a_n} \right| + \left| \frac{a_0}{a_n} \right| \right\}$;

(b) $|z| \leqslant \max \left\{ \left| \frac{a_0}{a_n} \right|, 1 + \max_{1 \leqslant i \leqslant n-1} \left| \frac{a_i}{a_n} \right| \right\}$.

8.4.8*. Let A_0 be a matrix of simple structure. Prove that there is a matrix norm $\| A \|$ such that when $A = A_0$ (8.4.1) becomes an equality relation.

8.4.9*. Let A_0 be an arbitrary matrix. Prove that for any positive number ε, there is a matrix norm $\| A \|$ for which $\| A_0 \| < \rho(A_0) + \varepsilon$.

8.4.10. Prove that for a normal matrix A_0, $\| A_0 \|_2 \leqslant M(A_0)$ for any matrix norm $M(A)$.

8.4.11. Prove that for an arbitrary matrix A_0 and any matrix norm $M(A)$, $\| A_0 \|_2 \leqslant \sqrt{M(A_0) M(A_0^*)}$.

8.4.12*. Let A be a matrix of order n with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove the following *Schur inequality*

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \| A \|_E^2. \tag{8.4.2}$$

8.4.13*. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ (see Problem 8.4.12) be the real and imaginary parts, respectively, of the eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that

$$\text{(a)}\ 4 \sum_{i=1}^{n} \alpha_i^2 \leqslant \| A + A^* \|_E^2; \quad \text{(b)}\ 4 \sum_{i=1}^{n} \beta_i^2 \leqslant \| A - A^* \|_E^2. \tag{8.4.3}$$

8.4.14*. Prove that equality occurs in (8.4.2) if and only if A is a normal matrix. The same is true for each of relations (8.4.3).

8.4.15*. Let A be an $n \times n$ matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$, and let P be an arbitrary nondegenerate matrix. Prove that

$$\inf_P \| P^{-1} A P \|_E^2 = \sum_{i=1}^{n} |\lambda_i|^2.$$

For which matrices A is the indicated lower bound reached?

8.4.16*. Using 8.4.14, prove that the normality of matrices A, B and AB implies the normality of BA.

8.4.17*. Suppose that a normal matrix A is partitioned into blocks A_{ij} so that

$$A=\begin{Vmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{Vmatrix},$$

and, moreover, the diagonal blocks A_{ii} are square, though perhaps, of different orders. Further, assume that the eigenvalues of the matrix A coincide with the set of the eigenvalues of the matrices A_{ii}. Prove that then all the off-diagonal blocks A_{ij} equal zero.

8.4.18*. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues, and let $\alpha_1, \dots, \alpha_n$ be the singular values of a matrix A. Prove that

$$|\lambda_1| + \dots + |\lambda_n| \leqslant \alpha_1 + \dots + \alpha_n.$$

8.4.19*. Using 8.4.18, prove that for any matrix A of order n,

$$\sum_{i=1}^{n} |\lambda_i| \leqslant \sum_{i,j=1}^{n} |a_{ij}|.$$

8.4.20*. Prove the following *Gershgorin theorem*: all the eigenvalues of an $n \times n$ matrix A lie in a region of the complex plane given by the union of n disks

$$|z-a_{ii}| \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad i=1, \dots, n.$$

8.4.21. Indicate a region on the complex plane containing all the eigenvalues of the matrix

$$\begin{Vmatrix} 1.23 & 0.03 & 0.04 \\ 0.03 & 2.17 & 0.01 \\ 0.02 & 0.04 & 3.06 \end{Vmatrix}.$$

8.4.22. These inequalities are valid for a matrix A

$$\operatorname{Re} a_{ii} < -\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad i=1, \dots, n.$$

Prove that A is a stable matrix.

8.4.23. Using the theorem about a perturbation of the eigenvalues, indicate a region on the complex plane containing all the eigenvalues of the matrix

$$\begin{Vmatrix} 2.001 & 1.499 & 0.001 \\ 0.499 & 1.001 & -0.001 \\ -0.001 & 0.001 & 0.999 \end{Vmatrix}.$$

8.4.24. Let

$$A=\begin{Vmatrix} -2 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad B=\begin{Vmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix}.$$

Find a region on the complex plane containing all the eigenvalues of the matrix $A + \varepsilon B$, using the theorem about a perturbation of the eigenvalues.

8.4.25*. Let A and B be Hermitian matrices, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix A. Prove that in each interval

$$-\|B\|_2 \leqslant x - \lambda_i \leqslant \|B\|_2, \quad i = 1, \ldots, n, \tag{8.4.4}$$

there is at least one eigenvalue of the matrix $A + B$.

8.4.26. Let $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 be the eigenvalues of the matrices A and B, respectively, where

$$A=\begin{Vmatrix} 2 & 3 & -2 \\ 3 & 1 & 0 \\ -2 & 0 & -1 \end{Vmatrix}, \quad B=\begin{Vmatrix} 2.1 & 2.9 & -2 \\ 2.9 & 0.9 & 0.1 \\ -2 & 0.1 & -1 \end{Vmatrix}.$$

Prove that for each λ_i, there is such μ_j that $|\lambda_i - \mu_j| \leqslant 0.3$.

8.4.27*. Find eigenvalues of the matrix

$$\begin{Vmatrix} 2\cdot10^{-4} & -3 & 4 & 9991 \\ -3\cdot10^{-8} & 1\cdot10^{-4} & -0.4993 & -6\cdot10^{-4} \\ 4\cdot10^{-8} & -0.4993 & 2\cdot10^{-4} & -2\cdot10^{-4} \\ 0.9991\cdot10^{-4} & -6\cdot10^{-4} & -2\cdot10^{-4} & 1\cdot10^{-4} \end{Vmatrix}$$

approximately, so that the error in each does not exceed 0.002.

8.4.28. Let an eigenvalue λ_i (see Problem 8.4.25) be of multiplicity k. Prove that then the interval

$$-\|B\|_2 \leqslant x - \lambda_i \leqslant \|B\|_2$$

contains at least k eigenvalues of the matrix $A + B$.

8.4.29*. Find approximations to the eigenvalues of the matrix

$$\begin{Vmatrix} 1.01 & -1.99 & 0.01 & 0.01 \\ -1.99 & 1.01 & -0.01 & -0.01 \\ 0.01 & -0.01 & -0.01 & -0.99 \\ 0.01 & -0.01 & -0.99 & -0.01 \end{Vmatrix}$$

such that the error in each eigenvalue does not exceed 0.02.

8.4.30. Let the region D, made up of the intervals (8.4.4), be broken into regions (i.e. intervals) having no common points. Prove that in each of these regions D_k, there are as many eigenvalues of the matrix $A + B$ (see Problem 8.4.25) as there are intervals in the set (8.4.4) that compose this region. Moreover, if λ_i is a multiple eigenvalue of A, then its corresponding interval is counted as many times as the multiplicity of λ_i.

8.4.31. A Hermitian matrix A is partitioned into blocks

$$\left\| \begin{matrix} A_{11} & A_{12} \\ A_{12}^{*} & A_{22} \end{matrix} \right\|$$

so that A_{11} and A_{22} are square and $\| A_{12} \|_2 = \varepsilon$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the matrix A numbered in descending order, and let $\xi_1, \ldots, \xi_r$ be the eigenvalues of A_{11}, and $\eta_1, \ldots, \eta_{n-r}$ the eigenvalues of A_{22}. Finally, let $\mu_1, \ldots, \mu_n$ be the numbers of the set $\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_{n-r}$ also numbered in descending order. Prove that $|\lambda_i - \mu_i| \leqslant \varepsilon$, $i = 1, \ldots, n$.

Thus, the eigenvalues of the diagonal blocks can be taken as approximations to the eigenvalues of the matrix A itself with an accuracy of ε.

8.4.32*. Prove that the following matrix A of order 8

$$\left\| \begin{array}{ccc|cc|ccc} 1 & 1/N & 0 & & & & & 1/N \\ 1/N & 1 & 2/N & & & & & \\ 0 & 2/N & 1 & & & & & \\ \hline & & & 2 & 1/N & & & \\ & & & 1/N & 2 & & & \\ \hline & & & & & -0.5 & 0.1 & -0.2 \\ & & & & & 0.1 & -1 & 0 \\ 1/N & & & & & -0.2 & 0 & 2 \end{array} \right\|$$

(the matrix A is quasi-diagonal to the accuracy of the elements positioned in (1.8) and (8.1)):

(a) for any $N > 0$, has *at least* one eigenvalue in the interval

$$-\frac{\sqrt{2}}{N} \leqslant \lambda - 1 \leqslant \frac{\sqrt{2}}{N};$$

(b) for $N \geqslant 10$ has *precisely* three eigenvalues in the interval

$$\frac{-3}{N} \leqslant \lambda - 1 \leqslant \frac{3}{N}.$$

In Problems 8.4.33-8.4.35, it is assumed that A is a normal matrix and $\tilde{x}$ is a column vector normed so that $\| \tilde{x} \|_2 = 1$.

8.4.33. Let $\| A\tilde{x} \|_2 = \varepsilon$. Prove that the matrix A has an eigenvalue λ for which $|\lambda| \leqslant \varepsilon$.

8.4.34. Assume for an arbitrary number μ that $\varepsilon = \| A\tilde{x} - \mu\tilde{x} \|_2$. Show that there is at least one eigenvalue of the matrix A in the disk on the complex plane $|z - \mu| \leqslant \varepsilon$.

8.4.35*. Let λ_1 be an eigenvalue of a matrix A lying in the disk $|z-\mu_0|\leqslant\varepsilon$ (ε is defined as in 8.4.34), and let for all the other eigenvalues $\lambda_2,\ldots,\lambda_n$, the condition be fulfilled

$$|\lambda_i-\mu_0|\geqslant a\gg\varepsilon.$$

If the normed eigenvector associated with the eigenvalue λ_1 is denoted by e_1 and

$$\tilde{x}=\alpha e_1+z, \tag{8.4.5}$$

where $z\perp e_1$, prove that

(a) $\|Az-\mu_0 z\|_2\geqslant a\|z\|_2$;

(b) $\|Az-\mu_0 z\|_2\leqslant\varepsilon,\quad \|z\|_2\leqslant\varepsilon/a$;

(c) $|\alpha|\geqslant\sqrt{1-\varepsilon^2/a^2}$;

(d) $|(Az,z)-\mu_0\|z\|_2^2|\leqslant\varepsilon^2/a$.

Thus, if ε is sufficiently small compared to a, then $\tilde{x}$ can be considered as an approximation to e_1.

8.4.36. Let A be a matrix of order n, x an arbitrary nonzero n-dimensional column vector. The number

$$r(x)=\frac{(Ax,x)}{(x,x)}$$

is called the *Rayleigh quotient* corresponding to the vector x. Prove that for any number μ

$$\|Ax-r(x)x\|_2\leqslant\|Ax-\mu x\|_2.$$

8.4.37. Prove that for a normal matrix A and any normed vector $\tilde{x}$, the disk

$$|z-r(\tilde{x})|\leqslant(\|A\tilde{x}\|_2^2-|r(\tilde{x})|^2)^{1/2}$$

contains an eigenvalue of the matrix A.

8.4.38*. Assume that μ_0 (see Problem 8.4.35) is the Rayleigh quotient corresponding to the vector $\tilde{x}$. Prove that the estimate is valid

$$|\lambda_1-\mu_0|\leqslant\frac{\varepsilon^2}{a}\left(1-\frac{\varepsilon^2}{a^2}\right)^{-1}. \tag{8.4.6}$$

8.4.39. For a symmetric matrix A

$$\begin{Vmatrix} 1 & 0.001 & 0.002 & 0.002\\ 0.001 & 2 & 0.002 & 0.002\\ 0.002 & 0.002 & 3 & 0.001\\ 0.002 & 0.002 & 0.001 & 4 \end{Vmatrix}$$

(a) with the aid of 8.4.25 find the eigenvalues to an accuracy of 0.005;

(b) show that the diagonal elements of A can be considered as the Rayleigh ratios if the corresponding vectors have been given;

(c) prove that the diagonal elements are approximations to the corresponding eigenvalues to an accuracy of 10^{-5}.

8.4.40. Let all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a matrix A be different and $d = \min_{i \neq j} |\lambda_i - \lambda_j|$. Prove that there is a matrix B for which $\| B \|_2 \leqslant d/2$ and the matrix $A + B$ has a multiple eigenvalue.

8.4.41*. Prove that (see Problem 8.4.40) for any number $\varepsilon > 0$, a matrix C_ε can be found such that $\| C_\varepsilon \|_2 < \frac{d}{2} + \varepsilon$ and the matrix $A + C_\varepsilon$ is not simply structured.

8.4.42*. All the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a matrix A are different. Let x_i be an eigenvector of the matrix A associated with λ_i, y_i an eigenvector of the matrix A^* corresponding to $\bar{\lambda}_i$. Put

$$s_i = \frac{(x_i, y_i)}{\| x_i \|_2 \| y_i \|_2}, \quad i = 1, \ldots, n.$$

For real x_i and y_i, the number s_i is the cosine of the angle between these vectors. It is obvious that $| s_i |$ is independent of the selection of a concrete pair of vectors x_i, y_i (for the given λ_i).

Prove that

(a) for any matrix X made up of the eigenvectors of the matrix A,

$$\text{cond}_2(X) \geqslant \frac{1}{|s_i|}, \quad i = 1, \ldots, n;$$

(b) a matrix X can be selected so that

$$\text{cond}_2(X) \leqslant \text{cond}_E(X) = \sum_{i=1}^{n} \frac{1}{|s_i|}.$$

Thus, the value of $| s_i |$, together with its condition number, can serve as a measure of the conditioning of the eigenvector matrix.

8.4.43. Let C be a triangular matrix

$$C = \begin{Vmatrix} \lambda_1 & c_{12} & \ldots & c_{1n} \\ 0 & \lambda_2 & \ldots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \lambda_n \end{Vmatrix},$$

and let the first component of some eigenvector y of the conjugate matrix C^* associated with the eigenvalue $\bar{\lambda}_1$ be equal to zero. Prove that λ_1 is a multiple eigenvalue of C.

8.4.44. Matrices A and A^* possess the eigenvectors x and y, associated with λ_1 and $\bar{\lambda}_1$, respectively, and, in addition, $(x, y) = 0$. Prove that λ_1 is a multiple eigenvalue of A.

8.4.45*. Write the matrix C (see Problem 8.4.43) in the partitioned form

$$C = \left\| \begin{matrix} \lambda_1 & c \\ 0 & C_{n-1} \end{matrix} \right\|.$$

We will assume that the eigenvector y of the matrix C^*, associated with the eigenvalue $\bar{\lambda}_1$, to be normed, and require instead of $\beta_1 = 0$, that $\beta_1 = \varepsilon$, $|\varepsilon| < 1$. Represent the vector y in the form

$$y = \binom{\varepsilon}{z}.$$

Prove that λ_1 is an eigenvalue of the matrix

$$\tilde{C}_{n-1} = C_{n-1} + \frac{\bar{\varepsilon}}{1 - |\varepsilon|^2} zc.$$

8.4.46. Prove that (see Problem 8.4.45) there 's a matrix $\tilde{C}$ such that

(a) $\| C - \tilde{C} \|_2 \leqslant \frac{|\varepsilon|}{\sqrt{1 - |\varepsilon|^2}} \| C \|_2$;

(b) $\tilde{C}$ has a multiple eigenvalue λ_1.

8.4.47. Let x and y be the normed eigenvectors of matrices A and A^* corresponding to λ_1 and $\bar{\lambda}_1$, respectively. Moreover, $|s| = |(x, y)| = \varepsilon \ll 1$. Prove that there is a matrix $\tilde{A}$ such that

(a) $\| A - \tilde{A} \|_2 \leqslant \frac{\varepsilon}{\sqrt{1 - \varepsilon^2}} \| A \|_2$;

(b) $\tilde{A}$ has a multiple eigenvalue λ_1. Thereby, the fact that conjugate matrices possess a pair of almost orthogonal eigenvectors associated with the conjugate eigenvalues, testifies that there exists a close matrix with a multiple eigenvalue.

Hints

1.1.18. Using only the distributivity and the existence of the opposite element, prove that $0 \cdot x = 0$ for any vector x. Hence, deduce that $(-1) \cdot x = -x$. Finally, using the associativity of addition, prove that $x + y = y + x$.

1.2.28. Write a linear combination $\lambda_1 x_1 + \ldots + \lambda_s x_s$ of the vectors $x_1, \ldots, x_s$. If we assume that there are nonzeroes among the coefficients $\lambda_1, \ldots, \lambda_s$, and that λ_j has the maximum modulus, then we have to show that the j-th component of the vector $\lambda_1 x_1 + \ldots + \lambda_s x_s$ is different from zero.

1.3.16. Use Theorem 15.1 (see the text-book by V. Voyevodin, p. 50).

1.3.25. Use 1.3.17 and 1.3.19.

1.3.26. Show that each elementary transformation leads to an equivalent vector set.

1.3.34. Let the rank of the vector set $x_1, \ldots, x_s$ be r. Then, the first r rows of the matrix, obtained from the reduction to trapezoidal form (see the solution of Problem 1.2.18), will be nonzero. Let them correspond to the rows numbered $i_1, \ldots, i_r$ in the original matrix. Prove that the vectors $x_{i_1}, \ldots, x_{i_r}$ make a base for the given set.

1.3.36. Set up the reduction so that the zero elements are positioned in the lower right corner of the matrix.

1.3.39. If $x_j = \alpha_1 x_{i_1} + \ldots + \alpha_r x_{i_r}$, then any vector, for which the coefficient α_l in this decomposition is different from zero, can be taken as x_{i_l}.

1.3.44. Use 1.3.23.

1.4.41. Extend an arbitrary basis for the subspace L to form a basis $e_1, \ldots, e_n$ of the space V. Obtain the basis fulfilling the conditions of the problem by elementary transformations of the set $e_1, \ldots, e_n$.

1.5.16. Use 1.5.14. 1.5.18. Use 1.5.16.

2.1.2. Let $e_1, \ldots, e_n$ be a basis for the given linear space. But

$$(x, y) = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n$$

for arbitrary vectors $x = \alpha_1 e_1 + \ldots + \alpha_n e_n$ and $y = \beta_1 e_1 + \ldots + \beta_n e_n$. Verify that all the requirements for a scalar product are fulfilled.

2.1.8. Prove the necessary condition $ac > b^2$ by considering the scalar square (x, x) of a vector of the form $x = (\alpha_1, 1)$ as a quadratic trinomial of α_1.

2.1.9. Derive the representation

$$(x, x) = \alpha_1^2 + (3\alpha_1 + \alpha_2)^2 + (\alpha_2 + \alpha_3)^2$$

for the scalar square of the vector $x = (\alpha_1, \alpha_2, \alpha_3)$.

2.1.10. Use the inequality $2 \mid a_{ij} \mid \mid \alpha_i \mid \mid \alpha_j \mid \leqslant \mid a_{ij} \mid \mid \alpha_i \mid^2 + \mid a_{ij} \mid \alpha_j \mid^2$ to verify the fourth requirement for a scalar product.

2.1.15. Define a scalar product arbitrarily on the subspace, complementary to L. Then use 2.1.13.

2.1.16. See Voyevodin, Theorem 27.2, p. 93.

2.1.18. (d) Use 2.1.16.

2.2.23. For each i, $1 \leqslant i \leqslant k$, the vectors $y_1, \ldots, y_i$ and $z_1, \ldots, z_i$ form an orthogonal basis for the span drawn on the vectors $x_1, \ldots, x_i$. Therefore $(y_l, z_m) = 0$ when $l \neq m$.

2.2.25. See the hint to Problem 2.1.2.

2.3.7. (a) Interpret each equation of the system as a condition of the orthogonality of the vector $z = (\alpha_1, \ldots \alpha_n)$ to the vector made up of the coefficients of the equation.

2.3.9. Use the basis for the orthogonal complement derived in 2.3.6.

2.3.11. See the solution of Problem 2.3.10.

2.3.14. The coefficients of the equations of the system give the coordinates of the vectors on which $L^{\perp}$ is drawn. By the method of Problem 2.3.10, find the perpendicular z, and then y as the difference $x - z$.

2.3.27. Set up a basis for V as the union of bases for the subspaces $L_1, \ldots, L_p$, and define a scalar product on V by 2.2.25.

2.4.16. Show that in the decomposition of the vector x, $x = y + z$ where $y \in L$, $z \perp L$, the vector $y \in L_2$ and, therefore, the perpendicular from x to L_2 coincides with the perpendicular z from x to L.

2.4.17. The perpendicular z from the vector x to L is collinear with the vector a.

2.4.19. Use 2.4.17.

2.4.20. In evaluating the cosine of the angle between x and an arbitrary vector u of the subspace L use the decomposition $x = y + z$ where $y \in L$, $z \perp L$.

2.4.23. See the hint for 2.4.16.

2.5.2. As in Problem 2.1.2 (see the hint), fix a basis $e_1, \ldots, e_n$ and for arbitrary vectors x and y assume $(x, y) = \alpha_1\overline{\beta}_1 + \ldots + \alpha_n\overline{\beta}_n$.

2.5.5. See the hint for 2.1.10.

2.5.13. (c) Show that if $e_1, \ldots, e_n$ is a basis for the space R, then any vector from C is a linear combination of the vectors $e_1 + i0, \ldots, e_n + i0$.

3.1.21. Derive the iterative formula $m_n = m_{n-1} + 1$ for the number m_n of nonzero terms in the determinant of order n. Here $m_1 = 1$.

3.1.22. Derive the iterative formula $m_n = m_{n-1} + m_{n-2}$ for the number m_n of nonzero terms in the determinant of order n. The general solution of such an equation is (see Sec. 3.0) $m_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right) + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$. The constants c_1 and c_2 are determined from the equalities $m_1 = 1$, $m_2 = 2$.

3.1.23. Derive the iterative formula $m_n = 2m_{n-1}$ for the number m_n of nonzero terms in the determinant of order n. In this case $m_1 = 1$.

3.1.25. Let $P_n(t)$ be a determinant of order n of the indicated form. Derive the following iterative formula: $P_n(t) = tP_{n-1}(t) + a_1$.

3.1.34. Show that the given transformation of the determinant is equivalent to multiplication of its rows by $\alpha, \alpha^2, \ldots, \alpha^n$, respectively, and its columns by $\alpha^{-1}, \alpha^{-2}, \ldots, \alpha^{-n}$, respectively.

3.1.35. Use 3.1.34. 3.1.36. Transpose the determinant.

3.1.37. Transpose the determinant. 3.1.40. Use 3.1.38.

3.1.42. Transpose the determinant and use 3.1.40.

3.1.43. The indicated transformation of the determinant can be replaced by transposing it about the principal diagonal and reversing the rows.

3.1.44. A polynomial of the fourth degree cannot have more than four different roots.

3.1.56. Differentiate the general term of the determinant.

3.2.6. Use 3.1.35.

3.2.20. The given determinant has an almost triangular form. When this determinant is expanded by the first two rows, the sum contains only three terms.

3.2.26. Expand the determinant by the first three columns.

3.2.33. Subtract the first row from the second, third and fourth.

3.2.45. Show that for a determinant d_n of the indicated form, the iterative formula $d_n = 2\cos\alpha d_{n-1} - d_{n-2}$ is valid and $d_1 = \cos\alpha$, $d_2 = 2\cos^2\alpha - 1 = \cos 2\alpha$.

3.3.1. The vector b_l is obtained from a_l by subtracting a linear combination of the vectors $a_1, \ldots, a_{l-1}$.
3.3.15. See V. Voyevodin, Theorem 41.2, p. 134.
3.3.16. Any principal minor of a Gram determinant is itself a Gram determinant with respect to a subset of a given vector set.
3.3.17. Use 3.3.14. 3.3.18. Use 3.3.17 and 3.3.13.
3.3.23. Use 3.3.17 and 3.3.13. 3.3.24. Use 3.3.18.
3.3.25. The length of the perpendicular dropped from the vector x_{l+1} to the span of the vectors $x_1, \ldots, x_l$ does not exceed the length of the vector itself; the length of the perpendicular dropped from the vector x_j, $l + 1 < j \leqslant k$, to the span of the vectors $x_1, \ldots, x_{j-1}$, does not exceed the length of the perpendicular dropped from the same vector to the span of the vectors $x_{l+1}, \ldots, x_{j-1}$.
3.3.32. The element positioned in $(n, 1)$ should be replaced by $(-1)^n \cdot 2^{-(n-1)}$.
3.4.3. Use 1.2.28.
3.4.4. Use 3.3.25 and 3.4.3 to prove the latter statement.
3.4.8. Transpose the minor M to the upper left corner and use the Gauss elimination.
3.4.9. Use 3.2.11, recalling that the Gauss elimination consists of a sequence of elementary row and column transformation.
3.4.16. Before applying the Gauss elimination, decrease the value of the terms of the determinant by elementary transformations.
3.4.17. Reduce the elements of each row to their common denominator and use 3.4.16.
3.4.19. Take the common factor of the elements in each row outside the bracket.
3.4.20. See the hint for 3.4.16.
3.4.24. The determinant is obtained by enclosing the determinant of Problem 3.4.10.
3.4.26. The determinant is obtained by enclosing the determinant of Problem 3.4.24.
3.4.35. (b) Use the formulae of the $(k + 1)$th stage of the elimination method, recalling that the moduli of ratios $\frac{a_{i,k+1}^{(k)}}{a_{k,k+1}^{(k)}}$, $i > k + 1$, are bounded by unity.
3.4.41. Carry out the transformation that reduces the matrix A to triangular form over each of the n sets from the n rows of the determinant D. The matrix of the determinant obtained will contain m^2 triangular blocks. Using the Laplace theorem, this determinant can be expanded as in 3.2.27, (b).

4.1.2. Prove that the rank of the set of columns is $n - 1$.
4.1.3. Using 4.1.2, prove that the columns of the matrix A, containing the minor M, form a base of the vector set.
4.1.4. Use 1.3.39.
4.1.6. Consider the submatrix formed by the given r linearly independent columns. Show that the rows, in which the given minor is placed, are the basis rows for the submatrix.
4.1.9. The rank of a Gram matrix equals the highest order of nonzero principal minors of this matrix. For the principal minors of a Gram matrix see 3.3.16.
4.1.11. Use 3.1.36.
4.1.12. The first r columns contain at least one nonzero minor of order r.
4.1.20. The indicated increase in the rank can be achieved by changing the elements of the minor that is complementary to the basis minor.
4.1.22. Use 4.1.19. 4.1.29. The rows of the matrix are orthogonal.
4.1.30. See 1.2.28.
4.1.36. Prove that the minor of order k placed in the left-hand corner is nonzero.
4.2.5. Use 4.2.4. 4.2.9. See 1.4.38.

4.2.19. Set the isomorphism between M and an arbitrary subspace, complementary to L.

4.2.33. That the intersection is a plane follows from 4.2.14. Moreover, if $L_1, \ldots, L_k$ are the directional subspaces of the given hyperplanes, then

$$\dim(\pi_1 \cap \ldots \cap \pi_k) = \dim(L_1 \cap \ldots \cap L_k).$$

Now, prove by the method of induction that in an n-dimensional space the dimension of the intersection of $k(n-1)$-dimensional subspaces is not less than $n-k$.

4.3.3. If $\pi = x_0 + L_{n-1}$ is a given hyperplane, then any nonzero vector from $L_{n-1}^{\perp}$ can be taken as the vector n, writing it as $(n, x) = b$. In this case $b = (n, x_0)$.

4.3.9. (a) follows from 4.3.8; (b) follows from 4.2.34 and 4.3.7.

4.3.11. Use 4.2.6.

4.3.17. It is obvious that length in a Euclidean space possesses the property $\rho(x, u) = \rho(x - x_0, u - x_0)$.

4.3.20. See the hint for 4.3.17.

4.3.24. Note that $L(p_1, p_2, q_1, q_2)$ can be described by the equation $\alpha_3 = 0$.

4.3.25. The vector $x_0 - y_0$ is orthogonal to the subspace $L(p_1, p_2, q_1, q_2)$.

4.3.27. Define a scalar product on the space so that the given basis may become orthonormal.

4.3.28. See the hint for 4.3.27.

4.3.29. Let $e_1, \ldots, e_k$ be a basis for the directional subspace of the plane P. Extend the linearly independent set $e_1, \ldots, e_k, x$ to form a basis for the space, and then define the scalar product with the aid of this basis.

4.4.2. The subspaces $L(u_1, \ldots, u_m)$ and $L(v_1, \ldots, v_l)$ must coincide.

4.4.11. See 4.4.10 and 4.4.3. 4.4.12. Use 4.4.10.

4.4.24. Change the variables by putting $t_1 = 3x_1$ and $t_2 = 2x_2$.

4.4.28. Use 4.1.36.

4.4.30. Find a basis for the orthogonal complement to $L(y_1, y_2, y_3)$.

4.4.32. If an arbitrary n-th row is adjoined to the system matrix then in the obtained square matrix, the numbers $(-1)^i A_i$ are (to the accuracy of the sign which is the same for all n numbers) cofactors of the elements of the n-th row.

4.4.34. Use 4.4.32. 4.5.3. Use 4.5.2. 4.5.10. See 4.4.14.

4.5.18. Change the variables by putting $t_1 = 6x_1$, $t_2 = 3x_2$, $t_3 = 11x_3$, $t_4 = -5x_4$.

4.5.19. Multiply the third equation of the system by 10, the fourth by 10^{-1}, and then make the substitutions: $t_1 = 1000x_1$, $t_2 = 0.001x_2$, $t_3 = 0.1x_3$, $t_4 = 10x_4$.

4.5.34. See 4.4.28.

4.5.36. Construct the general solution of the given system of equations and find the fundamental system of solutions of the reduced homogeneous system. Note that a normal solution should be orthogonal to this fundamental system.

4.5.48. Express the polynomial $f(t)$ in terms of the basis $1, t - a_1, (t - a_1)^2, \ldots, (t - a_1)^n$.

4.5.50. Prove that only the null polynomial satisfies the corresponding conditions for the homogeneous case.

4.5.52. Use the Cramer formulae and 3.1.56.

5.1.8. $Ax = (a, b)x - (a, x)b$. 5.1.49. Use 5.1.43.

5.1.56. Use 5.1.43.

5.1.58. Match each vector from T_A with the plane of its pre-images.

5.1.60. According to 5.1.59, the subspace T_A is isomorphic to the factor-space of the space X with respect to the subspace N_A.

5.1.63. Use 5.1.43.

5.1.65. Let $y_1 = Ax_1, \ldots, y_k = Ax_k$ be an arbitrary basis for the subspace L.

Show that the complete pre-image of L is the direct sum of the subspaces N_A and $L(x_1, \ldots, x_k)$. 5.2.3. Use 5.1.46.

5.2.9. The set of all operators mapping an n-dimensional space X to a one-dimensional space has n dimensions (see 5.2.3).

5.2.14. Show that if M is an arbitrary subspace complementary to N, then the spaces ω_{MY} and K_N are isomorphic.

5.2.15. (a) Let $e_1, \ldots, e_n$ be some basis for X. Given some operator A from ω_{XL}, expressions of the vectors $Ae_1, \ldots, Ae_n$ can be written in terms of the subspaces L_1 and L_2: $Ae_i = u_i + v_i$, $u_i \in L_1$, $v_i \in L_2$. Then $A = A_1 + A_2$, where $A_1e_i = u_i$, $A_2e_i = v_i$, $i = 1, \ldots, n$.

5.2.16. Use 1.5.16. 5.2.17. Use 5.2.4.

5.2.18. Prove that $T_{A+B} = X$.

5.2.24. It follows from the data that $Ax = \lambda_x Bx$ and $Ay = \lambda_y B_y$ for any nonzero vectors x and y. Show that $\lambda_x = \lambda_y$.

5.2.25. Use 5.2.14.

5.3.1. (a) Use the relations $T_{BA} \subset T_B$ and $T_{BA} = BT_A$.

5.3.2. (a) Use the equality: $(BA)X = BT_A$.

5.3.3. Use the relations

$$r_{BAC} = r_{AC} - \dim(T_{AC} \cap N_B), \quad r_{BA} = r_A - \dim(T_A \cap N_B).$$

5.3.8. Use 5.3.6. 5.3.11. Use 5.3.10.

5.3.14. If $A^2 = 0$, then $\alpha = 0$. When $A^2 \neq 0$, use 5.2.25.

5.3.17. Show that the intersection of N_P and T_P contains only the null vector and that $PT_P = T_P$.

5.3.18. (a) Use 5.3.17.

5.3.20. The operators $E, A, A^2, \ldots, A^{n^2}$ are linearly dependent.

5.3.23. Use 5.3.16, 5.3.15. 5.3.24. Use 5.3.14.

5.3.29. See 5.2.25. 5.3.33. Use 5.3.30.

5.3.34. Use 5.2.24.

5.3.47. If x is a nonzero vector from N_A, then $f(A)x \neq 0$, which is contrary to the condition that $f(t)$ is the annihilator.

5.3.48. If the free term is equal to zero, then a polynomial of a lesser degree can be found that also annihilates the given operator.

5.3.49. Use 5.3.20.

5.4.8. First evaluate BC. 5.4.9. First evaluate BC.

5.4.28. Use the theorem stating that any permutation can be factorized into the product of transpositions.

5.4.35. Represent the matrix J_λ as $J_\lambda = \lambda E + A$, where A is a Jordan block corresponding to zero; then use the result of Problem 5.4.33.

5.4.36. (b), (c) For the given diagonal matrix Λ, construct an interpolation polynomial $f(t)$ so that $f(d_{ii}) = \lambda_{ii}$, $i = 1, \ldots, n$.

5.4.40. Use 5.4.39. 5.4.49. Use 5.4.33. 5.4.52. Use 5.4.34.

5.4.56. Use 5.4.23.

5.4.57. The columns of AB are linear combinations of the columns of A, the rows of AB are linear combinations of the rows of B.

5.4.59. See 4.1.14.

5.4.69. Partition the matrices A and B into four square blocks of order 2 and apply the Strassen formulae to these blocks. Use the Strassen algorithm to evaluate the products of the blocks.

5.4.73. (d) Use the multiplication of partitioned matrices.

5.4.77. (b) See 3.4.41.

5.5.12. Using the properties of (skew) symmetry about the principal and secondary diagonals, only four minors can be evaluated. Use the orthogonality of its rows to evaluate the determinant.

5.5.15. Use the result of Problem 5.5.12.

5.5.17. For example, use the statement in Problem 5.3.49, according to which the inverse matrix A^{-1} is a polynomial of the matrix A.

5.5.18. Use 5.4.49. 5.5.19. Use 5.4.52. 5.5.20. Find the sum of the elements of the i-th row of the product $A^{-1}A = E$ in two ways.

5.5.27. Represent the matrix as $a\left(E + \frac{1}{a} J_0\right)$ and use 5.3.45. Here J is the Jordan block corresponding to zero.

5.5.28. According to 5.5.18, it suffices to compute the elements of the upper row of the inverse matrix only.

5.5.32. If P is a permutation matrix of the following form

$$P = \begin{Vmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{Vmatrix},$$

then PA is an upper triangular matrix.

5.5.39. Make all the leading principal minors nonzero by interchanging rows.

5.5.46. Show that $J_n^2 = nJ_n$.

5.5.49. Use 5.5.47.

5.5.54. Use 5.5.53.

5.5.56. Use the result of Problem 5.5.51 for the matrix 5.5.55.

5.5.57. Use 5.5.53.

5.5.61. Represent the matrix M as the product

$$\begin{Vmatrix} A & 0 \\ C & E_l \end{Vmatrix} \begin{Vmatrix} E_k & A^{-1}B \\ 0 & D - CA^{-1}B \end{Vmatrix},$$

where k is the order of the matrix A, and $k + l$ is the order of matrix M.

5.5.65. Use 5.4.73, (d). 5.5.66. See 5.5.60.

5.5.67. Use the formulae of Problem 5.5.62.

5.5.68. Use the formulae of Problem 5.5.64.

5.5.69. Use 5.5.65.

5.5.72. Differentiate the equality $AA^{-1} = E$.

5.5.77. Use the formula of Problem 5.5.75.

5.5.79. (d) Use the formula of Problem 5.5.75.

5.6.12. Use 5.6.9, (c). 5.6.27. Use 5.6.16.

5.6.29. Consider the operator which the matrix A defines with respect to an arbitrary pair of bases for the spaces X and Y. 5.6.30. Use 5.6.29.

5.6.32. Let

$$P^{-1}AP = A \quad \text{or} \quad AP = PA$$

for the matrix A and for any nondegenerate matrix P. Verify that the Schur lemma (see 5.4.40) remains valid in the case when A only commutes with all nondegenerate matrices.

5.6.36. Show that the mirror reflection of a matrix in its centre is a similarity transformation with the matrix P (see the hint to 5.5.32).

5.6.37. The equality of the traces of similar matrices can be deduced from 5.4.22, (c).

5.6.42. Use 5.6.22.

6.1.17. The matrix A is a polynomial of the matrix J_n of Problem 6.1.16.

6.1.19. See V. Voyevodin, Theorem 65.1, p. 204.

6.1.24. Use the test of the direct sum 1.5.18.

6.1.25. Use 6.1.24 to prove the necessary condition.

6.1.33. See 5.4.37. 6.1.34. See 5.4.39. 6.1.35. See 5.4.36.

6.1.38. Rewrite the condition $P^{-1}AP = \Lambda$ as $AP = P\Lambda$, and write the latter with respect to the columns.

6.1.40. Use the property of the Kronecker product 5.4.73, (c).

6.1.41. See 5.6.42. 6.1.43. See 5.6.43. 6.2.2. See 3.2.4.

6.2.3. The rank of the matrix is unity.

6.2.4. The rank of the matrix equals two. 6.2.7. Use 5.5.77.

6.2.13. Show that $m_l(A) = \operatorname{tr}(A^l)$.

6.2.19. Use the matrix of the operator constructed in 5.6.2.

6.2.20. Use the matrix of the operator constructed in 5.6.3, (a).

6.2.21. See 6.1.8.

6.2.41. Consider the matrix of the operator with respect to a basis whose first vectors form a basis for the eigensubspace, associated with λ. Using this matrix, compute the characteristic polynomial of the operator.

6.2.49. Show that for any eigenvalue λ_0 the rank of the matrix $\lambda_0 E - C(f(\lambda))$ equals $n-1$.

6.2.56. The matrix P^T is the companion of the polynomial $f(\lambda) = \lambda^n - 1$.

6.2.60. Use the matrix equality

$$\left\|\begin{matrix}\lambda E_m - AB & A \\ 0 & \lambda E_n\end{matrix}\right\| \left\|\begin{matrix}E_m & 0 \\ B & E_n\end{matrix}\right\| = \left\|\begin{matrix}E_m & 0 \\ B & E_n\end{matrix}\right\| \left\|\begin{matrix}\lambda E_m & A \\ 0 & \lambda E_n - BA\end{matrix}\right\|.$$

6.2.61. Use the matrix equality

$$\left\|\begin{matrix}E & E \\ 0 & E\end{matrix}\right\| \left\|\begin{matrix}\lambda E - A & -B \\ -B & \lambda E - A\end{matrix}\right\| = \left\|\begin{matrix}\lambda E - (A + B) & 0 \\ -B & \lambda E - (A - B)\end{matrix}\right\| \left\|\begin{matrix}E & E \\ 0 & E\end{matrix}\right\|.$$

6.2.64. Use the matrix equality

$$\left\|\begin{matrix}E & iE \\ 0 & E\end{matrix}\right\| \left\|\begin{matrix}\lambda E - B & C \\ -C & \lambda E - B\end{matrix}\right\| = \left\|\begin{matrix}\lambda E - A & 0 \\ -C & \lambda E - \bar{A}\end{matrix}\right\| \left\|\begin{matrix}E & iE \\ 0 & E\end{matrix}\right\|,$$

$A = B + iC$, $\bar{A} = B - iC$.

6.3.5. See 6.1.25.

6.3.6. Any subspace of dimension $k-1$ can be represented as the intersection of two subspaces of dimension k. Thereby, any subspace of dimension $k-1$ is also A-invariant.

6.3.9. Use 6.3.3 for the operator $A - \lambda_0 E$ where λ_0 is an eigenvalue of A.

6.3.21. Use 6.3.18, (a).

6.3.26. Use 6.3.14 and 6.3.25. 6.3.27. Use 6.3.14.

6.3.28. If n is the dimension of the space, then there are not more than n^2 linearly independent operators; therefore, it suffices to prove the statement for a finite number of commuting operators, e.g., by induction over this number.

6.3.30. In each pair of invariant subspaces with respect to the differential operator, one is contained in the other.

6.3.32. Let all roots of the characteristic polynomial of the operator A be complex; each is an eigenvalue of the corresponding operator $\hat{A}$. Show that if λ is an arbitrary eigenvalue of $\hat{A}$ and $z = x + iy$ is an eigenvector associated with λ, then the subspace drawn on the real vectors x and y has 2 dimensions and is A-invariant.

6.3.36. Use 6.3.9. 6.3.38. Use 6.3.19.

6.3.39. Use 6.3.19.

6.3.42. Show that for each eigenvalue of $\lambda_1, \ldots, \lambda_m$, the defect of the matrix $B - \lambda_i E$ equals k_i.

6.3.46. Repeat the construction performed in Problem 6.3.38, taking into account Problem 6.3.32.

6.3.49. See 6.3.48, (b).

6.4.1. Prove that for the number q, mentioned in 5.3.10, the subspaces N_q and T_q intersect in the null vector only.

6.4.2. Let $X = N \dot{+} T$ be the decomposition derived in 6.4.1, where N is the kernel and T is the image of the operator A^q. If $X = N_1 \dot{+} T_1$ is any other decomposition such that A/N_1 is nilpotent and A/T_1 is nondegenerate, then show that $N_1 \subset N$, $T_1 \subset T$.

6.4.3. The characteristic polynomial of the operator A equals the product of the characteristic polynomials of the operators A/N and A/T.

6.4.4. Apply 6.4.1 to the operator $A - \lambda_1 E$ and show that in the decomposition $X = N_1 \dot{+} T_1$, the subspace N_1 possesses all the properties required for K_{λ_1}. Then, decompose the subspace T_1 emanating from the operator $A - \lambda_2 E$, etc.

6.4.5. Use 6.4.2 and 6.3.43. 6.4.9. See 6.3.50.

6.4.10. Use the decomposition (6.4.2). 6.4.11. Use 6.4.10.

6.4.14. Use 6.2.61 to find the eigenvalues of the matrix.

6.4.37. Contrast with 6.3.17. 6.4.38. Use 6.4.17, (e), and 6.4.37.

6.4.41. Select linearly independent vectors $x_1, \ldots, x_p$ so that their span in the direct sum with H_{t-1} produces the whole space X.

6.4.49. (d) $m_{t-1} - m_{t-2} = p_2$.

6.4.57. According to 6.4.56 the sequence of numbers $p_1, \ldots, p_t$ is nondecreasing.

6.4.62. The matrix is reduced to quasi-diagonal form by the same transformations of the columns and rows.

6.4.72. Use 6.4.18. 6.4.75. Use 6.4.34, 6.4.48, 6.4.55.

6.4.76. Use the Jordan form of the operator.

6.4.80. Square the matrix $J - \lambda_0 E$ and calculate the increase in the defect; multiply the matrix $(J - \lambda_0 E)^2$ by $J - \lambda_0 E$ and calculate the increase in the defect again, etc.

6.4.86. Note that for the matrix

$$B = \begin{Vmatrix} -1 & 0 & 1 \\ 1 & -1 & -3 \\ 0 & 1 & 2 \end{Vmatrix},$$

the equality $B^3 = 0$ holds. When raising the matrix $A - E$ to a power, remember that the matrix is quasi-triangular.

6.4.87. See the hint for 6.4.86. 6.4.88. The defect of the operator equals unity.

6.4.91. Verify the equality of the ranks and traces of the matrices A, B and C.

6.4.98. Use 6.4.39.

6.4.100. Let $A = P\Lambda P^{-1}$, where Λ is a diagonal matrix. Then the matrix $A \times B$ is similar to the matrix $\Lambda \times J$, and $A \times E_n + E_m \times B$ is similar to $\Lambda \times E_n + E_m \times J$.

6.4.102. Use 6.4.32.

7.1.9. Consider the matrix of the operator in Cartesian coordinate system.

7.1.12. Note that the basis (b) mentioned in Problem 7.1.11 is orthonormal with respect to the scalar product defined in (7.1.2).

7.1.13. Note that the basis (b) mentioned in Problem 7.1.11 is orthogonal with respect to the scalar product defined in (7.1.3), and use the result of 7.1.6.

7.1.22. Use 6.4.77.

7.1.23. Use the correspondence between the conjugate operators and conjugate matrices.

7.1.34. Use the result of Problem 6.3.17.

7.1.40. Use the existence of a common eigenvector of commuting operators A^* and B^*, and therefore, of a common invariant subspace of dimension $n - 1$ to the operators A and B. Here n is the dimension of the space.

7.1.41. Use the Schur theorem. 7.1.45. Use 7.1.7. 7.2.8. See 7.1.10. 7.2.9. Construct the matrices for the operators with respect to the orthonormal basis $1, t, t^2, \ldots, t^n$.

7.2.10. Use 5.4.52. 7.2.14. Use 7.1.16. 7.2.16. Use 7.1.15.

7.2.19. Show that $N_A = N_{A^*}$.

7.2.20. Deduce from the indicated data that the root subspaces of the operator A coincide with its eigensubspaces, and that they are mutually orthogonal.

7.2.23. Use 7.2.18.

7.2.24. Prove the existence of an orthonormal basis containing the eigenvectors of the operator A following the procedure for the construction of the Schur basis.

7.2.25. Use 6.3.25. Use 7.2.13 for another possible solution.

7.2.26. Use 7.2.25.

7.2.32. The given matrix differs from a real one by an addend $-iE$.

7.2.36. Define a scalar product using the eigenvector basis of the operator A.

7.2.38. To prove the necessary condition, construct an interpolation polynomial $f(\lambda)$ so that for each eigenvalue λ_i of the operator A, the condition $f(\lambda_i) = \bar{\lambda}_i$ may be fulfilled.

7.2.40. See 7.1.40.

7.2.47. Use the decomposition of vector x in terms of the orthonormal eigenvector basis of the operator A.

7.2.48. Use 7.2.47 for the vector $x = (1\ 1\ 1\ \ldots\ 1)^T$.

7.2.50. To prove the latter statement, show that a basis can be selected that consists of "real vectors", i.e. vectors of the form $x + i0$ for the eigensubspace of the operator $\widehat{A}$ associated with an eigenvalue λ.

7.3.9. Consider the operator matrix with respect to the orthonormal eigenvector basis and remember that a circumference can be drawn through the points $\lambda_1, \lambda_2, \lambda_3$ on the complex plane.

7.3.13. Verify that $A^2 = E$.

7.3.16. The effect of the operator in the polynomial $1 - 2t + t^2$, which is orthogonal to the two given polynomials $1 + t + t^2$ and $1 - t^2$ may be determined by the data given. Construct the matrix of the operator Q with respect to the basis formed by these polynomials, and then transform it to the required basis $1, t, t^2$.

7.3.18. See 7.3.19.

7.3.20. Use the relation $(x, y) = \dfrac{|x+y|^2 - |x-y|^2}{4}$ for the real case and $(x, y) = \dfrac{|x+y|^2 - |x-y|^2 + i\,|x+iy|^2 - i\,|x-iy|^2}{4}$ for the complex case.

7.3.39. Consider a sequence of matrices $T_{12}, T_{13}, \ldots, T_{1n}$ whose parameters are selected in accordance with 7.3.38.

7.4.8. Assume the vectors x and y to be a basis for the space.

7.4.10. A skew-symmetric operator on the three-dimensional space is degenerate. Consider the matrix of the operator K with respect to an orthonormal basis, one of whose vectors belongs to the kernel of K.

7.4.11. Verify that $(x_i, y_j) = (y_i, x_j)$ when $i \neq j$.

7.4.15. Find a polynomial $f_3(t)$, orthogonal to the given polynomials $f_1(t) = 2 + 2t - t^3$ and $f_2(t) = 2 - t + 2t^2$ and having the same length. The matrix of the operator S with respect to the orthogonal basis $f_1(t), f_2(t), f_3(t)$ can be determined from the data of the problem. Then transform the matrix to the required basis $1, t, t^2$.

7.4.27. Consider the matrix of the operator A with respect to the orthonormal eigenvector basis. Draw a straight line through the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the complex plane.

7.4.32. According to 7.4.30, the i-th column e_i is the eigenvector associated with the eigenvalue λ_1.

7.4.41. Show that the eigenvalues of an irreducible Hermitian matrix cannot be multiple.

7.4.43. (a) A common root of the polynomials $f_{i+1}(\lambda)$ and $f_i(\lambda)$ is also a root of the polynomial $f_{i-1}(\lambda)$, etc. But the polynomial $f_0(\lambda) \equiv 1$ has no roots; (b) use (a) and 7.4.35; (c) use iterative formulae.

7.4.48. To find the sequence (7.4.8), employ the iterative formulae connecting the polynomials $f_i(\lambda)$.

7.4.51. Show that the matrix A is similar to a tridiagonal irreducible Hermitian matrix.

7.4.52. Use 7.2.50 to prove (b).

7.5.9. Without loss of generality, the principal submatrix of order k under consideration can be assumed to lie in the left-hand upper corner of the given matrix H. Then the scalar product (Hx, x) for the column vectors, in which only the first k components can be different from zero, should be considered.

7.5.10. Let G be the Gram matrix, $x = (\alpha_1, \ldots, \alpha_k)^T$ an arbitrary k-dimensional column vector. Show that $(Gx, x) = |\bar{\alpha}_1 x_1 + \ldots + \bar{\alpha}_k x_k|$.

7.5.16. (a) Use the decomposition of the vector x in terms of the eigenvectors of the operator H.

7.5.23. Use 7.5.22. 7.5.24. Use the test 7.5.18.

7.5.25. Use 7.4.53.

7.5.27. Consider the associated matrix H_k.

7.5.29. Show that the Schur product of the matrices H_1 and H_2 is the principal submatrix of the Kronecker product $H_1 \times H_2$.

7.5.32. Use 7.4.38.

7.5.33. Use the decomposition in terms of the eigenvectors of the operator H.

7.5.34. (a) Use 7.5.23; (b) use 7.5.32.

7.5.36. To prove the sufficient condition, use 7.4.35.

7.5.41. Use 7.5.24. 7.5.43. Use 7.5.30.

7.5.44. Use the Sylvester criterion.

7.5.45. See V. Voyevodin, p. 261. 7.5.49. $H^2 = 4H$.

7.5.50. Use Hadamard's inequality (see 3.3.3) for the square root of the matrix H.

7.5.51. Use 3.3.25. 7.5.55. Use 7.5.52.

7.5.56. Show that the operator HS has the same eigenvalues as the Hermitian operator $S^{1/2}HS^{1/2}$.

7.5.62. See V. Voyevodin, Sec. 78.

7.6.8. Consider the matrices of the differential operator and its conjugate with respect to the orthonormal basis $1, t, t^2, \ldots, t^n$.

7.6.9. To evaluate the singular values, use the matrix of the conjugate operator obtained in 7.1.12, with respect to the basis $1, t, t^2$.

7.6.10. Let X and Y be arbitrary Euclidean (unitary) spaces of dimensions n and m, respectively. Consider the operator generated by the matrix A with respect to an arbitrary pair of bases for the spaces X and Y, and use the singular bases of this operator.

7.6.17. If $A = U\Lambda V$ is the singular-value decomposition of the matrix A, then U^* and V^* are convenient matrices.

7.6.26. Use 6.3.51. 7.6.27. Use 7.6.26.

7.6.29. Using 7.6.28, prove that in the Schur form for the operator A, all the off-diagonal elements of the row and column, in whose intersection λ_1 lies, equal zero.

7.6.30. Using 7.6.29, prove the existence of a basis of the eigenvectors common to the operators A and A^*.

7.6.33. The proof is similar to that of 7.4.38.

7.6.39. The columns of the matrix are orthogonal.

7.6.40. The rank of the matrix equals unity. 7.6.42. Use 7.6.20.

7.6.43. The matrix is symmetric.

7.6.45. Verify that the matrix to the accuracy of the numerical factor 2 is unitary.

7.6.46. Use 7.6.36. 7.6.50. See the solution of 7.6.49.

7.6.51. Use the equality $A/T_{A^*} = (H/T_A)(U/T_{A^*})$ or $(H/T_A)^{-1} \times (A/T_{A^*}) = U/T_{A^*}$.

7.6.62. Use 7.6.59. 7.7.2. Use 7.4.24.

7.7.8. Use 7.4.16.

7.7.13. Use an orthonormal basis of the eigenvectors, common to the commuting normal operators (see 7.2.40).

7.7.17. (b) AH is similar to the matrix $H^{1/2}AH^{1/2} = H^{1/2}H_1H^{1/2} + iH^{1/2}H_2H^{1/2}$.

7.7.18. Perform a similar transformation $\widetilde{A} = DAD^{-1}$, where D is a diagonal matrix with positive diagonal elements selected so that for the matrix $\widetilde{A}$ the equality is valid $\widetilde{a}_{i\,i+1} = -\widetilde{a}_{i+1\,i}$, $i = 1, 2, \ldots, n-1$. Then use the Bendixson theorem (see 7.7.15).

7.7.19. Use the Bendixson theorem.

7.7.20. Consider the equality $AH_1^{-1} = E + iH_2H_1^{-1}$ and show that $|\det(AH_1^{-1})| \geqslant 1$.

7.7.21. Consider the Hermitian decomposition of the operator A with respect to the Schur basis and use 7.5.50.

7.8.8. Use 7.5.62.

7.8.11. The vector $b = (1\ 1\ 1)^T$ is orthogonal to T_A.

7.8.15. The matrix of the system is nonnegative.

7.8.18. The system splits into two, in x_1, x_4 and in x_2, x_3, x_5.

7.8.22. Use 7.8.5. 7.8.23. Use 7.8.6. 7.8.32. Use 7.8.7.

7.8.37. Use 7.1.32 and 7.8.26.

7.8.38. It follows from the data that $T_{BA} = T_B$, $N_{BA} = N_A$. To prove the required relation, use 7.8.26.

7.8.39. Show that the effect of both operators on the vectors of the singular basis is the same.

7.8.42. (a), (b). Use 7.8.41; (c) first show that the operator X has the same image and kernel as the operator A^*, and then deduce from the equation $A^*AX = A^*$ that the effect of X on the two subspaces T_A and T_{A^*} is inverse to the effect of the operator A.

7.8.43. Use 7.8.42, (a).

7.9.6. The proof is given in the same way as the law of inertia.

7.9.11. Use the law of inertia.

7.9.14. Note that in transforming from D_{k-1} to D_{k+1}, the number of coincidences and changes of sign increases by one each, irrespective of the sign ascribed to D_k. Moreover, the number of positive and the number of negative eigenvalues of the submatrix A_{k+1} each is one greater than the corresponding number for the submatrix A_{k-1}.

7.9.39. See 7.9.40. 7.9.40. Use 5.6.36.

7.9.42. Show that the roots of a z-equation are unaltered in a nondegenerate transformation of both forms.

7.9.44. Use 7.2.40.

7.9.50. Let A and B be the matrices of the quadratic forms F and G, and let $B = S^TS$ be the triangular decomposition of the matrix B. The roots of the z-equation $|A - zB| = 0$ are the eigenvalues of the symmetric matrix $(S^{-1})^T AS^{-1}$. Hence, 7.4.30 can be used.

8.1.2. (d) Use the Minkowski inequality.

8.1.20. Show that all the subsequences have the same limit a and that a is the limit of the whole sequence.

8.1.22. Use 8.1.21.

8.1.23. For a fixed basis for the space, the coordinates of all the vectors of the given sequence are bounded.

8.1.32. Use the equivalence of the convergence, with respect to any norm, to the coordinate convergence.

8.1.35. Consider the values of each norm on the unit ball determined by another norm.

8.1.38. Use 8.1.35. 8.1.50. Use 8.1.49.

8.2.6. Prove the statement (b) for the subordinate operator norm, and then use the equivalence of the norms.

8.2.18. Use 7.1.17. 8.2.21. (b), (c). Use 7.6.34.

8.2.22. Use the relations

$$H_1=\frac{1}{2}(A+A^*),\quad H_2=\frac{1}{2i}(A-A^*).$$

8.2.27. Use 7.6.64 and 7.6.34.

8.2.28. Show that for a positive semidefinite matrix A: $S(A) = \operatorname{tr} A$.

8.2.29. Use 8.1.34. 8.2.37. Use 8.2.34.

8.2.39. Use the representation of the subordinate norm from 8.2.38.

8.2.41. Put for the given norm $M(A)$, $m(x) = M(X)$ where X is a matrix with x as the first column, and whose other columns are zero. Show that $m(x)$ is a norm on the arithmetic space and that $M(A)$ is consistent with $m(x)$.

8.2.44. Use 8.2.42 and 8.2.39.

8.2.46. Use 8.2.45.

8.3.3. Use 8.3.2.

8.3.5. Use 7.6.33.

8.3.6. Use 3.3.32 and 8.3.5.

8.3.7. Use 7.6.33. The solution is similar to that of 8.3.5.

8.3.8. Represent A as $A = D(E + D^{-1}B)$, where D is a diagonal matrix made up of the diagonal elements of A.

8.3.10. Apply the test of Problem 8.3.9 to the transpose A^T.

8.3.11. See the hint for 8.3.8.

8.3.25. Verify that $|\det A| = 1$. Therefore (see 8.3.24) an increase in the condition number is only possible when the norm of the matrix increases. Show that matrices with a large Euclidean norm fulfilling the conditions of the problem possess a smaller condition number.

8.3.27. Use the expression $\operatorname{cond}_2(A + \alpha E)$ in terms of the eigenvalues of the matrix A.

8.3.28. See 7.4.35.

8.3.30. To estimate the condition number, use the inequalities of Problem 7.6.28. If the first row of the system is multiplied by 10^{-1}, second by 10, and third by 100, then the matrix of the derived system will be symmetric.

8.3.35. Show that the solution of the system $D(x) = b$ may be taken as an approximation to the exact solution of the given system, where

$$D=\left\|\begin{matrix}2.5 & 0 & 0 & 0\\ 0 & -3 & 0 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 0 & 4\end{matrix}\right\|,\quad b=\left\|\begin{matrix}5\\ 3\\ 10\\ 4\end{matrix}\right\|.$$

8.3.36. Show that the solution of the system $Bx = b$ may be taken as an approximation to the exact solution of the given system, where

$$B=\left\|\begin{matrix}0.5 & -0.5 & 0 & 0\\ 0.5 & 0.5 & 0 & 0\\ 0 & 0 & 3 & -2\\ 0 & 0 & -2 & 1\end{matrix}\right\|,\quad b=\left\|\begin{matrix}0.5\\ 0.5\\ 0\\ 0\end{matrix}\right\|.$$

8.3.37. Use the identity $B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1}$.
8.4.1. See 8.2.41, and also V. Voyevodin, p. 275.
8.4.4. Prove that the matrix is positive definite.
8.4.7. Apply 8.4.1 to the companion matrix of the polynomial $f(z)/a_n$.
8.4.9. Use 6.4.102. 8.4.12. Use the Schur theorem. 8.4.15. Use 6.4.102.
8.4.17. Use the Schur inequality and the statement 8.4.14.
8.4.23. Consider the given matrix as a perturbation of the matrix

$$\begin{Vmatrix} 2 & 1.5 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

8.4.25. Use the inequalities of Problem 7.4.38.
8.4.27. The given matrix is similar to the symmetric matrix

$$\begin{Vmatrix} 2\cdot10^{-4} & -3\cdot10^{-4} & 4\cdot10^{-4} & 0.9991 \\ -3\cdot10^{-4} & 1\cdot10^{-4} & -0.4993 & -6\cdot10^{-4} \\ 4\cdot10^{-4} & -0.4993 & 2\cdot10^{-4} & -2\cdot10^{-4} \\ 0.9991 & -6\cdot10^{-4} & -2\cdot10^{-4} & 1\cdot10^{-4} \end{Vmatrix},$$

which can be considered as a perturbation of a symmetric matrix B such that $b_{14} = b_{41} = 1$, $b_{23} = b_{32} = -0.5$, and the other elements b_{ij} equal zero.

8.4.28. Use the inequalities of Problem 7.4.38.

8.4.29. Consider the given matrix as a perturbation of a symmetric matrix B such that $b_{11} = b_{22} = 1$, $b_{12} = b_{21} = -2$, $b_{34} = b_{43} = -1$ and the other elements b_{ij} are zeroes.

8.4.30. Use 7.4.38. 8.4.33. See 7.6.23.

8.4.34. From the normality of the matrix A, the normality of $A - \mu E$ follows.

8.4.35. (a) On the orthogonal complement to the vector e_1, the moduli of the eigenvalues of the normal matrix $A - \mu_0 E$ are not less than a; (b) use the condition $\| A\tilde{x} - \mu_0\tilde{x} \|_2 = \varepsilon$; (d) use the Cauchy-Buniakowski inequality.

8.4.37. See 8.4.34.

8.4.39. (c) Use the estimate (8.4.6).

8.4.40. Use the Schur theorem. 8.4.41. Use the Schur theorem.

8.4.46. Add the matrix $\Delta_{n-1} = \frac{\bar{\varepsilon}}{1 - |\varepsilon|^2} zc$ to C_{n-1} and estimate $\| \Delta_{n-1} \|_2$.

8.4.47. This statement may be derived from 8.4.46 in the same way 8.4.44 is derived from 8.4.43.

Answers and Solutions

1.1.2. Yes, if the straight line passes through the point O; otherwise no. 1.1.3. No. 1.1.4. No.

1.1.7. No. 1.1.8. Yes.

1.1.10. 2^k.

1.1.11. Yes. 1.1.12. Yes.

1.1.13. Yes. 1.1.14. Yes. 1.1.15. No.

1.1.16. (a) No; (b), (c), (d) yes.

1.1.17. Let G be an abelian group under addition containing more than one element. Fix some field P, and for any $x \in G$ and any $\lambda \in P$, set $\lambda x = 0$.

Thus, the use of this axiom $1 \cdot x = x$ means that by multiplying vectors of the given space by arbitrary numbers every vector could be obtained.

1.2.9. (a) Yes; (b) yes; (c) no.

1.2.11. The set is linearly independent.

1.2.12. The set is linearly dependent.

1.2.13. $5t^3 - 5t^2 - 4t + 6$ in both cases. The set is linearly dependent.

1.2.18. Let $x_1, \ldots, x_s$ be an arbitrary set of vectors of the arithmetic space. Set up the coordinate matrix of these vectors

$$\left\| \begin{matrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1k} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2k} \\ \cdot & \cdot & \cdots & \cdot \\ \beta_{s1} & \beta_{s2} & \cdots & \beta_{sk} \end{matrix} \right\| .$$

Let m be the first column in the matrix with nonzero numbers. By interchanging the rows of the matrix, which corresponds to interchanging the vectors of the set, we can make $\beta_{1m} \neq 0$. Subtracting the appropriate multiples of the first row from all the following rows zeroes can be obtained at every place in the m-th column except the first. These row transformations are obviously equivalent to a sequence of elementary transformations of type (c) over the vector set $x_1, \ldots, x_s$. Considering now all the rows of the matrix except the first, repeat the procedure, etc.

1.2.19. The set is linearly independent. 1.2.20. The set is linearly dependent. 1.2.21. The set is linearly independent. 1.2.22. The set is linearly dependent. 1.2.23. The set is linearly dependent. 1.2.24. The set is linearly independent. 1.2.25. The set is linearly independent. 1.2.26. The set is linearly independent. 1.2.27. The set is linearly independent.

1.3.1. All vectors of the form $(\alpha, 0, \beta, 0, \gamma)$. 1.3.2. All vectors of the form $(\alpha, \beta, \gamma, \beta, \alpha)$. 1.3.3. All vectors $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ satisfying the condition $\sum_{i=1}^{5} \alpha_i = 0$.

1.3.4. All polynomials of degree $\leqslant 2$ and the null polynomial. 1.3.5. The same answer as to 1.3.4. 1.3.6. All polynomials of degree $\leqslant 2$, in which the coefficient sum equals zero, and the null polynomial. 1.3.7. The same as in 1.3.6.

1.3.8. No.

1.3.11. $z_1 = 6x_2 + 4x_3$, $z_2 = 2x_1 - 10x_2 + 8x_3$.

1.3.14. Yes. 1.3.15. No.

1.3.28. 2. 1.3.29. 2. 1.3.30. 4. 1.3.31. 3. 1.3.32. 3. 1.3.33. 4.

1.3.35. For example, x_1, x_2. 1.3.36. For example, x_1, x_2, x_4. 1.3.37. For example, x_1, x_2, x_4. 1.3.38. For example, x_1, x_2.

1.3.40. (a) Precisely r vectors of the set are different from zero; (b) $r + 1$ vectors of the set are different from zero, two of them being collinear; (c) either $r + 2$ vectors of the set are different from zero, three of them being collinear, or $r + 1$ vectors of the set are nonzero and there exist three linearly dependent vectors, two of which are not collinear.

1.3.41. x_1, x_2; x_1, x_4; x_2, x_3; x_3, x_4.

1.3.42. Any two vectors. 1.3.43. x_1, x_2; x_2, x_3; x_2, x_4.

1.3.45. Yes. 1.3.46. Yes.

1.4.1. The space is one-dimensional, its basis being any number other than one. 1.4.2. The dimension of the space equals k. 1.4.3. The space is infinite-dimensional. 1.4.4. The dimension of the space equals 2. 1.4.5. The space is infinite-dimensional. 1.4.6. The dimension of the space equals $n + 1$.

1.4.7. (a) 1; (b) 2. 1.4.8. (a) n; (b) $2n$.

1.4.13. The basis is the set (b).

1.4.21. The base can be made up, for example, of the 1st, 3rd and 4th polynomials. 1.4.22. The base can be made up, for example, of the 1st and 2nd polynomials.

1.4.23. 1/3, 1/3, 1/3. 1.4.24. 0, −5.4. 1.4.25. 0, 2, 1, 2. 1.4.26. 67, −51, −3, 11.

1.4.27. (a) 1, −1, −1, 1, −1, 1; (b) 2, −1, −1, 1, −1, 1; (c) 1, −1, −1, 2, −1, 1.

1.4.28. $e = e_1 - e_2$.

1.4.35. x_1, x_2, x_3, x_4. 1.4.36. For example, x_1, x_2, x_3.

1.4.37. The dimension of L equals $n - 1$.

1.4.38. (a), (b), (c) n; (d) $n - 1$.

1.4.39. The basis can be made up, for example, of the 1st, 2nd and 3rd polynomials.

1.4.43. No. 1.5.2. No. 1.5.3. No.

1.5.7. $L_2 \subset L_1$. The vectors x_1, x_2, x_3 are linearly independent. 1.5.8. The basis for the sum can be made up, for example, of the vectors x_1, x_2, x_3, y_1. The dimension of the intersection is 2.

1.5.10. The basis for the sum can be made up, for example, of the vectors x_1, x_2, y_1. The basis for the intersection is the vector $z = (3, 5, 7)$.

1.5.11. The basis for the sum can be made up, for example, of the vectors x_1, x_2, x_3, y_1; the basis of the intersection, for example, of $z_1 = (1, -1, 1, -1)$, $z_2 = (2, 0, 2, 0)$. 1.5.12. The basis for the sum could be, for example, x_1, x_2, x_3, y_2. The basis for the intersection could be, for example, $z_1 = (0, 4, 1, 3)$, $z_2 = (2, 0, 1, -1)$.

1.5.20. $x = (-1, -2, -6, -3) + (3, 2, 6, 6)$.

1.5.23. The subspace L is two-dimensional. As the complementary subspace, for example, the spans of the vectors $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$ and $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ can be taken.

1.5.24. For example, the set of polynomials of the form $c \cdot t^n$.

2.1.5. A change in the scale unit for measuring lengths.

2.1.7. (a) −1; (b) 4; (c) 0.

2.1.11. Yes, if $\alpha = 0$; no, when $\alpha \neq 0$.

2.1.14. 0. 2.1.19. No.

2.2.5. $y_1 = x_1 = (1, -2, 2)$, $y_2 = \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$, $y_3 = (6, -3, -6)$.

2.2.6. $y_1 = x_1 = (1, 1, 1, 1)$, $y_2 = (2, 2, -2, -2)$, $y_3 = (-1, 1, -1, 1)$.

2.2.10. For example, add the vectors $x_3 = (1, 1, 1, 0)$ and $x_4 = (-1, 1, 0, 1)$.

2.2.11. For example, add the vectors $x_3 = (2, 3, 1, 0)$ and $x_4 = (1, -1, 1, 1)$.

2.2.12. Add, for example, the vector $x_3 = (2/3, -1/3, 2/3)$.

2.2.13. Add, for example, the vector $x_3 = (1/2,\ 1/2,\ 1/2,\ 1/2)$, $x_4 = (1/2,\ 1/2,\ -1/2,\ -1/2)$.

2.2.16. $n - 1$, where n is the dimension of the given Euclidean space.

2.2.17. (a) $(x, y) = \lambda_1^2\alpha_1\beta_1 + \lambda_2^2\alpha_2\beta_2 + \ldots + \lambda_n^2\alpha_n\beta_n$;

(b) $(x, y) = (2\alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_2\beta_1) + (\alpha_2\beta_2 + \alpha_3\beta_3 + \ldots + \alpha_n\beta_n)$.
Here $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are the coordinates of the vectors x and y with respect to the corresponding bases.

2.2.19. $y_1 = x_1 = (2, 3, -4, -6)$, $y_2 = (-3, 2, 6, -4)$, $y_3 = (4, 6, 2, 3)$.

2.2.20. $y_1 = x_1 = (1, 1, -1, -2)$, $y_2 = (2, 5, 1, 3)$, $y_3 = (2, -1, 1, 0)$.

2.2.22. Let $n \geqslant 3$. Compile by row the coordinate matrix of the vectors $e_1, \ldots, e_n$. Note that if the signs of all elements of an arbitrary column of this matrix are changed, then its rows produce the coordinates of a new and still orthogonal set of vectors. We shall assume, therefore, that the first row of the matrix consists of units only, and that in the first three rows, the columns of only one of the following forms are possible

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{matrix}.$$

Denote the number of columns of each of the indicated forms by x, y, z, w, respectively. Then, obviously,

$$x + y + z + w = n.$$

It follows from the orthogonality of the first three vectors that

$$x + y - z - w = 0,$$
$$x - y + z - w = 0,$$
$$x - y - z + w = 0.$$

We obtain from this system that $x = y = z = w = n/4$. Thus, n must be a multiple of 4.

2.2.26. If $f(t) = a_0 + a_1t + a_2t^2 + \ldots + a_nt^n$, $g(t) = b_0 + b_1t + b_2t^2 + \ldots + b_nt^n$, then

$$(f, g) = a_0b_0 + a_1b_1 + (2!)^2 a_2b_2 + \ldots + (n!)^2 a_nb_n.$$

2.3.6. For example, $y_1 = (-3, 1, -2, 0)$, $y_2 = (1, -1, -2, 1)$.

2.3.8. (a) The subspace of polynomials of dimension one all of whose coefficients are equal; (b) the subspace of all odd polynomials.

2.3.9. For example,

$$3\alpha_1 - \alpha_2 + 2\alpha_3 = 0$$
$$\alpha_1 - \alpha_2 - 2\alpha_3 + \alpha_4 = 0$$

for the subspace L, and

$$\alpha_1 + 3\alpha_2 + 2\alpha_4 = 0$$
$$3\alpha_1 + 7\alpha_2 - \alpha_3 + 2\alpha_4 = 0$$

for its orthogonal complement.

2.3.10. Let L be the span of the vector set $a_1, \ldots, a_k$, not necessarily linearly independent. The required vector y can be represented as a linear combination $y = \alpha_1a_1 + \ldots + \alpha_ka_k$. Since $(z, a_i) = 0$, $i = 1, \ldots, k$, to determine the coefficients $\alpha_1, \ldots, \alpha_k$ the following system of linear equations can be formed

$$(a_1, a_1)\alpha_1 + (a_2, a_1)\alpha_2 + \ldots + (a_k, a_1)\alpha_k = (x, a_1),$$
$$(a_1, a_2)\alpha_1 + (a_2, a_2)\alpha_2 + \ldots + (a_k, a_2)\alpha_k = (x, a_2),$$
$$\cdots\cdots\cdots\cdots$$
$$(a_1, a_k)\alpha_1 + (a_2, a_k)\alpha_2 + \ldots + (a_k, a_k)\alpha_k = (x, a_k).$$

To construct the vector y, any solution of the system can be used. The vector z is determined as the difference $x - y$.

2.3.11. If the set $x_1, \ldots, x_k$ is linearly independent.

2.3.12. $y = (5, 2, -9, -8)$, $z = (9, -5, 3, 1)$.

2.3.13. $y = (0, -3, 5, 2)$, $z = (2, -2, -2, 2)$.

2.3.14. $y = (1, 2, -5, 1)$, $z = (-4, -2, 0, 8)$.

2.3.16. Suppose $e_1, \ldots, e_n$ is the given, and $f_1, \ldots, f_n$ is the required, biorthogonal basis. The conditions

$$(e_i, f_j) = 0, \quad i = 1, \ldots, j-1, j+1, \ldots, n,$$

stipulate that the vector f_j should belong to the orthogonal complement of the span of the vectors $e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n$. In this one-dimensional subspace, the condition $(e_j, f_j) = 1$ uniquely determines the vector f_j.

2.3.18. $f_1 = (1, 0, 0, 0)$, $f_2 = \left(0, \frac{1}{2}, 0, 0\right)$, $f_3 = \left(0, 0, \frac{1}{3}, 0\right)$, $f_4 = \left(0, 0, 0, \frac{1}{4}\right)$.

2.3.19. $f_1 = (1, 0, 0, 0)$, $f_2 = (0, 1, 0, 0)$, $f_3 = (-1, -2, 1, -3)$, $f_4 = (0, 0, 0, 1)$.

2.3.20. $f_1 = (1, 0, 0, 0)$, $f_2 = (-1, 1, 0, 0)$, $f_3 = (0, -1, 1, 0)$, $f_4 = (0, 0, -1, 1)$.

2.3.21. $f_1 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$, $f_2 = \left(\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$, $f_3 = \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right)$, $f_4 = \left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)$.

2.4.2. (a) It remains unaltered; (b) it is reduced to the supplementary angle (up to π); (c) it remains unaltered.

2.4.4. $|x| = 3\sqrt{2}$, $|y| = 6$, $|x - y| = 3\sqrt{2}$. Thus, the triangle is isosceles. $\widehat{x, (x - y)} = \frac{\pi}{2}$, hence, the triangle is right-angled. $\widehat{x, y} = \frac{\pi}{4}$ and is an interior angle of the triangle. $\widehat{y, (x - y)} = \frac{3\pi}{4}$ and, hence, an interior angle of the triangle is the angle, $\widehat{y, (y - x)}$.

2.4.6. (a) $|f|^2 = 10$, $|g|^2 = 9$, $|f - g|^2 = 3$, $|f|^2 < |g|^2 + |f - g|^2$, and, hence, the triangle is acute-angled; (b) $|f|^2 = 19$, $|g|^2 = 13$, $|f - g|^2 = 4$; $|f|^2 > |g|^2 + |f - g|^2$, and the triangle is obtuse-angled.

2.4.10. For a parallelogram, the conditions for the equality of the lengths of the sides and for the perpendicularity of the diagonals are equivalent.

2.4.14. (a) $t^2 + 3t + 3$; (b) 3; (c) $(3 + m_3^2 + \ldots + m_n^2)^{1/2}$.

2.4.18. (a) 1; (b) 1; (c) α.

2.4.24. $\frac{\pi}{4}$. 2.4.25. $\frac{\pi}{3}$.

2.5.8. The equality $|x - y|^2 = |x|^2 + |y|^2$ means that the scalar product of the vectors x and y is a purely imaginary number.

2.5.10. From the equality of the lengths of the vectors x and y, it does not follow that the vectors $x + y$ and $x - y$ are orthogonal.

2.5.15. The complex arithmetic space C_n.

2.5.19. The vector $(-\beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_n)$ corresponds to the vector iz. The natural scalar product (2.2.1) is induced on R_{2n}.

3.1.1. The term is positive. 3.1.2. It is not a term of the determinant. 3.1.3. The term has a minus sign. 3.1.4. It is not a term of the determinant.

3.1.5. (a) $a_{13}a_{24}a_{35}a_{46}a_{57}a_{61}a_{72}$; (b) $a_{13}a_{24}a_{35}a_{46}\, a_{57}\, a_{42}\, a_{71}$.
3.1.6. (a) $i = j$; (b) $i < j$; (c) $i > j$.
3.1.7. The plus sign.
3.1.8. $a_{11}a_{22} \ldots a_{nn}$.
3.1.9. (a) $i + j = n + 1$; (b) $i + j < n + 1$; (c) $i + j > n + 1$.
3.1.10. The sign is $(-1)^{n(n-1)/2}$.
3.1.11. $(-1)^{n(n-1)/2} \cdot a_{1n}a_{2,\, n-1} \ldots a_{n1}$.
3.1.12. $(-1)^{n-1}$. 3.1.13. $(-1)^{(n-2)(n-1)/2}$. 3.1.14. 1. 3.1.15. 1.
3.1.17. 0. 3.1.18. 0. 3.1.19. 16.
3.1.21. n.
3.1.22. If we put

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

then the number of nonzero terms in a determinant of order n of the indicated form equals

$$\frac{r_2-1}{r_2-r_1} r_1^n + \frac{1-r_1}{r_2-r_1} r_2^n.$$

3.1.23. 2^{n-1}.
3.1.24. $(-1)^n (t^n - a_1a_2 \ldots a_n)$.
3.1.25. $t^n + a_n t^{n-1} + a_{n-1}t^{n-2} + \ldots + a_2 t + a_1$.
3.1.26. n. 3.1.27. n.
3.1.29. The determinant

$$\begin{Vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{Vmatrix}$$

must be equal to zero.

3.1.30. The free term is then determinant

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix}.$$

3.1.31. The determinant obtained is a complex number, conjugate to the original determinant.

3.1.32. The determinant is multiplied by $(-1)^n$. 3.1.33. The determinant is multiplied by α^n. 3.1.34. The determinant is unaltered.

3.1.38. The determinant is multiplied by $(-1)^{n(n-1)/2}$. The element $a_{n+1-i\, j}$ of the original determinant is at (i, j) in the obtained determinant.

3.1.39. $a_{n+1-i,\, n+1-j}$.
3.1.40. The determinant is unaltered.
3.1.41. $a_{n+1-j\; n+1-i}$.
3.1.42. The determinant is unaltered.
3.1.43. The determinant is multiplied by $(-1)^{n(n-1)/2}$.
3.1.44. The roots of the equation are the numbers $-2, -1, 1, 2$.
3.1.45. The roots of the equation are the numbers 0 and -1.
3.1.46. x_1y_1 when $n = 1$, 0 when $n > 1$. 3.1.47. 1 when $n = 1$, -2 when $n = 2$, 0 when $n > 2$.

3.1.48. The polynomials $f_1(t), \ldots, f_n(t)$ are linearly dependent. Let, for definiteness, $f_n(t) = \alpha_1 f_1(t) + \ldots + \alpha_{n-1} f_{n-1}(t)$. Then, for any number a, $f_n(a) = \alpha_1 f_1(a) + \ldots + \alpha_{n-1} f_{n-1}(a)$. Therefore, the rows of the indicated determinant are linearly dependent.

3.1.49. (a) The determinant is unaltered; (b) the determinant becomes zero.

3.1.51. $1 + x_1y_1$ when $n = 1$, $(x_2 - x_1)(y_2 - y_1)$ when $n = 2$, 0 when $n > 2$.

3.1.52. $\cos(\alpha_1 - \beta_1)$ when $n = 1$, $\sin(\alpha_1 - \alpha_2)\sin(\beta_1 - \beta_2)$ when $n = 2$, 0 when $n > 2$.

3.1.53. $1 + x_1y_1 + x_2y_2 + \ldots + x_ny_n$.

3.1.54. $1 - 2(w_1^2 + w_2^2 + \ldots + w_n^2) = -1$.

3.1.57. The permanent of the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

equals two even though its rows are linearly dependent. However, the permanent of the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

with linearly independent rows equals 0.

3.2.1. (a) ^nC_k; (b) $(^nC_k)^2$. 3.2.3. ^nC_k.

3.2.4. Let p_i be the sum of all the principal minors of order i of the determinant

$$\begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{Vmatrix}.$$

Then $f(t) = t^n + p_1t^{n-1} + \ldots + p_{n-1}t + p_n$.

3.2.5. $k + 1$.

3.2.8. If D is of odd order, then D' is symmetric; if D is of even order, then D' is skew-symmetric.

3.2.9. Let $i > j$. Then there is a matrix comprising only zeroes and having $n - j$ columns in the first j rows of the minor M_{ij} complementary to a_{ij}. Since $j + (n - j) = n > n - 1$, by 3.1.20, $M_{ij} = 0$.

3.2.10. $D' = D^{n-1}$.

3.2.11. (a) The i-th row of D' is unaltered and all the others are multiplied by α. The whole determinant D' is multiplied by α^{n-1}; (b) the i-th and j-th rows are interchanged, and then all the rows are multiplied by (-1). The overall change of the determinant, thereby, has been that it was multiplied by $(-1)^{n+1}$; (c) all the rows of D', except the i-th, are unaltered; the corresponding elements of the j-th, premultiplied by α, have been subtracted from the elements of the i-th row. The determinant D' is unaltered; (d) D' is transposed.

3.2.16. 216. 3.2.17. −106. 3.2.18. 1. 3.2.19. 120. 3.2.20. −11. 3.2.21. −2. 3.2.22. −13. 3.2.23. 1. 3.2.24. 15. 3.2.25. 3. 3.2.26. 7.

3.2.28. −12. 3.2.29. 16. 3.2.30. 1. 3.2.31. −400. 3.2.32. −36. 3.2.33. 0. 3.2.34. 8. 3.2.35. −1.

3.2.37. $\frac{1}{3}(4^{n+1} - 1)$. 3.2.38. $4^{n+1} - 3^{n+1}$. 3.2.39. $2^{n+1} - 1$. 3.2.40. 5^n.

3.2.41. $\frac{i^n}{2}(1 + (-1)^n)$. 3.2.42. $\frac{1}{2}(1 + (-1)^n)$. 3.2.43. $1 + n$. 3.2.44. $6^n(1 + ln)$.

3.2.46. $f_{i+1}(\lambda) = (\lambda - a_{i+1}) f_i(\lambda) - b_{i+1}c_{i+1}f_{i-1}(\lambda)$.

3.3.2. The property indicated is possessed by an orientation volume of a parallelepiped in any Euclidean or unitary space.

3.3.5. (a) It follows from Hadamard's inequality; (b) let ε be the nth root of unity, i.e. $\varepsilon = \cos 2\pi/n + i \sin 2\pi/n$. Then the estimate indicated in (a) is reached by the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{vmatrix};$$

(c) for $n = 2$ the estimate occurs for the determinant

$$\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}.$$

If the estimate is reached by the determinant of the matrix A_n for $n = 2^k$, then for $m = 2^{k+1}$, the determinant of the following matrix of order $2n$ must be considered

$$A_{2n} = \begin{pmatrix} A_n & -A_n \\ A_n & A_n \end{pmatrix}.$$

3.3.6. If the modulus of the element a_{ij} is less than 1 for a certain pair of indices i, j in the given determinant d_n, then d_n is increased on replacing a_{ij} by 1 if $A_{ij} > 0$, and by replacing a_{ij} by -1 if $A_{ij} < 0$. Finally, if $A_{ij} = 0$, then the determinant is unaltered when a_{ij} is replaced by either of these numbers. Similar reasoning, with respect to the minimum of the determinant, shows that the determinants of a given order with the maximum modulus is a determinant made up of 1's and -1's.

3.3.7. To prove the inequality $h_{n-1} \leqslant h_n$, it suffices to enclose the determinant d_{n-1} of order $n-1$ made up of 0 and 1 and whose modulus equals h_{n-1},

$$d = \left|\begin{array}{c|c} d_{n-1} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} \\ \hline 0 \ \dots \ 0 & 1 \end{array}\right|,$$

so that a determinant of order n, also made up of 0 and 1 and whose modulus equals h_{n-1}, is obtained.

To prove the inequality $h_n \leqslant g_{n-1}$, consider the extremal determinant made up of 0's and 1's; interchange its rows so that the (1, 1) position contains a 1 and make all the other elements of the first column equal to zero by subtracting from the subsequent rows. Then a determinant of order $n-1$, whose elements are 0, 1, -1 and modulus equals to h_n, will be obtained in the bottom right-hand corner. By 3.3.6, the modulus of such a determinant does not exceed g_{n-1}.

To prove the inequality $g_{n-1} \leqslant g_n$, it suffices to enclose the extremal determinant $\tilde{d}_{n-1}$, made up of 1's and -1's, in the way indicated in the proof of the first inequality, and then to use 3.3.6.

For the proof of the last inequality, consider the extremal determinant $\tilde{d}_n$ with the elements 1 and -1. Multiplying, if necessary, the rows and columns of this determinant by -1, make all the elements of the first row equal to 1, and all the elements of the first column, beginning with the second element, equal to -1. Now, adding the first row to all the others, we shall obtain, in the bottom right-hand corner, a determinant of order $n-1$ made up of 0's and 2's and equal in modulus to g_n. Factoring out the 2's we shall see that this determinant is the product of 2^{n-1} and some determinant of order $n-1$ made up of 0's and 1's. Hence, the required inequality.

3.3.8. According to 3.3.7, $h_3 \leqslant g_3 = 2$. That $h_3 = 2$ is shown, for example, by the determinant

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}.$$

Since obviously $h_2 = 1$, by 3.3.7, $g_3 \leqslant 4$. Since $g_3 \geqslant g_2 = 2$ and g_3 is a multiple of 4, $g_3 = 4$.

3.3.9. Let $\tilde{d}_{n-1}$ be the extremal determinant of order $n - 1$ made up of 1's and -1's. Denote the columns of this determinant by $a_1, a_2, \ldots, a_{n-1}$ and construct the determinant of order n, which also contains only 1's and -1's:

$$d = \left| \begin{array}{cccc|c} a_1 & a_2 & \ldots & a_{n-1} & a_1 \\ \hline -1 & 1 & \ldots & 1 & 1 \end{array} \right|.$$

Only two of the minors of order $n - 1$ in the first $n - 1$ rows of the determinant are nonzero. Hence, expanding by the last row, we find that $d = 2d_{n-1}$.

3.3.10. According to 3.3.5, 3.3.7 and 3.3.9, g_5 is a multiple of 16 and $g_5 \geqslant 2g_4 = 32$. On the other hand, by Hadamard's inequality

$$g_5 \leqslant (\sqrt{5})^5 = 25\sqrt{5} < 64.$$

Therefore, g_5 is either equal to 32 or 48. The method of enclosing the extremal determinant of order 4, indicated in the solution to Problem 3.3.9, makes it possible to obtain, for a determinant of order 5, an estimate of 32. Hence, enclose the determinant in a different way:

$$\left| \begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \hline -1 & 1 & -1 & -1 & 1 \end{array} \right|.$$

This determinant equals 48. Thus, $g_5 = 48$.

3.3.11. Assume that $M = 1$ and enclose a determinant d of order n in the following way:

$$\tilde{d} = \left| \begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \hline 0 & & & \\ \vdots & & d & \\ 0 & & & \end{array} \right|.$$

The new determinant $\tilde{d}$ of order $n + 1$ equals $d/2$. Now subtract the first row of $\tilde{d}$ from all the others. Then the modulus of all elements of the determinant does not exceed 1/2 and, according to 3.3.5, (a),

$$d/2 \leqslant (1/2)^{n+1} (n + 1)^{(n+1)/2}. \text{ Q.E.D.}$$

3.3.13. (a) The determinant $G(x_1, \ldots, x_k)$ has a diagonal form and equals $|x_1|^2 \cdot |x_2|^2 \ldots |x_k|^2$; (b) the determinant $G(x_1, \ldots, x_k)$ has a "quasi-diagonal form" and equals $G(x_1, \ldots, x_l)\, G(x_{l+1}, \ldots, x_k)$.

3.3.14. (a) The determinant is unaltered; (b) the determinant is multiplied by $|\alpha|^2$; (c) the determinant is unaltered.

3.3.20. According to 3.3.18, $G^{1/2}(a_1, \ldots, a_n)$ is the volume of the parallelepiped drawn on the vectors $a_1, \ldots, a_n$; det A has the same meaning.

3.3.25. The Gram determinant does not exceed the product of two of its minors, complementary to one another, and is equal to this product if and only if at least one of the minors is zero or all the elements of the determinant, outside these minors, are equal to zero.

3.3.27. When $k = 3$, the inequality assumes the following form

$$V^2(x_1, x_2, x_3) \leqslant S(x_1, x_2)\,S(x_1, x_3)\,S(x_2, x_3).$$

Thus, the square of the volume of the parallelepiped drawn on the vectors x_1, x_2, x_3 does not exceed the product of the areas of its faces.

3.3.29. If all the elements of some row of an orthogonal determinant are replaced by ε_j, $j = 1, \ldots, n$ so that $\varepsilon^2 = \sum_{j=1}^{n} |\varepsilon_j|^2 < 1$, then the new determinant d' satisfies the inequalities

$$1 - |\varepsilon| \leqslant |d'| \leqslant 1 + |\varepsilon|.$$

3.3.20. The modulus of the minor at the intersection of the rows numbered $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$ is the volume of the parallelepiped obtained by projecting the indicated rows on the coordinate subspace of the vectors $e_{j_1}, \ldots, e_{j_k}$ where $e_1, \ldots, e_n$ is the natural basis for the arithmetic space.

3.3.32. If the base of the corresponding n-dimensional parallelepiped P_n is reckoned to be the $(n-1)$-dimensional parallelepiped P_{n-1} drawn on the first $n-1$ rows, then P_n has a very small height and a very large volume of the base P_{n-1}.

3.4.2. $$a_{pp}^{(p-1)} = \frac{A(1, \ldots, p)}{A(1, \ldots, p-1)}.$$

3.4.6. $$a_{pj}^{(p-1)} = \frac{A\begin{pmatrix}1 \ldots p-1 & p\\ 1 \ldots p-1 & j\end{pmatrix}}{A(1, \ldots, p-1)}, \quad p \leqslant j \leqslant n.$$

3.4.10. 4. 3.4.11. $-16i$. 3.4.12. -12. 3.4.13. 5. 3.4.14. 0. 3.4.15. 80. 3.4.16. 3. 3.4.17. $2^{-9}\cdot 3^{-3}\cdot 5^{-4}\cdot 7^{-2}$. 3.4.18. 240. 3.4.19. $-1/2$. 3.4.20. $-18\,016$. 3.4.21. 2. 3.4.22. -1. 3.4.23. 5. 3.4.24. 16. 3.4.25. 63. 3.4.26. 32. 3.4.27. 1. 3.4.28. 13.

3.4.29. This number is a polynomial in n with the higher-order term equal to $n^3/3$.

3.4.30. (a) The higher-order term of the number of operations equals $n^2/2$; (b) the higher-order term equals $3n$.

3.4.31. Evaluate the determinant d_{n+1} so that d_n remains the leading principal minor of d_{n+1}. If the matrix of the determinant is triangular, start the operations at the upper corner.

3.4.32. It follows from the condition of nondegeneracy that all the leading principal minors may be made nonzero by interchanging the rows only; interchanging the first $n-1$ columns is also possible. Then the Gauss method is performed for the last columns of all the k determinants.

3.4.33. For example, place the first row last and carry out the same operation over the first column.

3.4.36. For example, the determinant

$$\begin{vmatrix} 1 & 0 & 0 \ldots 1 \\ -1 & 1 & 0 \ldots 1 \\ -1 & -1 & 1 \ldots 1 \\ \cdot & \cdot & \cdots \\ -1 & -1 & -1 \ldots 1 \end{vmatrix}.$$

3.4.38. The statement does not hold for the Gauss method with partial pivoting; an example is

$$\begin{vmatrix} 1 & 1000 \\ 1 & 2000 \end{vmatrix}.$$

3.4.39. Assume that $\max_{i,j} | a_{ij} | = 1$. Let $\alpha = | a_{22}^{(1)} |$, $\beta = | a_{33}^{(2)} |$. Then $\beta \leqslant 2\alpha$, $\alpha\beta \leqslant 4$, whence $\beta \leqslant 2\sqrt{2} < 3$.

3.4.42. The determinant equals 1 and the length of each of its row equals 1. According to 3.3.4, the rows of the determinant are orthogonal each to each.

3.4.43. (a) $2^n (\det A)^2$; (b) 0; (c) $(\det A)^2$; (d) $(-1)^n (\det A)^2$.

4.1.4. All the matrix elements not in the basis minor are zeroes.

4.1.5. See the answer to Problem 4.1.4.

4.1.14. If $b_1, \ldots, b_m, c_1, \ldots, c_n$ is a convenient set of numbers, then for any number α, $\alpha \neq 0$, the set $\alpha b_1, \ldots, \alpha b_m, \frac{1}{\alpha} c_1, \ldots, \frac{1}{\alpha} c_n$ is also convenient.

4.1.17. No. A counter example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

4.1.18. The rank is either unaltered or changed by unity.

4.1.19. The rank is changed by not more than unity; not more than k.

4.1.21. 0, 1, 2.

4.1.28. 1. 4.1.29. 4. 4.1.30. 3. 4.1.31. 3. 4.1.32. 4. 4.1.33. 4.

4.1.34. The dimension equals three.

4.1.35. (a) Yes; (b) yes; (c) no.

4.2.7. Such a plane consists of only one vector.

4.2.8. Such a plane coincides with the whole space.

4.2.9. n.

4.2.12. If $x_0, x_1, \ldots, x_k$ is the given set of vectors, then x_0 may be chosen as the translation vector of the required plane and the span of the vectors $x_1 - x_0, \ldots, x_k - x_0$ as the direction subspace.

4.2.15. $L_1 + L_2$.

4.2.16. L if $\lambda \neq 0$, O if $\lambda = 0$.

4.2.17. Yes, if $L = O$; in this case M coincides with V. No, if $L \neq O$, since multiplication by a number, as it is defined in 4.2.16, may yield a result which lies outside M.

4.2.18. Retain the definition of multiplication by a number for nonzero numbers λ. Put $0 \cdot P = L$ for any plane $P = x_0 + L$. Then L is the null element of the space M.

4.2.19. $\dim M = n - k$.

4.2.20. The indicated plane contains the vector z, but does not contain the vector v.

4.2.22. (a) The straight line does not intersect the plane; (b) the straight line has only the vector $z_0 = (2, 1, -2, 2)$ in common with the plane; (c) the straight line lies in the plane.

4.2.23. The straight lines have one common vector $z_0 = (-5, 11, -16, -11, 7)$. The plane passing through this vector and having the span of the vectors q_1 and q_2 as its directional subspace, contains both the given straight lines.

4.2.24. Draw a plane through x_1 parallel to the span of the vectors $x_2 - x_1$, q_1, q_2.

4.2.25. The planes have only the vector $z_0 = (1, 2, 1, 0, 1)$ in common.

4.2.26. The planes do not meet, whereas their directional subspaces intersect only in a null vector.

4.2.27. The planes do not meet, whereas their directional subspaces intersect in a one-dimensional subspace drawn on the vector $2p_1 + p_2 = q_2 - q_1 = (5, 1, 0, 0, 5)$.

4.2.28. The planes intersect in the straight line $x = z_0 + q_2 t$, where $z_0 = (-2, -1, 6, 6, 7)$.

4.2.29. The planes do not meet. Meanwhile, their directional subspaces coincide.

4.2.30. The planes coincide.

4.2.34. Let $P = x_0 + L_k$ and $e_1, \ldots, e_k$ be the basis for L_k. Extend it to form a basis for the whole space: $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$. Then the following can be taken as the required hyperplanes

$$\pi_1 = x_0 + L(e_1, \ldots e_k, e_{k+2}, e_{k+3}, \ldots, e_n)$$

$$\pi_2 = x_0 + L(e_1, \ldots, e_k, e_{k+1}, e_{k+3}, \ldots, e_n)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\pi_{n-k} = x_0 + L(e_1, \ldots, e_k, e_{k+1}, e_{k+2}, \ldots, e_{n-1}).$$

4.3.1. If x_0 is an arbitrary vector fulfilling the condition $(n, x_0) = b$ (the vector $\alpha_0 n$, $\alpha_0 = b/(n, n)$, for example, can be chosen as x_0), then the given set is a hyperplane of vectors of the form $x_0 + y$ where y is any vector orthogonal to n. This hyperplane is a subspace if and only if $b = 0$.

4.3.5. $n(t) = 1 + ct + c^2t^2 + \ldots + c^n t^n$, $b = d$.

4.3.8. Let x_0 be an arbitrary vector of the intersection. Rewrite the equations of the hyperplanes in the form

$$(n_1, x - x_0) = 0,$$

$$(n_2, x - x_0) = 0,$$

$$\cdots\cdots$$

$$(n_k, x - x_0) = 0.$$

Hence, the intersection of the given hyperplanes is the plane $P = x_0 + L$, where L is the orthogonal complement of the span of the vectors $n_1, \ldots, n_k$.

4.3.10. The orthogonal complement of L is drawn on the vectors $z_1 = (-3, 1, -2, 0)$, $z_2 = (1, -1, -2, 1)$ (see 2.3.6). Therefore, P can be described, for example, by the system of equations:

$$(z_1, x) = (z_1, x_0)$$

$$(z_2, x) = (z_2, x_0),$$

i.e.

$$-3\alpha_1 + \alpha_2 - 2\alpha_3 = -4,$$

$$\alpha_1 - \alpha_2 - 2\alpha_3 + \alpha_4 = -1.$$

4.3.14. $z_0 = \alpha_0 n$, where $\alpha_0 = b/(n, n)$.

4.3.16. $f(t) \equiv 1$.

4.3.19. 5. 4.3.22. 2. 4.3.23. 2. 4.3.24. 150. 4.3.25. 5. 4.3.26. 5. 4.4.4. 0, 7.

4.4.5. When $\lambda \neq 1, 2$ the system has a unique solution; when $\lambda = 1$ it has a one-dimensional, and when $\lambda = 2$, a two-dimensional solution subspace.

4.4.6. When $\lambda \neq -1, -2$ the system has a unique solution; when $\lambda = -1$ it has a one-dimensional, and when $\lambda = -2$, a three-dimensional solution subspace.

4.4.8. The pivots are equal to the ratios of the corner minors.

4.4.10. The linear dependence of the vectors $z_1, \ldots, z_k$ is obviously due to the linear dependence of the vectors $y_1, \ldots, y_k$. Conversely, let now at least one of the coefficients in the equality $\alpha_1 z_1 + \ldots + \alpha_k z_k = 0$ be different from zero. Then two solutions of system (4.4.1), the zero solution and $\alpha_1 y_1 + \ldots + \alpha_k y_k$, have the same values for the last $n - r$ components. Therefore,

$\alpha_1 y_1 + \ldots + \alpha_k y_k = 0$, i.e. the vectors $y_1, \ldots, y_k$ are linearly dependent.

4.4.14. When the Gauss method is used and the subsequent formulae are derived for the general solution, only elementary transformations over the rows of the indicated submatrix are performed. The final result is the matrix C (the zero rows from the $(r+1)$th to the m-th are omitted).

4.4.16. Any vector of the 4-dimensional arithmetic space is a solution of the system.

4.4.17. For example, a general solution is: $x_1 = -\frac{7}{3}x_2 + \frac{5}{3}x_3$, $x_4 = 0$. The fundamental system of solutions is: $y_1 = (-7, 3, 0, 0)$, $y_2 = (5, 0, 3, 0)$.

4.4.18. The general solution is: $x_3 = 2x_1 + 5x_2 - 9x_4$. The fundamental system of solutions is: $y_1 = (1, 0, 2, 0)$, $y_2 = (0, 1, 5, 0)$, $y_3 = (0, 0, -9, 1)$.

4.4.19. The system has only a zero solution.

4.4.20. The general solution is: $x_1 = x_4$, $x_2 = x_4$, $x_3 = -x_4$. The fundamental system of solutions has only one vector, e.g. $y = (1, 1, -1, 1)$.

4.4.21. The general solution is: $x_1 = 2x_3 + 8x_4$, $x_2 = -x_3 - 2x_4$, $x_5 = 0$. The fundamental system of solutions is: $y_1 = (2, -1, 1, 0, 0)$, $y_2 = (8, -2, 0, 1, 0)$.

4.4.22. The general solution is: $x_1 = -\frac{1}{2}x_3 - 12x_4 - \frac{41}{2}x_5$, $x_2 = x_3 - 9x_4 - 18x_5$. The fundamental system of solutions is: $y_1 = (-1, 2, 2, 0, 0)$, $y_2 = (12, 9, 0, -1, 0)$, $y_3 = (41, 36, 0, 0, -2)$.

4.4.23. The general solution is: $x_1 = x_2 = \frac{1}{7}x_5$, $x_3 = x_4 = -\frac{3}{7}x_5$. The fundamental system of solutions consists of a unique vector, e.g. $y = (1, 1, -3, -3, 7)$.

4.4.24. The general solution is: $x_1 = -\frac{1}{3}x_3 - \frac{1}{3}x_5$, $x_2 = -\frac{2}{3}x_3 + \frac{1}{2}x_5$, $x_4 = 0$. The fundamental system of solutions is: $y_1 = (2, 9, -6, 0, 0)$, $y_2 = (-2, 3, 0, 0, 6)$.

4.4.25. The general solution is: $x_1 = -x_2 = x_3 = -x_4 + 3x_5$. The fundamental system of solutions is: $y_1 = (-1, 1, -1, 1, 0)$, $y_2 = (3, -3, 3, 0, 1)$.

4.4.26. The first three columns of the matrix are linearly dependent; the fourth column is not linearly dependent on the rest, therefore $x_4 = 0$; the same is valid for the fifth column, thus $x_5 = 0$.

4.4.27. x_4, x_5; x_1, x_4; x_3, x_4; x_2, x_5; x_1, x_2; x_2, x_3.

4.4.28. $n + 1 - k$.

4.4.29. The basis for the subspace is formed, for example, by the polynomials $f_1(t) = t^4 - 6t^3 + 11t^2 - 6t$ and $f_2(t) = t^5 - 25t^3 + 60t^2 - 36t$.

4.4.30. (a) For example,

$$70x_1 - 16x_2 + 4x_3 + x_4 = 0,$$
$$-5x_1 + x_2 - x_3 + x_5 = 0.$$

In order to answer (b), (c), any linear combination (two linear combinations) of the equations of (a) can be added to (a).

4.4.31. No. The given systems are not equivalent.

4.4.33. (44, −11, −31, −6).

4.5.6. When $\lambda \neq 0.6$ the system is defined; when $\lambda = 0$ it is inconsistent; when $\lambda = 6$ the system has a two-dimensional solution plane.

4.5.7. When $\lambda \neq -1, 2$ the system is defined; when $\lambda = 2$ it is inconsistent; when $\lambda = -1$ the system has a two-dimensional solution plane.

4.5.12. For example, the general solution could be:

$$x_1 = \frac{45}{19} + \frac{37}{19}x_2 - \frac{23}{19}x_3 - \frac{42}{19}x_4.$$

4.5.13. The system is inconsistent.

4.5.14. For example, the general solution could be: $x_1 = -1 + x_3 + 2x_4$, $x_2 = -3 + x_3 + 2x_4$.

4.5.15. The system has the unique solution: $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = 1$.

4.5.16. The general solution is: $x_1 = 6 - x_5$, $x_2 = -5 + x_5$, $x_3 = 3$, $x_4 = -1 - x_5$.

4.5.17. The general solution is: $x_1 = -\frac{3}{2} + \frac{3}{2}x_2 - \frac{39}{2}x_5$, $x_3 = \frac{1}{3} + x_5$, $x_4 = -\frac{2}{3} - 2x_5$.

4.5.18. The general solution is: $x_1 = \frac{7}{8} - \frac{3}{8}x_2 - \frac{11}{8}x_3 + \frac{5}{8}x_4$, $x_5 = 0$.

4.5.19. The system is inconsistent.

4.5.20. The system has the unique solution: $x_1 = x_2 = x_3 = x_4 = 1$, $x_5 = 2$.

4.5.21. The general solution is: $x_1 = \frac{5}{2} - \frac{3}{2}x_2$, $x_3 = x_4 = 0$, $x_5 = \frac{11}{5} - \frac{6}{5}x_6$.

4.5.22. The system is inconsistent.

4.5.23. When $\lambda \neq 5$ the system is inconsistent. When $\lambda = 5$ the system is consistent and its general solution could be, for example, $x_1 = -4 + x_3$, $x_2 = \frac{11}{2} - 2x_3$.

4.5.24. When $\lambda \neq -3$ the system has a unique solution

$$x_1 = -\frac{1}{\lambda+3}, \quad x_2 = \frac{4\lambda+11}{3(\lambda+3)}, \quad x_3 = -\frac{\lambda+11}{3(\lambda+3)}.$$

When $\lambda = -3$ the system is inconsistent.

4.5.25. The system is consistent for any value of λ. When $\lambda \neq -95$ the general solution is of the form $x_2 = 0$, $x_1 = \frac{13}{12} - \frac{23}{12}x_3$. When $\lambda = -95$ the general solution is: $x_1 = \frac{13}{12} + \frac{19}{12}x_2 - \frac{23}{12}x_3$.

4.5.26. When $\lambda \neq 1, -2$ the system has the unique solution

$$x_1 = x_2 = x_3 = \frac{1}{\lambda+2}.$$

When $\lambda = 1$ the general solution is: $x_1 = 1 - x_2 - x_3$. When $\lambda = -2$ the system is inconsistent.

4.5.27. When $\lambda \neq 1, -2$ the system has the unique solution

$$x_1 = x_2 = -\frac{1}{\lambda-1}, \quad x_3 = \frac{2}{\lambda-1}.$$

When $\lambda = 1$ the system is inconsistent. When $\lambda = -2$ the system is consistent and its general solution is: $x_1 = x_2 = -1 + x_3$.

4.5.28. When $\lambda \neq 1, -2$ the system has the unique solution

$$x_1 = x_2 = -\frac{3}{(\lambda-1)(\lambda+2)}, \quad x_3 = \frac{3(\lambda+1)}{(\lambda-1)(\lambda+2)}.$$

When $\lambda = 1$ and $\lambda = -2$ the system is inconsistent.

4.5.29. When $\lambda \neq 1, 3$ the system has the unique solution

$$x_1 = -1, \quad x_2 = \frac{\lambda-4}{\lambda-3}, \quad x_3 = -\frac{1}{\lambda-3}.$$

When $\lambda=1$ the general solution is: $x_1 = 1 - x_2 - x_3$. When $\lambda = 3$ the system is inconsistent.

4.5.30. When $\lambda \neq 1, 3$ the system has the unique solution

$$x_1=\frac{2}{3-\lambda}, \quad x_2=x_3=0, \quad x_4=\frac{3-7\lambda}{(\lambda-1)(3-\lambda)}.$$

When $\lambda = 1$ the system is inconsistent. When $\lambda = 3$ the general solution is:

$$x_1=-\frac{17}{9}-\frac{1}{3}x_3-\frac{2}{9}x_4, \quad x_2=2.$$

4.5.31. The third column of the matrix of the system is linearly independent of the remaining columns; the fifth column is linearly independent of the remaining columns of the augmented matrix of the system.

4.5.32. No. The formulae are not equivalent.

4.5.33. Yes. 4.5.34. $n + 1 - k$.

4.5.35. The indicated conditions determine a two-dimensional plane. If $f_0(t)$ is a polynomial in this plane, then the polynomials $f_0(t)$, $f_0(t) + f_1(t)$, $f_0(t) + f_2(t)$, where $f_1(t)$ and $f_2(t)$ form a basis for the directional subspace, are linearly independent. For $f_1(t)$ and $f_2(t)$, the polynomials of Problem 4.4.29 can be taken.

4.5.36. $x_1 = 2,\ x_2 = 1,\ x_3 = 1,\ x_4 = 3$.

4.5.40. $$x_1=\frac{ap-bq-cr-ds}{A}, \quad x_2=\frac{bp+aq-dr+cs}{A},$$

$$x_3=\frac{cp+dq+ar-bs}{A}, \quad x_4=\frac{dp-cq+br+as}{A}.$$

4.5.42. $f(t) = t^3 - 4t^2 + 3t - 2$.

4.5.43. $f(t) = -3t^3 + 7t$.

4.5.47. $f(t) = t^4 - 4t^3 + 3t - 1$.

4.5.49. $f(t) = 2t^4 - t^3 - 3t^2 - 2t + 1$.

4.5.51. $f(t) = 2t^5 - 4t^4 - 3t^2 + 5t - 2$.

4.5.53. For the function f and its derivatives, write the system of equations:

$$\begin{aligned}
&hf &&= g\\
&h'f+hf' &&= g'\\
&h^{(2)}f+2h'f'+hf^{(2)} &&= g^{(2)}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&h^{(n)}f + nh^{(n-1)}f' + C_n^2h^{(n-2)}f^{(2)} + \dots + hf^{(n)} &&= g^{(n)}.
\end{aligned}$$

Using Cramer's formula for $f^{(n)}$, we obtain the required relation.

4.5.54. $f^{(5)}(1) = -2$.

5.1.1. Yes, if $a = 0$. No, if $a \neq 0$. 5.1.2. See the answer to 5.1.1. 5.1.3. Yes. 5.1.4. Yes. 5.1.5. Yes. 5.1.6. No. 5.1.7. Yes. 5.1.8. Yes.

5.1.9. Yes, if $\alpha = 0$. No, if $\alpha \neq 0$. 5.1.10. Yes. 5.1.11. No. 5.1.12. No. 5.1.13. Yes. 5.1.14. No.

5.1.15. No. 5.1.16. Yes. 5.1.17. No. 5.1.18. Yes.

5.1.19. Yes. 5.1.20. Yes. 5.1.21. Yes. 5.1.22. Yes. 5.1.23. Yes. 5.1.24. Yes. 5.1.25. No. 5.1.26. No. 5.1.27. No.

5.1.34. Any operator on the space R^+ raises all the numbers of this space to a power with a fixed (for a given operator) real exponent.

5.1.36. No. 5.1.37. Yes.

5.1.40. (a) Yes; (b) no.

5.1.44. No, if the set $x_1, \ldots, x_k$ is linearly dependent.

5.1.45. Yes.

5.1.47. For linear functional φ on the space M_n to be given by the formula $\varphi f(t) = f(a_0)$, it is necessary and sufficient that the numbers

$$c_i = \varphi(t^i), \quad i = 0, 1, \ldots, n,$$

satisfy the relations

$$c_0 = 1, \quad \frac{c_{i+1}}{c_i} = \text{const}, \quad i = 0, 1, \ldots, n-1.$$

5.1.52. No. Any operator on the space C, obtained in the indicated way, converts "real" vectors (i.e. vectors of the form $x + i0$) back into real vectors.

5.1.53. No, if this functional is not zero.

5.1.55. Yes, if $\dim Y \geqslant \dim X$; no, if $\dim Y < \dim X$.

5.1.56. No, if $\dim Y > \dim X$; yes, if $\dim Y \leqslant \dim X$.

5.1.59. N_A; 0.

5.1.66 n, if $f = 0$; $n - 1$, if $f \neq 0$.

5.1.67. A two-dimensional space of vectors orthogonal to a; a two-dimensional vector subspace coplanar with a and b.

5.1.68. N_A is a straight line drawn on the vector a; T_A is a plane perpendicular to the vector a.

5.1.69. If $(a, b) = 0$, then N_A is a plane perpendicular to the vector a and T_A is a straight line drawn on the vector b. If, however, $(a, b) \neq 0$, then N_A is a straight line drawn on the vector b and T_A is a plane perpendicular to the vector a.

5.1.70. $r_A = 1$, the basis for the image is $y = (1, 1, 1)$; $n_A = 2$, the basis for the kernel is: $Z_1 = (1, -1, 0)$, $Z_2 r = (1, 0, -1)$.

5.1.71. $r_A = 2$, the basis for the image is: $y_1 = (2, 1, 1)$, $y_2 = (-1, -2, 1)$; $n_A = 1$, the basis for the kernel is: $Z = (1, 1, 1)$.

5.1.72. $r_A = 3$; $n_A = 0$.

5.1.73. The image is: M_{n-1}; the kernel: M_0.

5.1.74. See the answer to 5.1.73.

5.1.75. $n + 1 - k$, if $k < n + 1$; 0, if $k \geqslant n + 1$.

5.1.76. $N_P = L_2$; $T_P = L_1$.

5.2.7. Let $e_1, \ldots, e_n$ be a basis for the space X, and let for the given operator A from ω_{XY}

$$Ae_1 = a_{11}q_1 + a_{21}q_2 + \ldots + a_{m1}q_m,$$
$$Ae_2 = a_{12}q_1 + a_{22}q_2 + \ldots + a_{m2}q_m,$$
$$\cdots\cdots\cdots\cdots$$
$$Ae_n = a_{1n}q_1 + a_{2n}q_2 + \ldots + a_{mn}q_m.$$

Put

$$B_1e_1 = a_{11}q_1, \quad B_2e_1 = a_{21}q_2, \ldots, B_me_1 = a_{m1}q_m,$$
$$B_1e_2 = a_{12}q_1, \quad B_2e_2 = a_{22}q_2, \ldots, B_me_2 = a_{m2}q_m,$$
$$\cdots\cdots\cdots\cdots$$
$$B_1e_n = a_{1n}q_1, \quad B_2e_n = a_{2n}q_2, \ldots, B_me_n = a_{mn}q_m.$$

It is obvious that the operators B_i, $i = 1, \ldots, m$ satisfy the conditions of the problem.

5.2.11. $\dim \omega_{XY} = mn$.

5.2.12 (a) No, if $T \neq O$; (b) no, if $N \neq X$.

5.2.13. $\dim \omega_{XT} = kn$.

5.2.14. $\dim K_N = m(n - l)$.

5.2.20. Let $e_1, \ldots, e_d$ $(d = n - r)$ be a basis for N_A; $e_1, \ldots, e_d, e_{d+1}, \ldots, e_n$ a basis for X. Then the vectors $y_1 = Ae_{d+1}, \ldots, y_r = Ae_n$ make up a basis for T_A. The required representation of the operator A is given by the operators $B_1, \ldots, B_r$ determined by

$$B_i e_k = \begin{cases} 0, & k \neq d ; \iota \\ y_i, & k = d + \iota. \end{cases}$$

5.2.21. Either $N_A = N_B$ or $T_A = T_B$.

5.2.22. Let all operators of L be of rank $\leqslant 1$, and let A be an arbitrary operator of rank 1 from L. Consider a subset L_1, in L, of all operators B for which $N_B \supset N_A$, and a subset L_2 of all operators C for which $T_C \subset T_A$. According to 5.2.13, 5.2.14, these subsets are the subspaces of L of dimension $\leqslant n$. Therefore $L_1 \neq L$, $L_2 \neq L$, and there exists an operator D from L such that $D \notin L_1$, $D \notin L_2$, i.e. $T_D \neq T_A$, $N_D \neq N_A$. But then (see 5.2.21) $A + D$ is of rank 2.

5.2.23. No (see 5.2.8).

5.2.24. Yes.

5.2.26. $N_{E-P} = T_P$, $T_{E-P} = N_P$.

5.3.4. No.

5.3.6. $n(n-r)$. 5.3.7. $n(n-r)$. 5.3.8. The rank equals nr, the defect $n(n-r)$.

5.3.10. Let $x \in N_{q+k}$. Then

$$A^{q+k}x = 0 = A^{q+1}(A^{k-1}x).$$

Since $N_q = N_{q+1}$,

$$A^q(A^{k-1}x) = 0 = A^{q+k-1}x,$$

i.e. $x \in N_{q+k-1}$. Therefore, $N_{q+k} = N_{q+k-1}$. Continuing to reason in the same way, we obtain

$$N_{q+k} = N_{q+k-1} = N_{q+k-2} = \ldots = N_{q+1} = N_q.$$

5.3.12 $n + 1$.

5.3.19. If D is a differential operator, then

$$A = E + \frac{1}{1!}D + \frac{1}{2!}D^2 + \ldots + \frac{1}{n!}D^n.$$

5.3.21. Let $\varphi(A) = 0$ and $\varphi(t) = q(t)\,m(t) + r(t)$, where the degree of $r(t)$ is less than that of $m(t)$, or $r(t) \equiv 0$. If $r(t)$ is a nonzero polynomial, then $r(A) = \varphi(A) - q(A)\,m(A) = 0$, which is contrary to the definition of the polynomial $m(t)$.

5.3.22 Let $m_1(t)$ and $m_2(t)$ be two annihilating polynomials of the least degree. Moreover, we may assume that the higher-order coefficients of both the polynomials equal 1. If $p(t) = m_1(t) - m_2(t)$ is a nonzero polynomial, then it is also an A-annihilator.

5.3.23. (a) $m(t) = t^2 - t$ if $P \neq 0, E$; $m(t) = t$ if $P = 0$; $m(t) = t - 1$, when $P = E$; (b) $m(t) = t^2 - 1$; (c) $m(t) = t^q$.

5.3.25. No.

5.3.31. That P_1P_2 is a projection operator follows (see 5.3.17) from the equality

$$(P_1P_2)^2 = P_1P_2P_1P_2 = P_1^2P_2^2 = P_1P_2.$$

It also follows from the commutativity of P_1 and P_2 that $T_{P_1P_2} \subset T_{P_1} \cap T_{P_2}$. If, conversely, $x \in T_{P_1} \cap T_{P_2}$, then $P_1x = P_2x = x$ and $P_1P_2x = x$, i.e. $x \in T_{P_1P_2}$.

Using the commutativity of P_1 and P_2 again, we obtain that $N_{P_1} \subset N_{P_1P_2}$ and $N_{P_2} \subset N_{P_1P_2}$, i.e. $N_{P_1} + N_{P_2} \subset N_{P_1P_2}$. Now, if $x \in N_{P_1P_2}$, then $P_2x \in N_{P_1}$, and $(E - P_2)x \in N_{P_2}$. The identity $x = P_2x + (E - P_2)x$ proves the reverse inclusion relation: $N_{P_1P_2} \subset N_{P_1} + N_{P_2}$

5.3.32. It is easy to verify that it follows from $P_1P_2 = P_2P_1 = 0$ that $(P_1 + P_2)^2 = P_1 + P_2$, i.e. $P_1 + P_2$ is a projection operator. Conversely, let $(P_1 + P_2)^2 = P_1 + P_2$, whence

$$P_2P_1 + P_1P_2 = 0.$$

Premultiplying and postmultiplying this equality by P_1, we obtain

$$P_1P_2P_1 + P_1P_2 = 0, \quad P_2P_1 + P_1P_2P_1 = 0,$$

and hence

$$P_2P_1 - P_1P_2 = 0.$$

Therefore,

$$P_1P_2 = P_2P_1 = 0.$$

The inclusion $T_{P_1+P_2} \subset T_{P_1} + T_{P_2}$ is obvious. It follows from the equality $P_1P_2 = 0$ that $T_{P_2} \subset N_{P_1}$. Since the sum $T_{P_1} + N_{P_1}$ is direct, the same il true for the sum $T_{P_1} + T_{P_2}$. Now, if $x \in T_{P_1} + T_{P_2}$, i.e. $x = x_1 + x_2$ (where $x_1 \in T_{P_1}$, $x_2 \in T_{P_2}$), then

$$(P_1 + P_2)\,x = (P_1 + P_2)\,x_1 + (P_1 + P_2)\,x_2$$
$$= (P_1 + P_2)\,P_1x_1 + (P_1 + P_2)\,P_2x_2 = P_1^2x_1 + P_2^2x_2 = x_1 + x_2 = x,$$

s.e. $T_{P_1} \dot{+} T_{P_2} \subset T_{P_1+P_2}$.

Since $T_{P_1} \cap T_{P_2} = 0$, it follows from $x \in N_{P_1+P_2}$ (i.e. from $P_1x = -P_2x$), that $x \in N_{P_1} \cap N_{P_2}$.

5.3.38. The operator that matches each function with its (unique) antiderivative belonging to the given space.

5.3.39. $R^{-1} = R$.

5.3.41. The kernel of each of the operators $E + A$ and $E - A$, in case they are degenerate, should coincide with the image of the operator A. But for a nonzero vector x, the equalities $-Ax = x$ and $Ax = x$ cannot be both true.

5.3.51. Yes, if $\dim Y = \dim X$; no, if $\dim Y > \dim X$. The case of $\dim Y < \dim X$ is impossible.

5.4.1. $AB = -1$. $\quad BA = \begin{Vmatrix} 8 & -12 & 0 \\ 6 & -9 & 0 \\ 2 & -3 & 0 \end{Vmatrix}$

5.4.2. $AB = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}$. $\quad BA = \begin{Vmatrix} -4 & 6 & 0 & 2 \\ -3 & 2 & -2 & 2 \\ 4 & -1 & 4 & -3 \\ 1 & 6 & 6 & -2 \end{Vmatrix}$.

5.4.3. $AB = \begin{Vmatrix} -52 \\ 78 \\ 69 \end{Vmatrix}$.

5.4.4. $AB = (-15 \quad 97 \quad 78 \quad -112)$

5.4.5. $AB = \begin{Vmatrix} 1 \\ -3 \\ 3 \\ 1 \end{Vmatrix}$.

5.4.6. $AB = (-1 \;\; 0 \;\; -1 \;\; 4)$.

5.4.7. $AB = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$.

5.4.8. $ABC = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{Vmatrix}$. 5.4.9. $ABCD = \begin{Vmatrix} 3 & -6 & 3 \\ 5 & -10 & 5 \\ 7 & -14 & 7 \end{Vmatrix}$.

5.4.12. $X = \begin{Vmatrix} 1 & 2 \\ 1 & -1 \end{Vmatrix}$. 5.4.13. $X = \begin{Vmatrix} 5 & 8 \\ 3 & 5 \end{Vmatrix}$.

5.4.14. $X = \frac{1}{5} \begin{Vmatrix} 7 & -1 \\ -3 & -1 \end{Vmatrix}$. 5.4.15. $X = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$.

5.4.16. $X = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{Vmatrix}$. 5.4.17. $X = \begin{Vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{Vmatrix}$.

5.4.19. There are mpn multiplications and $mp(n-1)$ additions.

5.4.20. $mp(n+q)$ multiplications are required for the product $(AB)C$, and $nq(m+p)$ multiplications for $A(BC)$.

5.4.25. (a) The i-th row is multiplied by α_{i+1} and added to the $(i+1)$th row of A then, premultiplied by $\alpha_{i+2,\,i}$ instead of $\alpha_{i+1,\,i}$, the i-th row is added to the $(i+2)$th row. This sequence is repeated until the i-th row, premultiplied by α_{ni}, is added to the n-th row; (b) as in (a) the i-th row is premultiplied this time by $\alpha_{1i}, \ldots, \alpha_{i-1,\,i}, \alpha_{i+1,\,i}, \ldots, \alpha_{ni}$ and added to the i-th, $(i+1)$th, $\ldots$, n-th rows, respectively.

In the postmultiplication case: (a) each of the subsequent rows premultiplied by α_{ki}, $k = i+1, \ldots, n$, respectively, and in turn is added to the i-th column of A; (b) each of the subsequent columns premultiplied by α_{ki}, $k \neq i$, respectively, and in turn is added to the i-th column of A.

5.4.29. $\begin{Vmatrix} 1 & k \\ 0 & 1 \end{Vmatrix}$.

5.4.30. $\begin{Vmatrix} c_k & c_{k-1} \\ c_{k-1} & c_{k-2} \end{Vmatrix}$,

where c_i are the Fibonacci numbers, $c_{-1} = 0$, $c_0 = 1$, $c_k = c_{k-1} + c_{k-2}$.

5.4.31. $\begin{Vmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & & \ddots \\ 0 & & \lambda_n^k \end{Vmatrix}$.

5.4.32. $\begin{Vmatrix} \lambda_1^m \lambda_n^m & & & 0 \\ & \lambda_2^m \lambda_{n-1}^m & & \\ & & \ddots & \\ & & \lambda_{n-1}^m \lambda_2^m & \\ 0 & & & \lambda_n^m \lambda_1^m \end{Vmatrix}$

when $k = 2m$ and

$$\left\| \begin{matrix} 0 & & & \lambda_1^{m+1}\lambda_n^m \\ & & & \lambda_2^{m+1}\lambda_{n-1}^m \\ & & \cdot^{\cdot^{\cdot}} & \\ & \lambda_{n-1}^{m+1}\lambda_2^m & & \\ \lambda_n^{m+1}\lambda_1^m & & & 0 \end{matrix} \right\|$$

when $k = 2m + 1$.

5.4.33. If the given matrix is denoted by A, then for $B = A^k$ and $k < n$, we obtain $b_{i,\ i+k} = 1$, $i = 1, \ldots, n - k$, the other elements b_{ij} being equal to zero. When $k \geqslant n$, $B = 0$.

5.4.34. Let the given matrix be A. Then for $B = A^k$ and $k < n$, we obtain $b_{i,\ i+k} = 1$, $i = 1, \ldots, n - k$; $b_{i,\ i+k-n} = 1$, $i = n - k + 1, \ldots, n$; the other elements b_{ij} equal zero. For $k = n$, we obtain $A^n = E$. If, however, $k > n$, then representing k as $k = np + m$, we obtain $A^k = A^m$.

5.4.42. $\left\| \begin{matrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{matrix} \right\|$.

5.4.43. $\left\| \begin{matrix} a & b & c & \ldots & h \\ & a & b & \ldots & \cdot \\ & & \ldots & & \\ & & a & \ldots & \cdot \\ & & & \ldots & c \\ & & & \cdot & b \\ 0 & \cdot & & & a \end{matrix} \right\|$

5.4.44. n, the order of the Jordan block.

5.4.46. $\dfrac{n(n+1)(n+2)}{6}$.

5.4.54. n^2. It is required to evaluate only n elements fully determining the circulant.

5.4.55. $2(m_1 + m_2) + 1$.

5.4.60. $\alpha = x_1y_1 + x_2y_2 + \ldots + x_ny_n$.

5.4.61. $n(n + 2)$. Represent the matrix AB as $AB = (\beta x)\, v$ where $\beta = y_1u_1 + \ldots + y_nu_n$. To evaluate β, n multiplications are required; to evaluate the column vector βx, again n multiplications, but n^2 operations for $(\beta x)\, v$.

5.4.62. Construct the matrix B from a set of the basis columns of A, the columns of B being the decomposition coefficients of the corresponding columns of A in terms of the set.

5.4.63. For example, $B = \left\| \begin{matrix} 2 & 2 \\ 2 & 0 \\ 0 & 2 \end{matrix} \right\|$, $C = \left\| \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \right\|$.

5.4.64. For example, $B = \left\| \begin{matrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{matrix} \right\|$, $C = \left\| \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{matrix} \right\|$.

5.4.67. An arbitrary quasi-diagonal matrix whose diagonal blocks are of orders $k_1, \ldots, k_r$, respectively.

5.4.75. Let $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} \alpha_{ij} f_j \times e_i = 0$. Then

$$\sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \alpha_{ij} f_i \Big) \times e_i = 0.$$

Due to the linear independence of the set $e_1, \ldots, e_m$, we obtain

$$\sum_{j=1}^{n} \alpha_{ij} f_j = 0, \quad i = 1, \ldots, m,$$

whence for all i, j, $\alpha_{ij} = 0$.

5.5.1. $-\frac{1}{2}\begin{Vmatrix} 6 & 4 \\ 8 & 5 \end{Vmatrix}$. 5.5.2. $\begin{Vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{Vmatrix}$.

5.5.3. $\frac{1}{a^2+b^2}\begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}$. 5.5.4. $\frac{1}{ad-bc}\begin{Vmatrix} d & -b \\ -c & a \end{Vmatrix}$.

5.5.5. $-\frac{1}{7}\begin{Vmatrix} 2 & -1 & 0 \\ -1 & -3 & 7 \\ 0 & 7 & -21 \end{Vmatrix}$. 5.5.6. $\frac{1}{9}\begin{Vmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{Vmatrix}$.

5.5.7. $\begin{Vmatrix} 32 & 14 & -1 \\ 2 & 1 & 0 \\ 25 & 11 & -1 \end{Vmatrix}$. 5.5.8. $-\frac{1}{2}\begin{Vmatrix} 1 & -1 & -1 \\ -5 & 3 & 1 \\ -1 & 1 & -1 \end{Vmatrix}$.

5.5.9. $\begin{Vmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{Vmatrix}$. 5.5.10. $-\frac{1}{6}\begin{Vmatrix} -2 & 3 & -1 & 0 \\ 2 & 0 & -2 & 0 \\ -6 & -3 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{Vmatrix}$.

5.5.11. $\begin{Vmatrix} 5 & -2 & -5 & 4 \\ -7 & 3 & 16 & -13 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & -11 & 9 \end{Vmatrix}$.

5.5.12. $\frac{1}{a^2+b^2+c^2+d^2}\begin{Vmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{Vmatrix}$.

5.5.16. (a) Yes. E.g. the set of matrices of the form

$$\begin{Vmatrix} a & 0 \\ 0 & 0 \end{Vmatrix},$$

where $a \neq 0$; (b) no; the equation $Ax = B$, where A is a degenerate matrix and B is a nondegenerate matrix, is inconsistent.

5.5.22. $\begin{Vmatrix} \frac{1}{\lambda_1} & & & 0 \\ & \frac{1}{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\lambda_n} \end{Vmatrix}$. 5.5.23. $\begin{Vmatrix} 0 & & & \frac{1}{\lambda_n} \\ & & \frac{1}{\lambda_{n-1}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ \frac{1}{\lambda_1} & & & 0 \end{Vmatrix}$.

5.5.24. $\begin{Vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{Vmatrix}$.

5.5.25. $\begin{Vmatrix} 1 & 2 & 4 & 8 & \dots & 2^{n-1} \\ 0 & 1 & 2 & 4 & \dots & 2^{n-2} \\ 0 & 0 & 1 & 2 & \dots & 2^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 \end{Vmatrix}$.

5.5.26. $$\begin{Vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a & 1 & 0 & 0 & \dots & 0 & 0 \\ a^2 & -a & 1 & 0 & \dots & 0 & 0 \\ -a^3 & a^2 & -a & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (-a)^{n-1} & (-a)^{n-2} & (-a)^{n-3} & (-a)^{n-4} & \dots & -a & 1 \end{Vmatrix}$$

5.5.27. $$\begin{Vmatrix} \frac{1}{a} & \frac{-1}{a^2} & \frac{1}{a^3} & \frac{-1}{a^4} & \dots & (-1)^{n-1}\frac{1}{a^n} \\ 0 & \frac{1}{a} & \frac{-1}{a^2} & \frac{1}{a^3} & \dots & (-1)^{n-2}\frac{1}{a^{n-1}} \\ 0 & 0 & \frac{1}{a} & \frac{-1}{a^2} & \dots & (-1)^{n-3}\frac{1}{a^{n-2}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \frac{1}{a} \end{Vmatrix}.$$

5.5.28. $$\begin{Vmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 \end{Vmatrix}.$$

5.5.29. $P_{ij}^{-1} = P_{ij}^T = P_{ij}$;

$$D_i^{-1} = \begin{Vmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \frac{1}{\alpha} & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{Vmatrix}; \quad L_{ij}^{-1} = \begin{Vmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & \dots & -\alpha \\ & & & \ddots & \vdots \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{Vmatrix}.$$

5.5.30. In the inverse matrix: (a) the i-th and j-th columns are interchanged; (b) the i-th column is multiplied by the number $1/\alpha$; (c) the i-th columns multiplied by the number α is subtracted from the j-th.

$$N_i^{-1}=\begin{Vmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -\alpha_{i+1,\,i} & \ddots & \\ & & \vdots & & \\ & & -\alpha_{ni} & & 1 \end{Vmatrix}, \qquad S_i^{-1}=\begin{Vmatrix} 1 & & -\alpha_{1i} & & \\ & \ddots & \vdots & & \\ & 1 & -\alpha_{i-1,\,i} & & \\ & & 1 & & \\ & & -\alpha_{i+1,\,i} & 1 & \\ & & \vdots & & \ddots \\ & & -\alpha_{ni} & & 1 \end{Vmatrix}.$$

5.5.33. $n!$

5.5.35. $\begin{Vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 1 & -2 \\ -1 & 1 & 0 & 1 \end{Vmatrix}.$ 5.5.36. $\begin{Vmatrix} 15 & 10 & -6 & -4 \\ 10 & 5 & -4 & -2 \\ -9 & -6 & 3 & 2 \\ -6 & -3 & 2 & 1 \end{Vmatrix}.$

5.5.37. Reduction of the matrix A, to triangular form mentioned in the problem, only needs elementary transformations of type (c). Each stage of the Gauss elimination can then be interpreted as left multiplication of the current matrix by the sequence of matrices L_{ki} or, equivalently, by the corresponding matrix N_i. Finally, we obtain

$$N_{n-1} \dots N_i \dots N_1 A = R,$$

where R is an upper triangular matrix. Hence,

$$A=(N_1^{-1} \dots N_i^{-1} \dots N_{n-1}^{-1})\, R.$$

All the matrices N_i^{-1} are lower triangular with units on the principal diagonal which is also true for their product.

5.5.40. Reduce A by the Gauss elimination to an upper triangular matrix with units on the principal diagonal. Then, subtract appropriate multiples of the last row from the previous rows to make all the off-diagonal elements of the last column equal zero. The same is done for the next row and so on to the last row.

5.5.43. Let M_i be the matrices of elementary transformations involved in the reduction of A to the identity matrix, then

$$M_k \dots M_1 A = E,$$

i.e.

$$A^{-1} = M_k \dots M_1.$$

5.5.44. $\begin{Vmatrix} -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 2 & 1 \end{Vmatrix}.$ 5.5.45. $\begin{Vmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -\frac{4}{5} \end{Vmatrix}.$

5.5.50. Perform the calculations in the following order: 1. $A^{-1}x$. 2. yA^{-1}. 3. $\alpha = y\,(A^{-1}x)$. 4. $\beta = \dfrac{1}{1+\alpha}$. 5. $\beta\,(A^{-1}x)$. 6. $\beta A^{-1}BA^{-1} = (\beta A^{-1}x)\,(yA^{-1})$. 7. $(A+B)^{-1}$. Then $3n^2+2n+1$ operations of multiplication and division are required.

5.5.51. $\tilde{A}^{-1} = A^{-1} - \dfrac{\gamma}{1+\gamma c_{ji}}\, r_i s_j.$

Here c_{ji} is the (j, i) element of the matrix $C = A^{-1}$, r_i is the i-th column, and s_j is the j-th row of A^{-1}.

5.5.52. Let $v = (\gamma_1, \ldots, \gamma_n)$, $s = vA^{-1}$, r_n the last column of A^{-1}. Then

$$\widetilde{A}^{-1} = A^{-1} - \frac{a}{1 + vr_n} r_n s.$$

5.5.53. Let e be a column vector (of the same order as A) all of whose elements are equal to unity. Put $t = A^{-1} e$, $u = e' A^{-1}$. Then

$$\widetilde{A}^{-1} = A^{-1} - \frac{a}{1 + aS} tu,$$

where S is the sum of all elements of A^{-1}.

5.5.54.

$$\frac{1}{(a-b)(a+b(n-1))} \begin{Vmatrix} \alpha & -b & -b & \ldots & -b \\ -b & \alpha & -b & \ldots & -b \\ -b & -b & \alpha & \ldots & -b \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -b & -b & -b & \ldots & \alpha \end{Vmatrix},$$

$\alpha = a + b(n-2)$.

5.5.55.

$$\frac{1}{n-1} \begin{Vmatrix} 2-n & 1 & 1 & \ldots & 1 \\ 1 & 2-n & 1 & \ldots & 1 \\ 1 & 1 & 2-n & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 2-n \end{Vmatrix}.$$

5.5.56. $\begin{Vmatrix} 2-n & 1 & 1 & \ldots & 1 \\ 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \ldots & -1 \end{Vmatrix}.$

5.5.57.

$$-\frac{1}{t} \begin{Vmatrix} \frac{1-a_1 t}{a_1^2} & \frac{1}{a_1 a_2} & \frac{1}{a_1 a_3} & \ldots & \frac{1}{a_1 a_n} \\ \frac{1}{a_1 a_2} & \frac{1-a_2 t}{a_2^2} & \frac{1}{a_2 a_3} & \ldots & \frac{1}{a_2 a_n} \\ \frac{1}{a_1 a_3} & \frac{1}{a_2 a_3} & \frac{1-a_3 t}{a_3^2} & \ldots & \frac{1}{a_3 a_n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{a_1 a_n} & \frac{1}{a_2 a_n} & \frac{1}{a_3 a_n} & \ldots & \frac{1-a_n t}{a_n^2} \end{Vmatrix},$$

where $t = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$.

5.5.60. $\begin{Vmatrix} E_k & -B \\ 0 & E_l \end{Vmatrix}.$

5.5.62. Look for the partitioned matrix A_n^{-1}

$$A_n^{-1}=\left\|\begin{matrix} P_{n-1} & r_{n-1} \\ q_{n-1} & b \end{matrix}\right\|,$$

where P_{n-1} is a square matrix of order $n-1$. From the condition $A_nA_n^{-1}=E_n$ we derive

$$A_{n-1}P_{n-1}+u_{n-1}q_{n-1}=E_{n-1} \qquad (\alpha)$$

$$A_{n-1}r_{n-1}+bu_{n-1}=0, \qquad (\beta)$$

$$v_{n-1}P_{n-1}+aq_{n-1}=0, \qquad (\gamma)$$

$$v_{n-1}r_{n-1}+ab=1. \qquad (\delta)$$

It follows from (β) that

$$r_{n-1}=-bA_{n-1}^{-1}u_{n-1}. \qquad (\varepsilon)$$

Substituting in (δ), we find b:

$$b=\frac{1}{a-v_{n-1}A_{n-1}^{-1}u_{n-1}}.$$

Now, r_{n-1} is determined from (ε).
Substitute the expression for P_{n-1}, derived from (α),

$$P_{n-1}=A_{n-1}^{-1}-A_{n-1}^{-1}u_{n-1}q_{n-1}$$

in (γ):

$$v_{n-1}A_{n-1}^{-1}-v_{n-1}A_{n-1}^{-1}u_{n-1}q_{n-1}+aq_{n-1}=0.$$

Hence

$$q_{n-1}=\frac{v_{n-1}A_{n-1}^{-1}}{v_{n-1}A_{n-1}^{-1}u_{n-1}-a}=-bv_{n-1}A_{n-1}^{-1}.$$

Finally, find P_{n-1}

$$P_{n-1}=A_{n-1}^{-1}\cdot bA_{n-1}^{-1}u_{n-1}v_{n-1}A_{n-1}^{-1}.$$

5.5.63. Evaluate in the following sequence:

1. $A_{n-1}^{-1}u_{n-1}$. 2. $v_{n-1}A_{n-1}^{-1}$. 3. $v_{n-1}(A_{n-1}^{-1}u_{n-1})$.
4. b. 5. r_{n-1}. 6. q_{-n1}. 7. $r_{n-1}(v_{n-1}A_{n-1}^{-1})$. 8. P_{n-1}.

Then $3n^3-3n+1$ operations of multiplication and division are required.

5.5.66. $$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -83 & 47 & 1 & 0 & 0 \\ 55 & -94 & 0 & 1 & 0 \\ -62 & 71 & 0 & 0 & 1 \end{matrix}\right\|.$$

5.5.67. $$\left\|\begin{matrix} 1 & 4 & -2 & 2 & -1 \\ 2 & 8 & -3 & 2 & -2 \\ 2 & 9 & -4 & 2 & -2 \\ 0 & -4 & 2 & -1 & 1 \\ -1 & -4 & 2 & -1 & 1 \end{matrix}\right\|.$$

5.5.68. $$\left\|\begin{matrix} 1 & 0 & -3 & -9 \\ 0 & 1 & -7 & -21 \\ 3 & 12 & -92 & -279 \\ -1 & -4 & 31 & 94 \end{matrix}\right\|.$$

5.5.69. $$\left\|\begin{matrix} 14 & -8 & -21 & 12 \\ -10 & 6 & 15 & -9 \\ -35 & 20 & 56 & -32 \\ 25 & -15 & -40 & 24 \end{matrix}\right\|.$$

5.5.80. Consider the equality

$$A_pB_p=(E_n)_p \qquad (\alpha)$$

as a system of equations in the elements of the matrix B_p. This system is determinate.

Apply the Laplace theorem to the determinant of the matrix A:

$$\sum_{1\leqslant i_1<i_2\ldots<i_p\leqslant n} A\begin{pmatrix} j_1 j_2 \ldots j_p \\ i_1 i_2 \ldots i_p\end{pmatrix}\cdot(-1)^{\sum_{s=1}^{p}(i_s+k_s)}\cdot A\begin{pmatrix} k_1' k_2' \ldots k_{n-p}' \\ i_1' i_2' \ldots i_{n-p}'\end{pmatrix}$$

$$=\begin{cases} |A|, \text{ when } \sum_{s=1}^{p}(j_s-k_s)^2=0, \\ 0, \text{ when } \sum_{s=1}^{p}(j_s-k_s)^2\neq 0.\end{cases}$$

Obtaining the above decompositions for all the sets $j_1, j_2, \ldots, j_p$ and $k_1, k_2, \ldots, k_p$ such that $1\leqslant j_1<j_2<\ldots<j_p\leqslant n$, $1\leqslant k_1<k_2<\ldots<k_p\leqslant n$, we obtain that the numbers

$$\frac{(-1)^{\sum_{s=1}^{p}(i_s+k_s)} A\begin{pmatrix} k_1' k_2' \ldots k_{n-p}' \\ i_1' i_2' \ldots i_{n-p}'\end{pmatrix}}{|A|}$$

are the solutions of the system (α).

5.6.1. $\begin{Vmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{Vmatrix}$. 5.6.2. $\begin{Vmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0\end{Vmatrix}$.

5.6.3. (a) $\begin{Vmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & 0 & \ldots & 0\end{Vmatrix}$; (b) $\begin{Vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 2 & 3 & \ldots & n \\ 0 & 0 & 0 & 3 & \ldots & C_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & 0 & \ldots & 0\end{Vmatrix}$.

5.6.4. (a) $\begin{Vmatrix} 0 & 1 & 0 & 0 \ldots 0 \\ 0 & 0 & 2 & 0 \ldots 0 \\ 0 & 0 & 0 & 3 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \ldots n\end{Vmatrix}$.

The matrix is of order $n\times(n+1)$.

(b) $\begin{Vmatrix} 0 & 0 & 0 \ldots 0 \\ 1 & 0 & 0 \ldots 0 \\ 0 & \frac{1}{2} & 0 \ldots 0 \\ 0 & 0 & \frac{1}{3} \ldots 0 \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \ldots \frac{1}{n}\end{Vmatrix}$.

The matrix is of order $(n+1)\times n$.

5.6.5. (a) $\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$; (b) $\begin{Vmatrix} a & b \\ -b & a \end{Vmatrix}$.

5.6.6. (a) The first r elements of the principal diagonal are units; all the other elements are zeroes; (b) the last $n-r$ elements of the principal diagonal are units; all the other elements are zeroes; (c) the matrix is diagonal, the first r elements of the principal diagonal being equal to unity, and the others to -1.

5.6.8. (a) $\begin{Vmatrix} -5 & -10 & -7 \\ 6 & 13 & -10 \\ 17 & 36 & -27 \end{Vmatrix}$; (b) $\begin{Vmatrix} 5 & -20 & 33 \\ 7 & -24 & 38 \\ 0 & 0 & 0 \end{Vmatrix}$.

5.6.9. (a) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$; (b) $A\times B^T$; (c) $A\times E+E\times B^T$.

When the basis matrices are interchanged, the matrix (a) is unaltered, the matrix (b) is replaced by $B^T\times A$, the matrix (c) is replaced by $E\times A+$ $+B^T\times E$.

5.6.10. (a) $B^T\times A$; (b) $E_n\times A+B^T\times E_m$. 5.6.12. 2. 5.6.13. 3.

5.6.14. The last $n-r$ columns of the operator matrix contain zeroes whereas the first r are linearly independent.

5.6.18. (a) The i-th and j-th rows are interchanged; (b) the k-th and l-th columns are interchanged.

5.6.19. $\begin{Vmatrix} 1 & 0 & 0 \\ -\frac{15}{4} & -4 & -5 \\ \frac{9}{4} & 3 & 4 \end{Vmatrix}$.

5.6.20. $AB=\begin{Vmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{Vmatrix}$, $BA=\begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{Vmatrix}$.

5.6.21. (a) $P\times Q^T$; (b) $P^{-1}\times (Q^{-1})^T$.

5.6.22. The operator G_{AB} possesses the matrix

$$(P^{-1}AP)\times (QBQ^{-1})^T$$

with respect to the basis $F_{11}, \ldots, F_{mn}$, and the operator F_{AB} the matrix

$$(P^{-1}AP)\times E_n+E_m\times (QBQ^{-1})^T.$$

5.6.31. No; when $B=P^{-1}AP$, $B=(\alpha P)^{-1}A(\alpha P)$ for any nonzero number α.

5.6.38. For example, for the matrices

$$A=\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix},\quad B=\begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}$$

we obtain

$$AB=\left\|\begin{matrix}0 & 1\\ 0 & 0\end{matrix}\right\|,\quad BA=\left\|\begin{matrix}0 & 0\\ 0 & 0\end{matrix}\right\|.$$

5.6.40. No. For example, $A^2 = 0$, $B^2 = B$ for the matrices A and B in the solution to Problem 5.6.38, even though A and B are equivalent.

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the operator A, then:
6.1.3. The operator A^{-1} has the eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$.
6.1.5. The operator $A - \lambda_0 E$ has the eigenvalues $\lambda_1 - \lambda_0, \ldots, \lambda_n - \lambda_0$.
6.1.6. (c) The operator $f(A)$ has the eigenvalues $f(\lambda_1), \ldots, f(\lambda_n)$.
6.1.7. No.
6.1.10. The eigenvectors are those collinear with a. The associated eigenvalue is zero.
6.1.11. The eigenvectors are polynomials of zero degree; the associated eigenvalue is zero.
6.1.12. There are no eigenvectors.
6.1.14. $(1\ 1\ \ldots\ 1)^T$.
6.1.15. The matrix $A = xy$ always has the eigenvalue $\lambda = x_1y_1 + \ldots + x_ny_n$. If the order of the matrix is greater than unity, then there is a zero eigenvalue.
6.1.16. The nonzero eigenvalue is n, the associated eigenvector is $(1\ 1\ \ldots\ \ldots\ 1)^T$. The following equation for the components of the eigenvectors is associated with the zero eigenvalue:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0.$$

6.1.17. The eigenvectors are the same as those of the matrix J_n in Problem 6.1.16. The eigenvalues are $a + b(n-1)$ and $a - b$.
6.1.18. If $B = T^{-1}AT$ and x is an eigenvector of the matrix A associated with an eigenvalue λ, then $T^{-1}x$ is an eigenvector of the matrix B associated with the same eigenvalue.
6.1.22. (a) The projection operator has eigenvalues 1 and 0, L_1 being the eigensubspace associated with $\lambda = 1$, L_2 the eigensubspace associated with $\lambda = 0$: (b) the reflection operator has eigenvalues 1 and -1, where L_1 is the eigensubspace for $\lambda = 1$, L_2 is the eigensubspace for $\lambda = -1$.
6.1.27. An operator of simple structure "stretches" the space in n linearly independent directions (n being the dimension of the space). The matrix of this operator with respect to the eigenvector basis is diagonal.
6.2.1. (a) $\lambda - a_{11}$; (b) $\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$; (c) $\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})\lambda - |A|$.
6.2.3. $\lambda^n - (x_1y_1 + x_2y_2 + \ldots + x_ny_n)\lambda^{n-1}$.
6.2.4. $\lambda^n - a\lambda^{n-1} - (b_1c_1 + b_2c_2 + \ldots + b_{n-1}c_{n-1})\lambda^{n-2}$.
6.2.9. The sum of all principal minors of order k of the matrix A^{-1} equals the sum of all principal minors of order $n-k$ of the matrix A divided by the determinant $|A|$ ($k = 1, \ldots, n-1$). The determinant $|A^{-1}|$ is reciprocal of the determinant $|A|$.
6.2.10. For example, the matrices

$$A=\left\|\begin{matrix}0 & 0\\ 0 & 0\end{matrix}\right\|,\quad B=\left\|\begin{matrix}0 & 1\\ 0 & 0\end{matrix}\right\|$$

are not similar.
6.2.12. $m(A) = \sum_{i=1}^{n} \lambda_i^2$, $\lambda_1, \ldots, \lambda_n$ being the eigenvalues of the matrix A.

6.2.15. The eigenvalues are the diagonal elements $a_{11}, \ldots, a_{nn}$.
6.2.18. $\lambda^2 - (2\cos\alpha)\lambda + 1 = 0$.
6.2.19. $\lambda^3 + |a|^2\lambda = 0$.
6.2.20. λ^{n+1}.
6.2.21. λ^n.
6.2.24. $\lambda_1 = \lambda_2 = 2$. The eigenvectors are all nonzero two-dimensional column vectors.
6.2.25. $\lambda_1 = \lambda_2 = 2$. The eigenvectors are of the form $\alpha\,(1\;\;1+i)^T$, $\alpha \neq 0$.
6.2.26. $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The eigenvectors for $\lambda = 1$ are of the form $\alpha\,(1\;1\;1)^T$, for $\lambda = 2$ of the form $\alpha\,(1\;0\;1)^T$, and for $\lambda = 3$ of the form $\alpha\,(1\;1\;0)^T$; $\alpha \neq 0$.
6.2.27. $\lambda_1 = \lambda_2 = 3$; $\lambda_3 = 6$. The eigenvectors for $\lambda = 3$ are of the form $\alpha\,(0\;1\;-1)^T$, for $\lambda = 6$ of the form $\alpha\,(3\;4\;-2)^T$; $\alpha \neq 0$.
6.2.28. $\lambda_1 = \lambda_2 = 3$; $\lambda_3 = 6$. The eigenvectors for $\lambda = 3$ are of the form $\alpha\,(-7\;5\;-6)^T + \beta\,(6\;-3\;3)^T$ (α and β are not both zero), for $\lambda = 6$ of the form $\alpha\,(1\;1\;-3)^T$ where $\alpha \neq 0$.
6.2.29. $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The eigenvectors are of the form $\alpha\,(1\;1\;0)^T + \beta\,(0\;1\;2)^T$, where α and β are not both zero.
6.2.30. $\lambda_1 = -3$, $\lambda_2 = -1$, $\lambda_3 = 1$, $\lambda_4 = 3$. The eigenvectors for $\lambda = -3$ are of the form $\alpha\,(1\;-3\;3\;-1)^T$; for $\lambda = -1$ of the form $\alpha\,(1\;-1\;-1\;1)^T$; for $\lambda = 1$ of the form $\alpha\,(1\;1\;-1\;-1)^T$; for $\lambda = 3$ of the form $\alpha\,(1\;-3\;3\;-1)^T$; $\alpha \neq 0$.
6.2.31. $\lambda_1 = \lambda_2 = 0$; $\lambda_3 = \lambda_4 = 2$. The eigenvectors for $\lambda = 0$ are of the form $\alpha\,(0\;1\;0\;-1)^T$, for $\lambda = 2$ of the form $\alpha\,(0\;1\;0\;1)^T$; $\alpha \neq 0$.
6.2.32. $\lambda_1 = \lambda_2 = 0$; $\lambda_3 = \lambda_4 = 2$. The eigenvectors for $\lambda = 0$ are: $\alpha\,(2\;-1\;0\;0)^T + \beta\,(3\;0\;0\;-1)^T$, for $\lambda = 2$: $\alpha\,(1\;-1\;0\;1)^T + \beta\,(0\;0\;1\;0)^T$; α and β are not both zero.
6.2.33. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 3$. The eigenvectors are of the following form: $\alpha\,(1\;0\;0\;-1)^T + \beta\,(0\;0\;1\;0)^T$; α and β are not both zero.
6.2.35. (a) There are no eigenvalues; (b) $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$.
6.2.36. (a) $\lambda_1 = 2$; (b) $\lambda_1 = 2$; $\lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.
6.2.37. (a) $\lambda_1 = -1$, $\lambda_2 = 5$; (b) $\lambda_1 = -1$, $\lambda_2 = 5$, $\lambda_3 = 2 + i$, $\lambda_4 = 2 - i$.
6.2.38. (a) There are no eigenvalues; (b) $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1 + i$, $\lambda_4 = 1 - i$.
6.2.42. In the complex case the sum of the algebraic multiplicities of the operator eigenvalues equals the dimension of the space. In the real case this may not be true.

6.2.43 $\left\|\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \sqrt{6} & -\sqrt{6} \\ 2 & -2 & 0 & 0 \end{matrix}\right\|$, $\left\|\begin{matrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & -\sqrt{6} \end{matrix}\right\|$.

6.2.44. The matrix is not of simple structure.

6.2.45. $\left\|\begin{matrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{matrix}\right\|$, $\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{matrix}\right\|$.

6.2.46. The matrix is not of simple structure.

6.2.47. $\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \end{Vmatrix}.$

6.2.48. The matrix is not of simple structure.

6.2.52. $\lambda^n - 1$.

6.2.53. Let ε be an arbitrary eigenvalue of P, i.e. an arbitrary n-th root of unity. The eigenvector associated with ε is, given collinearity, of the form $(1\ \varepsilon\ \varepsilon^2 \ldots \varepsilon^{n-1})^T$.

6.2.54. According to 5.4.52 any circulant is a polynomial in the matrix P (see Problem 6.2.52). A circulant of order n is determined by n numbers a_0, $a_1, \ldots, a_{n-1}$. Given the polynomial $f(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1}t^{n-1}$, the eigenvalues of the circulant are the numbers $f(\varepsilon_1), \ldots, f(\varepsilon_n)$ where $\varepsilon_1, \ldots, \ldots, \varepsilon_n$ are all the n-th roots of unity.

6.2.55. The eigenvectors for λ_i $(i = 1, \ldots, m)$ are

$$\alpha\,(\lambda_i^{n-1}\,\lambda_i^{n-2} \ldots \lambda_i^2 \lambda_i\ 1)^T \quad \alpha \neq 0.$$

6.2.57. The only case that needs proving is when λ_0 is an eigenvalue of A. Let its multiplicity be k. Then the rank of $A - \lambda_0 E$ is $n - k$ (the matrix is of simple structure!). Since the characteristic polynomial of the matrix $A - \lambda_0 E$ has a zero root of multiplicity k, the coefficient of λ^k of this polynomial is nonzero, and there is a nonzero minor among the principal minors of order $n - k$.

6.2.59. $(\lambda - \lambda_1) \ldots (\lambda - \lambda_m)$.

6.3.5. A is a scalar operator.

6.3.11. The converse statement is not true.

6.3.16. The nontrivial invariant subspaces are: the straight line whose direction vector is a (the zero operator is induced on it), and the plane orthogonal to a. The operator induced on this plane is the operator of rotation through 90°.

6.3.17. The spaces M_k, $0 \leqslant k \leqslant n$, and the zero subspace.

6.3.19. Rewriting the condition $B = P^{-1}AP$ in the form $PB = AP$ and equalizing the first columns in the derived matrix relation, we see that b_{11} is an eigenvalue of A, and the first column of P is its associated eigenvector. Hence to construct the transforming matrix P find an eigenvector of A and extend it arbitrarily to form a nondegenerate matrix.

6.3.24. Select a basis $e_1, \ldots, e_n$ for the space so that the first vectors of this basis, $e_1, \ldots, e_k$, form a basis for L. Then the matrix of the operator A is

$$A_e = \begin{Vmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{Vmatrix},$$

A_{11} being the matrix of the induced operator A/L with respect to the basis $e_1, \ldots \ldots, e$. Assume that A_{11} is not of simple structure, and for a certain eigenvalue λ of this matrix of multiplicity p, $r_1 = r_{A_{11}-\lambda E_k} > k - p$. Let λ be of algebraic multiplicity q as of an eigenvalue of A_{22}; then $r_2 = r_{A_{22}-\lambda E_{n-k}} \geqslant (n-k) - q$. Thus, λ is an eigenvalue of A_e of multiplicity $p + q$, but

$$r_{A_e - \lambda E_n} \geqslant r_1 + r_2 > k - p + (n - k) - q = n - (p + q),$$

which is contrary to the assumption that A_e is of simple structure.

6.3.33. The two-dimensional invariant subspace is drawn on the vectors $x = (0\ 1\ 1)^T$ and $y = (2\ 1\ 0)^T$.

6.3.36. The eigenvalues of the operator are on the diagonal of the matrix.

6.3.40. $\begin{Vmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{Vmatrix}$, $\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{Vmatrix}$.

6.3.41. $\begin{Vmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$, $\begin{Vmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 0 \end{Vmatrix}$.

6.3.44. In accordance with Problems 6.3.19 and 6.3.38, construct a matrix P reducing A to triangular form in $n-1$ stages. At the first stage, take the eigenvector common to the matrices A and B to be the first column of the transforming matrix $P^{(1)}$. Then $A^{(1)} = (P^{(1)})^{-1}AP^{(1)}$ and $B^{(1)} = (P^{(1)})^{-1}BP^{(1)}$ are of the form

$$A^{(1)} = \begin{Vmatrix} \alpha & a \\ 0 & A_{n-1} \end{Vmatrix}, \quad B^{(1)} = \begin{Vmatrix} \beta & b \\ 0 & B_{n-1} \end{Vmatrix},$$

A_{n-1} and B_{n-1} being square matrices of order $n-1$. Construct the matrix $P^{(2)}$

$$P^{(2)} = \begin{Vmatrix} 1 & 0 \\ 0 & P_{n-1} \end{Vmatrix}$$

so that the first column of P_{n-1} is the eigenvector common to the (commuting) matrices A_{n-1} and B_{n-1}. Continuing in this manner, determine P as the product $P^{(1)}P^{(2)} \ldots P^{(n-1)}$, both matrices $P^{-1}AP$ and $P^{-1}BP$ being upper triangular.

6.3.45. For the commuting operators A and B, there is a basis for the space with respect to which the matrices of both the operators are triangular of similar form.

6.3.48. Let a matrix A be similar to an upper triangular matrix R, and a matrix B to an upper triangular matrix T. Then $A \times B$ is similar to $R \times T$, the latter also being upper triangular with all the possible products $\lambda_i\mu_j$ placed on its principal diagonal. Similarly, the matrix $A \times E_n + E_m \times B$ is similar to the upper triangular matrix $R \times E_n + E_m \times T$ on whose principal diagonal all the possible sums $\lambda_i + \mu_j$ are placed.

6.3.50. Select a basis $e_1, \ldots, e_n$ for the space so that $e_1, \ldots, e_k$ form a basis for L_1, and $e_{k+1}, \ldots, e_n$ a basis for L_2. Then the matrix of the operator A is quasi-diagonal:

$$A_e = \begin{Vmatrix} A_{11} & 0 \\ 0 & A_{22} \end{Vmatrix}.$$

Partition the matrix B_e of the operator B thus:

$$B_e = \begin{Vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{Vmatrix}.$$

From the condition $A_eB_e - B_eA_e = 0$, we obtain

$$A_{11}B_{12} - B_{12}A_{22} = 0, \quad A_{22}B_{21} - B_{21}A_{11} = 0.$$

Now, it follows from 6.3.49 that $B_{12} = 0$, $B_{21} = 0$.

6.3.51. If A is similar to a triangular matrix R, then A_p is similar to the triangular matrix R_p.

6.4.12. The basis for the root subspace for $\lambda = 0$ is the vector $(0\ 1\ -1)^T$. The basis for the root subspace for $\lambda = 1$ is formed by the vectors $(1\ 0\ 1)^T$ and $(0\ 1\ 0)^T$.

6.4.13. The only eigenvalue is $\lambda = 1$. The root subspace coincides with the three-dimensional arithmetic space.

6.4.14. The basis for the root subspace for $\lambda = 2$ is made up of the vectors $(2\ -1\ 0\ 0)^T$, $(1\ 0\ 1\ 0)^T$, $(2\ 0\ 0\ 1)^T$. The basis for the root subspace when $\lambda = -2$ is the vector $(0\ 1\ 0\ -1)^T$.

6.4.15. The basis for the root subspace when $\lambda = -1$ is formed by the vectors $(1\ 1\ 0\ 0)^T$, $(0\ 0\ 1\ 1)^T$. The basis for the root subspace when $\lambda = 1$ is formed by the vectors $(3\ 1\ 0\ 0)^T$, $(0\ -2\ 3\ 1)^T$.

6.4.17. (a) Assume that the vector $(A - \lambda_j E)\,x$ has height k, $k < h$. Then

$$(A - \lambda_i E)^k\,(A - \lambda_j E)\,x = 0 = (A - \lambda_j E)\,(A - \lambda_i E)^k x.$$

Thereby, the nonzero vector $(A - \lambda_i E)^k\,x$ is an eigenvector associated with the eigenvalue λ_j, $\lambda_j \neq \lambda_i$, which is impossible since the root subspaces intersect only in the zero vector;

(c) as with (b), show that for any number α other than λ_i, the height of the vector $(A - \alpha E)\,x$ is the same as that of the vector x.

6.4.22. The transpose of the Jordan block of order n for the number λ_0.

6.4.23. The canonical basis can comprise, for example, the vectors $e_1 = (4\ 3)^T$, $e_2 = (0\ 1)^T$. The Jordan form is as follows

$$J = \begin{Vmatrix} 7 & 1 \\ 0 & 7 \end{Vmatrix}.$$

6.4.24. $e_1 = (1\ \ 1\ \ -1)^T$, $e_2 = (-4\ \ -5\ \ 6)^T$, $e_3 = (0\ \ 0\ \ 1)^T$; $\quad J = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix}.$

6.4.25. $e_1 = (1\ \ -1\ \ 0\ \ 0)^T$. $e_2 = (0\ \ 1\ \ -1\ \ 0)^T$. $e_3 = (0\ \ 0\ \ 1\ \ -1)^T$. $e_4 = (0\ \ 0\ \ 0\ \ 1)^T$; $\quad J = \begin{Vmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{Vmatrix}.$

6.4.26. $e_1 = (1\ 1\ 1\ 1\ \ 1)^T$, $e_2 = (3\ 2\ 1\ 0\ -1)^T$, $e_3 = (3\ 1\ 0\ 0\ \ 1)^T$, $e_4 = (1\ 0\ 0\ 0\ -1)^T$, $e_5 = (0\ 0\ 0\ 0\ \ 1)^T$; $\quad J = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}.$

6.4.27. $\begin{Vmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 \end{Vmatrix}.$

6.4.28. $\begin{Vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{Vmatrix}.$

6.4.29. $\begin{Vmatrix} 9 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 9 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 9 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 9 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 9 \end{Vmatrix}.$

6.4.30. $\begin{Vmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{Vmatrix}.$

6.4.31. $\begin{Vmatrix} 2 & 1 & 0 & 0 \dots 0 & 0 \\ 0 & 2 & 1 & 0 \dots 0 & 0 \\ 0 & 0 & 2 & 1 \dots 0 & 0 \\ \dots & & & & \\ 0 & 0 & 0 & 0 \dots 2 & 1 \\ 0 & 0 & 0 & 0 \dots 0 & 2 \end{Vmatrix}$. 6.4.32. $\begin{Vmatrix} \alpha & 1 & 0 & 0 \dots 0 & 0 \\ 0 & \alpha & 1 & 0 \dots 0 & 0 \\ 0 & 0 & \alpha & 1 \dots 0 & 0 \\ \dots & & & & \\ 0 & 0 & 0 & 0 \dots \alpha & 1 \\ 0 & 0 & 0 & 0 \dots 0 & \alpha \end{Vmatrix}$.

6.4.33. The Jordan form is the Jordan block of order $n + 1$ corresponding to zero. The canonical basis is $1, t, \frac{1}{2!} t^2, \dots, \frac{1}{n!} t^n$.

6.4.39. Both are equal to $(\lambda - \lambda_0)^n$, n being the dimension of the space.

6.4.40. If

$$\alpha_1 (A - \lambda_0 E)^k x_1 + \dots + \alpha_p (A - \lambda_0 E)^k x_p = 0,$$

then

$$(A - \lambda_0 E)^k (\alpha_1 x_1 + \dots + \alpha_p x_p) = 0,$$

whence (since $k < t$)

$$\alpha_1 x_1 + \dots + \alpha_p x_p = 0,$$

i.e., $\alpha_1 = \dots = \alpha_p = 0$.

Now, let $y = \alpha_1 (A - \lambda_0 E)^k x_1 + \dots + \alpha_p (A - \lambda_0 E)^k x_p \in H_{t-k-1}$. Then

$$0 = (A - \lambda_0 E)^{t-k-1} y = (A - \lambda_0 E)^{t-1} (\alpha_1 x_1 + \dots + \alpha_p x_p).$$

Therefore

$$\alpha_1 x_1 + \dots + \alpha_p x_p = 0$$

and $\alpha_1 = \dots = \alpha_p = 0$.

6.4.42. Applying the operator $(A - \lambda_0 E)^{t-1}$ to both sides of the equality

$$\alpha_1 x_1 + \dots + \alpha_p x_p + \beta_1 (A - \lambda_0 E) x_1 + \dots + \beta_p (A - \lambda_0 E) x_p + \dots$$
$$\dots + \gamma_1 (A - \lambda_0 E)^{t-1} x_1 + \dots + \gamma_p (A - \lambda_0 E)^{t-1} x_p = 0 \quad (\alpha),$$

we obtain

$$(A - \lambda_0 E)^{t-1} (\alpha_1 x_1 + \dots + \alpha_p x_p) = 0,$$

whence $\alpha_1 = \dots = \alpha_p = 0$. Similarly, applying the operator $(A - \lambda_0 E)^{t-2}$ to (α), show that $\beta_1 = \dots = \beta_p = 0$, etc.

6.4.44. A canonical basis is, for example,

$$e_1 = (-2 \; 2 \; 1 \; 2)^T, \quad e_2 = (0 \; 0 \; 1 \; 1)^T, \quad e_3 = (1 \; 2 \; 1 \; -1)^T, \quad e_4 = (1 \; 1 \; 0 \; 0)^T;$$

and the Jordan form

$$J = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

6.4.45.

$$e_1 = (0 \; 0 \; 101 \; 0)^T, \quad e_2 = (0 \; 1 \; 0 \; 0)^T, \quad e_3 = (101 \; 0 \; 0 \; 0)^T, \quad e_4 = (0 \; 0 \; 0 \; 1)^T;$$

$$J = \begin{Vmatrix} 99 & 1 & 0 & 0 \\ 0 & 99 & 0 & 0 \\ 0 & 0 & 99 & 1 \\ 0 & 0 & 0 & 99 \end{Vmatrix}.$$

6.4.46.

$$\begin{aligned}
e_1&=(\ \ 1\ \ \ 2\ \ \ 1\ 0\ 0\ 0)^T,\\
e_2&=(-2\ \ -3\ \ -1\ 0\ 0\ 0)^T,\\
e_3&=(\ \ 1\ \ \ 0\ \ \ 0\ 0\ 0\ 0)^T,\\
e_4&=(\ \ 0\ \ \ 0\ \ \ 0\ 1\ 2\ 1)^T,\\
e_5&=(\ \ 0\ \ \ 0\ \ \ 0\ 1\ 1\ 0)^T,\\
e_6&=(\ \ 0\ \ \ 0\ \ \ 0\ 1\ 0\ 0)^T;
\end{aligned}
\qquad
J=\begin{Vmatrix}
-1 & 1 & 0 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 0 & -1 & 1\\
0 & 0 & 0 & 0 & 0 & -1
\end{Vmatrix}.$$

6.4.47.

$$\begin{aligned}
e_1&=(0\ 0\ 0\ \ \ 0\ -1\ \ \ 0)^T,\\
e_2&=(1\ 0\ 0\ \ \ 0\ \ \ 0\ \ \ 0)^T,\\
e_3&=(0\ 0\ 0\ -3\ \ \ 0\ \ \ 0)^T,\\
e_4&=(0\ 1\ 0\ \ \ 0\ \ \ 0\ \ \ 0)^T,\\
e_5&=(0\ 0\ 0\ \ \ 0\ \ \ 0\ -5)^T,\\
e_6&=(0\ 0\ 1\ \ \ 0\ \ \ 0\ \ \ 0)^T;
\end{aligned}
\qquad
J=\begin{Vmatrix}
2 & 1 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 1 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 2
\end{Vmatrix}.$$

6.4.48. The Jordan form consists of two Jordan blocks of order k corresponding to zero. A canonical basis could be, for example,

$$1,\ \frac{1}{2!}t^2,\ \frac{1}{4!}t^4,\ \ldots,\ \frac{1}{(2k-2)!}t^{2k-2},\ t,\ \frac{1}{3!}t^3,\ \frac{1}{5!}t^5,\ \ldots,\ \frac{1}{(2k-1)!}t^{2k-1}.$$

6.4.50. $n=(m_t-m_{t-1})\,t+2\,(m_{t-1}-m_t-m_{t-2})\,(t-1)=p_1t+(p_2-p_1)(t-1)$. The Jordan form consists of p_1 blocks of order t and p_2-p_1 blocks of order $t-1$.

6.4.51.
$$\begin{Vmatrix}
e_1=(1 & -2 & 1)^T,\\
e_2=(1 & 0 & 0)^T,\\
e_3=(0 & 1 & -1)^T;
\end{Vmatrix}
\qquad
J=\begin{Vmatrix}
3 & 1 & 0\\
0 & 3 & 0\\
0 & 0 & 3
\end{Vmatrix}.$$

6.4.52.
$$\begin{Vmatrix}
e_1=(1 & 1 & 1 & 1)^T,\\
e_2=(1 & 0 & 0 & 0)^T,\\
e_3=(0 & 1 & 1 & 0)^T,\\
e_4=(0 & 0 & 1 & -1)^T;
\end{Vmatrix}
\qquad
J=\begin{Vmatrix}
4 & 1 & 0 & 0\\
0 & 4 & 0 & 0\\
0 & 0 & 4 & 0\\
0 & 0 & 0 & 4
\end{Vmatrix}.$$

6.4.53.
$$\begin{aligned}
e_1&=(24\ 0\ 0\ 0\ 0)^T,\\
e_2&=(\ 5\ 7\ 8\ 0\ 0)^T,\\
e_3&=(\ 0\ 0\ 0\ 0\ 1)^T,\\
e_4&=(\ 4\ 6\ 0\ 0\ 0)^T,\\
e_5&=(\ 0\ 0\ 0\ 1\ 0)^T;
\end{aligned}
\qquad
J=\begin{Vmatrix}
-2 & 1 & 0 & 0 & 0\\
0 & -2 & 1 & 0 & 0\\
0 & 0 & -2 & 0 & 0\\
0 & 0 & 0 & -2 & 1\\
0 & 0 & 0 & 0 & -2
\end{Vmatrix}.$$

6.4.54.
$$\begin{aligned} e_1&=(1\ \ 1\ \ 1\ \ 1\ \ 1)^T,\\ e_2&=(0\ \ -1\ \ 0\ \ 0\ \ 0)^T,\\ e_3&=(1\ \ 1\ \ 0\ \ 1\ \ 1)^T,\\ e_4&=(0\ \ 0\ \ 0\ \ -1\ \ 0)^T,\\ e_5&=(0\ \ 0\ \ 0\ \ 0\ \ 1)^T; \end{aligned}\qquad J=\begin{Vmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix}.$$

6.4.55. The Jordan form consists of two Jordan blocks corresponding to zero, one being of order $k+1$, and the other of order k. A canonical basis could be, for example,

$$1,\ \frac{1}{2!}t^2,\ \frac{1}{4!}t^4,\ \ldots,\ \frac{1}{2k!}t^{2k},\ t,\ \frac{1}{3!}t^3,\ \frac{1}{5!}t^5,\ \ldots,\ \frac{1}{(2k-1)!}t^{2k-1}.$$

6.4.56. The Jordan form consists of p_1 blocks of order t, p_2-p_1 blocks of order $t-1$, and in general, $p_{t-k+1}-p_{t-k}$ blocks of order k, $0<k<t$.

6.4.58. No; otherwise

$$m_4-m_3=2>m_3-m_2=1.$$

6.4.59.
$$\begin{aligned} e_1&=(2\ \ 2\ \ -2\ \ -2)^T,\\ e_2&=(0\ \ 1\ \ 1\ \ 0)^T,\\ e_3&=(0\ \ 0\ \ 0\ \ 1)^T,\\ e_4&=(1\ \ 0\ \ 0\ \ 1)^T; \end{aligned}\qquad J=\begin{Vmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

6.4.60.
$$\begin{aligned} e_1&=(-3\ \ 1\ \ 1\ \ 1\ \ 1)^T,\\ e_2&=(-2\ \ 0\ \ 1\ \ 1\ \ 1)^T,\\ e_3&=(1\ \ 0\ \ 0\ \ 0\ \ 0)^T,\\ e_4&=(1\ \ 0\ \ 0\ \ -1\ \ 0)^T,\\ e_5&=(1\ \ 0\ \ 0\ \ 0\ \ -1)^T; \end{aligned}\qquad J=\begin{Vmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix}.$$

6.4.61.
$$\begin{aligned} e_1&=(24\ \ -12\ \ 0\ \ 0\ \ 0\ \ 0)^T,\\ e_2&=(6\ \ 0\ \ -2\ \ 8\ \ -4\ \ 0)^T,\\ e_3&=(1\ \ 0\ \ 0\ \ 3\ \ 0\ \ -1)^T,\\ e_4&=(0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^T,\\ e_5&=(3\ \ 0\ \ -1\ \ -8\ \ 4\ \ 0)^T,\\ e_6&=(2\ \ 0\ \ 0\ \ -3\ \ 0\ \ 1)^T; \end{aligned}\qquad J=\begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

6.4.62.
$$\begin{aligned} e_1&=(-2\ \ 0\ \ 2\ \ 0\ \ 2\ \ 0)^T,\\ e_2&=(0\ \ 0\ \ 0\ \ 0\ \ 2\ \ 0)^T,\\ e_3&=(1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0)^T,\\ e_4&=(0\ \ 0\ \ 0\ \ 3\ \ 0\ \ 1)^T,\\ e_5&=(0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^T,\\ e_6&=(0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0)^T; \end{aligned}\qquad J=\begin{Vmatrix} 2 & 1 & 0 & 0 & 0 & 0\\ 0 & 2 & 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 2 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 2 \end{Vmatrix}.$$

6.4.63. $e_1=(-2\ \ 2\ \ \ 2)^T$, $e_2=(\ \ 1\ \ 1\ \ -1)^T$, $e_3=(\ \ 0\ \ 1\ \ \ 1)^T$; $J=\begin{Vmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & 3\end{Vmatrix}$.

6.4.64. $e_1=(\ \ 1\ \ 1\ \ 0)^T$, $e_2=(\ \ 0\ \ 1\ \ 1)^T$, $e_3=(-1\ \ 2\ \ 2)^T$; $J=\begin{Vmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{Vmatrix}$.

6.4.65. $e_1=(-4\ \ -3\ \ -4)^T$, $e_2=(\ \ 2\ \ \ 2\ \ -1)^T$, $e_3=(\ \ 1\ \ \ 0\ \ \ 1)^T$; $J=\begin{Vmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{Vmatrix}$.

6.4.66. $e_1=(1\ \ -1\ \ \ 2)^T$, $e_2=(0\ \ \ 0\ \ -1)^T$, $e_3=(0\ \ \ 1\ \ \ 0)^T$; $J=\begin{Vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{Vmatrix}$.

6.4.67. $e_1=(\ \ 3\ \ \ 3\ \ -3\ \ -3)^T$, $e_2=(\ \ 1\ \ \ 0\ \ -1\ \ \ 0)^T$, $e_3=(-3\ \ -3\ \ -3\ \ -3)^T$, $e_4=(\ \ 1\ \ \ 0\ \ \ 1\ \ \ 0)^T$; $J=\begin{Vmatrix}2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & -2 & 1\\ 0 & 0 & 0 & -2\end{Vmatrix}$.

6.4.68. $e_1=(\ \ 2\ \ \ 1\ \ \ 0\ \ \ 0)^T$, $e_2=(-21\ \ -10\ \ \ 0\ \ \ 0)^T$, $e_3=(\ \ 0\ \ \ 0\ \ \ 3\ \ -2)^T$, $e_4=(\ \ 8\ \ \ 3\ \ -1\ \ \ 1)^T$; $J=\begin{Vmatrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{Vmatrix}$.

6.4.69. $e_1=(-2\ \ -1\ \ \ 0\ \ 0)^T$, $e_2=(\ \ 1\ \ \ 0\ \ -2\ \ 3)^T$, $e_3=(\ \ 0\ \ \ 0\ \ \ 1\ \ 0)^T$, $e_4=(\ \ 0\ \ \ 0\ \ \ 0\ \ 1)^T$; $J=\begin{Vmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{Vmatrix}$.

6.4.70. $e_1=(2\ \ -2\ \ 2\ \ -2)^T$, $e_2=(1\ \ \ 0\ \ 1\ \ \ 0)^T$, $e_3=(0\ \ -1\ \ 0\ \ \ 1)^T$, $e_4=(0\ \ \ 0\ \ 1\ \ -1)^T$; $J=\begin{Vmatrix}3 & 1 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & -2\end{Vmatrix}$.

6.4.71. Each block is replaced by its transpose; the order of the blocks on the principal diagonal is reversed.

6.4.72. The diagonal elements $\lambda_1, \ldots, \lambda_m$ of the Jordan form of the operator A are changed to $\lambda_1-\lambda_0, \ldots, \lambda_m-\lambda_0$ in (a), and to $1/\lambda_1, \ldots, 1/\lambda_m$ in (b).

6.4.75. The Jordan form of the operator A^2 can be obtained from that of the operator A as follows: replace λ by λ^2 in each block corresponding to $\lambda \neq 0$; replace each block of order k corresponding to 0 by two blocks of order l if $k=2l$, and by two blocks of orders $l+1$ and l, respectively, if $k=2l+1$.

6.4.77. It follows from the condition $A^2 = E$ that the eigenvalues of the operator A can only be equal to 1 and -1. Verifying the equality $J^2 = E$ for the Jordan form of the operator A, we find that J is a diagonal matrix, i.e. A is an operator of simple structure whereas both 1 and -1 must be the eigenvalues of A, otherwise $A = -E$ or $A = E$. Denoting the eigensubspaces of the operator A associated with 1 by L_1 and L_2, respectively, we obtain that A is a reflection operator in L_1 parallel to L_2.

6.4.79. The defect of the operator $A - \lambda_0 E$ can be found using the matrix $J - \lambda_0 E$, where J is the Jordan form of the operator A. Using this matrix each block of J corresponding to λ_0 is transformed into a block of $J - \lambda_0 E$ corresponding to 0, the defect of the latter being equal to unity. The other blocks of $J - \lambda_0 E$ are nondegenerate, and thus the defect of $J - \lambda_0 E$ equals the number of Jordan blocks of J corresponding to λ_0.

6.4.82. $\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix}$.
6.4.83. $\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix}$.

6.4.84. $\begin{Vmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 19 \end{Vmatrix}$.
6.4.85. $\begin{Vmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{Vmatrix}$.

6.4.86. $\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$.
6.4.87. $\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$.

6.4.88. The Jordan form consists of one block of order $n+1$ corresponding to zero.

6.4.89. The Jordan forms of both the operators coincide and consist of three Jordan blocks of order 3 corresponding to zero.

6.4.91. There is not a single pair of the matrices A, B and C that contains similar matrices.

6.4.92. A and C are similar to one another and not similar to B.

6.4.93. A and B are similar to one another and not similar to C.

6.4.95. If λ is an eigenvalue of A not 1 or -1, then $1/\lambda$ is also an eigenvalue; moreover, to both eigenvalues there corresponds the same number of Jordan blocks of respectively equal orders.

6.4.98. Only one Jordan block in the Jordan form can correspond to each eigenvalue.

6.4.100. Write a quasi-diagonal matrix of order mn whose diagonal is the matrix J repeated m times. Then the Jordan form of the matrices $A \times B$ and $A \times E_n + E_m \times B$ is obtained as follows: (a) for each nonzero eigenvalue λ_i of A, multiply the diagonal elements of the i-th block of J by λ_i; and if $\lambda_i = 0$, then the corresponding block of J is replaced by the zero matrix; (b) add λ_i

to all the diagonal elements of the i-th block of J. The Jordan form of the operators G_{AB} and F_{AB}, respectively, is obtained in the same way.

6.4.101. If α is the n-th primitive root of unity, $r=\sqrt[n]{\bar{e}}$, then the Jordan form of A is as follows:

$$\left\| \begin{matrix} 1+r & & & & 0 \\ & 1+r\alpha & & & \\ & & 1+r\alpha^2 & & \\ & & & \cdot & \\ 0 & & & & 1+r\alpha^{n-1} \end{matrix} \right\|.$$

7.1.6. If Λ is a diagonal matrix such that $\lambda_{ii} = (e_i, e_i)$, then

$$(A^*)_e = \Lambda^{-1} (A_e)^* \Lambda,$$

where $(A_e)^*$ is a matrix conjugate to A_e. In particular, if the lengths of all the vectors e_i are the same, then $(A^*)_e = (A_e)^*$.

7.1.7. For elements a_{ij} of the matrix A_e of the operator A, the equalities must hold

$$a_{ij} = (Ae_j, f_i);$$

similarly, for elements a_{ij}^* of the matrix A_f^* of the conjugate operator A^*:

$$a_{ij}^* = (A^* f_j, e_i).$$

Therefore

$$a_{ij}^* = \bar{a}_{ji}.$$

7.1.8. Any operator on a one-dimensional space multiplies each vector in the space by a fixed (for the given operator) number α. If the space is unitary, then the conjugate operator multiplies by the conjugate number $\bar{\alpha}$. Any operator on a Euclidean one-dimensional space coincides with its conjugate.

7.1.9. Rotation through the angle α in the opposite direction.

7.1.10. $A^* = -A$.

7.1.11. (a) $\left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{matrix} \right\|$, $\left\| \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{matrix} \right\|$;

(b) $\frac{1}{2}\left\| \begin{matrix} -3 & -4 & -1 \\ 1 & 0 & -1 \\ 1 & 4 & 3 \end{matrix} \right\|$, $\left\| \begin{matrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{matrix} \right\|$;

(c) $\left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{matrix} \right\|$, $\left\| \begin{matrix} 0 & \frac{2}{3} & 0 \\ 1 & 0 & -\frac{1}{2} \\ 0 & \frac{4}{3} & 0 \end{matrix} \right\|$.

7.1.12. (a) $\begin{Vmatrix} 0 & -4 & 0 \\ \frac{3}{2} & 0 & 1 \\ 0 & 6 & 0 \end{Vmatrix}$; (b) $\frac{1}{2}\begin{Vmatrix} -3 & 1 & 1 \\ -4 & 0 & 4 \\ -1 & -1 & 3 \end{Vmatrix}$; (c) $\begin{Vmatrix} 0 & -2 & 0 \\ \frac{3}{2} & 0 & \frac{3}{4} \\ 0 & 4 & 0 \end{Vmatrix}$.

7.1.13. (a) $\begin{Vmatrix} 0 & -5/2 & 0 \\ 3 & 0 & 1 \\ 0 & 15/2 & 0 \end{Vmatrix}$; (b) $\begin{Vmatrix} -3 & 2 & 2 \\ -\frac{5}{4} & 0 & \frac{5}{4} \\ -2 & -2 & 3 \end{Vmatrix}$; (c) $\begin{Vmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 5 & 0 \end{Vmatrix}$.

7.1.19. In the complex case $\beta_i = \operatorname{tr}(A_i^* B)$.

7.1.20. If $e_1, \ldots, e_n$ is an orthonormal basis for the space X, then the vector f is one whose coordinates with respect to this basis are the numbers $\overline{f(e_1)}, \ldots, \overline{f(e_n)}$.

7.1.25. A^* is the projection operator on the plane $x + y + z = 0$ parallel to the axis Oz.

7.1.26. (a) The basis for the kernel is the polynomial t^2, and the basis for the image is made up of the polynomials t and t^2; (b) the basis for the kernel is the polynomial $3t^2 - 2$, and the basis for the image consists of the polynomials t and $3t^2 - 2$; (c) the basis for the kernel is the polynomial $3t^2 - 1$, and the basis for the image contains t and $3t^2 - 1$.

7.1.32. The inclusion $T_{A+B} \subset T_A + T_B$ is always fulfilled. Show that for the data given, $T_{A+B} = T_A + T_B$, for which it suffices to show that $T_A \subset T_{A+B}$ and $T_B \subset T_{A+B}$.

Let $x \in T_{B^*}$; then $Ax = 0$ (by the condition $AB^* = 0$) and $(A + B)x = Bx$. If x ranges over T_{B^*}, then Bx ranges over T_B; thus, $T_B \subset T_{A+B}$. In the same way deduce that $T_A \subset T_{A+B}$ by rewriting the condition $AB^* = 0$ as $BA^* = 0$.

By the second condition in the problem and Problem 7.1.31, the sum $T_{A+B} = T_A + T_B$ is orthogonal. Therefore

$$r_{A+B} = r_A + r_B.$$

Similarly, it can be shown that $T_{(A+B)^*} = T_{A^*} + T_{B^*}$, whence by transferring to the orthogonal complements, we obtain the second statement of the problem.

7.1.34. The null subspace and spans of the sets of polynomials $t^k, t^{k+1}, \ldots, t^n$ $(k = 0, 1, \ldots, n)$.

7.1.35. The required subspace is determined by the condition

$$\sum_{k=0}^{n} f(k) = 0.$$

7.1.36. The required subspace is determined by the condition

$$\int_{-1}^{1} f(t)\, dt = 0.$$

7.1.39. (a) $1, t, t^2$; (b) $1/\sqrt{3}$, $t/\sqrt{2}$, $(3t^2-2)/\sqrt{6}$; (c) $1/\sqrt{2}$, $\sqrt{3/2}t$, $\sqrt{5/8}(3t^2-1)$.

7.1.41. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the operator A, then the eigenvalues of the operator A^* are $\bar{\lambda}_1, \ldots, \bar{\lambda}_n$.

7.1.44. Let k be the dimension of K_λ. Then for any vector x from K_λ: $(A - \lambda E)^k x = 0$. If y is an arbitrary vector from K_μ^* then

$$0 = ((A - \lambda E)^k x, y) = (x, (A^* - \bar{\lambda} E)^k y).$$

The operator $A^* - \bar{\lambda}E$ is nondegenerate on the invariant subspace K_μ^*. Therefore the obtained equality means that $K_\lambda \perp K_\mu^*$.

7.1.45. The Jordan form of the operator A^* is obtained from the Jordan form of A by replacing the diagonal elements with the conjugate complex numbers.

7.1.46. A canonical basis for the differential operator can be constructed, for example, by the polynomials $2, 2t, t^2$; a canonical basis for the conjugate operator by the polynomials $t^2, t/2, 1/2$.

7.1.47. Let the given order of the eigenvalues be as follows $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$, and let it be required to construct an upper Schur form. Then take the normed eigenvector of the operator A^*, associated with the eigenvalue $\bar{\lambda}_{i_n}$, as the vector e_n and consider both the operator A_1 induced on the orthogonal complement to e_n which is A-invariant, and its conjugate A_1^*. Take the normed eigenvector A_1^* associated with $\bar{\lambda}_{i_{n-1}}$ as e_{n-1}, and then consider the orthogonal complement to the span of the vectors e_{n-1} and e_n, etc. The construction may also be performed "in the reverse order": select the normed eigenvector of A associated with λ_{i_1} as the vector e_1 and then consider the orthogonal complement of e_1 which is A^*-invariant, etc.

7.2.20. This statement about a Euclidean space does not hold. Any operator which has no eigenvalues and is not normal will serve as a counter-example.

7.2.22. Yes, it follows.

7.2.29. No, if all the eigenvalues of the operator are of multiplicity unity; yes, if at least one of them is multiple.

7.2.30. $\lambda_1 = 1 + i$, $\lambda_2 = 1 - i$. A basis could be, for example, the vectors $e_1 = \frac{1}{\sqrt{2}}(1\ 1)^T$, $e_2 = \frac{1}{\sqrt{2}}(1\ -1)^T$.

7.2.31. $\lambda_1 = 0$, $\lambda_2 = 3i$, $\lambda_3 = -3i$. A basis could be, for example, the vectors $e_1 = \frac{1}{3}(2\ 1\ -2)^T$, $e_2 = \frac{1}{3\sqrt{10}}(4-3i\ \ 2+6i\ \ 5)^T$, $e_3 = \frac{1}{3\sqrt{10}}(4 + 3i\ \ 2-6i\ \ 5)^T$.

7.2.32. $\lambda_1 = -i$, $\lambda_2 = 2-i$, $\lambda_3 = 3-i$. The basis is $e_1 = \frac{1}{\sqrt{6}}(1\ 2\ -1)^T$, $e_2 = \frac{1}{\sqrt{2}}(1\ 0\ 1)^T$, $e_3 = \frac{1}{\sqrt{3}}(-1\ 1\ 1)^T$.

7.2.33. $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 2i$, $\lambda_4 = -2i$. The basis is $e_1 = \frac{1}{\sqrt{2}}(1\ 1\ 0\ 0)^T$, $e_2 = \frac{1}{\sqrt{2}}(0\ 0\ 1\ -1)^T$; $e_3 = \frac{1}{2}(1\ -1\ i\ i)^T$, $e_4 = \frac{1}{2}(1\ -1\ -i\ -i)^T$.

7.2.34. No. The differential operator is not an operator of simple structure.

7.2.35. No, if $a \neq 0$. When $a = 0$ the identity operator is obtained.

7.2.37. If $x = (\alpha_1, \alpha_2, \alpha_3)$ and $y = (\beta_1, \beta_2, \beta_3)$ are arbitrary vectors from R_3, then the scalar product can be given by the formula

$$(x, y) = \alpha_1\beta_1 + \alpha_1\beta_2 + \alpha_1\beta_3 + \alpha_2\beta_1 + 2\alpha_2\beta_2 + 2\alpha_2\beta_3 + \alpha_3\beta_1 + 2\alpha_3\beta_2 + 3\alpha_3\beta_3$$

7.2.41. $e_1=\frac{1}{\sqrt{3}}(1\ 1\ 1)^T$, $e_2=\frac{1}{\sqrt{3}}\left(1-\frac{1}{2}+i\frac{\sqrt{3}}{2}-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^T$, $e_3=\frac{1}{\sqrt{3}}\left(1-\frac{1}{2}-i\frac{\sqrt{3}}{2}-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^T$.

7.2.42. For example, $e_1=\frac{1}{\sqrt{6}}(1\ -2\ 1)^T$, $e_2=\frac{1}{\sqrt{2}}(1\ 0\ -1)^T$, $e_3=\frac{1}{\sqrt{3}}(1\ 1\ 1)^T$.

7.2.44. Let all the eigenvalues of the operator A be different in modulus $|\lambda_1|>|\lambda_2|>\ldots>|\lambda_n|$, and let $e_1, \ldots, e_n$ be the corresponding orthonormal eigenvector basis. The matrix of the operator AB with respect to this basis is normal and equal to the product of the matrices A_e and B_e. Equalizing (in accordance with 7.2.12) the sums of the squares of the moduli of the elements of the first row to those of the first column of the matrix A_eB_e, we obtain

$$|\lambda_1|^2(|b_{11}|^2+|b_{12}|^2+\ldots+|b_{1n}|^2)=|\lambda_1|^2|b_{11}|^2+|\lambda_2||b_{21}|^2+\ldots+|\lambda_n||b_{n1}|^2$$

Since B_e is also a normal matrix,

$$|b_{11}|^2+|b_{21}|^2+\ldots+|b_{n1}|^2=|b_{11}|^2+|b_{12}|^2+\ldots+|b_{1n}|^2.$$

These equalities are all true only if

$$b_{21}=\ldots=b_{n1}=b_{12}=\ldots=b_{1n}=0.$$

Similarly it can be shown that all the other off-diagonal elements of the matrix B_e are zeroes. Thus B_e is diagonal matrix, and therefore the operators A and B commute.

7.2.45. Reasoning in the same way as for the proof of 7.2.44, show that the matrix of the operator B is quasi-diagonal with respect to the orthonormal eigenvector basis of the operator A (if it satisfies the conditions of the problem), and that its diagonal blocks of order >1 correspond to multiples of the eigenvalues of the operator A. Hence the matrices of the operators commute.

7.2.47. Any vector for which this maximum occurs is an eigenvector of the operator A associated with the eigenvalue with maximum modulus.

7.2.49. No. For example, for a unitary operator U, the ratio $|Ux|/|x|$ equals unity for any nonzero vector x.

7.3.4. The operators of multiplication by a number whose modulus equals unity.

7.3.6. No. The operator A is degenerate.

7.3.8. (a) Yes; (b) no.

7.3.10. No, if the operator is not the identity operator.

7.3.12. (a) The eigensubspace for $\lambda=1$ coincides with the set of all even polynomials and the eigensubspace for $\lambda=-1$ coincides with the set of all odd polynomials; (c) the eigensubspace for $\lambda=1$ is drawn on the set of polynomials t^n+1, $t^{n-1}+t, \ldots$, and the eigensubspace for $\lambda=-1$ is drawn on the polynomials t^n-1, $t^{n-1}-t, \ldots$ If $n=2k-1$, then both the subspaces are of dimension k; if, however, $n=2k$, then the dimension of the first is $k+1$ and of the second k.

7.3.13. The scalar product of polynomials $f(t)=a_0+a_1t+a_2t^2$ and $g(t)=b_0+b_1t+b_2t^2$ may be evaluated by the formula

$$\begin{aligned}(f, g) = 3a_0b_0 - 2a_0b_1 - 2a_0b_2\\ -2a_1b_0 + 2a_1b_1 + a_1b_2\\ -2a_2b_0 + a_2b_1 + 2a_2b_2.\end{aligned}$$

7.3.16.

$$Q_e=\left\|\begin{matrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{matrix}\right\|.$$

7.3.18. Yes, it is.

7.3.21. Let A be the given operator and let $e_1, \ldots, e_n$ be an arbitrary orthonormal basis. By the data, the vectors $Ae_1, \ldots, Ae_n$ are orthogonal each to each. Show that they have equal length. If, for example, $\alpha_1 = |Ae_1| \neq \alpha_2 = |Ae_2|$, then the vectors $e_1 + e_2$ and $e_1 - e_2$ are orthogonal, and $A(e_1 + e_2)$ and $A(e_1 - e_2)$ are not:

$$(A(e_1+e_2),\ A(e_1-e_2)) = (Ae_1, Ae_1) - (Ae_2, Ae_2) = \alpha_1^2 - \alpha_2^2.$$

Therefore $|Ae_i| = \alpha$ for all $i = 1, \ldots, n$, and then $A = \alpha U$, where U is a unitary operator transforming the vectors e_i into the vectors $(1/\alpha)\, Ae_i$.

7.3.34. Interchanging the rows and columns of a matrix in the reverse order is a unitarily similar transformation.

7.3.37. $\psi_1 - \psi_2 = (\psi_3 - \psi_4) + 2k\pi$.

7.3.38. $\psi_2 = -\psi_3 = \arg a_{ii} - \arg a_{ji}$,

$$\cos\varphi = \frac{|a_{ii}|}{\sqrt{|a_{ii}|^2 + |a_{ji}|^2}}, \quad \sin\varphi = -\frac{|a_{ji}|}{\sqrt{|a_{ii}|^2 + |a_{ji}|^2}}.$$

7.3.40. Multiply the given matrix A on the left by the sequence of elementary unitary matrices $T_{12}, T_{13}, \ldots, T_{1n}, T_{23}, \ldots, T_{n-1,n}$ so as to make all the subdiagonal elements equal to zero one by one. The derived upper triangular matrix is one of the factors of the required decomposition, and the other is the product $T_{12}^*T_{13}^*, \ldots, T_{n-1,n}^*$.

7.3.44. The length of the vector w must be equal to unity.

7.3.46. The eigenvalues equal 1 and -1. Moreover, $\lambda = -1$ is an eigenvalue of multiplicity unity, and its corresponding vectors are collinear with w. The eigenvectors for $\lambda = 1$ (and the zero vector) make up the orthogonal complement of w.

7.3.47. The determinant equals -1.

7.3.49. $w = \left(-\sin\frac{\varphi}{2}\ \cos\frac{\varphi}{2}\right)^T.$

7.3.50. The product Hx should be evaluated by the formula

$$Hx = x - 2(x, w)\, w.$$

The scalar product (x, w) is computed by (7.1.4).

7.3.51. $w = \frac{1}{|x - ke_1|}(x - ke_1)$, where $|k| = |x| = (x, x)^{1/2}$, as to the rest, the choice of k is arbitrary.

7.3.52. In accordance with 7.3.51 select the matrix H_1 so that for the given matrix A of order n, the matrix $A_1 = H_1A$ may be of the form

$$A_1 = \left\|\begin{matrix} k & \times \ldots \times \\ 0 & \widetilde{A}_1 \end{matrix}\right\|,$$

such that $\widetilde{A}_1$ is a submatrix of order $n - 1$. Now construct the matrix H_2:

$$H_2 = \left\|\begin{matrix} 1 & 0 \ldots 0 \\ 0 & \widetilde{H}_2 \end{matrix}\right\|,$$

where $\widetilde{H}_2$ is a reflection matrix of order $n-1$ selected so that all the subdiagonal elements of the first column of the matrix $\widetilde{H}_2\widetilde{A}_1$ are zero. The first two columns of the matrix H_2H_1A will now coincide with the columns of a triangular matrix. In a similar way after $n-1$ steps an upper triangular matrix is obtained.

The unitary factor of the required presentation is the product $H_1H_2 \ldots H_{n-1}$.

7.3.54. If the column vector $(a_{21}\ a_{31} \ldots a_{n1})^T$ is denoted by $\widetilde{a}_1$, then a matrix of the form

$$\begin{Vmatrix} 1 & 0 \ldots 0 \\ 0 & \widetilde{H} \end{Vmatrix}$$

may be taken as H, where $\widetilde{H}$ is a reflection matrix transforming $\widetilde{a}_1$ into a vector collinear with the unit column e_1 of order $n-1$. Moreover, H is itself also a reflection matrix.

7.3.55. For any operator there exists an orthonormal basis for the space with respect to which the matrix of this operator is upper (lower) almost triangular.

7.4.7. In the complex case, these are operators of multiplication by a real number. All linear operators on a one-dimensional Euclidean space are symmetric.

7.4.11. Yes.

7.4.15. $$S_e = \frac{1}{9}\begin{Vmatrix} 17 & 5 & -1 \\ 5 & -7 & 5 \\ -1 & 5 & 17 \end{Vmatrix}.$$

7.4.24. $H = 0$.

7.4.34. Let L_k be an arbitrary k-dimensional subspace. Consider the span M_{n-k+1} of the vectors $e_k, e_{k+1}, \ldots, e_n$ together with L_k. The intersection of L_k and M_{n-k+1} is at least one-dimensional; let x_0 be a nonzero vector from this intersection. Then according to (7.4.3),

$$\frac{(Hx_0, x_0)}{(x_0, x_0)} \leqslant \lambda_k.$$

Therefore

$$\min_{\substack{x \neq 0 \\ x \in L_k}} \frac{(Hx, x)}{(x, x)} \leqslant \lambda_k,$$

so that also

$$\max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{(Hx, x)}{(x, x)} \leqslant \lambda_k.$$

That the equality in the relation (7.4.4) occurs is demonstrated by the k-dimensional span of the vectors $e_1, \ldots, e_k$.

(7.4.5) is proved similarly.

7.4.35. Without loss of generality, the submatrix H_{n-1} may be assumed to be in the upper left-hand corner of the matrix H. Let $f_1, \ldots, f_{n-1}$ be an orthonormal basis of the eigenvectors of the matrix H_{n-1} associated with $\mu_1, \ldots, \mu_{n-1}$, respectively, where $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1}$. According to (7.4.3),

$$\max_{\substack{y \neq 0 \\ y \in \widetilde{M}_{n-k}}} \frac{(H_{n-1}y, y)}{(y, y)} = \mu_k = \min_{\substack{y \neq 0 \\ y \in \widetilde{M}_k}} \frac{(H_{n-1}y, y)}{(y, y)},$$

where $\widetilde{M}_k$ is drawn on $f_1, \ldots, f_k$, and $\widetilde{M}_{n-k}$ on $f_k, \ldots, f_{n-1}$. Now, match each $(n-1)$-dimensional column y with an n-dimensional column vector x such that

$$x = \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Then

$$\frac{(H_{n-1}y, y)}{(y, y)} = \frac{(Hx, x)}{(x, x)}$$

for the corresponding vectors y and x. To the subspaces $\widetilde{M}_{n-k}$ and $\widetilde{M}_k$ there correspond, in the n-dimensional space, M_{n-k} and M_k of the same dimension. Therefore, it follows from the Courant-Fischer theorem that

$$\lambda_{k+1} \leqslant \mu_k \leqslant \lambda_k.$$

7.4.36. One positive and one negative eigenvalue.

7.4.40. For any Hermitian operator, there is an orthonormal basis for the space with respect to which the matrix of this operator is tridiagonal.

7.4.42. $f_0(\lambda) = 1$, $f_1(\lambda) = \lambda$, $f_2(\lambda) = \lambda^2 - 1$, $f_3(\lambda) = \lambda^3 - 2\lambda$, $f_4(\lambda) = \lambda^4 - 3\lambda^2 + 1$, $f_5(\lambda) = \lambda^5 - 4\lambda^3 + 3\lambda$.

7.4.44. Use the induction method over the number of polynomials in the set $f_0(\lambda), f_1(\lambda), \ldots, f_k(\lambda)$. Let $k = 1$. Then $f_1(\mu)$ is greater or less than zero according as the number μ is greater or less than the unique root $\lambda_1^{(1)}$ of the polynomial $f_1(\lambda)$. In the first case, the sequence

$$f_0(\mu) = 1, \quad f_1(\mu)$$

has no sign changes, and in the second it has.

Assume that the proposition in the question is true for all $k \leqslant r$. The values of the polynomials $f_r(\lambda)$ and $f_{r+1}(\lambda)$ at the point μ may then be calculated by the formulae

$$f_r(\mu) = \prod_{j=1}^{r} (\mu - \lambda_j^{(r)}), \quad f_{r+1}(\mu) = \prod_{j=1}^{r+1} (\mu - \lambda_j^{(n+1)}).$$

Hence the sign of each of the numbers $f_r(\mu)$ and $f_{r+1}(\mu)$ is determined by the number of negative brackets in the corresponding product. The number of roots of the polynomial $f_{r+1}(\lambda)$ occurring to the right of the point μ is equal to, or one greater than, that of the polynomial $f_r(\lambda)$ (see 7.4.43 (b)). In the former case, the sign of $f_{r+1}(\mu)$ coincides with that of $f_r(\mu)$, and the sequence

$$f_0(\mu), f_1(\mu), \ldots, f_r(\mu), f_{r+1}(\mu)$$

has the same number of the sign changes as the sequence

$$f_0(\mu), f_1(\mu), \ldots, f_r(\mu).$$

In the latter case, the signs of $f_{r+1}(\mu)$ and $f_r(\mu)$ are opposite, and in the first sequence there is one more sign change than in the second.

7.4.45. The proof, as in 7.4.44, uses the induction method. First, let $k = 1$. If $\mu = \lambda_1^{(1)}$, then the same sign as of $f_0(\mu) = 1$ is ascribed to the zero value of $f_1(\mu)$ and the sequence

$$f_0(\mu), \quad f_1(\mu)$$

has no sign changes.

Now, let the statement be true for all $k \leqslant r$. If μ is not a root of the polynomials $f_r(\lambda)$ and $f_{r+1}(\lambda)$, then the induction follows as in proof of 7.4.44. Consider the two remaining possibilities:

(a) μ is a root of the polynomial $f_r(\lambda)$. Then by problem 7.4.43 (b), the number of roots of the polynomial $f_{r+1}(\lambda)$ lying to the right of μ is one greater than that of $f_r(\lambda)$. The numbers $f_{r+1}(\mu)$ and $f_{r-1}(\mu)$ have opposite signs and there is one more sign change in the sequence

$$f_0(\mu), f_1(\mu), \ldots, f_{r-1}(\mu), f_r(\mu), f_{r+1}(\mu)$$

than in the sequence

$$f_0(\mu), f_1(\mu), \ldots, f_{r-1}(\mu), f_r(\mu);$$

(b) μ is a root of the polynomial $f_{r+1}(\lambda)$. In this case, the rule described above to ascribe a sign to a zero value is used, and so the indicated sequences both have the same number of sign changes. In addition, both the polynomials $f_r(\lambda)$ and $f_{r+1}(\lambda)$ have the same number of the roots lying to the right of μ.

7.4.46. Let $S(x)$ be the number of sign changes in the sequence of numbers

$$f_0(x), f_1(x), \ldots, f_n(x).$$

According to the condition $S(a) \geqslant k$, $S(b) < k$. Put $c = \frac{a+b}{2}$ and set up the sequence

$$f_0(c), f_1(c) \ldots, f_n(c).$$

If $S(c) \geqslant k$, then λ_k lies in the interval (c, b). If, however, $S(c) < k$, then either $\lambda_k = c$ or λ_k lies in the interval (a, c).

7.4.49. The required approximation to λ_1 is 27/16.

7.4.52. (b) Any real symmetric matrix is orthogonally similar to a diagonal matrix.

7.5.1. No.

7.5.20. Let λ be any eigenvalue of the matrix A and x the corresponding eigenvector. Then

$$0 > (Cx, x) = (A^*Bx, x) + (BAx, x) = (Bx, Ax) + (Ax, Bx)$$
$$= (\bar{\lambda} + \lambda)(Bx, x) = 2\operatorname{Re}\lambda\cdot(Bx, x),$$

whence $\operatorname{Re}\lambda < 0$. Now, the uniqueness of the solution of the Lyapunov equation for the matrix A follows from 6.3.49.

7.5.21. $H = 0$.

7.5.26. The proposition in the problem follows from 7.4.19 and 6.3.51.

7.5.28. The proposition in the problem follows from 7.4.20 and 6.3.48.

7.5.30. The matrix S is the Schur product of the positive-definite matrices H and H^T.

7.5.36. The necessary condition follows from 7.5.9. Now, let the matrix H fulfil the Sylvester criterion. Prove by induction that the leading principal submatrix H_k is positive definite.

For $k = 1$ it is obvious. Further if H_k is positive definite, then the eigenvalues $\mu_1 \geqslant \ldots \geqslant \mu_k$ of this matrix are positive. It follows from 7.4.35 that at least $\lambda_1, \ldots, \lambda_k$ of the eigenvalues $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant \lambda_{k+1}$ of the submatrix H_{k+1} are positive. But in view of the condition $\det H_{k+1} > 0$, λ_{k+1} is also positive, hence H_{k+1} is positive definite.

7.5.39. The matrix is not positive semidefinite. 7.5.40. The matrix is not positive semidefinite. 7.5.41. The matrix is positive definite. 7.5.42. For any $\varepsilon > 0$ the matrix $H + \varepsilon E$ satisfies condition (7.5.2), therefore H is, at least, positive semidefinite. However, it can be shown that the determinant of the matrix H is positive by evaluating it using the iterative formulae which relate the principal minors in the lower right-hand corner. Therefore H is positive definite.

7.5.43. The matrix is positive semidefinite. 7.5.44. The matrix is positive semidefinite.

7.5.46. $\frac{1}{\sqrt{2}}\begin{Vmatrix} 3 & -1 \\ -1 & 3 \end{Vmatrix}$. 7.5.47. $\frac{1}{3}\begin{Vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{Vmatrix}$.

7.5.48. $\frac{1}{3}\begin{Vmatrix} 14 & 2 & -4 \\ 2 & 17 & 2 \\ -4 & 2 & 14 \end{Vmatrix}$. 7.5.49. $\frac{1}{2}\begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{Vmatrix}$.

7.5.53. It follows from $H \geqslant S$ that $S^{-1/2}HS^{-1/2} \geqslant E$, where $S^{-1/2} = (S^{1/2})^{-1}$. Then, according to 7.5.33, $S^{1/2}H^{-1}S^{1/2} \leqslant E$ or $H^{-1} \leqslant S^{-1}$.

7.5.60. Let x be an orthonormal eigenvector of the operator HS corresponding to the eigenvalue γ_1. Then

$$\gamma_1 = (HSx, x) = (Sx, Hx) \leqslant |Sx| \cdot |Hx| \leqslant \alpha_1\beta_1.$$

In the last part of the relation, relation (7.4.2) is used.

7.5.61. (a) It follows from $H = S + iK$ that $iS^{-1}K = S^{-1}H - E$. Since the eigenvalues of the matrix $S^{-1}H$ are positive, the eigenvalues of the matrix $iS^{-1}K$ are real and greater than -1. Note that $S^{-1}K$ is similar to the skew-symmetric matrix $S^{-1/2}KS^{-1/2}$, therefore its eigenvalues are placed symmetrically about zero. Hence the eigenvalues of $iS^{-1}K$ are less than 1;

(b) to prove the proposition it suffices to verify that $\det(S^{-1}H) \leqslant 1$. It follows from (a) that the eigenvalues of the matrix $S^{-1}H$ lie in the interval (0, 2) and are symmetric about its midpoint. The product of each symmetric pair of the eigenvalues, $1 + x$ and $1 - x$, does not exceed unity, hence the required inequality. When $\det S = \det H$, the matrix $S^{-1}H$ equals the unit matrix;

(c) it follows from (a) that $|\det(S^{-1}K)| < 1$ whence $\det K < \det S$.

7.6.1. If $\alpha_1, \ldots, \alpha_s$ are the singular values of the operator A, then (a) A^* has the same singular values; (b) αA has the singular values $|\alpha|\alpha_1, \ldots, |\alpha|\alpha_s$.

7.6.5. The singular values of A^{-1} are reciprocal to the singular values of A.

7.6.8. $n, n-1, \ldots, 2, 1, 0$. 7.6.9. $2\sqrt{3}$, $\sqrt{\frac{3}{2}}$, 0.

7.6.12. The columns of U make up an orthonormal set of the eigenvectors of the matrix AA^*, and the columns of V^* form an orthonormal set of the eigenvectors of the matrix A^*A.

7.6.16. (a) $A' = V'\Lambda U$; (b) $A^* = V^*\Lambda U^*$; (c) $A^{-1} = (PV)^*P\Lambda^{-1}P(UP)^*$, where P is the following permutation matrix:

$$P = \begin{Vmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{Vmatrix}.$$

7.6.19. The unique nonzero singular value equals

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}.$$

7.6.28. These estimates are obtained from 7.6.23 if the unit column vectors are taken as x.

7.6.31. Without loss of generality, we can assume that $\tilde{A}$ is in the upper left-hand corner of A, since this can be achieved by interchanging appropriate rows and columns which evidently leave the singular values unaltered. Let A be of the following partitioned form

$$A = \begin{Vmatrix} \tilde{A} & B \\ C & D \end{Vmatrix}.$$

Then the matrix $F = \tilde{A}\tilde{A}^* + BB^*$ is the principal submatrix of AA^*, and its eigenvalues, enumerated in descending order, do not exceed the corresponding eigenvalues of AA^*. Since BB^* is a positive semidefinite matrix, the eigenvalues of $\tilde{A}\tilde{A}^*$ also do not exceed the corresponding eigenvalues of F. Hence the required statement.

7.6.34. Let L_k^0 be a subspace of X such that

$$\sigma_k = \min_{\substack{x \neq 0 \\ x \in L_k^0}} \frac{|ABx|}{|x|}.$$

Since

$$\frac{|ABx|}{|x|} \leqslant \alpha_1 \frac{|Bx|}{|x|},$$

$$\sigma_k \leqslant \alpha_1 \cdot \min_{\substack{x \neq 0 \\ x \in L_k^0}} \frac{|Bx|}{|x|} \leqslant \alpha_1 \max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{|Bx|}{|x|} = \alpha_1 \beta_k.$$

If there is a nonzero vector x in the subspace L_k^0 such that $Bx = 0$, then $\delta_k = 0$, and the inequality $\sigma_k \leqslant \alpha_k \beta_1$ becomes evident. (Note that in this case $\beta_n = 0$ also, so that the fourth inequality is also valid.) Otherwise the subspace BL_k^0 is of dimension k and for all nonzero vectors from L_k^0:

$$\frac{|ABx|}{|x|} = \frac{|ABx|}{|Bx|} \cdot \frac{|Bx|}{|x|} \leqslant \beta_1 \frac{|A(Bx)|}{|Bx|}.$$

Hence

$$\delta_k \leqslant \beta_1 \cdot \min_{\substack{x \neq 0 \\ x \in L_k^0}} \frac{|A(Bx)|}{|Bx|} = \beta_1 \cdot \min_{\substack{y \neq 0 \\ y \in BL_k^0}} \frac{|Ay|}{|y|} \leqslant \beta_1 \cdot \max_{L_k} \min_{\substack{x \neq 0 \\ x \in L_k}} \frac{|Ax|}{|x|} = \beta_1 \alpha_k.$$

The other two inequalities are proved in a similar way.

7.6.36. All the possible products $\alpha_i \beta_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

7.6.37. $\alpha_1 = \alpha_2 = 2$, $\alpha_3 = 1$. 7.6.38. $\alpha_1 = 3$, $\alpha_2 = 2$, $\alpha_3 = 1$. 7.6.39. $\alpha_1 = \alpha_2 = 6$, $\alpha_3 = 3$. 7.6.40. $\alpha_1 = 9$, $\alpha_2 = \alpha_3 = 0$. 7.6.41. $\alpha_1 = \alpha_2 = 5$, $\alpha_3 = 3$. 7.6.42. $\alpha_1 = \alpha_2 = 2\sqrt{2}$, $\alpha_3 = \sqrt{2}$, $\alpha_4 = 0$. 7.6.43. $\alpha_1 = \alpha_2 = 3$, $\alpha_3 = \alpha_4 = 1$. 7.6.44. $\alpha_1 = 4$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$. 7.6.45. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2$. 7.6.46. $\alpha_1 = \alpha_2 = 2\sqrt{10}$, $\alpha_3 = \alpha_4 = \sqrt{10}$.

7.6.47. When $n = 1$ we obtain the trigonometric form of a complex number.

7.6.48. $H = (A^*A)^{1/2}$.

7.6.49. It follows from the polar representation $A = HU$ that $AA^* = H^2$, $A^*A = U^*H^2U$. Let $A^*Ae_i = U^*H^2Ue_i = \alpha_i^2 e_i$. Then $H^2(Ue_i) = \alpha_i^2 (Ue_i)$. Q.E.D.

7.6.53. $H_1 = (A^*A)^{1/2}$.

7.6.54. If H and U commute, then $A^*A = AA^* = H^2$, and the operator A is normal. Conversely let A be a normal operator, i.e. $A^*A = AA^*$, and let

$e_1, \ldots, e_n$ be the orthonormal basis of the eigenvectors of the operator AA^*. The same vectors $e_1, \ldots, e_n$ are also the eigenvectors of the operator H, since $AA^* = H^2$, therefore

$$(UH)\, e_i = U\,(He_i) = \alpha_i U e_i, \quad i = 1, \ldots, n. \qquad (\alpha)$$

On the other hand, it follows from 7.6.49 that

$$H^2\,(Ue_i) = \alpha_i^2 U e_i, \quad i = 1, \ldots, n$$

or

$$(HU)\, e_i = H(Ue_i) = \alpha_i U e_i, \quad i = 1, \ldots, n. \qquad (\beta)$$

The relations (α) and (β) show that $UH = HU$.

7.6.56. $H = -S$, $U = -E$.

7.6.57. If the matrix of the differential operator is considered with respect to the basis $1, t, t^2, \ldots, t^n$, then the operator H has a diagonal matrix having elements $1, 2, 3, \ldots, n, 0$ with respect to the same basis, and the operator U has the matrix

$$\begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ \pm 1 & 0 & 0 & \ldots & 0 \end{Vmatrix}.$$

Thus, U is either a cyclic permutation operator: $1 \to t^n$, $t \to 1$, $t^2 \to t, \ldots, \ldots, t^n \to t^{n-1}$, or an operator both of cyclic permutation and of reflection: $1 \to -t^n$.

7.6.58. $A_p = H_p U_p$.

7.6.59. $A \times B = (H \times K)\,(U \times V)$.

7.6.60. $H = \begin{Vmatrix} 5 & -5 \\ -5 & 5 \end{Vmatrix}, \qquad U = \frac{1}{5} \begin{Vmatrix} 3 & -4 \\ 4 & 3 \end{Vmatrix}.$

7.6.61. $H = \begin{Vmatrix} 2\sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & 3\sqrt{2} & 0 \\ 0 & 0 & 5 \end{Vmatrix}, \qquad U = \begin{Vmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -i & 0 & 0 \end{Vmatrix}.$

7.6.62. $H = \frac{\sqrt{10}}{2} \begin{Vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{Vmatrix}, \qquad U = \frac{1}{\sqrt{10}} \begin{Vmatrix} 1 & -1 & 2 & -2 \\ 1 & 1 & 2 & 2 \\ -2 & 2 & 1 & -1 \\ -2 & -2 & 1 & 1 \end{Vmatrix}.$

7.6.63. Let $A = P\Lambda P^{-1}$, where Λ is a diagonal matrix, and let $P = KU$ be the polar representation of the matrix P. Then

$$A = KU\Lambda U^* K^{-1} = (KU\Lambda U^* K)\,(K^{-1})^2.$$

Assuming $H = KU\Lambda U^* K$, $S = (K^{-1})^2$, we obtain the required representation.

7.6.64. Let $A = U\Lambda V$ be the singular decomposition of the matrix A. Then

$$\operatorname{tr}(AW) = \operatorname{tr}(U\Lambda VW) = \operatorname{tr}(\Lambda VWU) = \operatorname{tr}(\Lambda Z),$$

where $Z = VWU$ and together with W ranges over the whole set of unitary matrices. It is obvious that

$$|\operatorname{tr}(\Lambda Z)| \leqslant \alpha_1 + \ldots + \alpha_n,$$

and the equality occurs when, for example, $Z = E$, i.e. $W = V^*U^*$.

7.7.1. When $n = 1$ we obtain the ordinary form $z = a + ib$ of a complex number z.

7.7.2. $A = 0$. 7.7.3. (a) $A = B$; (b) $A^* = B$.

7.7.7. $A = 0$.

7.7.9. $A^* = H_1 - iH_2$.

7.7.20. The equality $|\det A| = \det H_1$ occurs if and only if $A = H_1$.

7.7.23. $S = \frac{1}{2}(A + A^*)$, $K = \frac{1}{2}(A - A^*)$.

7.7.25. A is a skew-symmetric operator.

7.8.4. If the polynomial $g(t)$ is represented in the form $g(t) = at^n + g_{n-1}(t)$, where $g_{n-1}(t)$ is a polynomial of degree $\leqslant n - 1$, then the pseudosolutions of the equation $Af = g$ are pre-images of the polynomial $g_{n-1}(t)$, i.e. all its antiderivatives. The normal pseudosolution is the antiderivative with the free term equal to zero.

7.8.5. If the plane of the pseudosolutions of the equation $Ax = b$ is given in the form $x = x_0 + N_A$, where x_0 is the normal pseudosolution, then for (a), (b), (c) the corresponding planes are: (a) $x = \frac{1}{\alpha}x_0 + N_A$; (b) $x = \alpha x_0 + N_A$; (c) $x = x_0 + N_A$.

7.8.6. Let x_0 be the normal pseudosolution of the equation $Ax = b$. Then: (a) x_0 is the normal pseudosolution of the equation $UAx = Ub$; (b) V^*x_0 is the normal pseudosolution of the equation $AVx = b$.

7.8.7. Let r be the rank of the operator A, and let the eigenvectors $e_1, \ldots, e_r$ be associated with nonzero eigenvalues $\lambda_1, \ldots, \lambda_r$. If

$$b = \alpha_1 e_1 + \ldots + \alpha_r e_r + \alpha_{r+1} e_{r+1} + \ldots + \alpha_n e_n,$$

then the pseudosolutions of the equation $Ax = b$ are vectors of the form

$$x = \frac{\alpha_1}{\lambda_1} e_1 + \ldots + \frac{\alpha_r}{\lambda_r} e_r + \beta_{r+1} e_{r+1} + \ldots + \beta_n e_n,$$

where $\beta_{r+1}, \ldots, \beta_n$ are arbitrary numbers. The normal pseudosolution is

$$x_0 = \frac{\alpha_1}{\lambda_1} e_1 + \ldots + \frac{\alpha_r}{\lambda_r} e_r.$$

7.8.10. $x_0 = (0\ 0\ 0)^T$. 7.8.11. $x_0 = (0\ 0)^T$.

7.8.12. $x_0 = \frac{3}{4}(1\ 1\ 1\ 1)^T$. 7.8.13. $x_0 = -\frac{1}{75}(1\ 2)^T$.

7.8.14. $x_0 = \frac{1}{7}(5\ 6)^T$. 7.8.15. $x_0 = \frac{1}{2}(1\ 0\ 1)^T$.

7.8.16. $x_0 = \frac{1}{2}(1\ 0\ 1)^T$. 7.8.17. $x_0 = (1\ 1\ 0)^T$.

7.8.18. $x_0 = \left(1\ \frac{1}{2}\ -\frac{1}{2}\ 1\ 1\right)^T$.

7.8.19. The null operator from Y to X.

7.8.21. The effect of the pseudoinverse operator on M_{n-1} is similar to that of the integration operator. Polynomials of the form at^n constitute the kernel of the pseudoinverse operator.

7.8.27. Let $B = (A^+)_{ef}$. Then B is an $n \times m$ matrix in which $b_{11} = 1/\alpha_1$, $b_{22} = 1/\alpha_2, \ldots, b_{rr} = 1/\alpha_r$, and all the other entries are zeroes.

7.8.32. Nonzero eigenvalues of the operators A and A^+ are reciprocal to one another.

7.8.35. $A^+ = U^*H^+ = H_1^+U_1^*$.

7.8.45. The operators A and X are reciprocally inverse on the pair of the subspaces T_{A^*} and T_A.

7.8.47. The operator X must have the same rank as A. Consequently, the subspace T_{A^*} is the image of this operator.

7.8.49. In addition to the conditions in Problem 7.8.47, the equation $(AX)^* = AX$ shows that the kernel of the operator X must be orthogonal to the subspace T_A. Thus, the image and kernel of X coincide with the image and kernel of A^*, the operators A and X being reciprocally inverse on the pair of the subspaces T_{A^*} and T_A and by 7.8.26 $X = A^+$.

In Problems 7.9.1-7.9.5, the transformation of the unknowns is not uniquely determined.

7.9.1. $y_1^2+7y_2^2+y_3^2$; $x_1=\frac{1}{\sqrt{2}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{1}{\sqrt{3}}y_3$,

$$x_2=-\frac{2}{\sqrt{6}}y_2+\frac{1}{\sqrt{3}}y_3,\quad x_3=\frac{1}{\sqrt{2}}y_1-\frac{1}{\sqrt{6}}y_2-\frac{1}{\sqrt{3}}y_3.$$

7.9.2. $-y_1^2-7y_2^2+5y_3^2$; $x_1=\frac{1}{\sqrt{6}}y_1+\frac{1}{\sqrt{3}}y_2+\frac{1}{\sqrt{2}}y_3$,

$$x_2=\frac{2}{\sqrt{6}}y_1-\frac{1}{\sqrt{3}}y_2,\quad x_3=-\frac{1}{\sqrt{6}}y_1-\frac{1}{\sqrt{3}}y_2+\frac{1}{\sqrt{2}}y_3.$$

7.9.3. $-7y_1^2+2y_2^2$; $x_1=\frac{2}{\sqrt{21}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{3}{\sqrt{14}}y_3$,

$$x_2=\frac{1}{\sqrt{21}}y_1+\frac{2}{\sqrt{6}}y_2-\frac{2}{\sqrt{14}}y_3,\quad x_3=-\frac{4}{\sqrt{21}}y_1+\frac{1}{\sqrt{6}}y_2+\frac{1}{\sqrt{14}}y_3.$$

7.9.4. $y_1^2+3y_2^2-3y_3^2-y_4^2$; $x_1=\frac{1}{\sqrt{2}}y_1+\frac{1}{\sqrt{2}}y_4$,

$$x_2=\frac{1}{\sqrt{2}}y_2+\frac{1}{\sqrt{2}}y_3,\quad x_3=\frac{1}{\sqrt{2}}y_2-\frac{1}{\sqrt{2}}y_3,\quad x_4=\frac{1}{\sqrt{2}}y_1-\frac{1}{\sqrt{2}}y_4.$$

7.9.5. $10y_1^2$; $x_1=\frac{1}{\sqrt{10}}y_1+\frac{2}{\sqrt{10}}y_2+\frac{1}{\sqrt{10}}y_3+\frac{2}{\sqrt{10}}y_4$,

$$x_2=\frac{2}{\sqrt{10}}y_1-\frac{1}{\sqrt{10}}y_2+\frac{2}{\sqrt{10}}y_3-\frac{1}{\sqrt{10}}y_4,$$

$$x_3=\frac{1}{\sqrt{10}}y_1+\frac{2}{\sqrt{10}}y_2-\frac{1}{\sqrt{10}}y_3-\frac{2}{\sqrt{10}}y_4.$$

$$x_4=\frac{2}{\sqrt{10}}y_1-\frac{1}{\sqrt{10}}y_2-\frac{2}{\sqrt{10}}y_3+\frac{1}{\sqrt{10}}y_4.$$

7.9.12. We give a proof by induction over n. When $n = 1$ the proposition is obviously true. Let it be true for $n = k$. Consider the form in $k + 1$ unknowns, and let A_{k+1} be its matrix, A_k the leading principal submatrix of order k. Since A_{k+1} and A_k are nondegenerate, by 7.4.35 the matrix A_{k+1} has either one positive or one negative eigenvalue more than the matrix A_k. In the first case, D_{k+1} has the same sign as D_k, and the sequence $1, D_1, \ldots, D_k, D_{k+1}$ has one more coincidence of sign than the sequence $1, D_1, \ldots, D_k$. In the second case, D_{k+1} has the sign opposite to that of D_k, and thus there is another sign change.

7.9.13. Since $D_{k-1} \neq 0$, $\lambda = 0$ is an eigenvalue of multiplicity one of the leading principal submatrix A_k. Let l be the number of its negative eigenvalues. Then according to 7.4.35, the number of negative eigenvalues equals l for A_{k-1}, and $l + 1$ for A_{k+1}. Hence $D_{k-1}D_{k+1} < 0$.

7.9.16. Each of the indices of inertia equals 2. 7.9.17. The positive index of inertia is 1, and negative 3.

7.9.18. The form is positive definite.

7.9.19. The form F is reduced to $F = y_1^2 + G$ where G is, however, a quadratic form in the unknowns $y_2, \ldots, y_n$.

7.9.21. For example, $y_1 = x_1 + x_2 + x_3$, $y_2 = x_2 + x_3$, $y_3 = x_3$.

7.9.22. For example, $y_1 = x_1 + x_2$, $y_2 = x_2 + x_3$, $y_3 = x_3$.

7.9.23. For example, $y_1 = x_1 - x_3 - 2x_4$, $y_2 = 2x_2 + x_3$, $y_3 = 3x_3 + 2x_4$, $y_4 = 4x_4$.

7.9.28. $S = \begin{Vmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{Vmatrix}$. 7.9.29. $S = \begin{Vmatrix} 3 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{Vmatrix}$.

7.9.30. $S = \begin{Vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$. 7.9.32. $S = \begin{Vmatrix} 1 & \sqrt{2} & & & \\ & 1 & \sqrt{2} & & 0 \\ & & \ddots & \ddots & \\ 0 & & & 1 & \sqrt{2} \\ & & & & 1 \end{Vmatrix}$.

7.9.34. $S = \begin{Vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$.

7.9.36. There are n extractions of the square root. The number of operations of multiplication and division is expressed by a polynomial in n whose higher-order term equals $n^3/6$.

7.9.37. The solution of the system $Ax = b$ is reduced to the solution of the two triangular systems of equations $S'y = b$ and $Sx = y$.

7.9.38. The solution of the two triangular systems requires $O(n^2)$ of operations of multiplication and division. Taking into account 7.9.36, we see that the square roots method is approximately twice more economical than the Gauss method.

7.9.43. For an appropriate numeration, $\lambda_i \mu_i = 1$, $i = 1, \ldots, n$.

7.9.45. The form F is positive definite. The transformation of the unknowns

$$z_1=\frac{1}{\sqrt{3}}x_1+\frac{2}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3,\quad z_2=\frac{1}{\sqrt{6}}x_1+\frac{2}{\sqrt{6}}x_2-\frac{2}{\sqrt{6}}x_3,$$

$$z_3=\frac{1}{\sqrt{2}}x_1-\sqrt{2}\,x_3$$

makes the form F normal, and the form G canonical, i.e. $5z_1^2+2z_2^2$.

7.9.46. The matrices F and G are commuting. The orthogonal transformation of the unknowns

$$y_1=\frac{1}{\sqrt{3}}x_1-\frac{1}{\sqrt{3}}x_2+\frac{1}{\sqrt{3}}x_3,\quad y_2=\frac{1}{\sqrt{2}}x_1-\frac{1}{\sqrt{2}}x_3,$$

$$y_3=\frac{1}{\sqrt{6}}x_1+\frac{2}{\sqrt{6}}x_2+\frac{1}{\sqrt{6}}x_3$$

makes the form F canonical, $3y_1^2-2y_2^2+6y_3^2$, and the form G canonical also, $(-6)y_1^2+6y_2^2$.

7.9.47. The form F is negative definite. The transformation of the unknowns

$$z_1=\frac{1}{3}x_1-\frac{4}{3}x_2+x_3,\quad z_2=\frac{2}{3}x_1-\frac{5}{3}x_2-2x_3,\quad z_3=\frac{2}{3}x_1-\frac{2}{3}x_2-3x_3$$

makes the form F normal, and the form G canonical, $(-5)z_1^2-2z_2^2+z_3^2$.

7.9.48. The form G is positive definite. The transformation of the unknowns $y_1=x_1-x_2$, $y_2=x_2-x_3$, $y_3=x_3-x_4$, $y_4=x_4$ makes the form G normal, and the form F canonical, $y_1^2+2y_2^2-y_3^2$.

7.9.49. The matrices for the forms F and G commute. The orthogonal transformation of the unknowns $y_1=\frac{1}{2}x_1+\frac{1}{2}x_2+\frac{1}{2}x_3+\frac{1}{2}x_4$, $y_2=\frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{2}x_3-\frac{1}{2}x_4$, $y_3=\frac{1}{\sqrt{2}}x_1-\frac{1}{\sqrt{2}}x_3$, $y_4=\frac{1}{\sqrt{2}}x_2-\frac{1}{\sqrt{2}}x_4$ makes the form F canonical, $5y_1^2+y_2^2-y_3^2-y_4^2$, and the form G canonical also, $y_1^2+5y_2^2+y_3^2+y_4^2$.

8.1.29. Let $\rho(x, M)=\|x-y_0\|=\|x-y_0'\|$, then

$$\rho(x, M)\leqslant\left\|x-\frac{y_0+y'}{2}\right\|\leqslant\frac{1}{2}(\|x-y_0\|+\|x-y_0'\|)=\rho(x, M).$$

Therefore

$$\left\|\frac{x-y_0}{2}+\frac{x-y_0'}{2}\right\|=\left\|\frac{x-y_0}{2}\right\|+\left\|\frac{x-y_0'}{2}\right\|.$$

According to 2.4.13,

$$x-y_0=\lambda(x-y_0')$$

where $\lambda>0$. Hence

$$\lambda=\frac{\|x-y_0\|}{\|x-y_0'\|}=1$$

and $y_0=y_0'$.

8.1.33. If the number of c does not exist, then there is such a sequence $\{x_k\}$, $x_k\in M$ such that $|F(x_k)|>k$. Single out of $\{x_k\}$ a subsequence $\{x_{k_j}\}$, convergent to a certain $x_0\in M$. Then by the continuity of the functional F, the relation $F(x_{k_j}) \to F(x_0)$ must hold, and thus is contrary to the assumption that $F(x_{k_j}) \to \infty$.

8.1.34. Put

$$C = \sup_{x \in M} |F(x)|.$$

According to 8.1.33, the number C is finite, so if, for any x from M, this bound is not reached, then the functional

$$G(x) = \frac{1}{G - |F(x)|}$$

must be continuous on M and its values bounded, which is contrary to the definition of the number C.

8.1.35. $c_1^{-1} = \max_{\substack{m(x) \leqslant 1 \\ x \neq 0}} n(x)$, $c_2 = \max_{\substack{n(x) \leqslant 1 \\ x \neq 0}} m(x)$.

8.1.36. $\| x \|_2 \leqslant \| x \|_1 \leqslant \sqrt{n} \| x \|_2$,

$\| x \|_\infty \leqslant \| x \|_1 \leqslant n \| x \|_\infty$,

$\| x \|_\infty \leqslant \| x \|_2 \leqslant \sqrt{n} \| x \|_\infty$.

8.1.37. Put c_1 equal to the least, and c_2 to the greatest singular value of the matrix P.

8.1.41. Define the scalar product on X so that L_1 and L_2 may be orthogonal. Let z_0 be boundary point of N and $\{z_k\}$, a sequence of vectors from N, convergent to z_0. If $z_k = x_k + y_k$, $x_k \in M_1$, $y_k \in M_2$ and $z_0 = x_0 + y_0$ is the expression of the vector z_0 in terms of the subspaces L_1 and L_2, then

$$| z_k - z_0 |^2 = | x_k - x_0 |^2 + | y_k - y_0 |^2,$$

whence $x_k \to x_0$ and $y_k \to y_0$. Since M_1 and M_2 are closed, $x_0 \in M_1$, $y_0 \in M_2$, $z_0 \in N$

8.1.45. The length of a vector is dual to itself with respect to the scalar product generating it.

8.1.46. $m^*(x) = \| x \|_1 = |\alpha_1| + \ldots + |\alpha_n|$.

8.1.47. The inequality (8.1.4) for the two norms $\| x \|_p$ and $\| x \|_q$ is actually the Hölder inequality.

8.1.49. It suffices to consider vectors x_0 such that $m(x_0) = 1$. Each of these vectors is a boundary point of the unit sphere determined by the norm $m(x)$. It is proved in the course of convex analysis that for any boundary point x_0 of a convex set M, there exists the so-called "supporting hyperplane" determined by the equality $\mathrm{Re}(x, y) = c$ (where y is a fixed vector) such that it possesses the property that $\mathrm{Re}(x_0, y)$ is equal to c and $\mathrm{Re}(x, y)$ is less than or equal to c for all the other x from M. Applying this theorem to the case under consideration, construct the supporting hyperplane $\mathrm{Re}(x, y) = c$ for the given vector x_0. The vector y determining this hyperplane is the one required.

8.2.2. Yes, if the operator in nondegenerate.

8.2.3. In case the operator is degenerate, the statement may be invalid.

8.2.5. Let $M_1 = M \cap T_A$. Being the intersection of closed sets, M_1 is also closed. Define the scalar products on the spaces under consideration. The complete pre-image of M (or M_1, which is the same) is the set of the planes $x + N_A$, where x ranges over the set A^+M_1. Since the operator A^+, considered only on T_A, is nondegenerate, A^+M_1 is a closed set (see 8.2.3). Now the required statement follows from 8.1.41.

8.2.15. (a) The spectral norm of a diagonal matrix is equal to the greatest modulus of the diagonal elements; (b) the spectral norm of a quasi-diagonal matrix equals the greatest spectral norm of the diagonal blocks.

8.2.17. $\sqrt{n}$. 8.2.18. $\| A \|_E^2 = \alpha_1^2 + \ldots + \alpha_n^2$.

8.2.23. The real (imaginary) part of a complex number z is the point nearest to z on the axis of reals (imaginaries).

8.2.24. This equality is similar to the formula for the modulus of a complex number, $z = x + iy$.

8.2.25. Let U be an arbitrary unitary matrix. Then

$$\| H - U \|_E^2 = \operatorname{tr}((H - U)^* (H - U)) = \operatorname{tr} H^2 + n - 2 \operatorname{Re} \operatorname{tr}(HU).$$

According to 7.6.64,

$$-\operatorname{tr} H \leqslant \operatorname{Re} \operatorname{tr}(HU) \leqslant \operatorname{tr} H,$$

the equality on the right occurring only if $U = E$, and on the left only if $U = -E$.

When H is a positive-semidefinite matrix, the statement remains valid. However, the closest and farthest unitary matrices may not be unique.

8.2.26. For a nonzero complex number $z = \rho(\cos\varphi + i\sin\varphi)$, the number $r_1 = \cos\varphi + i\sin\varphi$ is the closest, and the number $r_2 = -(\cos\varphi + i\sin\varphi)$ is the farthest point of the unit circle.

8.2.30. (a) $\| A \|_1 = \max_j \sum_{i=1}^{n} | a_{ij} |$; (b) $\| A \|_\infty = \max_i \sum_{j=1}^{n} | a_{ij} |$. The values of both the norms on a diagonal matrix D equal the greatest of the moduli of the diagonal elements d_{ii}.

8.2.33. $N(A) = M(PAP^{-1})$.

8.2.35. If $x = (\alpha_1, \ldots, \alpha_n)^T$, $y = (\beta_1, \ldots, \beta_n)^T$, then

$$\| A \|_\infty = (\max_i |\alpha_i|) \cdot \Big(\sum_{i=1}^{n} |\beta_i|\Big).$$

8.2.38. Since any matrix B of rank 1 may be represented as the product xy^* where x and y are column vectors, then with the aid of (8.1.4), we obtain

$$\max_{r_B=1} \frac{|\operatorname{tr}(AB)|}{M(B)} = \max_{x,\, y\neq 0} \frac{|\operatorname{tr}(Axy^*)|}{M(xy^*)} = \max_{x,\, y\neq 0} \frac{|(Ax, y)|}{m(x)\, m^*(y)}$$

$$\leqslant \max_{x,\, y\neq 0} \frac{m(Ax)\, m^*(y)}{m(x)\, m^*(y)} = \max_{x\neq 0} \frac{m(Ax)}{m(x)} = M(A).$$

According to 8.1.49, for a fixed vector x, a vector y can be found such that

$$|(Ax, y)| = m(Ax)\, m^*(y).$$

Selecting appropriate x, y, we can convert the above relations into equalities.

8.2.40. The proof is given by the following chain of equalities:

$$M^*(A^*) = \max_{m^*(y)=1} m^*(A^*y) = \max_{m^*(y)=1} \max_{m(x)=1} |(A^*y, x)|$$

$$= \max_{m(x)=1} \max_{m^*(y)=1} |(Ax, y)| = \max_{m(x)=1} m(Ax) = M(A).$$

Here the statement 8.1.50 is used, i.e. $m(x)$ coincides with the norm dual to $m^*(y)$.

8.2.43. Let the given norm $\| A \|$ be consistent with vector norms $m(x)$ and $n(x)$. From 8.2.42 and 8.2.39, it follows that $\| A \|$ must be subordinate both to $m(x)$ and $n(x)$, so that

$$\| A \| = \max_{x\neq 0} \frac{m(Ax)}{m(x)}, \qquad (\alpha)$$

$$\| A \| = \max_{x\neq 0} \frac{n(Ax)}{n(x)}. \qquad (\beta)$$

Suppose that there is no constant c such that $m(x) = cn(x)$ for any vector x. The norm $m(x)$ may be made less than or equal to $n(x)$ for any x by multiplying one of the norms by an appropriate number (according to 8.2.32, this leaves a subordinate norm unaltered). Moreover, $m(x_0) = n(x_0)$ for a certain vector x_0. Since the norms $m(x)$ and $n(x)$ are not identical by assumption, there exists a vector x_1 such that $m(x_1) < n(x_1)$. We may assume that $m(x_0) = m(x_1) = 1$. According to 8.1.49, there is a vector y such that

$$(x_0, y) = m(x_0)\, m^*(y) = m^*(y).$$

The vector y can also be normed by the condition $m^*(y) = 1$. Now, for the matrix $A = x_1 y^*$, we have

$$Ax_0 = x_1 y^* x_0 = (x_0, y)\, x_1 = x_1,$$

$$\| A \| = m(x_1)\, m^*(y^*) = 1.$$

However, if the representation (β) is used for the evaluation of $\| A \|$, then

$$\| A \| \geqslant \frac{n(Ax_0)}{n(x_0)} = n(x_1) > m(x_1) = 1.$$

This contradiction shows that the norms $m(x)$ and $n(x)$ must be proportional.

8.2.45. If $\| A \|$ is consistent with another norm $n(x)$ and $N(A)$ is the corresponding subordinate norm, then $M(A) \geqslant N(A)$ on the set of matrices of rank 1. Using representation (8.2.5), we obtain that $M(A) = N(A)$ for all A, whence (see 8.2.43) it follows that the norms $m(x)$ and $n(x)$ are proportional.

8.2.47. No. For example, the norm

$$M(A) = \max \{\| A \|_1, \| A \|_\infty\}$$

satisfies the conditions of the problem, but is consistent with the two non-proportional norms $\| x \|_1$ and $\| x \|_\infty$ and so it cannot be subordinate.

8.3.3. $\operatorname{cond}_\infty (A) \geqslant \frac{3}{2} e^{-1}$.

8.3.5. It follows from 7.6.33 that if $\| B \|_2 < \alpha_n$, then the matrix $A + B$ is nondegenerate. Construct now a matrix B such that $\| B \|_2 = \alpha_n$ and $A + B$ is a degenerate matrix. Let $A = U \Lambda V$ be the singular value decomposition of the matrix A; as usual $\lambda_{11} \geqslant \lambda_{22} \geqslant \ldots \geqslant \lambda_{nn}$, and $\lambda_{nn} = \alpha_n$. Then the matrix B is of the form: $B = U \widetilde{\Lambda} V$, where $\widetilde{\lambda}_{11} = \ldots = \widetilde{\lambda}_{n-1,\, n-1} = 0, \widetilde{\lambda}_{nn} = -\alpha_n$.

8.3.9. Let A be degenerate and $Ax = 0$ for a nonzero vector x. Partition the vector x in accordance with the partitioning of the matrix A:

$$x = \begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{Vmatrix}.$$

Assume that $\| x_i \| = \max \{ \| x_1 \|, \| x_2 \|, \ldots, \| x_k \| \}$. Then from the equality

$$-A_{ii} x_i = A_{i1} x_1 + \ldots + A_{i,i-1} x_{i-1} + A_{i,\,i+1} x_{i+1} + \ldots + A_{ik} x_k,$$

we obtain

$$\|x_i\| = \Big\|A_{ii}^{-1}\sum_{\substack{j=1\\ j\neq i}}^{k} A_{ij}x_j\Big\| \leqslant \|A_{ii}^{-1}\| \sum_{\substack{j=1\\ j\neq i}}^{k} \|A_{ij}\|\,\|x_j\|$$

$$\leqslant \Big(\|A_{ii}^{-1}\| \sum_{\substack{j=1\\ j\neq i}}^{k} \|A_{ij}\|\Big)\|x_i\| < \|x_i\|.$$

This contradiction proves that A is nondegenerate.

When $m = 1$ a criterion of diagonal dominance with respect to the rows is obtained.

8.3.10. The matrix A is nondegenerate.

8.3.12. If D is a diagonal matrix made up of the diagonal elements of the matrix A, then

$$\mathrm{cond}_\infty(D)\frac{1}{1+\alpha} \leqslant \mathrm{cond}_\infty(A) \leqslant \mathrm{cond}_\infty(D)\frac{1+\alpha}{1-\alpha}.$$

8.3.13. Using the inequalities derived in 8.3.12, we obtain

$$0.9n \leqslant \mathrm{cond}_\infty(A) \leqslant 1.25n.$$

8.3.14. The maximum condition number is reached for the matrix

$$R_0 = \begin{Vmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & 1 & -1 & \dots & -1 \\ 0 & 0 & 1 & \dots & -1 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{Vmatrix},$$

for which

$$R_0^{-1} = \begin{Vmatrix} 1 & 1 & 2 & 4 & \dots & 2^{n-3} & 2^{n-2} \\ 0 & 1 & 1 & 2 & \dots & 2^{n-4} & 2^{n-3} \\ 0 & 0 & 1 & 1 & \dots & 2^{n-5} & 2^{n-4} \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{Vmatrix}.$$

Therefore $\mathrm{cond}_\infty(R_0) = n2^{n-1}$.

8.3.15. Since $\|A_k\| = 1$, the elements of all the matrices A_k are bounded in absolute value. All minors of order $n-1$ of these matrices are therefore also bounded. Hence an increase in the condition number is possible only if $\det A_k$ tends to zero.

8.3.18. If $|\lambda_1| \geqslant |\lambda_2| \geqslant \dots \geqslant |\lambda_n|$, then

$$\mathrm{cond}_2(A) = \frac{|\lambda_1|}{|\lambda_n|}.$$

8.3.19. $\mathrm{cond}_2(A) = \dfrac{\alpha_1}{\alpha_n}$.

8.3.24. $\mathrm{cond}_E(A) = \dfrac{\|A\|_E^2}{|\det A|} = \dfrac{|a_{11}|^2+|a_{12}|^2+|a_{21}|^2+|a_{22}|^2}{|a_{11}a_{22}-a_{12}a_{21}|}$.

8.3.29. $\mathrm{cond}_2(A) = \mathrm{cond}_2^2(S)$.

8.3.30. For the original system of equations, $\text{cond}_2(A) \geqslant 1000$. The solution is: $x_1 = 1.5$, $x_2 = 1$, $x_3 = -1$.

8.3.31. The estimate $\text{cond}_2(A) > 363$ for matrix A of the original system of equations using the inequalities of Problem 7.6.28. To decrease the condition number, multiply the second equation of the system by 10, and the third by 100, then substitute the variables: $y_1 = x_1$, $y_2 = 10x_2$, $y_3 = 100x_3$. A system with a symmetric matrix will be obtained, its solution being $y_1 = -1$, $y_2 = -1$, $y_3 = -1$. Therefore the solution of the original system is: $x_1 = -1$, $x_2 = -0.1$, $x_3 = -0.01$.

8.3.33. The components of the solution can be changed by 6.01. The solution of the original system is: $x = -1$, $y = 0$. The solution of the perturbed system is: $\widetilde{x} = 1$, $\widetilde{y} = 1$.

8.3.34. $\text{cond}_\infty(A) = 10\,967$. The solution of the original sistem is: $x=1$, $y=1$. The solution of the perturbed system is: $\widetilde{x} = -12.9$, $\widetilde{y} = -20$. The perturbation of the system: $x - \widetilde{x} = 13.9$, $y - \widetilde{y} = 21$.

8.3.35. For example, $x_1 = 2$, $x_2 = -1$, $x_3 = 2$, $x_4 = 1$.

8.3.36. For example, $x_1 = 1$, $x_2 = x_3 = x_4 = 0$.

8.4.2. For example, the circle $|z| \leqslant \sqrt{5} + \sqrt{2}$.

8.4.6. This inequality determines an interval within which the eigenvalues are located, and the bisection method may be started with it.

8.4.8. Let $P^{-1}A_0P = \Lambda$, where Λ is a diagonal matrix made up of the eigenvalues of the matrix A_0. Any of the norms $\| P^{-1}AP \|_{1,2,\infty}$ (see 8.2.10, (c)) may be taken as the required norm $\| A \|$.

8.4.13. Let $A = H_1 + iH_2$ be the Hermitian decomposition, and $B = U^*AU$ the Schur form for the matrix A. Then the Hermitian decomposition of the matrix B is:

$$B = U^*H_1U + iU^*H_2U = \widetilde{H}_1 + i\widetilde{H}_2.$$

The principal diagonal of the matrices $\widetilde{H}_1$ and $\widetilde{H}_2$ contains the numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively. Therefore $\sum_{i=1}^{n} \alpha_i^2 \leqslant \| U^*H_1U \|_E^2 = \| H_1 \|_E^2 = \frac{1}{4} \| A + A^* \|_E^2$. The numbers $\beta_1, \ldots, \beta_n$ satisfy a similar relation.

8.4.14. As regards the relations (8.4.3), the equality $4 \sum_{i=1}^{n} \alpha_i^2 = \| A + A^* \|_E^2$ means (see the solution to 8.4.13) that $\widetilde{H}_1$ is a diagonal matrix. Since $\widetilde{H}_1 = \frac{1}{2}(B + B^*)$ and B is a triangular matrix, it follows from the off-diagonal elements of $\widetilde{H}_1$ being zero that the same holds for B. Hence A is a normal matrix.

8.4.15. For matrices A of simple structure.

8.4.16. According to 6.2.7, the matrices AB and BA have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. Since AB is a normal matrix,

$$\| AB \|_E^2 = \sum_{i=1}^{n} \| \lambda_i \|^2.$$

Show that $\| BA \|_E = \| AB \|_E$, whence (due to 8.4.14) the normality of the matrix BA follows. Really,

$$\| BA \|_E^2 = \text{tr}\,(BA\,(BA)^*) = \text{tr}\,(BAA^*B^*) = \text{tr}\,(AA^*B^*B)$$
$$= \text{tr}\,(A^*ABB^*) = \text{tr}\,(B^*A^*AB) = \text{tr}\,((AB)^*AB) = \| AB \|_E^2.$$

Here both the normality of the matrices A and B and the equality $\text{tr}\,(XY) = \text{tr}\,(YX)$ are used.

8.4.18. In 7.6.64, it was derived that $\alpha_1 + \ldots + \alpha_n = \max_W |\operatorname{tr}(AW)|$, where W is an arbitrary unitary matrix. Let $B = U^*AU$ be the Schur form of the matrix A. Then $\operatorname{tr}(AW) = \operatorname{tr}(UBU^*W) = \operatorname{tr}(BU^*WU)$. Determine W_0 from the relation $U^*W_0U = D$, where D is a diagonal unitary matrix such that $b_{ii}d_{ii} = |b_{ii}| = |\lambda_i|$. For the matrix W_0 (which is not unique if there is a zero among the λ_i), $\operatorname{tr}(AW_0) = |\lambda_1| + \ldots + |\lambda_n|$, whence the required inequality.

8.4.19. The statement follows from 8.2.13, 8.2.27 and 8.4.18.

8.4.20. If

$$|z - a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n,$$

then the matrix $zE - A$ is diagonally dominant and therefore nondegenerate. Hence z may not be an eigenvalue of the matrix A.

8.4.21. This region consists of the three discs: $|z - 1.23| \leqslant 0.07$, $|z - 2.17| \leqslant 0.04$, $|z - 3.06| \leqslant 0.06$.

8.4.23. This region consists of the three discs: $|z - \lambda_i| \leqslant 0.012$, $i = 1, 2, 3$, where $\lambda_1 = 0.5$, $\lambda_2 = 1$, $\lambda_3 = 2.5$.

8.4.24. For example, the region consisting of the three discs: $|z - \lambda_i| \leqslant 45e$, $i = 1, 2, 3$, where $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$.

8.4.27. For example, $\tilde{\lambda}_1 = -0.5$, $\tilde{\lambda}_2 = -1$, $\tilde{\lambda}_3 = 0.5$, $\tilde{\lambda}_4 = 1$.

8.4.29. For example, $\tilde{\lambda}_1 = \tilde{\lambda}_2 = -1$, $\tilde{\lambda}_3 = 1$, $\tilde{\lambda}_4 = 3$.

8.4.32. To prove (a), replace the elements a_{12}, a_{21} and a_{18}, a_{81} by zeroes. The spectral norm of the corresponding perturbation matrix equals $\sqrt{2}/N$, whence (a) follows.

To prove (b), consider the matrix A as a perturbation of the quasi-diagonal matrix D with the diagonal blocks

$$D_{11} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad D_{22} = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix}, \quad D_{33} = \begin{Vmatrix} -0.5 & 0.1 & -0.2 \\ 0.1 & -1 & 0 \\ -0.2 & 0 & 2 \end{Vmatrix}.$$

For the perturbation matrix $B = A - D$ $\|B\|_2 < \|B\|_\infty = 3/N$. Therefore, at least three eigenvalues of the matrix lie in the interval

$$-3/N \leqslant \lambda - 1 \leqslant 3/N. \qquad (\alpha)$$

To show that there are precisely three eigenvalues, prove that when $N \geqslant 10$, the interval (α) does not intersect the other intervals of the system $|x - \lambda_i| \leqslant 3/N$, $i = 1, \ldots, n$, λ_i being the eigenvalues of the matrix D.

This is clear for the interval $|x - 2| \leqslant 3/N \leqslant 0.3$. Note now that according to the Gershgorin theorem, the eigenvalues λ_6, λ_7, λ_8 of the matrix D_{33} lie in the intervals $[-1.3, -0.7]$, $[-0.8, -0.2]$, and $[1.7, 2.3]$. Hence when $N \geqslant 10$ the intervals $|x - \lambda_i| \leqslant 3/N \leqslant 0.3$, $i = 6, 7, 8$ remain separate from the interval (α).

8.4.36. The vector $r(x)\,x$ is the projection of the vector Ax on $L(x)$.

8.4.38. Since $|\alpha|^2 + \|z\|_2^2 = 1$, $\mu_0 = \mu_0|\alpha|^2 + \mu_0\|z\|_2^2$. On the other hand, $\mu_0 = (A\tilde{x}, \tilde{x}) = \lambda_1|\alpha|^2 + (Az, z)$. Hence $|\lambda_1 - \mu_0|\,|\alpha|^2 = |(Az, z) - \mu_0\|z\|_2^2| \leqslant e^2/a$. Since $|\alpha| \geqslant \sqrt{1 - e^2/a^2}$, the required estimate is proved.

8.4.39. (a) For example, $\tilde{\lambda}_1 = 1$, $\tilde{\lambda}_2 = 2$, $\tilde{\lambda}_3 = 3$, $\tilde{\lambda}_4 = 4$; (b) the unit column vectors.

8.4.42. (a) If $Z = X^{-1}$ and z_i is the i-th row of the matrix Z, then $z_i^* = y_i$ is the eigenvalue of the matrix A^* associated with the eigenvalue λ_i. Moreover, it follows from the matrix equality $XZ = E$ that $(x_i, y_i) = 1$ and $|s_i|^{-1} = \| x_i \|_2 \| y_i \|_2$. Now, from 7.6.28 deduce that $\text{cond}_2(X) = \| X \|_2 \| X^{-1} \|_2 \geqslant \| x_i \|_2 \| y_i \|_2 = 1/|s_i|$;

(b) select vectors x_i such that $\| x_i \|_2 = 1/\sqrt{|s_i|}$. Then for the rows z_i of the matrix X^{-1}, we obtain $\| z_i \|_2 = 1/\sqrt{|s_i|}$. Therefore

$$\text{cond}_E(X) = \| X \|_E \| X^{-1} \|_E = \| X \|_E^2 = \sum_{i=1}^{n} \frac{1}{|s_i|}.$$

8.4.44. Without loss of generality, the vectors x and y may be assumed to be normed. Let $C = Q^*AQ$ be an upper Schur form of the matrix A selected so that $c_{11} = \lambda_1$. According to 7.1.47, such a form may be constructed in which the vector x may be chosen as the first column of the matrix Q. Then the vector $z = Q^*y$ is an eigenvector of C^*, and $(e_1, z) = (Q^*x, Q^*y) = (x, y) = 0$. Thus, the first component of the vector z equals zero, and the required statement follows from 8.4.43.

8.4.45. From the condition $C^*y = \bar{\lambda}_1 y$, we obtain that $\varepsilon c^* + C_{n-1}^* z = \bar{\lambda}_1 z$ or

$$C_{n-1}^* z + \varepsilon c^* \frac{z^* z}{1 - |\varepsilon|^2} = \left(C_{n-1}^* + \frac{\varepsilon}{1 - |\varepsilon|^2} c^* z^* \right) z = \bar{\lambda}_1 z.$$

Hence the statement of the problem follows.

Index

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