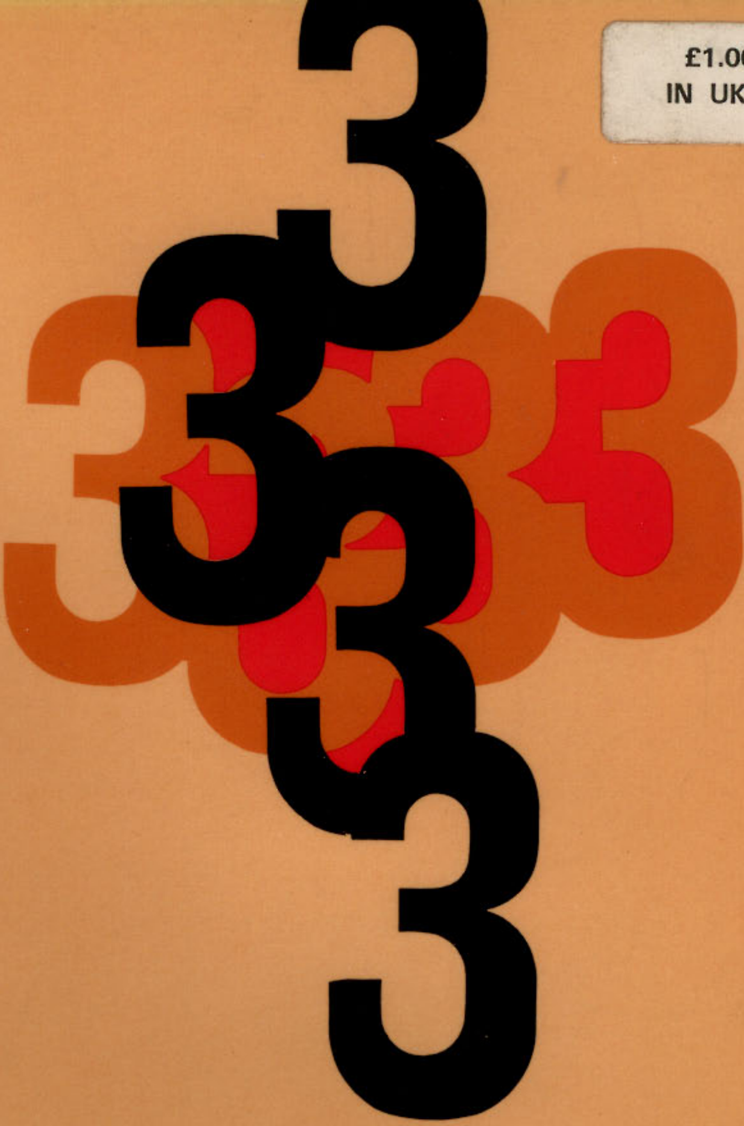


ELEMENTS OF NUMBER THEORY

I. M. VINOGRADOV

VINOGRADOV ELEMENTS OF NUMBER THEORY DOVER S259

DOVER BOOKS ON INTERMEDIATE AND ADVANCED MATHEMATICS

Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Milton Abramowitz and Irene A. Stegun. $4.00

Analysis of Straight Line Data, Forman S. Acton. $2.00

Elementary Concepts of Topology, Paul Alexandroff. $1.00

The Mathematical Analysis of Electrical and Optical Wave-Motion, Harry Bateman. $1.75

A Collection of Modern Mathematical Classics: Analysis, Richard Bellman. $2.00

Selected Papers on Mathematical Trends in Control Theory, Volume I, edited by Richard Bellman and Robert Kalaba. $2.25

Selected Papers on Mathematical Trends in Control Theory, Volume II, edited by Richard Bellman and Robert Kalaba. $2.25

Numerical Integration of Differential Equations, Albert A. Bennett, William E. Milne and Harry Bateman. $1.35

Almost Periodic Functions, A. S. Besicovitch. $1.75

Collected Mathematical Papers, George D. Birkhoff. Clothbound. Three-volume set $20.00 (tent.)

Algebraic Functions, Gilbert A. Bliss. $1.85

Introduction to Higher Algebra, Maxime Bôcher. $2.25

Lectures on the Calculus of Variations, Oskar Bolza. $1.75

Non-Euclidean Geometry: A Critical and Historical Study of Its Developments, by Roberto Bonola; *The Science of Absolute Space*, by John Bolyai; and *The Theory of Parallels*, by Nicholas Lobachevski. $2.00

An Investigation of the Laws of Thought, George Boole. $2.25

A Treatise on the Calculus of Finite Differences, George Boole. $1.85

Introduction to Bessel Functions, Frank Bowman. $1.50

Introduction to Elliptic Functions with Applications, Frank Bowman. $1.25

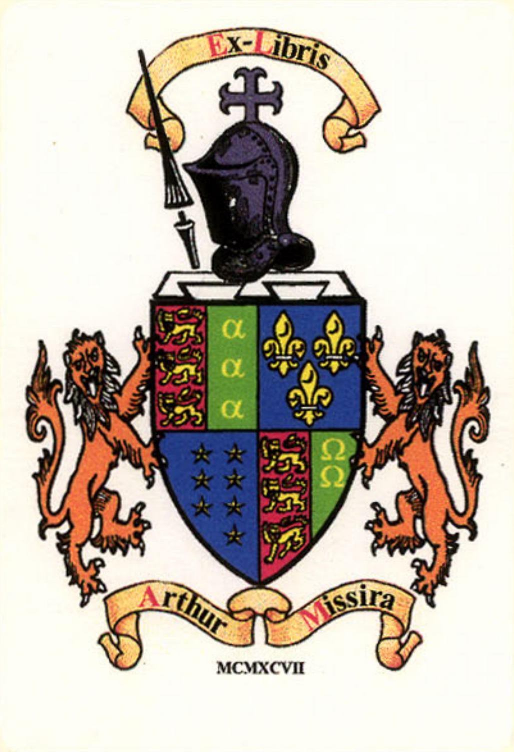
The History of the Calculus and Its Conceptual Development, Carl B. Boyer. $2.00

Theory of Groups of Finite Order, William S. Burnside. $2.75

Theory of Probability, William S. Burnside. $1.00

The Theory of Equations; with an Introduction to the Theory of Binary Algebraic Forms, William S. Burnside and Arthur W. Panton. Two-volume set $3.70

(*continued on back flap*)


Ex-Libris
α
α
α
Ω
Ω
Arthur Missira
MCMXCVII

ELEMENTS OF NUMBER THEORY

BY I. M. VINOGRADOV

Translated from the Fifth Revised Edition by

Saul Kravetz

DOVER PUBLICATIONS, INC.

Published in Canada by General Publishing Company, Ltd., 30 Lesmill Road, Don Mills, Toronto, Ontario.

Published in the United Kingdom by Constable and Company, Ltd., 10 Orange Street, London WC 2.

This Dover edition, first published in 1954, is an unabridged English translation of the fifth Russian edition of 1949.

Library of Congress Catalog Card Number: 54-3691

Manufactured in the United States of America
Dover Publications, Inc.
180 Varick Street
New York, N. Y. 10014

CONTENTS

Chapter IV

CONGRUENCES IN ONE UNKNOWN

Chapter V

CONGRUENCES OF SECOND DEGREE

Chapter VI

PRIMITIVE ROOTS AND INDICES

SOLUTIONS OF THE PROBLEMS

PRINTER'S NOTE

Throughout this book the *a* which takes the following form in the formula (a) assumes the following form (${\scriptstyle a}$) whenever it is used in the smallest type size, as a subscript or a superscript. Both forms are meant to be the same.

PREFACE

A series of Russian mathematicians—Chebyshev, Korkin, Zolotaryov, Markov, Voronoi and others—have worked on the theory of numbers. One can become acquainted with the content of the classical work of these notable mathematicians in B. N. Delone's book "The Petersburg School of the Theory of Numbers" ("Peterburgskaya shkola teorii chisel," in Russian, 1947).

Soviet mathematicians, working in the field of number theory, have continued the great tradition of their predecessors and have created powerful new methods which have been used to obtain a series of first-class results; in the number theory section of the book "Mathematics in the USSR after 30 years" ("Matematika v SSSR za 30 let," in Russian, 1948) one can find a report on the attainments of Soviet mathematicians in the field of number theory, and the corresponding bibliographical references.

In my book I present a systematic exposition of the fundamentals of number theory within the scope of a university course. A large collection of problems introduces the reader to some of the new ideas in number theory.

This fifth edition of my book differs considerably from the fourth. A series of changes, allowing a simpler exposition, have been made in all the chapters of the book. The most important changes are the merging of the old chapters IV and V into one chapter IV (reducing the number of chapters to six) and the new, simpler proof of the existence of primitive roots.

The problems at the end of each chapter have been essentially revised. The order of the problems is now in complete correspondence with the order of the presentation of the theoretical material. Some new problems have been added; but the number of numbered problems has been substantially reduced. This was accomplished by the unification, under the letters a, b, c, . . . , of previously separate problems which were related by the method of solution or by content. All the solutions of the problems have been reviewed; in many cases these solutions have been simplified or replaced by better ones. Particularly essential changes have been made in the solutions of the problems relating to the distribution of n-th power residues and non-residues, and primitive roots, as well as in the estimations of the corresponding trigonometric sums.

I. M. Vinogradov

ELEMENTS OF NUMBER THEORY

CHAPTER I

DIVISIBILITY THEORY

§1. *Basic Concepts and Theorems*

a. The theory of numbers is concerned with the study of the properties of integers. By integers we mean not only the numbers of the natural number sequence 1, 2, 3, ... (the positive integers) but also zero and the negative integers: -1, -2, -3, ...

As a rule, in presenting the theoretical material, we will use letters only to denote integers. In the cases in which letters may denote non-integers, if this is not clear in itself, we will mention it specifically.

The sum, difference and product of two integers a and b are also integers, but the quotient resulting from the division of a by b (if b is different from zero) may be an integer or a non-integer.

b. In the case in which the quotient resulting from the division of a by b is an integer, denoting it by q, we have $a = bq$, i.e. *a is equal to the product of b by an integer.* We will then say that a is *divisible by b* or that b *divides a.* Here a is said to be a *multiple* of b and b is said to be a *divisor* of the number a. The fact that b divides a is written as: $b\backslash a$.

We have the following two theorems.

1. *If a is a multiple of m, and m is a multiple of b, then a is a multiple of b.*

Indeed, it follows from $a = a_1m$, $m = m_1b$ that $a = a_1m_1b$, where a_1m_1 is an integer. But this proves the theorem.

2. *If we know that in an equation of the form $k + l + \ldots + n = p + q + \ldots + s$, all terms except one are multiples of b, then this one term is also a multiple of b.*

Indeed, let the exceptional term be k. We have

$$l = l_1 b, \ldots, n = n_1 b, p = p_1 b, q = q_1 b, \ldots, s = s_1 b,$$
$$k = p + q + \ldots + s - l - \ldots - n$$
$$= (p_1 + q_1 + \ldots + s_1 - l_1 - \ldots - n_1)b,$$

proving our theorem.

c. In the general case, which includes the particular case in which a is divisible by b, we have the theorem:

Every integer a is uniquely representable in terms of the positive integer b in the form

$$a = bq + r, \qquad 0 < r < b$$

Indeed, we obtain one such representation of a by taking bq to be equal to the largest multiple of b which does not exceed a. Assuming that we also have $a = bq_1 + r_1$, $0 < r_1 < b$, we find that $0 = b(q - q_1) + r - r_1$, from which it follows (**2, b**) that $r - r_1$ is a multiple of b. But since $|r - r_1| < b$, the latter is only possible if $r - r_1 = 0$, i.e. if $r = r_1$, from which it also follows that $q = q_1$.

The number q is called the *partial quotient* and the number r is called the *remainder* resulting from the division of a by b.

Examples. Let $b = 14$. We have

$$177 = 14\ 12 + 9, \qquad 0 < 9 < 14;$$
$$-64 = 14\ (-5) + 6, \qquad 0 < 6 < 14;$$
$$154 = 14\ 11 + 0, \qquad 0 = 0 < 14.$$

§2. *The Greatest Common Divisor*

a. In what follows we shall consider only the positive divisors of numbers. Every integer which divides all the integers $a, b, \ldots, l$ is said to be a *common divisor* of them. The largest of these common divisors is said to be their *greatest common divisor* and is denoted by the symbol $(a, b, \ldots, l)$.

In view of the finiteness of the number of common divisors the existence of the greatest common divisor is evident. If $(a, b, \ldots, l) = 1$, then $a, b, \ldots, l$ are said to be *relatively prime*. If each of the numbers $a, b, \ldots, l$ is relatively prime to any other of them, then $a, b, \ldots, l$ are said to be *pairwise prime*. It is evident that pairwise prime numbers are also relatively prime; in the case of two numbers the concepts of "pairwise prime" and "relatively prime" coincide.

Examples. The numbers 6, 10, 15 are relatively prime since $(6, 10, 15) = 1$. The numbers 8, 13, 21 are pairwise prime since $(8, 13) = (8, 21) = (13, 21) = 1$.

b. We first consider the common divisors of two numbers.

1. *If a is a multiple of b, then the set of common divisors of the numbers a and b coincides with the set of divisors of b; in particular, $(a, b) = b$.*

Indeed, every common divisor of the numbers a and b is a divisor of b. Conversely, if a is a multiple of b, then (**1**, **b**, §**1**) every divisor of the number b is also a divisor of the number a, i.e. it is a common divisor of the numbers a and b. Thus the set of common divisors of the numbers a and b coincides with the set of divisors of b, but since the greatest divisor of the number b is b itself, we have $(a, b) = b$.

2. *If*

$$a = bq + c,$$

then the set of common divisors of the numbers a and b coincides with the set of common divisors of the numbers b and c; in particular, $(a, b) = (b, c)$.

Indeed, the above equation shows that every common divisor of the numbers a and b divides c (**2**, **b**, §**1**) and therefore is a common divisor of the numbers b and c. Conversely, the same equation shows that every common divisor of the numbers b and c divides a and consequently is a common divisor of the numbers a and b. Therefore the common divisors of the numbers a and b are just those numbers which are also common divisors of the numbers b and c; in particular, the greatest of these divisors must also coincide, i.e. $(a, b) = (b, c)$.

c. In order to obtain the least common divisor as well as to deduce its most important properties, *Euclid's algorithm* is applied. The latter consists of the following process. Let a and b be positive integers. By **c, §1,** we find the sequence of equations:

$$(1)\qquad \left\{\begin{array}{ll} a = bq_2 + r_2, & 0 < r_2 < b, \\ b = r_2q_3 + r_3, & 0 < r_3 < r_2, \\ r_2 = r_3q_4 + r_4, & 0 < r_4 < r_3, \\ \dots\dots\dots\dots\dots & \dots\dots\dots \\ r_{n-2} = r_{n-1}q_n + r_n, & 0 < r_n < r_{n-1} \\ r_{n-1} = r_nq_{n+1}, & \end{array}\right.$$

which terminates when we obtain some $r_{n+1} = 0$. The latter must occur since the sequence $b, r_2, r_3, \dots$ as a decreasing sequence of integers cannot contain more than b positive integers.

d. Considering the equations of (1), proceeding from the top down, (**b**) shows that the common divisors of the numbers a and b are identical with the common divisors of the numbers b and r_2, are moreover identical with the common divisors of the numbers r_2 and r_3, of the numbers r_3 and r_4, $\dots$, of the numbers r_{n-1} and r_n, and finally with the divisors of the number r_n. Along with this, we have

$$(a, b) = (b, r_2) = (r_2, r_3) = \dots = (r_{n-1}, r_n) = r_n.$$

We arrive at the following results.

1. *The set of common divisors of the numbers a and b coincides with the set of divisors of their greatest common divisor.*

2. *This greatest common divisor is equal to* r_n, i.e. *the last non-zero remainder in Euclid's algorithm.*

Example. We apply Euclid's algorithm to the evaluation of (525, 231). We find (the auxiliary calculations are given on the left)

```
      525|231
      462|2
   231|63
   189|3
  63|42
  42|1
42|21
42|2
```

$$\begin{aligned} 525 &= 231\cdot 2 + 63,\\ 231 &= 63\cdot 3 + 42,\\ 63 &= 42\cdot 1 + 21,\\ 42 &= 21\cdot 2. \end{aligned}$$

Here the last positive remainder is $r_4 = 21$. This means that $(525, 231) = 21$.

e.1. *If m denotes any positive integer, we have $(am, bm) = (a, b)m$.*

2. *If δ is any common divisor of the numbers a and b, then $\left(\frac{a}{\delta}, \frac{b}{\delta}\right) = \frac{(a, b)}{\delta}$; in particular, $\left(\frac{a}{(a, b)}, \frac{b}{(a, b)}\right) = 1$, i.e. the quotients resulting from the division of two numbers by their greatest common divisor are relatively prime numbers.*

Indeed, multiply each of the terms of the equations (1) by m. We obtain new equations, where $a, b, r_2, \ldots, r_n$ are replaced by $am, bm, r_2m, \ldots, r_nm$. Therefore $(am, bm) = r_nm$, showing that proposition 1 is true.

Applying proposition 1, we find that

$$(a, b) = \left(\frac{a}{\delta}\delta, \frac{b}{\delta}\delta\right) = \left(\frac{a}{\delta}, \frac{b}{\delta}\right)\delta;$$

and this proves proposition 2.

f.1. *If $(a, b) = 1$, then $(ac, b) = (c, b)$.*

Indeed, (ac, b) divides ac and bc, which implies (**1, d**) that it also divides (ac, bc) which is equal to c by **1, e**; but (ac, b) also divides b and therefore also divides (c, b). Conversely, (c, b) divides ac and b, and therefore also divides (ac, b). Thus (ac, b) and (c, b) divide each other and are therefore equal to one another.

2. *If $(a, b) = 1$ and ac is divisible by b, then c is divisible by b.*

Indeed, since $(a, b) = 1$, we have $(ac, b) = (c, b)$. But if ac is a multiple of b, then (**1, b**) we have $(ac, b) = b$, which means that $(c, b) = b$, i.e. c is a multiple of b.

3. *If each* $a_1, a_2, \ldots, a_m$ *is relatively prime to each* $b_1, b_2, \ldots, b_n$, *then the product* $a_1a_2\ldots a_m$ *is relatively prime to the product* $b_1b_2\ldots b_n$.

Indeed (theorem **1**), we have

$$(a_1a_2a_3\ldots a_m, b_k) = (a_2a_3\ldots a_m, b_k)$$
$$= (a_3\ldots a_m, b_k) = \ldots = (a_m, b_k) = 1,$$

and moreover, setting $a_1a_2\ldots a_m = A$, in the same way we find

$$(b_1b_2b_3\ldots b_n, A) = (b_2b_3\ldots b_n, A)$$
$$= (b_3\ldots b_n, A) = \ldots = (b_n, A) = 1.$$

g. The problem of finding the greatest common divisor of more than two numbers reduces to the same problem for two numbers. Indeed, in order to find the greatest common divisor of the numbers $a_1, a_2, \ldots, a_n$ we form the sequence of numbers:

$$(a_1, a_2) = d_2, \ (d_2, a_3) = d_3, \ (d_3, a_4) = d_4, \ \ldots, \ (d_{n-1}, a_n) = d_n.$$

The number d_n is also the greatest common divisor of all the given numbers.

Indeed (**1, d**), the common divisors of the numbers a_1 and a_2 coincide with the divisors of d_2; therefore the common divisors of the numbers a_1, a_2 and a_3 coincide with the common divisors of the numbers d_2 and a_3, i.e. coincide with the divisors of d_3. Moreover, we can verify that the common divisors of the numbers a_1, a_2, a_3, a_4 coincide with the divisors of d_4, and so forth, and finally, that the common divisors of the numbers $a_1, a_2, \ldots, a_n$ coincide with the divisors of d_n. But since the largest divisor of d_n is d_n itself, it is the greatest common divisor of the numbers $a_1, a_2, \ldots, a_n$.

Considering the above proof, we can see that theorem **1, d** is true for more than two numbers also. Theorems **1, e** and **2, e** are also true, because multiplication by m or division by

δ of all the numbers $a_1, a_2, \ldots, a_n$ causes all the numbers $d_1, d_2, \ldots, d_n$ to be multiplied by m or to be divided by δ.

§3. *The Least Common Multiple*

a. Any integer which is a multiple of each of a set of given numbers, is said to be their *common multiple*. The smallest positive common multiple is called the *least common multiple*.

b. We first consider the least common multiple of two numbers. Let M be any common multiple of the integers a and b. Since it is a multiple of a, $M = ak$, where k is an integer. But M is also a multiple of b, and hence

$$\frac{ak}{b}$$

must also be an integer which, setting $(a, b) = d$, $a = a_1 d$, $b = b_1 d$, can be represented in the form $\frac{a_1 k}{b_1}$, where $(a_1, b_1) = 1$ (**2**, **e**, §**2**). Therefore (**2**, **f**, §**2**) k must be divisible by b_1, $k = b_1 t = \frac{b}{d} t$, where t is an integer. Hence

$$M = \frac{ab}{d} t.$$

Conversely, it is evident that every M of this form is a multiple of a as well as b, and therefore, this form gives all the common multiples of the numbers a and b.

The smallest positive one of these multiples, i.e. the least common multiple, is obtained for $t = 1$. It is

$$m = \frac{ab}{d}.$$

Introducing m, we can rewrite the formula we have obtained for M as:

$$M = mt.$$

The last and the next to the last equations lead to the theorems:

1. *The common multiples of two numbers are identical with the multiples of their least common multiple.*

2. *The least common multiple of two numbers is equal to their product divided by their greatest common divisor.*

c. Assume that we are now required to find the least common multiple of more than two numbers $a_1, a_2, \ldots, a_n$. Letting the symbol $m(a, b)$ denote the least common multiple of the numbers a and b, we form the sequence of numbers:

$$m(a_1, a_2) = m_2,\ m(m_2, a_3) = m_3,\ \ldots,\ m(m_{n-1}, a_n) = m_n.$$

The m_n obtained in this way will be the least common multiple of all the given numbers.

Indeed (**1, b**), the common multiples of the numbers a_1 and a_2 coincide with the multiples of m_2, and hence the common multiples of the numbers a_1, a_2 and a_3 coincide with the common multiples of m_2 and a_3, i.e. they coincide with the multiples of m_3. It is then clear that the common multiples of the numbers a_1, a_2, a_3, a_4 coincide with the multiples of m_4, and so forth, and finally, that the common multiples of the numbers $a_1, a_2, \ldots, a_n$ coincide with the multiples of m_n, and since the smallest positive multiple of m_n is m_n itself, it is also the least common multiple of the numbers $a_1, a_2, \ldots, a_n$.

Considering the proof given above, we see that theorem **1, b** is also true for more than two numbers. Moreover, we have shown the validity of the following theorem:

The least common multiple of pairwise prime numbers is equal to their product.

§4. *The Relation of Euclid's Algorithm to Continued Fractions*

a. Let α be an arbitrary real number. Let q be the largest integer which does not exceed α.

For a non-integer α, we have

$$\alpha = q_1 + \frac{1}{\alpha_2}; \quad \alpha_2 > 1,$$

Similarly, for non-integers $\alpha_2, \ldots, \alpha_{s-1}$ we have

$$\alpha_2 = q_2 + \frac{1}{\alpha_3}; \quad \alpha_3 > 1;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{s-1} = q_{s-1} + \frac{1}{\alpha_s}; \quad \alpha_s > 1,$$

from which we obtain the following *development of* α *in a continued fraction*:

$$(1) \qquad \alpha = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdot \; \cdot \; \cdot + \cfrac{1}{q_{s-1} + \cfrac{1}{\alpha_s}}}}.$$

If α is irrational, then it is evident that there can be no integers in the sequence $\alpha, \alpha_2, \ldots$, and the above process can be continued indefinitely.

If α is rational, then, as we shall see later (**b**), there will eventually be an integer in the sequence $\alpha, \alpha_2, \ldots$, and the above process will be terminated.

b. If α is an irreducible rational fraction, then the development of α in a continued fraction is closely connected with Euclid's algorithm. Indeed, we have

$$a = bq_1 + r_2; \qquad \frac{a}{b} = q_1 + \frac{r_2}{b},$$

$$b = r_2q_2 + r_3; \qquad \frac{b}{r_2} = q_2 + \frac{r_3}{r_2},$$

$$r_2 = r_3q_3 + r_4; \qquad \frac{r_2}{r_3} = q_3 + \frac{r_4}{r_3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$r_{n-2} = r_{n-1}q_{n-1} + r_n; \frac{r_{n-2}}{r_{n-1}} = q_{n-1} + \frac{r_n}{r_{n-1}},$$

$$r_{n-1} = r_n q_n; \qquad \frac{r_{n-1}}{r_n} = q_n,$$

from which we find

$$\frac{a}{b} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdot \cdot \cdot + \cfrac{1}{q_n}}}.$$

c. The numbers $q_1, q_2, \ldots,$ which occur in the expansion of the number α in a continued fraction, are called the *partial quotients* (for the case of rational α these are, by **b**, the partial quotients of the successive divisions of the Euclidean algorithm), and the fractions

$$\delta_1 = q_1, \ \delta_2 = q_1 + \frac{1}{q_2}, \ \delta_3 = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3}}, \ \ldots$$

are called the *convergents*.

d. The very simple rule for the formation of the convergents is easily obtained by noting that $\delta_s (s > 1)$ is obtained from δ_{s-1} by replacing q_{s-1} in the expression for δ_{s-1} by $q_{s-1} + \frac{1}{q_s}$.

Indeed, setting $P_0 = 1$, $Q_0 = 0$, for the sake of uniformity, we can represent the convergents recursively in the following way (when the equation $\frac{A}{B} = \frac{P_s}{Q_s}$ is written here, it means that A is denoted by the symbol P_s, and B by the symbol Q_s):

$$\delta_1 = \frac{q_1}{1} = \frac{P_1}{Q_1}, \quad \delta_2 = \frac{q_1 + \frac{1}{q_2}}{1} = \frac{q_2 q_1 + 1}{q_2 \cdot 1 + 0} = \frac{q_2 P_1 + P_0}{q_2 Q_1 + Q_0} = \frac{P_2}{Q_2}$$

$$\delta_3 = \frac{\left(q_2 + \frac{1}{q_3}\right) P_1 + P_0}{\left(q_2 + \frac{1}{q_3}\right) Q_1 + Q_0} = \frac{q_3 P_2 + P_1}{q_3 Q_2 + Q_1} = \frac{P_3}{Q_3}$$

etc., and in general

$$\delta_s = \frac{q_s P_{s-1} + P_{s-2}}{q_s Q_{s-1} + Q_{s-2}} = \frac{P_s}{Q_s}$$

Thus the numerators and the denominators of the convergents can be recursively calculated by means of the formulae

$$(2) \qquad \begin{cases} P_s = q_s P_{s-1} + P_{s-2}, \\ Q_s = q_s Q_{s-1} + Q_{s-2}. \end{cases}$$

These calculations can easily be carried out by means of the following schema:

q_s		q_1	q_2	....			q_s	....		q_n
P_s	1	q_1	P_2	....	P_{s-2}	P_{s-1}	P_s	....	P_{n-1}	a
Q_s	0	1	Q_2	....	Q_{s-2}	Q_{s-1}	Q_s	....	Q_{n-1}	b

Example. Develop the number $\frac{105}{38}$ in a continuous fraction.

Here

```
                105 | 38
                 76 | 2
            38 | 29
            29 | 1
        29 |  9
        27 |  3
     9 |  2
     8 |  4
  2 |  1
  2 |  2
```

$$\frac{105}{38} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Therefore the aforementioned schema gives:

q_s		2	1	3	4	2
P_s	1	2	3	11	47	105
Q_s	0	1	1	4	17	38

e. We now consider the difference $\delta_s - \delta_{s-1}$ of successive convergents. For $s > 1$, we find

$$\delta_s - \delta_{s-1} = \frac{P_s}{Q_s} - \frac{P_{s-1}}{Q_{s-1}} = \frac{h_s}{Q_s Q_{s-1}}$$

where $h_s = P_s Q_{s-1} - Q_s P_{s-1}$; replacing P_s and Q_s by their expressions in (2) and making the evident simplifications, we find that $h_s = -h_{s-1}$. The latter, in conjunction with $h_1 = q_1 . 0 - 1 \cdot 1 = -1$, gives $h_s = (-1)^s$. Thus

(3) $$P_s Q_{s-1} - Q_s P_{s-1} = (-1)^s \qquad (s > 0),$$

(4) $$\delta_s - \delta_{s-1} = \frac{(-1)^s}{Q_s Q_{s-1}} \quad (s > 1).$$

Example. In the table of the example given in **d**, we have

$$105 \cdot 17 - 38 \cdot 47 = (-1)^5 = -1.$$

f. It follows from (3) that (P_s, Q_s) divides $(-1)^s = \pm 1$ (2, **b**, §1). Hence $(P_s, Q_s) = 1$, i.e. *the convergents* $\frac{P_s}{Q_s}$ *are irreducible.*

g. We now investigate the sign of the difference $\delta_s - \alpha$ for δ_s which are not equal to α (i.e. we exclude the case in which δ_s is the last convergent for rational α). It is evident that δ_s is obtained by replacing α_s by q_s in the expression (1) for α. But, as is evident from **a**, as a result of this replacement

$$\begin{array}{ll} \alpha_s & \text{is decreased,} \\ \alpha_{s-1} & \text{is increased,} \\ \alpha_{s-2} & \text{is decreased,} \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ \alpha & \begin{cases} \text{is decreased for odd } s, \\ \text{is increased for even } s. \end{cases} \end{array}$$

Therefore $\delta_s - \alpha < 0$ for odd s and $\delta_s - \alpha > 0$ for even s, and consequently, the sign of $\delta_s - \alpha$ coincides with the sign of $(-1)^s$.

h. *We have*

$$|\alpha - \delta_{s-1}| \leqslant \frac{1}{Q_s Q_{s-1}}$$

Indeed, for $\delta_s = \alpha$ this assertion follows (with the equality sign) from (4). For δ_s unequal to α, it follows (with the inequality sign) from (4) and from the fact that, $\delta_s - \alpha$ and $\delta_{s-1} - \alpha$ have different signs, because of **g.**

§5. *Prime Numbers*

a. The number 1 has only one positive divisor, namely 1. In this respect the number 1 stands alone in the sequence of natural numbers.

Every integer, greater than 1, has no fewer than two divisors, namely 1 and itself; if these divisors exhaust all the positive divisors of an integer, then it is said to be *prime*. An integer > 1 which has positive divisors other than 1 and itself, is said to be *composite*.

b. *The smallest divisor, different from one, of an integer greater than one, is a prime number.*

Indeed, let q be the smallest divisor, different from one, of the integer $a > 1$. If q were composite, then it would have some divisor q_1 such that $1 < q_1 < q$; but the number a, being divisible by q, would also be divisible by q_1 (**1**, **b**, **§1**), and this contradicts our hypothesis concerning the number q.

c. *The smallest divisor, different from* 1, *of a composite number a (by* **b**, *it will be prime) does not exceed* $\sqrt{a}$.

Indeed, let q be this divisor; then $a = qa_1$, $a_1 \geqslant q$ from which, multiplying by q, we obtain $a \geqslant q^2$, $q \leqslant \sqrt{a}$.

d. *The number of primes is infinite.*

The validity of this theorem follows from the fact that no matter what different primes $p_1, p_2, \ldots, p_k$ are considered, we can obtain a new prime which is not among them. Such a prime is any prime divisor of the sum $p_1 p_2 \ldots p_k + 1$ which, dividing the whole sum, cannot be equal to any of the primes $p_1, p_2, \ldots, p_k$ (**2**, **b**, **§1**).

e. There is a simple method, called the *sieve of Eratosthenes*, for the formation of a table of the primes not exceeding a given N. It consists of the following.

We write down the numbers

$$(1) \qquad 1, 2, \ldots, N.$$

The first number of this sequence greater than one is 2; it is only divisible by 1 and itself, and hence it is a prime.

We delete from the sequence (1) (since they are composite numbers) all the numbers which are multiples of 2, except 2 itself. The first number following 2 which is not deleted is 3; it is not divisible by 2 (otherwise it would have been deleted), and hence 3 is divisible only by 1 and itself, and hence it is also prime.

Delete from the sequence (1) all the numbers which are multiples of 3, except 3 itself. The first number following 3 which is not deleted is 5; it is not divisible by either 2 or 3 (otherwise it would have been deleted). Therefore 5 is divisible only by 1 and itself, and therefore it is also prime.

And so forth.

When this process has deleted all the numbers which are multiples of primes less than the prime p, then all the numbers remaining which are less than p^2 are primes. Indeed, every composite number a which is less than p^2 has already been deleted since it is a multiple of its smallest prime divisor which is $\leqslant \sqrt{a} < p$. This implies:

1. *In the process of deleting the multiples of the prime* p*, this set of deleted numbers must start with* p^2.

2. *The formation of the table of primes* $\leqslant N$ *is completed once we have deleted all the composite multiples of primes not exceeding* $\sqrt{N}$.

§6. *The Unicity of Prime Decomposition*

a. *Every integer* a *is either relatively prime to a given prime* p*, or is divisible by* p.

Indeed, (a, p), being a divisor of p, is either 1 or p. In the first case, a is relatively prime to p, and in the second, a is divisible by p.

b. *If the product of several factors is divisible by* p*, then at least one of the factors is divisible by* p.

Indeed (**a**), every factor is either divisible by p or is relatively prime to p. If all the factors were relatively prime to p, then their product (**3**, **f**, §**2**) would be relatively prime to p; therefore at least one factor is divisible by p.

c. *Every integer greater than one can be decomposed into the product of prime factors and uniquely, if we disregard the order of the factors.*

Indeed, let a be an integer greater than unity; if p_1 is its smallest prime divisor, then $a = p_1a_1$. If $a_1 > 1$, then if p_2 is its smallest prime divisor, we have $a_1 = p_2a_2$. If $a_2 > 1$, then, in exactly the same way, we find $a_2 = p_3a_3$, etc. until we come to some a_n equal to one. Then $a_{n-1} = p_n$. Multiplying all these equations together, and simplifying, we obtain the following decomposition of a into prime factors:

$$a = p_1p_2 \dots p_n.$$

Assume that there exists a second decomposition of the same a into prime factors $a = q_1q_2 \dots q_s$. Then

$$p_1p_2 \dots p_n = q_1q_2 \dots q_s.$$

The right side of this equation is divisible by q_1. Therefore (**b**), at least one of the factors of the left side must be divisible by q_1. For example, let p_1 be divisible by q_1 (in the order of enumeration in our arrangement) then $p_1 = q_1$ (p_1 is divisible only by p_1 except for 1). Dividing both sides of the equation by $p_1 = q_1$, we have $p_2p_3 \dots p_n = q_2q_3 \dots q_s$. Repeating the preceding argumentation applied to this equation, we find $p_3 \dots p_n = q_3 \dots q_s$, etc., until we finally find that all the factors on one side, say the left side, are divided out. But all the factors on the right side must be cancelled simultaneously since the equation $1 = q_{n+1} \dots q_s$ for $q_{n+1}, \dots, q_s$ greater than 1, is impossible.

Therefore the second decomposition into prime factors is identical with the first.

d. In the decomposition of the number a into prime factors, several of them may be repeated. Letting $p_1, p_2, \dots, p_k$ be the different primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ be the multiplicity of their occurrence in a, we obtain the so-called *canonical decomposition of a into factors*:

$$a = p_1^{\alpha_1}p_2^{\alpha_2} \dots p_k^{\alpha_k}.$$

Example. The canonical decomposition of the number 588 000 is: 588 000 = $2^5 \cdot 3 \cdot 5^3 \cdot 7^2$.

e. Let $a = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$ be the canonical decomposition of the number a. Then all the divisors of a are just all the numbers of the form

$$d = p_1^{\beta_1}p_2^{\beta_2} \dots p_k^{\beta_k};$$

(1)

$$0 \leqslant \beta_1 \leqslant \alpha_1,\ 0 \leqslant \beta_2 \leqslant \alpha_2,\ \dots,\ 0 \leqslant \beta_k \leqslant \alpha_k.$$

Indeed, let d divide a. Then (**b**, §1) $a = dq$, and therefore all the prime divisors of d enter into the canonical decomposition of a with indices no smaller than those with which they enter into the canonical decomposition of d. Therefore d is of the form (1).

Conversely, every d of the form (1) evidently divides a.

Example. All the divisors of the number $720 = 2^4 \cdot 3^2 \cdot 5$ can be obtained if we let β_1, β_2, β_3 in $2^{\beta_1}3^{\beta_2}5^{\beta_3}$ run independently through the values $\beta_1 = 0, 1, 2, 3, 4$; $\beta_2 = 0, 1, 2$; $\beta_3 = 0, 1$. Therefore these divisors are: 1, 2, 4, 8, 16, 3, 6, 12, 24, 48, 9, 18, 36, 72, 144, 5, 10, 20, 40, 80, 15, 30, 60, 120, 240, 45, 90, 180, 360, 720.

Problems for Chapter I

1. Let a and b be integers which are not both zero, and let $d = ax_0 + by_0$ be the smallest positive number of the form $ax + by$ (x and y integers). Prove that $d = (a, b)$. From this deduce theorem **1, d,** §**2** and the theorems of **e,** §**2.** Generalize these results by considering numbers of the form $ax + by + \dots + fu$.

2. Prove that, of all the rational numbers with denominators $\leqslant Q_s$, the convergent $\delta_s = \frac{P_s}{Q_s}$ represents the number α most exactly.

3. Let the real number α be developed in a continued fraction; let N be a positive integer, let k be the number of decimal digits in it, and let n be the largest integer such that $Q_n \leqslant N$. Prove that $n \leqslant 5k + 1$. In order to prove this, compare the expressions for $Q_2, Q_3, Q_4, \ldots, Q_n$ with those which would occur if all the q_s were equal to 1, and compare the latter with the numbers $1, \xi, \xi^2, \ldots, \xi^{n-2}$ where ξ is the positive root of the equation $\xi^2 = \xi + 1$.

4. Let $\tau \geqslant 1$. The sequence of irreducible rational fractions with positive denominators not exceeding τ, arranged in increasing order, is called the *Farey series corresponding to* τ.

a. Prove that the part of the Farey series corresponding to τ, containing fractions α such that $0 \leqslant \alpha \leqslant 1$, can be obtained in the following way: we write down the fractions $\frac{0}{1}, \frac{1}{1}$. If $2 \leqslant \tau$, then we insert the fraction $\frac{0+1}{1+1} = \frac{1}{2}$ between these fractions, and then in the resulting sequence $\frac{0}{1}$, $\frac{1}{2}, \frac{1}{1}$ between every two neighboring fractions $\frac{a_1}{b_1}$ and $\frac{c_1}{d_1}$ with $b_1 + d_1 \leqslant \tau$ we insert the fraction $\frac{a_1 + c_1}{b_1 + d_1}$, and so forth as long as this is possible. First prove that for any two pairs of neighboring fractions $\frac{a}{b}$ and $\frac{c}{d}$ of the sequence, obtained in the above manner, we have $ad - bc = -1$.

b. Considering the Farey series, prove the theorem: let $\tau \geqslant 1$, then every real number α can be represented in the form

$$\alpha = \frac{P}{Q} + \frac{\theta}{Q\tau}; \quad 0 < Q \leqslant \tau, \ (P, Q) = 1, \ |\theta| < 1.$$

c. Prove the theorem of problem **b** using **h,** §4.

5, a. Prove that there are an infinite number of primes of the form $4m + 3$.

b. Prove there are an infinite number of primes of the form $6m + 5$.

6. Prove that there exist an infinite number of primes by counting the number of integers, not exceeding N, whose canonical decomposition does not contain prime numbers different from $p_1, p_2, \ldots, p_k$.

7. Let K be a positive integer. Prove that the sequence of natural numbers contains an infinite set of sequences $M, M + 1, \ldots, M + K - 1$, not containing primes.

8. Prove that there are an infinite number of composite numbers among the numbers represented by the polynomial $a_0x^n + a_1x^{n-1} + \ldots + a_n$, where $n > 0$, $a_0, a_1, \ldots, a_n$ are integers and $a_0 > 0$.

9, a. Prove that the indeterminate equation (1) $x^2 + y^2 = z^2$, $x > 0, y > 0, z > 0, (x, y, z) = 1$ is satisfied by those, and only those, systems x, y, z for which one of the numbers x and y is of the form $2uv$, the other of the form $u^2 - v^2$, and finally z is of the form $u^2 + v^2$; here $u > v > 0$, $(u, v) = 1$, uv is even.

b. Using the theorem of problem **a**, prove that the equation $x^4 + y^4 = z^4$ cannot be solved in positive integers x, y, z.

10. Prove the theorem: if the equation $x^n + a_1x^{n-1} + \ldots + a_n = 0$, where $n > 0$ and $a_1, a_2, \ldots, a_n$ are integers, has a rational root then this root is an integer.

11, a. Let $S = \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}$; $n > 1$. Prove that S is not an integer.

b. Let $S = \dfrac{1}{3} + \dfrac{1}{5} + \ldots + \dfrac{1}{2n + 1}$; $n > 0$. Prove that S is not an integer.

12. Let n be an integer, $n > 0$. Prove that all the coefficients of the expansion of the Newtonian binomial $(a + b)^n$ are odd if and only if n is of the form $2^k - 1$.

Numerical Exercises for Chapter I

1, a. Applying the Euclidean algorithm, find (6188, 4709).

b. Find (81 719, 52 003, 33 649, 30 107).

2, a. Expanding $\alpha = \dfrac{125}{92}$ in a continuous fraction and forming the table of convergents (**d, §4**), find: α) δ_4; β) the representation of α in the form considered in problem **4, b,** with $\tau = 20$.

b. Expanding $\alpha = \dfrac{5391}{3976}$ in a continuous fraction and forming the table of convergents, find: α) δ_6; β) the representation of α in the form considered in problem **4, b,** with $\tau = 1000$.

3. Form the Farey series (problem **4**) from 0 to 1, excluding 1, with denominators not exceeding 8.

4. Form the table of primes less than 100.

5, a. Find the canonical decomposition of the number 82 798 848.

b. Find the canonical decomposition of the number 81 057 226 635 000.

CHAPTER II

IMPORTANT NUMBER-THEORETICAL FUNCTIONS

§1. *The Functions* $[x]$, $\{x\}$

a. The function $[x]$ plays an important role in number theory; it is defined for all real numbers x and is the largest integer not exceeding x. This function is called *the integral part of* x.

Examples.

$$[7] = 7; [2.6] = 2; [-4.75] = -5.$$

The function $\{x\} = x - [x]$ is also considered sometimes. This function is called *the fractional part of* x.

Examples.

$$\{7\} = 0; \{2.6\} = 0.6; \{-4.75\} = 0.25.$$

b. In order to show the usefulness of the functions we have introduced, we prove the theorem:

The power with which a given prime p *enters into the product* $n!$ *is equal to*

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$$

Indeed, the number of factors of the product $n!$ which are multiples of p is $\left[\frac{n}{p}\right]$; of these the number of multiples of p^2 is $\left[\frac{n}{p^2}\right]$; of the latter the number of multiples of p^3 is $\left[\frac{n}{p^3}\right]$, etc. The sum of the latter numbers gives the required power since each factor of the product $n!$ which is a multiple of the maximal p^m is counted m times by the above process, as a multiple of $p, p^2, p^3, \ldots$, and finally, p^m.

Example. The power to which the number 3 enters into the product 40! is

$$\left[\frac{40}{3}\right] + \left[\frac{40}{9}\right] + \left[\frac{40}{27}\right] = 13 + 4 + 1 = 18.$$

§2. *Sums Extended over the Divisors of a Number*

a. Multiplicative functions play an important role in number theory. A function $\theta(a)$ is said to be *multiplicative* if the following conditions are satisfied:

1. *The function $\theta(a)$ is defined for all positive integers a and is not equal to zero except possibly for at most one such a.*

2. *For any two relatively prime positive integers a_1 and a_2, we have*

$$\theta(a_1a_2) = \theta(a_1)\,\theta(a_2).$$

Example. It is not difficult to see that the function $\theta(a) = a^s$, where s is any real or complex number, is multiplicative.

b. From the aforementioned properties of the function $\theta(a)$ it follows in particular that $\theta(1) = 1$. Indeed, let $\theta(a_0)$ be different from zero, then $\theta(a_0) = \theta(1 \cdot a_0) = \theta(1)\theta(a_0)$, i.e. $\theta(1) = 1$. Moreover we have the following important property: if $\theta_1(a)$ and $\theta_2(a)$ are multiplicative functions, then $\theta_0(a) = \theta_1(a)\theta_2(a)$ is also a multiplicative function. Indeed, we find that

$$\theta_0(1) = \theta_1(1)\theta_2(1) = 1.$$

Moreover, for $(a_1, a_2) = 1$, we find

$$\theta_0(a_1a_2) = \theta_1(a_1a_2)\theta_2(a_1a_2) = \theta_1(a_1)\theta_1(a_2)\theta_2(a_1)\theta_2(a_2) =$$
$$= \theta_1(a_1)\theta_2(a_1)\theta_1(a_2)\theta_2(a_2) = \theta_0(a_1)\theta_0(a_2).$$

c. *Let $\theta(a)$ be a multiplicative function and let $a = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$ be the canonical decomposition of the number a. Then, denoting by the symbol $\sum\limits_{d\backslash a}$ the sum extended over all the divisors d of the integer a, we have*

$$\sum_{d\backslash a}\theta(d) = (1 + \theta(p_1) + \theta(p_1^2) + \dots + \theta(p_1^{\alpha_1}))\dots$$
$$\dots(1 + \theta(p_k) + \theta(p_k^2) + \dots + \theta(p_k^{\alpha_k}))$$

(*if $a = 1$ the right side is considered to be equal to* 1).

In order to prove this identity, we multiply-out the right side. Then we obtain a sum of terms of the form

$$\theta(p_1^{\beta_1})\theta(p_2^{\beta_2})\dots\theta(p_k^{\beta_k}) = \theta(p_1^{\beta_1}p_2^{\beta_2}\dots p_k^{\beta_k});$$
$$0 \leqslant \beta_1 \leqslant \alpha_1,\ 0 \leqslant \beta_2 \leqslant \alpha_2,\ \dots,\ 0 \leqslant \beta_k \leqslant \alpha_k,$$

where no terms are lacking and there are no repeated terms, and this is exactly the situation on the left (**e**, **§6**, **ch. I**).

d. For $\theta(a) = a^s$ the identity of **c** takes on the form

$$(1)\quad \sum_{d\backslash a} d^s = (1 + p_1^s + p_1^{2s} + \dots + p_1^{\alpha_1 s})\dots$$
$$\dots(1 + p_k^s + p_k^{2s} + \dots + p_k^{\alpha_k s}).$$

In particular, for $s = 1$, the left side of (1) represents *the sum of the divisors* $S(a)$ of the number a. Simplifying the right side we find

$$S(a) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1}\cdot\frac{p_2^{\alpha_2+1} - 1}{p_2 - 1}\dots\frac{p_k^{\alpha_k+1} - 1}{p_k - 1}.$$

Example.

$$S(720) = S(2^4 \cdot 3^2 \cdot 5) =$$

$$= \frac{2^{4+1} - 1}{2 - 1} \cdot \frac{3^{2+1} - 1}{3 - 1} \cdot \frac{5^{1+1} - 1}{5 - 1} = 2418.$$

For $s = 0$, the left side of (1) represents the number of divisors $\tau(a)$ of the number a and we find

$$\tau(a) = (\alpha_1 + 1)(\alpha_2 + 1)\ldots(\alpha_k + 1).$$

Example.

$$\tau(720) = (4 + 1)(2 + 1)(1 + 1) = 30.$$

§3. *The Möbius Function*

a. The *Möbius function* $\mu(a)$ is defined for all positive integers a. It is given by the equations: $\mu(a) = 0$, if a is divisible by a square different from unity; $\mu(a) = (-1)^k$ if a is not divisible by a square different from unity, where k denotes the number of prime divisors of the number a; in particular, for $a = 1$, we let $k = 0$, and hence we take $\mu(1) = 1$.

Examples.

$$\begin{array}{lll} \mu(1) = 1, & \mu(5) = -1, & \mu(9) = 0, \\ \mu(2) = -1, & \mu(6) = 1, & \mu(10) = 1, \\ \mu(3) = -1, & \mu(7) = -1, & \mu(11) = -1, \\ \mu(4) = 0, & \mu(8) = 0, & \mu(12) = 0. \end{array}$$

b. *Let $\theta(a)$ be a multiplicative function and let*

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

be the canonical decomposition of the number a. Then

$$\sum_{d \backslash a} \mu(d)\theta(d) = (1 - \theta(p_1))(1 - \theta(p_2)) \ldots (1 - \theta(p_k)).$$

(*If* $a = 1$ *the right side is taken to be equal to* 1.)

Indeed it is evident that the function $\mu(a)$ is multiplicative. Therefore the function $\theta_1(a) = \mu(a)\theta(a)$ is also multiplicative. Applying the identity of **c**, §2 to the latter, and noting that $\theta_1(p) = -\theta(p)$; $\theta_1(p^s) = 0$ for $s > 1$, we have proved the validity of our theorem.

c. In particular, setting $\theta(a) = 1$, we obtain from **b**,

$$\sum_{d\backslash a} \mu(d) \begin{cases} = 0, \text{ if } a > 1, \\ = 1, \text{ if } a = 1. \end{cases}$$

Setting $\theta(d) = \dfrac{1}{d}$, we find

$$\sum_{d\backslash a} \frac{\mu(d)}{d} \begin{cases} = \left(1 - \dfrac{1}{p_1}\right)\left(1 - \dfrac{1}{p_2}\right)\cdots\left(1 - \dfrac{1}{p_k}\right), & \text{if } a > 1, \\ = 1, & \text{if } a = 1. \end{cases}$$

d. *Let the real or complex* $f = f_1, f_2, \ldots, f_n$ *correspond to the positive integers* $\delta = \delta_1, \delta_2, \ldots, \delta_n$. *Then, letting* S' *be the sum of the values of* f *corresponding to the values of* δ *equal to* 1, *and letting* S_d *be the sum of the values of* f *corresponding to the values of* δ *which are multiples of* d, *we have*

$$S' = \sum \mu(d) S_d,$$

where d *runs through all the positive integers dividing at least one value of* δ.

Indeed, in view of **c** we have

$$S' = f_1 \sum_{d\backslash \delta_1} \mu(d) + f_2 \sum_{d\backslash \delta_2} \mu(d) + \ldots + f_n \sum_{d\backslash \delta_n} \mu(d).$$

Gathering those terms with the same value of d and bracketing the coefficient of this $\mu(d)$, the bracket contains those and only those f whose corresponding δ are multiples of d, and this is just S_d.

§4. *The Euler Function*

a. *Euler's function* $\varphi(a)$ is defined for all positive integers a and represents the number of numbers of the sequence

$$(1)\qquad 0,\ 1,\ \ldots,\ a-1$$

which are relatively prime to a.

Examples.

$$\begin{aligned} \varphi(1) &= 1, & \varphi(4) &= 2,\\ \varphi(2) &= 1, & \varphi(5) &= 4,\\ \varphi(3) &= 2, & \varphi(6) &= 2. \end{aligned}$$

b. *Let*

$$(2)\qquad a = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$$

be the canonical decomposition of the number a. Then

$$(3)\qquad \varphi(a) = a\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right)$$

or also

$$(4)\qquad \varphi(a) = (p_1^{\alpha_1}-p_1^{\alpha_1-1})(p_2^{\alpha_2}-p_2^{\alpha_2-1})\ldots(p_k^{\alpha_k}-p_k^{\alpha_k-1});$$

in particular,

$$(5)\qquad \varphi(p^{\alpha}) = p^{\alpha}-p^{\alpha-1},\ \varphi(p) = p-1.$$

Indeed we apply the theorem of **d, §3.** Here the numbers δ and the numbers f are defined as follows: let x run through the numbers of the sequence (1); to each value of x let the number $\delta = (x, a)$ and the number $f = 1$ correspond.

Then S' becomes the number of values $\delta = (x, a)$ equal to 1, i.e. becomes $\varphi(a)$. Moreover S_d becomes the number of values $\delta = (x, a)$ which are multiples of d. But (x, a) can be a multiple of d only if d is a divisor of the number a. On the strength of these conditions S_d reduces to the number of values of x which are multiples of d, i.e. to $\frac{a}{d}$. Thus we find

$$\varphi(a) = \sum_{d \backslash a} \mu(d) \frac{a}{d}$$

from which formula (3) follows in view of **c**, **§3**, and formula (4) follows from (3) in view of (2).

Examples.

$$\varphi(60) = 60 \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 16;$$

$$\varphi(81) = 81 - 27 = 54;$$

$$\varphi(5) = 5 - 1 = 4.$$

c. *The function* $\varphi(a)$ *is multiplicative function.*

Indeed, for $(a_1, a_2) = 1$, it follows evidently from **b** that

$$\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2).$$

Example. $\varphi(405) = \varphi(81)\varphi(5) = 54 \cdot 4 = 216.$

d. $\sum\limits_{d \backslash a} \varphi(d) = a.$

In order to prove the validity of this formula we apply the identity of **c**, **§2**, which for $\theta(a) = \varphi(a)$ gives

$$\varphi(d) = (1 + \varphi(p_1) + \varphi(p_1^2) + \ldots + \varphi(p_1^{\alpha_1}))\ldots$$
$$\ldots(1 + \varphi(p_k) + \varphi(p_k^2) + \ldots + \varphi(p_k^{\alpha_k})),$$

In view of (5), the right side can be rewritten as

$$(1 + (p_1 - 1) + (p_1^2 - p_1) + \dots + (p_1^{\alpha_1} - p_1^{\alpha_1 - 1}))\dots$$
$$\dots(1 + (p_k - 1) + (p_k^2 - p_k) + \dots + (p_k^{\alpha_k} - p_k^{\alpha_k - 1})),$$

which turns out to be equal to $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k} = a$ after gathering similar terms in each large parenthesis.

Example. Setting $a = 12$, we find

$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(4) + \varphi(6) + \varphi(12) =$$
$$= 1 + 1 + 2 + 2 + 2 + 4 = 12.$$

Problems for Chapter II

1, a. Let the function $f(x)$ be continuous and non-negative in the interval $Q \leqslant x \leqslant R$. Prove that the sum

$$\sum_{Q<x\leqslant R} [f(x)]$$

is equal to the number of lattice points (points with integer coordinates) in the plane region: $Q < x \leqslant R,\ 0 < y \leqslant f(x)$.

b. Let P and Q be positive odd relatively prime integers. Prove that

$$\sum_{0<x<\frac{P}{2}} \left[\frac{P}{Q}x\right] + \sum_{0<y<\frac{Q}{2}} \left[\frac{Q}{P}y\right] = \frac{P-1}{2} \cdot \frac{Q-1}{2}.$$

c. Let $r > 0$ and let T be the number of lattice points in the region $x^2 + y^2 \leqslant r^2$. Prove that

$$T = 1 + 4[r] + 8 \sum_{0<x\leqslant\frac{r}{\sqrt{2}}} [\sqrt{r^2 - x^2}] - 4\left[\frac{r}{\sqrt{2}}\right]^2.$$

d. Let $n > 0$ and let T be the number of lattice points of the region $x > 0$, $y > 0$, $xy \leqslant n$. Prove that

$$T = 2 \sum_{0<x\leqslant\sqrt{n}} \left[\frac{n}{x}\right] - [\sqrt{n}\,]^2 .$$

2. Let $n > 0$, m an integer, $m > 1$, and let x run through the positive integers which are not divisible by the m-th power of an integer exceeding 1. Prove that

$$\sum_x \left[\sqrt[m]{\frac{n}{x}}\,\right] = [n].$$

3. Let the positive numbers α and β be such that

$$[\alpha x],\ x = 1, 2, \ldots;\ [\beta y],\ y = 1, 2, \ldots$$

form, taken together, all the natural numbers without repetitions. Prove that this occurs if and only if α is irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

4, a. Let $r \geqslant 1$, $t = [r]$, and let $x_1, x_2, \ldots, x_t$ be the numbers $1, 2, \ldots, t$ in some order so that the numbers

$$0,\ \{\alpha x_1\},\ \{\alpha x_2\},\ \ldots,\ \{\alpha x_t\},\ 1$$

are non-decreasing. Prove the theorem of problem **4, b, ch. I,** by considering the differences of neighboring numbers of the latter sequence.

b. Let $X, Y, \ldots, Z$ be real numbers, each of which is not less than 1; let $\alpha, \beta, \ldots, \gamma$ be real numbers. Prove that there exist integers $x, y, \ldots, z$, not all zero, and an integer u, satisfying the conditions:

$$|x| \leqslant X, \; |y| \leqslant Y, \; \ldots, \; |z| \leqslant Z,$$

$$(x, y, \ldots, z) = 1, \; |\alpha x + \beta y + \ldots + \gamma z - u| < \frac{1}{XY\ldots Z}$$

5. Let α be a real number, c an integer, $c > 0$. Prove that

$$\left[\frac{[\alpha]}{c}\right] = \left[\frac{\alpha}{c}\right]$$

6, a. Let $\alpha, \beta, \ldots, \lambda$ be real numbers. Prove that

$$[\alpha + \beta + \ldots + \lambda] \geqslant [\alpha] + [\beta] + \ldots + [\lambda].$$

b. Let $a, b, \ldots, l$ be positive integers, and let $a + b + \ldots + l = n$. Applying **b, § 1**, prove that

$$\frac{n!}{a!b!\ldots l!}$$

is an integer.

7. Let h be a positive integer, p a prime and

$$u_s = \frac{p^{s+1} - 1}{p - 1}$$

Representing h in the form $h = p_m u_m + p_{m-1} u_{m-1} + \ldots + p_1 u_1 + p_0$, where u_m is the largest u_s not exceeding h, $p_m u_m$ is the largest multiple of u_m which does not exceed h, $p_{m-1} u_{m-1}$ is the largest multiple of u_{m-1} which does not exceed $h - p_m u_m$, $p_{m-2} u_{m-2}$ is the largest multiple of u_{m-2} which does not exceed $h - p_m u_m - p_{m-1} u_{m-1}$ etc., prove that numbers a such that the number p enters into the canonical representation of $a!$ with the power h, exist if and only if all the $p_m, p_{m-1}, \ldots, p_1, p_0$ are less than p, while, if this occurs, the numbers a are just all the numbers of the form

$$a = p_m p^{m+1} + p_{m-1}p^m + \ldots + p_1p^2 + p_0p + p';$$

where p' has the values $0, 1, \ldots, p - 1$.

8, a. Let the function $f(x)$ have a continuous second derivative in the interval $Q \leqslant x \leqslant R$. Setting

$$\rho(x) = \frac{1}{2} - \{x\},\ \sigma(x) = \int_0^x \rho(z)dz,$$

prove (Sonin's formula)

$$\sum_{Q<x\leqslant R} f(x) = \int_Q^R f(x)dx + \rho(R)f(R) - \rho(Q)f(Q) -$$

$$- \sigma(R)f'(R) + \sigma(Q)f'(Q) + \int_Q^R \sigma(x)f''(x)dx.$$

b. Let the conditions of problem **a** be satisfied for arbitrarily large R, while $\int_R^\infty |f''(x)|\,dx$ converges. Prove that

$$\sum_{Q<x\leqslant R} f(x) = C + \int_Q^R f(x)dx + \rho(R)f(R) -$$

$$- \sigma(R)f'(R) - \int_R^\infty \sigma(x)f''(x)dx,$$

where C does not depend on R.

c. If B takes on only positive values and the ratio $\dfrac{|A|}{B}$ is bounded above, then we write $A = O(B)$.

Let n be an integer, $n > 1$. Prove that

$$\ln(n!) = n\ln n - n + O(\ln n)$$

9, a. Let $n \geqslant 2$, $\Theta(z, z_0) = \sum_{z_0 < p \leqslant z} \ln p$, where p runs through the primes. Moreover, let $\Theta(z) = \Theta(z, 0)$ and for $x > 0$,

$$\psi(x) = \Theta(x) + \Theta(\sqrt{x}) + \Theta(\sqrt[3]{x}) + \ldots$$

Prove that

$$\alpha)\ \ln([n]!) = \psi(n) + \psi\left(\frac{n}{2}\right) + \psi\left(\frac{n}{3}\right) + \ldots$$

$$\beta)\ \psi(n) < 2n$$

$$\gamma)\ \Theta\left(n, \frac{n}{2}\right) + \Theta\left(\frac{n}{3}, \frac{n}{4}\right) + \Theta\left(\frac{n}{5}, \frac{n}{6}\right) + \ldots =$$

$$= n \ln 2 + O(\sqrt{n}).$$

b. For $n > 2$, prove that

$$\sum_{p \leqslant n} \frac{\ln p}{p} = \ln n + O(1),$$

where p runs through the primes.

c. Let ϵ be an arbitrary positive constant. Prove that the sequence of natural numbers contains an infinite number of pairs p_n, p_{n+1} of prime numbers such that

$$p_{n+1} < p_n(1 + \epsilon).$$

d. Let $n > 2$. Prove that

$$\sum_{p \leqslant n} \frac{1}{p} = C + \ln \ln n + O\left(\frac{1}{\ln n}\right),$$

where p runs through the primes and C does not depend on n.

e. Let $n > 2$. Prove that

$$\prod_{p \leqslant n} \left(1 - \frac{1}{p}\right) = \frac{C_0}{\ln n}\left(1 + O\left(\frac{1}{\ln n}\right)\right)$$

where p runs through the primes and C_0 does not depend on n.

10, a. Let $\theta(a)$ be a multiplicative function. Prove that $\theta_1(a) = \sum_{d \backslash a} \theta(d)$ is also a multiplicative function.

b. Let the function $\theta(a)$ be defined for all positive integers a and let the function $\psi(a) = \sum_{d \backslash a} \theta(a)$ be multiplicative. Prove that the function $\theta(a)$ is also multiplicative.

11. For $m > 0$, let $\tau_m(a)$ denote the number of solutions of the indeterminate equation $x_1 x_2 \ldots x_m = a(x_1, x_2, \ldots, x_m$ run through the positive integers independently of one-another); in particular, it is evident that $\tau_1(a) = 1$, $\tau_2(a) = \tau(a)$. Prove that

a. $\tau_m(a)$ is a multiplicative function.

b. If the canonical decomposition of the number a is of the form $a = p_1 p_2 p_3 \ldots p_k$, then $\tau_m(a) = m^k$.

c. If ϵ is an arbitrary positive constant, then

$$\lim_{a \to \infty} \frac{\tau_m(a)}{a^{\epsilon}} = 0.$$

d. $\sum_{0 < a \leqslant n} \tau_m(a)$ is equal to the number of solutions of the inequality $x_1 x_2 \ldots x_m \leqslant a$ in positive integers $x_1, x_2, \ldots, x_m$.

12. Let $R(s)$ be the real part of the complex number s. For $R(s) > 1$, we set $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Let m be a positive integer. Prove that

$$(\zeta(s))^m = \sum_{n=1}^{\infty} \frac{\tau_m(n)}{n^s}$$

13, a. For $R(s) > 1$, prove that

$$\zeta(s) = \Pi \frac{1}{1 - \frac{1}{p^s}}$$

where p runs through all the primes.

b. Prove that there exist an infinite number of primes, starting from the fact that the harmonic series diverges.

c. Prove that there exist an infinite number of primes, starting from the fact that $\zeta(2) = \frac{\pi^2}{6}$ is an irrational number.

14. Let $\Lambda(a) = \ln p$ for $a = p^l$, where p is a prime and l is a positive integer; and let $\Lambda(a) = 0$ for all other positive integers a. For $R(s) > 1$, prove that

$$\frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

15. Let $R(s) > 1$. Prove that

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where p runs through all the primes.

16, a. Let $n \geqslant 1$. Applying **d, §3,** prove that

$$1 = \sum_{0<d\leqslant n} \mu(d) \left[\frac{n}{d}\right].$$

b. Let $M(z, z_0) = \sum_{z_0<a\leqslant z} \mu(a)$; $M(x) = M(x, 0)$. Prove that

α) $M(n) + M\left(\frac{n}{2}\right) + M\left(\frac{n}{3}\right) + \ldots = 1, n \geqslant 1.$

β) $M\left(n, \frac{n}{2}\right) + M\left(\frac{n}{3}, \frac{n}{4}\right) + M\left(\frac{n}{5}, \frac{n}{6}\right) + \ldots = -1, n \geqslant 2.$

c. Let $n \geqslant 1$, let l be an integer, $l > 1$, and let $T_{l,n}$ be the number of integers x, such that $0 < x \leqslant n$, which are not divisible by the l-th power of an integer exceeding 1. Applying **d, §3,** prove that

$$T_{l,n} = \sum_{d=1}^{\infty} \mu(d) \left[\frac{n}{d^l}\right].$$

17, a. Let a be a positive integer and let the function $f(x)$ be uniquely defined for the integers $x_1, x_2, \ldots, x_n$. Prove

$$S' = \sum_{d\backslash a} \mu(d) S_d,$$

where S' is the sum of the values of $f(x)$ extended over those values of x which are relatively prime to a, and S_d is the sum of the values of $f(x)$ extended over those values of x which are multiples of d.

b. Let $k > 1$ and consider the systems

$$x_1', x_2', \ldots, x_k';\ x_1'', x_2'', \ldots, x_k'';\ \ldots;\ x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)},$$

each of which consists of integers, not all zero. Moreover, let the function $f(x_1, x_2, \ldots, x_k)$ be uniquely defined for these systems. Prove that

$$S' = \sum \mu(d) S_d,$$

where S' is the sum of the values of $f(x_1, x_2, \ldots, x_k)$ extended over systems of relative prime numbers, and S_d is the sum of the values of $f(x_1, x_2, \ldots, x_k)$ extended over systems of numbers which are all multiples of d. Here d runs through positive integers.

c. Let a be a positive integer, and let $F(\delta)$ be uniquely defined for the divisors δ of the number a. Setting

$$G(\delta) = \sum_{d\backslash\delta} F(d),$$

prove (the inversion law for number-theoretic functions)

$$F(a) = \sum_{d\backslash a} \mu(d)G\left(\frac{a}{d}\right).$$

d. Associate with the positive integers

$$\delta_1, \delta_2, \ldots, \delta_n$$

arbitrary real or complex numbers

$$f_1, f_2, \ldots, f_n$$

different from zero. Prove that

$$P' = \Pi P_d^{\mu(d)}$$

where P' is the product of the values f associated with values of δ equal to one, and P_d is the product of the values f associated with values of δ which are multiples of d, where d runs through all the positive integers which divide at least one δ.

18. Let a be an integer, $a > 1$, $\sigma_m(n) = 1^m + 2^m + \ldots + n^m$; let $\psi_m(a)$ be the sum of the m-th powers of the numbers of the sequence $1, 2, \ldots, a$ which are relatively prime to a; let $p_1, p_2, \ldots, p_k$ be all the prime divisors of the number a.

a. Applying the theorem of problem **17, a,** prove that

$$\psi_m(a) = \sum_{d\backslash a} \mu(d)d^m\sigma_m\left(\frac{a}{d}\right).$$

b. Prove that

$$\psi_1(a) = \frac{a}{2}\varphi(a).$$

c. Prove that

$$\psi_2(a) = \left(\frac{a^2}{3} + \frac{(-1)^k}{6}p_1p_2\ldots p_k\right)\varphi(a).$$

19. Let $z > 1$, let a be a positive integer; let T_z be the number of numbers x such that $0 < x \leqslant z$, $(x, a) = 1$; let ϵ be an arbitrary positive constant.

a. Prove that

$$T_z = \sum_{d \backslash a} \mu(d)\left[\frac{z}{d}\right].$$

b. Prove that

$$T_z = \frac{z}{a}\varphi(a) + O(a^{\epsilon}).$$

c. Let $z > 1$; let $\pi(z)$ be the number of prime numbers not exceeding z; let a be the product of the primes not exceeding $\sqrt{z}$. Prove that

$$\pi(z) = \pi(\sqrt{z}) - 1 + \sum_{d \backslash a} \mu(d)\left[\frac{z}{d}\right].$$

20. Let $R(s) > 1$ and let a be a positive integer. Prove that

$$\sum{}' \frac{1}{n^s} = \Pi\left(1 - \frac{1}{p^s}\right)\zeta(s),$$

where, on the left side, n runs through the positive integers relatively prime to a, while, on the right side, p runs through all the prime divisors of the number a.

21, a. The probability P that k positive integers $x_1, x_2, \ldots, x_k$ are relatively prime is defined as the limit, as $N \longrightarrow \infty$, of the probability P_N that the k numbers $x_1, x_2, \ldots, x_k$ are relatively prime, when these k numbers take on the values 1, 2, ..., N independently and with equal probability. Applying the theorem of problem **17, b,** prove that $P = (\zeta(k))^{-1}$.

b. Defining the probability of the irreducibility of the fraction $\frac{x}{y}$ as in problem **a** for $k = 2$, prove that $P = \frac{6}{\pi^2}$.

22, a. Let $r > 2$ and let T be the number of lattice points (x, y) with relatively prime coordinates in the region $x^2 + y^2 \leqslant \leqslant r$. Prove that

$$T = \frac{6}{\pi} r^2 + O(r \ln r).$$

b. Let $r \geqslant 2$ and let T be the number of lattice points (x, y, z) with relatively prime coordinates lying in the region $x^2 + + y^2 + z^2 \leqslant r^2$. Prove that

$$T = \frac{4\pi}{3\zeta(3)} r^3 + O(r^2)$$

23, a. Prove the first theorem of **c, §3,** by considering the divisors of the number a which are not divisible by the square of an integer exceeding 1, and having 1, 2, ... prime divisors.

b. Let a be an integer, $a > 1$, and let d run through the divisors of the number a having no more than m prime divisors; Prove that $\sum \mu(d) \geqslant 0$ for m even, and $\sum \mu(d) \leqslant 0$ for m odd.

c. Under the conditions of the theorem of **d, §3,** assuming all the f to be non-negative and letting d run only through the numbers having no more than m prime divisors, prove that

$$S' \leqslant \sum \mu(d) S_d, \quad S' \geqslant \sum \mu(d) S_d$$

according as m is even or odd.

d. Prove the validity of the same inequalities as in problem **c**, under the conditions of problem **17**, **a**, assuming all the values of $f(x)$ are non-negative, as well as under the conditions of **17**, **b**, assuming all the values of $f(x_1, x_2, \ldots, x_k)$ are non-negative.

24. Let ϵ be an arbitrary constant such that $0 < \epsilon < \frac{1}{6}$; let N be an integer, $r = \ln N$, $0 < q \leqslant N^{1-\epsilon}$, $0 \leqslant l < q$, $(q, l) = 1$; let $\pi(N, q, l)$ be the number of primes such that $p \leqslant N$, $p = qt + l$, where t is an integer. Prove that

$$\pi(N, q, l) = O(\Delta);\ \Delta = \frac{N(qr)^{\epsilon}}{qr}$$

In order to prove this, setting $h = r^{1-\epsilon}$, the primes satisfying the above condition can be considered to be among all numbers satisfying these conditions relatively prime to a, where a is the product of all primes which do not exceed e^h and do not divide q. We can then apply the theorem of problem **23**, **d** (under the conditions of problem **17**, **a**) with the above a and $m = 2[2\ln r + 1]$.

25. Let k be a positive even number, let the canonical decomposition of the number a be of the form $a = p_1 p_2 \ldots p_k$ and let d run through the divisors of the number a such that $0 < d < \sqrt{a}$. Prove that

$$\sum_{d} \mu(d) = 0.$$

26. Let k be a positive integer, let d run through the positive integers such that $\varphi(d) = k$. Prove that

$$\sum \mu(d) = 0.$$

27. Using the expression for $\varphi(a)$, prove that there exist an infinite number of primes.

28, a. Prove the theorem of **d,** §**4** by showing that the number of integers of the sequence 1, 2, ..., a which have the same greatest common divisor δ with a, is equal to $\varphi\left(\frac{a}{\delta}\right)$.

b. Deduce expressions for $\varphi(a)$:

α) using the theorem of problem **10, b;**

β) using the theorem of problem **17, c.**

29. Let $R(s) > 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

30. Let n be an integer, $n \geqslant 2$. Prove that

$$\sum_{m=1}^{n} \varphi(m) = \frac{3}{\pi^2}n^2 + O(n \ln n).$$

Numerical Exercises for Chapter II

1, a. Find the exact power with which 5 enters into the canonical decomposition of 5258! (problem **5**).

b. Find the canonical decomposition of the number 125!

2, a. Find τ(2 800) and S(2 800).

b. Find τ(232 848) and S(232 848).

3. Form the table of values of the function $\mu(a)$ for all a = 1, 2, ..., 100.

4. Find α) φ(5040); β) φ(1 294 700).

5. Form the table of values of the function $\varphi(a)$ for all a = 1, 2, ..., 50, using only formula (5), §**4**, and theorem **c,** §**4.**

CHAPTER III

CONGRUENCES

§1. *Basic Concepts*

a. We will consider integers in relation to the remainders resulting from their division by a given positive integer m which we call the *modulus*.

To each integer corresponds a unique remainder resulting from its division by m (**c, §1, ch. I**); if the same remainder r corresponds to two integers a and b, then they are said to be *congruent* modulo m.

b. The congruence of the numbers a and b modulo m is written as

$$a \equiv b (\text{mod } m),$$

which is read: a is congruent to b modulo m.

c. *The congruence of the numbers a and b modulo m is equivalent to*:

1. *The possibility of representing a in the form $a = b + mt$, where t is an integer.*

2. *The divisibility of $a - b$ by m.*

Indeed, it follows from $a \equiv b (\text{mod } m)$ that

$$a = mq + r, \; b = mq_1 + r; \; 0 \leqslant r < m,$$

and hence

$$a - b = m(q - q_1), \; a = b + mt, \; t = (q - q_1).$$

Conversely, from $a = b + mt$, representing b in the form

$$b = mq_1 + r,\ 0 \leqslant r < m,$$

we deduce

$$a = mq + r;\ q = q_1 + t,$$

i.e.

$$a \equiv b (\text{mod } m)$$

proving assertion **1.**

Assertion **2** follows immediately from assertion **1.**

§2. *Properties of Congruences similar to those of Equations*

a. *Two numbers which are congruent to a third are congruent to each other.*

This follows from **a, §1.**

b. *Congruences can be added termwise.*

Indeed, let

(1) $a_1 \equiv b_1 (\text{mod } m),\ a_2 \equiv b_2 (\text{mod } m),\ \ldots,\ a_k \equiv b_k (\text{mod } m)$

Then (**1, c, §1**)

(2) $a_1 = b_1 + mt_1,\ a_2 = b_2 + mt_2,\ \ldots,\ a_k = b_k + mt_k,$

and hence

$$a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k + m(t_1 + t_2 + \ldots + t_k),$$

or (**1, c, §1**)

$$a_1 + a_2 + \ldots + a_k \equiv b_1 + b_2 + \ldots + b_k (\text{mod } m)$$

A summand on either side of a congruence can be put on the other side by changing its sign.

Indeed, adding the congruence $a + b \equiv c(\mathrm{mod}\ m)$ to the evident congruence $-b \equiv -b(\mathrm{mod}\ m)$, we find $a \equiv c - b(\mathrm{mod}\ m)$.

Any number which is a multiple of the modulus can be added to (or subtracted from) any side of a congruence.

Indeed, adding the congruence $a \equiv b(\mathrm{mod}\ m)$ to the evident congruence $mk \equiv 0(\mathrm{mod}\ m)$, we obtain $a + mk \equiv b(\mathrm{mod}\ m)$.

c. *Congruences can be multiplied termwise.*

Indeed, we again consider the congruences (1) and deduce from them the equations (2). Multiplying equations (2) together termwise we find

$$a_1a_2\dots a_k = b_1b_2\dots b_k + mN,$$

where N is an integer. Consequently (**1, c, §1**),

$$a_1a_2\dots a_k \equiv b_1b_2\dots b_k(\mathrm{mod}\ m).$$

Both sides of a congruence can be raised to the same power.

This follows from the preceding theorem.

Both sides of a congruence can be multiplied by the same integer.

Indeed, mutliplying the congruence $a \equiv b(\mathrm{mod}\ m)$ by the evident congruence $k \equiv k(\mathrm{mod}\ m)$, we find $ak \equiv bk(\mathrm{mod}\ m)$.

d. Properties **b** and **c** (addition and multiplication of congruences) can be generalized to the following theorem.

If we replace A, x_1, x_2, ..., x_k in the expression of an integral rational function $S = \sum Ax_1^{\alpha_1}x_2^{\alpha_2}\dots x_k^{\alpha_k}$ with integral coefficients, by the numbers B, y_1, y_2, ..., y_k which are congruent to the preceding ones modulo m, then the new expression S will be congruent to the old one modulo m.

Indeed, from

$$A \equiv B(\mathrm{mod}\ m),\ x_1 \equiv y_1(\mathrm{mod}\ m),$$

$$x_2 \equiv y_2(\mathrm{mod}\ m),\ \dots,\ x_k \equiv y_k(\mathrm{mod}\ m)$$

we find (c)

$$A \equiv B(\text{mod } m),\ x_1^{\alpha_1} \equiv y_1^{\alpha_1}(\text{mod } m)$$
$$x_2^{\alpha_2} \equiv y_2^{\alpha_2}(\text{mod } m),\ \ldots,\ x_k^{\alpha_k} \equiv y_k^{\alpha_k}(\text{mod } m),$$
$$Ax_1^{\alpha_1}x_2^{\alpha_2}\ldots x_k^{\alpha_k} \equiv By_1^{\alpha_1}y_2^{\alpha_2}\ldots y_k^{\alpha_k}(\text{mod } m)$$

from which, summing, we find

$$\sum Ax_1^{\alpha_1}x_2^{\alpha_2}\ldots x_k^{\alpha_k} = \sum By_1^{\alpha_1}y_2^{\alpha_2}\ldots y_k^{\alpha_k}(\text{mod } m).$$

If

$$a \equiv b(\text{mod } m),\ a_1 \equiv b_1(\text{mod } m),\ \ldots,\ a_n \equiv b_n(\text{mod } m),$$
$$x \equiv x_1(\text{mod } m),$$

then

$$ax^n + a_1x^{n-1} + \ldots + a_n \equiv bx_1^n + b_1x_1^{n-1} + \ldots + b_n(\text{mod } m).$$

This result is a special case of the preceding one.

e. *Both sides of a congruence can be divided by one of their common divisors if it is relatively prime to the modulus.*

Indeed, it follows from $a \equiv b(\text{mod } m)$, $a = a_1d$, $b = b_1d$, $(d, m) = 1$ that the difference $a - b$, which is equal to $(a_1 - b_1)d$, is divisible by m. Therefore (2, f, §2, ch. I) $a_1 - b_1$ is divisible by m, i.e. $a_1 \equiv b_1(\text{mod } m)$.

§3. *Further Properties of Congruences*

a. *Both sides of a congruence and the modulus can be multiplied by the same integer.*

Indeed, it follows from $a \equiv b(\text{mod } m)$ that

$$a = b + mt,\ ak = bk + mkt$$

and hence, $ak \equiv bk(\text{mod } mk)$.

b. *Both sides of a congruence and the modulus can be divided by any one of their common divisors.*

Indeed, let

$$a \equiv b(\text{mod } m), \ a = a_1d, \ b = b_1d, \ m = m_1d.$$

We have

$$a = b + mt, \ a_1d = b_1d + m_1dt, \ a_1 = b_1 + m_1t$$

and hence $a_1 \equiv b_1(\text{mod } m_1)$.

c. *If the congruence $a \equiv b$ holds for several moduli, then it also holds for the modulus equal to the least common multiple of these moduli.*

Indeed, it follows from $a \equiv b(\text{mod } m_1)$, $a \equiv b(\text{mod } m_2)$, ..., $a \equiv b(\text{mod } m_k)$ that the difference $a - b$ is divisible by all the moduli $m_1, m_2, \ldots, m_k$. Therefore (**c, §3, ch. I**) it must be divisible by the least common multiple m of these moduli, i.e. $a \equiv b(\text{mod } m)$.

d. *If a congruence holds modulo m, then it also holds modulo d, which is equal to any divisor of the number m.*

Indeed, it follows from $a \equiv b(\text{mod } m)$ that the difference $a - b$ must be divisible by m; therefore (**1, b, §1, ch. I**) it must be divisible by any divisor d of the number m, i.e. $a \equiv b(\text{mod } d)$.

e. *If one side of a congruence and the modulus are divisible by some number then the other side of the congruence must also be divisible by the same number.*

Indeed, it follows from $a \equiv b(\text{mod } m)$ that $a = b + mt$, and if a and m are multiples of d, then (**2, b, §1, ch. I**) b must also be a multiple of d, as was to be proven.

f. *If $a \equiv b(\text{mod } m)$, then $(a, m) = (b, m)$.*

Indeed, in view of **2, b, §2, ch. I** this equation follows immediately from $a = b + mt$.

§4. *Complete Systems of Residues*

a. Numbers which are congruent modulo m form an *equivalence class modulo m.*

It follows from this definition that all the numbers of an equivalence class have the same remainder r, and we obtain all the numbers of an equivalence class if we let q in the form $mq + r$ run through all the integers.

Corresponding to the m different values of r we have m equivalence classes of numbers modulo m.

b. Any number of an equivalence class is said to be a *residue* modulo m with respect to all the numbers of the equivalence class. The residue obtained for $q = 0$ is equal to the remainder r itself, and is called the *least non-negative residue*.

The residue ρ of smallest absolute value is called *the absolutely least residue.*

It is evident that we have $\rho = r$ for $r < \frac{m}{2}$; for $r > \frac{m}{2}$ we have $\rho = r - m$; finally, if m is even and $r = \frac{m}{2}$, then we can take for ρ either of the two numbers $\frac{m}{2}$ and $\frac{m}{2} - m = -\frac{m}{2}$.

Taking one residue from each equivalence class, we obtain a *complete system of residues modulo m.* Frequently, as a complete system of residues we use the least non-negative residues $0, 1, \ldots, m - 1$ or the absolutely least residues; the latter, as follows from our above discussion, is represented in the case of odd m by the sequence

$$-\frac{m-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{m-1}{2},$$

and in the case of even m by either of the two sequences

$$-\frac{m}{2} + 1, \ldots, -1, 0, 1, \ldots, \frac{m}{2},$$
$$-\frac{m}{2}, \ldots, -1, 0, 1, \ldots, \frac{m}{2} + 1.$$

c. *Any m numbers which are pairwise incongruent modulo m form a complete system of residues modulo m.*

Indeed, being incongruent, these numbers must belong to different equivalence classes, and since there are m of them, i.e. as many as there are classes, it follows that one number falls into each class.

d. *If* $(a, m) = 1$ *and x runs over a complete system of residues modulo m, then $ax + b$, where b is any integer also runs over a complete system of residues modulo m.*

Indeed, there are as many numbers $ax + b$ as there are numbers x, i.e. m. Accordingly, it only remains to prove that any two numbers $ax_1 + b$ and $ax_2 + b$ corresponding to incongruent x_1 and x_2 will also be incongruent modulo m.

But, assuming that $ax_1 + b \equiv ax_2 + b (\text{mod } m)$, we arrive at the congruence $ax_1 \equiv ax_2 \pmod{m}$, from which we obtain $x_1 \equiv x_2 \pmod{m}$ as a consequence of $(a, m) = 1$, and this contradicts the assumption of the incongruence of the numbers x_1 and x_2.

§5. *Reduced Systems of Residues*

a. By **f**, **§3**, the numbers of an equivalence class modulo m all have the same greatest common divisor relative to the modulus. Particularly important are the equivalence classes for which this divisor is equal to unity, i.e. the classes containing numbers relatively prime to the modulus.

Taking one residue from each such class we obtain a *reduced system of residues modulo m.* A reduced system of residues therefore consists of the numbers of a complete system which are relatively prime to the modulus. A reduced system of residues is usually chosen from among the numbers of the system of least non-negative residues $0, 1, \ldots, m - 1$. Since the number of these numbers which are relatively prime to m is $\varphi(m)$, the number of numbers of a reduced system, which is equal to the number of equivalence classes containing numbers relatively prime to the modulus, is $\varphi(m)$.

Example. A reduced system of residues modulo 42 is

$$1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41.$$

b. *Any* $\varphi(m)$ *numbers which are pairwise incongruent modulo* m *and relatively prime to the modulus form a reduced system of residues modulo* m.

Indeed, being incongruent and relatively prime to the modulus, these numbers belong to different equivalence classes which contain numbers relatively prime to the modulus, and since there are $\varphi(m)$ of them, i.e. as many as there are classes of the above kind, it follows that there is one number in each class.

c. *If* $(a, m) = 1$ *and* x *runs through a reduced system of residues modulo* m, *then* ax *also runs through a reduced system of residues modulo* m.

Indeed, there are as many numbers ax as there are numbers x, i.e. $\varphi(m)$. By **b**, it only remains to prove that the numbers ax are incongruent modulo m and are relatively prime to the modulus. But the first was proved in **d**, §**4** for the numbers of the more general form $ax + b$, and the second follows from $(a, m) = 1$, $(x, m) = 1$.

§6. *The Theorems of Euler and Fermat*

a. *For* $m > 1$ *and* $(a, m) = 1$, *we have* (*Euler's theorem*):

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Indeed, if x runs through a reduced system of residues

$$x = r_1, r_2, \ldots, r_c; \; c = \varphi(m),$$

which consists of the least non-negative residues, then the least non-negative residues $\rho_1, \rho_2, \ldots, \rho_c$ of the numbers ax will run through the same system, but, generally speaking, in a different order (**c**, §**5**).

Multiplying the congruences

$$ar_1 \equiv \rho_1 \pmod{m}, \; ar_2 \equiv \rho_2 \pmod{m}, \; \ldots, \; ar_c \equiv \rho_c \pmod{m}$$

together termwise, we find

$$a^c r_1 r_2 \ldots r_c \equiv \rho_1 \rho_2 \ldots \rho_c \pmod{m},$$

from which we find

$$a^c \equiv 1 \pmod{m}$$

by dividing both sides by the product $r_1 r_2 \ldots r_c = \rho_1 \rho_2 \ldots \rho_c$.

b. *If* p *is a prime and* a *is not divisible by* p, *then we have* (*Fermat's theorem*):

$$a^{p-1} \equiv 1 \pmod{p}. \tag{1}$$

This theorem is a consequence of theorem **a** for $m = p$. The latter theorem can be put in better form. Indeed, multiplying both sides of the congruence (1) by a, we obtain the congruence

$$a^p \equiv a \pmod{p},$$

which is valid for all integers a, since it is valid for integers a which are multiples of p.

Problems for Chapter III

1, a. Representing an integer in the ordinary decimal system, deduce criteria for divisibility by 3, 9, 11.

b. Representing an integer in the calculational system to the base 100, deduce a criterion for divisibility by 101.

c. Representing an integer in the calculational system to the base 1000, deduce criteria for divisibility by 37, 7, 11, 13.

2, a. Let $m > 0$, $(a, m) > 1$, let b be an integer, let x run through a complete, while ξ runs through a reduced, system of residues modulo m. Prove that

$$\alpha)\ \sum_{x}\left\{\frac{ax+b}{m}\right\} = \frac{1}{2}(m-1),$$

$$\beta)\ \sum_{\xi}\left\{\frac{a\xi}{m}\right\} = \frac{1}{2}\varphi(m).$$

b. Let $m > 0$, $(a, m) = 1$; let b, N, t be integers, $t > 0$; let $f(x) = \dfrac{ax+b}{m}$, $f(N) > 0$, $f(N+mt) > 0$. Prove, for the trapezoid bounded by the lines $x = N$, $x = N + mt$, $y = 0$, $y = f(x)$, that

$$S = \sum \delta \tag{1}$$

where S is the area of the trapezoid, while the sum on the right is extended over all the lattice points of the trapezoid where $\delta = 1$ for the interior points, $\delta = \dfrac{1}{4}$ for the vertices, $\delta = \dfrac{1}{2}$ for the remaining points of the contour.

c. Letting, in contradistinction to problem **b**, $\delta = \dfrac{1}{6}$ for the vertices, prove formula (1) for a triangle with lattice point vertices.

3, a. Let $m > 0$, $(a, m) = 1$, $h \geqslant 0$, let c be a real number, let

$$S = \sum_{x=0}^{m-1}\left\{\frac{ax+\psi(x)}{m}\right\}$$

where $\psi(x)$ takes on values such that $c \leqslant \psi(x) \leqslant c + h$ for the values of x considered in the sum. Prove that

$$\left|S - \frac{1}{2}m\right| \leqslant h + \frac{1}{2}.$$

b. Let M be an integer, $m > 0$, $(a, m) = 1$, let A and B be real numbers, let

$$A = \frac{a}{m} + \frac{\lambda}{m^2}, \quad S = \sum_{x=M}^{M+m-1} \{Ax + B\}.$$

Prove that

$$\left|S - \frac{1}{2}m\right| \leqslant |\lambda| + \frac{1}{2}.$$

c. Let M be an integer, $m > 0$, $(a, m) = 1$,

$$S = \sum_{x=M}^{M+m-1} \{f(x)\},$$

where the function $f(x)$ has continuous derivatives $f'(x)$ and $f''(x)$ in the interval $M \leqslant x \leqslant M + m - 1$, while

$$f'(M) = \frac{a}{m} + \frac{\theta}{m^2}; \ (a, m) = 1; \ |\theta| < 1; \frac{1}{A} \leqslant |f''(x)| \leqslant \frac{k}{A},$$

where

$$1 \leqslant m \leqslant \tau, \ \tau = A^3, \ A \geqslant 2, \ k \geqslant 1.$$

Prove that

$$\left|S - \frac{1}{2}m\right| < \frac{k+3}{2}$$

4. Let all the partial quotients in the continued fraction development of the irrational number A be bounded, let M be an integer, let m be a positive integer, and let B be a real number. Prove that

$$\sum_{x=M}^{M+m-1} \{Ax + B\} = \frac{1}{2}m + O(\ln m).$$

5, a. Let $A > 2$, $k \geqslant 1$ and let the function $f(x)$ have a continuous second derivative satisfying the condition

$$\frac{1}{A} \leqslant |f''(x)| \leqslant \frac{k}{A}$$

on the interval $Q \leqslant x \leqslant R$. Prove that

$$\sum_{Q<x\leqslant R} \{f(x)\} = \frac{1}{2}(R - Q) + \theta\Delta; \ |\theta| < 1,$$

$$\Delta = (2k^2(R - Q) \ln A + 8kA)A^{-1/3}.$$

b. Let Q and R be integers, and let $0 < \sigma \leqslant 1$. Under the assumptions of problem **a**, prove that the number $\psi(\sigma)$ of fractions $\{f(x)\}$; $x = Q + 1, \ldots, R$ such that $0 \leqslant \{f(\sigma)\} < \sigma$ is given by the formula

$$\psi(\sigma) = \sigma(R - Q) + \theta' \cdot 2\Delta; \ |\theta'| < 1.$$

6, a. Let T be the number of lattice points (x, y) of the region $x^2 + y^2 \leqslant r^2$ $(r \geqslant 2)$. Prove that

$$T = \pi r^2 + O(r^{2/3} \ln r).$$

b. Let n be an integer, $n > 2$, and let E be Euler's constant. Prove that

$$\tau(1) + \tau(2) + \ldots + \tau(n) = n(\ln n + 2E - 1) + O(n^{1/3} (\ln n)^2).$$

7. A system of n positive integers, each of which is represented to the base 2, is said to be proper if for every nonnegative integer s, the number of integers in whose representation 2^s occurs, is even, and is said to be improper if this number is odd for at least one s.

Prove that an improper system can be made proper by decreasing or completely deleting some one of its members, while a proper system can be made improper by decreasing or completely deleting any one of its members.

8. a. Prove that the form

$$3^n x_n + 3^{n-1} x_{n-1} + \dots + 3x_1 + x_0,$$

where $x_n, x_{n-1}, \dots, x_1, x_0$ run through the values -1, 0, 1 independently of one another, represents the numbers

$$-H, \dots, -1, 0, 1, \dots, H; \ H = \frac{3^{n+1} - 1}{3 - 1}$$

and represents each of them uniquely.

b. Let $m_1, m_2, \dots, m_k$ be positive integers which are relatively prime in pairs. Using **c, §4,** prove that we obtain a complete residue system modulo $m_1 m_2 \dots m_n$, by inserting in the form

$$x_1 + m_1 x_2 + m_1 m_2 x_3 + \dots + m_1 m_2 \dots m_{k-1} x_k$$

the numbers $x_1, x_2, \dots, x_k$ which run through complete residue systems modulo $m_1, m_2, \dots, m_k$.

9. Let $m_1, m_2, \dots, m_k$ be integers which are relatively prime in pairs, and let

$$m_1 m_2 \dots m_k = m_1 M_1 = m_2 M_2 = \dots = m_k M_k.$$

a. Applying **c, §4,** prove that we obtain a complete system modulo $m_1 m_2 \dots m_k$ by inserting in the form

$$M_1 x_1 + M_2 x_2 + \dots + M_k x_k$$

the numbers $x_1, x_2, \ldots, x_k$ which run through a complete system of residues modulo $m_1, m_2, \ldots, m_k$.

b. Applying **c, §4, ch. II** and **b, §5,** prove that we obtain a reduced system of residues modulo $m_1 m_2 \ldots m_k$ by inserting in the form

$$M_1x_1 + M_2x_2 + \ldots + M_kx_k$$

the numbers $x_1, x_2, \ldots, x_k$ which run through a reduced residue system modulo $m_1, m_2, \ldots, m_k$.

c. Prove the theorem of problem **b** independently of theorem **c, §4, ch. II**, and then deduce the latter theorem from the former one.

d. Find an expression for $\varphi(p^{\alpha})$ by an elementary method, and using the equation in **c, §4, ch. II,** deduce an expression for $\varphi(a)$.

10. Let $m_1, m_2, \ldots, m_k$ be integers greater than 1, which are relatively prime in pairs, and let $m = m_1 m_2 \ldots m_k$, $m_s M_s = m$.

a. Let $x_1, x_2, \ldots, x_k, x$ run through complete residue systems, while $\xi_1, \xi_2, \ldots, \xi_k, \xi$ run through reduced residue systems modulo $m_1, m_2, \ldots, m_k, m$. Prove that the fractions

$$\left\{\frac{x_1}{m_1} + \frac{x_2}{m_2} + \ldots + \frac{x_k}{m_k}\right\}$$

coincide with the fractions $\left\{\frac{x}{m}\right\}$, while the fractions

$\left\{\frac{\xi_1}{m_1} + \frac{\xi_2}{m_2} + \ldots + \frac{\xi_k}{m_k}\right\}$ coincide with the fractions $\left\{\frac{\xi}{m}\right\}$.

b. Consider k entire rational functions with integral coefficients of the r variables $x, \ldots, w$ $(r \geqslant 1)$:

$$f_s(x, \ldots, w) = \sum_{\alpha, \ldots, \delta} c^{(s)}_{\alpha, \ldots, \delta}\, x^{\alpha} \ldots w^{\delta};\ s = 1, \ldots, k,$$

and let

$$f(x, \ldots, w) = \sum_{\alpha, \ldots, \delta} c_{\alpha, \ldots, \delta} x^{\alpha} \ldots w^{\delta};\ c_{\alpha, \ldots, \delta} =$$

$$= \sum_{s=1}^{k} M_s c^{(s)}_{\alpha, \ldots, \delta};$$

$x_s, \ldots, w_s$ run through complete residue systems, while $\xi_s, \ldots, \omega_s$ run through reduced residue systems modulo m_s; $x, \ldots, w$ run through complete residue systems, while $\xi, \ldots, \omega$ run through reduced residue systems modulo m. Prove that the fractions

$$\left\{\frac{f_1(x_1, \ldots, w_1)}{m_1} + \ldots + \frac{f_k(x_k, \ldots, w_k)}{m_k}\right\}$$

coincide with the fractions $\left\{\frac{f(x, \ldots, w)}{m}\right\}$, while the fractions

$$\left\{\frac{f_1(\xi_1, \ldots, \omega_1)}{m_1} + \ldots + \frac{f_k(\xi_k, \ldots, \omega_k)}{m_k}\right\}$$

coincide with the fractions $\left\{\frac{f(\xi, \ldots, \omega)}{m}\right\}$ (a generalization of the theorem of problem **a**).

11, a. Let m be a positive integer, let a be an integer, and let x run through a complete residue system modulo m. Prove that

$$\sum e^{2\pi i \frac{ax}{m}} = \begin{cases} m, \text{ if } a \text{ is a multiple of } m \\ 0, \text{ otherwise.} \end{cases}$$

b. Let α be a real number, and let M and P be integers with $P > 0$. Letting (α) denote the numerical value of the difference between α and the integer closest to α (the distance from α to the nearest integer), prove that

$$\left|\sum_{x=M}^{M+P-1} e^{2\pi i\alpha x}\right| \leqslant \min\left(P, \frac{1}{h(\alpha)}\right); \; h \geqslant \begin{cases} 2 \text{ always} \\ 3, \text{ for } (\alpha) \leqslant \dfrac{1}{6}. \end{cases}$$

c. Let m be an integer, $m > 1$, and let the functions $M(a)$ and $P(a)$ take on integral values such that $P(a) > 0$ for the values $a = 1, 2, \ldots, m-1$. Prove that

$$\sum_{a=1}^{m-1}\left|\sum_{x=M(a)}^{M(a)+P(a)-1} e^{2\pi i\frac{a}{m}x}\right| < \begin{cases} m \ln m - \dfrac{m}{3}\ln\left(2\left[\dfrac{m}{6}\right]+1\right) \text{for } m \geqslant 6 \\ m \ln m - \dfrac{m}{2}, \text{ for } m \geqslant 12, \\ m \ln m - m, \quad \text{for } m \geqslant 60. \end{cases}$$

12, a. Let m be a positive integer, and let ξ run through a reduced residue system modulo m. Prove that

$$\mu(m) = \sum_{\xi} e^{2\pi i\frac{\xi}{m}}$$

b. Using the theorem of problem **a,** prove the first of the theorems of **c, §3, ch. II** (cf. solution of problem **28, a, ch. II**).

c. Deduce the theorem of problem **a,** using the theorem of problem **17, a, ch. II.**

d. Let

$$f(x, \ldots, w) = \sum_{\alpha, \ldots, \delta} c_{\alpha, \ldots, \delta}\, x^{\alpha} \ldots w^{\delta}$$

be an entire rational function with integral coefficients of the r variables $x, \ldots, w (r \geqslant 1)$ and let a, m be integers with $m > 0$; $x, \ldots, w$ run through complete residue systems, while $\xi, \ldots, \omega$ run through reduced residue systems modulo m. We introduce the symbols

$$S_{a,m} = \sum_x \cdots \sum_w e^{2\pi i \frac{af(x,\ldots,w)}{m}},$$

$$S'_{a,m} = \sum_\xi \cdots \sum_\omega \exp(af(\xi,\ldots,\omega)/m)$$

Moreover, let $m = m_1 m_2 \ldots m_k$, where $m_1, \ldots, m_k$ are integers exceeding 1 which are relatively prime in pairs, and let $m_s M_s = m$. Prove that

$$S_{a_1,m_1} \ldots S_{a_k,m_k} = S_{M_1a_1 + \ldots + M_ka_k, m},$$

$$S'_{a_1,m_1} \ldots S'_{a_k,m_k} = S'_{M_1a_1 + \ldots + M_ka_k, m}.$$

e. Using the notation of problem **d** we set

$$A(m) = m^{-r} \sum_a S_{a,m}, \quad A'(m) = m^{-r} \sum_a S'_{a,m},$$

where a runs through a reduced residue system modulo m. Prove that

$$A(m_1) \ldots A(m_k) = A(m), \quad A'(m_1) \ldots A'(m_k) = A'(m).$$

13, a. Prove that

$$\varphi(a) = \sum_{n=0}^{a-1} \prod_p \left(1 - \frac{1}{p} \sum_{x=0}^{n-1} e^{2\pi i \frac{nx}{p}}\right)$$

where p runs through the prime divisors of the number a.

b. Deduce the well-known expression for $\varphi(a)$ from the identity of problem **a.**

14. Prove that

$$r(a) = \lim_{\epsilon \to 0} 2\epsilon \sum_{0<x<\sqrt{a}} \sum_{k=1}^{\infty} k^{-(1+\epsilon)} \exp(2\pi i a k/x) + \delta$$

where $\delta = 1$ or $\delta = 0$, according as a is or is not the square of an integer.

15, a. Let p be a prime and let $h_1, h_2, \ldots, h_a$ be integers. Prove that

$$(h_1 + h_2 + \ldots + h_a)^p \equiv h_1^p + h_2^p + \ldots + h_a^p \pmod{p}.$$

b. Deduce Fermat's theorem from the theorem of problem **a.**

c. Deduce Euler's theorem from Fermat's theorem.

Numerical Exercises for Chapter III.

1, a. Find the remainder resulting from the division of $(12\,371^{56} + 34)^{28}$ by 111.

b. Is the number $2^{1093} - 2$ divisible by $1\,093^2$?

2, a. Applying the divisibility criteria of problem **1,** find the canonical decomposition of the number 244 943 325.

b. Find the canonical decomposition of the number 282 321 246 671 737.

CHAPTER IV

CONGRUENCES IN ONE UNKNOWN

§1. *Basic Concepts*

Our immediate problem is the study of congruences of the general form:

(1) $f(x) \equiv 0 (\text{mod } m); f(x) = ax^n + a_1x^{n-1} + \ldots + a_n.$

If a is not divisible by m, then n is said to be the *degree of the congruence.*

Solving a congruence means finding all the values of x which satisfy it. Two congruences which are satisfied by the same values of x are said to be *equivalent.*

If the congruence (1) is satisfied by some $x = x_1$, then (**d, §2, ch. III**) this congruence will also be satisfied by all numbers which are congruent to x_1 modulo m: $x \equiv x_1 \pmod m$. This whole class of numbers is considered to be *one solution.* In accordance with this convention, *congruence* (1) *has as many solutions as residues of a complete system satisfying it.*

Example. The congruence

$$x^5 + x + 1 \equiv 0 (\text{mod } 7)$$

is satisfied by two numbers $x = 2$ and $x = 4$ among the numbers 0, 1, 2, 3, 4, 5, 6 of a complete residue system modulo 7.

Therefore the above congruence has the two solutions:

$$x \equiv 2 (\text{mod } 7), \quad x \equiv 4 (\text{mod } 7).$$

§2. *Congruences of the First Degree*

a. A congruence of the first degree whose constant term has been placed on the right side (with opposite sign) can be put in the form

$$ax \equiv b (\text{mod } m). \tag{1}$$

b. Turning to the investigation of the number of solutions, we first restrict the congruence by the condition $(a, m) = 1$. According to **§1**, our congruence has as many solutions as residues of a complete system satisfy it. But when x runs through a complete system of residues modulo m, ax also runs through a complete residue system (**d, 4, ch. III**). Therefore, in particular, ax will be congruent to b for one and only one value of x taken from the complete residue system. Therefore congruence (1) has one solution for $(a, m) = 1$.

c. Now let $(a, m) = d > 1$. Then, in order that the congruence (1) have a solution it is necessary (**e, §3, ch. III**) that b be divisible by d, for otherwise the congruence (1) is impossible for all integers x. Assuming then that b is a multiple of d, we set $a = a_1d$, $b = b_1d$, $m = m_1d$. Then the congruence (1) is equivalent to the following one (obtained by dividing through by d): $a_1x \equiv b_1 (\text{mod } m_1)$, in which $(a_1, m_1) = 1$, and therefore it will have one solution modulo m_1. Let x_1 be the least non-negative residue of this solution modulo m_1, then all the numbers x which are solutions of this equation are found to be of the form

$$x \equiv x_1 (\text{mod } m_1). \tag{2}$$

But modulo m the numbers of (2) do not form one solution, but many solutions, and indeed as many solutions as there are numbers of (2) in the sequence 0, 1, 2, ..., $m - 1$ of least non-negative residues modulo m. But these consist of the following numbers of (2):

$$x_1, x_1 + m_1, x_1 + 2m_1, \ldots, x_1 + (d - 1)m_1,$$

i.e. d numbers of the form (2), and hence the congruence (1) has d solutions.

d. Gathering together our results, we obtain the following theorem:

Let $(a, m) = d$. *The congruence* $ax \equiv b(\text{mod } m)$ *is impossible if* b *is not divisible by* d. *For* b *a multiple of* d, *the congruence has* d *solutions.*

e. Turning to the finding of solutions of the congruence (1), we shall only consider a method which is based on the theory of continued fractions, where it is sufficient to restrict ourselves to the case in which $(a, m) = 1$.

Developing the fraction m/a in a continued fraction,

$$\frac{m}{a} = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ddots + \cfrac{1}{q_n}}}$$

and considering the last two convergents:

$$\frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n} = \frac{m}{a}$$

by the properties of continued fractions (e, §4, **ch. I**) we have

$$mQ_{n-1} - aP_{n-1} = (-1)^n,$$

$$aP_{n-1} \equiv (-1)^{n-1} (\text{mod } m),$$

$$a \cdot (-1)^{n-1} P_{n-1} b \equiv b(\text{mod } m).$$

Hence, our congruence has the solution

$$x \equiv (-1)^{n-1} P_{n-1} b \pmod{m},$$

for whose calculation it is sufficient to calculate P_{n-1} by the method described in **d, §4, ch. I.**

Example. We solve the congruence

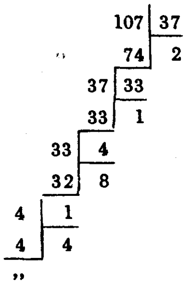
$$(3) \qquad 111x \equiv 75 \pmod{321}.$$

Here (111, 321) = 3, while 75 is a multiple of 3. Therefore the congruence has three solutions.

Dividing both sides of the congruence and the modulus by 3, we obtain the congruence

$$(4) \qquad 37x \equiv 25 \pmod{107},$$

which we must first solve. We have



q		2	1	8	4
P_s	1	2	3	26	107

Hence $n = 4$, $P_{n-1} = 26$, $b = 25$, and we have the solution of congruence (4) in the form

$$x \equiv -26 \cdot 25 \equiv 99 \pmod{107}.$$

From this the solutions of congruence (3) can be represented in the form:

$$x \equiv 99,\ 99 + 107,\ 99 + 2 \cdot 107 (\operatorname{mod} 321),$$

i.e.

$$x \equiv 99,\ 206,\ 313 (\operatorname{mod} 321).$$

§3. *Systems of Congruences of the First Degree*

a. We shall only consider the simplest system of congruences

(1) $x \equiv b_1 (\operatorname{mod} m_1),\ x \equiv b_2 (\operatorname{mod} m_2),\ \ldots,\ x \equiv b_k (\operatorname{mod} m_k)$

in one unknown, but with different and pairwise prime moduli.

b. It is possible to solve the system (1), i.e. find all values of x satisfying it, by applying the following theorem:

Let the numbers M_s and M_s' be defined by the conditions

$$m_1 m_2 \ldots m_s = M_s m_s,\ M_s M_s' \equiv 1 (\operatorname{mod} m_s)$$

and let

$$x_0 = M_1 M_1' b_1 + M_2 M_2' b_2 + \ldots + M_k M_k' b_k.$$

Then the set of values of x satisfying the system (1) *are defined by the congruence*

(2) $$x \equiv x_0 (\operatorname{mod} m_1 m_2 \ldots m_k)$$

Indeed, in view of the fact that all the M_j which are different from M_s are divisible by m_s, for any $s = 1, 2, \ldots, k$, we have

$$x_0 \equiv M_s M_s' b_s \equiv b_s (\operatorname{mod} m_s),$$

and therefore system (1) is satisfied by $x = x_0$. It follows immediately from this, that the system (1) is equivalent to the system

$$(3)\ x \equiv x_0 \pmod{m_1},\ x \equiv x_0 \pmod{m_2},\ \ldots,\ x \equiv x_0 \pmod{m_k}$$

(i.e. the systems (1) and (3) are satisfied by the same values of x). But the system (3), in view of the theorems of **c, §3, ch. III** and **d, §3, ch. III**, is satisfied by those and only those values of x which satisfy the congruence (2).

c. *If $b_1, b_2, \ldots, b_k$ independently run through complete residue systems modulo $m_1, m_2, \ldots, m_k$, then x_0 runs through a complete residue system modulo $m_1m_2\ldots m_k$.*

Indeed, x_0 runs through $m_1m_2\ldots m_k$ values which are incongruent modulo $m_1m_2\ldots m_k$, in view of **d, §3, ch. III.**

d. *Example.* We solve the system

$$x \equiv b_1 \pmod{4},\ x \equiv b_2 \pmod{5},\ x \equiv b_3 \pmod{7}.$$

Here $4 \cdot 5 \cdot 7 = 4 \cdot 35 = 5 \cdot 28 = 7 \cdot 20$, while

$$35 \cdot 3 \equiv 1 \pmod{4},\ 28 \cdot 2 \equiv 1 \pmod{5},\ 20 \cdot 6 \equiv 1 \pmod{7}.$$

Therefore

$$x = 35 \cdot 3b_1 + 28 \cdot 2b_2 + 20 \cdot 6b_3 = 105b_1 + 56b_2 + 120b_3$$

and hence the set of values of x satisfying the system, can be represented in the form

$$x \equiv 105b_1 + 56b_2 + 120b_3 \pmod{140}.$$

Thus, for example, the set of values satisfying the system

$$x \equiv 1 \pmod{4},\ x \equiv 3 \pmod{5},\ x \equiv 2 \pmod{7},$$

is

$$x \equiv 105 \cdot 1 + 56 \cdot 3 + 120 \cdot 2 \equiv 93 \pmod{140}$$

while the set of values of x satisfying the system

$$x \equiv 3 \pmod 4), \quad x \equiv 2 \pmod 5), \quad x \equiv 6 \pmod 7),$$

is

$$x \equiv 105 \cdot 3 + 56 \cdot 2 + 120 \cdot 6 \equiv 27 \pmod{140}.$$

§4. *Congruences of Arbitrary Degree with Prime Modulus*

a. Let p be a prime. We shall prove general theorems relating to congruences of the form

$$(1) \qquad f(x) \equiv 0 \pmod p; \; f(x) = ax^n + a_1x^{n-1} + \ldots + a_n.$$

b. *A congruence of the form* (1) *is equivalent to a congruence of degree not higher than* p − 1.

Indeed, dividing $f(x)$ by $x^p - x$, we have

$$f(x) = (x^p - x)Q(x) + R(x),$$

where the degree of $R(x)$ is not higher than $p - 1$. But $x^p - x \equiv 0 \pmod p$ implies that $f(x) \equiv R(x) \pmod p$, from which our theorem follows.

c. *If the congruence* (1) *has more than n solutions, then all the coefficients of $f(x)$ are multiples of* p.

Indeed, let the congruence (1) have at least $n + 1$ solutions. Letting $x_1, x_2, \ldots, x_n, x_{n+1}$ be the residues of these solutions, we can represent $f(x)$ in the form

$$\begin{aligned}(2) \quad f(x) = {} & a(x - x_1)(x - x_2)\ldots(x - x_{n-2})(x - x_{n-1})(x - x_n) + \\ & + b(x - x_1)(x - x_2)\ldots(x - x_{n-2})(x - x_{n-1}) + \\ & + c(x - x_1)(x - x_2)\ldots(x - x_{n-2}) + \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ & + k(x - x_1)(x - x_2) + \\ & + l(x - x_1) + \\ & + m.\end{aligned}$$

To this end, develop the summands on the right side into polynomials, and then choose b so that the sum of the coefficients of x^{n-1} in the first two polynomials coincide with a_1; knowing b, we choose c so that the sum of the coefficients of x^{n-2} in the first three polynomials coincides with a_2, etc.

Putting $x = x_1, x_2, \ldots, x_n, x_{n+1}$ successively in (2), we find that all the numbers $m, l, k, \ldots, c, b, a$ are multiples of p. This means that all the coefficients $a, a_1 \ldots, a_n$ are multiples of p (since they are sums of numbers which are multiples of p).

d. *For prime p, we have the congruence (Wilson's theorem)*

$$(3) \qquad 1 \cdot 2 \ldots (p - 1) + 1 \equiv 0 (\text{mod } p).$$

Indeed, if $p = 2$, then the theorem is evident. If $p > 2$, then we consider the congruence

$$(x - 1)(x - 2)\ldots(x - (p - 1)) - (x^{p-1} - 1) \equiv 0 (\text{mod } p);$$

its degree is not higher than $p - 2$ and it has $p - 1$ solutions, indeed solutions with residues $1, 2, \ldots, p - 1$. Therefore, by theorem **c**, all its coefficients are multiples of p; in particular the constant term is also divisible by p and the constant term is just equal to the left side of the congruence (3).

Example. We have $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1 = 721 \equiv 0 (\text{mod } 7)$.

§5. *Congruences of Arbitrary Degree with Composite Modulus*

a. *If $m_1, m_2, \ldots, m_k$ are pairwise prime, then the congruence*

$$(1) \qquad f(x) \equiv 0 (\text{mod } m_1 m_2 \ldots m_k)$$

is equivalent to the system

$$f(x) \equiv 0 (\text{mod } m_1),$$
$$f(x) \equiv 0 (\text{mod } m_2), \ \ldots, \ f(x) \equiv 0 (\text{mod } m_k).$$

Letting $T_1, T_2, \ldots, T_k$ be the numbers of solutions of the individual congruences of this system with respect to the corresponding moduli, and letting T be the number of solutions of the congruence (1), *we have*

$$T = T_1 T_2 \ldots T_k.$$

Indeed, the first part of the theorem follows from **c** and **d, §3, ch. III.** The second part of the theorem follows from the fact that each congruence

$$(2) \qquad f(x) \equiv 0 \pmod{m_s}$$

is satisfied if and only if one of the T_s congruences of the form

$$x \equiv b_s \pmod{m_s},$$

where b_s runs through the residues of the solutions of the congruence (2), is satisfied, while all $T_1 T_2 \ldots T_k$ different combinations of the form

$$x \equiv b_1 \pmod{m_1}, \; x \equiv b_2 \pmod{m_2}, \; \ldots, \; x \equiv b_k \pmod{m_k},$$

are possible, which leads (**c, §3**) to different classes modulo $m_1 m_2 \ldots m_k$.

Example. The congruence

$$(3) \qquad f(x) \equiv 0 \pmod{35}, \; f(x) = x^4 + 2x^3 + 8x + 9$$

is equivalent to the system

$$f(x) \equiv 0 \pmod 5, \; f(x) \equiv 0 \pmod 7.$$

It is easy (**§1**) to verify that the first congruence of this system has two solutions: $x \equiv 1; 4 \pmod 5$, the second congruence has three solutions: $x \equiv 3; 5; 6 \pmod 7$. Hence the congruence (3) has $2\cdot 3 = 6$ solutions. In order to find these six solutions, we must solve six systems of the form

$$x \equiv b_1 \pmod{5},\ x \equiv b_2 \pmod{7}, \tag{4}$$

which we obtain by letting b_1 run through the values $b_1 \equiv 1; 4$, while b_2 runs through the values $b_2 \equiv 3; 5; 6$. But since

$$35 = 5\cdot 7 = 7\cdot 5,\ 7\cdot 3 \equiv 1 \pmod{5},\ 5\cdot 3 \equiv 1 \pmod{7},$$

the set of values of x satisfying the system (4) can be represented in the form (**b**, §3)

$$x \equiv 21b_1 + 15b_2 \pmod{35}.$$

Therefore the solutions of congruence (3) are

$$x \equiv 31; 26; 6; 24; 19; 34 \pmod{35}.$$

b. In view of theorem **a** the investigation and solution of congruences of the form

$$f(x) \equiv 0 \pmod{p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}}$$

reduces to the investigation and solution of congruences of the form

$$f(x) \equiv 0 \pmod{p^{\alpha}}; \tag{5}$$

this last congruence reduces in general, as we shall soon see, to the congruence

$$f(x) \equiv 0 \pmod{p} \tag{6}$$

Indeed, every x satisfying the congruence (5) must necessarily satisfy the congruence (6). Let

$$x \equiv x_1 \pmod{p}$$

be any solution of the congruence (6). Then $x = x_1 + pt_1$, where t_1 is an integer. Inserting this value of x in the congruence

$$f(x) \equiv 0 \pmod{p^2}$$

and developing the left side by means of the Taylor formula, we find (noting that $\frac{1}{k!} f^{(k)}(x_1)$ is an integer, and deleting the terms which are multiples of p^2)

$$f(x_1) + pt_1f'(x_1) \equiv 0 \pmod{p^2}, \quad \frac{f(x_1)}{p} + t_1f'(x_1) \equiv 0 \pmod{p}.$$

Restricting ourselves to the case in which $f'(x_1)$ is not divisible by p, we have one solution:

$$t_1 \equiv t_1' \pmod{p}; \; t_1 = t_1' + pt_2.$$

The expression for x takes on the form

$$x = x_1 + p^2t_1' + p^2t_2 = x_2 + p^2t_2;$$

inserting it in the congruence

$$f(x) \equiv 0 \pmod{p^3},$$

we find

$$f(x_2) + p^2t_2f'(x_2) \equiv 0 \pmod{p^3}$$

$$\frac{f(x_2)}{p^2} + t_2f'(x_2) \equiv 0 \pmod{p}.$$

Here $f'(x_2)$ is not divisible by p since

$$x_2 \equiv x_1 \pmod{p},$$

$$f'(x_2) \equiv f'(x_1) \pmod{p},$$

and hence the latter equation has one solution:

$$t_2 \equiv t_2' \pmod{p},$$

$$t_2 = t_2' + pt_3.$$

The expression for x takes on the form

$$x = x_2 + p^2t_2' + p^3t_3 = x_3 + p^3t_3;$$

and so forth. In this way, given a solution of the congruence (6) we can find a solution of the congruence (5) which is congruent to it. *Hence, if $f'(x_1)$ is not divisible by p, each solution $x \equiv x_1 \pmod{p}$ of the congruence* (6) *gives a solution of the congruence* (5):

$$x = x_\alpha + p^\alpha t_\alpha;$$

$$x \equiv x_\alpha \pmod{p^\alpha}.$$

Example. We solve the congruence

$$(7) \qquad \begin{cases} f(x) \equiv 0 \pmod{27}; \\ f(x) = x^4 + 7x + 4. \end{cases}$$

The congruence $f(x) \equiv 0 \pmod{3}$ has one solution $x \equiv 1 \pmod{3}$; here $f'(1) \equiv 2 \pmod{3}$, and hence, is not divisible by 3. We find

$$x = 1 + 3t_1,$$

$$f(1) + 3t_1f'(1) \equiv 0 \pmod{9},\ 3 + 3t_1 \cdot 2 \equiv 0 \pmod{9},$$

$$2t_1 + 1 \equiv 0 \pmod{3},\ t_1 \equiv 1 \pmod{3},\ t_1 = 1 + 3t_2,$$

$$x = 4 + 9t_2,$$

$$f(4) + 9t_2f'(4) \equiv 0 \pmod{27},\ 18 + 9t_2 \cdot 2 \equiv 0 \pmod{27},$$

$$2t_2 + 2 \equiv 0 \pmod{3},\ t_2 \equiv 2 \pmod{3},\ t = 2 + 3t_3,$$

$$x = 22 + 27t_3.$$

Therefore, the congruence (7) has one solution:

$$x \equiv 22 \pmod{27}.$$

Problems for Chapter IV

1, a. Let m be a positive integer and let $f(x, \ldots, w)$ be an entire rational function with integral coefficients of the r variables $x, \ldots, w (r \geqslant 1)$. If the system $x = x_0, \ldots, w = w_0$ satisfies the congruence

$$(1) \qquad f(x, \ldots, w) \equiv 0 \pmod{m},$$

then (generalizing the definition of §**1**) the system of classes of integers modulo m:

$$x \equiv x_0 \pmod{m}, \quad \ldots, \quad w \equiv w_0 \pmod{m}$$

will be considered to be one solution of the congruence (1).

Let T be the number of solutions of the congruence (1). Prove that

$$Tm = \sum_{a=0}^{m-1} \sum_{x=0}^{m-1} \cdots \sum_{w=0}^{m-1} e^{2\pi i \frac{af(x, \ldots, w)}{m}}$$

b. Using the notation of problem **a** and problem **12, e, ch. III**, prove that

$$Tm = m^r \sum_{m_0 \backslash m} A(m_0).$$

c. Apply the equation of problem **a** to the proof of the theorem on the number of solutions of a congruence of the first degree.

d. Let m be a positive integer; let $a, \ldots, f, g$ be $r + l$ $(r > 0)$ integers; $d = (a, \ldots, f, m)$; let T be the number of solutions of the congruence

$$ax + \ldots + fw + g \equiv 0 \pmod{m}.$$

Using the equation of problem **a**, prove that

$$T = \begin{cases} m^{r-1}d, & \text{if } g \text{ is a multiple of } d, \\ 0, & \text{otherwise.} \end{cases}$$

e. Prove the theorem of problem **d**, starting from the theorem on the number of solutions of the congruence $ax \equiv b \pmod{m}$.

2, a. Let $m > 1$, $(a, m) = 1$. Prove that the congruence $ax \equiv b \pmod{m}$ has the solution $x \equiv ba^{\varphi(m)-1} \pmod{m}$.

b. Let p be a prime, $0 < a < p$. Prove that the congruence $ax \equiv b \pmod{p}$ has the solution

$$x \equiv b(-1)^{a-1} \frac{(p-1)(p-2)\ldots(p-a+1)}{1 \cdot 2 \ldots a} \pmod{p}.$$

c, α) Find the simplest possible method of solving a congruence of the form

$$2^k x \equiv b \pmod{m};\ (2, m) = 1.$$

β) Find the simplest possible method of solving a congruence of the form

$$3^k x \equiv b \pmod{m};\ (3, m) = 1.$$

γ) Let $(a, m) = 1$, $1 < a < m$. Applying the methods used in problems α) and β), prove that finding the solutions of the congruence $ax \equiv b \pmod{m}$ can be reduced to finding the solutions of a congruence of the form $b + mt \equiv 0 \pmod{p}$ where p is a prime divisor of the number a.

3. Let m be an integer, $m > 1$, $1 < r < m$, $(a, m) = 1$. Using the theory of congruences prove the existence of integers x and y such that

$$ax \equiv y \pmod{m}, \ 0 < x \leqslant r, \ 0 < |y| < \frac{m}{r}.$$

4, a. For $(a, m) = 1$, we will consider the symbolic fraction $\frac{b}{a}$ modulo m, which denotes any residue of a solution of the congruence $ax \equiv b \pmod{m}$. Prove that (the congruences are taken modulo m)

α) For $a \equiv a_1$, $b \equiv b_1$ we have $\frac{b}{a} \equiv \frac{b_1}{a_1}$.

β) The numerator b of the symbolic fraction $\frac{b}{a}$ can be replaced by a congruent b_0 which is a multiple of a. Then the symbolic fraction $\frac{b}{a}$ is congruent to the ordinary fraction $\frac{b_0}{a}$, where the congruence is taken with ordinary integers.

γ) $\frac{b}{a} + \frac{d}{c} \equiv \frac{bc + ad}{ac}$

δ) $\frac{b}{a} \cdot \frac{d}{c} \equiv \frac{bd}{ac}$

b, α) Let p be a prime, $p > 2$, and let a be an integer, $0 < a < p - 1$. Prove that

$$\binom{p-1}{a} \equiv (-1)^a \pmod{p}.$$

β) Let p be a prime, $p > 2$. Prove that

$$\frac{2^p - 2}{p} \equiv 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{p-1} \pmod{p}.$$

5, a. Let d be a divisor of the number a which is not divisible by primes smaller than n, and let κ be the number of different divisors of the number d. Prove that the number of multiples of d in the sequence

(1) $1 \cdot 2 \ldots n,\ 2 \cdot 3 \ldots (n+1),\ \ldots,\ a(a+1) \ldots (a+n-1)$

is $\dfrac{n^{\kappa} a}{d}$.

b. Let $p_1, p_2, \ldots, p_k$ be the different prime divisors of the number a which are not smaller than n. Prove that the number of integers of the sequence (1) relatively prime to a is

$$a\left(1 - \frac{n}{p_1}\right)\left(1 - \frac{n}{p_2}\right)\cdots\left(1 - \frac{n}{p_k}\right)$$

6. Let $m_{1,2,\ldots,k}$ be the least common multiple of the numbers $m_1, m_2, \ldots, m_k$.

a. Let $d = (m_1, m_2)$. Prove that the system

$$x \equiv b_1 \pmod{m_1},\quad x \equiv b_2 \pmod{m_2}$$

is solvable if and only if $b_2 - b_1$ is a multiple of d, and if the system is solvable, the set of values of x satisfying this system is determined by a congruence of the form

$$x \equiv x_{1,2} \pmod{m_{1,2}}.$$

b. Prove that, if the system

$$x \equiv b_1 \pmod{m_1},\quad x \equiv b_2 \pmod{m_2},\quad \ldots,\quad x \equiv b_k \pmod{m_k}$$

is solvable, the set of values of x satisfying it is determined by a congruence of the form

$$x \equiv x_{1,2,\ldots,k} \pmod{m_{1,2,\ldots,k}}.$$

7. Let m be an integer, $m > 1$, let a and b be integers, and let

$$\left(\frac{a, b}{m}\right) = \sum_x \exp\left(2\pi i \frac{ax + bx'}{m}\right)$$

where x runs through a reduced residue system modulo m, while $x' \equiv \frac{1}{x} \pmod m$ (in the sense of problem **4, a**). Prove the following properties of the symbol $\left(\frac{a, b}{m}\right)$:

α) $\left(\frac{a, b}{m}\right)$ is real.

β) $\left(\frac{a, b}{m}\right) = \left(\frac{b, a}{m}\right)$.

γ) For $(h, m) = 1$, we have $\left(\frac{a, bh}{m}\right) = \left(\frac{ah, b}{m}\right)$.

δ) For $m_1, m_2, \ldots, m_k$ relatively prime in pairs, setting $m_1 m_2 \ldots m_k = m$, $m = m_s M_s$, we have

$$\left(\frac{a_1, 1}{m_1}\right)\left(\frac{a_2, 1}{m_2}\right) \cdots \left(\frac{a_k, 1}{m_k}\right) =$$

$$= \left(\frac{M_1^2 a_1 + M_2^2 a_2 + \ldots + M_k^2 a_k, 1}{m}\right).$$

8. Let the congruence

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n \equiv 0 \pmod{p}$$

have the n solutions

$$x \equiv x_1, x_2, \ldots, x_n \pmod{p}.$$

Prove that

$$\begin{aligned} a_1 &\equiv -a_0S_1 \pmod p, \\ a_2 &\equiv a_0S_2 \pmod p, \\ a_3 &\equiv -a_0S_3 \pmod p, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n &\equiv (-1)^n a_0S_n \pmod p, \end{aligned}$$

where S_1 is the sum of all the x_s, S_2 is the sum of the products of pairs of the x_s, S_3 is the sum of the products of triples of the x_s, etc.

9, a. Prove Wilson's theorem by considering pairs x, x' of numbers of the sequence $2, 3, \ldots, p - 2$, satisfying the condition $xx' \equiv 1 \pmod p$.

b. Let P be an integer, $P > 1$, $1 \cdot 2 \ldots (P - 1) + 1 \equiv \equiv 0 \pmod P$. Prove that P is a prime.

10, a. Let $(a_0, m) = 1$. Find a congruence of degree $n (n > 0)$ with leading coefficient 1, equivalent to the congruence

$$a_0x^n + a_1x^{n-1} + \ldots + a_n \equiv 0 \pmod m.$$

b. Prove that a necessary and sufficient condition in order that the congruence $f(x) \equiv 0 \pmod p$; $f(x) = x^n + a_1x^{n-1} + + \ldots + a_n$; $n \leqslant p$; has n solutions, is the divisibility by p of all the coefficients of the remainder after the division of $x^p - x$ by $f(x)$.

c. Let n be a divisor of $p - 1$; $n > 1$; $(A, p) = 1$. Prove that a necessary and sufficient condition for the solvability of the congruence $x^n \equiv A \pmod p$ is $A^{\frac{p-1}{n}} \equiv 1 \pmod p$, while if the congruence is solvable, it has n solutions.

11. Let n be a positive integer, $(A, m) = 1$, we assume that we know a solution $x \equiv x_0 \pmod m$ of the congruence

$x^n \equiv A \pmod{m}$. Prove that all the solutions of this congruence can be represented as the product of x_0 and a residue of a solution of the congruence $y^n \equiv 1 \pmod{m}$.

Numerical Exercises for Chapter IV

1, a. Solve the congruence $256x \equiv 179 \pmod{337}$.

b. Solve the congruence $1215x \equiv 560 \pmod{2755}$.

2, a. Solve the congruences of exercises **1, a** and **1, b** by the method of problem **2, c.**

b. Solve the congruence $1296x \equiv 1105 \pmod{2413}$ by the method of problem **2, c.**

3. Find all pairs x, y satisfying the indeterminate equation $1245x - 1603y = 999$.

4, a. Find a general solution of the system

$$x \equiv b_1 \pmod{13}, \quad x \equiv b_2 \pmod{17}.$$

Using this general solution, find three numbers whose division by 13 and 17 gives the respective remainders 1 and 12, 6 and 8, 11 and 4.

b. Find a general solution for the system

$$x \equiv b_1 \pmod{25}, \quad x \equiv b_2 \pmod{27}, \quad x \equiv b_3 \pmod{59}.$$

5, a. Solve the system of congruences

$$x \equiv 3 \pmod{8}, \quad x \equiv 11 \pmod{20}, \quad x \equiv 1 \pmod{15}.$$

b. Solve the system of congruences

$$x \equiv 1 \pmod{3}, \quad x \equiv 4 \pmod{5}, \quad x \equiv 2 \pmod{7},$$
$$x \equiv 9 \pmod{11}, \quad x \equiv 3 \pmod{13}.$$

6. Solve the system of congruences

$$x + 4y - 29 \equiv 0 \pmod{143}, \quad 2x - 9y + 84 \equiv 0 \pmod{143}.$$

7, a. What congruence of degree smaller than 5 is equivalent to the congruence

$$3x^{14} + 4x^{13} + 3x^{12} + 2x^{11} + x^9 + 2x^8 + 4x^7 + x^6 + {} + 3x^4 + x^3 + 4x^2 + 2x \equiv 0 \pmod 5?$$

b. What congruence of degree smaller than 7 is equivalent to the congruence

$$2x^{17} + 6x^{16} + x^{14} + 5x^{12} + 3x^{11} + 2x^{10} + x^9 + 5x^8 + {} + 2x^7 + 3x^5 + 4x^4 + 6x^3 + 4x^2 + x + 4 \equiv 0 \pmod 7?$$

8. What congruence with leading coefficient 1 is equivalent to the congruence (problem **10, a**)

$$70x^6 + 78x^5 + 25x^4 + 68x^3 + 52x^2 + 4x + 3 \equiv 0 \pmod{101}?$$

9, a. Solve the congruence

$$f(x) \equiv 0 \pmod{27}, \quad f(x) = 7x^4 + 19x + 25,$$

by first finding all the solutions of the congruence

$$f(x) \equiv 0 \pmod 3$$

by trial.

b. Solve the congruence $9x^2 + 29x + 62 \equiv 0 \pmod{64}$.

10, a. Solve the congruence $x^3 + 2x + 2 \equiv 0 \pmod{125}$.

b. Solve the congruence $x^4 + 4x^3 + 2x^2 + 2x + 12 \equiv 0 \pmod{625}$.

11, a. Solve the congruence $6x^3 + 27x^2 + 17x + 20 \equiv 0 \pmod{30}$.

b. Solve the congruence $31x^4 + 57x^3 + 96x + 191 \equiv 0 \pmod{225}$.

CHAPTER V

CONGRUENCES OF THE SECOND DEGREE

§1. *General Theorems*

a. We shall only consider the simplest of the congruences of degree $n > 1$, i.e. *the two-term congruences*:

$$(1) \qquad x^n \equiv a \pmod{m}; \quad (a, m) = 1$$

If the congruence (1) has solutions, then a is said to be an *n-th power residue*, otherwise a is said to be an *n-th power non-residue*. In particular, for $n = 2$ the residues or non-residues are said to be *quadratic*, for $n = 3$ *cubic*, for $n = 4$ *biquadratic*.

In this chapter we shall consider the case $n = 2$ in detail and we first consider the two-term congruences of the second degree for odd prime modulus p:

$$(2) \qquad x^2 \equiv a \pmod{p}; \quad (a, p) = 1.$$

c. *If a is a quadratic residue modulo* p, *then the congruence* (2) *has two solutions.*

Indeed, if a is a quadratic residue, then the congruence (2) has at least one solution $x \equiv x_1 \pmod{p}$. But since $(-x_1)^2 = x_1^2$, the same congruence also has the second solution $x \equiv -x_1 \pmod{p}$. This second solution is different from the first since $x_1 \equiv -x_1 \pmod{p}$ would imply $2x_1 \equiv 0 \pmod{p}$, which is impossible since $(2, p) = (x_1, p) = 1$.

These two solutions exhaust all the solutions of the congruence (2) since the latter, being a congruence of the second degree, cannot have more than two solutions (**c**, §**4**, **ch. IV**).

d. *A reduced residue system modulo p consists of* $\frac{p-1}{2}$ *quadratic residues which are congruent to the numbers*

$$(3) \qquad 1^2,\ 2^2,\ \ldots,\ \left(\frac{p-1}{2}\right)^2$$

and $\frac{p-1}{2}$ *quadratic non-residues.*

Indeed, among the residues of a reduced system modulo p, the quadratic residues are those and only those which are squares of the numbers (a reduced system of residues)

$$(4) \qquad -\frac{p-1}{2},\ \ldots,\ -2,\ -1,\ 1,\ 2,\ \ldots,\ \frac{p-1}{2}$$

i.e. with the numbers of (3). Here the numbers of (3) are incongruent modulo p, since $k^2 \equiv l^2 \pmod{p}$, $0 < k < 1 \leqslant \frac{p-1}{2}$, it would follow that the congruence $x^2 \equiv l^2 \pmod{p}$ is satisfied by four numbers: $x = -l, -k, k, l$ among the numbers of (4), contradicting **c**.

e. *If a is a quadratic residue modulo p, then*

$$(5) \qquad a^{\frac{p-1}{2}} \equiv 1 \pmod{p};$$

if a is a quadratic non-residue modulo p, then

$$(6) \qquad a^{\frac{p-1}{2}} \equiv -1 \pmod{p}.$$

Indeed, by Fermat's theorem,

$$a^{p-1} \equiv 1 \pmod p; \quad \left(a^{\frac{p-1}{2}} - 1\right)\left(a^{\frac{p-1}{2}} + 1\right) \equiv 0 \pmod p.$$

One and only one of the factors of the left side of the latter congruence is divisible by p (both factors cannot be divisible by p, for if they were, then 2 would be divisible by p). Therefore one and only one of the congruences (5) and (6) can hold.

But every quadratic residue a satisfies the congruence

$$a \equiv x^2 \pmod p \tag{7}$$

for some x, and therefore also satisfies the congruence (5), which can be obtained by raising each side of (7) to the power $\frac{p-1}{2}$. Here the quadratic residues exhaust all the solutions of the congruence (5), since it cannot have more than $\frac{p-1}{2}$ solutions because it is a congruence of degree $\frac{p-1}{2}$.

Therefore the quadratic non-residues satisfy the congruence (6).

§2. *The Legendre Symbol*

a. We now consider *Legendre's symbol* $\left(\frac{a}{p}\right)$ (read as: the symbol of a with respect to p). This symbol is defined for all a which are not divisible by p; it is equal to 1 if a is a quadratic residue, and equal to -1 if a is a quadratic non-residue. The number a is said to be the numerator, the number p the denominator, of the symbol.

b. In view of **e**, **§1**, it is evident that we have

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod p.$$

c. Here we deduce the most important properties of the Legendre symbol and in the next paragraph, the properties of the generalization of this symbol—Jacobi's symbol, which is useful for the rapid calculation of this symbol, and hence solves the problem of the possibility of the congruence

$$x^2 \equiv a \pmod p.$$

d. *If* $a \equiv a_1 \pmod p$, *then* $\left(\dfrac{a}{p}\right) = \left(\dfrac{a_1}{p}\right)$.

This property follows from the fact that the numbers of an equivalence class are all either quadratic residues or non-residues.

e. $\left(\dfrac{1}{p}\right) = 1.$

Indeed, $1 = 1^2$ and hence 1 is a quadratic residue.

f. $\left(\dfrac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

This property follows from **b** for $a = -1$.

Since $\dfrac{p-1}{2}$ is even for p of the form $4m + 1$ and odd for p of the form $4m + 3$, it follows that -1 is a quadratic residue of primes of the form $4m + 1$ and a quadratic non-residue of primes of the form $4m + 3$.

g. $\left(\dfrac{ab\ldots l}{p}\right) = \left(\dfrac{a}{p}\right)\left(\dfrac{b}{p}\right)\cdots\left(\dfrac{l}{p}\right)$.

Indeed, we have

$$\left(\frac{ab\ldots l}{p}\right) \equiv (ab\ldots l)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} b^{\frac{p-1}{2}} \ldots l^{\frac{p-1}{2}} \equiv$$

$$\equiv \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)\cdots\left(\frac{l}{p}\right) \pmod p,$$

from which it follows that our assertion is true. A consequence of our result is

$$\left(\frac{ab^2}{p}\right) = \left(\frac{a}{p}\right)$$

i.e. we can delete any square factor from the numerator of a symbol.

h. In order to deduce further properties of Legendre's symbol, we first give it another interpretation. Setting $p_1 = \frac{p-1}{2}$, we consider the congruences

$$(1)\qquad \left\{\begin{array}{l} a \cdot 1 \equiv \epsilon_1 r_1 \,(\text{mod } p) \\ a \cdot 2 \equiv \epsilon_2 r_2 \,(\text{mod } p) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a \cdot p_1 \equiv \epsilon_{p_1} r_{p_1} (\text{mod p});\ p_1 = \dfrac{p-1}{2}. \end{array}\right.$$

where $\epsilon_x r_x$ is the absolutely least residue of ax and r_x is its modulus so that $\epsilon_x = \pm 1$.

The numbers $a \cdot 1$, $-a \cdot 1$, $a \cdot 2$, $-a \cdot 2$, $\ldots$, $a \cdot p_1$, $-a \cdot p_1$ form a reduced residue system modulo p (**c, §5, ch. III**); their absolutely least residues are just $\epsilon_1 r_1$, $-\epsilon_1 r_1$, $\epsilon_2 r_2$, $-\epsilon_2 r_2$, $\ldots$, $\epsilon_{p_1} r_{p_1}$, $-\epsilon_{p_1} r_{p_1}$. Those which are positive i.e. $r_1, r_2, \ldots, r_{p_1}$, must coincide with the numbers $1, 2, \ldots, p_1$ (**b, §4, ch. III**).

Multiplying together the congruences (1) and dividing through by

$$1 \cdot 2 \ldots p_1 = r_1 r_2 \ldots r_{p_1},$$

we find $a^{\frac{p-1}{2}} \equiv \epsilon_1 \epsilon_2 \ldots \epsilon_{p_1}$ (mod p), from which (**b**) we have

$$(2)\qquad \left(\frac{a}{p}\right) = \epsilon_1 \epsilon_2 \ldots \epsilon_{p_1}$$

i. The expression for Legendre's symbol which we have found can be put in a more concise form. We have

$$\left[\frac{2ax}{p}\right] = \left[2\left[\frac{ax}{p}\right] + 2\left\{\frac{ax}{p}\right\}\right] = 2\left[\frac{ax}{p}\right] + \left[2\left\{\frac{ax}{p}\right\}\right],$$

which is even or odd according as the least positive residue of the number ax is less or greater than $\frac{1}{2}p$, i.e. according as $\epsilon_x = 1$ or $\epsilon_x = -1$. It is evident from this that

$$\epsilon_x = (-1)^{\left[\frac{2ax}{p}\right]}$$

and therefore we find from (2) that

$$\left(\frac{a}{p}\right) = (-1)^{\sum\limits_{x=1}^{p_1}\left[\frac{2ax}{p}\right]}$$

j. Assuming a to be odd, we transform the latter equation. We have ($a + p$ is even)

$$\left(\frac{2a}{p}\right) = \left(\frac{2a + 2p}{p}\right) = \left(\frac{4\cdot\frac{a+p}{2}}{p}\right) = \left(\frac{\frac{a+p}{2}}{p}\right) =$$

$$= (-1)^{\sum\limits_{x=1}^{p_1}\left[\frac{(a+p)x}{p}\right]} = (-1)^{\sum\limits_{x=1}^{p_1}\left[\frac{ax}{p}\right] + \sum\limits_{x=1}^{p_1} x}$$

and hence

$$\left(\frac{2}{p}\right)\left(\frac{a}{p}\right) = (-1)^{\sum\limits_{x=1}^{p_1}\left[\frac{ax}{p}\right] + \frac{p^2-1}{8}} \tag{3}$$

The formula (3) allows us to deduce two very important properties of the Legendre symbol.

k. $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

This follows from formula (3) for $a = 1$. Moreover, since

$$\frac{(8m \pm 1)^2 - 1}{8} = 8m^2 \pm 2m, \text{ even}$$

while

$$\frac{(8m \pm 3)^2 - 1}{8} = 8m^2 \pm 6m + 1, \text{ odd},$$

it follows that 2 is a quadratic residue of primes of the form $8m \pm 1$ ($8m + 1$, $8m + 7$) and a quadratic non-residue of primes of the form $8m \pm 3$ ($8m + 3$, $8m + 5$).

l. *If p and q are odd primes, then (the quadratic reciprocity law)*

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}\left(\frac{p}{q}\right)$$

Since $\frac{p-1}{2} \cdot \frac{q-1}{2}$ is odd only in the case in which both numbers p and q are of the form $4m + 3$ and even if one of these numbers is of the form $4m + 1$, the above property can be formulated as follows:

If both the numbers p and q are of the form $4m + 3$, then

$$\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right);$$

if one of them is of the form $4m + 1$, then

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right).$$

In order to prove our results, we note that, in view of **k**, formula (3) takes on the form

$$(4) \qquad \left(\frac{a}{p}\right) = (-1)^{\sum\limits_{x=1}^{p_1}\left[\frac{ax}{p}\right]}$$

Setting $\frac{q-1}{2} = q_1$, we consider $p_1 q_1$ pairs of numbers which are obtained when the numbers a and y in the expressions qx, py run through the systems of values

$$x = 1, 2, \ldots, p_1, \; y = 1, 2, \ldots, q_1,$$

independently.

We can never have $qx = py$, because it would follow from this equation that py is a multiple of q which is impossible because $(p, q) = (y, q) = 1$ (since $0 < y < q$). Therefore we can set $p_1 q_1 = S_1 + S_2$, where S_1 is the number of pairs with $qx < py$ and S_2 is the number of pairs with $py < qx$.

It is evident that S_1 is also the number of pairs with $x < \frac{p}{q}y$. For given y we can take $x = 1, 2, \ldots, \left[\frac{p}{q}y\right]$.

(Since $\frac{p}{q}y \leqslant \frac{p}{q}q_1 < \frac{p}{2}$ we have $\left[\frac{p}{q}y\right] \leqslant p_1$.).

Consequently,

$$S_1 = \sum_{y=1}^{q_1} \left[\frac{p}{q}y\right].$$

Analogously, we can prove that

$$S_2 = \sum_{x=1}^{p_1} \left[\frac{q}{p}x\right].$$

But then equation (4) gives

$$\left(\frac{p}{q}\right) = (-1)^{S_1}, \quad \left(\frac{q}{p}\right) = (-1)^{S_2}$$

and hence

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{S_1+S_2} = (-1)^{p_1q_1}$$

from which the required property follows.

§3. *The Jacobi Symbol*

a. In order to evaluate Legendre's symbol most quickly, we consider the more general *Jacobi symbol.* Let P be an odd number greater than unity, and let $P = p_1p_2\ldots p_r$ be its decomposition into prime factors (some of which may be equal). Moreover, let $(a, P) = 1$. Then Jacobi's symbol is defined by the equation

$$\left(\frac{a}{P}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_r}\right)$$

The well-known properties of the Legendre symbol allow us to establish analogous properties for the Jacobi symbol.

b. *If* $a \equiv a_1 \pmod{P}$, *then* $\left(\frac{a}{P}\right) = \left(\frac{a_1}{P}\right)$.

Indeed,

$$\left(\frac{a}{P}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_r}\right) =$$

$$= \left(\frac{a_1}{p_1}\right)\left(\frac{a_1}{p_2}\right)\cdots\left(\frac{a_1}{p_r}\right) = \left(\frac{a_1}{P}\right)$$

so that a, being congruent to a_1 modulo P, is also congruent to a_1 modulo $p_1, p_2, \ldots, p_r$, which are the divisors of P.

c. $\left(\frac{1}{P}\right) = 1.$

Indeed,

$$\left(\frac{1}{P}\right) = \left(\frac{1}{p_1}\right)\left(\frac{1}{p_2}\right)\cdots\left(\frac{1}{p_r}\right) = 1.$$

d. $\left(\frac{-1}{P}\right) = (-1)^{\frac{P-1}{2}}.$

In order to establish this, we note that

$$\left(\frac{-1}{P}\right) = \left(\frac{-1}{p_1}\right)\left(\frac{-1}{p_2}\right)\cdots\left(\frac{-1}{p_r}\right) = \tag{1}$$

$$= (-1)^{\frac{p_1-1}{2}+\frac{p_2-1}{2}+\cdots+\frac{p_r-1}{2}};$$

but

$$\frac{P-1}{2} = \frac{p_1p_2\cdots p_r - 1}{2} =$$

$$= \frac{\left(1+2\frac{p_1-1}{2}\right)\left(1+2\frac{p_2-1}{2}\right)\cdots\left(1+2\frac{p_r-1}{2}\right)-1}{2} =$$

$$= \frac{p_1-1}{2} + \frac{p_2-1}{2} + \cdots + \frac{p_r-1}{2} + 2N$$

and hence from formula (1) we deduce

$$\left(\frac{-1}{P}\right) = (-1)^{\frac{P-1}{2}}.$$

e. $\left(\frac{ab\ldots l}{P}\right) = \left(\frac{a}{P}\right)\left(\frac{b}{P}\right)\ldots\left(\frac{l}{P}\right).$

Indeed,

$$\left(\frac{ab\ldots l}{P}\right) = \left(\frac{ab\ldots l}{p_1}\right)\ldots\left(\frac{ab\ldots l}{p_r}\right) =$$

$$= \left(\frac{a}{p_1}\right)\left(\frac{b}{p_1}\right)\ldots\left(\frac{l}{p_1}\right)\ldots\left(\frac{a}{p_r}\right)\left(\frac{b}{p_r}\right)\ldots\left(\frac{l}{p_r}\right);$$

and multiplying the symbols with the same numerators, we obtain the required property. From this we obtain the corollary:

$$\left(\frac{ab^2}{P}\right) = \left(\frac{a}{P}\right).$$

f. $\left(\frac{2}{P}\right) = (-1)^{\frac{P^2-1}{8}}.$

Indeed,

$$\left(\frac{2}{P}\right) = \left(\frac{2}{p_1}\right)\left(\frac{2}{p_2}\right)\ldots\left(\frac{2}{p_r}\right) = \tag{2}$$

$$= (-1)^{\frac{p_1^2-1}{8} + \frac{p_2^2-1}{8} + \ldots + \frac{p_r^2-1}{8}}.$$

But

$$\frac{P^2-1}{8} = \frac{p_1^2p_2^2\ldots p_r^2 - 1}{8} =$$

$$= \frac{\left(1 + 8\frac{p_1^2-1}{8}\right)\left(1 + 8\frac{p_2^2-1}{8}\right)\ldots\left(1 + 8\frac{p_r^2-1}{8}\right) - 1}{8} =$$

$$= \frac{p_1^2-1}{8} + \frac{p_2^2-1}{8} + \ldots + \frac{p_r^2-1}{8} + 2N$$

and hence we deduce from formula (2)

$$\left(\frac{2}{P}\right) = (-1)^{\frac{P^2-1}{8}}.$$

g. *If P and Q are positive relatively prime odd numbers*, then

$$\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}\left(\frac{P}{Q}\right).$$

Indeed, let $Q = q_1 q_2 \ldots q_s$ be the decomposition of Q into prime factors (some of them may be equal). We have

$$\left(\frac{Q}{P}\right) = \left(\frac{Q}{p_1}\right)\left(\frac{Q}{p_2}\right)\cdots\left(\frac{Q}{p_r}\right) = \prod_{\alpha=1}^{r}\prod_{\beta=1}^{s}\left(\frac{q_\beta}{p_\alpha}\right) =$$

$$= (-1)^{\sum\limits_{\alpha=1}^{r}\sum\limits_{\beta=1}^{s}\frac{p_\alpha-1}{2}\cdot\frac{q_\beta-1}{2}}\prod_{\alpha=1}^{r}\prod_{\beta=1}^{s}\left(\frac{p_\alpha}{q_\beta}\right) =$$

$$= (-1)^{\left(\sum\limits_{\alpha=1}^{r}\frac{p_\alpha-1}{2}\right)\left(\sum\limits_{\beta=1}^{s}\frac{q_\beta-1}{2}\right)}\left(\frac{P}{Q}\right).$$

But, as in **d**, we find

$$\frac{P-1}{2} = \sum_{\alpha=1}^{r}\frac{p_\alpha-1}{2} + 2N, \quad \frac{Q-1}{2} = \sum_{\beta=1}^{s}\frac{q_\beta-1}{2} + 2N_1,$$

and hence

$$\left(\frac{Q}{P}\right) = (-1)^{\frac{P-1}{2}\cdot\frac{Q-1}{2}}\left(\frac{P}{Q}\right).$$

Example. As an example of the calculation of the Legendre symbol (we will consider it to be a particular case of the Jacobi symbol) we investigate the solutions of the congruence

$$x^2 \equiv 219 \pmod{383}.$$

We have (applying in sequence the properties **g**, **b**, the corollary of **e**, **g**, **b**, **e**, **f**, **g**, **b**, **d**):

$$\left(\frac{219}{383}\right) = -\left(\frac{383}{219}\right) = -\left(\frac{164}{219}\right) = -\left(\frac{41}{219}\right) =$$

$$= -\left(\frac{219}{41}\right) = -\left(\frac{14}{41}\right) = -\left(\frac{2}{41}\right)\left(\frac{7}{41}\right) =$$

$$= -\left(\frac{7}{41}\right) = -\left(\frac{41}{7}\right) = -\left(\frac{-1}{7}\right) = 1,$$

and hence the congruence under consideration has two solutions.

§4. *The Case of Composite Moduli*

a. Congruences of the second degree with composite moduli are investigated and solved in accordance with the general methods of **§5, ch. IV.**

b. We start with a congruence of the form

$$x^2 \equiv a \pmod{p^{\alpha}};\ \alpha > 0,\ (a,\ p) = 1, \tag{1}$$

where p is an odd prime.

Setting $f(x) = x^2 - a$, we have $f'(x) = 2x$, and if $x \equiv x_1 \pmod{p}$ is a solution of the congruence

$$x^2 \equiv a \pmod{p} \tag{2}$$

then since $(a, p) = 1$ we also have $(x_1, p) = 1$, and since p is odd, $(2x_1, p) = 1$, i.e. $f'(x_1)$ is not divisible by p. Therefore to find the solutions of the congruence (1) we can apply the argument of **b, §5, ch. IV,** while each solution of the congruence (2) gives one solution of the congruence (1). It follows from this that

The congruence (1) *has two solutions or none according as a is a quadratic residue or a quadratic non-residue modulo* p.

c. We now consider the congruence

$$x^2 \equiv a \pmod{2^\alpha}; \ \alpha > 0, \ (a, 2) = 1. \tag{3}$$

Here $f'(x_1) = 2x_1$ is divisible by 2, and hence the argument of **b, §5, ch. IV** is inapplicable; it can be changed in the following way:

d. If the congruence (3) is solvable, then, since $(a, 2) = 1$, we have $(x, 2) = 1$, i.e. $x = 1 + 2t$, where t is an integer. The congruence (2) takes on the form

$$1 + 4t(t + 1) \equiv a \pmod{2^\alpha}.$$

But one of the numbers t, $t + 1$ is even and hence $4t(t + 1)$ is a multiple of 8. Therefore, for the solvability of the latter congruence, and along with it also the congruence (3), it is necessary that

$$a \equiv 1 \pmod 4 \text{ for } \alpha = 2; \ a \equiv 1 \pmod 8 \text{ for } \alpha \geqslant 3. \tag{4}$$

e. In the cases in which condition (4) is satisfied, we consider the question of finding solutions and the number of solutions.

For $\alpha \leqslant 3$, all the odd numbers satisfy the congruence in view of **d.** Therefore the congruence $x^2 \equiv a \pmod 2$ has one solution: $x \equiv 1 \pmod 2$, the congruence $x^2 \equiv a \pmod 4$ has two solutions: $x \equiv 1; 3 \pmod 4$, the congruence $x^2 \equiv a \pmod 8$ has four solutions: $x \equiv 1, 3, 5, 7 \pmod 8$.

In order to consider the cases $\alpha = 4, 5, \ldots$ all the odd numbers are put in the two arithmetic progressions:

$$x = \pm(1 + 4t_3) \tag{5}$$

$$(1 + 4t_3 \equiv 1 \pmod 4;\ -1 - 4t_3 \equiv -1 \equiv 3 \pmod 4))$$

We now decide which of the latter numbers satisfy the congruence $x^2 \equiv a \pmod{16}$. We find

$$(1 + 4t_3)^2 \equiv a \pmod{16},\ t_3 \equiv \frac{a-1}{8} \pmod 2,$$

$$t_3 = t_3' + 2t_4,\ x = \pm(1 + 4t_3' + 8t_4) = \pm(x_4 + 8t_4).$$

We now decide which of the latter numbers satisfy the congruence $x^2 \equiv a \pmod{32}$. We find

$$(x_4 + 8t_4)^2 \equiv a \pmod{32},\ t_4 = t_4' + 2t_5,\ x = \pm(x_5 + 16t_5),$$

etc. In this way we find that the values of x satisfying the congruence (3) for $\alpha > 3$, are representable in the form

$$x = \pm(x_\alpha + 2^{\alpha-1}t_\alpha).$$

These values of x form four different solutions of the congruence (3)

$$x \equiv x_\alpha;\ x_\alpha + 2^{\alpha-1};\ -x_\alpha;\ -x_\alpha - 2^{\alpha-1} \pmod{2^\alpha}$$

(modulo 4 the first two are congruent to 1 while the second two are congruent to -1).

Example. The congruence

$$x^2 \equiv 57 \pmod{64} \tag{6}$$

has four solutions since $57 \equiv 1 \pmod 8$. Representing x in the form $x = \pm(1 + 4t_3)$, we find

$$(1 + 4t_3)^2 \equiv 57 \pmod{16},\ 8t_3 \equiv 56 \pmod{16},$$

$$t_3 \equiv 1 \pmod 2,\ t_3 = 1 + 2t_4,\ x = \pm(5 + 8t_4),$$

$$(5 + 8t_4)^2 \equiv 57 \pmod{32},\ 5 \cdot 16t_4 \equiv 32 \pmod{32},$$

$$t_4 \equiv 0 \pmod 2,\ t_4 = 2t_5,\ x = \pm(5 + 16t_5),$$

$$(5 + 16t_5)^2 \equiv 57 \pmod{64},\ 5 \cdot 32t_5 \equiv 32 \pmod{64},$$

$$t_5 \equiv 1 \pmod 2,\ t_5 = 1 + 2t_6,\ x = \pm(21 + 32t_6).$$

Therefore the solutions of the congruence (6) are:

$$x \equiv \pm 21;\ \pm 53 \pmod{64}.$$

f. It follows from **c**, **d**, and **e** that:

The necessary conditions for the solvability of the congruence

$$x^2 \equiv a \pmod{2^\alpha};\ (a, 2) = 1$$

are: $a \equiv 1 \pmod 4$ *for* $\alpha = 2$, $a \equiv 1 \pmod 8$ *for* $\alpha \geqslant 3$. *If these conditions are satisfied, then the number of solutions is*: 1 *for* $\alpha = 1$; 2 *for* $\alpha = 2$; 4 *for* $\alpha \geqslant 3$.

g. It follows from **b**, **f** and **a**, **§5**, **ch. IV** that:

Necessary conditions for the solvability of congruences of the form

$$x^2 \equiv a \pmod m;\ m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k};\ (a, m) = 1$$

are:

$$a \equiv 1 \pmod 4 \text{ for } \alpha = 2,\ a \equiv 1 \pmod 8 \text{ for } \alpha \geqslant 3,$$

$$\left(\frac{a}{p_1}\right) = 1,\ \left(\frac{a}{p_2}\right) = 1,\ \ldots,\ \left(\frac{a}{p_k}\right) = 1.$$

If all of these conditions are satisfied, the number of solutions is: 2^k *for* $\alpha = 0$ *and* $\alpha = 1$; 2^{k+1} *for* $\alpha = 2$; 2^{k+2} *for* $\alpha \geqslant 3$.

Problems for Chapter V

Here p will always denote an odd prime.

1. Prove that finding the solutions of a congruence of the form

$$ax^2 + bx + c \equiv 0 \pmod{m}, \quad (2a, m) = 1$$

reduces to finding the solutions of a congruence of the form $x^2 \equiv q \pmod{m}$.

2, a. Using **e**, §**1**, find the solutions of the congruence (when they exist)

$$x^2 \equiv a \pmod{p}; \quad p = 4m + 3.$$

b. Using **b** and **k**, §**2**, obtain a method of finding the solutions of the congruence

$$x^2 \equiv a \pmod{p}; \quad p = 8m + 5.$$

c. Find the simplest possible method of finding the solutions of a congruence of the form

$$x^2 \equiv a \pmod{p}; \quad p = 8m + 1$$

when we know some quadratic non-residue N modulo p.

d. Using Wilson's theorem, prove that the solutions of the congruence

$$x^2 + 1 \equiv 0 \pmod{p}; \quad p = 4m + 1$$

are

$$x \equiv \pm 1 \cdot 2 \ldots 2m \pmod{p}.$$

3, a. Prove that the congruence

$$x^2 + 1 \equiv 0 \pmod{p} \qquad (1)$$

is solvable if and only if p is of the form $4m + 1$; the congruence

$$(2) \qquad x^2 + 2 \equiv 0 (\bmod p)$$

is solvable if and only if p is of the form $8m + 1$ or $8m + 3$; the congruence

$$x^2 + 3 \equiv 0 (\bmod p)$$

is solvable if and only if p is of the form $6m + 1$.

b. Prove that there are an infinite number of primes of the form $4m + 1$.

c. Prove that there are an infinite number of primes of the form $6m + 1$.

4. Dividing the numbers $1, 2, \ldots, p - 1$ into two sets, the second of which contains at least one number, we assume that the product of two numbers of the same set are congruent to a number of the first set modulo p, while the product of two elements of different sets is congruent to a number of the second set modulo p. Prove that this occurs if and only if the first set consists of quadratic residues, while the second set consists of quadratic non-residues modulo p.

5, a. Deduce the theory of congruences of the form

$$x^2 \equiv a (\bmod p^\alpha); \ (a, p) = 1,$$

by representing a and x in the calculational system to the base p.

b. Deduce the theory of congruences of the form

$$x^2 \equiv a (\bmod 2^\alpha); \ (a, 2) = 1,$$

by representing a and x in the calculational system to the base 2.

6. Prove that the solutions of the congruence

$$x^2 \equiv a (\bmod p^\alpha); \ (a, p) = 1$$

are $x \equiv \pm PQ'$ (mod p^{α}), where

$$P = \frac{(z+\sqrt{a})^{\alpha} + (z-\sqrt{a})^{\alpha}}{2}, \quad Q = \frac{(z+\sqrt{a})^{\alpha} - (z-\sqrt{a})^{\alpha}}{2\sqrt{a}}$$

$$z^2 \equiv a \pmod{p}, \quad QQ' \equiv 1 \pmod{p^{\alpha}}.$$

7. Find a method of solving the congruence $x^2 \equiv 1 \pmod{m}$ based on the fact that this congruence is equivalent to the congruence $(x-1)(x+1) \equiv 0 \pmod{m}$.

8. Let $\left(\frac{a}{p}\right) = 0$ for $(a, p) = p$.

a. For $(k, p) = 1$, prove that

$$\sum_{x=0}^{p-1} \left(\frac{x(x+k)}{p}\right) = -1.$$

b. Let each of the numbers ϵ and η have one of the values ± 1, let T be the number of pairs $x, x+1$, where $x = 1, 2, \ldots, p-2$, such that $\left(\frac{x}{p}\right) = \epsilon$, $\left(\frac{x+1}{p}\right) = \eta$. Prove that

$$T = \frac{1}{4}\left(p - 2 - \epsilon\left(\frac{-1}{p}\right) - \eta - \epsilon\eta\right)$$

c. Let $(k, p) = 1$, and let

$$S = \sum_x \sum_y \left(\frac{xy+k}{p}\right)$$

where x and y run through increasing sequences consisting, respectively, of X and Y residues of a complete system modulo

p. Prove that

$$|S| < \sqrt{2XYp}$$

In the proof use the inequality

$$|S|^2 \leqslant X \sum_x \left| \sum_y \left(\frac{xy + k}{p} \right) \right|^2$$

d. Let Q be an integer, $1 < Q < p$,

$$S = \sum_{x=0}^{p-1} S_x^2; \ S_x = \sum_{z=0}^{Q-1} \left(\frac{x + z}{p} \right).$$

α) Prove that $S = (p - Q)Q$.

β) Let λ be a constant, $0 < \lambda < 1$. Prove that the number T of integers $x = 0, 1, \ldots, p - 1$ for which the condition $S_x < Q^{0.5+0.5\lambda}$ is not satisfied, satisfies the condition $T \leqslant pQ^{-\lambda}$.

γ) Let $p > 25$, and let M be an integer. Prove that the sequence

$$M, M + 1, \ldots, M + 3[\sqrt{p}] - 1$$

contains a quadratic non-residue modulo p.

9, a. Prove that the number of representations of an integer $m > 1$ in the form

(1) $$m = x^2 + y^2, \ (x, y) = 1, \ x > 0, \ y > 0$$

is equal to the number of solutions of the congruence

(2) $$z^2 + 1 \equiv 0 \pmod{m}.$$

In proving this, set $r = \sqrt{m}$ and use the representation of $\alpha = \dfrac{z}{m}$ given in the theorem of problem **4, b, ch. I,** and then consider the congruence obtained by multiplying (2) termwise by Q^2.

b. Let a be one of the numbers 2, 3. Prove that the number of representations of a prime $p > a$ in the form

$$p = x^2 + ay^2, \; x > 0, \; y > 0 \tag{3}$$

is equal to half the number of solutions of the congruence

$$z^2 + a \equiv 0 (\operatorname{mod} p). \tag{4}$$

c. Let p be of the form $4m + 1$, $(k, p) = 1$,

$$S(k) = \sum_{x=0}^{p-1} \left(\frac{x(x^2 + k)}{p}\right)$$

Prove that (D. S. Gorshkov)

α) $S(k)$ is an even number.

β) $S(kt) = \left(\dfrac{t}{p}\right) S(k)$.

γ) For $\left(\dfrac{r}{p}\right) = 1$, $\left(\dfrac{n}{p}\right) = -1$, we have (cf. problem **a.**)

$$p = \left(\frac{1}{2}S(r)\right)^2 + \left(\frac{1}{2}S(n)\right)^2.$$

10. Let D be a positive integer which is not the square of an integer. Prove that:

a. If two pairs $x = x_1$, $y = y_1$ and $x = x_2$, $y = y_2$ of integers satisfy the equation

$$x^2 - Dy^2 = k$$

for a given integer k, then the equation

$$X^2 - DY^2 = k^2$$

is satisfied by integers X, Y defined by the equation (the ± sign can be taken arbitrarily)

$$X + Y\sqrt{D} = (x_1 + y_1\sqrt{D})(x_2 \pm y_2\sqrt{D}).$$

b. The equation (Pell's equation)

$$x^2 - Dy^2 = 1 \tag{1}$$

is solvable in positive integers x, y.

c. If x_0, y_0 is a pair of positive x, y with minimal x (or, equivalently, with minimal $x + y\sqrt{D}$) satisfying equation (1), then all pairs of positive x, y satisfying this equation are defined by the equations

$$x + y\sqrt{D} = (x_0 + y_0\sqrt{D})^r; \; r = 1, 2, \ldots \tag{2}$$

11, a. Let a be an integer. Let

$$U_{a,p} = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e^{2\pi i \frac{ax}{p}}.$$

α) For $(a, p) = 1$, prove that $|U_{a,p}| = \sqrt{p}$.

In proving this, multiply $U_{a,p}$ by its conjugate, which is obtained by replacing i by $-i$. Letting the letters x_1 and x be the summation variables of the original and conjugate sums, we then gather together the terms of the product such that

$$x_1 \equiv xt \pmod{p},$$

or

$$x_1 \equiv x + t \pmod{p}$$

for fixed t.

β) Prove that

$$\left(\frac{a}{p}\right) = \frac{U_{a,p}}{U_{1,p}}$$

b. Let $m > 2$, $(a, m) = 1$,

$$S_{a,m} = \sum_{x=0}^{m-1} e^{2\pi i \frac{ax^2}{m}}$$

α) Prove that $S_{a,p} = U_{a,p}$ (problem **a**).

β) It follows from the theorems of problems α) and **a**, α) that $S_{a,p} = \sqrt{p^*}$. Prove the following more general result:

$$\begin{aligned} |S_{a,m}| &= \sqrt{m}, &&\text{if } m \equiv 1 \pmod 2, \\ |S_{a,m}| &= 0, &&\text{if } m \equiv 2 \pmod 4, \\ |S_{a,m}| &= \sqrt{2m}, &&\text{if } m \equiv 0 \pmod 4. \end{aligned}$$

γ) Let $m > 1$, $(2A, m) = 1$, and let a be an arbitrary integer. Prove that

$$\left|\sum_{x=0}^{m-1} \exp\left(2\pi i \frac{Ax^2 + ax}{m}\right)\right| = \sqrt{m}.$$

12, a. Let m be an integer exceeding 1, let M and Q be integers such that $0 \leqslant M < M + Q \leqslant m$, and let $\sum_{z}$ denote a sum extended over the z in a given set of integers, while $\sum_{z}{}'$ denotes a sum extended over the z in this set which are congruent modulo m to the numbers

$$M, M+1, \ldots, M+Q-1.$$

Moreover, let the function $\Phi(z)$ be such that, for some Δ and any $a = 1, 2, \ldots, m-1$, we have

$$\left|\sum_z \Phi(z) \exp\left(2\pi i \frac{az}{m}\right)\right| \leqslant \Delta.$$

Prove that

$$\sum_z{}' \Phi(z) = \frac{Q}{m}\sum_z \Phi(z) + \theta\Delta(\ln m - \delta),$$

where $|\theta| < 1$, $\delta > 0$ always, $\delta > \frac{1}{2}$ for $m \geqslant 12$, $\delta > 1$ for $m \geqslant 60$.

b. Let M and Q be integers such that $0 < M < M + Q \leqslant p$.

α) Prove that

$$\left|\sum_{x=M}^{M+Q-1}\left(\frac{x}{p}\right)\right| < \sqrt{p}\ln p.$$

β) Let R be the number of quadratic residues and let N be the number of quadratic non-residues in the sequence $M, M+1, \ldots, M+Q-1$. Prove that

$$R = \frac{1}{2}Q + \frac{\theta}{2}\sqrt{p}\ln p,\ N = \frac{1}{2}Q - \frac{\theta}{2}\sqrt{p}\ln p;\ |\theta| < 1.$$

γ) Deduce the formulae of problem β), using the theorem of problem **11, b,** β) and the theorem of problem **a.**

δ) Let $m > 2$, $(2A, m) = 1$, and let M_0 and Q_0 be integers such that $0 < M_0 < M_0 + Q_0 \leqslant m$. Prove that

$$\left|\sum_{x=M_0}^{M_0+Q_0-1} \exp\left(2\pi i \frac{Ax^2}{m}\right)\right| < \sqrt{m}\ \ln m.$$

ϵ) Let $p > 2$, $(A, p) = 1$, let M_0 and Q_0 be integers such that $0 < M_0 < M_0 + Q_0 \leqslant p$ and let T be the number of integers of the sequence Ax^2; $x = M_0, M_0 + 1, \ldots, M_0 + Q_0 - 1$, which are congruent modulo p to the numbers of the sequence $M, M + 1, \ldots, M + Q - 1$. Prove that

$$T = \frac{Q_0 Q}{p} + \theta\sqrt{p}\ (\ln p)^2.$$

c. Deduce the formulae of problem **b**, β) by considering the sum

$$\sum_{a=0}^{p-1}\sum_{\alpha=1}^{p-1}\sum_{x=M}^{M+Q-1}\sum_{y=M}^{M+Q-1}\left(\frac{\alpha}{p}\right)\exp\left(2\pi i\frac{a(x - \alpha y)}{p}\right)$$

Numerical Exercises for Chapter V

1, a. Find the quadratic residues in a reduced residue system modulo 23.

b. Find the quadratic non-residues in a reduced residue system modulo 37.

2, a. Applying **e**, § **1**, find the number of solutions of the congruences:

$$\alpha)\ x^2 \equiv 3 \pmod{31};\quad \beta)\ x^2 \equiv 2 \pmod{31}.$$

b. Find the number of solutions of the congruences:

$$\alpha)\ x^2 \equiv 5 \pmod{73};\quad \beta)\ x^2 \equiv 3 \pmod{73}.$$

3, a. Using the Jacobi symbol, find the number of solutions of the congruences:

$$\alpha)\ x^2 \equiv 226 \pmod{563};\quad \beta)\ x^2 \equiv 429 \pmod{563}.$$

b. Find the number of solutions of the congruences:

$$\alpha)\ x^2 \equiv 3766 \pmod{5987};\quad \beta)\ x^2 \equiv 3149 \pmod{5987}.$$

4, a. Applying the methods of problems **2, a; 2, b; 2, c,** solve the congruences:

$$\alpha)\ x^2 \equiv 5 \pmod{19};\quad \beta)\ x^2 \equiv 5 \pmod{29};\quad \gamma)\ x^2 \equiv 2 \pmod{97}.$$

b. Solve the congruences:

$$\alpha)\ x^2 \equiv 2 \pmod{311};\quad \beta)\ x^2 \equiv 3 \pmod{277};$$

$$\gamma)\ x^2 \equiv 11 \pmod{353}.$$

5, a. Solve the congruence $x^2 \equiv 59 \pmod{125}$ by the methods of:

α) **b, §4**; β) problem **5, a**; γ) problem **6.**

b. Solve the congruence $x^2 \equiv 91 \pmod{243}$.

6, a. Solve the congruence $x^2 \equiv 41 \pmod{64}$ by the methods of:

α) **e, §4**; β) problem **5, b.**

b. Solve the congruence $x^2 \equiv 145 \pmod{256}$.

CHAPTER VI

PRIMITIVE ROOTS AND INDICES

§1. *General Theorems*

a. For $(a, m) = 1$ there exist positive γ such that $a^{\gamma} \equiv 1$ $(\operatorname{mod} m)$, for example (by Euler's theorem) $\gamma = \varphi(m)$. The smallest of these is called: *the exponent to which a belongs modulo m.*

b. *If a belongs to the exponent δ modulo m, then the numbers $1 = a^0, a^1, \ldots, a^{\delta-1}$ are incongruent modulo m.*

Indeed, it would follow from $a^l \equiv a^k (\operatorname{mod} m)$, $0 \leqslant k < l < \delta$ that $a^{l-k} \equiv 1 (\operatorname{mod} m)$, $0 < l - k < \delta$, which contradicts the definition of δ.

c. *If a belongs to the exponent δ modulo m, then $a^{\gamma} \equiv a^{\gamma'}$ $(\operatorname{mod} m)$ if and only if $\gamma \equiv \gamma' (\operatorname{mod} \delta)$; in particular (for $\gamma' = 0$), $a^{\gamma} \equiv 1 (\operatorname{mod} m)$ if and only if γ is divisible by δ.*

Indeed, let r and r_1 be the least non-negative residues of the numbers γ and γ' modulo δ; then for some q and q_1 we have $\gamma = \delta q + r$, $\gamma' = \delta q_1 + r$. From this and from $a^{\delta} \equiv 1$ $(\operatorname{mod} m)$ it follows that

$$a^{\gamma} = (a^{\delta})^{q} a^{r} \equiv a^{r} (\operatorname{mod} m),$$

$$a^{\gamma'} = (a^{\delta})^{q_1} a^{r_1} \equiv a^{r_1} (\operatorname{mod} m).$$

Therefore $a^{\gamma} \equiv a^{\gamma_1} (\operatorname{mod} m)$ if and only if $a^{r} \equiv a^{r_1} (\operatorname{mod} m)$, i.e. (**b**), when $r = r_1$.

d. It follows from $a^{\varphi(m)} \equiv 1 \pmod{m}$ and from **c** ($\gamma' = 0$) that $\varphi(m)$ is divisible by δ. Thus *the exponents to which numbers belong modulo m are just the divisors of* $\varphi(m)$. The largest of these divisors is $\varphi(m)$. The numbers belonging to the exponent $\varphi(m)$ (if such exist) are called the *primitive roots modulo m.*

§2. *Primitive Roots Modulo p^α and $2p^\alpha$*

a. Let p be an odd prime and let $\alpha \geqslant 1$. We shall prove the existence of primitive roots modulo p^α and $2p^\alpha$.

b. *If x belongs to the exponent ab modulo m, then x^a belongs to the exponent b.*

Indeed, let x^a belong to the exponent δ. Then $x^{a\delta} \equiv 1 \pmod{m}$, and hence (**c**, §**1**) $a\delta$ is divisible by ab, i.e. δ is divisible by b. On the other hand, $(x^a)^b \equiv 1 \pmod{m}$ implies (**c**, §**1**) that b is divisible by δ. Hence $\delta = b$.

c. *If x belongs to the exponent a, and y belongs to the exponent b modulo m, where* $(a, b) = 1$, *then xy belongs to the exponent ab.*

Indeed, let xy belong to the exponent δ. Then $(xy)^\delta \equiv 1 \pmod{m}$. Hence $x^{b\delta}y^{b\delta} \equiv 1 \pmod{m}$ and (**c**, §**1**) $x^{b\delta} \equiv 1 \pmod{m}$. Hence (**c**, §**1**) $b\delta$ is divisible by a, and since $(b, a) = 1$, δ is divisible by a. In the same way we find that δ is divisible by b. Since $(a, b) = 1$, being divisible by a and b, δ is also divisible by ab. On the other hand, $(xy)^{ab} \equiv 1 \pmod{m}$ implies (**c**, §**1**) that ab is divisible by δ. Hence $\delta = ab$.

d. *There exist primitive roots modulo p.*

Indeed, let τ be the least common multiple of all those exponents

$$\delta_1, \delta_2, \ldots, \delta_r, \tag{1}$$

to each of which belongs at least one number of the sequence $1, 2, \ldots, p-1$ modulo p, and let $\tau = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$ be the canonical decomposition of the number τ. Then for each s, among the numbers (1) there exists some δ which is divisible by $q_s^{\alpha_s}$ and is therefore representable in the form $\delta = aq_s^{\alpha_s}$. If x is a number belonging to the exponent δ, then, by **b**, $x_s = x^a$ belongs to the exponent $q_s^{\alpha_s}$. This holds for $s = 1, 2, \ldots, k$; by **c**, the number $g = x_1 x_2 \ldots x_k$ belongs to the exponent $q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k} = \tau$.

But since the exponents (1) are just the divisors of the number τ, all the numbers $1, 2, \ldots, p - 1$ satisfy (**c, §1**) the congruence $x^\tau \equiv 1 \pmod p$. This means (**c, §4, ch. IV**) that $p - 1 \leqslant \tau$. But τ is a divisor of $p - 1$. Hence $\tau = p - 1$, i.e. g is a primitive root.

e. *Let g be a primitive root modulo p. We can find a t such that u, which is defined by the equation $(g + pt)^{p-1} = 1 + pu$, is not divisible by p. The corresponding $g + pt$ is a primitive root modulo p^α for any $\alpha > 1$.*

Indeed, we have

$$(2) \qquad \begin{gathered} g^{p-1} = 1 + pT, \\ (g + pt)^{p-1} = 1 + p(T_0 - g^{p-2}t +. pT) = 1 + pu, \end{gathered}$$

where, along with t, u runs through a complete residue system modulo p. Therefore, we can find a t such that u is not divisible by p. For this t, we deduce from (2) the equations

$$(3) \qquad \begin{cases} (g + pt)^{p(p-1)} = (1 + pu)^p = 1 + p^2 u_2, \\ (g + pt)^{p^2(p-1)} = (1 + p^2 u_2)^p = 1 + p^3 u_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

where $u_2, u_3, \ldots$ are not divisible by p.

Let $g + pt$ belong to the exponent δ modulo p^α. Then

$$(4) \qquad (g + pt)^\delta \equiv 1 \pmod{p^\alpha}.$$

Hence $(g + pt)^\delta \equiv 1 \pmod p$; and consequently δ is a multiple of $p - 1$, and since δ divides $\varphi(p^\alpha) = p^{\alpha-1}(p - 1)$, it follows that $\delta = p^{r-1}(p - 1)$, where r is one of the numbers $1, 2, \ldots, \alpha$. Replacing the left side of the congruence (4) by its expression in the appropriate equation of (2) or (3), we find ($u = u_1$)

$$1 + p^r u_r \equiv 1 \pmod{p^\alpha},\ p^r \equiv 0 \pmod{p^\alpha},\ r = \alpha,\ \delta = \varphi(p^\alpha),$$

i.e. $g + pt$ is a primitive root modulo p^α.

f. *Let $\alpha \geqslant 1$ and let g be a primitive root modulo p^α. Whichever of the numbers g and $g + p^\alpha$ is odd, is a primitive root modulo $2p^\alpha$.*

Indeed, every odd x which satisfies one of the congruences $x^\gamma \equiv 1 \pmod{p^\alpha}$ and $x^\gamma \equiv 1 \pmod{2p^\alpha}$ obviously satisfies the other also. Hence, since $\varphi(p^\alpha) = \varphi(2p^\alpha)$ for all odd x, a primitive root for one of the moduli p^α and $2p^\alpha$, is also a primitive root for the other. But, of the two primitive roots g and $g + p^\alpha$ modulo p^α, at least one is odd; and consequently, it will be a primitive root modulo $2p^\alpha$.

§3. *Evaluation of the Primitive Roots for the Moduli* p^α and $2p^\alpha$

The primitive roots for the moduli p^α and $2p^\alpha$ where p is an odd prime and $\alpha \geqslant 1$, can be found by using the following general theorem:

Let $c = \varphi(m)$ and let $q_1, q_2, \ldots, q_k$ be the different prime divisors of the number c. In order that a number g, which is relatively prime to m, be a primitive root modulo m, it is necessary and sufficient that this g satisfy none of the congruences

$$g^{\frac{c}{q_1}} \equiv 1 \pmod{m},\ g^{\frac{c}{q_2}} \equiv 1 \pmod{m},$$

$$\ldots,\ g^{\frac{c}{q_k}} \equiv 1 \pmod{m}. \tag{1}$$

Indeed, if g is a primitive root, then a fortiori it belongs to the exponent c and hence none of the congruences of (1) can be satisfied.

Conversely, we now assume that g satisfies none of the congruences of (1). If the exponent δ to which g belongs, turns out to be less than c, then, letting q be one of the prime divisors of $\frac{c}{\delta}$, we would have $\frac{c}{\delta} = qu$, $\frac{c}{q} = \delta u$, $g^{\frac{c}{q}} \equiv 1$ (mod p), which contradicts our assumption. Hence $\delta = c$ and g is a primitive root.

Example 1. Let $m = 41$. We have $\varphi(41) = 40 = 2^3 \cdot 5$, $\frac{40}{5} = 8$, $\frac{40}{2} = 20$. Therefore, in order that the number g, not divisible by 41, be a primitive root modulo 41, it is necessary and sufficient that this g satisfy neither of the congruences

$$(2) \qquad g^8 \equiv 1 \,(\text{mod } 41), \quad g^{20} \equiv 1 \,(\text{mod } 41).$$

But going through the numbers 2, 3, 4, ... we find (modulo 41)

$$\begin{array}{lllll} 2^8 \equiv 10, & 3^8 \equiv 1, & 4^8 \equiv 18, & 5^8 \equiv 18, & 6^8 \equiv 10, \\ 2^{20} \equiv 1, & & 4^{20} \equiv 1, & 5^{20} \equiv 1, & 6^{20} \equiv 40. \end{array}$$

From this we see that the numbers 2, 3, 4, 5 are not primitive roots since each of them satisfies at least one of the congruences (2). The number 6 is a primitive root since it satisfies neither of the congruences of (2).

Example 2. Let $m = 1681 = 41^2$. A primitive root can also be obtained here by using the general theorem. But we can find it more simply by applying theorem e, §2. Knowing (example 1) that 6 is a primitive root modulo 41, we find

$$6^{40} = 1 + 41(3 + 41l)$$

$$(6 + 41t)^{40} = 1 + 41(3 + 41l - 6^{39}t + 41T) = 1 + 41u.$$

In order that u be non-divisible by 41, it is sufficient to take $t = 0$. We can therefore take the number $6 + 41 \cdot 0 = 6$ as a primitive root modulo 1681.

Example 3. Let $m = 3362 = 2\ 1681$. The primitive root can also be obtained here by using the general theorem. But we can find it more simply by applying theorem **f**, §2. Knowing (example 2) that 6 is a primitive root modulo 1681, we can take as a primitive root modulo 3362 the odd number in the pair 6, 6 + 1681, i.e. the number 1687.

§4. *Indices for the Moduli p^α and $2p^\alpha$*

a. Let p be an odd prime, $\alpha \geqslant 1$; let m be one of the numbers p^α and $2p^\alpha$; $c = \varphi(m)$, and let g be a primitive root modulo m.

b. *If γ runs through the least non-negative residues $\gamma = 0, 1, \ldots, c - 1$ modulo c, then g^γ runs through a reduced residue system modulo m.*

Indeed, g^γ runs through c numbers which are relatively prime to m, and by **b**, §1, incongruent modulo m.

c. For numbers a, which are relatively prime to m, we introduce the concept of index, which is analogous to the concept of logarithm; here, a primitive root plays a role analogous to the role of the base of a logarithm:

If

$$a \equiv g^\gamma \pmod{m}$$

(we assume that $\gamma \geqslant 0$), then γ is said to be the *index of the number a modulo m to the base g* and is denoted by the symbol $\gamma = \operatorname{ind} a$ (more precisely: $\gamma = \operatorname{ind}_g a$).

In view of **b**, every a, relatively prime to m, has some unique index γ' among the numbers of the sequence

$$\gamma = 0, 1, \ldots, c - 1.$$

Knowing γ', we can find all the indices of the number a; by **c**, §1, these are all the non-negative numbers of the class

$$\gamma \equiv \gamma' \pmod{c}.$$

It follows immediately from the definition of the index which we have given here that the numbers with a given index γ form an equivalence class of numbers modulo m.

d. *We have*

$$\text{ind } ab \ldots l \equiv \text{ind } a + \text{ind } b + \ldots + \text{ind } l \pmod{c}$$

and in particular,

$$\text{ind } a^n \equiv n \text{ ind } a \pmod{c}.$$

Indeed,

$$a \equiv g^{\text{ind } a} \pmod{m}, \ b \equiv g^{\text{ind } b} \pmod{m},$$
$$\ldots, \ l \equiv g^{\text{ind } l} \pmod{m},$$

and multiplying the latter together, we find

$$ab \ldots l \equiv g^{\text{ind } a + \text{ind } b + \ldots + \text{ind } l} \pmod{m}.$$

Therefore, ind a + ind b + ... + ind l is one of the indices of the product $ab \ldots l$.

e. In view of the practical use of indices, for each prime modulus p (which is not too large) tables of indices have been constructed. There are two tables: one for finding the index from the number, and the other for finding the number from the index. The tables contain the least non-negative residues of the numbers (a reduced residue system) and their smallest indices (a complete system) corresponding to a modulus p and $c = \varphi(p) = p - 1$.

Example. We construct the preceding table for the modulus $p = 41$. It was shown above (example 1, §3) that $g = 6$ is a primitive root modulo 41; we take it as the basis of the indices. We find (congruences are taken modulo 41):

$6^0 \equiv 1$	$6^8 \equiv 10$	$6^{16} \equiv 18$	$6^{24} \equiv 16$	$6^{32} \equiv 37$
$6^1 \equiv 6$	$6^9 \equiv 19$	$6^{17} \equiv 26$	$6^{25} \equiv 14$	$6^{33} \equiv 17$
$6^2 \equiv 36$	$6^{10} \equiv 32$	$6^{18} \equiv 33$	$6^{26} \equiv 2$	$6^{34} \equiv 20$
$6^3 \equiv 11$	$6^{11} \equiv 28$	$6^{19} \equiv 34$	$6^{27} \equiv 12$	$6^{35} \equiv 38$
$6^4 \equiv 25$	$6^{12} \equiv 4$	$6^{20} \equiv 40$	$6^{28} \equiv 31$	$6^{36} \equiv 23$
$6^5 \equiv 27$	$6^{13} \equiv 24$	$6^{21} \equiv 35$	$6^{29} \equiv 22$	$6^{37} \equiv 15$
$6^6 \equiv 39$	$6^{14} \equiv 21$	$6^{22} \equiv 5$	$6^{30} \equiv 9$	$6^{38} \equiv 8$
$6^7 \equiv 29$	$6^{15} \equiv 3$	$6^{23} \equiv 30$	$6^{31} \equiv 13$	$6^{39} \equiv 7$

and hence our tables are:

N	0	1	2	3	4	5	6	7	8	9
0		0	26	15	12	22	1	39	38	30
1	8	3	27	31	25	37	24	33	16	9
2	34	14	29	36	13	4	17	5	11	7
3	23	28	10	18	19	21	2	32	35	6
4	20									

I	0	1	2	3	4	5	6	7	8	9
0	1	6	36	11	25	27	39	29	10	19
1	32	28	4	24	21	3	18	26	33	34
2	40	35	5	30	16	14	2	12	31	22
3	9	13	37	17	20	38	23	15	8	7

Here the row number is the first digit and the column number is the second digit of the number (index). At the place common to the given row and given column we place the corresponding index (number).

For example, we find the *ind 25* at the place in the first table common to the *2*-nd row and the *5*-th column, i.e. ind 25 = 4. The number whose index is 33 is found in the place in the second table common to the *3*-rd row and the *3*-rd column, i.e. 33 = ind 17.

§5. *Consequences of the Preceding Theory*

a. Let p be an odd prime; $\alpha \geqslant 1$, let m be one of the numbers p^{α}, $2p^{\alpha}$, and finally, let $c = \varphi(m)$.

b. *Let* $(n, c) = d$; *then*:

1. *The congruence*

$$x^n \equiv a \pmod{m} \tag{1}$$

is solvable (and hence a is an n-th power residue modulo m) if and only if ind a is a multiple of d.

In the case of solvability the congruence has d solutions.

2. *The number of n-th power residues in a reduced residue system modulo m is* $\frac{c}{d}$.

Indeed, the congruence (1) is equivalent to the congruence

$$n \text{ ind } x \equiv \text{ind } a \pmod{c} \tag{2}$$

which is solvable if and only if *ind a* is a multiple of d (**d, §2, ch. IV**).

If the congruence (2) is solvable, we find d values of *ind x* which are incongruent modulo c; corresponding to them we find d values of x which are incongruent modulo m, proving assertion **1.**

Among the numbers $0, 1, \ldots, c-1$, which are the smallest indices of a reduced residue system modulo m, there are $\frac{c}{d}$ which are multiples of d, proving assertion **2.**

Example **1.** For the congruence

$$x^8 \equiv 23 \pmod{41} \tag{3}$$

we have $(8, 40) = 8$, while ind $23 = 36$ is not divisible by 8. Therefore the congruence (3) is unsolvable.

Example 2. For the congruence

$$x^{12} \equiv 37 \pmod{41} \tag{4}$$

we have $(12, 40) = 4$, while ind $37 = 32$ is divisible by 4. Therefore the congruence (4) is solvable and has 4 solutions. These solutions are obtained in the following way:

The congruence (4) is equivalent to the following ones:

$$12 \operatorname{ind} x \equiv 32 \pmod{40}, \quad \operatorname{ind} x \equiv 6 \pmod{10}.$$

Hence we find 4 values of *ind x* which are incongruent modulo 40:

$$\operatorname{ind} x = 6,\ 16,\ 26,\ 36,$$

from which we obtain the 4 solutions of the congruence (4)

$$x \equiv 39,\ 18,\ 2,\ 23 \pmod{41}.$$

Example 3. The numbers

$$1,\ 4,\ 10,\ 16,\ 18,\ 23,\ 25,\ 31,\ 37,\ 40 \tag{5}$$

whose indices are multiples of 4, are just all the biquadratic residues (or the residues of any power $n = 12, 28, 36, \ldots,$ where $(n, 40) = 4$), among the least positive residues modulo 41. The number of integers in the sequence (5) is $10 = \dfrac{40}{4}$.

c. Along with assertion **b, 1,** we shall also find the following one useful:

The number a is an n-th power residue modulo m if and only if

$$a^{\frac{c}{d}} \equiv 1 \pmod{m}. \tag{6}$$

Indeed, the condition ind $a \equiv 0 \pmod{d}$ is equivalent to the condition: $\frac{c}{d}$ ind $a \equiv 0 \pmod{c}$. The latter is equivalent to condition (6).

Example. By the theorem of §3, the impossibility of the congruence $g^{\frac{c}{q}} \equiv 1 \pmod{m}$ is equivalent to the statement that g is a q-th power non-residue modulo m. In particular, the impossibility of the congruence $g^{\frac{c}{2}} \equiv 1 \pmod{m}$ is equivalent to the statement that g is a quadratic non-residue modulo m (cf. **e**, §**1**, **ch. V**).

d, 1. *The exponent δ to which a belongs modulo m is defined by the equation* (ind a, c) $= \frac{c}{\delta}$; *in particular, the fact that a belongs to a number of primitive roots modulo m is equivalent to the equation* (ind a, c) $= 1$.

2. *In a reduced residue system modulo m, the number of numbers belonging to the exponent δ is* $\varphi(\delta)$; *in particular, the number of primitive roots is* $\varphi(c)$.

Indeed, δ is the smallest divisor of c such that $a^{\delta} \equiv 1 \pmod{m}$. This condition is equivalent to

$$\delta \text{ ind } a \equiv 0 \pmod{c},$$

or

$$\text{ind } a \equiv 0 \left(\text{mod } \frac{c}{\delta}\right).$$

This means that δ is the smallest divisor of c for which $\frac{c}{\delta}$ divides *ind a*, from which it follows that $\frac{c}{\delta}$ is the largest divisor of c which divides *ind a*, i.e. $\frac{c}{\delta} =$ (ind a, c), proving assertion **1**.

Among the numbers 0, 1, ..., $c - 1$, which are the smallest indices of a reduced residue system modulo m, the multiples of $\frac{c}{\delta}$ are the numbers of the form $\frac{c}{\delta}y$, where $y = 0, 1, \ldots, \delta - 1$. The condition $\left(\frac{c}{\delta}y, c\right) = \frac{c}{\delta}$ is equivalent to the condition $(y, \delta) = 1$; and the latter is satisfied by $\varphi(\delta)$ values of y, proving assertion 2.

Example 1. In a reduced residue system modulo 41, the numbers belonging to the exponent 10 are the numbers a such that $(\text{ind } a, 40) = \frac{40}{10} = 4$, i.e. the numbers

$$4, 23, 25, 31.$$

The number of these numbers is $4 = \varphi(10)$.

Example 2. In a reduced residue system modulo 41, the primitive roots are the numbers a such that $(\text{ind } a, 40) = 1$, i.e. the numbers

$$6, 7, 11, 12, 13, 15, 17, 19, 22, 24, 26, 28, 29, 30, 34, 35.$$

The number of these primitive roots is $16 = \varphi(40)$.

§6. *Indices Modulo* 2^α

a. The preceding theory is replaced, for the modulus 2^α, by a somewhat more complicated one.

b. Let $\alpha = 1$. Then $2^\alpha = 2$. We have $\varphi(2) = 1$. A primitive root modulo 2 is, for example, $1 \equiv -1 \pmod 2$. The number $1^0 = (-1)^0 = 1$ forms a reduced residue system modulo 2.

c. Let $\alpha = 2$. Then $2^2 = 4$. We have $\varphi(4) = 2$. The number $3 \equiv -1 \pmod 4$ is a primitive root modulo 4. The numbers $(-1)^0 = 1$, $(-1)^1 \equiv 3 \pmod 4$ form a reduced residue system modulo 4.

d. Let $\alpha \geqslant 3$. Then $2^{\alpha} \geqslant 8$. We have $\varphi(2^{\alpha}) = 2^{\alpha-1}$. It is easy to see that there are no primitive roots in this case; more precisely: the exponent to which the odd number x belongs modulo 2^{α} does not exceed $2^{\alpha-2} = \frac{1}{2}\varphi(2^{\alpha})$. Indeed, we have

$$\begin{aligned} x^2 &= 1 + 8t_1, \\ x^4 &= 1 + 16t_2, \\ &\ldots\ldots\ldots\ldots\ldots \\ x^{2^{\alpha-2}} &= 1 + 2^{\alpha}t_{\alpha-2} \equiv 1 \pmod{2^{\alpha}}. \end{aligned}$$

Therefore, there exist numbers belonging to the exponent $2^{\alpha-2}$. For example, 5 would be such a number. Indeed,

$$\begin{aligned} 5 &= 1 + 4, \\ 5^2 &= 1 + 8 + 16, \\ 5^4 &= 1 + 16 + 32u_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 5^{2^{\alpha-3}} &= 1 + 2^{\alpha-2} + 2^{\alpha}u_{\alpha-3}, \end{aligned}$$

from which we see that none of the powers $5^1, 5^2, 5^4, \ldots, 5^{2^{\alpha-3}}$ is congruent to 1 modulo 2^{α}.

It is not difficult to see that the numbers of the following two rows:

$$\begin{aligned} &5^0, \ 5^1, \ \ldots, \ 5^{2^{\alpha-2}-1}, \\ -&5^0, -5^1, \ldots, -5^{2^{\alpha-2}-1} \end{aligned}$$

form a reduced residue system modulo 2^{α}. Indeed, the number of these numbers is $2 \cdot 2^{\alpha-2} = \varphi(2^{\alpha})$; the numbers of each individual row are incongruent among themselves modulo 2^{α} (**b**, **§ 1**); finally, the numbers of the upper row are incongruent to the numbers of the lower row since the former are congruent to 1, while the latter are congruent to -1 modulo 4.

e. For convenience in later investigations, we express the results of **b**, **c**, **d** in more unified form, which is also applicable in the case $\alpha = 0$.

Let

$$c = 1;\ c_0 = 1,\ \textit{if}\ \alpha = 0,\ \textit{or}\ \alpha = 1;$$

$$c = 2;\ c_0 = 2^{\alpha-2},\ \textit{if}\ \alpha \geqslant 2$$

(therefore $cc_0 = \varphi(2^{\alpha})$*) and let* γ *and* γ_0 *run independently through the least non-negative residues*

$$\gamma = 0, \ldots, c - 1;\ \gamma_0 = 0, \ldots, c_0 - 1$$

modulo c *and* c_0*. Then* $(-1)^{\gamma}5^{\gamma_0}$ *runs through a reduced residue system modulo* 2^{α}*.*

f. *The congruence*

$$(-1)^{\gamma}5^{\gamma_0'} \equiv (-1)^{\gamma'}5^{\gamma_0'} \pmod{2^{\alpha}} \tag{1}$$

holds if and only if

$$\gamma \equiv \gamma' \pmod{c},\ \gamma_0 \equiv \gamma_0' \pmod{c_0}.$$

Indeed, the theorem is evident for $\alpha = 0$. We therefore assume that $\alpha > 0$. Let the least non-negative residues of the numbers γ and γ_0 be r and r_0, and of the numbers γ' and γ_0' be r' and r_0' modulo c and c_0. In view of **c**, §1 (-1 belongs to the exponent c, while 5 belongs to the exponent c_0), the congruence (1) holds if and only if $(1)^{r}5^{r_0} \equiv (-1)^{r'}5^{r_0'} \pmod{2^{\alpha}}$, i.e. if and only if $r = r'$; $r_0 = r_0'$ (in view of **e**).

g. If

$$a \equiv (-1)^{\gamma}5^{\gamma_0} \pmod{2^{\alpha}},$$

then the system γ, γ_0 is called an *index system of the number a modulo* 2^α.

In view of **e**, every a relatively prime to 2^α (i.e. every odd a) has a unique index system γ', γ_0' in the $cc_0 = \varphi(2^\alpha)$ pairs of values γ, γ_0 considered in **e**.

Knowing a system γ', γ_0', we can also find all index systems of the number a; according to **f**, these will be all pairs γ, γ_0 consisting of the non-negative numbers of the equivalence classes

$$\gamma \equiv \gamma' \pmod{c}, \quad \gamma_0 \equiv \gamma_0' \pmod{c_0}.$$

It follows immediately from the definition we have given of index systems that the numbers with a given index system γ, γ_0 forms an equivalence class of numbers modulo 2^α.

h. *The indices of a product are congruent modulo c and* c_0 *with the sums of the indices of the factors.*

Indeed, let $\gamma(a), \gamma_0(a); \ldots; \gamma(l), \gamma_0(l)$ be index systems for the numbers $a, \ldots, l$. We have

$$a \ldots l \equiv (-1)^{\gamma(a)+ \ldots + \gamma(l)} 5^{\gamma_0(a)+ \ldots + \gamma_0(l)}$$

Therefore $\gamma(a) + \ldots + \gamma(l), \gamma_0(a) + \ldots + \gamma_0(l)$ are the indices of the product $a \ldots l$.

§7. *Indices for Arbitrary Composite Modulus*

a. Let $m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the canonical decomposition of the number m. Moreover let c and c_0 have the values considered in **e**, §6; $c_s = \varphi(p_s^{\alpha_s})$; and let g_s be the smallest primitive root modulo $p_s^{\alpha_s}$.

b. If

$$a \equiv (-1)^\gamma 5^{\gamma_0} \pmod{2^\alpha},$$
(1)
$$a \equiv g_1^{\gamma_1} \pmod{p_1^{\alpha_1}}, \ \ldots, \ a \equiv g_k^{\gamma_k} \pmod{p_k^{\alpha_k}},$$

then $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ is called an *index system of the number a modulo m.*

It follows from this definition that γ, γ_0 is an index system of the number a modulo 2^α, while $\gamma_1, \ldots, \gamma_k$ are indices of the number a for the moduli $p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}$. Hence (**g, §6; c, §4**) every a which is relatively prime to m (and hence also relatively prime to all the numbers $2^\alpha, p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}$), has a unique index system $\gamma', \gamma_0', \gamma_1', \ldots, \gamma_k'$ in the $cc_0c_1 \ldots c_k = \varphi(m)$ systems which are obtained by letting $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ run independently through the least non-negative residues for the moduli $c, c_0, c_1, \ldots, c_k$, while all the index systems of the number a are just all the systems $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ consisting of the non-negative numbers of the equivalence classes

$$\gamma \equiv \gamma' \pmod{c}, \ \gamma_0 \equiv \gamma_0' \pmod{c_0},$$

$$\gamma_1 \equiv \gamma_1' \pmod{c_1}, \ \ldots, \ \gamma_k \equiv \gamma_k' \pmod{c_k}.$$

The numbers a with a given index system $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ can be found by solving the system (1), and hence they form an equivalence class of numbers modulo m (**b, §3, ch. IV**).

c. Since the indices $\gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ of the number a modulo m are the indices for the respective moduli $2^\alpha, p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}$, we have the theorem:

The indices of a product are congruent modulo $c, c_0, \ldots, c_k$ to the sums of the indices of the factors.

d. Let $r = \varphi(2^\alpha)$ for $\alpha \leqslant 2$ and $r = \dfrac{1}{2}\varphi(2^\alpha)$ for $\alpha > 2$ and let h be the least common multiple of the numbers $r, c_1, \ldots, c_k$. For every a which is relatively prime to m the congruence $a^h \equiv 1$ holds for all the moduli $2^\alpha, p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}$, which means that this congruence also holds for the modulus m. Hence a cannot be a primitive root modulo m in those cases in which $h < \varphi(m)$. But the latter holds for $\alpha > 2$, for $k > 1$, and for $\alpha = 2, k = 1$. Hence for $m > 1$, primitive roots can only exist if $m = 2, 4, p_1^{\alpha_1}, 2p_1^{\alpha_1}$. But the existence of primitive roots in these cases was proven above (**§6, §2**).

Hence

All the cases in which primitive roots modulo m, exceeding 1, *exist are just the cases in which*

$$m = 2, 4, p^{\alpha}, 2p^{\alpha}.$$

Problems for Chapter VI

The letter p always denotes an odd prime, except in problem **11, b** where we also allow the value 2.

1, a. Let a be an integer, $a > 1$. Prove the odd prime divisors of the number $a^p - 1$ divide $a - 1$ or are of the form $2px + 1$.

b. Let a be an integer, $a > 1$. Prove that the odd prime divisors of the number $a^p + 1$ divide $a + 1$ or are of the form $2px + 1$.

c. Prove that there are an infinite number of primes of the form $2px + 1$.

d. Let n be a positive integer. Prove that the prime divisors of the number $2^{2^n} + 1$ are of the form $2^{n+1}x + 1$.

2. Let a be an integer, $a > 1$, and let n be a positive integer. Prove that $\varphi(a^n - 1)$ is a multiple of n.

3, a. Let n be an integer, $n > 1$. Starting from the sequence $1, 2, \ldots, n$ we form, for odd n, the permutations

$$1, 3, 5, \ldots, n-2, n, n-1, n-3, \ldots, 4, 2;$$

$$1, 5, 9, \ldots, 7, 3$$

etc., while for even n we form the permutations

$$1, 3, 5, \ldots, n-1, n, n-2, \ldots, 4, 2;$$

$$1, 5, 9, \ldots, 7, 3,$$

etc. Prove that the k-th operation gives the original sequence if and only if $2^k \equiv \pm 1 \pmod{2n-1}$.

b. Let n be an integer, $n > 1$, and let m be an integer, $m > 1$. We consider the numbers $1, 2, \ldots. n$ in direct order from 1 to n, then in reverse order from n to 2, then in direct order from 1 to n, then in reverse order from n to 2, etc. From this sequence we take the 1-st, $(m + 1)$-st, $(2m + 1)$-st, etc., until we obtain n numbers. We repeat the same operation with this new sequence of n numbers, etc. Prove that the k-th operation gives the original sequence if and only if

$$m^k \equiv \pm 1 \pmod{2n - 1}$$

4. Prove that there exist $\varphi(\delta)$ numbers belonging to the index δ, by considering the congruence $x^\delta \equiv 1 \pmod p$ (problem **10, c, ch. IV**) and applying **d, § 3, ch. II.**

5, a. Prove that 3 is a primitive root of any prime of the form $2^n + 1$, $n > 1$.

b. Prove that 2 is a primitive root of any prime of the form $2p + 1$ if p is of the form $4n + 1$, while -2 is a primitive root of any prime of the form $2p + 1$ if p is of the form $4n + 3$.

c. Prove that 2 is a primitive root of any prime of the form $4p + 1$.

d. Prove that 3 is a primitive root of any prime of the form

$$2^n p + 1 \text{ for } n > 1 \text{ and } p > \frac{3^{2^{n-1}}}{2^n}.$$

6, a, α) Let n be a positive integer and let $S = 1^n + 2^n + \dots + (p - 1)^n$. Prove that

$$S \equiv -1 \pmod p, \text{ if } n \text{ is a multiple of } p - 1,$$

$$S \equiv 0 \pmod p, \text{ otherwise.}$$

β) Using the notation of problem **9, c, ch. V**, prove that

$$S(1) \equiv -\left(\frac{\frac{p-1}{2}}{\frac{p-1}{4}}\right).$$

b. Prove Wilson's theorem by applying **b**, §4.

7. Let g and g_1 be primitive roots modulo p, and let a ind$_g$ g. $\equiv 1 \pmod{p-1}$.

a. Let $(a, p) = 1$. Prove that

$$\text{ind}_{g_1} a \equiv \alpha \text{ ind}_g a \pmod{p-1}.$$

b. Let n be a divisor of $p - 1$, $1 < n < p - 1$. The numbers relatively prime to p can be divided into n sets by putting those numbers such that ind $a \equiv s \pmod{n}$ in the s-th set ($s = 0, 1, \ldots, n-1$). Prove that that the s-th set for the base g is identical with the s_1-th set for the base g_1, where $s_1 \equiv \alpha s \pmod{n}$.

8. Find the simplest possible method of solving the congruence $x^n \equiv a \pmod{p}$ (convenient for $(n, p-1)$ not too large) when we know some primitive root g modulo p.

9. Let $m, a, c, c_0, c_1, \ldots, c_k, \gamma, \gamma_0, \gamma_1, \ldots, \gamma_k$ have the values considered in §7. Considering any roots $R, R_0, R_1, \ldots, R_k$ of the equations

$$R^c = 1,\ R_0^{c_0} = 1,\ R_1^{c_1} = 1,\ \ldots,\ R_k^{c_k} = 1,$$

we set

$$\chi(a) = R^{\gamma} R_0^{\gamma_0} R_1^{\gamma_1} \ldots R_k^{\gamma_k}.$$

If $(a, m) > 1$, then we set $\chi(a) = 0$.

A function defined in this way for all integers a is said to be a *character*. If $R = R_0 = R_1 = \ldots = R_k = 1$, then we say that the character is *principal*; it has the value 1 for $(a, m) = 1$, and the value 0 for $(a, m) > 1$.

a. Prove that we obtain $\varphi(m)$ different characters in this way (two characters are said to be different if they are not equal for at least one value of a).

b. Deduce the following properties of characters:

α) $\chi(1) = 1$,

β) $\chi(a_1a_2) = \chi(a_1)\chi(a_2)$,

γ) $\chi(a_1) = \chi(a_2)$, if $a_1 \equiv a_2 \pmod{m}$.

c. Prove that

$$\sum_{a=0}^{m-1} \chi(a) = \begin{cases} \varphi(m), \text{ for the principal character,} \\ 0, \text{ for other characters.} \end{cases}$$

d. Prove that, for given a, summing all $\varphi(m)$ characters, we find

$$\sum_{\chi} \chi(a) = \begin{cases} \varphi(m), \text{ if } a \equiv 1 \pmod{m} \\ 0, \quad \text{otherwise.} \end{cases}$$

e. By considering the sum

$$H = \sum_{\chi} \sum_{a} \frac{\chi(a)}{\psi(a)}$$

where a runs through a reduced residue system modulo m, prove that a function $\psi(a)$ defined for all integers a and satisfying the conditions

$$\psi(a) = 0, \text{ if } (a, m) = 1,$$

$$\psi(a) \text{ is not identically equal to } 0,$$

$$\psi(a_1a_2) = \psi(a_1)\psi(a_2),$$

$$\psi(a_1) = \psi(a_2), \text{ if } a_1 \equiv a_2 \pmod{m},$$

is a character.

f. Prove the following theorems.

α) If $\chi_1(a)$ and $\chi_2(a)$ are characters, then $\chi_1(a)\chi_2(a)$ is also a character.

β) If $\chi_1(a)$ is a character and $\chi(a)$ runs through all the characters, then $\chi_1(a)\chi(a)$ also runs through all the characters.

γ) For $(l, m) = 1$, we have

$$\sum_{\chi} \frac{\chi(a)}{\chi(l)} = \begin{cases} \varphi(m), & \text{if } a \equiv l \pmod m \\ 0, & \text{otherwise.} \end{cases}$$

10, a. Let n be a divisor of $p - 1$, $1 < n \leqslant p - 1$, and let l be an integer which is not divisible by n. The number $R_1 = e^{2\pi i \frac{l}{n}}$ is a root of the equation $R_1^n = 1$, and hence the power $e^{2\pi i \frac{l \operatorname{ind} x}{n}}$, which is assumed to be equal to 0 for x a multiple of p, is a character modulo p.

α) For $(k, p) = 1$, prove that

$$\sum_{x=1}^{p-1} \exp\left(2\pi i \frac{l \operatorname{ind}(x + k) - l \operatorname{ind} x}{n}\right) = -1.$$

β) Let Q be an integer, $1 < Q < p$, and let

$$S = \sum_{x=0}^{p-1} |S_{l,n,x}|^2; \quad S_{l,n,x} = \sum_{z=0}^{Q-1} \exp\left(2\pi i \frac{l \operatorname{ind}(x + z)}{n}\right)$$

Prove that $S = (p - Q)Q$.

γ) Let M be an integer, $p > 4n^2$, $n > 2$. Prove that the sequence $M, M + 1, \ldots, M + 2[n\sqrt{p}\,] - 1$ contains a number of the s-th set of problem 7, b.

b. Let $p > 4\left(\dfrac{p - 1}{\varphi(p - 1)}\right)^2 2^{2k}$, let k be the number of different prime divisors of $p - 1$, and let M be an integer.

Prove that the sequence $M, M+1, \ldots, M +$
$+ 2\left[\frac{p-1}{\varphi(p-1)} 2^k\sqrt{p}\right] - 1$ contains a primitive root modulo p.

11, a. Let a be an integer, let n be a divisor of $p - 1$, $1 < n \leqslant p - 1$, and let k be an integer which is not divisible by n,

$$U_{a,p} = \sum_{x=1}^{p-1} \exp\left(2\pi i \frac{k \operatorname{ind} x}{n}\right) \exp\left(2\pi i \frac{ax}{p}\right)$$

α) For $(a, p) = 1$, prove that $|U_{a,p}| = \sqrt{p}$.

β) Prove that

$$\exp\left(2\pi i \frac{-k \operatorname{ind} a}{n}\right) = \frac{U_{a,p}}{U_{1,p}}.$$

γ) Let p be of the form $4m + 1$, and let

$$S = \sum_{x=1}^{p-2} \exp\left(2\pi i \frac{\operatorname{ind}(x^2 + x)}{4}\right).$$

Prove that (cf. problems **9, a** and **9, c, ch. V**) $p = A^2 + B^2$, where A and B are integers defined by the equation $S = A + Bi$.

b. Let n be an integer, $n > 2$, $m > 1$, $(a, m) = 1$,

$$S_{a,m} = \sum_{x} \exp\left(2\pi i \frac{ax^n}{m}\right), \quad S'_{a,m} = \sum_{\xi}{}' \exp\left(2\pi i \frac{a\xi^n}{m}\right),$$

where x runs through a complete residue system, while ξ runs through a reduced residue system modulo m (cf. problem **12, d, ch. III** and problem **11, b, ch. V**).

α) Let $\delta = (n, p - 1)$. Prove that

$$|S_{a,p}| \leqslant (\delta - 1)\sqrt{p}.$$

β) Let $(n, p) = 1$ and let s be an integer, $1 < s \leqslant n$. Prove that

$$S_{a,p^s} = p^{s-1},\ S'_{a,p^s} = 0.$$

γ) Let s be an integer, $s > n$. Prove that

$$S_{a,p^s} = p^{n-1}S_{a,p^{s-n}},\ S'_{a,p^s} = 0.$$

δ) Prove that

$$|S_{a,m}| < Cm^{1-\frac{1}{n}},$$

where C only depends on n.

12. Let M and Q be integers such that $0 < M < M + Q \leqslant p$.

a. Let n be a divisor of $p - 1$, $1 < n < p - 1$, and let k be an integer which is not divisible by n. Prove that

$$\left|\sum_{x=M}^{M+Q-1} \exp\left(2\pi i\frac{k \operatorname{ind} x}{n}\right)\right| < \sqrt{p}\ \ln p.$$

b. Let T be the number of integers of the s-th set of problem 7, **b**, contained in the set of numbers M, $M + 1$, ..., ..., $M + Q - 1$. Prove that

$$T = \frac{Q}{n} + \theta\sqrt{p}\ \ln p;\ |\theta| < 1.$$

c. Let k be the number of prime divisors of $p - 1$, and let H be the number of primitive roots modulo p in the set of numbers M, $M + 1$, ..., $M + Q - 1$. Prove that

$$H = \frac{\varphi(p - 1)}{p - 1}Q + \theta 2^k\sqrt{p}\ \ln p;\ |\theta| < 1.$$

d. Let M_1 and Q_1be integers, $0 \leqslant M_1 < M_1 + Q_1 \leqslant p - 1$, and let J be the number of integers of the sequence ind M, ind $(M + 1)$, ..., ind $(M + Q - 1)$ in the sequence $M_1, M_1 + 1, \ldots, M_1 + Q - 1$. Prove that

$$J = \frac{QQ_1}{p-1} + \theta\sqrt{p}\,(\ln p)^2;\ |\theta| < 1.$$

13. Prove that there exists a constant p_0 such that: if $p > p_0$, n is a divisor of $p - 1$, $1 < n < p - 1$, then the smallest of the positive non-residues of degree n modulo p is

$$<h;\ h = p^{\frac{1}{c}}(\ln p)^2;\ c = 2\exp\left(1 - \frac{1}{n}\right).$$

14, a. Let $m > 1$, $(a, m) = 1$,

$$S = \sum_{x=0}^{m-1}\sum_{y=0}^{m-1} \nu(x)\rho(y)\exp\left(2\pi i\frac{axy}{m}\right);$$

$$\sum_{x=0}^{m-1} |\nu(x)|^2 = X,\quad \sum_{y=0}^{m-1} |\rho(y)|^2 = Y.$$

Prove that $|S| \leqslant \sqrt{XYm}$.

b, α) Let $m > 1$, $(a, m) = 1$, let n be a positive integer, let K be the number of solutions of the congruence $x^n \equiv 1 \pmod{m}$, and let

$$S = \sum_{x=1}^{m-1} \chi(x)e^{2\pi i\frac{ax^n}{m}}.$$

Prove that $|S| \leqslant K\sqrt{m}$.

β) Let ϵ be an arbitrary positive constant. For constant n prove that $K = O(m^{\epsilon})$ where K is the number considered in problem α).

15, a. Let $(a, p) = (b, p) = 1$ and let n be an integer, $|n| = n_1$, $0 < n_1 < p$,

$$S = \sum_{x=1}^{p-1} \exp\left(2\pi i \frac{ax^n + bx}{p}\right).$$

Prove that

$$|S| < \frac{3}{2} n_1^{\frac{1}{2}} p^{\frac{3}{4}}.$$

b. Let $(A, p) = 1$, let n be an integer, $|n| = n_1$, $0 < n_1 < p$, and let M_0 and Q_0 be integers such that $0 < M_0 < M_0 + Q_0 < p$.

α) Let

$$S = \sum_{x=M_0}^{M_0+Q_0-1} \exp\left(2\pi i \frac{Ax^n}{p}\right).$$

Prove that $|S| < \frac{3}{2} n_1^{\frac{1}{2}} p^{\frac{3}{4}} \ln p$.

β) Let M and Q be integers such that $0 < M < M + Q \leqslant p$, and let T be the number of integers of the sequence Ax^n; $x = M_0, M_0 + 1, \ldots, M_0 + Q_0 - 1$, congruent to numbers of the sequence $M, M + 1, \ldots, M + Q_1 - 1$ modulo p.

Prove that

$$T = \frac{Q_1 Q}{p} + \theta \frac{3}{2} n_1^{\frac{1}{2}} p^{\frac{3}{4}} (\ln p)^2; \quad |\theta| < 1.$$

c. Let b and c be integers, $(a, p) = 1$, $(b^2 - 4ac, p) = 1$.

α) Let γ be an integer,

$$S = \sum_{x=0}^{p-1} \left(\frac{ax^2 + bx + c}{p}\right) \exp\left(2\pi i \frac{\gamma x}{p}\right).$$

Prove that $|S| < \frac{3}{2}p^{\frac{3}{4}}$.

β) Let M and Q be integers such that $0 < M < M + Q < p$, and let

$$S = \sum_{x=M}^{M+Q-1} \left(\frac{ax^2 + bx + c}{p}\right).$$

Prove that $|S| < \frac{3}{2}p^{\frac{3}{4}} \ln p$.

Numerical Exercises for Chapter VI

1, a. Find (in the simplest possible way) the exponent to which 7 belongs modulo 43.

b. Find the exponent to which 5 belongs modulo 108.

2, a. Find the primitive roots modulo 17, 289, 578.

b. Find the primitive roots modulo 41, 1681, 3362.

c. Find the smallest primitive roots modulo:

α) 1682; β) 3362.

3, a. Form the table of indices modulo 17.

b. Form the table of indices modulo 41.

4, a. Find a primitive root modulo 71, using the method of the example of **c**, §5.

b. Find a primitive root modulo 191.

5, a. Using the table of indices find the number of solutions of the congruences:

$$\alpha)\ x^{60} \equiv 79 \pmod{97};\quad \beta)\ x^{55} \equiv 17 \pmod{97};$$

$$\gamma)\ x^{15} \equiv 46 \pmod{97}.$$

b. Find the number of solutions of the congruences:

$$\alpha)\ 3x^{12} \equiv 31 \pmod{41};\quad \beta)\ 7x^{7} \equiv 11 \pmod{41};$$

$$\gamma)\ 5x^{30} \equiv 37 \pmod{41}.$$

6, a. Using the table of indices, solve the congruences:

$$\alpha)\ x^{2} \equiv 59 \pmod{67};\quad \beta)\ x^{35} \equiv 17 \pmod{67};$$

$$\gamma)\ x^{30} \equiv 14 \pmod{67}.$$

b. Solve the congruences:

$$\alpha)\ 23x^{5} \equiv 15 \pmod{73};\quad \beta)\ 37x^{6} \equiv 69 \pmod{73};$$

$$\gamma)\ 44x^{21} \equiv 53 \pmod{73}.$$

7, a. Using the theorem of **c**, **§5**, determine the number of solutions of the congruences:

$$\alpha)\ x^{3} \equiv 2 \pmod{37};\quad \beta)\ x^{16} \equiv 10 \pmod{37}.$$

b. Determine the number of solutions of the congruences:

$$\alpha)\ x^{5} \equiv 3 \pmod{71};\quad \beta)\ x^{21} \equiv 5 \pmod{71}.$$

8, a. Applying the methods of problem **8**, solve the congruences (in the solution of the second congruence use the table of primitive roots at the end of the book):

$$\alpha)\ x^{7} \equiv 37 \pmod{101};\quad \beta)\ x^{5} \equiv 44 \pmod{101}.$$

b. Solve the congruence

$$x^{3} \equiv 23 \pmod{109}.$$

9, a. Using the table of indices, in a reduced residue system modulo 19 find: α) the quadratic residues; β) the cubic residues.

b. In a reduced residue system modulo 37, find: α) the residues of degree 15; β) the residues of degree 8.

10, a. In a reduced residue system modulo 43, find: α) the numbers belonging to the exponent 6; β) the primitive roots.

b. In a reduced residue system modulo 61, find: α) the numbers belonging to the exponent 10; β) the primitive roots.

SOLUTIONS OF THE PROBLEMS

Solutions of the Problems for Chapter I.

1. The remainder resulting from the division of $ax + by$ by d, being of the form $ax' + by'$ and less than d, must be equal to zero. Therefore d is a divisor of all numbers of the form $ax + by$, and in particular is a common divisor of the numbers $a \cdot 1 + b \cdot 0 = a$ and $a \cdot 0 + b \cdot 1 = b$. On the other hand, the expression for d shows that every common divisor of the numbers a and b divides d. Therefore $d = (a, b)$, and hence theorem **1, d,** § **2** is valid. The theorems of **e,** § **2** are deduced as follows: the smallest positive number of the form $amx + bmy$ is $amx_0 + bmy_0$; the smallest positive number of the form $\frac{a}{\delta}x + \frac{b}{\delta}y$ is $\frac{a}{\delta}x_0 + \frac{b}{\delta}y_0$.

The generalization of these results is trivial.

2. We first note that the difference of two unequal rational fractions $\frac{k}{l}$ and $\frac{m}{n}$ $(l > 0, n > 0)$ is numerically $\geqslant \frac{1}{ln}$. We restrict ourselves by the assumption $\delta_s < \delta_{s+1}$. Let $\frac{a}{b}$ be an irreducible fraction, which is not equal to δ_s, such that

$0 < b \leqslant Q_s$. We cannot have $\delta_s < \frac{a}{b} < \delta_{s+1}$; otherwise we would have

$$\frac{a}{b} - \delta_s \geqslant \frac{1}{bQ_s}$$

$$\delta_{s+1} - \frac{a}{b} \geqslant \frac{1}{bQ_{s+1}}$$

$$\delta_{s+1} - \delta_s > \frac{1}{Q_s Q_{s+1}}$$

Therefore $\frac{a}{b} < \delta_s$ or $\delta_{s+1} < \frac{a}{b}$. In both cases δ_s is closer to α than $\frac{a}{b}$.

3. For $n \leqslant 6$ the theorem is evident; we therefore assume $n > 6$. We have

$$\xi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots;\ \log_{10}\xi = 0.2\ldots;$$

$$\begin{aligned}
&Q_2 > 1 = g_1 = 1\\
&Q_3 > Q_2 + 1 > g_2 = 22 > \xi,\\
&Q_4 > Q_3 + Q_2 > g_3 = g_2 + g_1 > \xi + 1 = \xi^2,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&Q_n > Q_{n-1} + Q_{n-2} > g_{n-1} = g_{n-2} + g_{n-3} > \xi^{n-3} + \xi^{n-4} = \xi^{n-2}.
\end{aligned}$$

Hence

$$N > \xi^{n-2};\ n < \frac{\log_{10}N}{\log_{10}\xi} + 2 < 5k + 2;\ n \leqslant 5k + 1.$$

4, a. For the fractions $\frac{0}{1}$ and $\frac{1}{1}$ we have $0 \cdot 1 - 1 \cdot 1 = -1$. Between the fractions $\frac{A}{B}$ and $\frac{C}{D}$ with $AD - BC = -1$, we insert the fraction $\frac{A + C}{B + D}$, and hence $A(B + D) - B(A + C) = (A + C)D - (B + D)C = -1$. Therefore the assertion at the end of the problem is true. The existence of a fraction $\frac{k}{l}$ such that $\frac{a}{b} < \frac{k}{l} < \frac{c}{d}$, $l < r$ is impossible. Otherwise we would have

$$\frac{k}{l} - \frac{a}{b} \geqslant \frac{1}{lb}; \frac{c}{d} - \frac{k}{l} \geqslant \frac{1}{ld}; \frac{c}{d} - \frac{a}{b} \geqslant \frac{b + d}{lbd} > \frac{1}{bd}$$

b. It is evident that it is sufficient to consider the case in which $0 \leqslant \alpha < 1$. Let $\frac{a}{b} \leqslant \alpha < \frac{c}{d}$, where $\frac{a}{b}$ and $\frac{c}{d}$ are neighboring fractions of the Farey series corresponding to r. There are two possible cases:

$$\frac{a}{b} \leqslant \alpha < \frac{a + c}{b + d}; \frac{a + c}{b + d} \leqslant \alpha < \frac{c}{d}$$

We therefore have one of the two inequalities

$$\left| \alpha - \frac{a}{b} \right| < \frac{1}{b(b + d)}; \left| \alpha - \frac{c}{d} \right| \leqslant \frac{1}{d(b + d)}$$

from which the required theorem follows because $b + d > r$.

c. For α irrational, the theorem follows from **h**, §4, if we take for $\frac{P}{Q}$ the convergent $\frac{P_{s-1}}{Q_{s-1}}$, where $Q_{s-1} \leqslant r < Q_s$.

In the case of rational $\alpha = \dfrac{a}{b}$, the above argument is only valid for $b > \tau$. But the theorem is true for $b \leqslant \tau$, since we can then take the fraction $\dfrac{a}{b}$ itself for $\dfrac{P}{Q}$, setting $\theta = 0$.

5, a. The remainder resulting from the division of an odd prime by 4 is either 1 or 3. The product of numbers of the form $4m + 1$ is of the form $4m + 1$. Therefore the number $4p_1 \dots p_k - 1$, where the $p_1, \dots, p_k$ are primes of the form $4m + 3$, has a prime divisor q of the form $4m + 3$. Moreover q is different from the primes $p_1, \dots, p_k$.

b. The primes greater than 3 are of the form $6m + 1$ or $6m + 5$. The number $6p_1 \dots p_k - 1$, where the $p_1, \dots, p_k$ are primes of the form $6m + 5$, has a prime divisor q of the form $6m + 5$. Moreover, q is different from the numbers $p_1, \dots, p_k$.

6. Let $p_1, \dots, p_k$ be any k primes, and let N be an integer such that $2 < N$, $(3 \ln N)^k < N$. The number of integers a of the sequence $1, 2, \dots, N$, whose canonical decomposition is of the form $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, is

$$\leqslant \left(\frac{\ln N}{\ln 2} + 1\right)^k < (3 \ln N)^k < N$$

since $\alpha_s < \dfrac{\ln N}{\ln 2}$. Therefore there are numbers in the sequence $1, 2, \dots, N$ whose canonical decomposition contains primes different from $p_1, \dots, p_k$.

7. For example, we obtain such sequences for

$$M = 2 \cdot 3 \cdots (K + 1)t + 2; \; t = 1, 2, \dots$$

8. Taking an integer x_0 such that $f(x) > 1$ and $f'(x) > 0$ for $x \geqslant x_0$, we set $f(x_0) = X$. All the numbers $f(x + Xt)$; $t = 1, 2, \dots$ are composite (multiples of X).

9, a. If (1) holds, one of the numbers x, y, say x, is even; it follows from

$$\left(\frac{x}{2}\right)^2 = \frac{z+y}{2}\,\frac{z-y}{2}$$

where, clearly, $\left(\frac{z+y}{2}, \frac{z-y}{2}\right) = 1$, that there exist positive integers u and v such that

$$\frac{x}{2} = uv, \quad \frac{z+y}{2} = u^2, \quad \frac{z-y}{2} = v^2.$$

This implies the necessity of the condition considered in the problem.

The sufficiency of these conditions is evident.

b. In the solution of this problem all letters denote positive integers. Assume the existence of systems x, y, z such that $x^4 + y^4 = z^2$, $x > 0$, $y > 0$, $z > 0$, $(x, y, z) = 1$, and choose the system with smallest z. Assuming x to be even we find $x^2 = 2uv$, $y^2 = u^2 - v^2$, $u > v \geqslant 1$, $(u, v) = 1$, where v is even (for even u we would have $y^2 = 4N + 1$, $u^2 = 4N_1$, $v^2 = 4N_2 + 1$, $4N + 1 = 4N_1 - 4N_2 - 1$, which is impossible). Hence $u = z_1^2$, $v = 2w^2$, $y^2 + 4w^2 = z_1^4$, $2w^2 = 2u_1v_1$, $u_1 = x_1^2$, $v_1 = y_1^2$, $x_1^4 + y_1^4 = z_1^2$, which is impossible since $z_1 < z$.

It follows from the non-solvability of the equation $x^4 + y^4 = z^2$ that the equation $x^4 + y^4 = t^4$ is not solvable in positive integers x, y, t.

10. Setting $x = \frac{k}{l}$; $(k, l) = 1$, we find

$$k^n + a_1k^{n-1}l + \ldots + a_nl^n = 0.$$

Therefore k^n is a multiple of l and hence $l = 1$.

11, a. Let k be the largest integer such that $2^k \leqslant n$ and let P be the product of all the odd numbers which do not exceed n. The number $2^{k-1}PS$ is a sum, all of whose terms, except $2^{k-1}P\dfrac{1}{2^k}$, are integers.

b. Let k be the largest integer such that $3^k \leqslant 2n + 1$ and let P be the product of all the integers relatively prime tᴖ 6 which do not exceed $2n + 1$. The number $3^{k-1}PS$ is a sum, all of whose terms, except $3^{k-1}P\dfrac{1}{3^k}$, are integers.

12. For $n \leqslant 8$, the theorem is immediately verifiable. It is therefore sufficient to assume that the theorem is true for the binomials $a + b$, $(a + b)^2, \ldots, (a + b)^{n-1}$ for $n > 8$, and prove that the theorem holds for $(a + b)^n$. But the coefficients of this binomial, except for the extreme ones, which are equal to 1, are just the numbers

$$\frac{n}{1}, \frac{n(n-1)}{1\cdot 2}, \ldots, \frac{n(n-1)\ldots 2}{1\cdot 2\ldots(n-1)}$$

A necessary and sufficient condition in order that all these numbers be odd is that the extreme numbers, both equal to n, be odd, and the numbers obtained by deleting the odd factors from the numerators and denominators of the remaining numbers be odd.

But, setting $n = 2n_1 + 1$, these numbers can be represented by the terms of the sequence

$$\frac{n_1}{1}, \frac{n_1(n_1-1)}{1\cdot 2}, \ldots, \frac{n_1(n_1-1)\ldots 2}{1\cdot 2\ldots(n_1-1)}.$$

Since $n_1 < n$, the latter are all odd if and only if n_1 is of the form $2^k - 1$, i.e. if and only if n is of the form $2(2^k - 1) + 1 = 2^{k+1} - 1$.

Solutions of the Problems for Chapter II

1, a. On the ordinate of the point of the curve $y = f(x)$ with abscissa x there are $[f(x)]$ lattice points of our region.

b. The required equation follows from $T_1 + T_2 = T$, where T_1, T_2, T denote the number of lattice points of the regions

$$0 < x < \frac{Q}{2},\ 0 < y < \frac{P}{Q}x,$$

$$0 < y < \frac{P}{2},\ 0 < x < \frac{Q}{P}y,$$

$$0 < x < \frac{Q}{2},\ 0 < y < \frac{P}{2}.$$

c. The required equation follows from

$$T = 1 + 4(T_1 + T_2 + T_3 - T_4),$$

where T_1, T_2, T_3, T_4 denote the number of lattice points of the regions

$$x = 0,\ 0 < y \leqslant r;$$

$$0 < x < \frac{r}{\sqrt{2}},\ 0 < y \leqslant \sqrt{r^2 - x^2};$$

$$0 < y \leqslant \frac{r}{\sqrt{2}},\ 0 < x \leqslant \sqrt{r^2 - y^2};$$

$$0 < x \leqslant \frac{r}{\sqrt{2}},\ 0 < y \leqslant \frac{r}{\sqrt{2}}.$$

d. The required equation follows from $T = T_1 + T_2 - T_3$, where T_1, T_2, T_3 denote the number of lattice points of the regions

$$0 < x \leqslant \sqrt{n}\ ,\ 0 < y \leqslant \frac{n}{x};$$

$$0 < y \leqslant \sqrt{n}\ ,\ 0 < x \leqslant \frac{n}{y};$$

$$0 < x \leqslant \sqrt{n}\ ,\ 0 < y \leqslant \sqrt{n}\ .$$

2. The number of positive integers which do not exceed n is equal to $[n]$. Each of them is uniquely representable in the form xk^m, where k is a positive integer; moreover, to a given x there correspond $\left[\sqrt[m]{\frac{n}{x}}\right]$ numbers of this form.

3. We prove the necessity of our conditions. Let N be an integer, $N > 1$. The number of values x such that $[\alpha x] \leqslant N$ can be represented in the form $\frac{N}{\alpha} + \lambda$; $0 \leqslant \lambda \leqslant C$, while the number of values y such that $[\beta y] \leqslant N$ can be represented in the form $\frac{N}{\beta} + \lambda_1$; $0 \leqslant \lambda_1 \leqslant C_1$, where C and C_1 do not depend on N. Dividing $\frac{N}{\alpha} + \lambda + \frac{N}{\beta} + \lambda_1 = N$ by N, and letting $N \longrightarrow \infty$, we find $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The latter equation for rational $\alpha = \frac{a}{b}\,(a > b > 0)$ would give $[\alpha b] = [\beta(a - b)]$.

Let our conditions be satisfied. Let c be a positive integer, and let $x_0 = \frac{c}{\alpha} + \xi$ and $y_0 = \frac{c}{\beta} + \eta$ be the smallest integers such that $x_0 \geqslant \frac{c}{\alpha}$, $y_0 \geqslant \frac{c}{\beta}$. Evidently, $[\alpha x] \gtrless c$ for $x \gtrless x_0$ and $[\beta y] \gtrless c$ for $y \gtrless y_0$, $0 < \xi < 1$, $0 < \eta < 1$, $\alpha\xi$ and $\beta\eta$ are irrational. Since $x_0 + y_0 = c + \eta + \xi$, we have $\xi + \eta = 1$, $\frac{\alpha\xi}{\alpha} + \frac{\beta\eta}{\beta} = 1$; therefore one and only one of the numbers $\alpha\xi$ and $\beta\eta$ is less than 1. Therefore, one and only one of the numbers $[\alpha x_0]$ and $[\beta y_0]$ is equal to c.

4, a. Our differences are equal to

$$\{\alpha x_1\},\ \{\alpha(x_2 - x_1)\},\ \ldots,\ \{\alpha(x_t - x_{t-1})\},\ \{-\alpha x_t\},$$

they are non-negative, their sum is equal to 1, there are $t + 1$ of them; therefore at least one of these differences does not exceed $\frac{1}{t+1} < \frac{1}{\tau}$, and hence there exists a number smaller than $\frac{1}{\tau}$ of the form $\{\pm\alpha Q\}$, where $0 < Q \leqslant \tau$. From $\pm\alpha Q = [\pm\alpha Q] + \{\pm\alpha Q\}$, setting $\pm[\pm\alpha Q] = P$, we find that $|\alpha Q - P| < \frac{1}{\tau}$, $\left|\alpha - \frac{P}{Q}\right| < \frac{1}{Q\tau}$.

b. Setting $X_0 = [X]$, $Y_0 = [Y]$, $\ldots$, $Z_0 = [Z]$, we consider the sequence formed by the numbers $\{\alpha x + \beta y + \ldots + \gamma z\}$ and the number 1 arranged in non-decreasing order, assuming that $x, y, \ldots, z$ run through the values:

$$x = 0, 1, \ldots, X_0;\ y = 0, 1, \ldots, Y_0;\ z = 0, 1, \ldots, Z_0.$$

We obtain $(X_0 + 1)(Y_0 + 1) \ldots (Z_0 + 1) + 1$ numbers, from which we obtain $(X_0 + 1)(Y_0 + 1) \ldots (Z_0 + 1)$ differences. At least one of these differences does not exceed

$$\frac{1}{(X_0 + 1)(Y_0 + 1) \ldots (Z_0 + 1)} < \frac{1}{XY \ldots Z}.$$

It is easy to obtain the required theorem from this.

5. We have $\alpha = cq + r + \{\alpha\};\ 0 \leqslant r < q$,

$$\left[\frac{[\alpha]}{c}\right] = \left[q + \frac{r}{c}\right] = q, \quad \left[\frac{\alpha}{c}\right] = \left[q + \frac{r + \{\alpha\}}{c}\right] = q.$$

6, a. We have $[\alpha + \beta + \ldots + \lambda] = [\alpha] + [\beta] + \ldots + [\lambda] = [\{\alpha\} + \{\beta\} + \ldots + \{\lambda\}]$.

b. The prime p divides $n!, a!, \ldots, l!$ to the exact powers

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots, \quad \left[\frac{a}{p}\right] + \left[\frac{a}{p^2}\right] + \ldots,$$

$$\ldots, \quad \left[\frac{l}{p}\right] + \left[\frac{l}{p^2}\right] + \ldots$$

Moreover

$$\left[\frac{n}{p^s}\right] \geqslant \left[\frac{a}{p^s}\right] + \ldots + \left[\frac{l}{p^s}\right].$$

7. Assuming that there exists a number a with the required properties, we represent it in the form

$$a = q_k p^{k+1} + q_{k-1}p^k + \ldots + q_1 p^2 + q_0 p + q';$$

$$0 < q_k < p,\ 0 \leqslant q_{k-1} < p,\ \ldots,$$
$$0 \leqslant q_1 < p,\ 0 \leqslant q_0 < p,\ 0 \leqslant q' < p.$$

By **b, §1,**

$$h = q_k u_k + q_{k-1}u_{k-1} + \ldots + q_1 u_1 + q_0 u_0.$$

Moreover, for any $s = 1, 2, \dots, m$, we have

$$q_{s-1}u_{s-1} + q_{s-2}u_{s-2} + \dots + q_1u_1 + q_0u_0 < u_s.$$

Therefore our expression for h must coincide completely with the one considered in the problem.

8, a. Letting x_1 be an integer, $Q \leqslant \alpha < \beta \leqslant R$, $x_1 < \alpha < \beta < x_1 + 1$, and integrating by parts, we find

$$-\int_{\alpha}^{\beta} f(x)dx = \int_{\alpha}^{\beta} \rho'(x)f(x)dx = \rho(\beta)f(\beta) - \rho(\alpha)f(\alpha) -$$

$$- \sigma(\beta)f'(\beta) + \sigma(\alpha)f'(\alpha) + \int_{\alpha}^{\beta} \sigma(x)f''(x)dx.$$

In particular, for $Q \leqslant x_1$, $x_1 + 1 \leqslant R$, passing to the limit, we have

$$-\int_{x_1}^{x_1+1} f(x)dx = -\frac{1}{2}f(x_1 + 1) - \frac{1}{2}f(x_1) + \int_{x_1}^{x_1+1} \sigma(x)f''(x)dx.$$

We can then obtain the required formula easily.

b. Rewriting the formula of problem **a** in the form

$$\sum_{Q<x\leqslant R} f(x) = \int^{R} f(x)dx - \int^{Q} f(x)dx + \rho(R)f(R) - \rho(Q)f(Q) -$$

$$- \sigma(R)f'(R) + \sigma(Q)f'(Q) + \int_{Q}^{\infty} \sigma(x)f''(x)dx - \int_{R}^{\infty} \sigma(x)f''(x)dx,$$

we obtain the required formula.

c. Applying the result of problem **b**, we find

$$\ln 1 + \ln 2 + \ldots + \ln n = C + n \ln n - n +$$

$$+ \frac{1}{2} \ln n + \int_n^\infty \frac{\sigma(x)}{x^2}\, dx = n \ln n - n + 0(\ln n).$$

9, a. α) We have (**b**, § **1** and problem **5**)

$$\ln([n]!) = \sum_{p \leqslant n} \left(\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] \right) + \ldots \quad \ln p. \tag{1}$$

The right side represents the sum of the values of the function $\ln p$, extended over the lattice points (p, s, u) with prime p of the region $p > 0$, $s > 0$, $0 < u \leqslant \frac{n}{p^s}$. The part of this sum corresponding to given s and u is equal to $\Theta\left(\sqrt[s]{\frac{n}{u}}\right)$; the part corresponding to given u is equal to $\psi\left(\frac{n}{u}\right)$.

β) Applying the result of problem α) for $n \geqslant 2$, we have

$$\ln([n]!) - 2 \ln \left(\left[\frac{n}{2}\right] ! \right) =$$

$$= \psi(n) - \psi\left(\frac{n}{2}\right) + \psi\left(\frac{n}{3}\right) - \psi\left(\frac{n}{4}\right) + \ldots > \psi(n) - \psi\left(\frac{n}{2}\right).$$

Setting $\left[\frac{n}{2}\right] = m$, we then find $([n] = 2m,\ [n] = 2m + 1)$

$$\psi(n) - \psi\left(\frac{n}{2}\right) \leqslant \ln \frac{(2m+1)!}{(m!)^2} \leqslant$$

$$\leqslant \ln\left(2^m \frac{3\cdot 5 \ldots (2m+1)}{1\cdot 2 \ldots m}\right) \leqslant \ln(2^m 3^m) < n,$$

$$\psi(n) = \psi(n) - \psi\left(\frac{n}{2}\right) + \psi\left(\frac{n}{2}\right) - \psi\left(\frac{n}{4}\right) +$$

$$+ \psi\left(\frac{n}{4}\right) - \psi\left(\frac{n}{8}\right) + \ldots < n + \frac{n}{2} + \frac{n}{4} + \ldots = 2n.$$

γ) We have (by the solution of problem β) and the result of problem **8, c**)

$$\psi(n) - \psi\left(\frac{n}{2}\right) + \psi\left(\frac{n}{3}\right) - \psi\left(\frac{n}{4}\right) + \ldots = \ln \frac{[n]!}{\left(\left[\frac{n}{2}\right]!\right)^2} =$$

$$= [n]\ln[n] - [n] - 2\left[\frac{n}{2}\right] \ln \left[\frac{n}{2}\right] + 2\left[\frac{n}{2}\right]. + O(\ln n) =$$

$$= n \ln 2 + O(\ln n).$$

Moreover, for $s \geqslant 2$ we find (problem β))

$$\Theta(\sqrt[s]{n}) - \Theta\left(\sqrt[s]{\frac{n}{2}}\right) +$$

$$+ \Theta\left(\sqrt[s]{\frac{n}{3}}\right) - \ldots \begin{cases} < 2\sqrt{n} \quad \text{always} \\ = 0 \text{ for } s > r;\ r = \left[\dfrac{\ln n}{\ln 2}\right]. \end{cases}$$

Therefore

$$0 \leqslant \psi(n) - \psi\left(\frac{n}{2}\right) + \psi\left(\frac{n}{3}\right) - \psi\left(\frac{n}{4}\right) +$$

$$+ \dots - \left(\Theta(n) - \Theta\left(\frac{n}{2}\right) + \Theta\left(\frac{n}{3}\right) - \Theta\left(\frac{n}{4}\right) + \dots\right) <$$

$$< 2\sqrt{n} + 2\sqrt[3]{n} + 2\sqrt[4]{n} +$$

$$+ \dots + 2\sqrt[r]{n} < 2(\sqrt{n} + r\sqrt[3]{n}) = O(\sqrt{n}).$$

b. The result follows from equation (1), the inequality of problem **a,** β) and the equation of problem **8, c.**

c. The equation of problem **b** for sufficiently large m gives

$$\sum_{m<p\leqslant m^2} \frac{\ln p}{p} = \ln m + 0(1) \geqslant \frac{\ln m}{2}, \quad \sum_{m<p\leqslant m^2} \frac{4}{p} > 1.$$

If $p_{n+1} > p_n(1 + \epsilon)$ for all pairs p_n, p_{n+1} such that $m < p_n < p_{n+1} \leqslant m^2$ then we would have

$$\sum_{r=0}^{\infty} \frac{4}{m(1+\epsilon)^r} > 1$$

which is impossible for sufficiently large m.

d. It is evidently sufficient to consider the case in which n is an integer.

Setting $\gamma(r) = \dfrac{\ln r}{r}$ for r prime and $\gamma(r) = 0$ for $r = 1$, and for r composite, we have (problem **b**)

$$\gamma(1) + \gamma(2) + \dots + \gamma(r) = \ln r + \alpha(r); \quad |\alpha(r)| < C_1,$$

where C_1 is a constant. Hence, for $r > 1$ (we consider $\alpha(1) = 1$)

$$\gamma(r) = \ln r - \ln(r-1) + \alpha(r) - \alpha(r-1),$$

$$\sum_{0<p\leqslant n} \frac{1}{p} = T_1 + T_2; \; T_1 = \sum_{1<r\leqslant n} \frac{\ln r - \ln(r-1)}{\ln \; r},$$

$$T_2 = \sum_{1<r\leqslant n} \frac{\alpha(r) - \alpha(r-1)}{\ln \; n}.$$

We have (8, b)

$$T = \sum_{1<r\leqslant n} \frac{1}{r \ln r} + \sum_{1<r\leqslant n} \left(\frac{1}{2r^2 \ln r} + \frac{1}{3r^2 \ln r} + \ldots\right) =$$

$$= C_2 + \ln \ln n + 0\left(\frac{1}{\ln \; n}\right),$$

where C_2 is a constant. Moreover we find

$$T = \alpha(2)\left(\frac{1}{\ln \; 2} - \frac{1}{\ln \; 3}\right) +$$

$$+ \ldots + \alpha(n-1)\left(\frac{1}{\ln \; (n-1)} - \frac{1}{\ln \; n}\right) + \frac{\alpha(n)}{\ln \; n}.$$

But, for an integer $m > 1$, we have

$$C_1\left(\frac{1}{\ln m} - \frac{1}{\ln(m+1)}\right) +$$

$$+ C_1\left(\frac{1}{\ln(m+1)} - \frac{1}{\ln(m+2)}\right) + \ldots = \frac{C_1}{\ln \; m}.$$

Therefore the series

$$\alpha(2)\left(\frac{1}{\ln\ 2}-\frac{1}{\ln\ 3}\right)+\alpha(3)\left(\frac{1}{\ln\ 3}-\frac{1}{\ln\ 4}\right)+\dots$$

converges; therefore, if C_3 is its sum, then

$$T_2 = C_3 + 0\left(\frac{1}{\ln\ n}\right) .$$

e. We have

$$\ln \prod_{p\leqslant n}\left(1-\frac{1}{p}\right) = -\sum_{p\leqslant n}\frac{1}{p} - \sum_{p\leqslant n}\left(\frac{1}{2p^2}+\frac{1}{3p^3}+\dots\right) =$$

$$= C' - \ln \ln n + 0\left(\frac{1}{\ln\ n}\right)$$

where C' is a constant. Setting $C' = \ln C_0$ in the latter equation, we obtain the required equation.

10, a. This result follows from **c**, §**2**.

b. Since $\theta(1) = \psi(1) = 1$, the function $\theta(a)$ satisfies condition **1, a**, §**2**. Let $a = a_1a_z$ be one of the decompositions of a into two relatively prime factors. We have

$$\sum_{d_1\backslash a_1}\sum_{d_2\backslash a_2}\theta(d_1d_2) = \psi(a) = \psi(a_1)\psi(a_z) =$$

(1)
$$= \sum_{d_1\backslash a_1}\sum_{d_2\backslash a_2}\theta(d_1)\theta(d_2).$$

If condition **2, a**, §**2** is satisfied for all products smaller than a, then, for $d_1d_2 < a$ we have $\theta(d_1d_2) = \theta(d_1)\theta(d_2)$, and equaticn (1) gives $\theta(a_1a_2) = \theta(a_1)\theta(a_2)$, i.e. condition **2, a**, §**2** is also satisfied for all products a_1a_2 equal to a. But condition **2, a, §2** is satisfied for the product $1 \cdot 1$ which is equal to 1. Therefore, it is satisfied for all products.

11, a. Let $m > 1$; for each given x_m dividing a, the indeterminate equation $x_1 \dots x_{m-1}x_m = a$ has $\tau_{m-1}\left(\frac{a}{x_m}\right)$ solutions. Therefore

$$\tau_m(a) = \sum_{x_m \backslash a} \tau_{m-1}\left(\frac{a}{x_m}\right);$$

but when x_m runs through all the divisors of the number a, the numbers $d = \frac{a}{x_m}$ run through all these same divisors in reverse order. Therefore

$$\tau_m(a) = \sum_{d \backslash a} \tau_{m-1}(d).$$

Hence (problem **10, a**), if the theorem is true for the function $\tau_{m-1}(a)$, then it is also true for the function $\tau_m(a)$. But the theorem is true for the function $\tau_1(a) = 1$, and hence it is always true.

b. If $m > 1$ and the theorem is true for the function $\tau_{m-1}(a)$, then

$$\tau_m(a) = \tau_m(p_1) \dots \tau_m(p_k) =$$
$$= (\tau_{m-1}(1) + \tau_{m-1}(p_1)) \dots (\tau_{m-1}(1) + \tau_{m-1}(p_k)) =$$
$$= (1 + m - 1)^k = m^k.$$

But the theorem is true for the function $\tau_1(a)$, and hence it is always true.

c. Let $\epsilon = m\epsilon_2$, $\epsilon_2 = 2\eta$, and let $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the canonical decomposition of the number a, where $p_1, \dots, p_k$

are arranged in increasing order. For the function $\tau_2(a) = \tau(a)$ we have

$$\frac{\tau(a)}{a^\eta} \leqslant \frac{\alpha_1 + 1}{2^{\alpha_1 \eta}} \frac{\alpha_2 + 1}{3^{\alpha_2 \eta}} \cdots \frac{\alpha_k + 1}{(k+1)^{\alpha_k \eta}} .$$

Each of the factors of the product on the right is smaller than $\frac{1}{\eta}$; the factors $\frac{\alpha_{r-1} + 1}{r^{\alpha_{r-1}\eta}}$ such that $r > 2^{\frac{1}{\eta}}$ is smaller than $\frac{\alpha_{r-1} + 1}{2^{\alpha_{r-1}}} \leqslant 1$. Therefore, setting $C = \left(\frac{1}{\eta}\right)^{2^{\frac{1}{\eta}}}$, we find

$$\frac{\tau(a)}{a^\eta} < C, \lim_{a\to\infty} \frac{\tau(a)}{a^{\epsilon_2}} \leqslant \lim_{a\to\infty} \frac{C}{a^\eta} = 0.$$

It is evident that $\tau_m(a) < (\tau(a))^m$ for $m > 2$. Therefore

$$\lim_{a\to\infty} \frac{\tau_m(a)}{a^\epsilon} < \lim_{a\to\infty} \left(\frac{\tau(a)}{a^{\epsilon_2}}\right)^m = 0.$$

d. We divide the systems of values $x_1, \ldots, x_m$ satisfying our inequality into $[n]$ sets with subscripts $1, 2, \ldots, [n]$. The systems such that $x_1 \ldots x_m = a$ are put in the set with subscript a; the number of these systems is $\tau_m(a)$.

12. The series defining $\zeta(s)$ converges absolutely for $R(s) > 1$. Therefore

$$(\zeta(s))^m = \sum_{n_1=1}^{\infty} \cdots \sum_{n_m=1}^{\infty} \frac{1}{(n_1 \ldots n_m)^s}$$

while, for given positive n, the number of systems $n_1, \ldots, n_m$ such that $n_1 \ldots n_m = n$ is equal to $\tau_m(n)$.

13, a. The product $P = \prod_p \frac{1}{1 - \frac{1}{p^s}}$ converges absolutely for $R(s) > 1$. Since $\frac{1}{1 - \frac{1}{p^s}} = \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots$ for $N > 2$, we have

$$\prod_{p \leqslant N} \frac{1}{1 - \frac{1}{p^s}} = \sum_{0<n\leqslant N} \frac{1}{n^s} + \sum{}' \frac{1}{n^s}$$

where the second sum on the right is extended over those numbers n which are not divisible by primes larger than N. As $N \longrightarrow \infty$, the left side tends to P, the first sum of the right side tends to $\zeta(s)$, while the second sum on the right tends to zero.

b. Let $N > 2$. Assuming that there are no primes other than $p_1, \ldots, p_k$, we find that (cf. the solution of problem **a**)

$$\prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j}} \geqslant \sum_{0<n\leqslant N} \frac{1}{n} .$$

This inequality is impossible for sufficiently large N because the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ diverges.

c. Assuming that there are no primes other than $p_1, \ldots, p_k$ we find (problem **a**)

$$\prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j^2}} = \zeta(2).$$

This equation is impossible in view of the irrationality of $\zeta(2) = \dfrac{\pi^2}{6}$.

14. The infinite product for $\zeta(s)$ of problem **13, a** converges absolutely for $R(s) > 1$. Therefore

$$\ln \zeta(s) = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots\right)$$

where p runs through all the primes. Differentiating, we find

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \left(-\frac{\ln p}{p^s} - \frac{\ln p}{p^{2s}} - \frac{\ln p}{p^{3s}} - \dots\right) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

15. Let $N > 2$. Applying theorem **b, §3,** we have

$$\prod_{p \leqslant N} \left(1 - \frac{1}{p^s}\right) = \sum_{0 < n \leqslant N} \frac{\mu(n)}{n^s} + \sum{}' \frac{\mu(n)}{n^s}$$

where the second sum on the right is extended over those numbers n larger than N which are not divisible by primes exceeding N. Taking the limit as $N \to \infty$, we obtain the required identity.

16, a. We apply **d, §3** to the case in which

$$\delta = 1, 2, \dots, [n], f = 1, 1, \dots, 1.$$

It is then evident that $S' = 1$. Moreover S_d is the number of values δ which are multiples of d, i.e. $\left[\dfrac{n}{d}\right]$.

b, α) The right side of the equation of problem **a** is the sum of the values of the function $\mu(d)$, extended over the lattice points (d, u) of the region $d > 0$, $0 < u \leqslant \frac{n}{d}$. The part of this sum corresponding to a given value of u, is equal to $M\left(\frac{n}{u}\right)$.

β) The required equation is obtained by termwise subtraction of the equations

$$M(n) + M\left(\frac{n}{2}\right) + M\left(\frac{n}{3}\right) + M\left(\frac{n}{4}\right) + \ldots = 1,$$

$$2M\left(\frac{n}{2}\right) + \qquad\qquad 2M\left(\frac{n}{4}\right) + \ldots = 2.$$

c. Let $n_1 = [n]$; let $\delta_1, \delta_2, \ldots, \delta_n$ be defined by the condition: δ_s is the largest integer whose l-th power divides s, $f_s = 1$. Then $S' = T_{l,n}$, S_d is equal to the number of multiples of d^l not exceeding n, i.e. $S_d = \left[\frac{n}{d^l}\right]$. From this we obtain the required expression for $T_{l,n}$.

In particular, since $\pi(2) = \frac{\pi^2}{6}$, we have

$$T_{2,n} = \frac{6}{\pi^2}n + O(\sqrt{n})$$

for the number $T_{2,n}$ of integers not exceeding n and not divisible by the square of an integer exceeding 1.

17, a. We obtain the required equation from **d**, §**3**, if we set

$$\delta_s = (x_s, a),\ f_s = f(x_s).$$

b. We obtain the required equation from **d**, §**3**, if we set

$$\delta_s = (x_1^{(s)}, \ldots, x_k^{(s)}),\ f_s = f(x_1^{(s)}, \ldots, x_k^{(s)}).$$

c. Applying **d**, §**3** to the case

$$\delta = \delta_1, \delta_2, \ldots, \delta_T,$$

$$f = F\left(\frac{a}{\delta_1}\right), F\left(\frac{a}{\delta_2}\right), \ldots, F\left(\frac{a}{\delta_T}\right),$$

where we have written down all the divisors of a in the first row, we have

$$S' = F(a),\ S_d = \sum_{D\backslash\frac{a}{d}} F\left(\frac{a}{dD}\right) = G\left(\frac{a}{d}\right).$$

d. The required equation follows from

$$P' = f_1^{\sum_{d\backslash\delta_1}\mu(d)}\ f_2^{\sum_{d\backslash\delta_2}\mu(d)}\ \ldots\ f_n^{\sum_{d\backslash\delta_n}\mu(d)}.$$

18, a. We apply the theorem of problem **17, a,** letting x run through the numbers 1, 2, ..., a and taking $f(x) = x^m$. Then

$$S' = \psi_m(a),\ S_d = d^m + 2^m d^m + \ldots + \left(\frac{a}{d}\right)^m d^m =$$

$$= d^m \sigma_m\left(\frac{a}{d}\right).$$

b. We have

$$\psi_1(a) = \sum_{d\backslash a} \mu(d)\left(\frac{a^2}{2d} + \frac{a}{2}\right) = \frac{a}{2}\varphi(a).$$

We can obtain the same result more simply. We first write down the numbers of the sequence 1, ..., a relatively prime to a in increasing order, and then in decreasing order. The sum of the terms of the two sequences equally distant from the initial terms, is equal to a; the number of terms in each sequence is equal to $\varphi(a)$.

c. We have

$$\psi_2(a) = \sum_{d \backslash a} \mu(d) \left(\frac{a^3}{3d} + \frac{a^2}{2} + \frac{a}{6} d \right) =$$

$$= \frac{a^2}{3} \varphi(a) + \frac{a}{6} (1 - p_1) \ldots (1 - p_k).$$

19, a. We apply the theorem of problem **17, a,** letting x run through the numbers 1, 2, ..., $[z]$ and taking $f(x) = 1$. Then $S' = T_z$, S_d is equal to the number of multiples of d which do not exceed z, i.e. $S_d = \left[\frac{z}{d} \right]$.

b. We have

$$T_z = \sum_{d \backslash a} \mu(d) \frac{z}{d} + O(\tau(a)) = \frac{z}{a} \varphi(a) + O(a^{\epsilon}).$$

c. This follows from the equation of problem **19, a.**

20. We apply the theorem of problem **17, a,** letting x run through the numbers 1, 2, ..., N, where $N > a$, and taking $f(x) = \frac{1}{x^s}$. We then find

$$\sum_{x \leqslant N} \frac{1}{x^s} = \sum_{d \backslash a} \mu(d) \sum_{0<x\leqslant \frac{N}{d}} \frac{1}{d^s x^s} = \sum_{d \backslash a} \frac{\mu(d)}{d^s} \sum_{0<x\leqslant \frac{N}{d}} \frac{1}{x^s}.$$

Taking the limit as $N \to \infty$, we obtain the required identity.

21, a. We apply the theorem of problem **17, b,** considering the systems of values $x_1, x_2, \ldots, x_k$ considered in the definition of the probabilities P_N, and taking $f(x_1\, x_2, \ldots, x_k) = 1$.

Then $P_N = \dfrac{S'}{N^k}$, $S_d = \left[\dfrac{N}{d}\right]^k$, and we obtain

$$P_N = \frac{\sum\limits_{d=1}^{N} \mu(d) \left[\dfrac{N}{d}\right]^k}{N^k} = \sum_{d=1}^{N} \frac{\mu(d)}{d^k} + O\left(\sum_{d=1}^{N} \frac{1}{Nd^{k-1}}\right).$$

Therefore

$$P_N = (\zeta(k)^{-1} + O(\Delta)\,;\ \Delta = \frac{1}{N} \text{ for } k > 2,$$

$$\Delta = \frac{\ln N}{N} \text{ for } k = 2.$$

b. We have $\zeta(2) = \dfrac{\pi^2}{6}$.

22, a. Elementary arguments show that the number of lattice points (u, v) of the region $u^2 + v^2 \leqslant \rho^2$; $\rho > 0$, not counting the point (0, 0), is equal to $\pi\rho^2 + O(\rho)$. We apply the theorem of problem **17, b,** considering the coordinates x, y of the lattice points of the region $x^2 + y^2 \leqslant r^2$, different from (0, 0), and setting $f(x, y) = 1$. Then $T = S' + 1$, S_d is equal to the number of lattice points of the region $u^2 + v^2 \leqslant \left(\dfrac{r}{d}\right)^2$, not considering the point (0, 0). Therefore

$$S_d = \pi \frac{r^2}{d^2} + O\left(\frac{r}{d}\right),$$

$$T = \sum_{d=1}^{[r]} \mu(d)\pi \frac{r^2}{d^2} + O\left(\sum_{d=1}^{[r]} \frac{r}{d}\right) = \frac{6}{\pi} r^2 + O(r \ln r).$$

b. Arguing in analogy to the above, we find

$$T = \sum_{d=1}^{[r]} \mu(d)\frac{4}{3}\pi \frac{r^3}{d^3} + O\left(\sum_{d=1}^{[r]} \frac{r^2}{d^2}\right) = \frac{4\pi r^3}{3\zeta(3)} + O(r^2).$$

23, a. The number of divisors d of the numbers $a = p_1^{a_1} \ldots p_k^{a_k}$ which are not divisible by the square of an integer exceeding 1, and having $\varkappa$ prime divisors, is equal to $\binom{k}{\varkappa}$; moreover $\mu(d) = (-1)^{\varkappa}$. Therefore

$$\sum_{d \backslash a} \mu(d) = \sum_{\varkappa=0}^{k} \binom{k}{\varkappa}(-1)^k = (1 - 1)^k = 0.$$

b. Let a be of the same form as in problem **a.** It is sufficient to consider the case $m < k$. For the sum under consideration we have two expressions

$$\sum \mu(d) = \binom{k}{0} - \binom{k}{1} + \ldots + (-1)^m \binom{k}{m}$$

$$= (-1)^m \left(\binom{k}{m+1} - \binom{k}{m+2} + \ldots\right).$$

If m is even, then for $m \leqslant \frac{k}{2}$, the first expression < 0, and for $m > \frac{k}{2}$ the second expression is $\geqslant 0$. If m is odd, then for $m \leqslant \frac{k}{2}$, the first expression <0, and for $m > \frac{k}{2}$ the second expression $\leqslant 0$.

c. The proof is almost the same as in **d**, **§3**, except that the result of problem **b** must be taken into account.

d. The proof is almost the same as in problems **17, a** and **17, b.**

24. Let d run through the divisors of the number a, let $\Omega(d)$ be the number of prime divisors of the number d, and let $\Omega(a) = s$. Following the process given in the problem, we have

$$(N, q, 1) \leqslant \sum_{\Omega(d) \leqslant m} \mu(d) \left(\frac{N}{q^d} + \theta_d \right) =$$

$$= T + T_0 - T_1; \ |\theta_d| < 1,$$

$$|T| \leqslant \sum_{\Omega(d) \leqslant m} 1, \ T_0 = \frac{N}{q} \sum_d \frac{\mu(d)}{d}, \ |T_1| = \sum_{\Omega(d) > m} \frac{N}{q^d}.$$

Moreover

$$|T| \leqslant \sum_{n=0}^{m} \binom{s}{n} = s^m \leqslant e^{nm} < \exp(5r^{1-\epsilon} \ln r) \frac{qr}{N} \frac{N}{qr} = O(\Delta),$$

$$T_0 = \frac{N}{q} \frac{\prod_{p \leqslant e^n} \left(1 - \frac{1}{p}\right)}{\prod_{p \backslash q} \left(1 - \frac{1}{p}\right)} = O(\Delta).$$

Finally, letting C_1, C_2, C_3 denote constants, we find

$$|T_1| \leqslant \frac{N}{q}\sum_{n=m+1}^{s}\sum_{\Omega(d)=n}\frac{1}{d} \leqslant$$

$$\leqslant \frac{N}{q}\sum_{n=m+1}^{s}\frac{\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{p_s}\right)^n}{n!} \leqslant$$

$$\leqslant \frac{N}{q}\sum_{n=m+1}^{s}\left(\frac{\frac{C_1+\ln r}{4\ln r-1}}{l}\right)^n \leqslant$$

$$\leqslant C_2\frac{N}{q}\sum_{n=m+1}^{s}\left(\frac{3}{4}\right)^n < C_3\frac{N}{q}r^{-4\ln\frac{4}{3}} = O(\Delta).$$

25. To every divisor d_1 of the number a such that $d_1 < \sqrt{a}$ there corresponds a divisor d_2 such that $d_2 > \sqrt{a}$, $d_1 d_2 = a$. Here $\mu(d_1) = \mu(d_2)$. Therefore

$$2\sum_{d_1}\mu(d_1) = \sum_{d_1}\mu(d_1) + \sum_{d_2}\mu(d_2) = \sum_{d\backslash a}\mu(d) = 0.$$

26. We consider pairs of numbers d which are not divisible by the square of an integer exceeding 1, and satisfying the condition $\varphi(d) = k$, such that each pair consists of some odd number d_1 and the even number $2d_1$. We have $\mu(d_1) + \mu(2d_1) = 0$.

27. Let $p_1, \ldots, p_k$ are distinct prime numbers. Setting $a = p_1 \ldots p_k$, we have

$$\varphi(a) = (p_1 - 1) \ldots (p_k - 1).$$

If there were no primes other than $p_1, \ldots, p_k$, we would have $\varphi(a) = 1$.

28, a. Our numbers are among the numbers $s\delta$; $s = 1, 2, \ldots, \frac{a}{\delta}$. But $(s\delta, a) = \delta$ if and only if $\left(s, \frac{a}{\delta}\right) = 1$ (**e, §2, ch. I**). Therefore the assertion in the problem is true and we have

$$a = \sum_{d\backslash a} \varphi\left(\frac{a}{d}\right) = \sum_{d\backslash a} \varphi(d).$$

b, α) Let $a = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$ be the canonical decomposition of the number a. By **a**, the function $\varphi(a)$ is multiplicative, while

$$p_s^{\alpha_s} = \sum_{d\backslash p^s} \varphi(d), \; p_s^{\alpha_s-1} = \sum_{d\backslash p^{s-1}} \varphi(d), \; p_s^{\alpha_s} - p_s^{\alpha_s-1} = \varphi(p_s^{\alpha_s}).$$

β) For a positive integer m, we have

$$m = \sum_{d\backslash m} (d).$$

Therefore

$$\varphi(a) = \sum_{d\backslash a} \mu(d)\frac{a}{d}.$$

29. We have (p runs through all the primes)

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \left(1 + \frac{\varphi(p)}{p^s} + \frac{\varphi(p^2)}{p^{2s}} + \ldots\right) =$$

$$= \prod_p \frac{1 - \frac{1}{p^s}}{1 - \frac{1}{p^{s-1}}} = \frac{\zeta(s-1)}{\zeta(s)}.$$

30. We have

$$\varphi(1) + \varphi(2) + \ldots + \varphi(n) =$$

$$= \sum_{d\backslash 1} \frac{\mu(d)}{d} + 2\sum_{d\backslash 2} \frac{\mu(d)}{d} + \ldots + n\sum_{d\backslash n} \frac{\mu(d)}{d} =$$

$$= \sum_{d=1}^{n} \mu(d)\left(1 + 2 + \ldots + \left[\frac{n}{d}\right]\right) =$$

$$= \sum_{d=1}^{n} \mu(d)\frac{n^2}{2d^2} + O(n \ln n) =$$

$$= \frac{n^2}{2}\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O(n \ln n) = \frac{3}{\pi^2} n^2 + O(n \ln n).$$

Solutions of the Problems for Chapter III

1, a. It follows from

$$P = a_n 10^{n-1} + a_{n-1} 10^{n-2} + \ldots + a_1,$$

that

$$P \equiv a_n + a_{n-1} + \ldots + a_1 \pmod 9$$

since $10 \equiv 1 \pmod 9$. Therefore P is a multiple of 3 if and only if the sum of its digits is a multiple of 3; it is a multiple of 9 if and only if this sum is a multiple of 9.

Noting that $10 \equiv -1 \pmod{11}$, we have

$$P \equiv (a_1 + a_3 + \ldots) - (a_2 + a_4 + \ldots) \pmod{11}.$$

Therefore P is a multiple of 3 if and only if the sum of its digits in the odd places minus the sum of its digits in the even places is a multiple of 11.

b. It follows from

$$P = b_n 100^{n-1} + b_{n-1} 100^{n-2} + \ldots + b_1$$

that

$$P \equiv (b_1 + b_3 + \ldots) - (b_2 + b_4 + \ldots) \pmod{101}$$

since $100 \equiv -1 \pmod{101}$. Therefore P is a multiple of 101 if and only if $(b_1 + b_3 + \ldots) - (b_2 + b_4 + \ldots)$ is a multiple of 101.

c. It follows from

$$P = c_n 1000^{n-1} + c_{n-1} 1000^{n-2} + \ldots + c_1$$

that

$$P \equiv c_n + c_{n-1} + \ldots + c_1 \pmod{37}$$

since $1000 \equiv 1 \pmod{37}$. Therefore P is a multiple of 37 if and only if $c_n + c_{n-1} + \ldots + c_1$ is a multiple of 37.

Since $1000 \equiv -1 \pmod{7 \cdot 11 \cdot 13}$, we have

$$P \equiv (c_1 + c_3 + \ldots) - (c_2 + c_4 + \ldots) \pmod{7 \cdot 11 \cdot 13}.$$

Therefore P is a multiple of one of the numbers 7, 11, 13 if and only if $(c_1 + c_3 + \ldots) - (c_2 + c_4 + \ldots)$ is a multiple of that number.

2, a. α) When x runs through a complete system of residues modulo m, then $ax + b$ also runs through a complete residue system; the smallest non-negative residue r of the numbers $ax + b$ also runs through the numbers $0, 1, \ldots, m - 1$.

Therefore

$$\sum_x \left\{\frac{ax+b}{m}\right\} = \sum_{r=0}^{m-1} \frac{r}{m} = \frac{1}{2}(m-1).$$

β) Applying the result of problem **18, b, ch. II,** we find

$$\sum_\xi \left\{\frac{a\xi}{m}\right\} = \frac{\psi_1(m)}{m} = \frac{1}{2}\varphi(m).$$

b. For $t = 1$, we have $[f(N+m)] - [f(N)] = a$,

$$\sum \delta = \sum_{x=N+1}^{N+m} [f(x)] - \frac{1}{2}[f(N+m)] + \frac{1}{2}[f(N)] - \frac{1}{2} + \frac{1}{2}m =$$

$$= \sum_{x=N+1}^{N+m} f(x) - \sum_{x=N+1}^{N+m} \{f(x)\} - \frac{1}{2}a + \frac{1}{2}(m-1) = S;$$

and the case in which $t > 1$ also reduces to this case trivially.

c. Let N, M, P_1, P_2 be integers, $M > 0$, $P_1 > 0$, $P_2 > 0$. The trapezoid with vertices $(N, 0)$, (N, P_1), $(N + M, 0)$, $(N + M, P_2)$ is a special case of the one considered in problem **b.** Therefore equation (1) is also valid for it. Equation (1) can also be obtained easily for such a trapezoid by considering the rectangle with vertices $(N, 0)$, $(N, P + P)$, $(N + M, 0)$, $(N + M, P + P)$, which is equal to two such trapezoids. For this rectangle, the equation

$$\sum{}' \delta = S',$$

analogous to equation (1), is evident. Since $\sum{}' \delta = 2\sum \delta$ this implies $S' = 2S$, so that we obtain equation (1).

The analogous formula for the triangle considered in the problem follows trivially from this result. But it is of some interest to consider the following derivation: our triangle can be obtained from a certain parallelogram with integral vertices by dividing it into two equal triangles. Let S be the area of the parallelogram and let $T = \sum \delta$ where the sum is extended over all the lattice points of the parallelogram and δ is defined as in problem **b.** We will have proven the property of the triangle that interests us, if we prove that $S = T$. We consider a square whose side A increases to infinity. The whole plane can be divided into an infinite number of parallelograms of the above type. Let k be the number of parallelograms completely within the interior of the square, and let R be the number of lattice points in the square. As $A \rightarrow \infty$, we find

$$\lim \frac{kS}{A^2} = 1, \ \lim \frac{A^2}{R} = 1, \ \lim \frac{R}{kT} = 1.$$

Multiplying these expressions termwise, we find

$$\lim \frac{S}{T} = 1, \ S = T.$$

3, a. Let r be the smallest positive residue of the number $ax + [c]$ modulo m. We have

$$S = \sum_{r=0}^{m-1} \left\{ \frac{r + \Phi(r)}{m} \right\}$$

where $\epsilon \leqslant \Phi(r) \leqslant \epsilon + h$; $\epsilon = \{c\}$. The theorem is evident for $m \leqslant 2h + 1$. We therefore only consider the case $m > 2h + 1$. Setting

$$\left\{ \frac{r + \Phi(r)}{m} \right\} - \frac{r}{m} = \delta(r),$$

we have $-1 + \frac{\epsilon}{m} \leqslant \delta(r) < \frac{h + \epsilon}{m}$ for $r = m - [h + \epsilon],$ $\ldots, m - 1$; in the other cases $\frac{\epsilon}{m} \leqslant \delta(r) \leqslant \frac{h + \epsilon}{m}$. Therefore

$$-[h + \epsilon] + \epsilon \leqslant S - \frac{m - 1}{2} \leqslant h + \epsilon, \left| S - \frac{1}{2} m \right| \leqslant h + \frac{1}{2}.$$

b. We have

$$S = \sum_{z=0}^{m-1} \left\{ \frac{az + \psi(z)}{m} \right\};\ \psi(z) = m(AM + B) + \frac{\lambda}{m} z.$$

We apply the theorem of problem **a**, setting $h = |\lambda|$. Then we obtain the required result.

c. We find

$$S = \sum_{z=0}^{m-1} \left\{ f(M) + \frac{az}{m} + \frac{\theta z}{m^2} + \frac{f''(M + z_0)}{2} z^2 \right\},$$

$$0 < z_0 < m - 1.$$

We apply the theorem of problem **a**, setting $h = 1 + \frac{k}{2}$. We then obtain the required result.

4. We develop A in a continued fraction. Let $Q_n = Q'$ be the largest of the denominators of the convergents which does not exceed m, and note (**h, §4, ch. I**)

$$A = \frac{P'}{Q'} + \frac{\theta'}{Q'm},\ (P', Q') = 1,\ |\theta'| < 1.$$

It follows from $m < Q_{n+1} < (q_{n+1} + 1)Q_n \leqslant CQ_n$, where C is a constant which is not larger than all the $q_s + 1$, that, for the largest integer H' such that $H'Q' < m$, we have $H' < C$. Applying the theorem of problem **3, b,** we find

$$\left| \sum_{x=M}^{M+H'Q'-1} \{Ax + B\} - \frac{1}{2}H'Q' \right| \leqslant \frac{3}{2}C.$$

Let $m_1 = m - H'Q'$. If $m_1 > 0$, then, choosing Q'' and H'' depending on m as we chose Q' and H' depending on m, we find

$$\left| \sum_{x=M_1}^{M_1+H''Q''-1} \{Ax + B\} - \frac{1}{2}H''Q'' \right| \leqslant \frac{3}{2}C.$$

Let $m_2 = m_1 - H''Q''$. If $m_2 > 0$, then, as above, we find

$$\left| \sum_{x=M_2}^{M_2+H'''Q'''-1} \{Ax + B\} - \frac{1}{2}H'''Q''' \right| \leqslant \frac{3}{2}C,$$

etc., until we find some $m_k = 0$. We then have $(H'Q' + + H''Q'' + \dots + H^{(k)}Q^{(k)} = m)$

$$\left| \sum_{x=M}^{M+m-1} \{Ax + B\} - \frac{1}{2}m \right| < \frac{3}{2}Ck.$$

The numbers $Q', Q'', \dots, Q^{(k)}$ satisfy the conditions

$$m \geqslant Q' > m_1 \geqslant Q'' > m_2 \geqslant \dots > m_{k-1} \geqslant Q^{(k)} \geqslant 1.$$

Therefore (problem **3, ch. I**) $k = O(\ln m)$, and hence the required formula is true.

5, a. Let the sum on the left be denoted by S. Let $\tau = A^{\frac{1}{3}}$. The theorem is evident for $\tau \leqslant 40$. We therefore assume that $\tau > 40$. Taking $M_1 = [Q + 1]$, we can find numbers a_1, m_1, θ such that

$$f'(M_1) = \frac{a_1}{m_1} + \frac{\theta_1}{m_1\tau};\ 0 < m \leqslant \tau,\ (a_1, m_1) = 1,\ |\theta_1| < 1.$$

Taking $M_2 = M_1 + m_1$, we find the numbers a_2, m_2, θ_2 analogously; taking $M_3 = M_2 + m_2$, we find the numbers a_3, m_3, θ_3; etc., until we come to $M_{s+1} = M_s + m_s$ such that $0 \leqslant [R] - M_{s+1} < [\tau]$. Applying the theorem of problem **3, c,** we find

$$\left| S - \frac{1}{2}(m_1 + m_2 + \ldots + m_s + [R] - M_{s+1}) \right| <$$

$$< s\frac{k + 3}{2} + \frac{1}{2}([R] - M_{s+1}),$$

$$\left| S - \frac{1}{2}(R - Q) \right| < s\frac{k + 3}{2} + \frac{\tau + 1}{2}.$$

The length of the interval for which $\frac{a}{m} - \frac{1}{m\tau} \leqslant f'(x) \leqslant$ $\leqslant \frac{a}{m} + \frac{1}{m\tau}$ does not exceed $\frac{2A}{m\tau}$. Therefore there are $<\frac{2A}{m^2\tau} + 1$ numbers $m_1, m_2, \ldots, m_s$ associated with the fraction $\frac{a}{m}$. Let a_1 and a_2 be the smallest and largest values of a associated with a given m.

We have

$$\frac{a_2 - a_1}{m} - \frac{2}{m\tau} \leqslant \frac{k(R - Q)}{A};$$

$$a_2 - a_1 + 1 < \frac{k(R - Q)m}{A} + 1.05.$$

Therefore, there are

$$\left(< \frac{2A}{m^2\tau} + 1\right)\left(\frac{k(R - Q)m}{A} + 1.05\right) =$$

$$= \frac{k(R - Q)}{\tau}\left(\frac{2}{m} + \frac{m}{\tau^2}\right) + \left(\frac{2A}{m^2\tau} + 1\right)1.05$$

numbers $m_1, m_2, \ldots, m_k$ associated with a given m. Summing the latter expression over all $m = 1, 2, \ldots, [\tau]$, we find

$$S < \frac{k(R - Q)}{\tau}\left(2 \ln \tau + 2 + \frac{\tau^2 + \tau}{2\tau^2}\right) + \frac{10A}{3\tau}1.05 <$$

$$< \frac{k(R - Q)}{\tau} \ln A + \frac{7}{2}\frac{A}{\tau},$$

$$\left|S - \frac{1}{2}(R - Q)\right| < 2\frac{k^2(R - Q)}{\tau} \ln A + 8k\frac{A}{\tau}.$$

b. We have

$$\left|\sum_{Q<x\leqslant R} \{f(x) + 1 - \sigma\} - \frac{1}{2}(R - Q)\right| < \Delta,$$

$$\left|\sum_{Q<x\leqslant R} \{f(x)\} - \frac{1}{2}(R - Q)\right| < \Delta,$$

from which, setting $\delta(x) = \{f(x) + 1 - \sigma\} - \{f(x)\}$, we find

$$\left| \sum_{Q<x\leqslant R} \delta(x) \right| < 2\Delta .$$

But, for $\{f(x)\} < \sigma$ we have $\delta(x) = 1 - \sigma$, while for $\{f(x)\} \geqslant \sigma$ we have $\delta(x) = -\sigma$, and hence $|(1 - \sigma)\psi(\sigma) - \sigma(R - Q - \psi(\sigma))| < 2\Delta$, from which we obtain the required formula.

6, a. We apply the formula of problem **1, c, ch. II.** Setting $f(x) = \sqrt{r^2 - x^2}$, we have

$$f'(x) = -\frac{x}{\sqrt{r^2 - x^2}}, \quad f''(x) = \frac{-r^2}{(r^2 - x^2)^{3/2}},$$

$$\frac{1}{r} \leqslant |f''(x)| \leqslant \frac{\sqrt{8}}{r}$$

in the interval $0 \leqslant x \leqslant \dfrac{r}{\sqrt{2}}$. Therefore (problem **8, a, ch. II,** problem **5, a**)

$$T = 4r + 8\int_0^{\frac{r}{\sqrt{2}}} \sqrt{r^2 - x^2}\,dx + 8\rho\left(\frac{r}{\sqrt{2}}\right)\frac{r}{\sqrt{2}} - 8\rho(0)r -$$

$$- 4\frac{r}{\sqrt{2}} - 4\frac{r^2}{2} + 8\frac{r}{\sqrt{2}}\left\{\frac{r}{\sqrt{2}}\right\} + 0(r^{\frac{2}{3}} \ln r) =$$

$$= \pi r^2 + 0(r^{\frac{2}{3}} \ln r).$$

b. We have (problems **11, d** and **1, d, ch. II**)

$$r(1) + r(2) + \ldots + r(n) = 2\sum_{0<x\leqslant\sqrt{n}} \left[\frac{n}{x}\right] - [\sqrt{n}\,]^2.$$

It is sufficient to consider the case $n > 64$. We divide the interval $X < x \leqslant \sqrt{n}$, where $X = 2n^{\frac{1}{3}}$, into $O(\ln n)$ intervals of the form $M < x \leqslant M'$, where $M' \leqslant 2M$. Setting $f(x) = \frac{n}{x}$, we have

$$f'(x) = -\frac{n}{x^2},\ f''(x) = \frac{2n}{x^3};\ \frac{n}{4M^3} \leqslant f''(x) \leqslant \frac{8n}{4M^3}$$

in the interval $M < x \leqslant M'$. Therefore (problem 5, a)

$$\sum_{M<x\leqslant M'} \left\{\frac{n}{x}\right\} = \frac{1}{2}(M' - M) + O(n^{\frac{1}{3}} \ln n),$$

$$\sum_{0<x\leqslant\sqrt{n}} \left\{\frac{n}{x}\right\} = \frac{1}{2}\sqrt{n} + O(n^{\frac{1}{3}} (\ln n)^2).$$

Moreover (problem 8, b, ch. II)

$$\sum_{0<x\leqslant\sqrt{n}} \frac{n}{x} = En + \frac{1}{2}n \ln n + \rho(\sqrt{n})\sqrt{n} + O(1).$$

Therefore

$$\begin{aligned}\tau(1) + \tau(2) + \ldots + \tau(n) &= \\ &= 2En + n \ln n + 2\rho(\sqrt{n})\sqrt{n} - \sqrt{n} - n + \\ &\quad + 2\sqrt{n}\{\sqrt{n}\} + O(n^{\frac{1}{3}} (\ln n)^2) = \\ &= n(\ln n + 2E - 1) + O(n^{\frac{1}{3}} (\ln n)^2).\end{aligned}$$

7. Let the system be improper and let s be the largest integer such that 2^s enters into an odd number of numbers of the system. We replace one of the latter numbers by the smallest number containing only those powers 2^s which enter into an odd number of integers of the rest of the system.

Let the system be proper. A number smaller than one of the numbers T of this system, differs from T in at least one digit in its representation to the base 2.

8, a. Adding the number $H = 3^n + 3^{n-1} + . . + 3 + 1$ to each of the numbers of the system represented in the aforementioned manner, we obtain numbers which we can obtain by letting $x_n, x_{n-1}, \ldots, x_1, x_0$ in the same form, run through the values 0, 1, 2, i.e. we obtain all the values $0, 1, \ldots, 2H$.

b. In this way we obtain $m_1m_2 \ldots m_k$ numbers which are incongruent to one-another modulo $m_1m_2 \ldots m_k$, since

$$x_1 + m_1x_2 + m_1m_2x_3 + \ldots + m_1m_2 \ldots m_{k-1}x_k \equiv$$

$$\equiv x_1' + m_1x_2' + m_1m_2x_3' + \ldots + m_1m_2 \ldots m_{k-1}x_k' \pmod{m_1m_2 \ldots m_k}$$

implies in sequence:

$$x_1 \equiv x_1' \pmod{m_1},\ x_1 = x_1';\ m_1x_2 \equiv m_1x_2' \pmod{m_1m_2},\ x_2 = x_2';$$

$$m_1m_2x_3 \equiv m_1m_2x_3' \pmod{m_1m_2m_3},\ x_3 = x_3',$$

etc.

9, a. In this way we obtain $m_1m_2 \ldots m_k$ numbers which are incongruent modulo $m_1m_2 \ldots m_k$, since

$$M_1x_1 + M_2x_2 + \ldots + M_kx_k \equiv M_1x_1' + M_2x_2' + \ldots + M_kx_k' \pmod{m_1m_2 \ldots m_k}$$

would imply (every M_j, different from M_s, is a multiple of m_s)

$$M_s x_s \equiv M_s x_s' \ (\mathrm{mod}\ m),\ x_s \equiv x_s' \ (\mathrm{mod}\ m_s),\ x_s = x_s'.$$

b. In this way we obtain $\varphi(m_1)\varphi(m_2) \ldots \varphi(m_k) = \varphi(m_1 m_2 \ldots m_k)$ numbers which are incongruent modulo $m_1 m_2 \ldots m_k$ by the theorem of problem **a**, and are relatively prime to $m_1 m_2 \ldots m_k$ since $(M_1 x_1 + M_2 x_2 + \ldots + M_k x_k, m_s) = (M_s x_s, m_s) = 1$.

c. By the theorem of problem **a**, the number $M_1 x_1 + M_2 x_2 + \ldots + M_k x_k$ runs through a complete residue system modulo $m_1 m_2 \ldots m_k$ when $x_1, x_2, \ldots, x_k$ run through complete residue systems modulo $m_1, m_2, \ldots, m_k$. This number is relatively prime to $m_1 m_2 \ldots m_k$ if and only if $(x_1, m_1) = (x_2, m_2) = \ldots = (x_k, m_k) = 1$. Therefore $\varphi(m_1 m_2 \ldots m_k) = \varphi(m_1)\varphi(m_2) \ldots \varphi(m_k)$.

d. To obtain the numbers of the sequence $1, 2, \ldots, p^\alpha$ relatively prime to p^α we delete the numbers of this sequence which are multiples of p, i.e. the numbers $p, 2p, \ldots, p^{\alpha-1}p$. Therefore $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1}$. The expression for $\varphi(a)$ follows from the latter and theorem **c**, §**4**, ch. **II**.

10, a. The first assertion follows from

$$\left\{\frac{x_1}{m_1} + \ldots + \frac{x_k}{m_k}\right\} = \left\{\frac{M_1 x_1 + \ldots + M_k x_k}{m}\right\};$$

the second assertion follows from

$$\left\{\frac{\xi_1}{m_1} + \ldots + \frac{\xi_k}{m_k}\right\} = \left\{\frac{M_1 \xi_1 + \ldots + M_k \xi_k}{m}\right\}.$$

b. The fractions

$$\left\{\frac{f_1(x_1, \ldots, w_1)}{m_1} + \ldots + \frac{f_k(x_k, \ldots, w_k)}{m_k}\right\}$$

coincide with the fractions

$$\left\{\frac{f_1(M_1x_1+\ldots+M_kx_k,\ \ldots,\ M_1w_1+\ldots+M_kw_k)}{m_1}+\ldots+\right.$$
$$\left.+\frac{f_k(M_1x_1+\ldots+M_kx_k,\ \ldots,\ M_1w_1+\ldots+M_kw_k)}{m_k}\right\},$$

i.e. with the fractions

$$\left\{\frac{f_1(x,\ \ldots,\ w)}{m_1}+\ldots+\frac{f_k(x,\ \ldots,\ w)}{m_k}\right\}.$$

The first assertion follows trivially from this. The second assertion is proved analogously.

11, a. If a is a multiple of m, we have

$$\sum_x \exp\left(2\pi i\frac{ax}{m}\right)=\sum_x 1=m.$$

If a is not a multiple of m, we have

$$\sum_x \exp\left(2\pi i\frac{ax}{m}\right)=\frac{\exp\left(2\pi i\frac{am}{m}\right)-1}{\exp\left(2\pi i\frac{a}{m}\right)-1}=0.$$

b. For non-integral α, the left side is equal to

$$\left|\frac{\exp(2\pi i\alpha(M+P))-\exp(2\pi i\alpha M)}{\exp(2\pi i\alpha)-1}\right|\leqslant\frac{1}{\sin\pi(\alpha)}<\frac{1}{h(\alpha)}.$$

c. By the theorem of problem **b**, the left side does not exceed T_m, where

$$T_m = \sum \frac{1}{h\left(\frac{a}{m}\right)} .$$

For odd m,

$$T_m < m \sum_{0<a<\frac{m}{2}} \ln \frac{2a+1}{2a-1} = m \ln m,$$

and for even m,

$$T_m < \frac{m}{2} \sum_{0<a\leqslant\frac{m}{2}} \ln \frac{2a+1}{2a-1} + \frac{m}{2} \sum_{0<a\leqslant\frac{m}{2}} \ln \frac{2a+1}{2a-1} < m \ln m.$$

For $m \geqslant 6$, since $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, the bound $m \ln m$ can be decreased to

$$2\frac{m}{6} \sum_{0<a\leqslant\frac{m}{6}} \ln \frac{2a+1}{2a-1} = \frac{m}{3} \ln \left(2\left[\frac{m}{6}\right] + 1\right) .$$

The latter expression is $> \frac{m}{2}$ for $m \geqslant 12$ and $> m$ for $m \geqslant 60$.

12, a. Let $m = p_1^{a_1} \ldots p_k^{a_k}$ be the canonical decomposition of the number m. Setting $p_1^{a_1} = m_1, \ldots, p_k^{a_k} = m_k$, and using the notation of problem **10, a,** we have

$$\sum_{\xi_1} \exp\left(2\pi i \frac{\xi_1}{m_1}\right) \ldots \sum_{\xi_k} \exp\left(2\pi i \frac{\xi_k}{m_k}\right) = \sum_{\xi} \exp\left(2\pi i \frac{\xi}{m}\right) .$$

For $\alpha_s = 1$, we find

$$\sum_{\xi_s} \exp\left(2\pi i \frac{\xi_s}{m_s}\right) = \sum_{x_s} \exp\left(2\pi i \frac{x_s}{m_s}\right) - 1 = -1.$$

For $\alpha_s > 1$, setting $m_s = p_s m_s'$, we find

$$\sum_{\xi_s} \exp\left(2\pi i \frac{\xi_s}{m_s}\right) =$$

$$= \sum_{x_s} \exp\left(2\pi i \frac{x_s}{m_s}\right) - \sum_{u=0}^{m_s'-1} \exp\left(2\pi i \frac{u}{m_s'}\right) = 0.$$

b. Let m be an integer, $m > 1$. We have $\sum\limits_{x=0}^{m-1} \exp\ 2\pi i \frac{x}{m} =$ $= 0$. By the theorem of problem **a**, the sum of the terms on the left side of this equation such that $(x, m) = d$, is equal to $\mu\left(\frac{m}{d}\right)$.

c. We find

$$\sum_{\xi} \exp\left(2\pi i \frac{\xi}{m}\right) = \sum_{d \backslash m} \mu(d) S_d,$$

where, setting $m = m_0 d$, we have

$$S_d = \sum_{u=0}^{m_0-1} \exp\left(2\pi i \frac{u}{m_0}\right).$$

The latter is equal to 0 for $d < m$ and equal to 1 for $d = m$. From this we obtain the theorem of problem **a**.

d. This equation follows from problem **10, b.**

e. We have

$$A(m_1)\ \ldots\ A(m_k) = m^{-t} \sum_{a_1} \cdots \sum_{a_k} S_{a_1, m_1} \cdots S_{a_k, m_k},$$

where $a_1, \ldots, a_k$ run through reduced residue systems modulo $m_1, \ldots, m_k$. Hence (problem **d**) the first equation of the problem follows immediately.

We also prove the second result analogously.

13, a. We have

$$\sum_{x=0}^{p-1} \exp\left(2\pi i \frac{nx}{p}\right) = \begin{cases} p, \text{ if } n \text{ is a multiple of } p, \\ 0, \text{ otherwise.} \end{cases}$$

b. Expanding the product corresponding to a given n, we find

$$\sum_{d \backslash a} \frac{\mu(d)}{d} \sum_{x=0}^{d-1} \exp\left(2\pi i \frac{nx}{d}\right).$$

Hence, summing over all the $n = 0, 1, \ldots, a-1$, we obtain the expression for $\varphi(a)$.

14. The part of the expression on the right corresponding to x dividing a, is equal to

$$\lim_{\epsilon \to 0} 2\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} =$$

$$= \lim_{\epsilon \to 0} \left(2\epsilon\left(\int_1^{\infty} \frac{dx}{x^{1+\epsilon}} + 0(1)\right)\right) = 2.$$

Setting $\Phi(K) = \sum_{k=1}^{K} \exp\left(2\pi i \frac{ak}{x}\right)$, the part corresponding to

x, not divisible by a, can be represented in the form

$$\lim_{\epsilon\to 0} 2\epsilon\left(\frac{\Phi(1)}{1} + \frac{\Phi(2)-\Phi(1)}{2^{1+\epsilon}} + \frac{\Phi(3)-\Phi(2)}{3^{1+\epsilon}} + \ldots\right) =$$

$$= \lim_{\epsilon\to 0} 2\epsilon\left(\Phi(1)\left(1 - \frac{1}{2^{1+\epsilon}}\right) + \Phi(2)\left(\frac{1}{2^{1+\epsilon}} - \frac{1}{3^{1+\epsilon}}\right) + \ldots\right).$$

The factor to the right of 2ϵ, is $< x$ in absolute value since $|\Phi(K)| < x$; here $\lim_{\epsilon\to 0} 2\epsilon x = 0$. Therefore the right side of the equation considered in the problem is equal to twice the number of divisors of the number a which are smaller than $\sqrt{a}$, multiplied by δ, i.e. equal to $\tau(a)$.

15, a. We have

$$(h_1 + h_2)^p =$$
$$= h_1^p + \binom{p}{1} h_1^{p-1}h_2 + \ldots + \binom{p}{p-1} h_1h_2^{p-1} + h_2^p \equiv$$
$$\equiv h_1^p + h_2^p \pmod{p};$$

$$(h_1 + h_2 + h_3)^p \equiv (h_1 + h_2)^p + h_3^p \equiv h_1^p + h_2^p + h_3^p \pmod{p},$$

etc.

b. Setting $h_1 = h_2 = \ldots = h_a = 1$, the theorem of problem **a** gives Fermat's theorem.

c. Let $(a, p) = 1$. For certain integers $N_1, N_2, \ldots, N_\alpha$ we have

$$a^{p-1} = 1 + N_1p,\; a^{p(p-1)} = (1 + N_1p)^p = 1 + N_2p^2,$$
$$a^{p^2(p-1)} = 1 + N_3p^3,\; \ldots,\; a^{p^{\alpha-1}(p-1)} = 1 + N_\alpha p^\alpha,$$
$$a^{\varphi(p^\alpha)} \equiv 1 \pmod{p^\alpha}.$$

Let $m = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the canonical decomposition of the number m. We have

$$a^{\varphi(p_1^{\alpha_1})} \equiv 1 \pmod{p_1^{\alpha_1}}, \ \ldots, \ a^{\varphi(p_k^{\alpha_k})} \equiv 1 \pmod{p_k^{\alpha_k}},$$

$$a^{\varphi(m)} \equiv 1 \pmod{p_1^{\alpha_1}}, \ \ldots, \ a^{\varphi(m)} \equiv 1 \pmod{p_k^{\alpha_k}},$$

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Solutions of the Problems for Chapter IV

1, a. The theorem follows immediately from the theorem of problem **11, a, ch. III.**

b. Let d be a divisor of the number m, $m = m_0 d$, and let H_d denote the sum of the terms such that $(a, m) = d$ in the expression for Tm in problem **a.** We find

$$H_d = \sum_{a_0} \sum_{x=0}^{m-1} \cdots \sum_{w=0}^{m-1} \exp\left(2\pi i \frac{a_0 f(x, \ldots, w)}{m_0}\right),$$

where a_0 runs through a reduced residue system modulo m_0. From this we deduce

$$H_d = d^r \sum_{a_0} \sum_{x_0=0}^{m_0-1} \cdots \sum_{w_0=0}^{m_0-1} \exp\left(2\pi i \frac{a_0 f(x_0, \ldots, w_0)}{m_0}\right) = m^r A(m_0).$$

c. Let $m > 0$, $(a, m) = d$, $a = a_0 d$, $m = m_0 d$, and let T be the number of solutions of the congruence $ax \equiv b \pmod{m}$. We have

$$\begin{aligned} Tm &= \sum_{\alpha=0}^{m-1} \sum_{x=0}^{m-1} \exp\left(2\pi i \frac{\alpha(ax - b)}{m}\right) \\ &= \sum_{\alpha=0}^{m-1} \sum_{x=0}^{m-1} \exp\left(2\pi i \frac{\alpha a_0}{m_0} x - 2\pi i \frac{b\alpha}{m}\right) \\ &= m \sum_{\alpha_1=0}^{d-1} \exp\left(-2\pi i \frac{b\alpha_1}{d}\right) = \begin{cases} md, \text{ if } b \text{ is a multiple of } d, \\ 0, \text{ otherwise} \end{cases} \end{aligned}$$

d. Setting $(a, m) = d_1$, $(b, d_1) = d_2, \dots, (f, d_{r-1}) = d_r$, $m = d_1m_1$, $d_1 = d_2m_2, \dots, d_{r-1} = d_rm_r$, we find $d = d_r$,

$$Tm = \sum_{\alpha=0}^{m-1}\sum_{x=0}^{m-1}\sum_{y=0}^{m-1}\cdots\sum_{w=0}^{m-1}\exp\left(2\pi i\frac{\alpha(ax+by+\dots+fw+g)}{m}\right)$$

$$= m\sum_{\alpha_1=0}^{d_1-1}\sum_{y=0}^{m-1}\cdots\sum_{w=0}^{m-1}\exp\left(2\pi i\frac{\alpha_1(by+\dots+fw+g)}{d_1}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= m^{r-1}\sum_{\alpha_{r-1}=0}^{d_{r-1}-1}\sum_{w=0}^{m-1}\exp\left(2\pi i\frac{\alpha_{r-1}(fw+g)}{d_{r-1}}\right) =$$

$$= m^r\sum_{\alpha_r=0}^{d_r-1}\exp\left(2\pi i\frac{\alpha_r g}{d_r}\right) .$$

e. We apply the method of induction. Using the notation of problem **d**, assume that the theorem is true for r variables. We consider the congruence

$$(2) \qquad lv + ax + \dots + fw + g \equiv 0 \pmod{m}.$$

Let $(l, m) = d_0$. Congruence (2) holds if and only if $ax + {} + \dots + fw + g \equiv 0 \pmod{d_0}$. The latter congruence holds if and only if g is a multiple of d', where $d' = (a, \dots, f, d_0) = (1, a, \dots, f, m)$, and it has $d_0^{r-1}d'$ solutions. Therefore the congruence (2) holds only if g is a multiple of d'; and it then has $d_0^{r-1}d'\left(\frac{m}{d_0}\right)^r d_0 = m^rd'$ solutions. Therefore the theorem is also true for $r + 1$ variables. But the theorem is true for one variable, and hence is always true.

2, a. We have $a^{\varphi(m)} \equiv 1 \pmod{m}$, $a \cdot ba^{\varphi(m)-1} \equiv b \pmod{m}$.

b. We have

$$1 \cdot 2 \ldots (a-1)\ ab\ (-1)^{a-1} \frac{(p-1)\ldots(p-a+1)}{1 \cdot 2 \ldots (a-1)} \equiv$$

$$\equiv b \cdot 1 \cdot 2 \ldots (a-1) \pmod{p},$$

and dividing by $1 \cdot 2 \ldots (a-1)$, we obtain the required theorem.

c. It is evidently sufficient to consider the case $(2, b) = 1$. For an appropriate choice of sign, $b \pm m \equiv 0 \pmod 4$. Let 2^δ be the largest power of 2 dividing $b \pm m$. For $\delta \geqslant k$, we have

$$x \equiv \frac{b \pm m}{2^k} \pmod{m}.$$

If $\delta < k$, then

$$2^{k-\delta}x \equiv \frac{b \pm m}{2^\delta} \pmod{m}.$$

We repeat the analogous operation with this congruence, etc.

β) We consider $(3, b) = 1$. For an appropriate choice of sign, we have $b \pm m \equiv 0 \pmod 3$. Let 3^δ be the largest power of 3 dividing $b \pm m$. For $\delta \geqslant k$, we have

$$x \equiv \frac{b \pm m}{2^k} \pmod{m}.$$

If $\delta < k$, then

$$3^{k-\delta}x \equiv \frac{b \pm m}{3^\delta} \pmod{m}.$$

We repeat the analogous operation with this congruence, etc.

γ) Let p be a prime divisor of the number a. Determine t by the condition $b + mt \equiv 0 \pmod{p}$. Let p^{δ} be the largest power of p dividing $(a, b + mt)$, and let $a = a_1 p^{\delta}$. We have

$$a_1 x \equiv \frac{b + mt}{p^{\delta}} \pmod{m}.$$

If $|a_1| > 1$, then we repeat this operation with the new congruence, etc.

This method is convenient for the case in which a has small prime factors.

3. Setting $t = [r]$, we write the congruences

$$\begin{aligned} a \cdot 0 &\equiv 0 \pmod{m}, \\ a \cdot 1 &\equiv y_1 \pmod{m}, \\ &\cdots\cdots\cdots\cdots\cdots \\ a \cdot t &\equiv y_t \pmod{m}, \\ a \cdot 0 &\equiv m \pmod{m}. \end{aligned}$$

Arranging these congruences so that their right sides are in order of increase (cf. problem **4, a, ch. II**) and multiplying termwise each congruence (except the last one) by its successor, we obtain $t + 1$ congruences of the form $az \equiv u$ $\pmod{m}$; $0 < |z| \leqslant r$. Here $0 < u < \frac{m}{r}$ in at least one congruence. Indeed, u has $t + 1 > r$ values, these values are positive, and their sum is equal to m.

4, a, α) This follows from the definition of symbolic fractions.

β) Here we can set $b_0 = b + mt$, where t is defined by the condition $b + mt \equiv 0 \pmod{a}$; then the congruence $ax \equiv b$ has as solution an integer which represents the ordinary fraction $\frac{b_0}{a}$.

γ) We have (b_0 is a multiple of a, d_0 is a multiple of c)

$$\frac{b}{a} + \frac{d}{c} \equiv \frac{b_0}{a} + \frac{d_0}{c} = \frac{b_0c + ad_0}{ac} \equiv \frac{bc + ad}{ac}.$$

δ) We have

$$\frac{b}{a} \cdot \frac{d}{c} \equiv \frac{b_0}{a} \cdot \frac{d_0}{c} = \frac{b_0d_0}{ac} \equiv \frac{bd}{ac}.$$

b, α) We have (the congruences are taken modulo p)

$$\binom{p-1}{a} = \frac{(p-1)(p-2)\ldots(p-a)}{1\cdot 2\ldots a} \equiv$$

$$\equiv \frac{(-1)^a 1\cdot 2\ldots a}{1\cdot 2\ldots a} \equiv (-1)^a.$$

Now problem 2, **b** is solved more simply as follows:

$$\frac{b}{a} \equiv \frac{b(-1)^{a-1}(p-1)\ldots(p-(a-1))}{1\cdot 2\ldots(a-1)^a} \pmod{p}.$$

β) We have

$$\frac{2^p - 2}{p} \equiv 1 + \frac{p-1}{1\cdot 2} + \frac{(p-1)(p-2)}{1\cdot 2\cdot 3} +$$

$$+ \ldots + \frac{(p-1)(p-2)\ldots(p-(p-2))}{1\cdot 2\ldots(p-1)} \pmod{p}.$$

5, a. The numbers $s, s+1, \ldots, s+n-1$ have no divisors in common with d. The products $s(s+1)\ldots(s+ + n-1)$ can be put in $n^{\varkappa}$ sets in a number of ways equal to the number of ways that d can be decomposed into n relatively prime factors, where order of the factors is taken into account (problem **11, b, ch. II**). Let $d = u_1 u_2 \ldots u_n$ be one of these decompositions. The number of products such that $s \equiv$ $\equiv 0 \pmod{u_1}$, $s + 1 \equiv 0 \pmod{u_2}, \ldots, s + n - 1 \equiv 0$ (mod u_n) is equal to $\frac{a}{d}$. Therefore the required number is equal to $n^{x}\frac{a}{d}$.

b. This number is equal to

$$\sum_{d \backslash a} \mu(d) S_d; \; S_d = \frac{n^k a}{d},$$

where k is the number of different prime divisors of the number d. But we have

$$\sum_{d \backslash a} \mu(d) \frac{n^k a}{d} = a \left(1 - \frac{n}{p_1}\right) \left(1 - \frac{n}{p_2}\right) \ldots \left(1 - \frac{n}{p_k}\right).$$

6, a. All the values of x satisfying the first congruence are given by the equation $x = b_1 + m_1 t$, where t is an integer. In order to choose from them those values which also satisfy the second congruence, it is only necessary to choose those values of t which satisfy the congruence

$$m_1 t \equiv b_2 - b_1 \pmod{m_2}.$$

But this congruence is solvable if and only if $b_2 - b_1$ is a multiple of d. Moreover, when the congruence is solvable, the set of values of t satisfying it is defined by an equation of the form $t = t_0 + \frac{m_2}{d} t'$, where t' is an integer; and hence the set of values of x satisfying the system considered in the problem is defined by the equation

$$x = b_1 + m_1 \left(t_0 + \frac{m_2}{d} t'\right) = x_{1,2} + m_{1,2} t'; \; x_{1,2} = b_1 + m_1 t_0.$$

b. If the system

$$x \equiv b_1 \pmod{m_1},\ x \equiv b_2 \pmod{m_2}$$

is solvable, the set of values of x satisfying it is representable in the form $x \equiv x_{1,2} \pmod{m_{1,2}}$. If the system

$$x \equiv x_{1,2} \pmod{m_{1,2}},\ x \equiv b_3 \pmod{m_3}$$

is solvable, the set of values of x satisfying it is representable in the form $x \equiv x_{1,2,3} \pmod{m_{1,2,3}}$. If the system

$$x \equiv x_{1,2,3} \pmod{m_{1,2,3}},\ x \equiv b_4 \pmod{m_4}$$

is solvable, the set of values of x satisfying it is representable in the form $x \equiv x_{1,2,3,4} \pmod{m_{1,2,3,4}}$, etc.

7, α) If x is replaced by $-x$ (and hence x' is replaced by $-x'$) the sum $\left(\dfrac{a,\ b}{m}\right)$ is not changed.

β) When x runs through a reduced residue system modulo m, x' also runs through a reduced residue system modulo m.

γ) Setting $x \equiv hz \pmod{m}$, we find

$$\left(\frac{a,\ bh}{m}\right) = \sum_z \exp\left(2\pi i\,\frac{ahz + bz'}{m}\right) = \left(\frac{ah,\ b}{m}\right).$$

δ) We have

$$\left(\frac{a_1,\ 1}{m_1}\right)\left(\frac{a_2,\ 1}{m_2}\right) =$$

$$= \sum_x \sum_y \exp\left(2\pi i\,\frac{a_1m_2x + a_2m_1y + m_2x' + m_1y'}{m_1m_2}\right).$$

Setting $m_2x' + m_1y' = z'$, we have

$$(a_1m_2x + a_2m_1y)(m_2x' + m_1y') \equiv a_1m_2^2 + a_2m_1^2 \pmod{m_1m_2},$$

$$\left(\frac{a_1, 1}{m_1}\right)\left(\frac{a_2, 1}{m_2}\right) = \left(\frac{m_2^2a_1 + m_1^2a_2, 1}{m_1m_2}\right)$$

which proves our property for the case of two factors. The generalization to the case of more than two factors is trivial.

8. The congruence

$$a_0x^n + a_1x^{n-1} + \dots + a_n - a_0(x - x_1)(x - x_2) \dots (x - x_n) \equiv$$
$$\equiv 0 \pmod p$$

has n solutions. Its degree is less than n. Therefore all its coefficients are multiples of p, and this is also expressed in the congruences considered in the problem.

9, a. Corresponding to x in the sequence $2, 3, \dots, p - 2$ we find a number x', different from it, in the same sequence such that $xx' \equiv 1 \pmod p$; indeed, it would follow from $x = x'$ that $(x - 1)(x + 1) \equiv 0 \pmod p$, and hence $x \equiv 1$ or $x \equiv p - 1$. Therefore

$$2 \cdot 3 \dots (p - 2) \equiv 1 \pmod p;\ 1 \cdot 2 \dots (p - 1) \equiv -1 \pmod p.$$

b. Let $P > 2$. Assuming that P has a divisor u such that $1 < u < P$, we would have $1 \cdot 2 \dots (P - 1) + 1 \equiv 1 \pmod u$.

10, a. We find h such that $a_0h \equiv 1 \pmod m$. The given congruence is equivalent to the following one:

$$x^n + a_1hx^{n-1} + \dots + a_nh \equiv 0 \pmod m.$$

b. Let $Q(x)$ be the quotient and let $R(x)$ be the remainder resulting from the division of $x^p - x$ by $f(x)$. All the coefficients of $Q(x)$ and $R(x)$ are integers, the degree of $Q(x)$ is $p - n$, the degree of $R(x)$ is less than n,

$$x^p - x = f(x)Q(x) + R(x).$$

Let the congruence $f(x) \equiv 0 \pmod{p}$ have n solutions. These solutions will also be solutions of the congruence $R(x) \equiv$ $\equiv 0 \pmod{p}$; therefore all the coefficients of $R(x)$ are multiples of p.

Conversely, let all the coefficients of $R(x)$ be multiples of p. Then $f(x)Q(x)$ is a multiple of p for those values of x for which $x^p - x$ is also a multiple of p; therefore the sum of the numbers of solutions of the congruences

$$f(x) \equiv 0 \pmod{p}, \quad Q(x) \equiv 0 \pmod{p}$$

is no smaller than p. Let the first have α, and let the second have β solutions. From

$$\alpha \leqslant n, \ \beta \leqslant p - n, \ p - n, \ p \leqslant \alpha + \beta$$

we deduce $\alpha = n$, $\beta = p - n$.

c. Raising the given congruence to the power $\dfrac{p-1}{n}$ termwise, we find that the given condition is necessary. Let this condition be satisfied; it follows from $x^p - x =$

$= x(x^{p-1} - A^{\frac{p-1}{n}} + A^{\frac{p-1}{n}} - 1)$ that the remainder resulting from the division of $x^p - x$ by $x^n - A$ is $(A^{\frac{p-1}{n}} - 1)x$, where $A^{\frac{p-1}{n}} - 1$ is a multiple of p.

11. It follows from $x_0^n \equiv A \pmod{m}$, $y^n \equiv 1 \pmod{m}$ that $(x_0y)^n \equiv A \pmod{m}$; here the products x_0y, corresponding to incongruent (modulo m)y, are incongruent. It follows from $x_0^n \equiv A \pmod{m}$, $x^n \equiv A \pmod{m}$ that $x^n \equiv x_0^n \pmod{m}$, while, defining y by the condition $x \equiv yx_0 \pmod{m}$, we have

$$y^n \equiv 1 \pmod{m}.$$

Solutions of the Problems for Chapter V

1. This congruence is equivalent to the following one: $(2ax + b)^2 \equiv b^2 - 4ac \pmod{m}$. Corresponding to each solution $z \equiv z_0 \pmod{m}$ of the congruence $z^2 \equiv b^2 - 4ac \pmod{m}$, from $2ax + b \equiv z_0 \pmod{m}$ we find a solution of the congruence under consideration.

2, a. For $\left(\dfrac{a}{p}\right) = 1$ we have $a^{2m+1} \equiv 1 \pmod{p}$,

$(a^{m+1})^2 \equiv a \pmod{p}$, $x \equiv \pm a^{m+1} \pmod{p}$.

b. For $\left(\dfrac{a}{p}\right) = 1$ we have $a^{4m+2} \equiv 1 \pmod{p}$, $a^{2m+1} \equiv \pm 1$

$\pmod{p}$, $a^{2m+2} \equiv \pm a \pmod{p}$. Since $\left(\dfrac{2}{p}\right) = -1$ we also

have $2^{4m+2} \equiv -1 \pmod{p}$. Therefore, for a certain s, having one of the values 0, 1, we find

$$a^{2m+2}2^{(4m+2)} \equiv a \pmod{p},\ x \equiv \pm a^{m+2}2^{(2m+1)s} \pmod{p}.$$

c. Let $p = 2^k h + 1$, where $k \geqslant 3$ and h is odd, $\left(\dfrac{a}{p}\right) = 1$.

We have

$$a^{2^{k-1}h} \equiv 1 \pmod{p},\ a^{2^{k-2}h} \equiv \pm 1 \pmod{p},\ N^{2^{k-1}h} \equiv -1 \pmod{p}.$$

Therefore, for some non-negative integer s_2 we find

$$a^{2^{k-2}h}N^{s_2 2^{k-1}} \equiv 1 \pmod{p},\ a^{2^{k-3}h}N^{s_2 2^{k-2}} \equiv \pm 1 \pmod{p};$$

and hence for some non-negative integer s_3 we find

$$a^{2^{k-3}h}N^{s_3 2^{k-2}} \equiv 1 \pmod{p},\ a^{2^{k-4}h}N^{s_3 2^{k-3}} \equiv \pm 1 \pmod{p},$$

etc.; finally we find

$$a^h N^{2s_k} \equiv 1 \pmod p, \quad x \equiv \pm a^{\frac{h+1}{2}} N^{s_k} \pmod p.$$

d. We have

$$1 \cdot 2 \ldots 2m(p - 2m) \ldots (p - 2)(p - 1) + 1 \equiv 0 \pmod p,$$

$$(1 \circ 2 \ldots 2m)^2 + 1 \equiv 0 \pmod p.$$

3, a. The conditions for the solvability of congruence (1) and (2) are deduced trivially (**f**, §2 and **k**, §2). The congruence (3) is solvable if and only if $\left(\frac{-3}{p}\right) = 1$. But

$\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$, while

$$\left(\frac{p}{3}\right) = \begin{cases} 1, \text{ if } p \text{ is of the form } 6m + 1, \\ -1, \text{ if } p \text{ is of the form } 6m + 5. \end{cases}$$

b. For any distinct primes $p_1, p_2, \ldots, p_k$ of the form $4m + 1$, the smallest prime divisor p of the number $(2p_1p_2 \ldots p_k)^2 + 1$ is different from $p_1, p_2, \ldots, p_k$, and since $(2p_1p_2 \ldots p_k)^2 + 1 \equiv 0 \pmod p$, it is of the form $4m + 1$.

c. For any distinct primes $p_1, p_2, \ldots, p_k$ of the form $6m + 1$, the smallest prime divisor p of the number $(2p_1p_2 \ldots p_k)^2 + 3$ is different from $p_1, p_2, \ldots, p_k$, and since $(2p_1p_2 \ldots p_k)^2 + 3 \equiv 0 \pmod p$, it is of the form $6m + 1$.

4. There are numbers in the first set which are congruent to $1 \cdot 1, 2 \cdot 2, \ldots, \frac{p-1}{2} \circ \frac{p-1}{2}$, i.e. all the quadratic residues of a complete system; a number in the second set is a quadratic non-residue, by definition. But the second set contains with this non-residue, all the products of the non-residue with residues, i.e. it contains all the quadratic non-residues.

5, a. In the number system to the base p, let

$$a = a_{\alpha-1}p^{\alpha-1} + \dots + a_1p + a_0$$

and let the required solution (the smallest non-negative residue) be

$$x = x_{\alpha-1}p^{\alpha-1} + \dots + x_1p + x_0.$$

We form the table:

$a_{\alpha-1}$	....	a_4	a_3	a_2	a_1	a_0
$2x_0x_{\alpha-2}$	....	$2x_0x_4$	$2x_0x_3$	$2x_0x_2$	$2x_0x_1$	x_0^2
$2x_1x_{\alpha-2}$	....	$2x_1x_3$	$2x_1x_2$	x_1^2		
$2x_2x_{\alpha-3}$	....	x_2^2				
....						

where the column under a_s consists of numbers whose sum is the coefficient of p^s in the decomposition of the square of the right side of (1) in powers of p. We determine x_0 by the condition

$$x_0^2 \equiv a \pmod{p}.$$

Setting $\dfrac{x_0^2 - a_0}{p} = p_1$, we determine x_1 by the condition

$$p_1 + 2x_0x_1 \equiv a_1 \pmod{p}.$$

Setting $\dfrac{p_1 + 2x_0x_1 - a_1}{p} = p_2$, we determine x_2 by the condition

$$p_2 + 2x_0x_2 + x_1^2 \equiv a_2 \pmod{p},$$

etc. For given x_0, the numbers $x_1, x_2, \ldots, x_{\alpha-1}$ are uniquely determined since $(x_0, p) = 1$.

b. Here

$$a = a_{\alpha-1}2^{\alpha-1} + \ldots + a_3 2^3 + a_2 2^2 + a_1 2 + a_0,$$

$$x = x_{\alpha-1}2^{\alpha-1} + \ldots + x_3 2^3 + x_2 2^2 + x_1 2 + x_0,$$

and we have the following table:

$a_{\alpha-1}$	....	a_4	a_3	a_2	a_1	a_0
$x_0x_{\alpha-2}$ $x_1x_{\alpha-3}$ $x_2x_{\alpha-4}$	....	x_0x_3 x_1x_2 x_2^2	x_0x_2	x_0x_1 x_1^2		x_0^2

We only consider the case $\alpha \geqslant 3$. Since $(a, 2) = 1$, it follows that $a_0 = 1$. Therefore $x_0 = 1$. Moreover $a_1 = 0$, and since $x_0x_1 + x_1^2 = x_1 + x_1^2 \equiv 0 \pmod{2}$, we must have $a_2 = 0$. For x_1 there are two possible values: 0 and 1. The numbers $x_2, x_3, \ldots, x_{\alpha-2}$ are uniquely determined, while for $x_{\alpha-2}$, there are two possible values: 0 and 1. Therefore, for $\alpha \geqslant 3$, we must have $a \equiv 1 \pmod{8}$, and then the congruence under consideration has 4 solutions.

6. It is evident that P and Q are integers, where Q is congruent modulo p to a number which we obtain by replacing a by z^2, for which it is sufficient to replace $\sqrt{a}$ by z. Therefore $Q \equiv 2^{\alpha-1}z^{\alpha-1} \pmod{p}$; therefore $(Q, p) = 1$ and Q' is determined by the congruence $QQ' \equiv 1 \pmod{p}$. We have

$$P^2 - aQ^2 = (z + \sqrt{a}\,)^\alpha(z - \sqrt{a}\,)^\alpha = (z^2 - a)^\alpha \equiv 0 \pmod{p^\alpha},$$

from which it follows that

$$(PQ')^2 \equiv a(QQ')^2 \equiv a \pmod{p^\alpha}.$$

7. Let $m = 2^\alpha p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the canonical decomposition of the number m. Then m can be represented in the form $m = 2^\alpha ab$, where $(a, b) = 1$, in 2^k ways.

Let $\alpha = 0$. It follows from $(x - 1)(x + 1) \equiv 0 \pmod{m}$, that for certain a and b

$$x \equiv 1 \pmod{a}; \; x \equiv -1 \pmod{b}.$$

Solving this system, we obtain $x \equiv x_0 \pmod{m}$. Therefore the congruence under consideration has 2^k solutions.

Let $\alpha = 1$. For certain a and b

$$x \equiv 1 \pmod{2a}; \; x \equiv -1 \pmod{2b}.$$

Solving this system, we obtain $x \equiv x_0 \pmod{m}$. Hence this congruence has 2^k solutions.

Let $\alpha = 2$. For certain a and b

$$x \equiv 1 \pmod{2a}; \; x \equiv -1 \pmod{2b}.$$

Solving this system, we obtain $x \equiv x_0 \left(\bmod \dfrac{m}{2}\right)$. Therefore our congrunece has 2^{k+1} solutions.

Let $\alpha \geqslant 3$. For certain a and b, one of the systems

$$x \equiv 1 \pmod{2a}; \; x \equiv -1 \pmod{2^{\alpha-1}b)}$$

$$x \equiv 1 \pmod{2^{\alpha-1}a}; \; x \equiv -1 \pmod{2b}$$

is satisfied. Solving one of these systems, we obtain

$x \equiv x_0 \left(\bmod \dfrac{m}{2}\right)$. Therefore our congruence has 2^{k+2} solutions.

8, a. Defining x' by the congruence $xx' \equiv 1 \pmod p$, we have

$$\sum_{x=1}^{p-1} \left(\frac{x(x+k)}{p}\right) = \sum_{x=1}^{p-1} \left(\frac{xx'(xx'+kx')}{p}\right) =$$

$$= \sum_{x=1}^{p-1} \left(\frac{1+kx'}{p}\right).$$

It is evident that $1 + kx'$ runs through all the residues of a complete system, except 1. The required theorem follows from this.

b. The required equation follows from

$$T = \frac{1}{4}\sum_{x=1}^{p-2} \left(1+\epsilon\left(\frac{x}{p}\right)\right); \left(1+\eta\left(\frac{x+1}{p}\right)\right) =$$

$$= \frac{1}{4}\sum_{x=1}^{p-2} \left(1+\epsilon\left(\frac{x}{p}\right)+\eta\left(\frac{x+1}{p}\right)+\epsilon\eta\left(\frac{x(x+1)}{p}\right)\right).$$

c. We have

$$S \leqslant X \sum_{x=0}^{p-1} \sum_{y_1} \sum_{y} \left(\frac{(xy_1+5)(xy+5)}{p}\right).$$

The part of the expression of the right corresponding to the case $y_1 = y$, does not exceed XpY. We consider the part corresponding to a pair of unequal values y_1 and y, where we assume that $y > 0$ for the sake of definiteness. Setting

$xy + k \equiv z \pmod{p}$, we reduce this part to the form

$$X \sum_{z=0}^{p-1} \left(\frac{z \quad \frac{y_1}{y} z + k \left(1 - \frac{y_1}{y}\right)}{p} \right)$$

from which (problem **a**) we find that it is $<X$ in absolute value. Therefore $S^2 < XpY + XY^2 \leqslant 2pXY$.

d, α) We have

$$S = \sum_{x=0}^{p-1} \sum_{z_1=0}^{Q-1} \sum_{z=0}^{Q-1} \left(\frac{(x + z_1)(x + z)}{p} \right).$$

For $z_1 = z$, summation with respect to x gives $p - 1$. For $z_1 \neq z$, summation with respect to x (problem **a**) gives -1. Therefore

$$S = (p - 1)Q - Q(Q - 1) = (p - Q)Q.$$

β) By the theorem of problem α) we have

$$T(Q^{0.5+0.5\lambda})^2 < pQ; \; T < pQ^{-\lambda}.$$

γ) Setting $[\sqrt{p}\,] = Q$, we apply the theorem of problem α). Assuming there are no quadratic non-residues in the sequence under consideration, we find that $|S_x| \geqslant Q - 1$ for $x = M, M + 1, \ldots, M + 2Q - 1$ and hence

$$2Q(Q - 1)^2 \leqslant (p - Q)Q, \; 2(Q - 1)^2 < (Q + 1)^2 - Q,$$
$$Q^2 - 5Q < 0,$$

which is impossible for $Q \geqslant 5$.

9, a. If m is representable in the form (1), then the solution

$$z \equiv z_0 \pmod{m} \tag{5}$$

of the congruence $x \equiv zy \pmod{m}$ is also a solution of the congruence (2). We will say that our representation is associated with the solution (5) of the congruence (2).

With each solution (5) of the congruence (2) is associated not less than one representation (1). Indeed, taking $r = \sqrt{m}$, we have

$$\frac{z_0}{m} = \frac{P}{Q} + \frac{\theta}{Q\sqrt{m}}; \ (P, Q) = 1, \ 0 < Q \leqslant \sqrt{m}, \ |\theta| < 1.$$

Therefore $z_0Q = mP + r$, where $|r| < \sqrt{m}$. Moreover, it follows from (2) that $|r|^2 + Q^2 \equiv 0 \pmod{m}$. From this and from $0 < |r|^2 + Q^2 < 2m$, we find

$$m = |r|^2 + Q^2. \tag{6}$$

Here $(|r|, Q) = 1$, since

$$1 = \frac{r^2 + Q^2}{m} = \frac{(z_0Q - mP)z_0Q - rmP + Q^2}{m} \equiv rP \pmod{Q}.$$

If $|r| = r$, then the representation (6) is associated with the solution (5) because $r \equiv z_0Q \pmod{m}$. If $|r| = -r$, then the representation $m = Q^2 + |r|^2$ is associated with the solution (5) because $z_0^2Q \equiv z_0r \pmod{m}$, $Q \equiv z_0|r| \pmod{m}$.

No more than one representation (1) is associated with each solution (5). Indeed, if there were two representations $m = x^2 + y^2$ and $m = x_0^2 + y_0^2$ of the number m in the form (1) associated with a single solution (5), then $x \equiv z_0y \pmod{m}$, $x_1 \equiv z_0y_1 \pmod{m}$ would imply that $xy_1 \equiv x_1y \pmod{m}$. Therefore $xy_1 = x_1y$, from which it follows that $x = x_1, y - y_1$ because $(x, y) = (x_1, y_1) = 1$.

b. If m is representable in the form (3), then the solution

$$z \equiv z_0 \pmod{p} \tag{7}$$

of the congruence $x \equiv zy \pmod{p}$ is also a solution of the congruence (4). We will say that this representation is associated with the solution (7) of the congruence (4).

Knowing a solution (7) of the congruence (4), there is no more than one representation (3). Indeed, taking $r = \sqrt{p}$, we have

$$\frac{z_0}{p} = \frac{P}{Q} + \frac{\theta}{Q\sqrt{p}}; \ (P, Q) = 1, \ 0 < Q \leqslant \sqrt{p}, \ |\theta| < 1.$$

Therefore $z_0Q \equiv r \pmod{p}$, where $|r| < p$. Moreover, it follows from (4) that $|r|^2 + aQ^2 \equiv 0 \pmod{p}$. From this and from $0 < |r|^2 + aQ^2 < (1 + a)p$ it follows that we must have $|r|^2 + 2Q^2 = p$ or $|r|^2 + 2Q^2 = 2p$ for $a = 2$. In the latter case, $|r|$ is even, $|r| = 2r_1$, $p = Q^2 + 2r_1^2$. For $a = 3$ we must have $|r|^2 + 3Q^2 = p$, or $|r|^2 + 3Q^2 = 2p$, or $|r|^2 + 3Q^2 = 3p$. The second case is impossible: modulo 4 the left side is congruent to 0 while the right side is congruent to 2. In the third case, $|r|$ is a multiple of 3, $|r| = 3r_1$, $p = Q^2 + 3r_1^2$.

Assuming that two representations $p = x^2 + ay^2$ and $p = x_1^2 + ay_1^2$ of the number p in the form (3) are associated with a single solution of the congruence (4), we find $x = x_1$, $y = y_1$. Assuming that these representations are associated with different solutions of the congruence (4), we find $x \equiv zy \pmod{p}$, $x_1 \equiv -zy \pmod{p}$ and hence $xy_1 + x_1y \equiv 0 \pmod{p}$, which is impossible because

$$0 < (xy_1 + x_1y)^2 < (x^2 + y^2)(x_1^2 + y_1^2) < p.$$

c, α) The terms of the sum $S(k)$ with $x = x_1$ and $x = -x_1$ are equal.

β) We have

$$S(kt^2) = \sum_{x=0}^{p-1} \left(\frac{xt(x^2t^2 + kt^2)}{p}\right) = \left(\frac{t}{p}\right) S(k).$$

γ) Setting $p - 1 = 2p_1$, we have

$$p_1(S(r))^2 + p_1(S(n))^2 = \sum_{t=1}^{p_1} (S(rt^2))^2 + \sum_{t=1}^{p_1} (S(nt^2))^2 =$$

$$= \sum_{k=1}^{p-1} (S(k))^2 = \sum_{x=1}^{p-1} \sum_{y=1}^{p-1} \sum_{k=1}^{p-1} \left(\frac{xy(x^2 + k)(y^2 + k)}{p}\right).$$

For y different from x and $p - x$, the result of summation with respect to k is $-2\left(\frac{xy}{p}\right)$; for $y = x$ and $y = p - x$ it is $(p - 2)\left(\frac{xy}{p}\right)$. Therefore

$$p_1(S(r))^2 + p_1(S(n))^2 = 4pp_1, \quad p = \left(\frac{1}{2}S(r)\right)^2 + \left(\frac{1}{2}S(n)\right)^2.$$

10, a. We have

$X^2 - DY^2 =$

$$= (x_1 + y_1\sqrt{D})(x_2 \pm y_2\sqrt{D})(x_1 - y_1\sqrt{D})(x_2 \mp y_2\sqrt{D}) = k^2.$$

b. Taking any τ_1 such that $\tau_1 > 1$, we find integers x_1, y_1 such that $|y_1\sqrt{D} - x_1| < \frac{1}{\tau_1}$, $0 < y \leqslant \tau_1$, and multiplying this termwise by $y_1\sqrt{D} + x_1 < 2y_1\sqrt{D} + 1$, we find $|x_1^2 - Dy_1^2| < 2\sqrt{D} + 1$. Taking $\tau_2 > \tau_1$ so that

$|y_1\sqrt{D} - x_1| > \dfrac{1}{\tau_2}$, we find new integers x_2, y_2 such that $|x_2^2 - Dy_2^2| < 2\sqrt{D} + 1$, etc.

It is evident that there exists an integer k, not equal to zero, in the interval $-2\sqrt{D} - 1 < k < 2\sqrt{D} + 1$ such that there is an infinite set of pairs x, y with $x^2 - Dy^2 = k$ among the pairs $x_1, y_1; x_2, y_2; \ldots$; among these pairs there are two pairs ξ_1, η_1 and ξ_2, η_2 such that $\xi_1 \equiv \xi_2 \pmod{|k|}$, $\eta_1 \equiv \eta_2 \pmod{|k|}$. Defining the integers ξ_0, η_0 by means of the equation $\xi_0 + \eta_0\sqrt{D} = (\xi_1 + \eta_1\sqrt{D})(\xi_2 + \eta_2\sqrt{D})$, we have (problem **a**)

$$\xi_0^2 - D\eta_0^2 = |k|^2;\ \xi_0 \equiv \xi_1^2 - D\eta_1^2 \equiv 0 \pmod{k};$$

$$\eta_0 \equiv -\xi_1\eta_1 + \xi_1\eta_1 \equiv 0 \pmod{|k|}.$$

Therefore $\xi_0 = \xi|k|$, $\eta_0 = \eta|k|$, where ξ and η are integers and $\xi^2 - D\eta^2 = 1$.

c. The numbers x, y defined by the equation (2) satisfy (problem **a**) the equation (1).

Assuming that there exist pairs of integers x, y satisfying equation (1), but different from the pairs determined by the equation (2), we have

$$(x_0 + y_0\sqrt{D})^r < x + y\sqrt{D} < (x_0 + y_0\sqrt{D})^{r+1}$$

for certain $r = 1, 2, \ldots$. Dividing this termwise by $(x_0 + y_0\sqrt{D})^r$, we find

$$1 < X + Y\sqrt{D} < x_0 + y_0\sqrt{D}, \tag{3}$$

where (problem **a**) X and Y are integers determined by the equation

$$X + Y\sqrt{D} = \frac{x + y\sqrt{D}}{(x_0 + y_0\sqrt{D})^r} = (x + y\sqrt{D})(x - y\sqrt{D})^r$$

and satisfying the equation

$$X^2 - DY^2 = 1. \tag{4}$$

But from (4) we obtain the inequality $0 < |X| - |Y\sqrt{D}| < 1$, which along with the first inequality of (3) shows that X and Y are positive. Therefore the second inequality of (3) contradicts the definition of x_0 and y_0.

11, a, α) We have

$$|U_{a,p}|^2 = U_{a,p}\overline{U}_{a,p} = \sum_{t=1}^{p-1}\sum_{x=1}^{p-1}\left(\frac{t}{p}\right)\exp\left(2\pi i\frac{ax(t-1)}{p}\right).$$

For $t = 1$, summation with respect to x gives $p - 1$; for $t > 1$ it gives $-\left(\frac{t}{p}\right)$. Therefore

$$|U_{a,p}|^2 = p - 1 - \sum_{t=2}^{p-1}\left(\frac{t}{p}\right) = p, \quad |U_{a,p}| = \sqrt{p},$$

or

$$|U_{a,p}|^2 = U_{a,p}\overline{U}_{a,p} = \sum_{t=0}^{p-1}\sum_{x=0}^{p-1}\left(\frac{x+t}{p}\right)\left(\frac{x}{p}\right)\exp\left(2\pi i\frac{at}{p}\right).$$

For $t = 0$ summation with respect to x gives $p - 1$; for $t > 0$ it gives $-\exp\left(2\pi i\frac{at}{p}\right)$. Therefore

$$|U_{a,p}|^2 = p - 1 - \sum_{t=1}^{p-1}\exp\left(2\pi i\frac{at}{p}\right) = p, \quad |U_{a,p}| = \sqrt{p}.$$

β) The theorem is evident for $(a, p) = p$. For $(a, p) = 1$ it follows from

$$U_{a,p} = \frac{a}{p}\sum_{x=1}^{p-1}\left(\frac{ax}{p}\right)\exp\left(2\pi i\frac{ax}{p}\right) = \left(\frac{a}{p}\right)U_{1,p}.$$

b, α) Let r run through the quadratic residues and let n run through the quadratic non-residues, in a complete system of residues. We have

$$S_{a,p} = 1 + 2\sum\exp\left(2\pi i\frac{ar}{p}\right).$$

Subtracting

$$0 = 1 + \sum_{r}\exp\left(2\pi i\frac{ar}{p}\right) + \sum_{n}\exp\left(2\pi i\frac{an}{p}\right)$$

from the latter termwise, we obtain the required equation.

β) We have

$$|S_{a,m}|^2 = \sum_{t=0}^{m-1}\sum_{x=0}^{m-1}\exp\left(2\pi i\frac{a(t^2 + 2tx)}{m}\right).$$

For given t, summation with respect to x gives $m\exp\left(2\pi i\frac{at^2}{m}\right)$ or 0 according as $2t$ is divisible by m or not. For odd m we have

$$|S_{a,m}|^2 = m\exp\left(2\pi i\frac{a\cdot 0}{m}\right) = m.$$

For even $m = 2m_1$ we have

$$|S_{a,m}|^2 = m \ \exp\left(2\pi i\frac{a\cdot 0^2}{m}\right) + \exp\left(2\pi i\frac{a\cdot m_1^2}{m}\right) \quad .$$

Here the right side is equal to zero for odd m_1 and equal to $2m$ for even m_1.

γ) For any integer b we have

$$|S_{A,m}| = \left|\sum_{x=0}^{m-1} \exp\left(2\pi i\frac{Ax^2 + 2Abx}{m}\right)\right|$$

and choosing b such that $2Ab \equiv a \pmod m$, we again obtain the result considered in problem β).

12, a. We have

$$m \sum{}' \Phi(z) = \sum_z \sum_{s=M}^{M+Q-1} \sum_{a=0}^{m-1} \Phi(z) \exp\left(2\pi i\frac{a(x-z)}{m}\right) \quad .$$

The part of the sum on the right corresponding to $a = 0$ is equal to $Q\sum\limits_z \Phi(z)$; the part corresponding to the remaining values of a is numerically (problem **11, c, ch. III**)

$$< \Delta \sum_{a=1}^{m-1} \left|\sum_{s=M}^{M+Q-1} \exp\left(2\pi i\frac{-as}{m}\right)\right| < \Delta m(\ln m - \delta).$$

b, α) This follows from the theorem of problem **11, a,** α) and the theorem of problem **a.**

β) The inequality of problem α) gives $R - N = \theta\sqrt{p}\,\ln p$. Moreover it is evident that $R + N = Q$.

γ) It follows from the theorem of problem **11, b,** β) that the conditions of the theorem of problem **a** are satisfied if we take $m = p$, $\Phi(z) = 1$, while z runs through the values $z = x^2$; $x = 0, 1, \ldots, p - 1$. But, among the values of z there is one which is congruent modulo p to 0 and two congruent modulo p with each quadratic residue of a complete residue system, and hence

$$\sum_{z}{}' \Phi(z) = 2R, \ \sum_{z} \Phi(z) = p,$$

and we obtain

$$2R = \frac{Q}{p}p + \theta\sqrt{p}\ \ln p.$$

δ) This follows from the theorem of problem **11, b,** γ) and the theorem of problem **a.**

ϵ) It follows from the theorem of problem δ) that the conditions of the theorem of problem **a** are satisfied if we set $m = p$, $\Phi(z) = 1$, while z runs through the values $z = Ax^2$; $x = M_0, M_0 + 1, \ldots, M_0 + Q_0 - 1$. Therefore

$$\sum_{z}{}' \Phi(z) = T, \ \sum_{z} \Phi(z) = Q_0,$$

from which the required formula follows.

c. The part of the sum containing the terms with $\left(\frac{\alpha}{p}\right) = 1$ is equal to $p(R^2 + N^2)$, the remaining part is equal to $-2pRN$. Therefore the whole sum is equal to $p(R - N)^2$.

The part of the sum containing the terms with $a = 0$ is equal to 0. The remaining part is numerically smaller than (problem **11, c, ch. III**)

$$\sum_{a=1}^{p-1}\left|\sum_{x=M}^{M+Q-1} \exp\left(2\pi i\frac{ax}{p}\right)\right|\left|\sum_{\alpha=1}^{p-1}\right|\sum_{y=M}^{M+Q-1} \exp\left(2\pi i\frac{-a\alpha y}{p}\right)\left|\right| <$$

$$< p^2 (\ln p)^2.$$

Therefore $p(R - N)^2 < p^2 (\ln p)^2$, $R - N < \sqrt{p}\ \ln p$.

Solutions of the Problems for Chapter VI

1, a. If q is an odd prime and $a^p \equiv 1 \pmod{q}$, then a belongs to one of the exponents $\delta = 1, p$ modulo q. For $\delta = 1$ we have $a \equiv 1 \pmod{q}$, for $\delta = p$ we have $q - 1 = 2px$ where x is an integer.

b. If q is an odd prime and $a^p + 1 \equiv 0 \pmod{q}$, then $a^{2p} \equiv 1 \pmod{q}$. Therefore a belongs to one of the exponents $\delta = 1, 2, p, 2p$ modulo q. The cases $\delta = 1, p$ are impossible. For $\delta = 2$ we have $a^2 \equiv 1 \pmod{q}$, $a + 1 \equiv 0 \pmod{q}$. For $\delta = 2p$ we have $q - 1 = 2px$ where x is an integer.

c. The prime divisors of $2^p - 1$ are primes of the form $2px + 1$. Let $p_1, p_2, \ldots, p_k$ be any k primes of the form $2px + 1$; the number $(p_1 p_2 \ldots p_k)^p - 1$ has a prime divisor of the form $2px + 1$ which is different from $p_1, p_2, \ldots, p_k$.

d. If q is a prime and $2^{2^n} + 1 \equiv 0 \pmod{q}$, then $2^{2^{n+1}} \equiv$ $\equiv 1 \pmod{q}$. Therefore 2 belongs to the exponent 2^{n+1} modulo q, and hence $q - 1 = 2^{n+1}x$ where x is an integer.

2. It is evident that a belongs to the exponent n modulo $a^n - 1$. Therefore n is a divisor of $\varphi(a^n - 1)$.

3, a. Assume that we arrive at the original sequence after k operations. It is evident that the k-th operation is equivalent to the following one: consider the numbers in the sequence

$$1, 2, \ldots, n - 1, n, n, n - 1, \ldots, 2, 1, 2, \ldots$$
$$\ldots, n - 1, n, n, n - 1, \ldots, 2, 1, 2, \ldots$$

in places $1, 1 + 2^k, 1 + 2 \cdot 2^k, \ldots$. Therefore the number 2 is in the $1 + 2^k$ place. Therefore the condition considered in the problem is necessary. But it is also sufficient, since it implies that we have the following congruences modulo $2n - 1$:

$$1 \equiv 1,\ 1 + 2^k \equiv 0,\ 1 + 2 \cdot 2^k \equiv -1, \ldots$$

or

$$1 \equiv 1,\ 1 + 2^k \equiv 2,\ 1 + 2\cdot 2^k \equiv 3,\ \ldots$$

b. The solution is analogous to the solution of problem **a.**

4. The solution of the congruence $x^{\delta} \equiv -1 \pmod p$ belongs to an exponent of the form $\frac{\delta}{\delta'}$ where δ' is a divisor of δ. Here δ' is a multiple of d if and only if $x^{\frac{\delta}{d}} \equiv 1 \pmod p$. Considering the δ values of δ' and taking $f = 1$, we find that $S' = \sum_{d\backslash\delta} \mu(d)S_d$, where S' is the required number and $S_d = \frac{\delta}{d}$.

5, a. Here (3; example **c**, §5) we must have $\left(\frac{g}{2^n + 1}\right) = -1$. This condition is satisfied for $g = 3$.

b. Here we cannot have $\left(\frac{g}{2p + 1}\right) = 1$, $g^2 \equiv 1 \pmod{2p + 1}$. This condition is satisfied for our values of g.

c. Here we cannot have $\left(\frac{g}{4p + 1}\right) = 1$, $g^4 \equiv 1 \pmod{4p + 1}$. This condition is satisfied for $g = 2$.

d. Here we cannot have $\left(\frac{g}{2^np + 1}\right) = 1$, $g^{2^n} \equiv 1 \pmod{2^np + 1}$. This condition is satisfied for $g = 3$.

6, a, α) The theorem is evident if n is a multiple of $p - 1$. Assume that n is not divisible by $p - 1$. If we disregard the order, the numbers $1, 2, \ldots, p - 1$ are congruent modulo p to the numbers $g, 2g, \ldots, (p - 1)g$, where g is a primitive root modulo p. Hence

$$S_n \equiv g^nS_n \pmod p,\ S_n \equiv 0 \pmod p.$$

β) We have

$$\sum_{x=1}^{p-1}\left(\frac{x(x^2+1)}{p}\right) \equiv \sum_{x=1}^{p-1} x^{\frac{p-1}{2}}(x^2+1)^{\frac{p-1}{2}} \pmod p$$

from which (problem α)) we obtain the required result.

b. For $p > 2$, we have

$$1\cdot 2\ldots(p-1)\equiv g^{1+2+\cdots+p-1}\equiv g^{\frac{p-1}{2}}\equiv -1 \pmod p.$$

7, a. We have $g^{\text{ind}_{g_1} a} \equiv a \pmod p$, $\text{ind}_{g_1} a\ \text{ind}_g g_1 \equiv$ $\equiv \text{ind}_g a \pmod{p-1}$, $\text{ind}_{g_1} a \equiv \alpha\ \text{ind}_g a \pmod{p-1}$.

b. It follows from $\text{ind}_g a \equiv s \pmod n$, $\text{ind}_{g_1} a \equiv \alpha\ \text{ind}_g a$ $\pmod{p-1}$ that $\text{ind}_{g_1} a \equiv \alpha s \equiv s_1 \pmod n$..

8. Let $(n, p-1) = 1$. Determining u by the condition $nu \equiv 1 \pmod{p-1}$ we find the solution $x \equiv a^u \pmod p$.

Let n be a prime, $p - 1 = n^{\alpha}t$, where α is a positive integer and $(t, n) = 1$. If the congruence is possible, then $a^{n^{\alpha-1}t} \equiv 1 \pmod p$; if $\alpha > 1$, then, noting that $x \equiv g^{n^{\alpha-1}tr}$ $\pmod p$, $r = 0, 1, \ldots, n-1$ are just all the solutions of the congruence $x^n \equiv 1 \pmod p$; for some $r_1 = 0, 1, \ldots, n-1$ we have

$$a^{n^{\alpha-2}t}g^{n^{\alpha-1}t} \equiv 1 \pmod p;$$

if $\alpha > 2$, then for certain $r_2 = 0, 1, \ldots, n-1$ we have

$$a^{n^{\alpha-3}t}g^{n^{\alpha-2}tr_1+n^{\alpha-1}tr_2} \equiv 1 \pmod p,$$

etc.; finally, for certain $r_{\alpha-1} = 0, 1, \ldots, n-1$ we have

$$a^t g^{ntr_1+n^2tr_2+\cdots+n^{\alpha-1}tr_{\alpha-1}} \equiv 1 \pmod p.$$

Determining u and v by the condition $tu - nv = -1$, we obtain n solutions:

$$x \equiv a^{v} g^{uf(r_1+nr_2+\dots+n^{\alpha-2}r_{\alpha-1})+n^{\alpha-1}tr} \equiv (\mathrm{mod}\ p);$$

$$r = 0,\ 1,\ \dots,\ n-1.$$

Let the prime n_1 divide $(n, p-1)$, $n = n_1 n_2$, $n_2 > 1$. Corresponding to each solution of the congruence $y^{n_1} \equiv a \pmod p$ we obtain a solution of the congruence $x^{n_2} \equiv y \pmod p$.

9, a. In this way we obtain $cc_0c_1\dots c_k = \varphi(m)$ characters.

b, α) We have $\chi(1) = R^0\dots R_k^0 = 1$.

β) Let $\gamma\ ;\ \dots,\ \gamma'_k;\ \gamma'',\ \dots,\ \gamma''_k$ be the index systems of the numbers a_1 and a_2; then $\gamma' + \gamma'',\ \dots,\ \gamma'_k + \gamma''_k$ is an index system for the number a_1a_2 (**c,** § 7).

γ) For $a_1 \equiv a_2 \pmod m$, the indices of the numbers a_1 and a_2 are congruent to one-another modulo $c, \dots, c_k$ respectively.

c. This property follows from

$$\sum_{a=0}^{m-1} \chi(a) = \sum_{\gamma=0}^{c-1} R^{\gamma} \dots \sum_{\gamma_k=0}^{c_k-1} R_k^{\gamma_k}.$$

d. This property follows from

$$\chi(a) = \sum_{R} R^{\gamma} \dots \sum_{R_k} R_k^{\gamma_k}.$$

e. Let $\psi(a_1) \gtrless 0$. Then $\psi(a_1) = \psi(a_1)\psi(1)$. Therefore $\psi(1) = 1$. Determining a' by the condition $aa' \equiv 1 \pmod m$, we have $\psi(a)\psi(a') = 1$. Therefore $\psi(a) \gtrless 0$ for $(a, m) = 1$.

For $(a_1, m) = 1$, we have

$$\sum_a{}' \frac{\chi(a)}{\psi(a)} = \sum_a \frac{\chi(a_1a)}{\psi(a_1a)} = \frac{\chi(a_1)}{\psi(a_1)} \sum_a{}' \frac{\chi(a)}{\psi(a)};$$

therefore, either $\sum_a{}' \dfrac{\chi(a)}{\psi(a)} = 0$, or $\psi(a_1) = \chi(a_1)$ for all a_1.

But the first cannot hold for all χ; if it did, then we would have $H = 0$, while $H = \varphi(m)$ since, summing over all characters for given a, we have

$$\sum_x \frac{\chi(a)}{\psi(a)} = \begin{cases} \varphi(m), & \text{if } a \equiv 1 \pmod{m}, \\ 0, & \text{otherwise.} \end{cases}$$

f, α) If $R', \ldots, R_k'$ and $R'', \ldots, R_k''$ are values of $R, \ldots, R_k$ corresponding to the characters $\chi_1(a)$ and $\chi_2(a)$, then $\chi_1(a)\chi_2(a)$ is a character corresponding to the values $R'R'', \ldots, R_k'R_k''$.

β) When $R, \ldots, R_k$ run through all the roots of the corresponding equations, then $R'R, \ldots, R_k'R_k$ run through the same roots in some order.

γ) Determining l' by the condition $ll' \equiv 1 \pmod{m}$, we have

$$\sum_\chi \frac{\chi(a)}{\chi(l)} = \sum_\chi \frac{\chi(al')}{\chi(ll')} = \sum_\chi \chi(al')$$

which is equal to $\varphi(m)$ or 0, according as $a \equiv l \pmod{m}$ or not.

10, a, α) Defining x' by the congruence $xx' \equiv 1 \pmod{p}$, we have

$$\sum_{x=1}^{p-1} \exp\left(2\pi i \frac{l \operatorname{ind}(x+k) - l \operatorname{ind} x}{n}\right) =$$

$$= \sum_{x=1}^{p-1} \exp\left(2\pi i \frac{l \operatorname{ind}(1+kx')}{n}\right) = -1.$$

β) We have

$$S = \sum_{x=0}^{p-1} \sum_{z_1=0}^{Q-1} \sum_{z=0}^{Q-1} \exp\left(2\pi i \frac{l \operatorname{ind}(x+z_1) - l \operatorname{ind}(x+z)}{n}\right).$$

For $z_1 = z$, summing over x gives $p - 1$, and for z_1 unequal to z, summation over x (problem α)) gives -1. Therefore

$$S = (p-1)Q - Q(Q-1) = (p-Q)Q.$$

γ) Let Q_x be the number of integers of the sequence $x + z$; $z = 0, 1, \ldots, Q-1$ which are not divisible by p, while $T_{n,x}$ is the number of integers of this sequence which are in the s-th set. Finally, let

$$U_{n,x} = -\frac{Q_x}{n} + T_{n,x}, \; S = \sum_{x=0}^{p-1} U_{n,x}^2.$$

We have

$$U_{n,x} = \frac{1}{n} \sum_{l=1}^{n-1} \sum_{z=1}^{Q-1} \exp\left(2\pi i \frac{l(\operatorname{ind}(x+z) - s)}{n}\right) =$$

$$= \frac{1}{n} \sum_{l=1}^{n-1} \exp\left(-2\pi i \frac{ls}{n}\right) S_{l,n,x},$$

$$U_{n,x}^2 \leqslant \frac{1}{n^2}(n-1) \sum_{l=1}^{n-1} \left|S_{l,n,x}\right|^2, \; S \leqslant \left(\frac{n-1}{n}\right)^2 (p-Q)Q.$$

Setting $Q = [n\sqrt{p}\,]$, and assuming that there are no numbers of the s-th set in our sequence, we find that $|U_{n,x}| \geqslant \dfrac{Q-1}{n}$

for $x = M, M + 1, \ldots, M + Q + 1$, and hence

$$Q\left(\frac{Q-1}{n}\right)^2 \leqslant \left(\frac{n-1}{n}\right)^2 (p - Q)Q,$$

$$(n\sqrt{p} - 2)^2 < (n\sqrt{p} - \sqrt{p})^2,$$

which is impossible.

b. Let p_0 be the product of the different prime divisors of the number $p - 1$, let Q_x be the number of integers of the sequence $x + z; z = 0, 1, \ldots, Q - 1$ which are not divisible by p, and let G_x be the number of integers of the same sequence which are primitive roots modulo p. Finally, let

$$P = \left(\sum_{d\backslash p_0} \frac{\mu(d)}{d}\right)^{-1} = \frac{p-1}{\varphi(p-1)}, \quad w_x = -\frac{1}{p} + G_x,$$

$$\Omega = \sum_{x=0}^{p-1} w_x^2 .$$

Taking $f(\xi) = 1$ and letting ξ run through the values $\xi = \text{ind}\,(x + z); z = 0, 1, \ldots, Q - 1$, we obtain

$S' = \sum_{d\backslash p_0} \mu(d) S_d$. Here S' is the number of values of ξ such

that $(\xi, p - 1) = 1$ and hence $S' = G_x$. Moreover, S_d is the number of values of ξ which are multiples of d and hence $S_d = T_{d,x}$ (problem **a**, γ) for $s = 0$). Therefore

$$w_x = -\frac{1}{p} + \sum_{d\backslash p_0} \mu(d) T_{d,x} = \sum_{d\backslash p_0} \mu(d) U_{d,x},$$

$$w_x^2 \leqslant 2^k \sum_{d\backslash p_0} U_{d,x}^2, \quad \Omega \leqslant 2^k (p - Q)Q.$$

Setting $Q = [P\,2^k\sqrt{p}\,]$ and assuming that there are no primitive roots in our sequence, we find that $|w_x| \geqslant \dfrac{Q-1}{P}$ for $x = M, M+1, \ldots, M+Q-1$ and hence

$$Q\left(\frac{Q-1}{P}\right)^2 < 2^{2k}(p-Q)Q,$$

$$(P2^k\sqrt{p}-2)^2 < \left(P2^k\sqrt{p} - \frac{P2^kQ}{2\sqrt{p}}\right)^2,$$

which is impossible.

11, a, α). We have

$$|U_{a,p}|^2 = \sum_{t=1}^{p-1}\sum_{x=1}^{p-1} \exp\left(2\pi i\frac{k \text{ ind } t}{n}\right) \exp\left(2\pi i\frac{a(t-1)x}{n}\right) =$$

$$= p - 1 - \sum_{t=z}^{p-1} \exp\left(2\pi i\frac{k \text{ ind } t}{n}\right) = p.$$

β) For $(a, p) = p$ the theorem is evident. For $(a, p) = 1$, it follows from

$$U_{a,p} =$$

$$= \exp\left(2\pi i\frac{-k \text{ ind } a}{n}\right) \sum_{x=1}^{p-1} \exp\left(2\pi i\frac{k \text{ ind } ax}{n}\right) \exp\left(2\pi i\frac{ax}{p}\right) =$$

$$= \exp\left(2\pi i\frac{-k \text{ ind } a}{n}\right) U_{1,p}.$$

γ) It is evident that A and B are integers with $|S|^2 = A^2 + B^2$.

For certain ϵ, ϵ'; ϵ'' such that $|\epsilon| = |\epsilon'| = |\epsilon''| = 1$ we have (problem β))

$$S = \frac{1}{\epsilon\sqrt{p}\,\epsilon\sqrt{p}} \times$$

$$\times \sum_{z_1=1}^{p-1}\sum_{z=1}^{p-1}\sum_{x=0}^{p-1} \exp\left(-2\pi i\frac{\operatorname{ind} z_1 + \operatorname{ind} z}{4}\right)\exp\left(2\pi i\frac{z_1 x + z(x+1)}{p}\right).$$

If $z_1 + z$ is not equal to p, then summing over x gives zero. Therefore

$$S = \epsilon' \sum_{z=1}^{p-1}\left(\frac{z}{p}\right)\exp\left(2\pi i\frac{z}{p}\right) = \epsilon''\sqrt{p}\,,\quad |S|^2 = p.$$

b, α) For given z, the congruence $x^n \equiv z \pmod p$ is solvable only if ind z is divisible by δ, and it then has δ solutions. Therefore, for $\delta = 1$ we have $S_{a,p} = 0$. If $\delta > 1$, then we have

$$S_{a,p} = 1 + \sum_{k=0}^{\delta-1}\sum_{z=1}^{p-1}\exp\left(2\pi i\frac{k \operatorname{ind} z}{\delta}\right)\exp\left(2\pi i\frac{az}{p}\right).$$

For $k = 0$, summation with respect to z gives -1; for $k > 0$ it gives a quantity whose modulus is equal to $\sqrt{p}$. The required result follows from this.

β) Setting

$$x = u + p^{s-1}v;\ u = 0, \ldots, p^{s-1} - 1.\ v = 0, \ldots, p-1,$$

we have

$$\exp\left(2\pi i\frac{ax^n}{p^s}\right) = \exp(2\pi i a(u^n p^{-s} + nu^{n-1}p^{-1}v).$$

For $(u, p) = 1$, summation with respect to v gives zero.

Therefore

$$S_{a,p^s} = \sum_{x_0=0}^{p^{s-1}-1} \exp(2\pi i a p^{n-s} x_0^n) = p^{s-1},\ S'_{a,p^s} = 0.$$

γ) Let p^τ be the largest power of p dividing n. We have $s \geqslant r + 3$. Setting

$$x = u + p^{s-1-\tau}v;\ u = 0, \dots, p^{s-1-\tau} - 1,\ v = 0, \dots, p^{\tau+1} - 1,$$

we have

$$\exp\left(2\pi i \frac{ax^n}{p^s}\right) = \exp(2\pi i a(u^n p^{-s} + n u^{n-1} p^{-\tau-1} v)).$$

For $(u, p) = 1$, summation with respect to v gives zero. Therefore

$$S_{a,p^s} = \sum_{x_0=0}^{p^{s-1}-1} \exp\left(2\pi i \frac{ax_0^n}{p^{s-n}}\right) = p^{n-1} S_{a,p^{s-n}},\ S'_{a,p^s} = 0.$$

δ) Let $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the canonical decomposition of the number m. Setting

$$T_{a,m} = m^{-1+\nu} S_{a,m};\ \nu = \frac{1}{n},\ m = p_1^{\alpha_1} M_1 = \dots = p_k^{\alpha_k} M_k$$

and defining $a_1, \dots, a_k$ by the condition $a \equiv a_1 M_1 + \dots + a_k M_k \pmod m$, we have (problem **12, d, ch. III**)

$$T_{a,m} = T_{a_1,p_1^{\alpha_1}} \dots T_{a_k,p_k^{\alpha_k}}.$$

For $s = 1$ we have

$$|T_{a,p^s}| < p^{-1+\nu} n\sqrt{p} < np^{-\frac{1}{6}}.$$

For $1 < s \leqslant n$, $(n, p) = 1$ we have

$$|T_{a,p^s}| = p^{-s+s\nu}p^{s-1} \leqslant 1.$$

For $1 < s \leqslant n$, $(n, p) = p$ we have

$$|T_{a,p^s}| \leqslant p^{-s+s\nu}p^{s} < p \leqslant n.$$

The case $s > n$ reduces to the case $s \leqslant n$ since $T_{a,p^s} = p^{-s+s\nu}p^{n-1}S_{a,p^{s-n}} = T_{a,p^{s-n}}$. Therefore

$$|T_{a,m}| \leqslant C = n^{n^6+n},$$

from which we obtain the required inequality.

12, a. This follows from the theorem of problem **11, a,** α) and the theorem of problem **12, a, ch. V.**

b. We have

$$Tn = \sum_{x=M}^{M+Q-1} \sum_{k=0}^{n-1} \exp\left(2\pi i \frac{k(\operatorname{ind} x - s)}{n}\right).$$

For $k = 0$, summing with respect to x we obtain Q; for $k > 0$, we obtain a number whose modulus is $<\sqrt{p}\ \ln p$. And this implies the required formula.

c. Taking $f(x) = 1$ and letting x run through the numbers $x = \operatorname{ind} M, \operatorname{ind}(M + 1), \ldots, \operatorname{ind}(M + Q - 1)$, we find (problem **17, a, ch. II**) $S' = \sum_{d \backslash p-1} \mu(d)S_d$. Here S' is the number of x such that $(x, p - 1) = 1$; therefore $S' = T$. Moreover, S_d is the number of values of x which are multiples of d, i.e. the number of residues of power d in the sequence $M, M + 1, \ldots, M + Q - 1$. Then

$$H = \sum_{d \backslash p-1} \mu(d)\left(\frac{Q}{d} + \theta_d\sqrt{p}\ \ln p\right); \quad |\theta_d| < 1,\ \theta_1 = 0.$$

d. It follows from the theorem of problem **a** that the conditions of problem **12, a, ch. V** are satisfied, if we set $m = p - 1$, $\Phi(z) = 1$, while we let z run through the values $z =$ ind x; $x = M, M + 1, \ldots, M + Q - 1$. We then find (with Q_1 in place of Q)

$$\sum_z{}' \Phi(z) = J, \ \sum_z \Phi(z) = Q, \ J = \frac{Q_1}{p-1} Q + \theta\sqrt{p}\ (\ln p)^2.$$

13. Assume that there are no non-residues not exceeding h. The number of n-th power non-residues among the numbers $1, 2, \ldots, Q$ where

$$Q = \sqrt{p}\ (\ln p)^2$$

can be estimated by two methods: starting from the formula of problem **12, b** and starting from the fact that the non-residues can only be numbers divisible by primes exceeding h. We find

$$1 - \frac{1}{n} < \ln \frac{\frac{1}{2}\ln p + 2\ln\ln p}{\frac{1}{c}\ln p + 2\ln\ln p} + O\left(\frac{1}{\ln p}\right),$$

$$0 < \ln \frac{1 + 4\frac{\ln\ln p}{\ln p}}{1 + 2c\frac{\ln\ln p}{\ln p}} + O\left(\frac{1}{\ln p}\right).$$

The impossibility of the latter inequality for all sufficiently large p proves the theorem.

14, a. We have

$$|S|^2 < X \sum_{x=0}^{m-1} \sum_{y_1=0}^{m-1} \sum_{y=0}^{m-1} \rho(y_1)\,\overline{\rho(y)} \exp\left(2\pi i \frac{ax(y_1 - y)}{m}\right).$$

For given y_1 and y summation with respect to x gives $Xm|\rho(y)|^2$ or zero according as $y_1 = y$ or not. Therefore

$$|S|^2 \leqslant XYm, \quad |S| \leqslant \sqrt{XYm}\,.$$

b, α) We have

$$S = \frac{1}{\varphi(m)} \sum_u \sum_v \chi(u)\chi(v) \exp\left(2\pi i \frac{au^n v^n}{m}\right)$$

where u and v run through reduced residue systems modulo m. Hence

$$S = \frac{1}{\varphi(m)} \sum_{x=0}^{m-1} \sum_{y=0}^{m-1} \nu(x)\rho(y) \exp\left(2\pi i \frac{axy}{m}\right);$$

$$\nu(x) = \sum_{u^n \equiv x \pmod m} \chi(u), \quad \rho(y) = \sum_{v^n \equiv y \pmod m} \chi(v).$$

But we have (problem **11, ch. IV**)

$$\sum_{x=0}^{m-1} |\nu(x)|^2 \leqslant K\varphi(m), \quad \sum_{y=0}^{m-1} |\rho(y)|^2 \leqslant K\varphi(m).$$

Therefore (problem **a**)

$$|S| \leqslant \frac{1}{\varphi(m)} \sqrt{K\varphi(m)\,K\varphi(m)m} = K\sqrt{m}\,.$$

β) Let $m = 2^{\alpha} p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be the canonical decomposition of the number m. The congruence $x^n \equiv 1 \pmod m$ is equivalent to the system

$$x^n \equiv 1 \pmod{2^{\alpha}}, \; x^n \equiv 1 \pmod{p_1^{\alpha_1}}, \; \ldots, \; x^n \equiv 1 \pmod{p_k^{\alpha_k}}.$$

Let $\gamma(x)$ and $\gamma_0(x)$ be the indices of the number x modulo 2^α (**g**, §**6**). The congruence $x^n \equiv 1 \pmod{2^\alpha}$ is equivalent to the system $n\gamma(x) \equiv 0 \pmod{c}$, $n\gamma_0(x) \equiv 0 \pmod{c_0}$. The first congruence of this system has at most 2 solutions, while the second has at most n solutions. Therefore the congruence $x^n \equiv 1 \pmod{2^\alpha}$ has at most $2n$ solutions. By **b**, §**5**, each of the congruences $x^n \equiv 1 \pmod{p_1^{\alpha_1}}, \ldots, x^n \equiv 1 \pmod{p_k^{\alpha_k}}$ have no more than n solutions. Therefore

$$K \leqslant 2n^{k+1} = 2(2^{k+1})^{\frac{\ln n}{\ln 2}} \leqslant 2(\tau(m))^{\frac{\ln n}{\ln 2}}; \; K = 0\,(m^\epsilon).$$

15, a. We have

$$|S|^2 = \sum_{t=1}^{p-1}\sum_{x=1}^{p-1} \exp\left(2\pi i \frac{a(t^n - 1)x^n + b(t - 1)x}{p}\right).$$

If $t^n \equiv 1 \pmod{p}$, then summation with respect to x gives $p - 1$ for $t \equiv 1 \pmod{p}$ and -1 in the remaining cases. Otherwise, taking $z(t - 1)^{-1}$ in place of x, we can represent the part of the double sum corresponding to given t in the form

$$\sum_{z=1}^{p-1} \exp\left(2\pi i \frac{bz}{p}\right) \exp\left(2\pi i \frac{a(t^n - 1)(t - 1)^{-n} z^n}{p}\right)$$

and hence

$$|S|^2 \leqslant p - 1 + \left|\sum_{u=1}^{p-1}\sum_{v=1}^{p-1} \nu(u)\rho(v) \exp\left(2\pi i \frac{auv}{p}\right)\right|,$$

where $\nu(u)$ is equal to the number of solutions of the congruence $(t^n - 1)(t - 1)^{-n} \equiv u \pmod{p}$, while $|\rho(v)|$ does not exceed the number of solutions of the congruence $z^n \equiv v \pmod{p}$. Therefore $\nu(u) \leqslant 2n_1$, $|\rho(v)| < n_1$,

$$\sum_{u=1}^{p-1} |\nu(u)|^2 \leqslant (p - 1)2n_1, \quad \sum_{v=1}^{p-1} |\rho(v)|^2 \leqslant (p - 1)n_1.$$

Applying the theorem of problem **14, a,** we find

$$|S|^2 < p - 1 + \sqrt{(p-1)2n_1(p-1)n_1p} < 2n_1p^{\frac{3}{2}}.$$

b, α) This follows from problem **a** and the theorem of problem **12, a, ch. V.**

β) It follows from the theorem of problem α) that the conditions of the theorem of problem **12, a, ch. V** are satisfied, if we set $m = p$, $\Phi(z) = 1$, where we let z run through the values $z = Ax^n$; $x = M_0, M_0 + 1, \ldots, M_0 + Q_0 - 1$. Then

$$\sum_z{}' \Phi(z) = T, \ \sum_z \Phi(z) = Q_0,$$

from which we obtain the required formula.

c, α) Let $y \equiv 4ay_1 (\text{mod } p)$. We have (problem **11, a, ch, V**)

$$\left(\frac{a}{p}\right) S = \sum_{x=0}^{p-1} \left(\frac{4a^2x^2 + 4abx + 4ac}{p}\right) \exp\left(2\pi i \frac{4ay_1x}{p}\right) =$$

$$= \frac{1}{U_{1,p}} \sum_{z=1}^{p-1} \left(\frac{z}{p}\right) \sum_{x=0}^{p-1} \exp\left(2\pi i \frac{z(4a^2x^2 + 4abx + 4ac + 4ay_1xz^{-1}}{p}\right) =$$

$$= \sum_{z=1}^{p-1} \exp\left(2\pi i \frac{-(b^2 - 4ac)z - 2by_1 - y_1^2z^{-1}}{p}\right).$$

The latter sum (problem **a**) is numerically $< \frac{3}{2} p^{\frac{3}{4}}$.

β) This follows from the theorem of problem α) and the theorem of problem **12, a, ch. V.**

ANSWERS FOR THE NUMERICAL EXERCISES

Answers for chapter I.

1, a. 17.
b. 23.

2, a, α) $\delta_4 = \dfrac{15}{11}$; β) $\alpha = \dfrac{19}{14} + \dfrac{\theta}{14 \cdot 20}$.
b, α) $\delta_6 = \dfrac{80}{59}$; β) $\alpha = \dfrac{1002}{739} + \dfrac{\theta}{739 \cdot 1000}$.

3. We obtain 22 fractions.

5, a. $2^8 \cdot 3^5 \cdot 11^3$.
b. $2^2 \cdot 3^3 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 17 \cdot 23 \cdot 37$.

Answers for chapter II.

1, a. 1312.
b. $2^{119} \cdot 3^{59} \cdot 5^{31} \cdot 7^{19} \cdot 11^{12} \cdot 13^9 \cdot 17^7 \cdot 19^6 \cdot 23^5 \cdot 29^4 \times$
$\times\ 31^4 \cdot 37^3 \cdot 41^3 \cdot 43^2 \cdot 47^2 \cdot 53^2 \cdot 59^2 \cdot 61^2 \cdot 67 \cdot 71 \times$
$\times\ 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113$.

2, a. $\tau(2800) = 30$; $S(2800) = 7688$.
b. $\tau(232\,848) = 120$; $S(232\,848) = 848\,160$.

3. The sum of all the values is equal to 1.

4. α) 1152; β) 466 400.

5. The sum of all the values is equal to 774.

Answers for chapter III.

1, a. 70.

b. It is divisible.

2, a. $3^3 \cdot 5^2 \cdot 11^2 \cdot 2999$.

b. $7 \cdot 13 \cdot 37 \cdot 73 \cdot 101 \cdot 137 \cdot 17 \cdot 19 \cdot 257$.

Answers for chapter IV.

1, a. $x \equiv 81 \pmod{337}$.

b. $x \equiv 200; 751; 1302; 1853; 2404 \pmod{2755}$.

2, b. $x \equiv 1630 \pmod{2413}$.

3. $x = 94 + 111t$; $y = 39 + 47t$, where t is any integer.

4, a. $x \equiv 170b_1 + 52b_2 \pmod{221}$; $x \equiv 131 \pmod{221}$; $x \equiv 110 \pmod{221}$; $x \equiv 89 \pmod{221}$.

b. $x \equiv 11\,151b_1 + 11\,800b_2 + 16\,875b_3 \pmod{39\,825}$.

5, a. $x \equiv 91 \pmod{120}$.

b. $x \equiv 8479 \pmod{15\,015}$.

6. $x \equiv 100 \pmod{143}$; $y \equiv 111 \pmod{143}$.

7, a. $3x^4 + 2x^3 + 3x^2 + 2x \equiv 0 \pmod{5}$.

b. $x^5 + 5x^4 + 3x^2 + 3x + 2 \equiv 0 \pmod{7}$.

8. $x^6 + 4x^5 + 22x^4 + 76x^3 + 70x^2 + 52x + 39 \equiv 0 \pmod{101}$.

9, a. $x \equiv 16 \pmod{27}$.

b. $x \equiv 22; 53 \pmod{64}$.

10, a. $x \equiv 113 \pmod{125}$.

b. $x \equiv 43, 123, 168, 248, 293, 373, 418, 498, 543, 623 \pmod{625}$.

11, a. $x \equiv 2, 5, 11, 17, 20, 26 \pmod{30}$.

b. $x \equiv 76, 22, 176, 122 \pmod{225}$.

Answers for chapter V.

1, a. 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18.

b. 2, 5, 6, 8, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 29, 31, 32, 35.

2, a. α) 0; β) 2.

b. α) 0; β) 2.

3, a. α) 0; β) 22.

b. α) 0; β) 2.

4, a. α) $x \equiv 9 \pmod{19}$; β) $x \equiv 11 \pmod{29}$; γ) $x \equiv 14 \pmod{97}$.

b. α) $x \equiv 66 \pmod{311}$; β) $x \equiv 130 \pmod{277}$; γ) $x \equiv 94 \pmod{353}$.

5, a. $x \equiv 72 \pmod{125}$.

b. $x \equiv 127 \pmod{243}$.

6, a. $x \equiv 13, 19, 45, 51 \pmod{64}$.

b. $x \equiv 41, 87, 169, 215 \pmod{256}$.

Answers for chapter VI.

1, a. 6.

b. 18.

2, a. 3, 3, 3.

b. 6, 6, 1687.

c. α) 3; β) 7.

5, a. α) 0; β) 1; γ) 3.

b. α) 0; β) 1; γ) 10.

6, a. α) $x \equiv 40; 27 \pmod{67}$, β) $x \equiv 33 \pmod{67}$, γ) $x \equiv 8, 36, 28, 59, 31, 39 \pmod{67}$.

b. α) $x \equiv 17 \pmod{73}$; β) $x \equiv 50, 12, 35, 23, 61, 38 \pmod{73}$, γ) $x \equiv 3, 24, 46 \pmod{73}$.

7, a. α) 0; β) 4.

b. α) 0; β) 7.

8, a. α) $x \equiv 54 \pmod{101}$. β) $x \equiv 53, 86, 90, 66, 8 \pmod{101}$.

b. $x \equiv 59, 11, 39 \pmod{109}$.

9, a. α) 1, 4, 5, 6, 7, 9, 11, 16, 17; β) 1, 7, 8, 11, 12, 18.

b. α) 1, 6, 8, 10, 11, 14, 23, 26, 27, 29, 31, 36; β) 1, 7, 9, 10, 12, 16, 26, 33, 34.

10, a. α) 7, 37; β) 3, 5, 12, 18, 19, 20, 26, 28, 29, 30, 33, 34.

b. α) 3, 27, 41, 52; β) 2, 6, 7, 10, 17, 18, 23, 26, 30, 31, 35, 43, 44, 51, 54, 55, 59.

TABLES OF INDICES

The Prime 3

N	0	1	2	3	4	5	6	7	8	9
0		0	1							

I	0	1	2	3	4	5	6	7	8	9
0	1	2								

The Prime 5

N	0	1	2	3	4	5	6	7	8	9
0		0	1	3	2					

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	3						

The Prime 7

N	0	1	2	3	4	5	6	7	8	9
0		0	2	1	4	5	3			

I	0	1	2	3	4	5	6	7	8	9
0	1	3	2	6	4	5				

The Prime 11

N	0	1	2	3	4	5	6	7	8	9
0		0	1	8	2	4	9	7	3	6
1	5									

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	5	10	9	7	3	6
1										

The Prime 13

N	0	1	2	3	4	5	6	7	8	9
0		0	1	4	2	9	5	11	3	8
1	10	7	6							

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	3	6	12	11	9	5
1	10	7								

The Prime 17

N	0	1	2	3	4	5	6	7	8	9
0		0	14	1	12	5	15	11	10	2
1	3	7	13	4	9	6	8			

I	0	1	2	3	4	5	6	7	8	9
0	1	3	9	10	13	5	15	11	16	14
1	8	7	4	12	2	6				

The Prime 19

N	0	1	2	3	4	5	6	7	8	9
0		0	1	13	2	16	14	6	3	8
1	17	12	15	5	7	11	4	10	9	

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	13	7	14	9	18
1	17	15	11	3	6	12	5	10		

The Prime 23

N	0	1	2	3	4	5	6	7	8	9
0		0	2	16	4	1	18	19	6	10
1	3	9	20	14	21	17	8	7	12	15
2	5	13	11							

I	0	1	2	3	4	5	6	7	8	9
0	1	5	2	10	4	20	8	17	16	11
1	9	22	18	21	13	19	3	15	6	7
2	12	14								

The Prime 29

N	0	1	2	3	4	5	6	7	8	9
0		0	1	5	2	22	6	12	3	10
1	23	25	7	18	13	27	4	21	11	9
2	24	17	26	20	8	16	19	15	14	

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	3	6	12	24	19
1	9	18	7	14	28	27	25	21	13	26
2	23	17	5	10	20	11	22	15		

The Prime 31

N	0	1	2	3	4	5	6	7	8	9
0		0	24	1	18	20	25	28	12	2
1	14	23	19	11	22	21	6	7	26	4
2	8	29	17	27	13	10	5	3	16	9
3	15									

I	0	1	2	3	4	5	6	7	8	9
0	1	3	9	27	19	26	16	17	20	29
1	25	13	8	24	10	30	28	22	4	12
2	5	15	14	11	2	6	18	23	7	21

The Prime 37

N	0	1	2	3	4	5	6	7	8	9
0		0	1	26	2	23	27	32	3	16
1	24	30	28	11	33	13	4	7	17	35
2	25	22	31	15	29	10	12	6	34	21
3	14	9	5	20	8	19	18			

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	27	17	34	31
1	25	13	26	15	30	23	9	18	36	35
2	33	29	21	5	10	20	3	6	12	24
3	11	22	7	14	28	19				

The Prime 41

N	0	1	2	3	4	5	6	7	8	9
0		0	26	15	12	22	1	39	38	30
1	8	3	27	31	25	37	24	33	16	9
2	34	14	29	36	13	4	17	5	11	7
3	23	28	10	18	19	21	2	32	35	6
4	20									

I	0	1	2	3	4	5	6	7	8	9
0	1	6	36	11	25	27	39	29	10	19
1	32	28	4	24	21	3	18	26	33	34
2	40	35	5	30	16	14	2	12	31	22
3	9	13	37	17	20	38	23	15	8	7

The Prime 43

N	0	1	2	3	4	5	6	7	8	9
0		0	27	1	12	25	28	35	39	2
1	10	30	13	32	20	26	24	38	29	19
2	37	36	15	16	40	8	17	3	5	41
3	11	34	9	31	23	18	14	7	4	33
4	22	6	21							

I	0	1	2	3	4	5	6	7	8	9
0	1	3	9	27	38	28	41	37	25	32
1	10	30	4	12	36	22	23	26	35	19
2	14	42	40	34	16	5	15	2	6	18
3	11	33	13	39	31	7	21	20	17	8
4	24	29								

The Prime 47

N	0	1	2	3	4	5	6	7	8	9
0		0	18	20	36	1	38	32	8	40
1	19	7	10	11	4	21	26	16	12	45
2	37	6	25	5	28	2	29	14	22	35
3	39	3	44	27	34	33	30	42	17	31
4	9	15	24	13	43	41	23			

I	0	1	2	3	4	5	6	7	8	9
0	1	5	25	31	14	23	21	11	8	40
1	12	13	18	43	27	41	17	38	2	10
2	3	15	28	46	42	22	16	33	24	26
3	36	39	7	35	34	29	4	20	6	30
4	9	45	37	44	32	19				

The Prime 53

N	0	1	2	3	4	5	6	7	8	9
0		0	1	17	2	47	18	14	3	34
1	48	6	19	24	15	12	4	10	35	37
2	49	31	7	39	20	42	25	51	16	46
3	13	33	5	23	11	9	36	30	38	41
4	50	45	32	22	8	29	40	44	21	28
5	43	27	26							

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	11	22	44	35
1	17	34	15	30	7	14	28	3	6	12
2	24	48	43	33	13	26	52	51	49	45
3	37	21	42	31	9	18	36	19	38	23
4	46	39	25	50	47	41	29	5	10	20
5	40	27								

The Prime 59

N	0	1	2	3	4	5	6	7	8	9
0		0	1	50	2	6	51	18	3	42
1	7	25	52	45	19	56	4	40	43	38
2	8	10	26	15	53	12	46	34	20	28
3	57	49	5	17	41	24	44	55	39	37
4	9	14	11	33	27	48	16	23	54	36
5	13	32	47	22	35	31	21	30	29	

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	5	10	20	40
1	21	42	25	50	41	23	46	33	7	14
2	28	56	53	47	35	11	22	44	29	58
3	57	55	51	43	27	54	49	39	19	38
4	17	34	9	18	36	13	26	52	45	31
5	3	6	12	24	48	37	15	30		

The Prime 61

N	0	1	2	3	4	5	6	7	8	9
0		0	1	6	2	22	7	49	3	12
1	23	15	8	40	50	28	4	47	13	26
2	24	55	16	57	9	44	41	18	51	35
3	29	59	5	21	48	11	14	39	27	46
4	25	54	56	43	17	34	58	20	10	38
5	45	53	42	33	19	37	52	32	36	31
6	30									

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	3	6	12	24
1	48	35	9	18	36	11	22	44	27	54
2	47	33	5	10	20	40	19	38	15	30
3	60	59	57	53	45	29	58	55	49	37
4	13	26	52	43	25	50	39	17	34	7
5	14	28	56	51	41	21	42	23	46	31

The Prime 67

N	0	1	2	3	4	5	6	7	8	9
0		0	1	39	2	15	40	23	3	12
1	16	59	41	19	24	54	4	64	13	10
2	17	62	60	28	42	30	20	51	25	44
3	55	47	5	32	65	38	14	22	11	58
4	18	53	63	9	61	27	29	50	43	46
5	31	37	21	57	52	8	26	49	45	36
6	56	7	48	35	6	34	33			

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	64	61	55	43
1	19	38	9	18	36	5	10	20	40	13
2	26	52	37	7	14	28	56	45	23	46
3	25	50	33	66	65	63	59	51	35	3
4	6	12	24	48	29	58	49	31	62	57
5	47	27	54	41	15	30	60	53	39	11
6	22	44	21	42	17	34				

The Prime 71

N	0	1	2	3	4	5	6	7	8	9
0		0	6	26	12	28	32	1	18	52
1	34	31	38	39	7	54	24	49	58	16
2	40	27	37	15	44	56	45	8	13	68
3	60	11	30	57	55	29	64	20	22	65
4	46	25	33	48	43	10	21	9	50	2
5	62	5	51	23	14	59	19	43	4	3
6	66	69	17	53	36	67	63	47	61	41
7	35									

I	0	1	2	3	4	5	6	7	8	9
0	1	7	49	59	58	51	2	14	27	47
1	45	31	4	28	54	23	19	62	8	56
2	37	46	38	53	16	41	3	21	5	35
3	32	11	6	42	10	70	64	22	12	13
4	20	69	57	44	24	26	40	67	43	17
5	48	52	9	63	15	34	25	33	18	55
6	30	68	50	66	36	39	60	65	29	61

The Prime 73

N	0	1	2	3	4	5	6	7	8	9
0		0	8	6	16	1	14	33	24	12
1	9	55	22	59	41	7	32	21	20	62
2	17	39	63	46	30	2	67	18	49	35
3	15	11	40	61	29	34	28	64	70	65
4	25	4	47	51	71	13	54	31	38	66
5	10	27	3	53	26	56	57	68	43	5
6	23	58	19	45	48	60	69	50	37	52
7	42	44	36							

I	0	1	2	3	4	5	6	7	8	9
0	1	5	25	52	41	59	3	15	2	10
1	50	31	9	45	6	30	4	20	27	62
2	18	17	12	60	8	40	54	51	36	34
3	24	47	16	7	35	29	72	68	48	21
4	32	14	70	58	71	63	23	42	64	28
5	67	43	69	53	46	11	55	56	61	13
6	65	33	19	22	37	39	49	26	57	66
7	38	44								

The Prime 79

N	0	1	2	3	4	5	6	7	8	9
0		0	4	1	8	62	5	53	12	2
1	66	68	9	34	57	63	16	21	6	32
2	70	54	72	26	13	46	38	3	61	11
3	67	56	20	69	25	37	10	19	36	35
4	74	75	58	49	76	64	30	59	17	28
5	50	22	42	77	7	52	65	33	15	31
6	71	45	60	55	24	18	73	48	29	27
7	41	51	14	44	23	47	40	43	39	

I	0	1	2	3	4	5	6	7	8	9
0	1	3	9	27	2	6	18	54	4	12
1	36	29	8	24	72	58	16	48	65	37
2	32	17	51	74	64	34	23	69	49	68
3	46	59	19	57	13	39	38	35	26	78
4	76	70	52	77	73	61	25	75	67	43
5	50	71	55	7	21	63	31	14	42	47
6	62	28	5	15	45	56	10	30	11	33
7	20	60	22	66	40	41	44	53		

The Prime 83

N	0	1	2	3	4	5	6	7	8	9
0		0	1	72	2	27	73	8	3	62
1	28	24	74	77	9	17	4	56	63	47
2	29	80	25	60	75	54	78	52	10	12
3	18	38	5	14	57	35	64	20	48	67
4	30	40	81	71	26	7	61	23	76	16
5	55	46	79	59	53	51	11	37	13	34
6	19	66	39	70	6	22	15	45	58	50
7	36	33	65	69	21	44	49	32	68	43
8	31	42	41							

I	0	1	2	3	4	5	6	7	8	9
0	1	2	4	8	16	32	64	45	7	14
1	28	56	29	58	33	66	49	15	30	60
2	37	74	65	47	11	22	44	5	10	20
3	40	80	77	71	59	35	70	57	31	62
4	41	82	81	79	75	67	51	19	38	76
5	69	55	27	54	25	50	17	34	68	53
6	23	46	9	18	36	72	61	39	78	73
7	63	43	3	6	12	24	48	13	26	52
8	21	42								

The Prime 89

N	0	1	2	3	4	5	6	7	8	9
0		0	16	1	32	70	17	81	48	2
1	86	84	33	23	9	71	64	6	18	35
2	14	82	12	57	49	52	39	3	25	59
3	87	31	80	85	22	63	34	11	51	24
4	30	21	10	29	28	72	73	54	65	74
5	68	7	55	78	19	66	41	36	75	43
6	15	69	47	83	8	5	13	56	38	58
7	79	62	50	20	27	53	67	77	40	42
8	46	4	37	61	26	76	45	60	44	

I	0	1	2	3	4	5	6	7	8	9
0	1	3	9	27	81	65	17	51	64	14
1	42	37	22	66	20	60	2	6	18	54
2	73	41	34	13	39	28	84	74	44	43
3	40	31	4	12	36	19	57	82	68	26
4	78	56	79	59	88	86	80	62	8	24
5	72	38	25	75	47	52	67	23	69	29
6	87	83	71	35	16	48	55	76	50	61
7	5	15	45	46	49	58	85	77	53	70
8	32	7	21	63	11	33	10	30		

The Prime 97

N	0	1	2	3	4	5	6	7	8	9
0		0	34	70	68	1	8	31	6	44
1	35	6	42	25	65	71	40	89	78	81
2	69	5	24	77	76	2	59	18	3	13
3	9	46	74	60	27	32	16	91	19	95
4	7	85	39	4	58	45	15	84	14	62
5	36	63	93	10	52	87	37	55	47	67
6	43	64	80	75	12	26	94	57	61	51
7	66	11	50	28	29	72	53	21	33	30
8	41	88	23	17	73	90	38	83	92	54
9	79	56	49	20	22	82	48			

I	0	1	2	3	4	5	6	7	8	9
0	1	5	25	28	43	21	8	40	6	30
1	53	71	64	29	48	46	36	83	27	38
2	93	77	94	82	22	13	65	34	73	74
3	79	7	35	78	2	10	50	56	86	42
4	16	80	12	60	9	45	31	58	96	92
5	72	69	54	76	89	57	91	67	44	26
6	33	68	49	51	61	14	70	59	4	20
7	3	15	75	84	32	63	24	23	18	90
8	62	19	95	87	47	41	11	55	81	17
9	85	37	88	52	66	39				

Table of primes < 4000 and their smallest primitive roots.

p	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*
2	1	179	2	419	2	661	2	947	2	1 229	2	1 523	2
3	2	181	2	421	2	673	5	953	3	1 231	3	1 531	2
5	2	191	19	431	7	677	2	967	5	1 237	2	1 543	5
7	3	193	5	433	5	683	5	971	6	1 249	7	1 549	2
11	2	197	2	439	15	691	3	977	3	1 259	2	1 553	3
13	2	199	3	443	2	701	2	983	5	1 277	2	1 559	19
17	3	211	2	449	3	709	2	991	6	1 279	3	1 567	3
19	2	223	3	457	13	719	11	997	7	1 283	2	1 571	2
23	5	227	2	461	2	727	5	1 009	11	1 289	6	1 579	3
29	2	229	6	463	3	733	6	1 013	3	1 291	2	1 583	5
31	3	233	3	467	2	739	3	1 019	2	1 297	10	1 597	11
37	2	239	7	479	13	743	5	1 021	10	1 301	2	1 601	3
41	6	241	7	487	3	751	3	1 031	14	1 303	6	1 607	5
43	3	251	6	491	2	757	2	1 033	5	1 307	2	1 609	7
47	5	257	3	499	7	761	6	1 039	3	1 319	13	1 613	3
53	2	263	5	503	5	769	11	1 049	3	1 321	13	1 619	2
59	2	269	2	509	2	773	2	1 051	7	1 327	3	1 621	2
61	2	271	6	521	3	787	2	1 061	2	1 361	3	1 627	3
67	2	277	5	523	2	797	2	1 063	3	1 367	5	1 637	2
71	7	281	3	541	2	809	3	1 069	6	1 373	2	1 657	11
73	5	283	3	547	2	811	3	1 087	3	1 381	2	1 663	3
79	3	293	2	557	2	821	2	1 091	2	1 399	13	1 667	2
83	2	307	5	563	2	823	3	1 093	5	1 409	3	1 669	2
89	3	311	17	569	3	827	2	1 097	3	1 423	3	1 693	2
97	5	313	10	571	3	829	2	1 103	5	1 427	2	1 697	3
101	2	317	2	577	5	839	11	1 109	2	1 429	6	1 699	3
103	5	331	3	587	2	853	2	1 117	2	1 433	3	1 709	3
107	2	337	10	593	3	857	3	1 123	2	1 439	7	1 721	3
109	6	347	2	599	7	859	2	1 129	11	1 447	3	1 723	3
113	3	349	2	601	7	863	5	1 151	17	1 451	2	1 733	2
127	3	353	3	607	3	877	2	1 153	5	1 453	2	1 741	2
131	2	359	7	613	2	881	3	1 163	5	1 459	5	1 747	2
137	3	367	6	617	3	883	2	1 171	2	1 471	6	1 753	7
139	2	373	2	619	2	887	5	1 181	7	1 481	3	1 759	6
149	2	379	2	631	3	907	2	1 187	2	1 483	2	1 777	5
151	6	383	5	641	3	911	17	1 193	3	1 487	5	1 783	10
157	5	389	2	643	11	919	7	1 201	11	1 489	14	1 787	2
163	2	397	5	647	5	929	3	1 213	2	1 493	2	1 789	6
167	5	401	3	653	2	937	5	1 217	3	1 499	2	1 801	11
173	2	409	21	659	2	941	2	1 223	5	1 511	11	1 811	6

(*continued*)

p	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*
1 823	5	2 129	3	2 417	3	2 729	3	3 049	11	3 373	5	3 691	2
1 831	3	2 131	2	2 423	5	2 731	3	3 061	6	3 389	3	3 697	5
1 847	5	2 137	10	2 437	2	2 741	2	3 067	2	3 391	3	3 701	2
1 861	2	2 141	2	2 441	6	2 749	6	3 079	6	3 407	5	3 709	2
1 867	2	2 143	3	2 447	5	2 753	3	3 083	2	3 413	2	3 719	7
1 871	14	2 153	3	2 459	2	2 767	3	3 089	3	3 433	5	3 727	3
1 873	10	2 161	23	2 467	2	2 777	3	3 109	6	3 449	3	3 733	2
1 877	2	2 179	7	2 473	5	2 789	2	3 119	7	3 457	7	3 739	7
1 879	6	2 203	5	2 477	2	2 791	6	3 121	7	3 461	2	3 761	3
1 889	3	2 207	5	2 503	3	2 797	2	3 137	3	3 463	3	3 767	5
1 901	2	2 213	2	2 521	17	2 801	3	3 163	3	3 467	2	3 769	7
1 907	2	2 221	2	2 531	2	2 803	2	3 167	5	3 469	2	3 779	2
1 913	3	2 237	2	2 539	2	2 819	2	3 169	7	3 491	2	3 793	5
1 931	2	2 239	3	2 543	5	2 833	5	3 181	7	3 499	2	3 797	2
1 933	5	2 243	2	2 549	2	2 837	2	3 187	2	3 511	7	3 803	2
1 949	2	2 251	7	2 551	6	2 843	2	3 191	11	3 517	2	3 821	3
1 951	3	2 267	2	2 557	2	2 851	2	3 203	2	3 527	5	3 823	3
1 973	2	2 269	2	2 579	2	2 857	11	3 209	3	3 529	17	3 833	3
1 979	2	2 273	3	2 591	7	2 861	2	3 217	5	3 533	2	3 847	5
1 987	2	2 281	7	2 593	7	2 879	7	3 221	10	3 539	2	3 851	2
1 993	5	2 287	19	2 609	3	2 887	5	3 229	6	3 541	7	3 853	2
1 997	2	2 293	2	2 617	5	2 897	3	3 251	6	3 547	2	3 863	5
1 999	3	2 297	5	2 621	2	2 903	5	3 253	2	3 557	2	3 877	2
2 003	5	2 309	2	2 633	3	2 909	2	3 257	3	3 559	3	3 881	13
2 011	3	2 311	3	2 647	3	2 917	5	3 259	3	3 571	2	3 889	11
2 017	5	2 333	2	2 657	3	2 927	5	3 271	3	3 581	2	3 907	2
2 027	2	2 339	2	2 659	2	2 939	2	3 299	2	3 583	3	3 911	13
2 029	2	2 341	7	2 663	5	2 953	13	3 301	6	3 593	3	3 917	2
2 039	7	2 347	3	2 671	7	2 957	2	3 307	2	3 607	5	3 919	3
2 053	2	2 351	13	2 677	2	2 963	2	3 313	10	3 613	2	3 923	2
2 063	5	2 357	2	2 683	2	2 969	3	3 319	6	3 617	3	3 929	3
2 069	2	2 371	2	2 687	5	2 971	10	3 323	2	3 623	5	3 931	2
2 081	3	2 377	5	2 689	19	2 999	17	3 329	3	3 631	21	3 943	3
2 083	2	2 381	3	2 693	2	3 001	14	3 331	3	3 637	2	3 947	2
2 087	5	2 383	5	2 699	2	3 011	2	3 343	5	3 643	2	3 967	6
2 089	7	2 389	2	2 707	2	3 019	2	3 347	2	3 659	2	3 989	2
2 099	2	2 393	3	2 711	7	3 023	5	3 359	11	3 671	13		
2 111	7	2 399	11	2 713	5	3 037	2	3 361	22	3 673	5		
2 113	5	2 411	6	2 719	3	3 041	3	3 371	2	3 677	2		

CATALOGUE OF DOVER BOOKS

MATHEMATICS, ELEMENTARY TO INTERMEDIATE

HOW TO CALCULATE QUICKLY, Henry Sticker. This handy volume offers a tried and true method for helping you in the basic mathematics of daily life—addition, subtraction, multiplication, division, fractions, etc. It is designed to awaken your "number sense" or the ability to see relationships between numbers as whole quantities. It is not a collection of tricks working only on special numbers, but a serious course of over 9,000 problems and their solutions, teaching special techniques not taught in schools: left-to-right multiplication, new fast ways of division, etc. 5 or 10 minutes daily use will double or triple your calculation speed. Excellent for the scientific worker who is at home in higher math, but is not satisfied with his speed and accuracy in lower mathematics. 256pp. 5 x 7¼. T295 Paperbound **$1.00**

TEACH YOURSELF books. For adult self-study, for refresher and supplementary study.

The most effective series of home study mathematics books on the market! With absolutely no outside help, they will teach you as much as any similar college or high-school course, or will helpfully supplement any such course. Each step leads directly to the next, each question is anticipated. Numerous lucid examples and carefully-wrought practice problems illustrate meanings. Not skimpy outlines, not surveys, not usual classroom texts, these 204- to 380-page books are packed with the finest instruction you'll find anywhere for adult self-study.

TEACH YOURSELF ALGEBRA, P. Abbott. Formulas, coordinates, factors, graphs of quadratic functions, quadratic equations, logarithms, ratio, irrational numbers, arithmetical, geometrical series, much more. 1241 problems, solutions. Tables. 52 illus. 307pp. 6⅞ x 4¼. Clothbound **$2.00**

TEACH YOURSELF GEOMETRY, P. Abbott. Solids, lines, points, surfaces, angle measurement, triangles, theorem of Pythagoras, polygons, loci, the circle, tangents, symmetry, solid geometry, prisms, pyramids, solids of revolution, etc. 343 problems, solutions. 268 illus. 334pp. 6⅞ x 4¼. Clothbound **$2.00**

TEACH YOURSELF TRIGONOMETRY, P. Abbott. Geometrical foundations, indices, logarithms, trigonometrical ratios, relations between sides, angles of triangle, circular measure, trig. ratios of angles of any magnitude, much more. Requires elementary algebra, geometry. 465 problems, solutions. Tables. 102 illus. 204pp. 6⅞ x 4¼. Clothbound **$2.00**

TEACH YOURSELF THE CALCULUS, P. Abbott. Variations in functions, differentiation, solids of revolution, series, elementary differential equations, areas by integral calculus, much more. Requires algebra, trigonometry. 970 problems, solutions. Tables. 89 illus. 380pp. 6⅞ x 4¼. Clothbound **$2.00**

TEACH YOURSELF THE SLIDE RULE, B. Snodgrass. Fractions, decimals, A-D scales, log-log scales, trigonometrical scales, indices, logarithms. Commercial, precision, electrical, dualistic, Brighton rules. 80 problems, solutions. 10 illus. 207pp. 6⅞ x 4¼. Clothbound **$2.00**

ARITHMETICAL EXCURSIONS: AN ENRICHMENT OF ELEMENTARY MATHEMATICS, H. Bowers and J. Bowers. For students who want unusual methods of arithmetic never taught in school; for adults who want to increase their number sense. Little known facts about the most simple numbers, arithmetical entertainments and puzzles, figurate numbers, number chains, mysteries and folklore of numbers, the "Hin-dog-abic" number system, etc. First publication. Index. 529 numbered problems and diversions, all with answers. Bibliography. 50 figures. xiv + 320pp. 5⅜ x 8. T770 Paperbound **$1.65**

HOW DO YOU USE A SLIDE RULE? by A. A. Merrill. Not a manual for mathematicians and engineers, but a lucid step-by-step explanation that presents the fundamental rules clearly enough to be understood by anyone who could benefit by the use of a slide rule in his work or business. This work concentrates on the 2 most important operations: multiplication and division. 10 easy lessons, each with a clear drawing, will save you countless hours in your banking, business, statistical, and other work. First publication. Index. 2 Appendixes. 10 illustrations. 78 problems, all with answers. vi + 36pp. 6⅛ x 9¼. T62 Paperbound **60¢**

THE THEORY AND OPERATION OF THE SLIDE RULE, J. P. Ellis. Not a skimpy "instruction manual", but an exhaustive treatment that will save you hours throughout your career. Supplies tull understanding ot every scale on the Log Log Duplex Decitrig type of slide rule. Shows the most time-saving methods, and provides practice useful in the widest variety of actual engineering situations. Each operation introduced in terms ot underlying logarithmic theory. Summary of prerequisite math. First publication. Index. 198 figures. Over 450 problems with answers. Bibliography. 12 Appendices. ix + 289pp. 5⅜ x 8. S727 Paperbound **$1.50**

COLLEGE ALGEBRA, H. B. Fine. Standard college text that gives a systematic and deductive structure to algebra; comprehensive, connected, with emphasis on theory. Discusses the commutative, associative, and distributive laws of number in unusual detail, and goes on with undetermined coefficients, quadratic equations, progressions, logarithms, permutations, probability, power series, and much more. Still most valuable elementary-intermediate text on the science and structure of algebra. Index. 1560 problems, all with answers. x + 631pp. 5⅜ x 8.
T211 Paperbound **$2.50**

COORDINATE GEOMETRY, L. P. Eisenhart. Thorough, unified introduction. Unusual for advancing in dimension within each topic (treats together circle, sphere; polar coordinates, 3-dimensional coordinate systems; conic sections, quadric surfaces), affording exceptional insight into subject. Extensive use made of determinants, though no previous knowledge of them is assumed. Algebraic equations of 1st degree, 2 and 3 unknowns, carried further than usual in algebra courses. Over 500 exercises. Introduction. Appendix. Index. Bibliography. 43 illustrations. 310pp. 5⅜ x 8.
S600 Paperbound **$1.65**

A TREATISE ON PLANE AND ADVANCED TRIGONOMETRY, E. W. Hobson. Extraordinarily wide coverage, going beyond usual college level trig, one of the few works covering advanced trig in full detail. By a great expositor with unerring anticipation and lucid clarification of potentially difficult points. Includes circular functions; expansion of functions of multiple angle; trig tables; relations between sides and angles of triangle; complex numbers; etc. Many problems solved completely. "The best work on the subject." Nature. Formerly entitled "A Treatise on Plane Trigonometry." 689 examples. 6 figures. xvi + 383pp. 5⅜ x 8.
S353 Paperbound **$2.25**

FAMOUS PROBLEMS OF ELEMENTARY GEOMETRY, Felix Klein. Expanded version of the 1894 Easter lectures at Göttingen. 3 problems of classical geometry, in an excellent mathematical treatment by a famous mathematician: squaring the circle, trisecting angle, doubling cube. Considered with full modern implications: transcendental numbers, pi, etc. Notes by R. Archibald. 16 figures. xi + 92pp. 5⅜ x 8.
T298 Paperbound **$1.00**

MONOGRAPHS ON TOPICS OF MODERN MATHEMATICS, edited by **J. W. A. Young.** Advanced mathematics for persons who haven't gone beyond or have forgotten high school algebra. 9 monographs on foundation of geometry, modern pure geometry, non-Euclidean geometry, fundamental propositions of algebra, algebraic equations, functions, calculus, theory of numbers, etc. Each monograph gives proofs of important results, and descriptions of leading methods, to provide wide coverage. New introduction by Prof. M. Kline, N. Y. University. 100 diagrams. xvi + 416pp. 6⅛ x 9¼.
S289 Paperbound **$2.00**

HIGHER MATHEMATICS FOR STUDENTS OF CHEMISTRY AND PHYSICS, J. W. Mellor. Not abstract, but practical, building its problems out of familiar laboratory material, this covers differential calculus, coordinate, analytical geometry, functions, integral calculus, infinite series, numerical equations, differential equations, Fourier's theorem, probability, theory of errors, calculus of variations, determinants. "If the reader is not familiar with this book, it will repay him to examine it," CHEM. & ENGINEERING NEWS. 800 problems. 189 figures. Bibliography. xxi + 641pp. 5⅜ x 8.
S193 Paperbound **$2.50**

TRIGONOMETRY REFRESHER FOR TECHNICAL MEN, A. Albert Klaf. 913 detailed questions and answers cover the most important aspects of plane and spherical trigonometry. They will help you to brush up or to clear up difficulties in special areas. The first portion of this book covers plane trigonometry, including angles, quadrants, trigonometrical functions, graphical representation, interpolation, equations, logarithms, solution of triangle, use of the slide rule and similar topics. 188 pages then discuss application of plane trigonometry to special problems in navigation, surveying, elasticity, architecture, and various fields of engineering. Small angles, periodic functions, vectors, polar coordinates, de Moivre's theorem are fully examined. The third section of the book then discusses spherical trigonometry and the solution of spherical triangles, with their applications to terrestrial and astronomical problems. Methods of saving time with numerical calculations, simplification of principal functions of angle, much practical information make this a most useful book. 913 questions answered. 1738 problems, answers to odd numbers. 494 figures. 24 pages of useful formulae, functions. Index. x + 629pp. 5⅜ x 8.
T371 Paperbound **$2.00**

TEXTBOOK OF ALGEBRA, G. Chrystal. One of the great mathematical textbooks, still about the best source for complete treatments of the topics of elementary algebra; a chief reference work for teachers and students of algebra in advanced high school and university courses, or for the mathematician working on problems of elementary algebra or looking for a background to more advanced topics. Ranges from basic laws and processes to extensive examination of such topics as limits, infinite series, general properties of integral numbers, and probability theory. Emphasis is on algebraic form, the foundation of analytical geometry and the key to modern developments in algebra. Prior course in algebra is desirable, but not absolutely necessary. Includes theory of quotients, distribution of products, arithmetical theory of surds, theory of interest, permutations and combinations, general expansion theorems, recurring fractions, and much, much more. Two volume set. Index in each volume. Over 1500 exercises, approximately half with answers. Total of xlviii + 1187pp. 5⅜ x 8.
S750 Vol I Paperbound **$2.35**
S751 Vol II Paperbound **$2.35**
The set **$4.70**

MATHEMATICS—INTERMEDIATE TO ADVANCED

General

INTRODUCTION TO APPLIED MATHEMATICS, Francis D. Murnaghan. A practical and thoroughly sound introduction to a number of advanced branches of higher mathematics. Among the selected topics covered in detail are: vector and matrix analysis, partial and differential equations, integral equations, calculus of variations, Laplace transform theory, the vector triple product, linear vector functions, quadratic and bilinear forms, Fourier series, spherical harmonics, Bessel functions, the Heaviside expansion formula, and many others. Extremely useful book for graduate students in physics, engineering, chemistry, and mathematics. Index. 111 study exercises with answers. 41 illustrations. ix + 389pp. 5⅜ x 8½.
S1042 Paperbound **$2.25**

OPERATIONAL METHODS IN APPLIED MATHEMATICS, H. S. Carslaw and J. C. Jaeger. Explanation of the application of the Laplace Transformation to differential equations, a simple and effective substitute for more difficult and obscure operational methods. Of great practical value to engineers and to all workers in applied mathematics. Chapters on: Ordinary Linear Differential Equations with Constant Coefficients;; Electric Circuit Theory; Dynamical Applications; The Inversion Theorem for the Laplace Transformation; Conduction of Heat; Vibrations of Continuous Mechanical Systems; Hydrodynamics; Impulsive Functions; Chains of Differential Equations; and other related matters. 3 appendices. 153 problems, many with answers. 22 figures. xvi + 359pp. 5⅜ x 8½. S1011 Paperbound **$2.25**

APPLIED MATHEMATICS FOR RADIO AND COMMUNICATIONS ENGINEERS, C. E. Smith. No extraneous material here!—only the theories, equations, and operations essential and immediately useful for radio work. Can be used as refresher, as handbook of applications and tables, or as full home-study course. Ranges from simplest arithmetic through calculus, series, and wave forms, hyperbolic trigonometry, simultaneous equations in mesh circuits, etc. Supplies applications right along with each math topic discussed. 22 useful tables of functions, formulas, logs, etc. Index. 166 exercises, 140 examples, all with answers. 95 diagrams. Bibliography. x + 336pp. 5⅜ x 8. S141 Paperbound **$1.75**

Algebra, group theory, determinants, sets, matrix theory

ALGEBRAS AND THEIR ARITHMETICS, L. E. Dickson. Provides the foundation and background necessary to any advanced undergraduate or graduate student studying abstract algebra. Begins with elementary introduction to linear transformations, matrices, field of complex numbers; proceeds to order, basal units, modulus, quaternions, etc.; develops calculus of linears sets, describes various examples of algebras including invariant, difference, nilpotent, semi-simple. "Makes the reader marvel at his genius for clear and profound analysis," Amer. Mathematical Monthly. Index. xii + 241pp. 5⅜ x 8. S616 Paperbound **$1.50**

THE THEORY OF EQUATIONS WITH AN INTRODUCTION TO THE THEORY OF BINARY ALGEBRAIC FORMS, W. S. Burnside and A. W. Panton. Extremely thorough and concrete discussion of the theory of equations, with extensive detailed treatment of many topics curtailed in later texts. Covers theory of algebraic equations, properties of polynomials, symmetric functions, derived functions, Horner's process, complex numbers and the complex variable, determinants and methods of elimination, invariant theory (nearly 100 pages), transformations, introduction to Galois theory, Abelian equations, and much more. Invaluable supplementary work for modern students and teachers. 759 examples and exercises. Index in each volume. Two volume set. Total of xxiv + 604pp. 5⅜ x 8.
S714 Vol I Paperbound **$1.85**
S715 Vol II Paperbound **$1.85**
The set **$3.70**

COMPUTATIONAL METHODS OF LINEAR ALGEBRA, V. N. Faddeeva, translated by **C. D. Benster.** First English translation of a unique and valuable work, the only work in English presenting a systematic exposition of the most important methods of linear algebra—classical and contemporary. Shows in detail how to derive numerical solutions of problems in mathematical physics which are frequently connected with those of linear algebra. Theory as well as individual practice. Part I surveys the mathematical background that is indispensable to what follows. Parts II and III, the conclusion, set forth the most important methods of solution, for both exact and iterative groups. One of the most outstanding and valuable features of this work is the 23 tables, double and triple checked for accuracy. These tables will not be found elsewhere. Author's preface. Translator's note. New bibliography and index. x + 252pp. 5⅜ x 8. S424 Paperbound **$2.00**

ALGEBRAIC EQUATIONS, E. Dehn. Careful and complete presentation of Galois' theory of algebraic equations; theories of Lagrange and Galois developed in logical rather than historical form, with a more thorough exposition than in most modern books. Many concrete applications and fully-worked-out examples. Discusses basic theory (very clear exposition of the symmetric group); isomorphic, transitive, and Abelian groups; applications of Lagrange's and Galois' theories; and much more. Newly revised by the author. Index. List of Theorems. xi + 208pp. 5⅜ x 8. S697 Paperbound **$1.45**

ALGEBRAIC THEORIES, L. E. Dickson. Best thorough introduction to classical topics in higher algebra develops theories centering around matrices, invariants, groups. Higher algebra, Galois theory, finite linear groups, Klein's icosahedron, algebraic invariants, linear transformations, elementary divisors, invariant factors; quadratic, bi-linear, Hermitian forms, singly and in pairs. Proofs rigorous, detailed; topics developed lucidly, in close connection with their most frequent mathematical applications. Formerly "Modern Algebraic Theories." 155 problems. Bibliography. 2 indexes. 285pp. 5⅜ x 8. S547 Paperbound **$1.50**

LECTURES ON THE ICOSAHEDRON AND THE SOLUTION OF EQUATIONS OF THE FIFTH DEGREE, Felix Klein. The solution of quintics in terms of rotation of a regular icosahedron around its axes of symmetry. A classic & indispensable source for those interested in higher algebra, geometry, crystallography. Considerable explanatory material included. 230 footnotes, mostly bibliographic. 2nd edition, xvi + 289pp. 5⅜ x 8. S314 Paperbound **$2.25**

LINEAR GROUPS, WITH AN EXPOSITION OF THE GALOIS FIELD THEORY, L. E. Dickson. The classic exposition of the theory of groups, well within the range of the graduate student. Part I contains the most extensive and thorough presentation of the theory of Galois Fields available, with a wealth of examples and theorems. Part II is a full discussion of linear groups of finite order. Much material in this work is based on Dickson's own contributions. Also includes expositions of Jordan, Lie, Abel, Betti-Mathieu, Hermite, etc. "A milestone in the development of modern algebra," W. Magnus, in his historical introduction to this edition. Index. xv + 312pp. 5⅜ x 8. S482 Paperbound **$1.95**

INTRODUCTION TO THE THEORY OF GROUPS OF FINITE ORDER, R. Carmichael. Examines fundamental theorems and their application. Beginning with sets, systems, permutations, etc., it progresses in easy stages through important types of groups: Abelian, prime power, permutation, etc. Except 1 chapter where matrices are desirable, no higher math needed. 783 exercises, problems. Index. xvi + 447pp. 5⅜ x 8. S300 Paperbound **$2.25**

THEORY OF GROUPS OF FINITE ORDER, W. Burnside. First published some 40 years ago, this is still one of the clearest introductory texts. Partial contents: permutations, groups independent of representation, composition series of a group, isomorphism of a group with itself, Abelian groups, prime power groups, permutation groups, invariants of groups of linear substitution, graphical representation, etc. 45pp. of notes. Indexes. xxiv + 512pp. 5⅜ x 8.
S38 Paperbound **$2.75**

CONTINUOUS GROUPS OF TRANSFORMATIONS, L. P. Eisenhart. Intensive study of the theory and geometrical applications of continuous groups of transformations; a standard work on the subject, called forth by the revolution in physics in the 1920's. Covers tensor analysis, Riemannian geometry, canonical parameters, transitivity, imprimitivity, differential invariants, the algebra of constants of structure, differential geometry, contact transformations, etc. "*Likely to remain one of the standard works on the subject for many years . . . principal theorems are proved clearly and concisely, and the arrangement of the whole is coherent,*" MATHEMATICAL GAZETTE. Index. 72-item bibliography. 185 exercises. ix + 301pp. 5⅜ x 8.
S781 Paperbound **$2.00**

THE THEORY OF GROUPS AND QUANTUM MECHANICS, H. Weyl. Discussions of Schroedinger's wave equation, de Broglie's waves of a particle, Jordan-Hoelder theorem, Lie's continuous groups of transformations, Pauli exclusion principle, quantization of Maxwell-Dirac field equations, etc. Unitary geometry, quantum theory, groups, application of groups to quantum mechanics, symmetry permutation group, algebra of symmetric transformation, etc. 2nd revised edition. Bibliography. Index. xxii + 422pp. 5⅜ x 8. S269 Paperbound **$2.35**

APPLIED GROUP-THEORETIC AND MATRIX METHODS, Bryan Higman. The first systematic treatment of group and matrix theory for the physical scientist. Contains a comprehensive, easily-followed exposition of the basic ideas of group theory (realized through matrices) and its applications in the various areas of physics and chem.stry: tensor analysis, relativity, quantum theory, molecular structure and spectra, and Eddington's quantum relativity. Includes rigorous proofs available only in works of a far more advanced character. 34 figures, numerous tables. Bibliography. Index. xiii + 454pp. 5⅜ x 8⅜.
S1147 Paperbound **$3.00**

THE THEORY OF GROUP REPRESENTATIONS, Francis D. Murnaghan. A comprehensive introduction to the theory of group representations. Particular attention is devoted to those groups—mainly the symmetric and rotation groups—which have proved to be of fundamental significance for quantum mechanics (esp. nuclear physics). Also a valuable contribution to the literature on matrices, since the usual representations of groups are groups of matrices. Covers the theory of group integration (as developed by Schur and Weyl), the theory of 2-valued or spin representations, the representations of the symmetric group, the crystallographic groups, the Lorentz group, reducibility (Schur's lemma, Burnside's Theorem, etc.), the alternating group, linear groups, the orthogonal group, etc. Index. List of references. xi + 369pp. 5⅜ x 8½. S1112 Paperbound **$2.35**

THEORY OF SETS, E. Kamke. Clearest, amplest introduction in English, well suited for independent study. Subdivision of main theory, such as theory of sets of points, are discussed, but emphasis is on general theory. Partial contents: rudiments of set theory, arbitrary sets and their cardinal numbers, ordered sets and their order types, well-ordered sets and their cardinal numbers. Bibliography. Key to symbols. Index. vii + 144pp. 5⅜ x 8.
S141 Paperbound **$1.35**

THEORY AND APPLICATIONS OF FINITE GROUPS, G. A. Miller, H. F. Blichfeldt, L. E. Dickson. Unusually accurate and authoritative work, each section prepared by a leading specialist: Miller on substitution and abstract groups, Blichfeldt on finite groups of linear homogeneous transformations, Dickson on applications of finite groups. Unlike more modern works, this gives the concrete basis from which abstract group theory arose. Includes Abelian groups, prime-power groups, isomorphisms, matrix forms of linear transformations, Sylow groups, Galois' theory of algebraic equations, duplication of a cube, trisection of an angle, etc. 2 Indexes. 267 problems. xvii + 390pp. 5⅜ x 8. S216 Paperbound **$2.00**

THE THEORY OF DETERMINANTS, MATRICES, AND INVARIANTS, H. W. Turnbull. Important study includes all salient features and major theories. 7 chapters on determinants and matrices cover fundamental properties, Laplace identities, multiplication, linear equations, rank and differentiation, etc. Sections on invariants gives general properties, symbolic and direct methods of reduction, binary and polar forms, general linear transformation, first fundamental theorem, multilinear forms. Following chapters study development and proof of Hilbert's Basis Theorem, Gordan-Hilbert Finiteness Theorem, Clebsch's Theorem, and include discussions of apolarity, canonical forms, geometrical interpretations of algebraic forms, complete system of the general quadric, etc. New preface and appendix. Bibliography. xviii + 374pp. 5⅜ x 8. S699 Paperbound **$2.25**

AN INTRODUCTION TO THE THEORY OF CANONICAL MATRICES, H. W. Turnbull and A. C. Aitken. All principal aspects of the theory of canonical matrices, from definitions and fundamental properties of matrices to the practical applications of their reduction to canonical form. Beginning with matrix multiplications, reciprocals, and partitioned matrices, the authors go on to elementary transformations and bilinear and quadratic forms. Also covers such topics as a rational canonical form for the collineatory group, congruent and conjunctive transformation for quadratic and hermitian forms, unitary and orthogonal transformations, canonical reduction of pencils of matrices, etc. Index. Appendix. Historical notes at chapter ends. Bibliographies. 275 problems. xiv + 200pp. 5⅜ x 8. S177 Paperbound **$1.55**

A TREATISE ON THE THEORY OF DETERMINANTS, T. Muir. Unequalled as an exhaustive compilation of nearly all the known facts about determinants up to the early 1930's. Covers notation and general properties, row and column transformation, symmetry, compound determinants, adjugates, rectangular arrays and matrices, linear dependence, gradients, Jacobians, Hessians, Wronskians, and much more. Invaluable for libraries of industrial and research organizations as well as for student, teacher, and mathematician; very useful in the field of computing machines. Revised and enlarged by W. H. Metzler. Index. 485 problems and scores of numerical examples. iv + 766pp. 5⅜ x 8. S670 Paperbound **$3.00**

THEORY OF DETERMINANTS IN THE HISTORICAL ORDER OF DEVELOPMENT, Sir Thomas Muir. Unabridged reprinting of this complete study of 1,859 papers on determinant theory written between 1693 and 1900. Most important and original sections reproduced, valuable commentary on each. No other work is necessary for determinant research: all types are covered—each subdivision of the theory treated separately; all papers dealing with each type are covered; you are told exactly what each paper is about and how important its contribution is. Each result, theory, extension, or modification is assigned its own identifying numeral so that the full history may be more easily followed. Includes papers on determinants in general, determinants and linear equations, symmetric determinants, alternants, recurrents, determinants having invariant factors, and all other major types. "A model of what such histories ought to be," NATURE. "Mathematicians must ever be grateful to Sir Thomas for his monumental work," AMERICAN MATH MONTHLY. Four volumes bound as two. Indices. Bibliographies. Total of lxxxiv + 1977pp. 5⅜ x 8. S672-3 The set, Clothbound **$12.50**

Calculus and function theory, Fourier theory, infinite series, calculus of variations, real and complex functions

FIVE VOLUME "THEORY OF FUNCTIONS' SET BY KONRAD KNOPP

This five-volume set, prepared by Konrad Knopp, provides a complete and readily followed account of theory of functions. Proofs are given concisely, yet without sacrifice of completeness or rigor. These volumes are used as texts by such universities as M.I.T., University of Chicago, N. Y. City College, and many others. "Excellent introduction . . . remarkably readable, concise, clear, rigorous," JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION.

ELEMENTS OF THE THEORY OF FUNCTIONS, Konrad Knopp. This book provides the student with background for further volumes in this set, or texts on a similar level. Partial contents: foundations, system of complex numbers and the Gaussian plane of numbers, Riemann sphere of numbers, mapping by linear functions, normal forms, the logarithm, the cyclometric functions and binomial series. "Not only for the young student, but also for the student who knows all about what is in it," MATHEMATICAL JOURNAL. Bibliography. Index. 140pp. 5⅜ x 8. S154 Paperbound **$1.50**

THEORY OF FUNCTIONS, PART I, Konrad Knopp. With volume II, this book provides coverage of basic concepts and theorems. Partial contents: numbers and points, functions of a complex variable, integral of a continuous function, Cauchy's integral theorem, Cauchy's integral formulae, series with variable terms, expansion of analytic functions in power series, analytic continuation and complete definition of analytic functions, entire transcendental functions, Laurent expansion, types of singularities. Bibliography. Index. vii + 146pp. 5⅜ x 8. S156 Paperbound **$1.35**

THEORY OF FUNCTIONS, PART II, Konrad Knopp. Application and further development of general theory, special topics. Single valued functions, entire, Weierstrass, Meromorphic functions. Riemann surfaces. Algebraic functions. Analytical configuration, Riemann surface. Bibliography. Index. x + 150pp. 5⅜ x 8. S157 Paperbound **$1.35**

PROBLEM BOOK IN THE THEORY OF FUNCTIONS, VOLUME 1, Konrad Knopp. Problems in elementary theory, for use with Knopp's THEORY OF FUNCTIONS, or any other text, arranged according to increasing difficulty. Fundamental concepts, sequences of numbers and infinite series, complex variable, integral theorems, development in series, conformal mapping. 182 problems. Answers. viii + 126pp. 5⅜ x 8. S158 Paperbound **$1.35**

PROBLEM BOOK IN THE THEORY OF FUNCTIONS, VOLUME 2, Konrad Knopp. Advanced theory of functions, to be used either with Knopp's THEORY OF FUNCTIONS, or any other comparable text. Singularities, entire & meromorphic functions, periodic, analytic, continuation, multiple-valued functions, Riemann surfaces, conformal mapping. Includes a section of additional elementary problems. "The difficult task of selecting from the immense material of the modern theory of functions the problems just within the reach of the beginner is here masterfully accomplished," AM. MATH. SOC. Answers. 138pp. 5⅜ x 8. S159 Paperbound **$1.35**

A COURSE IN MATHEMATICAL ANALYSIS, Edouard Goursat. Trans. by E. R. Hedrick, O. Dunkel. Classic study of fundamental material thoroughly treated. Exceptionally lucid exposition of wide range of subject matter for student with 1 year of calculus. Vol. 1: Derivatives and Differentials, Definite Integrals, Expansion in Series, Applications to Geometry. Problems. Index. 52 illus. 556pp. Vol. 2, Part I: Functions of a Complex Variable, Conformal Representations, Doubly Periodic Functions, Natural Boundaries, etc. Problems. Index. 38 illus. 269pp. Vol. 2, Part 2: Differential Equations, Cauchy-Lipschitz Method, Non-linear Differential Equations, Simultaneous Equations, etc. Problems. Index. 308pp. 5⅜ x 8.

Vol. 1 S554 Paperbound **$2.50**
Vol. 2 part 1 S555 Paperbound **$1.85**
Vol. 2 part 2 S556 Paperbound **$1.85**
3 vol. set **$6.20**

MODERN THEORIES OF INTEGRATION, H. Kestelman. Connected and concrete coverage, with *fully-worked-out proofs for every step. Ranges from elementary definitions through theory* of aggregates, sets of points, Riemann and Lebesgue integration, and much more. This new revised and enlarged edition contains a new chapter on Riemann-Stieltjes integration, as well as a supplementary section of 186 exercises. Ideal for the mathematician, student, teacher, or self-studier. Index of Definitions and Symbols. General Index. Bibliography. x + 310pp. 5⅝ x 8⅜. S572 Paperbound **$2.25**

THEORY OF MAXIMA AND MINIMA, H. Hancock. Fullest treatment ever written; only work in English with extended discussion of maxima and minima for functions of 1, 2, or n variables, problems with subsidiary constraints, and relevant quadratic forms. Detailed proof of each important theorem. Covers the Scheeffer and von Dantscher theories, homogeneous quadratic forms, reversion of series, fallacious establishment of maxima and minima, etc. Unsurpassed treatise for advanced students of calculus, mathematicians, economists, statisticians. Index. 24 diagrams. 39 problems, many examples. 193pp. 5⅜ x 8. S665 Paperbound **$1.50**

AN ELEMENTARY TREATISE ON ELLIPTIC FUNCTIONS, A. Cayley. *Still the fullest and clearest* text on the theories of Jacobi and Legendre for the advanced student (and an excellent supplement for the beginner). A masterpiece of exposition by the great 19th century British mathematician (creator of the theory of matrices and abstract geometry), it covers the addition-theory, Landen's theorem, the 3 kinds of elliptic integrals, transformations, the q-functions, reduction of a differential expression, and much more. Index. xii + 386pp. 5⅜ x 8. S728 Paperbound **$2.00**

THE APPLICATIONS OF ELLIPTIC FUNCTIONS, A. G. Greenhill. Modern books forego detail for sake of brevity—this book offers complete exposition necessary for proper understanding, use of elliptic integrals. Formulas developed from definite physical, geometric problems; examples representative enough to offer basic information in widely useable form. Elliptic integrals, addition theorem, algebraical form of addition theorem, elliptic integrals of 2nd, 3rd kind, double periodicity, resolution into factors, series, transformation, etc. Introduction. Index. 25 illus. xi + 357pp. 5⅜ x 8. S603 Paperbound **$1.75**

THE THEORY OF FUNCTIONS OF REAL VARIABLES, James Pierpont. A 2-volume authoritative exposition, by one of the foremost mathematicians of his time. Each theorem stated with all conditions, then followed by proof. No need to go through complicated reasoning to discover conditions added without specific mention. Includes a particularly complete, rigorous presentation of theory of measure; and Pierpont's own work on a theory of Lebesgue integrals, and treatment of area of a curved surface. Partial contents, Vol. 1: rational numbers, exponentials, logarithms, point aggregates, maxima, minima, proper integrals, improper integrals, multiple proper integrals, continuity, discontinuity, indeterminate forms. Vol. 2: point sets, proper integrals, series, power series, aggregates, ordinal numbers, discontinuous functions, sub-, infra-uniform convergence, much more. Index. 95 illustrations. 1229pp. 5⅜ x 8. S558-9, 2 volume set, paperbound **$5.20**

ELEMENTS OF THE THEORY OF REAL FUNCTIONS, J. E. Littlewood. Based on lectures given at Trinity College, Cambridge, this book has proved to be extremely successful in introducing graduate students to the modern theory of functions. It offers a full and concise coverage of classes and cardinal numbers, well-ordered series, other types of series, and elements of the theory of sets of points. 3rd revised edition. vii + 71pp. 5⅜ x 8.
S171 Clothbound **$2.85**
S172 Paperbound **$1.25**

TRANSCENDENTAL AND ALGEBRAIC NUMBERS, A. O. Gelfond. First English translation of work by leading Soviet mathematician. Thue-Siegel theorem, its p-adic analogue, on approximation of algebraic numbers by numbers in fixed algebraic field; Hermite-Lindemann theorem on transcendency of Bessel functions, solutions of other differential equations; Gelfond-Schneider theorem on transcendency of alpha to power beta; Schneider's work on elliptic functions, with method developed by Gelfond. Translated by L. F. Boron. Index. Bibliography. 200pp. 5⅜ x 8.
S615 Paperbound **$1.75**

ELLIPTIC INTEGRALS, H. Hancock. Invaluable in work involving differential equations containing cubics or quartics under the root sign, where elementary calculus methods are inadequate. Practical solutions to problems that occur in mathematics, engineering, physics: differential equations requiring integration of Lamé's, Briot's, or Bouquet's equations; determination of arc of ellipse, hyperbola, lemniscate; solutions of problems in elastica; motion of a projectile under resistance varying as the cube of the velocity; pendulums; many others. Exposition is in accordance with Legendre-Jacobi theory and includes rigorous discussion of Legendre transformations. 20 figures. 5 place table. Index. 104pp. 5⅛ x 8.
S484 Paperbound **$1.25**

LECTURES ON THE THEORY OF ELLIPTIC FUNCTIONS, H. Hancock. Reissue of the only book in English with so extensive a coverage, especially of Abel, Jacobi, Legendre, Weierstrasse, Hermite, Liouville, and Riemann. Unusual fullness of treatment, plus applications as well as theory, in discussing elliptic function (the universe of elliptic integrals originating in works of Abel and Jacobi), their existence, and ultimate meaning. Use is made of Riemann to provide the most general theory. 40 page table of formulas. 76 figures. xxiii + 498pp.
S483 Paperbound **$2.55**

THE THEORY AND FUNCTIONS OF A REAL VARIABLE AND THE THEORY OF FOURIER'S SERIES, E. W. Hobson. One of the best introductions to set theory and various aspects of functions and Fourier's series. Requires only a good background in calculus. Provides an exhaustive coverage of: metric and descriptive properties of sets of points; transfinite numbers and order types; functions of a real variable; the Riemann and Lebesgue integrals; sequences and series of numbers; power-series; functions representable by series sequences of continuous functions; trigonometrical series; representation of functions by Fourier's series; complete exposition (200pp.) on set theory; and much more. "The best possible guide," Nature. Vol. I: 88 detailed examples, 10 figures. Index. xv + 736pp. Vol. II: 117 detailed examples, 13 figures. Index. x + 780pp. 6⅛ x 9¼.
Vol. I: S387 Paperbound **$3.50**
Vol. II: S388 Paperbound **$3.00**

ALMOST PERIODIC FUNCTIONS, A. S. Besicovitch. This unique and important summary by a well-known mathematician covers in detail the two stages of development in Bohr's theory of almost periodic functions: (1) as a generalization of pure periodicity, with results and proofs; (2) the work done by Stepanoff, Wiener, Weyl, and Bohr in generalizing the theory. Bibliography. xi + 180pp. 5⅜ x 8.
S18 Paperbound **$1.75**

THE ANALYTICAL THEORY OF HEAT, Joseph Fourier. This book, which revolutionized mathematical physics, is listed in the Great Books program, and many other listings of great books. It has been used with profit by generations of mathematicians and physicists who are interested in either heat or in the application of the Fourier integral. Covers cause and reflection of rays of heat, radiant heating, heating of closed spaces, use of trigonometric series in the theory of heat, Fourier integral, etc. Translated by Alexander Freeman. 20 figures. xxii + 466pp. 5⅜ x 8.
S93 Paperbound **$2.50**

AN INTRODUCTION TO FOURIER METHODS AND THE LAPLACE TRANSFORMATION, Philip Franklin. Concentrates upon essentials, enabling the reader with only a working knowledge of calculus to gain an understanding of Fourier methods in a broad sense, suitable for most applications. This work covers complex qualities with methods of computing elementary functions for complex values of the argument and finding approximations by the use of charts; Fourier series and integrals with half-range and complex Fourier series; harmonic analysis; Fourier and Laplace transformations, etc.; partial differential equations with applications to transmission of electricity; etc. The methods developed are related to physical problems of heat flow, vibrations, electrical transmission, electromagnetic radiation, etc. 828 problems with answers. Formerly entitled "Fourier Methods." Bibliography. Index. x + 289pp. 5⅜ x 8.
S452 Paperbound **$2.00**

THE FOURIER INTEGRAL AND CERTAIN OF ITS APPLICATIONS, Norbert Wiener. The only book-length study of the Fourier integral as link between pure and applied math. An expansion of lectures given at Cambridge. Partial contents: Plancherel's theorem, general Tauberian theorem, special Tauberian theorems, generalized harmonic analysis. Bibliography. viii + 201pp. 5⅜ x 8.
S272 Paperbound **$1.50**

Differential equations, ordinary and partial; integral equations

INTRODUCTION TO THE DIFFERENTIAL EQUATIONS OF PHYSICS, L. Hopf. Especially valuable to the engineer with no math beyond elementary calculus. Emphasizing intuitive rather than formal aspects of concepts, the author covers an extensive territory. Partial contents: Law of causality, energy theorem, damped oscillations, coupling by friction, cylindrical and spherical coordinates, heat source, etc. Index. 48 figures. 160pp. 5⅜ x 8.
S120 Paperbound **$1.35**

INTRODUCTION TO THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS, E. G. Poole. Authoritative discussions of important topics, with methods of solution more detailed than usual, for students with background of elementary course in differential equations. Studies existence theorems, linearly independent solutions; equations with constant coefficients; with uniform analytic coefficients; regular singularities; the hypergeometric equation; conformal representation; etc. Exercises. Index. 210pp. 5⅜ x 8. S629 Paperbound **$1.65**

DIFFERENTIAL EQUATIONS FOR ENGINEERS, P. Franklin. Outgrowth of a course given 10 years at M. I. T. Makes most useful branch of pure math accessible for practical work. Theoretical basis of D.E.'s; solution of ordinary D.E.'s and partial derivatives arising from heat flow, steady-state temperature of a plate, wave equations; analytic functions; convergence of Fourier Series. 400 problems on electricity, vibratory systems, other topics. Formerly "Differential Equations for Electrical Engineers." Index 41 illus. 307pp. 5⅜ x 8.
S601 Paperbound **$1.65**

DIFFERENTIAL EQUATIONS, F. R. Moulton. A detailed, rigorous exposition of all the nonelementary processes of solving ordinary differential equations. Several chapters devoted to the treatment of practical problems, especially those of a physical nature, which are far more advanced than problems usually given as illustrations. Includes analytic differential equations; variations of a parameter; integrals of differential equations; analytic implicit functions; problems of elliptic motion; sine-amplitude functions; deviation of formal bodies; Cauchy-Lipschitz process; linear differential equations with periodic coefficients; differential equations in infinitely many variations; much more. Historical notes. 10 figures. 222 problems. Index. xv + 395pp. 5⅜ x 8. S451 Paperbound **$2.00**

DIFFERENTIAL AND INTEGRAL EQUATIONS OF MECHANICS AND PHYSICS (DIE DIFFERENTIAL- UND INTEGRALGLEICHUNGEN DER MECHANIK UND PHYSIK), edited by P. Frank and R. von Mises. Most comprehensive and authoritative work on the mathematics of mathematical physics available today in the United States: the standard, definitive reference for teachers, physicists, engineers, and mathematicians—now published (in the original German) at a relatively inexpensive price for the first time! Every chapter in this 2,000-page set is by an expert in his field: Carathéodory, Courant, Frank, Mises, and a dozen others. Vol I, on mathematics, gives concise but complete coverages of advanced calculus, differential equations, integral equations, and potential, and partial differential equations. Index. xxiii + 916pp. Vol. II (physics): classical mechanics, optics, continuous mechanics, heat conduction and diffusion, the stationary and quasi-stationary electromagnetic field, electromagnetic oscillations, and wave mechanics. Index. xxiv + 1106pp. Two volume set. Each volume available separately. 5⅝ x 8⅜. S787 Vol I Clothbound **$7.50**
S788 Vol II Clothbound **$7.50**
The set **$15.00**

LECTURES ON CAUCHY'S PROBLEM, J. Hadamard. Based on lectures given at Columbia, Rome, this discusses work of Riemann, Kirchhoff, Volterra, and the author's own research on the hyperbolic case in linear partial differential equations. It extends spherical and cylindrical waves to apply to all (normal) hyperbolic equations. Partial contents: Cauchy's problem, fundamental formula, equations with odd number, with even number of independent variables; method of descent. 32 figures. Index. iii + 316pp. 5⅜ x 8. S105 Paperbound **$1.75**

THEORY OF DIFFERENTIAL EQUATIONS, A. R. Forsyth. Out of print for over a decade, the complete 6 volumes (now bound as 3) of this monumental work represent the most comprehensive treatment of differential equations ever written. Historical presentation includes in 2500 pages every substantial development. Vol. 1, 2: EXACT EQUATIONS, PFAFF'S PROBLEM; ORDINARY EQUATIONS, NOT LINEAR: methods of Grassmann, Clebsch, Lie, Darboux; Cauchy's theorem; branch points; etc. Vol. 3, 4: ORDINARY EQUATIONS, NOT LINEAR; ORDINARY LINEAR EQUATIONS: Zeta Fuchsian functions, general theorems on algebraic integrals, Brun's theorem, equations with uniform periodic coffiecients, etc. Vol. 4, 5: PARTIAL DIFFERENTIAL EQUATIONS: 2 existence-theorems, equations of theoretical dynamics, Laplace transformations, general transformation of equations of the 2nd order, much more. Indexes. Total of 2766pp. 5⅜ x 8. S576-7-8 Clothbound: the set **$15.00**

PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS, A. G. Webster. A keystone work in the library of every mature physicist, engineer, researcher. Valuable sections on elasticity, compression theory, potential theory, theory of sound, heat conduction, wave propagation, vibration theory. Contents include: deduction of differential equations, vibrations, normal functions, Fourier's series, Cauchy's method, boundary problems, method of Riemann-Volterra. Spherical, cylindrical, ellipsoidal harmonics, applications, etc. 97 figures. vii + 440pp. 5⅜ x 8. S263 Paperbound **$2.25**

ORDINARY DIFFERENTIAL EQUATIONS, E. L. Ince. A most compendious analysis in real and complex domains. Existence and nature of solutions, continuous transformation groups, solutions in an infinite form, definite integrals, algebraic theory, Sturmian theory, boundary problems, existence theorems, 1st order, higher order, etc. "Deserves the highest praise, a notable addition to mathematical literature," BULLETIN, AM. MATH. SOC. Historical appendix. Bibliography. 18 figures. viii + 558pp. 5⅜ x 8. S349 Paperbound **$2.75**

INTRODUCTION TO NONLINEAR DIFFERENTIAL AND INTEGRAL EQUATIONS, Harold T. Davis. A thorough introduction to this important area, of increasing interest to mathematicians and scientists. First published by the United States Atomic Energy Commission, it includes chapters on the differential equation of the first order, the Riccati equation (as a bridge between linear and nonlinear equations), existence theorems, second order equations, elliptic integrals, elliptic functions, and theta functions, second order differential equations of polynomial class, continuous analytic continuation, the phase plane and its phenomena, nonlinear mechanics, the calculus of variations, etc. Appendices on Painlevé transcendents and Van der Pol and Volterra equations. Bibliography of 350 items. 137 problems. Index. xv + 566pp. 5⅜ x 8½. S971 Paperbound **$2.00**

THEORY OF FUNCTIONALS AND OF INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS, Vito Volterra. Unabridged republication of the only English translation. An exposition of the general theory of the functions depending on a continuous set of values of another function, based on the author's fundamental notion of the transition from a finite number of variables to a continually infinite number. Though dealing primarily with integral equations, much material on calculus of variations is included. The work makes no assumption of previous knowledge on the part of the reader. It begins with fundamental material and proceeds to Generalization of Analytic Functions, Integro-Differential Equations, Functional Derivative Equations, Applications, Other Directions of Theory of Functionals, etc. New introduction by G. C. Evans. Bibliography and criticism of Volterra's work by E. Whittaker. Bibliography. Index of authors cited. Index of subjects. xxxx + 226pp. 5⅜ x 8. S502 Paperbound **$1.75**

LINEAR INTEGRAL EQUATIONS, W. V. Lovitt. Systematic survey of general theory, with some application to differential equations, calculus of variations, problems of math, physics. Partial contents: integral equation of 2nd kind by successive substitutions; Fredholm's equation as ratio of 2 integral series in lambda, applications of the Fredholm theory, Hilbert-Schmidt theory of symmetric kernels, application, etc. Neumann, Dirichlet, vibratory problems. Index. ix + 253pp. 5⅜ x 8. S176 Paperbound **$2.00**

Foundations of mathematics

THE CONTINUUM AND OTHER TYPES OF SERIAL ORDER, E. V. Huntington. This famous book gives a systematic elementary account of the modern theory of the continuum as a type of serial order. Based on the Cantor-Dedekind ordinal theory, which requires no technical knowledge of higher mathematics, it offers an easily followed analysis of ordered classes, discrete and dense series, continuous series, Cantor's transfinite numbers. 2nd edition. Index. viii + 82pp. 5⅜ x 8. S130 Paperbound **$1.00**

CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS, Georg Cantor. These papers founded a new branch of mathematics. The famous articles of 1895-7 are translated, with an 82-page introduction by P. E. B. Jourdain dealing with Cantor, the background of his discoveries, their results, future possibilities. Bibliography. Index. Notes. ix + 211 pp. 5⅜ x 8. S45 Paperbound **$1.35**

ELEMENTARY MATHEMATICS FROM AN ADVANCED STANDPOINT, Felix Klein.

This classic text is an outgrowth of Klein's famous integration and survey course at Göttingen. Using one field of mathematics to interpret, adjust, illuminate another, it covers basic topics in each area, illustrating its discussion with extensive analysis. It is especially valuable in considering areas of modern mathematics. "Makes the reader feel the inspiration of . . . a great mathematician, inspiring teacher . . . with deep insight into the foundations and interrelations," BULLETIN, AMERICAN MATHEMATICAL SOCIETY.

Vol. 1. ARITHMETIC, ALGEBRA, ANALYSIS. Introducing the concept of function immediately, it enlivens abstract discussion with graphical and geometrically perceptual methods. Partial contents: natural numbers, extension of the notion of number, special properties, complex numbers. Real equations with real unknowns, complex quantities. Logarithmic, exponential functions, goniometric functions, infinitesimal calculus. Transcendence of e and pi, theory of assemblages. Index. 125 figures. ix + 274pp . 5⅜ x 8. S150 Paperbound **$1.85**

Vol. 2. GEOMETRY. A comprehensive view which accompanies the space perception inherent in geometry with analytic formulas which facilitate precise formulation. Partial contents: Simplest geometric manifolds: line segment, Grassmann determinant principles, classification of configurations of space, derivative manifolds. Geometric transformations: affine transformations, projective, higher point transformations, theory of the imaginary. Systematic discussion of geometry and its foundations. Indexes. 141 illustrations. ix + 214pp. 5⅜ x 8. S151 Paperbound **$1.75**

ESSAYS ON THE THEORY OF NUMBERS: 1. CONTINUITY AND IRRATIONAL NUMBERS; 2. THE NATURE AND MEANING OF NUMBERS, Richard Dedekind. The two most important essays on the logical foundations of the number system by the famous German mathematician. The first provides a purely arithmetic and perfectly rigorous foundation for irrational numbers and thereby a rigorous meaning to continuity in analysis. The second essay is an attempt to give a logical basis for transfinite numbers and properties of the natural numbers. Discusses the logical validity of mathematical induction. Authorized English translations by W. W. Deman of "Stetigkeit und irrationale Zahlen" and "Was sind und was sollen die Zahlen?" vii + 115pp. 5⅜ x 8. T1010 Paperbound **$1.00**

Geometry

THE FOUNDATIONS OF EUCLIDEAN GEOMETRY, H. G. Forder. The first rigorous account of Euclidean geometry, establishing propositions without recourse to empiricism, and without multiplying hypotheses. Corrects many traditional weaknesses of Euclidean proofs, and investigates the problems imposed on the axiom system by the discoveries of Bolyai and Lobachevsky. Some topics discussed are Classes and Relations; Axioms for Magnitudes; Congruence and Similarity; Algebra of Points; Hessenberg's Theorem; Continuity; Existence of Parallels; Reflections; Rotations; Isometries; etc. Invaluable for the light it throws on foundations of math. Lists: Axioms employed, Symbols, Constructions. 295pp. 5⅜ x 8. S481 Paperbound **$2.00**

ADVANCED EUCLIDEAN GEOMETRY, R. A. Johnson. For years the standard textbook on advanced Euclidean geometry, requires only high school geometry and trigonometry. Explores in unusual detail and gives proofs of hundreds of relatively recent theorems and corollaries, many formerly available only in widely scattered journals. Covers tangent circles, the theorem of Miquel, symmedian point, pedal triangles and circles, the Brocard configuration, and much more. Formerly "Modern Geometry." Index. 107 diagrams. xiii + 319pp. 5⅜ x 8. S669 Paperbound **$1.65**

HIGHER GEOMETRY: AN INTRODUCTION TO ADVANCED METHODS IN ANALYTIC GEOMETRY, F. S. Woods. Exceptionally thorough study of concepts and methods of advanced algebraic geometry (as distinguished from differential geometry). Exhaustive treatment of 1-, 2-, 3-, and 4-dimensional coordinate systems, leading to n-dimensional geometry in an abstract sense. Covers projectivity, tetracyclical coordinates, contact transformation, pentaspherical coordinates, much more. Based on M.I.T. lectures, requires sound preparation in analytic geometry and some knowledge of determinants. Index. Over 350 exercises. References. 60 figures. x + 423pp. 5⅜ x 8. S737 Paperbound **$2.00**

CONTEMPORARY GEOMETRY, André Delachet. Translated by Howard G. Bergmann. The recent developments in geometry covered in uncomplicated fashion. Clear discussions of modern thinking about the theory of groups, the concept of abstract geometry, projective geometry, algebraic geometry, vector spaces, new kinds of metric spaces, developments in differential geometry, etc. A large part of the book is devoted to problems, developments, and applications of topology. For advanced undergraduates and graduate students as well as mathematicians in other fields who want a brief introduction to current work in geometry. 39 figures. Index. xix + 94pp. 5⅜ x 8½. S988 Paperbound **$1.00**

ELEMENTS OF PROJECTIVE GEOMETRY, L. Cremona. Outstanding complete treatment of projective geometry by one of the foremost 19th century geometers. Detailed proofs of all fundamental principles, stress placed on the constructive aspects. Covers homology, law of duality, anharmonic ratios, theorems of Pascal and Brianchon, foci, polar reciprocal figures, etc. Only ordinary geometry necessary to understand this honored classic. Index. Over 150 fully worked out examples and problems. 252 diagrams. xx + 302pp. 5⅜ x 8. S668 Paperbound **$1.75**

AN INTRODUCTION TO PROJECTIVE GEOMETRY, R. M. Winger. One of the best introductory texts to an important area in modern mathematics. Contains full development of elementary concepts often omitted in other books. Employing the analytic method to capitalize on the student's collegiate training in algebra, analytic geometry and calculus, the author deals with such topics as Essential Constants, Duality, The Line at Infinity, Projective Properties and Double Ratio, Projective Coordinates, The Conic, Collineations and Involutions in One Dimension, Binary Forms, Algebraic Invariants, Analytic Treatment of the Conic, Collineations in the Plane, Cubic Involutions and the Rational Cubic Curve, and a clear discussion of Non-Euclidean Geometry. For senior-college students and graduates. "An excellent textbook . . . very clearly written . . . propositions stated concisely," A. Emch, Am. Math. Monthly. Corrected reprinting. 928 problems. Index. 116 figures. xii + 443pp. 5⅜ x 8. S949 Paperbound **$2.00**

ALGEBRAIC CURVES, Robert J. Walker, Professor of Mathematics, Cornell University. Fine introduction to algebraic geometry. Presents some of the recently developed algebraic methods of handling problems in algebraic geometry, shows how these methods are related to the older analytic and geometric problems, and applies them to those same geometric problems. Limited to the theory of curves, concentrating on birational transformations. Contents: Algebraic Preliminaries, Projective Spaces, Plane Algebraic Curves, Formal Power Series, Transformations of a Curve, Linear Series. 25 illustrations. Numerous exercises at ends of sections. Index. x + 201pp. 5⅜ x 8½. S336 Paperbound **$2.00**

ELEMENTARY CONCEPTS OF TOPOLOGY, P. Alexandroff. First English translation of the famous brief introduction to topology for the beginner or for the mathematician not undertaking extensive study. This unusually useful intuitive approach deals primarily with the concepts of complex, cycle, and homology, and is wholly consistent with current investigations. Ranges from basic concepts of set-theoretic topology to the concept of Betti groups. "Glowing example of harmony between intuition and thought," David Hilbert. Translated by A. E. Farley. Introduction by D. Hilbert. Index. 25 figures. 73pp. 5⅜ x 8. S747 Paperbound **$1.00**

Number theory

INTRODUCTION TO THE THEORY OF NUMBERS, L. E. Dickson. Thorough, comprehensive approach with adequate coverage of classical literature, an introductory volume beginners can follow. Chapters on divisibility, congruences, quadratic residues & reciprocity, Diophantine equations, etc. Full treatment of binary quadratic forms without usual restriction to integral coefficients. Covers infinitude of primes, least residues, Fermat's theorem, Euler's phi function, Legendre's symbol, Gauss's lemma, automorphs, reduced forms, recent theorems of Thue & Siegel, many more. Much material not readily available elsewhere. 239 problems. Index. I figure. viii + 183pp. 5⅜ x 8. S342 Paperbound **$1.75**

ELEMENTS OF NUMBER THEORY, I. M. Vinogradov. Detailed 1st course for persons without advanced mathematics; 95% of this book can be understood by readers who have gone no farther than high school algebra. Partial contents: divisibility theory, important number theoretical functions, congruences, primitive roots and indices, etc. Solutions to both problems and exercises. Tables of primes, indices, etc. Covers almost every essential formula in elementary number theory! Translated from Russian. 233 problems, 104 exercises. viii + 227pp. 5⅜ x 8. S259 Paperbound **$1.75**

THEORY OF NUMBERS and DIOPHANTINE ANALYSIS, R. D. Carmichael. These two complete works in one volume form one of the most lucid introductions to number theory, requiring only a firm foundation in high school mathematics. "Theory of Numbers," partial contents: Eratosthenes' sieve, Euclid's fundamental theorem, G.C.F. and L.C.M. of two or more integers, linear congruences, etc "Diophantine Analysis": rational triangles, Pythagorean triangles, equations of third, fourth, higher degrees, method of functional equations, much more. "Theory of Numbers": 76 problems. Index. 94pp. "Diophantine Analysis": 222 problems. Index. 118pp. 5⅜ x 8. S529 Paperbound **$1.35**

Numerical analysis, tables

MATHEMATICAL TABLES AND FORMULAS, Compiled by Robert D. Carmichael and Edwin R. Smith. Valuable collection for students, etc. Contains all tables necessary in college algebra and trigonometry, such as five-place common logarithms, logarithmic sines and tangents of small angles, logarithmic trigonometric functions, natural trigonometric functions, four-place antilogarithms, tables for changing from sexagesimal to circular and from circular to sexagesimal measure of angles, etc. Also many tables and formulas not ordinarily accessible, including powers, roots, and reciprocals, exponential and hyperbolic functions, ten-place logarithms of prime numbers, and formulas and theorems from analytical and elementary geometry and from calculus. Explanatory introduction. viii + 269pp. 5⅜ x 8½. S111 Paperbound **$1.25**

MATHEMATICAL TABLES, H. B. Dwight. Unique for its coverage in one volume of almost every function of importance in applied mathematics, engineering, and the physical sciences. Three extremely fine tables of the three trig functions and their inverse functions to thousandths of radians; natural and common logarithms; squares, cubes; hyperbolic functions and the inverse hyperbolic functions; $(a^2 + b^2)$ exp. ½a; complete elliptic integrals of the 1st and 2nd kind; sine and cosine integrals; exponential integrals Ei(x) and Ei(—x); binomial coefficients; factorials to 250; surface zonal harmonics and first derivatives; Bernoulli and Euler numbers and their logs to base of 10; Gamma function; normal probability integral; over 60 pages of Bessel functions; the Riemann Zeta function. Each table with formulae generally used, sources of more extensive tables, interpolation data, etc. Over half have columns of differences, to facilitate interpolation. Introduction. Index. viii + 231pp. 5⅜ x 8. S445 Paperbound **$2.00**

TABLES OF FUNCTIONS WITH FORMULAE AND CURVES, E. Jahnke & F. Emde. The world's most comprehensive 1-volume English-text collection of tables, formulae, curves of transcendent functions. 4th corrected edition, new 76-page section giving tables, formulae for elementary functions—not in other English editions. Partial contents: sine, cosine, logarithmic integral; factorial function; error integral; theta functions; elliptic integrals, functions; Legendre, Bessel, Riemann, Mathieu, hypergeometric functions, etc. Supplementary books. Bibliography. Indexed. "Out of the way functions for which we know no other source," SCIENTIFIC COMPUTING SERVICE, Ltd. 212 figures. 400pp. 5⅜ x 8. S133 Paperbound **$2.00**

MATHEMATICS, HISTORIES AND CLASSICS

HISTORY OF MATHEMATICS, D. E. Smith. Most comprehensive non-technical history of math in English. Discusses lives and works of over a thousand major and minor figures, with footnotes supplying technical information outside the book's scheme, and indicating disputed matters. Vol I: A chronological examination, from primitive concepts through Egypt, Babylonia, Greece, the Orient, Rome, the Middle Ages, the Renaissance, and up to 1900. Vol 2: The development of ideas in specific fields and problems, up through elementary calculus. Two volumes, total of 510 illustrations, 1355pp. 5⅜ x 8. Set boxed in attractive container. T429, 430 Paperbound, the set **$6.00**

A SHORT ACCOUNT OF THE HISTORY OF MATHEMATICS, W. W. R. Ball. Most readable non-technical history of mathematics treats lives, discoveries of every important figure from Egyptian, Phoenician mathematicians to late 19th century. Discusses schools of Ionia, Pythagoras, Athens, Cyzicus, Alexandria, Byzantium, systems of numeration; primitive arithmetic; Middle Ages, Renaissance, including Arabs, Bacon, Regiomontanus, Tartaglia, Cardan, Stevinus, Galileo, Kepler; modern mathematics of Descartes, Pascal, Wallis, Huygens, Newton, Leibnitz, d'Alembert, Euler, Lambert, Laplace, Legendre, Gauss, Hermite, Weierstrass, scores more. Index. 25 figures. 546pp. 5⅜ x 8. S630 Paperbound **$2.25**

A HISTORY OF GEOMETRICAL METHODS, J. L. Coolidge. Full, authoritative history of the techniques which men have employed in dealing with geometric questions . . . from ancient times to the modern development of projective geometry. Critical analyses of the original works. Contents: Synthetic Geometry—the early beginnings, Greek mathematics, non-Euclidean geometries, projective and descriptive geometry; Algebraic Geometry—extension of the system of linear coordinates, other systems of point coordinates, enumerative and birational geometry, etc.; and Differential Geometry—intrinsic geometry and moving axes, Gauss and the classical theory of surfaces, and projective and absolute differential geometry. The work of scores of geometers analyzed: Pythagoras, Archimedes, Newton, Descartes, Leibniz, Lobachevski, Riemann, Hilbert, Bernoulli, Schubert, Grassman, Klein, Cauchy, and many, many others. Extensive (24-page) bibliography. Index. 13 figures. xviii + 451pp. 5⅜ x 8½. S1006 Paperbound **$2.25**

THE MATHEMATICS OF GREAT AMATEURS, Julian Lowell Coolidge. Enlightening, often surprising, accounts of what can result from a non-professional preoccupation with mathematics. Chapters on Plato, Omar Khayyam and his work with cubic equations, Piero della Francesca, Albrecht Dürer, as the true discoverer of descriptive geometry, Leonardo da Vinci and his varied mathematical interests, John Napiei, Baron of Merchiston, inventor of logarithms, Pascal, Diderot, l'Hospital, and seven others known primarily for contributions in other fields. Bibliography. 56 figures. viii + 211pp. 5⅜ x 8½. S1009 Paperbound **$1.50**

ART AND GEOMETRY, Wm. M. Ivins, Jr. A controversial study which propounds the view that the ideas of Greek philosophy and culture served not to stimulate, but to stifle the development of Western thought. Through an examination of Greek art and geometrical inquiries and Renaissance experiments, this book offers a concise history of the evolution of mathematical perspective and projective geometry. Discusses the work of Alberti, Dürer, Pelerin, Nicholas of Cusa, Kepler, Desargues, etc. in a wholly readable text of interest to the art historian, philosopher, mathematician, historian of science, and others. x + 113pp. 5⅜ x 8⅜. T941 Paperbound **$1.25**

A SOURCE BOOK IN MATHEMATICS, D. E. Smith. Great discoveries in math, from Renaissance to end of 19th century, in English translation. Read announcements by Dedekind, Gauss, Delamain, Pascal, Fermat, Newton, Abel, Lobachevsky, Bolyai, Riemann, De Moivre, Legendre, Laplace, others of discoveries about imaginary numbers, number congruence, slide rule, equations, symbolism, cubic algebraic equations, non-Euclidean forms of geometry, calculus, function theory, quaternions, etc. Succinct selections from 125 different treatises, articles, most unavailable elsewhere in English. Each article preceded by biographical, historical introduction. Vol. I: Fields of Number, Algebra. Index. 32 illus. 338pp. 5⅜ x 8. Vol. II: Fields of Geometry, Probability, Calculus, Functions, Quaternions. 83 illus. 432pp. 5⅜ x 8.
Vol. 1: S552 Paperbound **$2.00**
Vol. 2: S553 Paperbound **$2.00**
2 vol. set, **$4.00**

A COLLECTION OF MODERN MATHEMATICAL CLASSICS, edited by R. Bellman. 13 classic papers, complete in their original languages, by Hermite, Hardy and Littlewood, Tchebychef, Fejér, Fredholm, Fuchs, Hurwitz, Weyl, van der Pol, Birkhoff, Kellogg, von Neumann, and Hilbert. Each of these papers, collected here for the first time, triggered a burst of mathematical activity, providing useful new generalizations or stimulating fresh investigations. Topics discussed include classical analysis, periodic and almost periodic functions, analysis and number theory, integral equations, theory of approximation, non-linear differential equations, and functional analysis. Brief introductions and bibliographies to each paper. xii + 292pp. 6 x 9. S730 Paperbound **$2.00**

THE WORKS OF ARCHIMEDES, edited by **T. L. Heath.** All the known works of the great Greek mathematician are contained in this one volume, including the recently discovered Method of Archimedes. Contains: On Sphere & Cylinder, Measurement of a Circle, Spirals, Conoids, Spheroids, etc. This is the definitive edition of the greatest mathematical intellect of the ancient world. 186-page study by Heath discusses Archimedes and the history of Greek mathematics. Bibliography. 563pp. 5⅜ x 8. S9 Paperbound **$2.45**

THE THIRTEEN BOOKS OF EUCLID'S ELEMENTS, edited by **Sir Thomas Heath.** Definitive edition of one of the very greatest classics of Western world. Complete English translation of Heiberg text, together with spurious Book XIV. Detailed 150-page introduction discussing aspects of Greek and Medieval mathematics. Euclid, texts, commentators, etc. Paralleling the text is an elaborate critical apparatus analyzing each definition, proposition, postulate, covering textual matters, mathematical analysis, commentators of all times, refutations, supports, extrapolations, etc. This is the full Euclid. Unabridged reproduction of Cambridge U. 2nd edition. 3 volumes. Total of 995 figures, 1426pp. 5⅜ x 8.
S88,89,90, 3 volume set, paperbound **$7.50**

A CONCISE HISTORY OF MATHEMATICS, D. Struik. Lucid study of development of mathematical ideas, techniques from Ancient Near East, Greece, Islamic science, Middle Ages, Renaissance, modern times. Important mathematicians are described in detail. Treatment is not anecdotal, but analytical development of ideas. "Rich in content, thoughtful in interpretation," U.S. QUARTERLY BOOKLIST. Non-technical; no mathematical training needed. Index. 60 illustrations, including Egyptian papyri, Greek mss., portraits of 31 eminent mathematicians. Bibliography. 2nd edition. xix + 299pp. 5⅜ x 8. T255 Paperbound **$1.75**

A HISTORY OF THE CALCULUS, AND ITS CONCEPTUAL DEVELOPMENT, Carl B. Boyer. Provides laymen and mathematicians a detailed history of the development of the calculus, from early beginning in antiquity to final elaboration as mathematical abstractions. Gives a sense of mathematics not as a technique, but as a habit of mind, in the progression of ideas of Zeno, Plato, Pythagoras, Eudoxus, Arabic and Scholastic mathematicians, Newton, Leibnitz, Taylor, Descartes, Euler, Lagrange, Cantor, Weierstrass, and others. This first comprehensive critical history of the calculus was originally titled "The Concepts of the Calculus." Foreword by R. Courant. Preface. 22 figures. 25-page bibliography. Index. v + 364pp. 5⅜ x 8. S509 Paperbound **$2.00**

A MANUAL OF GREEK MATHEMATICS, Sir Thomas L. Heath. A non-technical survey of Greek mathematics addressed to high school and college students and the layman who desires a sense of historical perspective in mathematics. Thorough exposition of early numerical notation and practical calculation, Pythagorean arithmetic and geometry, Thales and the earliest Greek geometrical measurements and theorems, the mathematical theories of Plato, Euclid's "Elements" and his other works (extensive discussion), Aristarchus, Archimedes, Eratosthenes and the measurement of the earth, trigonometry (Hipparchus, Menelaus, Ptolemy), Pappus and Heron of Alexandria, and detailed coverage of minor figures normally omitted from histories of this type. Presented in a refreshingly interesting and readable style. Appendix. 2 Indexes. xvi + 552pp. 5⅜ x 8. S279 Paperbound **$2.25**

THE GEOMETRY OF RENÉ DESCARTES. With this book Descartes founded analytical geometry. Excellent Smith-Latham translation, plus original French text with Descartes' own diagrams. Contains Problems the Construction of Which Requires Only Straight Lines and Circles; On the Nature of Curved Lines; On the Construction of Solid or Supersolid Problems. Notes. Diagrams. 258pp. 5⅜ x 8. S68 Paperbound **$2.00**

A PHILOSOPHICAL ESSAY ON PROBABILITIES, Marquis de Laplace. This famous essay explains without recourse to mathematics the principle of probability, and the application of probability to games of chance, natural philosophy, astronomy, many other fields. Translated from the 6th French edition by F. W. Truscott, F. L. Emory, with new introduction for this edition by E. T. Bell. 204pp. 5⅜ x 8. S166 Paperbound **$1.35**

Prices subject to change without notice.

Dover publishes books on art, music, philosophy, literature, languages, history, social sciences, psychology, handcrafts, orientalia, puzzles and entertainments, chess, pets and gardens, books explaining science, intermediate and higher mathematics, mathematical physics, engineering, biological sciences, earth sciences, classics of science, etc. Write to:

Dept. catrr.
Dover Publications, Inc.
180 Varick Street, N.Y. 14, N.Y.

ELEMENTS OF NUMBER THEORY

BY I. M. VINOGRADOV

This is the first English translation of the revised edition of this important modern book on number theory. Clear and detailed in its exposition, most of it can be understood by readers who have no background in advanced mathematics; only a small part requires a working knowledge of calculus. One of the most valuable characteristics of this book is its stress on learning number theory by means of demonstrations and problems. More than 200 problems, with full solutions, are presented in the text, while more than 100 numerical exercises afford further practice. Some of these exercises deal with estimation of trigonometric sums, and are especially valuable as introductions to more advanced studies.

Partial contents: Divisibility Theory; greatest common divisor, Euclid's algorithm, fundamental properties of fractions, unique factorization theorem, etc. Important Number-Theoretic Functions; the factorization of n!, with various functions, and general properties of multiplicative functions. Congruences; basic properties, reduced residue systems, theorems of Fermat, Euler. Congruences in One Unknown; continued fraction solution of linear congruence, congruences with composite and prime power moduli, Wilson's theorem. Congruences of the Second Degree; Legendre, Jacobi symbols, solution of the congruence $x^2 = a \pmod{m}$. Primitive Roots and Indices; determination of all moduli having primitive roots and corresponding theory of indices.

"A very welcome addition to books on number theory. . . ." *Bulletin, American Mathematical Society.*

1st English translation of the 5th (1949) Russian edition. Translated by S. Kravetz. Prefaces. Tables of Indices. viii + 227pp. $5\frac{3}{8} \times 8$.
S259 Paperbound **$2.00**

A DOVER EDITION DESIGNED FOR YEARS OF USE!

We have made every effort to make this the best book possible. Our paper is opaque, with minimal show-through; it will not discolor or become brittle with age. Pages are sewn in signatures, in the method traditionally used for the best books, and will not drop out, as often happens with paperbacks held together with glue. Books open flat for easy reference. The binding will not crack or split. This is a permanent book.

DOVER

VINOGRADOV ELEMENTS OF NUMBER THEORY DOVER S259