On the Theory of Groups.

By Prof. Cayley.

I refer to my papers on the theory of groups as depending on the symbolic equation $\theta^n = 1$, Phil. Mag., vol. VII (1854), pp. 40–47 and 408–409; also vol. XVIII (1859), pp. 34–37; and "On the Theory of Groups," *Amer. Journ. of Math.*, vol. I (1878), pp. 50–52, and "The Theory of Groups: Graphical Representation," id., pp. 174–176; also to Mr. Kempe's "Memoir on the Theory of Mathematical Form," Phil. Trans., vol. 177 (1886), pp. 1–70, see the section "Groups containing from one to twelve units," pp. 37–43, with the diagrams given therein. Mr. Kempe's paper has recalled my attention to the method of graphical representation explained in the second of the two papers of 1878, and has led me to consider, in place of a diagram as there given for the independent substitutions, a diagram such as those of his paper, for all the substitutions. I call this a colourgroup; viz. for the representation of a substitution-group of s substitutions upon the same number of letters, or say of the order s, we employ a figure of s points (in space or in a plane) connected together by coloured lines, and called a colourgroup.

I remark that up to $s = 11$, the first case of any difficulty is that of $s = 8$, and that the 5 groups of this order were determined in my papers of 1854 and 1859. For the order 12, Mr. Kempe has five groups, but one of these is non-existent, and there is a group omitted; the number is thus $= 5$.

The colourgroup consists of s points joined in pairs by $\frac{1}{2}s(s-1)$ coloured lines under prescribed conditions. A line joining two points is in general regarded as a vector drawn *from* one *to* the other of the two points; the currency is shown by an arrow, and in speaking of a line ab we mean the line from a to b. But we may have a line regarded as a double line, drawn from each to the other of the two points; the arrow is then omitted, and in speaking of such a line ab we mean the line from b to a and from a to b. A fresh condition is that for a given colour there shall be one and only one line *from* each of the points, and one and only one line *to* each of the points. We may have through two points a, b only the line ab of the given colour; this is then a double line regarded as drawn from a to b and from b to a; and there is thus one and only one line of the colour from each of these points and to each of these points. The condition implies that the lines of a given colour form either a single polygon or a set of polygons, with a continuous currency round each polygon; for instance, there may be a pentagon $abcde$, meaning thereby the pentagon formed by the lines drawn from a to b, from b to c, from c to d, from d to e, and from e to a. An arrow on one of the sides is sufficient to indicate the currency. In the case of a double line we have a polygon of two points, or say a digon.

There is a further condition which, after the necessary explanation of the meaning of the terms, may be concisely expressed as follows: Each route must be of independent effect, and (as will readily be seen) this implies that the lines of a given color must form either a single polygon or else two or more polygons each of the same number of points: thus if $s = ks_1$, they may form k s_1-gons; in particular, if s be even, they may form $\frac{1}{2}s$ digons.

To explain the foregoing statement, first as to the term "route." I denote the several colours by capital letters, $R =$ red, $G =$ green, $B =$ blue, etc. Any capital or combination of capitals determines a route; R means go along a red line; $RRBG$, go along a red line, a red line, a blue line, a green line, and so in other cases. Given the starting point, or initial, the route determines the several points passed through, and the point arrived at, or terminal, thus $aRRBG = abefk, = k$, means that the route $RRBG$ leads from a through b, e, f to k, viz. that the red line from a leads to b, the red line from b leads to e, the blue line from e leads to f, and the green line from f leads to k. We may give in this way the Itinerary, or write simply $aRRBG = k$, meaning that the route leads from a to k. We may of course write R^2 for RR, and so in other cases. A single capital, as already mentioned, is a route, but it may for distinction be called a stage. A stage, and thence also a route, may be *reversed;* R^{-1} means go along the red line drawn to the point; if $aR = b$, then $bR^{-1} = a$; and so if $aRRBG = abefk, = k$, then $kG^{-1}B^{-1}R^{-1}R^{-1} = kfeba, = a$; $R^{-1}R^{-1} = R^{-2}$, and so in other cases.

The effect of a route depends in general on the initial point: thus, a route may lead from a point a to itself, or say it may be a circuit from a; and it may not be a circuit from another point b. And similarly two different routes each leading from a point a, to one and the same point x, or say two routes equivalent for the initial point a, may not be equivalent for a different initial point b. Thus we cannot in general say simpliciter that a route is a circuit, or that two different routes are equivalent. But the figure may be such as to render either of these locutions, and if either, then each of them, admissible. For it is easy to see that if every route which is a circuit from any one initial point is also a circuit from every other initial point, then two routes which are equivalent for any one initial point will be equivalent for every other initial point. And conversely, if in every case where two different routes are equivalent for any one initial point, they are equivalent for every other initial point, then every route which is a circuit from any one initial point is a circuit from every other initial point; and we express this by saying that every route is of independent effect: this explains the meaning of the foregoing statement of the condition which is to be satisfied by a colourgroup.

It is at once evident that a colourgroup, quâ figure where each route is of independent effect, furnishes a graphical representation of the substitution-group and gives the square by which we define such group. For in the colourgroup of s points we have the route from a point to itself and the routes to each of the other $(s-1)$ points, in all s non-equivalent routes; and if starting from a given arrangement, say $abcd\ldots.$, of the s points, we go by one of these routes from the several points $a, b, c, d, \ldots.$ successively, we obtain a different arrangement of these points. Observe that this is so; the same point cannot occur twice, for if it did, there would be a route leading from two different points b, f to one and the same point x, or the reverse route from x would lead to two different points b, f. The route from a point to itself which leaves each point unaltered, and thus gives the primitive arrangement $abcd\ldots.$, may be called the route 1. Taking this route and the other $(s-1)$ routes successively, we obtain s different arrangements of the points, or say a square, each line of which is a different arrangement of the points. And not only are the arrangements different, but we cannot have the same point twice in any column, for this would mean that there were two different routes leading from a point to one and the same point x; hence each column of the square will be an arrangement of the s points. We have thus the substitution-group of the s points or letters; the s routes, or say the route 1 and the other $(s-1)$ routes, are the substitutions of the group.

The complete figure is called the colourgroup. As already mentioned, the lines of any colour form either a single polygon or two or more polygons each of the same number of points. The number of lines of a given color is thus $= s$, or when the polygons are digons (which implies s even), the number is $= \frac{1}{2} s$. The number of colours is thus $= \frac{1}{2}(s-1)$ at least, and $= (s-1)$ at most. A general description of the figure may be given as in the annexed Table. Thus for the group $6B$ we have

$$\begin{array}{r} R.\ 2\ 3\text{gons} = 6 \\ B, G, Y.\ (3\ 2\text{gons})^3 = \underline{9} \\ \underline{\underline{15}} \end{array}$$

; we have the red lines forming two trigons, 6 lines, and the blue, green and yellow lines each forming three digons, together $3 \times 3, = 9$ lines, in all $15, = \frac{1}{2} 6.5$ lines. Such description, however, does not indicate the currencies, and it is thus insufficient for the determination of the figure. But the figure is completely determined by means of the substitutions as given in the outside column of the square, thus $R = (abc)(dfe)$ shows that the red lines form the two triangles abc, dfe with these currencies, $G = (ad)(be)(cf)$, that the green lines form the three digons ad, be, cf, and so for the other two colours B and Y.

The lines of a colour may be spoken of as a colour, and the lines of a colour or of two or more colours as a colourset. The colourset either does not connect together all the points, and it is then a broken set; or it does connect together all the points, and it is then a bondset. A bondset not containing any superfluous colour is termed a bond, viz. a bond is a colourset which connects together all the points, but which is moreover such that if any one of the colours be omitted it becomes a broken set. The word colour is used as a prefix, colourset as above, colourbond, etc., and so also with a numeral, a twocolourbond is a bond with two colours, and so in other cases. Observe that we may very well have for instance a threecolourbond, and also a twocolour or a onecolourbond, only the colours or colour hereof must not be included among those of the threecolourbond, for this would then contain a superfluous colour or colours and would not be a bond.

A colourgroup may contain a onecolourbond, viz. this is the case when all the points form a single polygon; it is then said to be unibasic. If it contains no onecolourbond but contains a twocolourbond, it is bibasic; if it contains no onecolourbond or twocolourbond but contains a threecolourbond, it is tribasic, and so on. In all cases the number of bonds (onecolour-, twocolour-, etc.) may very well be and in general is greater than one; thus a unibasic colourgroup will in general contain several onecolourbonds, a bibasic colourgroup several two-colourbonds, and so on.

The bond of the proper number of colours completely determines the colour-group; in fact the colourbond gives the route from any one point to each of the other $(s-1)$ points; that is, it determines all the s routes, and consequently the colourgroup. The only type of onecolourbond is the polygon of the s points; we have thus for any value whatever of s a unibasic colourgroup which may be called sA. The theory is well known. If s be a prime number, the number of colours is $=\frac{1}{2}(s-1)$, each colour gives a polygon through the s points, so that we have here only onecolourbonds; but in other cases we have broken sets, and there will be in general (but not for all such values of s) twocolourbonds. Observe, moreover, that for s a prime number the only colourgroup is the foregoing unibasic group sA. I have just employed, and shall again do so, the word type; the sense in which it is used does not, I think, require explanation.

Passing next to the bibasic colourgroups sB: there will be in general for a given composite value of s several of these, and in the absence of a more complete classification they may be called $sB1$, $sB2$, etc. In regard hereto observe that supposing for a given value of s that we know all the different types of two-colourbond, each one of these gives rise to a group, but this is not in every case a group sB; any twocolourbond contained in the corresponding group sA would give rise to the group sA which contained it, and not to a group sB. We have thus in the first instance to reject those twocolourbonds which are contained in the group sA. But attending only to the remaining twocolourbonds, these give rise each of them to a group sB, but the groups thus obtained are not in every case distinct groups. For looking at the converse question, suppose that for a given value of s we know the group sA and also the several groups sB. In any one of these groups, combining in pairs the several colours hereof RG, RY, GY, etc., we ascertain how many of these combinations are distinct types of twocolourbond, and in this manner reproduce the whole series of types of two-colourbond, not in general singly, but in sets, those which arise from sA, those which arise from $sB1$, those which arise from $sB2$, etc.; and we thus have (it may be) several types of twocolourbond each leading to the unibasic group

ϑA, several types each leading to the bibasic group $\vartheta B1$, several each leading to $\vartheta B2$, and so on.

The like considerations would apply to the tribasic colourgroups ϑC. Supposing that we had for a given value of ϑ the several distinct types of three-colourbond, it would be necessary first to exclude from consideration those which give rise to a unibasic group ϑA or a bibasic group ϑB, and then to consider what sets out of the remaining types give rise to distinct tribasic groups ϑC. But in the table we have only one case $8C$ of a tribasic group.

I give now a table of the several groups $\vartheta = 2$ to 12, viz. these are as above: A, unibasic; B, bibasic; C, tribasic; the several groups being

$2A$,	$3A$,	$4A$,	$5A$,	$6A$,	$7A$,	$8A$,	$9A$,	$10A$,	$11A$,	$12A$,
		$4B$,		$6B$,		$8B1$,	$9B$,	$10B$,		$12B1$,
						$8B2$,				$12B2$,
						$8B3$,				$12B3$,
						$8C$,				$12B4$, .

in all 23 groups.

Table of the Groups 2 to 12.

$2A$

a	b
b	a

1 colour.

$1 \quad = 1$

$R = (ab) = R$

$R.$ 1 digon $\dfrac{1}{1}$

$3A$

a	b	c
b	c	a
c	a	b

1 colour.

$1 \ = 1 \quad = 1$

$R = (abc) = R$

$R^2 = (acb) = R^{-1}$

$R.$ 1 3gon $\dfrac{3}{3}$

$4A$

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

2 colours.

$1 \ = 1 \quad = 1$

$R = (abcd) \quad = R$

$R^2 = (ac)(bd) = G$

$R^3 = (adcb) \quad = R^{-1}$

$R.$ 1 4gon 4
$G.$ 2 digons 2
$\overline{\ 6\ }$

4B

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

3 colours.

$1 = 1 = 1$

$R = (ab)(cd) = R$

$G = (ac)(bd) = G$

$RG = (ad)(bc) = Y$

$R, G, Y.$ (2 digons)^3 6

6

5A

a	b	c	d	e
b	c	d	e	a
c	d	e	a	b
d	e	a	b	c
e	a	b	c	d

2 colours.

$1 = 1 = 1$

$R = (abcde) = R$

$R^2 = (acebd) = G$

$R^3 = (adbec) = G^{-1}$

$R^4 = (aedcb) = R^{-1}$

$R, G.$ (1 5gon)^2 10

10

6A

a	b	c	d	e	f
b	c	d	e	f	a
c	d	e	f	a	b
d	e	f	a	b	c
e	f	a	b	c	d
f	a	b	c	d	e

3 colours.

$1 = 1 = 1$

$R = (abcdef) = R$

$R^2 = (ace)(bdf) = G$

$R^3 = (ad)(be)(cf) = Y$

$R^4 = (aec)(bfd) = G^{-1}$

$R^5 = (afedcb) = Y^{-1}$

$R.$	1 6gon	6
$G.$	2 3gons	6
$Y.$	3 digons	3
		15

6B

a	b	c	d	e	f
b	c	a	f	d	e
c	a	b	e	f	d
d	e	f	a	b	c
e	f	d	c	a	b
f	d	e	b	c	a

4 colours.

$1 = 1 = 1$

$R = (abc)(dfe) = R$

$R^2 = (acb)(def) = R^{-1}$

$G = (ad)(be)(cf) = G$

$RG = (ae)(bf)(cd) = Y$

$R^2G = (af)(bd)(ce) = B$

$R.$	2 3gons	6
$G, Y, B.$	(3 digons)^3	9
		15

7A

a	b	c	d	e	f	g	$1 \;\; = 1$	3 colours. $= 1$
b	c	d	e	f	g	a	$R \;\; = (abcdefg)$	$= R$
c	d	e	f	g	a	b	$R^2 = (acegbdf)$	$= G$
d	e	f	g	a	b	c	$R^3 = (adgcfbe)$	$= Y$
e	f	g	a	b	c	d	$R^4 = (aebfcgd)$	$= Y^{-1}$
f	g	a	b	c	d	e	$R^5 = (afdbgec)$	$= G^{-1}$
g	a	b	c	d	e	f	$R^6 = (agfedcb)$	$= R^{-1}$

R, G, Y. $(1 \text{ 7gon})^3$ 21

21

8A

a	b	c	d	e	f	g	h	$1 \;\; = 1$	4 colours. $= 1$
b	c	d	e	f	g	h	a	$R \;\; = (abcdefgh)$	$= R$
c	d	e	f	g	h	a	b	$R^2 = (aceg)(bdef)$	$= Y$
d	e	f	g	h	a	b	c	$R^3 = (adgbehcf)$	$= G$
e	f	g	h	a	b	c	d	$R^4 = (ae)(bf)(cg)(dh)$	$= B$
f	g	h	a	b	c	d	e	$R^5 = (afchebgd)$	$= G^{-1}$
g	h	a	b	c	d	e	f	$R^6 = (agec)(bhfd)$	$= Y^{-1}$
h	a	b	c	d	e	f	g	$R^7 = (ahgfedcb)$	$= R^{-1}$

R, G.	$(1 \text{ 8gon})^2$	16
Y.	2 4gons	8
B.	4 digons	4
		28

8B1

5 colours.

a	b	c	d	e	f	g	h	1	$=1$	$=1$
b	c	d	a	f	g	h	e	R	$=(abcd)(efgh)$	$=R$
c	d	a	b	g	h	e	f	R^2	$=(ac)(bd)(eg)(fh)$	$=Y$
d	a	b	c	h	e	f	g	R^3	$=(adcb)(ehgf)$	$=R^{-1}$
e	f	g	h	a	b	c	d	G	$=(ae)(bf)(cg)(dh)$	$=G$
f	g	h	e	b	c	d	a	RG	$=(afch)(bgde)$	$=I$
g	h	e	f	c	d	a	b	R^2G	$=(ag)(bh)(ce)(df)$	$=B$
h	e	f	g	d	a	b	c	R^3G	$=(ahcf)(bedg)$	$=I^{-1}$

$R, I.$	$(2\ 4\text{gons})^2$	16
$Y, G, B.$	$(4\ \text{digons})^3$	12
		28

8B2

6 colours.

a	b	c	d	e	f	g	h	1	$=$	$=1$
b	c	d	a	h	e	f	g	R	$=(abcd)(ehgf)$	$=R$
c	d	a	b	g	h	e	f	R^2	$=(ac)(bd)(eg)(fh)$	$=Y$
d	a	b	c	f	g	h	e	R^3	$=(adcb)(efgh)$	$=R^{-1}$
e	f	g	h	a	b	c	d	G	$=(ae)(bf)(cg)(dh)$	$=G$
f	g	h	e	d	a	b	c	RG	$=(af)(bg)(ch)(de)$	$=I$
g	h	e	f	c	d	a	b	R^2G	$=(ag)(bh)(ce)(df)$	$=B$
h	e	f	g	b	c	d	a	R^3G	$=(ah)(be)(cf)(dg)$	$=O$

$R.$	$2\ 4\text{gons}$	8
$Y, G, I, B, O.$	$(4\ \text{digons})^5$	20
		28

8*B*3

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	1		$=1$	4 colours.
b	*c*	*d*	*a*	*h*	*e*	*f*	*g*	R	$=(abcd)(efgh)$	$=R$	
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*	R^2	$=(ac)(bd)(ef)(gh)$	$=Y$	
d	*a*	*b*	*c*	*f*	*g*	*h*	*e*	R^3	$=(adcb)(ehgf)$	$=R^{-1}$	
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*	G	$=(aecg)(bhdf)$	$=G$	
f	*g*	*h*	*e*	*d*	*a*	*b*	*c*	R^3G	$=(afch)(bedg)$	$=B$	
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*	R^2G	$=(agce)(bfdh)$	$=G^{-1}$	
h	*e*	*f*	*g*	*b*	*c*	*d*	*a*	RG	$=(ahcf)(bgde)$	$=B^{-1}$	

$R, G, B.$	$(2\ 4\text{gons})^3$	24
$Y.$	4 digons	4
		28

8*C*

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	1	$=1$	$=1$	7 colours.
b	*a*	*d*	*c*	*f*	*e*	*h*	*g*	R	$=(ab)(cd)(ef)(gh)$	$=R$	
c	*d*	*a*	*b*	*g*	*h*	*e*	*f*	G	$=(ac)(bd)(eg)(fh)$	$=G$	
d	*c*	*b*	*a*	*h*	*g*	*f*	*e*	RG	$=(ad)(bc)(eh)(fg)$	$=B$	
e	*f*	*g*	*h*	*a*	*b*	*c*	*d*	Y	$=(ae)(bf)(cg)(dh)$	$=Y$	
f	*e*	*h*	*g*	*b*	*a*	*d*	*c*	RY	$=(af)(be)(ch)(dg)$	$=I$	
g	*h*	*e*	*f*	*c*	*d*	*a*	*b*	GY	$=(ag)(bh)(ce)(df)$	$=O$	
h	*g*	*f*	*e*	*d*	*c*	*b*	*a*	RGY	$=(ah)(bg)(cf)(de)$	$=V$	

$R, G, B, Y, I, O, V.$	$(4\ \text{digons})^7$	28
		28

9A

4 colours.

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$1 \qquad = 1$
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*a*	$R = (abcdefghi) = R$
c	*d*	*e*	*f*	*g*	*h*	*i*	*a*	*b*	$R^2 = (acegibdfh) = G$
d	*e*	*f*	*g*	*h*	*i*	*a*	*b*	*c*	$R^3 = (adg)(beh)(cfi) = Y$
e	*f*	*g*	*h*	*i*	*a*	*b*	*c*	*d*	$R^4 = (aeidhcgbf) = B$
f	*g*	*h*	*i*	*a*	*b*	*c*	*d*	*e*	$R^5 = (afbgchdie) = B^{-1}$
g	*h*	*i*	*a*	*b*	*c*	*d*	*e*	*f*	$R^6 = (agd)(bhe)(cif) = Y^{-1}$
h	*i*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	$R^7 = (ahfdbigec) = G^{-1}$
i	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	$R^8 = (aihgfedcb) = R^{-1}$

R, *G*, *B*. (1 9gon)^3 27
Y. 3 3gons 9
36

9B

4 colours.

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$1 = 1 \qquad = 1$
b	*c*	*a*	*e*	*f*	*d*	*h*	*i*	*g*	$R = (abc)(def)(ghi) = R$
c	*a*	*b*	*f*	*d*	*e*	*i*	*g*	*h*	$R^2 = (acb)(dfe)(gih) = R^{-}$
d	*e*	*f*	*g*	*h*	*i*	*a*	*b*	*c*	$G = (adg)(beh)(cfi) = G$
e	*f*	*d*	*h*	*i*	*g*	*b*	*c*	*a*	$RG = (aei)(bfg)(cdh) = B$
f	*d*	*e*	*i*	*g*	*h*	*c*	*a*	*b*	$R^2G = (afh)(bdi)(ceg) = Y$
g	*h*	*i*	*a*	*b*	*c*	*d*	*e*	*f*	$G^2 = (agd)(bhe)(cif) = G^{-1}$
h	*i*	*g*	*b*	*c*	*a*	*e*	*f*	*d*	$RG^2 = (ahf)(bid)(cge) = Y^{-1}$
i	*g*	*h*	*c*	*a*	*b*	*f*	*d*	*e*	$R^2G^2 = (aie)(bgf)(chd) = B^{-1}$

R, *G*, *B*, *Y*. (3 3gons)^4 36
36

10*A*

5 colours.

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	1	$=1$
b	*c*	*d*	*e*	*f*	*g*	*h*		*j*	*a*	$R = (abcdefghij)$	$=R$
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	$R^2 = (acegi)(bdfhj)$	$=G$
d	*e*	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	$R^3 = (adgjcfibeh)$	$=Y$
e	*f*	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	$R^4 = (aeicg)(bfdjh)$	$=B$
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$R^5 = (af)(bg)(ch)(di)(ej)$	$=O$
g	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	$R^6 = (agcie)(bhjdf)$	$=B^{-1}$
h	*i*	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	$R^7 = (ahebifcjgd)$	$=Y^{-1}$
i	*j*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	$R^8 = (aigec)(bjhfd)$	$=G^{-1}$
j	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	$R^9 = (ajihgfedcb)$	$=R^{-1}$

$R, G, Y, B.$ $(1\ 10\text{gon})^4$ 40
$O.$ 5 digons 5
45

10*B*

7 colours.

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	$1 \quad =1$	$=1$
b	*c*	*d*	*e*	*a*	*j*	*f*	*g*	*h*	*i*	$R = (abcde)(fjihg)$	$=R$
c	*d*	*e*	*a*	*b*	*i*	*j*	*f*	*g*	*h*	$R^2 = (acebd)(figjh)$	$=Y$
d	*e*	*a*	*b*	*c*	*h*	*i*	*j*	*f*	*g*	$R^3 = (adbec)(fhjgi)$	$=Y^{-1}$
e	*a*	*b*	*c*	*d*	*g*	*h*	*i*	*j*	*f*	$R^4 = (aedcb)(fghij)$	$=R^{-1}$
f	*g*	*h*	*i*	*j*	*a*	*b*	*c*	*d*	*e*	$G = (af)(bg)(ch)(di)(ej)$	$=G$
g	*h*	*i*	*j*	*f*	*e*	*a*	*b*	*c*	*d*	$RG = (ag)(bh)(ci)(dj)(ef)$	$=B$
h	*i*	*j*	*f*	*g*	*d*	*e*	*a*	*b*	*c*	$R^2G = (ah)(bi)(cj)(df)(eg)$	$=O$
i	*j*	*f*	*g*	*h*	*c*	*d*	*e*	*a*	*b*	$R^3G = (ai)(bj)(cf)(dg)(eh)$	$=V$
j	*f*	*g*	*h*	*i*	*b*	*c*	*d*	*e*	*a*	$R^4G = (aj)(bf)(cg)(dh)(ei)$	$=I$

$R, Y.$ $(2\ 5\text{gons})^2$ 20
$G, B, O, V, I.$ $(5\ \text{digons})^5$ 25
45

11A

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*
e	*f*	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*
f	*g*	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*
g	*h*	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*
i	*j*	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
j	*k*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
k	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*

5 colours.

$1 = 1 = 1$ $\quad R, G, Y, B, O. \quad (1\ 11\text{gon})^5 \quad 55$

$\underline{55}$

$R = (abcdefghijk) = R$

$R^2 = (acegikbdfhj) = G$

$R^3 = (adgjbehkcfi) = Y$

$R^4 = (aeibfjcgkdh) = B$

$R^5 = (afkejdichbg) = O$

$R^6 = (agbhcidjekf) = O^{-1}$

$R^7 = (ahdkgcjfbie) = B^{-1}$

$R^8 = (aifckhebjgd) = Y^{-1}$

$R^9 = (ajhfdbkigec) = G^{-1}$

$R^{10} = (akjihgfedcb) = R^{-1}$

12A

a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*
b	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*
c	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*
d	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*
e	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*
f	*g*	*h*	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*
g	*h*	*i*	*f*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*
h	*i*	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*
i	*j*	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
j	*k*	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
k	*l*	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*
l	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*

6 colours.

$1 = 1$	$= 1$	$R, O.$	$(1\ 12\text{gon})^2$ 24
		$G.$	2 6gons 12
		$Y.$	3 4gons 12
		$B.$	4 3gons 12
		$V.$	6 digons $\underline{6}$
			66

$R = (abcdefghijkl) = R$

$R^2 = (acegik)(bdfhjl) = G$

$R^3 = (adgj)(behk)(cfil) = Y$

$R^4 = (aei)(bfj)(cgk)(dhl) = B$

$R^5 = (afkdibglejch) = O$

$R^6 = (ag)(bh)(ci)(dj)(ek)(fl) = V$

$R^7 = (ahcjelgbidkf) = O^{-1}$

$R^8 = (aie)(bjf)(ckg)(dlh) = B^{-1}$

$R^9 = (ajgd)(bkhe)(clif) = Y^{-1}$

$R^{10} = (akigec)(bljhfd) = G^{-1}$

$R^{11} = (alkjihgfedcb) = R^{-1}$

12B1

a	b	c	d	e	f	g	h	i	j	k	l
b	c	d	e	f	a	h	i	j	k	l	g
c	d	e	f	a	b	i	j	k	l	g	h
d	e	f	a	b	c	j	k	l	g	h	i
e	f	a	b	c	d	k	l	g	h	i	j
f	a	b	c	d	e	l	g	h	i	j	k
g	h	i	j	k	l	a	b	c	d	e	f
h	i	j	k	l	g	b	c	d	e	f	a
i	j	k	l	g	h	c	d	e	f	a	b
j	k	l	g	h	i	d	e	f	a	b	c
k	l	g	h	i	j	e	f	a	b	c	d
l	g	h	i	j	k	f	a	b	c	d	e

7 colours.

$1 \quad = 1 \quad = 1$

$R \quad = (abcdef)(ghijkl) \quad = R$

$R^2 \quad = (ace)(bdf)(gik)(hjl) \quad = Y$

$R^3 \quad = (ad)(be)(cf)(gj)(hk)(il) = B$

$R^4 \quad = (aec)(bfd)(gki)(hlj) \quad = Y^{-1}$

$R^5 \quad = (afedcba)(glkjih) \quad = R^{-1}$

$G \quad = (ag)(bh)(ci)(dj)(ek)(fl) = G$

$RG \quad = (ahcjel)(bidkfg) \quad = P$

$R^2G = (aiegck)(bjfhdl) \quad = O$

$R^3G = (aj)(bk)(cl)(dg)(eh)(fi) = V$

$R^4G = (akcgei)(bdlhfj) \quad = O^{-1}$

$R^5G = (alejch)(bgfkdi) \quad = P^{-1}$

$R, P, O.$	$(2\ 6\text{gons})^3$	36
$Y.$	4 3gons	12
$B, G, V.$	$(6\ \text{digons})^3$	18
		66

12B2

a	b	c	d	e	f	g	h	i	j	k	l
b	c	d	e	f	a	l	g	h	i	j	k
c	d	e	f	a	b	k	l	g	h	i	j
d	e	f	a	b	c	j	k	l	g	h	i
e	f	a	b	c	d	i	j	k	l	g	h
f	a	b	c	d	e	h	i	j	k	l	g
g	h	i	j	k	l	a	b	c	d	e	f
h	i	j	k	l	g	f	a	b	c	d	e
i	j	k	l	g	h	e	f	a	b	c	d
j	k	l	g	h	i	d	e	f	a	b	c
k	l	g	h	i	j	c	d	e	f	a	b
l	g	h	i	j	k	b	c	d	e	f	a

9 colours.

$1 \quad = 1 \quad = 1$

$R \quad = (abcdef)(glkjih) \quad = R$

$R^2 \quad = (ace)(bdf)(gki)(hlj) \quad = Y$

$R^3 \quad = (ad)(be)(cf)(gj)(hk)(il) = B$

$R^4 \quad = (aec)(bfd)(gik)(hjl) \quad = Y^{-1}$

$R^5 \quad = (afedcb)(ghijkl) \quad = R^{-1}$

$G \quad = (ag)(bh)(ci)(dj)(ek)(fl) = G$

$RG \quad = (ah)(bi)(cj)(dk)(el)(fg) = P$

$R^2G = (ai)(bj)(ck)(dl)(eg)(fh) = O$

$R^3G = (aj)(bk)(cl)(dg)(eh)(fi) = V$

$R^4G = (ak)(bl)(cg)(dh)(ei)(fj) = I$

$R^5G = (al)(bg)(ch)(di)(ej)(fk) = S$

$R.$	2 6gons	12
$Y.$	4 3gons	12
$B, G, P, O, V, I, S.$	$(6\ \text{digons})^7$	42
		66

12B3

a	b	c	d	e	f	g	h	i	j	k	l
b	c	d	e	f	a	l	g	h	i	j	k
c	d	e	f	a	b	k	l	g	h	i	j
d	e	f	a	b	c	j	k	l	g	h	i
e	f	a	b	c	d	i	j	k	l	g	h
f	a	b	c	d	e	h	i	j	k	l	g
g	h	i	j	k	l	d	e	f	a	b	c
h	i	j	k	l	g	c	d	e	f	a	b
i	j	k	l	g	h	b	c	d	e	f	a
j	k	l	g	h	i	a	b	c	d	e	f
k	l	g	h	i	j	f	a	b	c	d	e
l	g	h	i	j	k	e	f	a	b	c	d

6 colours.

1	$=1$	$=1$
R	$=(abcdef)(glkjih)$	$=R$
R^2	$=(ace)(bdf)(gki)(hlj)$	$=B$
R^3	$=(ad)(be)(cf)(gj)(hk)(il)$	$=Y$
R^4	$=(aec)(bfd)(gik)(hjl)$	$=B^{-1}$
R^5	$=(afedcb)(ghijkl)$	$=R^{-1}$
G	$=(agdj)(bhek)(cifl)$	$=G$
RG	$=(ahdk)(biel)(cjfg)$	$=O$
R^2G	$=(aidl)(bjeg)(ckfh)$	$=P$
R^3G	$=(ajdg)(bkeh)(clfi)$	$=G^{-1}$
R^4G	$=(akdh)(blei)(cgfj)$	$=O^{-1}$
R^5G	$=(aldi)(bgej)(chfk)$	$=P^{-1}$

R.	2 6gons	12
G, O, P.	(3 4gons)^3	36
B.	4 3gons	12
Y.	6 digons	6
		66

12B4

a	b	c	d	e	f	g	h	i	j	k	l
b	c	a	e	f	d	h	i	g	k	l	j
c	a	b	f	d	e	i	g	h	l	j	k
d	l	g	a	i	j	c	k	e	f	h	b
e	j	h	b	g	k	a	l	f	d	i	c
f	k	i	c	h	l	b	j	d	e	g	a
g	d	l	j	a	i	e	c	k	b	f	h
h	e	j	k	b	g	f	a	l	c	d	i
i	f	k	l	c	h	d	b	j	a	e	g
j	h	e	g	k	b	l	f	a	i	c	d
k	i	f	h	l	c	j	d	b	g	a	e
l	g	d	i	j	a	k	e	c	h	b	f

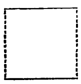
7 colours.

1	$=1$	$=1$
R	$=(abc)(def)(ghi)(jkl)$	$=R$
R^2	$=(acb)(dfe)(gih)(jlk)$	$=R^{-1}$
RGR^2	$=(ad)(bl)(cg)(ei)(fj)(hk)$	$=Y$
RG	$=(aeg)(bjd)(chl)(fki)$	$=B$
RGR	$=(afl)(bkg)(cid)(ehj)$	$=O$
GR^2	$=(age)(bdj)(clh)(fik)$	$=B^{-1}$
G	$=(ah)(be)(cj)(dk)(fg)(il)$	$=G$
GR	$=(aij)(bfh)(cke)(dlg)$	$=P$
R^2G	$=(aji)(bhf)(cek)(dgl)$	$=P^{-1}$
R^2GR	$=(ak)(bi)(cf)(dh)(el)(dj)$	$=V$
R^2GR^2	$=(alf)(bgk)(cdi)(ejh)$	$=O^{-1}$

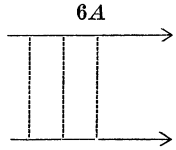
R, B, P, O.	(4 3gons)^4	48
Y, G, V.	(6 digons)^3	18
		66

Extracting from these colourgroups the twocolourbonds contained in them respectively, we have the twocolourbonds shown in the annexed series of figures. I have in each case given the number $4B$, $6A$, etc., of the colourgroup in which the bond is contained, and which colourgroup is given conversely by the two-colourbond. The several points may have letters a, b, c, d, etc., attached to them at pleasure, but as the particular letters are quite immaterial, it seemed to me better to give the several figures without any letters.

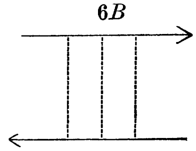
$4B$



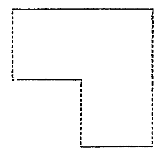
$6A$



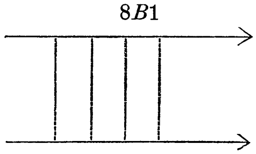
$6B$



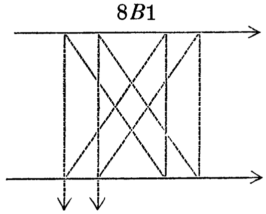
$6B$



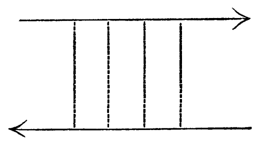
$8B1$



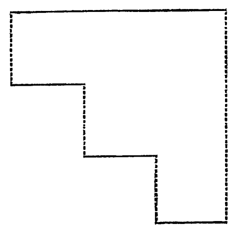
$8B1$



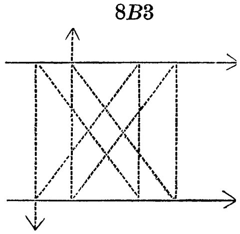
$8B2$



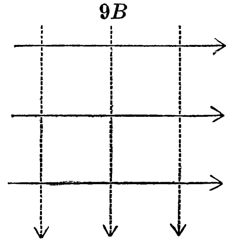
$8B2$



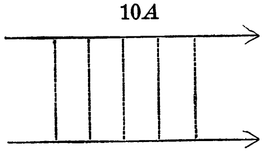
$8B3$



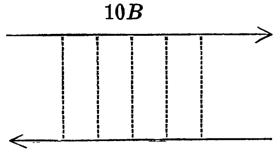
$9B$



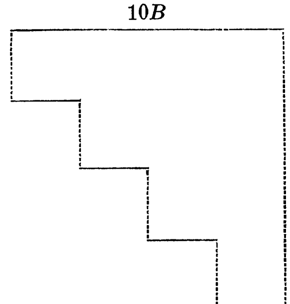
$10A$



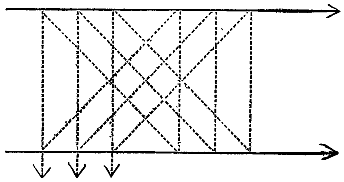
$10B$



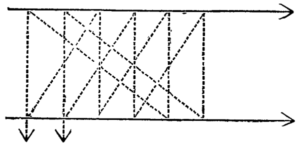
$10B$



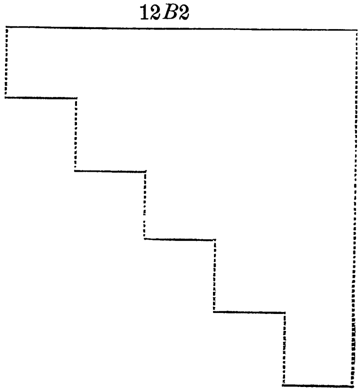
12A



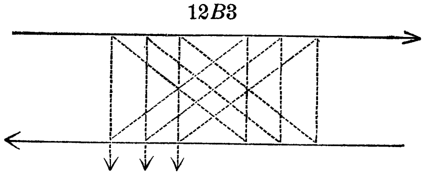
12B1



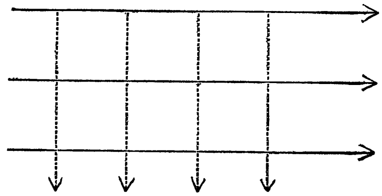
12B2



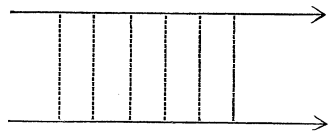
12B3



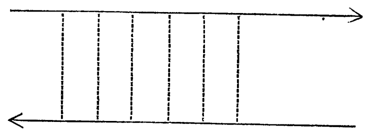
12A



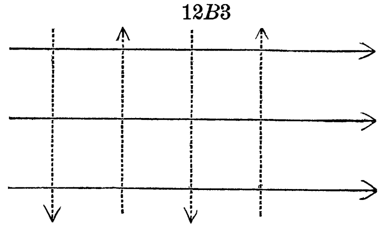
12B1



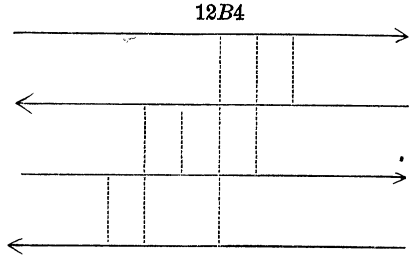
12B2



12B3

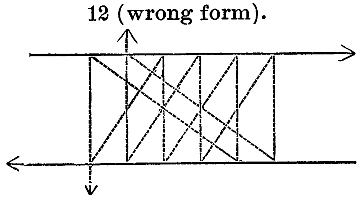


12B4

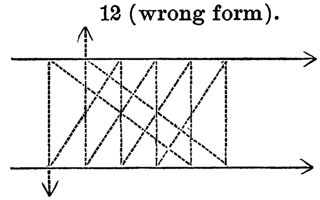


In any one of the foregoing forms of twocolourbond, each point is in its relations to the other points indistinguishable from each of the other points. This would seem to be a relation of symmetry equivalent to the before-mentioned condition that each route is of independent effect; and it would moreover seem as if the relation of symmetry were satisfied for each of the following forms:

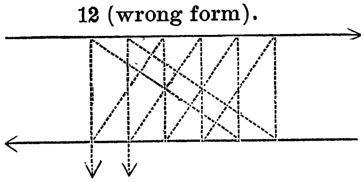
12 (wrong form).



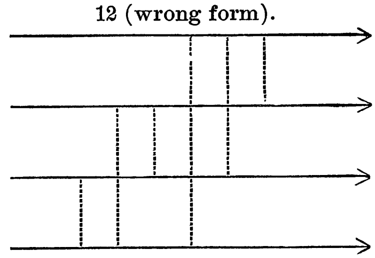
12 (wrong form).



12 (wrong form).



12 (wrong form).



Each of these is, however, a wrong form, not satisfying the condition that each route is of independent effect. As to this, observe that when the condition is satisfied, there are in all ($\vartheta =$) 12 non-equivalent routes, and there is thus a completely determinate square. When the condition is not satisfied, there are more than this number of non-equivalent routes, and there may very well be ϑ routes giving rise to a latin square, viz. a square each line of which, and also each column of which, contains all the letters, and which thus seems at first sight to represent a substitution-group; but the substitutions by which each line of the square is derived from itself and the other lines of the square are not the same as those by which each line is derived from the top line, and thus the square does not represent a group. Thus in one of the above wrong forms, starting from the routes $R = (abcdef)(glkjih)$ and $G = (agciek)(bhdjfl)$, we have

12 (wrong form).

a	b	c	d	e	f	g	h	i	j	k	l	$1 \quad = 1$
b	c	d	e	f	a	l	g	h	i	j	k	$R \quad = (abcdef)(glkjih)$
c	d	e	f	a	b	k	l	g	h	i	j	$R^2 \quad = (ace)(bdf)(gki)(hlj)$
d	e	f	a	b	c	j	k	l	g	h	i	$R^3 \quad = (ad)(be)(cf)(gj)(hk)(il)$
e	f	a	b	c	d	i	j	k	l	g	h	$R^4 \quad = (aec)(bfd)(gik)(hjl)$
f	a	b	c	d	e	h	i	j	k	l	g	$R^5 \quad = (afedcb)(ghijkl)$
g	h	i	j	k	l	c	d	e	f	a	b	$G \quad = (agciek)(bhdjfl)$
h	i	j	k	l	g	b	c	d	e	f	a	$RG \quad = (ahcjel)(bidkfg)$
i	j	k	l	g	h	a	b	c	d	e	f	$R^2G = (aickeg)(bjdlfh)$
j	k	l	g	h	i	f	a	b	c	d	e	$R^3G = (ajcleh)(bkdgfi)$
k	l	g	h	i	j	e	f	a	b	c	d	$R^4G = (akcgei)(bldhfj)$
l	g	h	i	j	k	d	e	f	a	b	c	$R^5G = (alchej)(bgdifk)$

	G	R^2	G
a	g	k	a
b	h	l	b
c	i	g	c
d	j	h	d
e	k	i	e
f	l	j	f
g	c	e	k
h	d	f	l
i	e	a	g
j	f	b	h
k	a	c	i
l	b	d	j

which is not a group; there is no substitution $G^{-1} = (akeicg)(blfjdh)$. And we see that in fact each route is not of independent effect; the route GR^2G leads as shown from the primitive arrangement *abcdefghijkl* to *abcdefklghij*, viz: it is a circuit from each of the points a, b, c, d, e, f, but not from any one of the remaining points g, h, i, j, k, l.