PROBLEMS AND EXERCISES IN INTEGRAL EQUATIONS

MIR PUBLISHERS

М. КРАСНОВ, А. КИСЕЛЕВ,
Г. МАКАРЕНКО

ИНТЕГРАЛЬНЫЕ УРАВНЕНИЯ

Задачи и упражнения

Издательство «Наука»

МОСКВА

M. KRASNOV, A. KISELEV, G. MAKARENKO

PROBLEMS AND EXERCISES IN INTEGRAL EQUATIONS

Translated from the Russian
by George Yankovsky

MIR PUBLISHERS . MOSCOW 1971

UDC 517.948(075.8)=20

REVISED FROM THE 1968 RUSSIAN EDITION

На английском языке

CONTENTS

1. *Measurable sets.* Let E be some set of points of an interval $S=[a, b]$. Denote the complement of E with respect to S by C_E; i.e., by definition C_E consists of points which do not belong to E.

There are a variety of ways in which the points of set E may be included in a finite or countable system of intervals

$$\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$$

We denote by $\sum\alpha$ the sum of the lengths of the intervals $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$. For any system of intervals covering E,

$$\sum\alpha > 0$$

The lower bound of $\sum\alpha$, which depends solely on the set E, is called the *exterior measure* and is denoted $m^* E$. From the definition of an exterior measure it follows that for any $\varepsilon > 0$ there exists a system of intervals $\alpha_1, \alpha_2, \ldots$ $\ldots, \alpha_n, \ldots$ which include all points of the set E such that

$$m^* E \leqslant \sum\alpha < m^* E + \varepsilon$$

The *interior measure* $m_* E$ of the set E is the difference between the length of the interval S and the exterior measure of the complement of the set; i.e.,

$$m_* E = b - a - m^* C_E$$

If the exterior and interior measures of E are equal, then the set E is called *measurable in the sense of Lebesgue* (Lebesgue measurable, or, simply, measurable), while the common value of the measures $m^* E$ and $m_* E$ is called the *Lebesgue measure of E* (or, simply, the measure of E) and is denoted by mE or mes E.

The measure of the interval (a, b) is its length: mes $(a, b) = b - a$. The set ω of points of the interval (a, b) is called a *set of measure zero* if ω can be covered by intervals the sum of whose lengths is arbitrarily small.

2. A function of a real variable $f(x)$ defined on a measurable set E is called *measurable* if for any number A the set $\mathscr{E}(f(x) > A)$, which consists of those points x belonging to the set E for which $f(x) > A$, is Lebesgue measurable.

Note. The requirement of measurability of the set $\mathscr{E}(f(x) > A)$ may be replaced by one of the following three conditions:

(a) the set $\mathscr{E}(f(x) \geqslant A)$ is measurable,
(b) the set $\mathscr{E}(f(x) < A)$ is measurable,
(c) the set $\mathscr{E}(f(x) \leqslant A)$ is measurable.

3. A function $f(x)$, nonnegative on the interval (a, b), is called *summable* on that interval if $\int_a^b f(x)\,dx$ is finite *.

A function $f(x)$ of arbitrary sign will be summable on an interval (a, b) if and only if the function $|f(x)|$ is summable, i.e., if the integral $\int_a^b |f(x)|\,dx$ has a finite value.

In the sequel we shall have to do with the basic interval $I = (a, b)$ (or $I_0 = (0, a)$) and the basic square

$$\Omega\,\{a \leqslant x, t \leqslant b\} \text{ (or } \Omega_0\,\{0 \leqslant x, t \leqslant a\})$$

4. *The* L_2 (a, b) *space*. We say that $f(x)$ is *a quadratically integrable function on* $[a, b]$ if the integral

$$\int_a^b f^2(x)\,dx$$

exists (is finite). The class of all quadratically integrable functions on $[a, b]$ is denoted by L_2 (a, b) or, simply, L_2.

Basic Properties of L_2 Functions

(a) The product of two quadratically integrable functions is an integrable function.

(b) The sum of two L_2 functions is also an L_2 function.

* The integral is to be understood in the sense of Lebesgue, but if the reader is not acquainted with the Lebesgue integral, the integrals may everywhere be understood in the sense of Riemann.

(c) If $f(x) \in L_2$ and λ is an arbitrary real number, then

$$\lambda f(x) \in L_2$$

(d) If $f(x) \in L_2$ and $g(x) \in L_2$, then we have the Bunyakovsky-Schwarz inequality

$$\left(\int_a^b f(x) g(x) \, dx \right)^2 \leqslant \int_a^b f^2(x) \, dx \int_a^b g^2(x) \, dx \tag{1}$$

The *scalar product* of two functions $f(x) \in L_2$ and $g(x) \in L_2$ is, by definition, the number

$$(f, g) = \int_a^b f(x) g(x) \, dx \tag{2}$$

The *norm* of an L_2 function $f(x)$ is the nonnegative number

$$\|f\| = \sqrt{(f, f)} = \sqrt{\int_a^b f^2(x) \, dx} \tag{3}$$

(e) For $f(x)$ and $g(x)$ taken in L_2 we have the triangle inequality

$$\|f + g\| \leqslant \|f\| + \|g\| \tag{4}$$

(f) *Convergence in the mean.* Let the functions $f(x)$ and $f_1(x), f_2(x), \ldots, f_n(x), \ldots$ be quadratically summable on (a, b). If

$$\lim_{n \to \infty} \int_a^b [f_n(x) - f(x)]^2 \, dx = 0$$

then we say that the sequence of functions $f_1(x), f_2(x) \ldots$ *converges in the mean* or, more precisely, in the mean square, to the function $f(x)$.

If a sequence $\{f_n(x)\}$ of L_2 functions converges uniformly to $f(x)$, then $f(x) \in L_2$ and $\{f_n(x)\}$ converges to $f(x)$ in the mean.

We say that a sequence $\{f_n(x)\}$ of functions in L_2 converges *in the mean in itself* if for any number $\varepsilon > 0$ there exists a number $N > 0$ such that

$$\int_a^b [f_n(x) - f_m(x)]^2 \, dx \leqslant \varepsilon$$

for $n > N$ and $m > N$. Sometimes sequences convergent in themselves are called *fundamental* sequences. For a sequence $\{f_n(x)\}$ to converge in the mean to some function, it is necessary and sufficient that this sequence be fundamental. The space L_2 is complete; i.e., any fundamental sequence of functions in L_2 converges to a function which also lies in L_2.

Two functions $f(x)$ and $g(x)$ in $L_2(a, b)$ are called *equivalent* on (a, b) if $f(x) \neq g(x)$ only on a set of measure zero. In this case we say that $f(x) = g(x)$ almost everywhere on (a, b).

5. *The space* $C^{(l)}(a, b)$. The elements of this space are all possible functions defined on the interval $[a, b]$ and having, on this interval, continuous derivatives up to the lth inclusive. The operations of addition of functions and multiplication of functions by a number are defined in the usual manner.

We determine the *norm* of an element $f(x) \in C^{(l)}(a, b)$ from the formula

$$\|f\| = \sum_{k=0}^{l} \max_{a \leqslant x \leqslant b} |f^{(k)}(x)|$$

$f^{(0)}(x)$ being equal to $f(x)$.

Convergence in $C^{(l)}(a, b)$ implies uniform convergence both of a sequence of the functions themselves and of the sequences of their derivatives of order k $(k = 1, 2, \ldots, l)$.

The concepts of a measurable set, a measurable function, a summable function, etc. are extended to the case of spaces of higher dimensionality. For example, the function $F(x, t)$ will be called quadratically summable on $\Omega\{a \leqslant x, t \leqslant b\}$ if the integral

$$\int_a^b \int_a^b F^2(x, t)\,dx\,dt < +\infty$$

In this case, the norm of the function $F(x, t)$ is defined by the equality

$$\|F\| = \sqrt{\int_a^b \int_a^b F^2(x, t)\,dx\,dt}$$

6. A function $f(z)$ of a complex variable z, differentiable at every point of a domain G in the plane of the complex variable z, is called *analytic* (regular) in that domain.

The function $f(z)$ is called *entire* (integral) if it is analytic throughout the plane (with the exception of the point at infinity).

The function $f(z)$ is called *meromorphic* (or fractional) if it can be represented as a quotient of two entire functions:

$$f(z)=\frac{g(z)}{h(z)}, \quad h(z) \not\equiv 0$$

In any bounded domain, the meromorphic function $f(z)$ can have only a finite number of poles.

A point $z=a$ is called an *isolated* singular point of the function $f(z)$ if there is a neighbourhood $0<|z-a|<\delta$ of that point in which $f(z)$ is analytic, while analyticity of the function breaks down at the point $z=a$ itself. The isolated singular point $z=a$ is called a *pole* of the function $f(z)$ if

$$\lim_{x \to a} f(z)=\infty$$

It is assumed that $f(z)$ is single-valued in the neighbourhood of the point $z=a$, $z \neq a$.

For the point $z=a$ to be a pole of the function $f(z)$ it is necessary and sufficient that it be a zero of the function $\varphi(z)=\frac{1}{f(z)}$, i.e., that $\varphi(a)=0$.

The *order* of a pole $z=a$ of the function $f(z)$ is the order of the zero $z=a$ of the function

$$\varphi(z)=\frac{1}{f(z)}$$

7. The *residue* of the function $f(z)$ at the isolated singular point $z=a$ is the number

$$\operatorname*{res}_{z=a} f(z)=\frac{1}{2\pi i}\int_c f(z)\,dz$$

where c is a circle $|z-a|=\rho$ of sufficiently small radius.

If the point $z=a$ is a pole of order n of the function $f(z)$, then

$$\operatorname*{res}_{z=a} f(z)=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$$

For a simple pole $(n=1)$

$$\operatorname*{res}_{z=a} f(z)=\lim_{z\to a}\{(z-a)f(z)\}$$

If $f(z)=\frac{\varphi(z)}{\psi(z)}$, $\varphi(a)\neq 0$, and, at the point $z=a$, $\psi(z)$ has a zero of order one, that is, $\psi(a)=0$, $\psi'(a)\neq 0$, then

$$\operatorname*{res}_{z=a} f(z)=\frac{\varphi(a)}{\psi'(a)}$$

8. *Jordan's lemma. If $f(z)$ is continuous in the domain $|z|\geqslant R_0$, $\operatorname{Im} z\geqslant\alpha$ (α is a fixed real number) and $\lim\limits_{z\to\infty} f(z)=0$, then for any $\lambda>0$*

$$\lim_{R\to\infty}\int_{c_R} e^{i\lambda z} f(z)\,dz=0$$

where c_R is an arc of the circle $|z|=R$ in that domain.

9. A function $f(x)$ is called *locally summable* if it is summable on any bounded set.

Let a complex-valued function $\varphi(t)$ of a real variable t be locally summable, equal to zero for $t<0$, and let it satisfy the condition $|\varphi(t)|<Me^{s_0 t}$ for all t $(M>0,\ s_0\geqslant 0)$. Such functions $\varphi(t)$ will be called *original functions*. The number s_0 is termed the *order of growth* of the function $\varphi(t)$.

The Laplace transform of the function $\varphi(t)$ is the function $\Phi(p)$ of the complex variable $p=s+i\sigma$ defined by the equality

$$\Phi(p)=\int_0^\infty e^{-pt}\varphi(t)\,dt$$

For any original function $\varphi(t)$, the function $\Phi(p)$ is defined in the half-plane $\operatorname{Re} p>s_0$ and is an analytic function in that half-plane. The fact that the function $\Phi(p)$ is the Laplace transform of the function $\varphi(t)$ is written as follows:

$$\varphi(t)\risingdotseq\Phi(p)$$

10. *Inversion theorem. If a function* $\varphi(t)$ *is the original function, and the function* $\Phi(p)$ *serves as its image (transform), then*

$$\varphi(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\Phi(p)\,dp,\ \gamma > s_0 \qquad (*)$$

where the integral is taken along the straight line $\operatorname{Re} p=\gamma$ *parallel to the imaginary axis and is understood in the sense of the principal value:*

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,\Phi(p)\,dp=\lim_{\omega\to\infty}\int_{\gamma-i\omega}^{\gamma+i\omega} e^{pt}\,\Phi(p)\,dp$$

Formula $(*)$ is called the *inversion formula* of the Laplace transformation. If

$$\Phi(p)=\frac{M(p)}{N(p)}$$

where $M(p)$ and $N(p)$ are polynomials in p, and the degree of the polynomial $M(p)$ is less than that of the polynomial $N(p)$, then for $\Phi(p)$ the original function will be

$$\varphi(t)=\sum_{k=1}^{l}\frac{1}{(n_k-1)!}\lim_{p\to a_k}\frac{d^{n_k-1}}{dp^{n_k-1}}\{(p-a_k)^{n_k}\Phi(p)\,e^{pt}\}$$

where a_k are the poles of $\Phi(p)$, n_k are their orders and the sum is taken over all poles of $\Phi(p)$.

When all poles a_k $(k=1,\ 2,\ \ldots,\ l)$ of the function $\Phi(p)=\frac{M(p)}{N(p)}$ are simple,

$$\frac{M(p)}{N(p)}\risingdotseq\sum_{k=1}^{l}\frac{M(a_k)}{N'(a_k)}\,e^{a_k t}=\varphi(t)$$

11. *Product theorem* (convolution theorem). *Let the functions* $f(t)$ *and* $\varphi(t)$ *be original functions, and let*

$$f(t)\risingdotseq F(p),$$
$$\varphi(t)\risingdotseq\Phi(p)$$

Then

$$F(p)\,\Phi(p) \doteqdot \int_0^t f(\tau)\,\varphi(t-\tau)\,d\tau \qquad (5)$$

The integral on the right of (5) is called the *convolution* of the functions $f(t)$ and $\varphi(t)$ and is denoted by the symbol $f(t) * \varphi(t)$.

Thus, a product of transforms is also a transform, namely, the transform of the convolution of the original functions:

$$F(p)\,\Phi(p) \doteqdot f(t) * \varphi(t)$$

12. Let the function $f(x)$ be absolutely integrable over the entire axis $-\infty < x < +\infty$. The function

$$\tilde{f}(\lambda) = \int_{-\infty}^{+\infty} f(x)\,e^{i\lambda x}\,dx$$

is called the *Fourier transform of the function* $f(x)$.

The inversion formula of the Fourier transformation is of the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\lambda)\,e^{-i\lambda x}\,d\lambda$$

In order to attain greater symmetry, the formulas of direct and inverse Fourier transformations are frequently written in the form

$$\tilde{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\,e^{i\lambda x}\,dx,$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(\lambda)\,e^{-i\lambda x}\,d\lambda$$

CHAPTER I

VOLTERRA INTEGRAL EQUATIONS

1. Basic Concepts

The equation

$$\varphi(x)=f(x)+\lambda\int_a^x K(x, t)\varphi(t)\,dt \tag{1}$$

where $f(x)$, $K(x, t)$ are known functions, $\varphi(x)$ is the unknown function and λ is a numerical parameter, is called *Volterra's linear integral equation of the second kind*. The function $K(x, t)$ is the *kernel* of Volterra's equation. If $f(x)\equiv 0$, then equation (1) takes the form

$$\varphi(x)=\lambda\int_a^x K(x, t)\varphi(t)\,dt \tag{2}$$

and is called a *homogeneous* Volterra equation of the second kind.

The equation

$$\int_a^x K(x, t)\varphi(t)\,dt=f(x) \tag{3}$$

where $\varphi(x)$ is the unknown function is called *Volterra's integral equation of the first kind*. Without loss of generality, we can consider the lower limit a as equal to zero (in the sequel we shall assume this to be the case).

A *solution* of the integral equation (1), (2) or (3) is a function $\varphi(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

Example. Show that the function $\varphi(x)=\frac{1}{(1+x^2)^{3/2}}$ is a solution of the Volterra integral equation

$$\varphi(x)=\frac{1}{1+x^2}-\int_0^x \frac{t}{1+x^2}\varphi(t)\,dt \tag{4}$$

Solution. Substituting the function $\frac{1}{(1+x^2)^{3/2}}$ in place of $\varphi(x)$ into the right member of (4), we obtain

$$\frac{1}{1+x^2}-\int_0^x \frac{t}{1+x^2}\,\frac{1}{(1+t^2)^{3/2}}\,dt=$$
$$=\frac{1}{1+x^2}-\frac{1}{1+x^2}\left(-\frac{1}{(1+t^2)^{1/2}}\right)\Bigg|_{t=0}^{t=x}=$$
$$=\frac{1}{1+x^2}+\frac{1}{(1+x^2)^{3/2}}-\frac{1}{1+x^2}=\frac{1}{(1+x^2)^{3/2}}=\varphi(x)$$

Thus, the substitution of $\varphi(x)=\frac{1}{(1+x^2)^{3/2}}$ into both sides of equation (4) reduces the equation to an identity with respect to x:

$$\frac{1}{(1+x^2)^{3/2}}\equiv\frac{1}{(1+x^2)^{3/2}}$$

According to the definition, this means that $\varphi(x)=\frac{1}{(1+x^2)^{3/2}}$ is a solution of the integral equation (4).

Verify that the given functions are solutions of the corresponding integral equations.

1. $\varphi(x)=\frac{x}{(1+x^2)^{5/2}}$;

$$\varphi(x)=\frac{3x+2x^3}{3(1+x^2)^2}-\int_0^x \frac{3x+2x^3-t}{(1+x^2)^2}\,\varphi(t)\,dt.$$

2. $\varphi(x)=e^x(\cos e^x-e^x\sin e^x)$;

$$\varphi(x)=(1-xe^{2x})\cos 1-e^{2x}\sin 1+\int_0^x [1-(x-t)\,e^{2x}]\,\varphi(t)\,dt.$$

3. $\varphi(x)=xe^x$; $\quad \varphi(x)=e^x\sin x+2\int_0^x \cos(x-t)\,\varphi(t)\,dt.$

4. $\varphi(x)=x-\frac{x^3}{6}$; $\quad \varphi(x)=x-\int_0^x \sinh(x-t)\,\varphi(t)\,dt.$

5. $\varphi(x) = 1 - x;\quad \int_0^x e^{x-t}\varphi(t)\,dt = x.$

6. $\varphi(x) = 3;\quad x^3 = \int_0^x (x-t)^2\,\varphi(t)\,dt.$

7. $\varphi(x) = \frac{1}{2};\quad \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}\,dt = \sqrt{x}.$

8. $\varphi(x) = \frac{1}{\pi\sqrt{x}};\quad \int_0^x \frac{\varphi(t)}{\sqrt{x-t}}\,dt = 1.$

Note. Volterra-type integral equations occur in problems of physics in which the independent variable varies in a preferential direction (for example, time, energy, etc.).

Consider a beam of X-rays traversing a substance in the direction of the x-axis. We will assume that the beam maintains that direction when scattered. Consider a collection of rays of specified wavelength. When passing through a thickness dx, some of the rays are absorbed; others undergo a change in wavelength due to scattering. On the other hand, the collection is augmented by those rays which, though originally of greater energy (i. e., shorter wavelength λ), lose part of their energy through scattering. Thus, if the function $f(\lambda, x)\,d\lambda$ gives the collection of rays whose wavelength lies in the interval from λ to $\lambda + d\lambda$, then

$$\frac{\partial f(\lambda, x)}{\partial x} = -\mu f(\lambda, x) + \int_0^\lambda P(\lambda, \tau) f(\tau, x)\,d\tau$$

where μ is the absorption coefficient and $P(\lambda, \tau)\,d\tau$ is the probability that in passing through a layer of unit thickness a ray of wavelength τ acquires a wavelength which lies within the interval between λ and $\lambda + d\lambda$.

What we have is an *integro-differential equation*, i. e., an equation in which the unknown function $f(\lambda, x)$ is under the sign of the derivative and the integral.

Putting

$$f(\lambda, x) = \int_0^\infty e^{-px} \psi(\lambda, p)\, dp$$

where $\psi(\lambda, p)$ is a new unknown function, we find that $\psi(\lambda, p)$ will satisfy the Volterra integral equation of the second kind

$$\psi(\lambda, p) = \frac{1}{\mu - p} \int_0^\lambda P(\lambda, \tau)\, \psi(\tau, p)\, d\tau$$

2. Relationship Between Linear Differential Equations and Volterra Integral Equations

The solution of the linear differential equation

$$\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n(x)\, y = F(x) \tag{1}$$

with continuous coefficients $a_i(x)$ $(i = 1, 2, \ldots, n)$, given the initial conditions

$$y(0) = C_0, \quad y'(0) = C_1, \quad \ldots, \quad y^{(n-1)}(0) = C_{n-1} \tag{2}$$

may be reduced to a solution of some Volterra integral equation of the second kind.

Let us demonstrate this in the case of a differential equation of the second order

$$\frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = F(x) \tag{1'}$$

$$y(0) = C_0, \quad y'(0) = C_1 \tag{2'}$$

Put

$$\frac{d^2 y}{dx^2} = \varphi(x) \tag{3}$$

Whence, taking into account the initial conditions (2′), we successively find

$$\frac{dy}{dx} = \int_0^x \varphi(t)\, dt + C_1, \quad y = \int_0^x (x - t)\, \varphi(t)\, dt + C_1 x + C_0 \tag{4}$$

Here, we utilized the formula

$$\underbrace{\int_{x_0}^{x} dx \int_{x_0}^{x} dx \ldots \int_{x_0}^{x}}_{n} f(x)\, dx = \frac{1}{(n-1)!} \int_{x_0}^{x} (x-z)^{n-1} f(z)\, dz$$

Taking into account (3) and (4), differential equation (1) may be written as follows:

$$\varphi(x) + \int_0^x a_1(x)\varphi(t)\, dt + C_1 a_1(x) + \int_0^x a_2(x)(x-t)\varphi(t)\, dt + \\ + C_1 x a_2(x) + C_0 a_2(x) = F(x)$$

or

$$\varphi(x) + \int_0^x [a_1(x) + a_2(x)(x-t)]\varphi(t)\, dt = \\ = F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x) \qquad (5)$$

Putting

$$K(x, t) = -[a_1(x) + a_2(x)(x-t)] \qquad (6)$$

$$f(x) = F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x) \qquad (7)$$

we reduce (5) to the form

$$\varphi(x) = \int_0^x K(x, t)\varphi(t)\, dt + f(x) \qquad (8)$$

which means that we arrive at a Volterra integral equation of the second kind.

The existence of a unique solution of equation (8) follows from the existence and uniqueness of solution of the Cauchy problem (1′)-(2′) for a linear differential equation with continuous coefficients in the neighbourhood of the point $x = 0$.

Conversely, solving the integral equation (8) with K and f determined from (6) and (7), and substituting the expression obtained for $\varphi(x)$ into the second equation of (4), we get a unique solution to equation (1′) which satisfies the initial conditions (2′).

Example. Form an integral equation corresponding to the differential equation

$$y'' + xy' + y = 0 \tag{1}$$

and the initial conditions

$$y(0) = 1, \quad y'(0) = 0 \tag{2}$$

Solution. Put

$$\frac{d^2y}{dx^2} = \varphi(x) \tag{3}$$

Then

$$\frac{dy}{dx} = \int_0^x \varphi(t)\,dt + y'(0) = \int_0^x \varphi(t)\,dt, \quad y = \int_0^x (x-t)\,\varphi(t)\,dt + 1 \tag{4}$$

Substituting (3) and (4) into the given differential equation, we get

$$\varphi(x) + \int_0^x x\varphi(t)\,dt + \int_0^x (x-t)\,\varphi(t)\,dt + 1 = 0$$

or

$$\varphi(x) = -1 - \int_0^x (2x-t)\,\varphi(t)\,dt$$

Form integral equations corresponding to the following differential equations with given initial conditions:

9. $y'' + y = 0$; $\quad y(0) = 0, \ y'(0) = 1$.
10. $y' - y = 0$; $\quad y(0) = 1$.
11. $y'' + y = \cos x$; $\quad y(0) = y'(0) = 0$.
12. $y'' - 5y' + 6y = 0$; $\quad y(0) = 0, \ y'(0) = 1$.
13. $y'' + y = \cos x$; $\quad y(0) = 0, \ y'(0) = 1$.
14. $y'' - y' \sin x + e^x y = x$; $\quad y(0) = 1, \ y'(0) = -1$.
15. $y'' + (1 + x^2)\,y = \cos x$; $\quad y(0) = 0, \ y'(0) = 2$.
16. $y''' + xy'' + (x^2 - x)\,y = xe^x + 1$;
    $y(0) = y'(0) = 1, \quad y''(0) = 0$.
17. $y''' - 2xy = 0$; $\quad y(0) = \frac{1}{2}, \ y'(0) = y''(0) = 1$.

18. Show that a linear differential equation with constant coefficients reduces, under any initial conditions, to a Volterra integral equation of the second kind with kernel dependent solely on the difference $(x-t)$ of arguments (integral equation of the closed cycle or equation of the Faltung type, or convolution type).

3. Resolvent Kernel of Volterra Integral Equation. Solution of Integral Equation by Resolvent Kernel

Suppose we have a Volterra integral equation of the second kind:

$$\varphi(x)=f(x)+\lambda\int_0^x K(x, t)\,\varphi(t)\,dt \tag{1}$$

where $K(x, t)$ is a continuous function for $0\leqslant x\leqslant a$, $0\leqslant t\leqslant x$, and $f(x)$ is continuous for $0\leqslant x\leqslant a$.

We shall seek the solution of integral equation (1) in the form of an infinite power in series λ:

$$\varphi(x)=\varphi_0(x)+\lambda\varphi_1(x)+\lambda^2\varphi_2(x)+\ldots+\lambda^n\varphi_n(x)+\ldots \tag{2}$$

Formally substituting this series into (1), we obtain

$$\varphi_0(x)+\lambda\varphi_1(x)+\ldots+\lambda^n\varphi_n(x)+\ldots=$$

$$=f(x)+\lambda\int_0^x K(x, t)\left[\varphi_0(t)+\lambda\varphi_1(t)+\ldots+\lambda^n\varphi_n(t)+\ldots\right]dt \tag{3}$$

Comparing coefficients of like powers of λ, we find

$$\varphi_0(x)=f(x),$$

$$\varphi_1(x)=\int_0^x K(x, t)\,\varphi_0(t)\,dt=\int_0^x K(x, t)\,f(t)\,dt, \tag{4}$$

$$\varphi_2(x)=\int_0^x K(x, t)\,\varphi_1(t)\,dt=\int_0^x K(x, t)\int_0^t K(t, t_1)\,f(t_1)\,dt_1\,dt$$

. . . .

The relations (4) yield a method for a successive determination of the functions $\varphi_n(x)$. It may be shown that under the assumptions made, with respect to $f(x)$ and $K(x, t)$, the series (2) thus obtained converges uniformly in x and λ for any λ and $x \in [0, a]$ and its sum is a unique solution of equation (1).

Further, it follows from (4) that

$$\varphi_1(x) = \int_0^x K(x, t) f(t)\, dt \tag{5}$$

$$\varphi_2(x) = \int_0^x K(x, t) \left[\int_0^t K(t, t_1) f(t_1)\, dt_1 \right] dt =$$

$$= \int_0^x f(t_1)\, dt_1 \int_{t_1}^x K(x, t) K(t, t_1)\, dt = \int_0^x K_2(x, t_1) f(t_1)\, dt_1 \tag{6}$$

where

$$K_2(x, t_1) = \int_{t_1}^x K(x, t) K(t, t_1)\, dt \tag{7}$$

Similarly, it is established that, generally,

$$\varphi_n(x) = \int_0^x K_n(x, t) f(t)\, dt \quad (n = 1, 2, \ldots) \tag{8}$$

The functions $K_n(x, t)$ are called *iterated kernels*. It can readily be shown that they are determined with the aid of the recursion formulas

$$K_1(x, t) = K(x, t),$$

$$K_{n+1}(x, t) = \int_t^x K(x, z) K_n(z, t)\, dz \quad (n = 1, 2, \ldots) \tag{9}$$

Utilizing (8) and (9), equality (2) may be written as

$$\varphi(x) = f(x) + \sum_{\nu=1}^{\infty} \lambda^{\nu} \int_0^x K_{\nu}(x, t) f(t)\, dt \tag{10}$$

The function $R(x, t; \lambda)$ defined by means of the series

$$R(x, t; \lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} K_{\nu+1}(x, t) \tag{11}$$

is called the *resolvent kernel* (or reciprocal kernel) for the integral equation (1). Series (11) converges absolutely and uniformly in the case of a continuous kernel $K(x, t)$.

Iterated kernels and also the resolvent kernel do not depend on the lower limit in an integral equation.

The resolvent kernel $R(x, t; \lambda)$ satisfies the following functional equation:

$$R(x, t; \lambda)=K(x, t)+\lambda \int_t^x K(x, s) R(s, t; \lambda) ds \tag{12}$$

With the aid of the resolvent kernel, the solution of integral equation (1) may be written in the form

$$\varphi(x)=f(x)+\lambda \int_0^x R(x, t; \lambda) f(t) dt \tag{13}$$

(see [5], [15]).

Example. Find the resolvent kernel of the Volterra integral equation with kernel $K(x, t) \equiv 1$.

Solution. We have $K_1(x, t)=K(x, t)=1$. Further, by formulas (9)

$$K_2(x, t)=\int_t^x K(x, z) K_1(z, t) dz=\int_t^x dz=x-t,$$

$$K_3(x, t)=\int_t^x 1\cdot(z-t)\, dz=\frac{(x-t)^2}{2},$$

$$K_4(x, t)=\int_t^x 1\cdot\frac{(z-t)^2}{2} dz=\frac{(x-t)^3}{3!},$$

. . . .

$$K_n(x, t)=\int_t^x 1\cdot K_{n-1}(z, t) dz=\int_t^x 1\cdot\frac{(z-t)^{n-2}}{(n-2)!} dz=\frac{(x-t)^{n-1}}{(n-1)!}$$

Thus, by the definition of the resolvent kernel,

$$R(x, t; \lambda)=\sum_{n=0}^{\infty} \lambda^n K_{n+1}(x, t)=\sum_{n=0}^{\infty} \frac{\lambda^n (x-t)^n}{n!}=e^{\lambda(x-t)}$$

Find the resolvent kernels for Volterra-type integral equations with the following kernels:

19. $K(x, t)=x-t$.
20. $K(x, t)=e^{x-t}$.
21. $K(x, t)=e^{x^2-t^2}$.
22. $K(x, t)=\frac{1+x^2}{1+t^2}$.
23. $K(x, t)=\frac{2+\cos x}{2+\cos t}$.
24. $K(x, t)=\frac{\cosh x}{\cosh t}$.
25. $K(x, t)=a^{x-t}\,(a>0)$.

Suppose that the kernel $K(x, t)$ is a polynomial of degree $n-1$ in t so that it may be represented in the form

$$K(x, t)=a_0(x)+a_1(x)(x-t)+\ldots+\frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1} \tag{14}$$

and the coefficients $a_k(x)$ are continuous in $[0, a]$. If the function $g(x, t; \lambda)$ is defined as a solution of the differential equation

$$\frac{d^n g}{dx^n}-\lambda\left[a_0(x)\frac{d^{n-1}g}{dx^{n-1}}+a_1(x)\frac{d^{n-2}g}{dx^{n-2}}+\ldots+a_{n-1}(x)g\right]=0 \tag{15}$$

satisfying the conditions

$$g\Big|_{x=t}=\frac{dg}{dx}\Big|_{x=t}=\ldots=\frac{d^{n-2}g}{dx^{n-2}}\Big|_{x=t}=0;\quad \frac{d^{n-1}g}{dx^{n-1}}\Big|_{x=t}=1 \tag{16}$$

then the resolvent kernel $R(x, t; \lambda)$ will be defined by the equality

$$R(x, t; \lambda)=\frac{1}{\lambda}\frac{d^n g(x, t; \lambda)}{dx^n} \tag{17}$$

and similarly when

$$K(x, t)=b_0(t)+b_1(t)(t-x)+\ldots+\frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1} \tag{18}$$

the resolvent kernel

$$R(x, t; \lambda)=-\frac{1}{\lambda}\frac{d^n g(t, x; \lambda)}{dt^n} \tag{19}$$

where $g(x,\ t;\ \lambda)$ is a solution of the equation

$$\frac{d^n g}{dt^n}+\lambda\left[b_0(t)\frac{d^{n-1}g}{dt^{n-1}}+\ldots+b_{n-1}(t)\,g\right]=0 \qquad (20)$$

which satisfies the conditions (16) (see [5]).

Example. Find the resolvent kernel for the integral equation

$$\varphi(x)=f(x)+\int_0^x (x-t)\,\varphi(t)\,dt$$

Solution. Here, $K(x,\ t)=x-t$; $\lambda=1$; hence, by (14), $a_1(x)=1$, and all the other $a_k(x)=0$.

In this case, equation (15) has the form

$$\frac{d^2 g(x,\ t;\ 1)}{dx^2}-g(x,\ t;\ 1)=0$$

whence

$$g(x,\ t;\ 1)=g(x,\ t)=C_1(t)\,e^x+C_2(t)\,e^{-x}$$

Conditions (16) yield

$$\begin{cases} C_1(t)\,e^t+C_2(t)\,e^{-t}=0, \\ C_1(t)\,e^t-C_2(t)\,e^{-t}=1 \end{cases} \qquad (21)$$

Solving the system (21), we find

$$C_1(t)=\frac{1}{2}e^{-t},\ C_2(t)=-\frac{1}{2}e^t$$

and, consequently,

$$g(x,\ t)=\frac{1}{2}\left(e^{x-t}-e^{-(x-t)}\right)=\sinh(x-t)$$

According to (17)

$$R(x,\ t;\ 1)=[\sinh(x-t)]''_x=\sinh(x-t)$$

Find the resolvent kernels of integral equations with the following kernels ($\lambda=1$):

26. $K(x,\ t)=2-(x-t)$.

27. $K(x,\ t)=-2+3(x-t)$.

28. $K(x,\ t)=2x$.

29. $K(x,\ t)=-\dfrac{4x-2}{2x+1}+\dfrac{8(x-t)}{2x+1}$.

30. Suppose we have a Volterra-type integral equation, the kernel of which is dependent solely on the difference of the arguments:

$$\varphi(x)=f(x)+\int_0^x K(x-t)\,\varphi(t)\,dt \quad (\lambda=1) \tag{22}$$

Show that for equation (22) all iterated kernels and the resolvent kernel are also dependent solely on the difference $x-t$.

Let the functions $f(x)$ and $K(x)$ in (22) be original functions. Taking the Laplace transform of both sides of (22) and employing the product theorem (transform of a convolution), we get

$$\Phi(p)=F(p)+\tilde{K}(p)\,\Phi(p)$$

where

$$\varphi(x)\risingdotseq\Phi(p),$$
$$f(x)\risingdotseq F(p),$$
$$K(x)\risingdotseq\tilde{K}(p)$$

Whence

$$\Phi(p)=\frac{F(p)}{1-\tilde{K}(p)}, \quad \tilde{K}(p)\neq 1 \tag{23}$$

Taking advantage of the results of Problem 30, we can write the solution of the integral equation (22) in the form

$$\varphi(x)=f(x)+\int_0^x R(x-t)\,f(t)\,dt \tag{24}$$

where $R(x-t)$ is the resolvent kernel for the integral equation (22).

Taking the Laplace transform of both sides of equation (24), we obtain

$$\Phi(p)=F(p)+\tilde{R}(p)\,F(p)$$

where

$$R(x)\risingdotseq\tilde{R}(p)$$

Whence

$$\tilde{R}(p)=\frac{\Phi(p)-F(p)}{F(p)} \tag{25}$$

Substituting into (25) the expression for $\Phi(p)$ from (23), we obtain

$$\tilde{R}(p)=\frac{\tilde{K}(p)}{1-\tilde{K}(p)} \tag{26}$$

The original function of $\tilde{R}(p)$ will be the resolvent kernel of the integral equation (22).

Example. Find the resolvent kernel for a Volterra integral equation with kernel $K(x, t)=\sin(x-t)$, $\lambda=1$.

Solution. We have $\tilde{K}(p)=\frac{1}{p^2+1}$. By (26)

$$\tilde{R}(p)=\frac{\frac{1}{p^2+1}}{1-\frac{1}{p^2+1}}=\frac{1}{p^2} \risingdotseq x$$

Hence, the required resolvent kernel for the integral equation is

$$R(x, t; 1)=x-t$$

Find the resolvent kernels for Volterra-type integral equations with the kernels ($\lambda=1$):

31. $K(x, t)=\sinh(x-t)$.

32. $K(x, t)=e^{-(x-t)}$.

33. $K(x, t)=e^{-(x-t)}\sin(x-t)$.

34. $K(x, t)=\cosh(x-t)$.

35. $K(x, t)=2\cos(x-t)$.

Example. With the aid of the resolvent kernel, find the solution of the integral equation

$$\varphi(x)=e^{x^2}+\int_0^x e^{x^2-t^2}\varphi(t)\,dt$$

Solution. The resolvent kernel of the kernel $K(x, t)=e^{x^2-t^2}$ for $\lambda=1$ is $R(x, t; 1)=e^{x-t}e^{x^2-t^2}$ (see No. 21). By formula (13), the solution of the given integral equation is

$$\varphi(x)=e^{x^2}+\int_0^x e^{x-t}e^{x^2-t^2}e^{t^2}\,dt=e^{x+x^2}$$

Using the results of the preceding examples, find (by means of resolvent kernels) solutions of the following integral equations:

36. $\varphi(x)=e^x+\int_0^x e^{x-t}\varphi(t)\,dt.$

37. $\varphi(x)=\sin x+2\int_0^x e^{x-t}\varphi(t)\,dt.$

38. $\varphi(x)=x3^x-\int_0^x 3^{x-t}\varphi(t)\,dt.$

39. $\varphi(x)=e^x\sin x+\int_0^x \frac{2+\cos x}{2+\cos t}\varphi(t)\,dt.$

40. $\varphi(x)=1-2x-\int_0^x e^{x^2-t^2}\varphi(t)\,dt.$

41. $\varphi(x)=e^{x^2+2x}+2\int_0^x e^{x^2-t^2}\varphi(t)\,dt.$

42. $\varphi(x)=1+x^2+\int_0^x \frac{1+x^2}{1+t^2}\varphi(t)\,dt.$

43. $\varphi(x)=\frac{1}{1+x^2}+\int_0^x \sin(x-t)\,\varphi(t)\,dt.$

44. $\varphi(x)=xe^{\frac{x^2}{2}}+\int_0^x e^{-(x-t)}\varphi(t)\,dt.$

45. $\varphi(x)=e^{-x}+\int_0^x e^{-(x-t)}\sin(x-t)\,\varphi(t)\,dt.$

Note 1. The unique solvability of Volterra-type integral equations of the second kind

$$\varphi(x)=f(x)+\lambda\int_0^x K(x,\ t)\,\varphi(t)\,dt \qquad (1)$$

holds under considerably more general assumptions with respect to the function $f(x)$ and the kernel $K(x, t)$ than their continuity.

Theorem. *The Volterra integral equation of the second kind* (1), *whose kernel* $K(x, t)$ *and function* $f(x)$ *belong, respectively, to spaces* $L_2(\Omega_0)$ *and* $L_2(0, a)$, *has one and only one solution in the space* $L_2(0, a)$.

This solution is given by the formula

$$\varphi(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t)\, dt \tag{2}$$

where the resolvent kernel $R(x, t; \lambda)$ is determined by means of the series

$$R(x, t; \lambda) = \sum_{\nu=0}^{\infty} \lambda^\nu K_{\nu+1}(x, t) \tag{3}$$

which is made up of iterated kernels and converges almost everywhere.

Note 2. In questions of uniqueness of solution of an integral equation, an essential role is played by the class of functions in which the solution is sought (the class of summable, quadratically summable, continuous, etc., functions).

Thus, if the kernel $K(x, t)$ of a Volterra equation is bounded when x varies in some finite interval (a, b) so that

$$|K(x, t)| \leqslant M, \quad M = \text{const}, \quad x \in (a, b)$$

and the constant term of $f(x)$ is summable in the interval (a, b), then the Volterra equation has, for any value of λ, a unique summable solution $\varphi(x)$ in the interval (a, b).

However, if we give up the requirement of summability of the solution, then the uniqueness theorem ceases to hold in the sense that the equation can have nonsummable solutions along with summable solutions.

P. S. Uryson ([29]) constructed elegant examples of integral equations (see Examples 1 and 2 below) which have summable and nonsummable solutions even when the kernel $K(x, t)$ and the function $f(x)$ are continuous.

For simplicity we consider $f(x) \equiv 0$ and examine the integral equation

$$\varphi(x) = \int_0^x K(x, t)\varphi(t)\,dt \tag{1}$$

where $K(x, t)$ is a continuous function.

The only summable solution of equation (1) is $\varphi(x) \equiv 0$.

Example 1. Let

$$K(x, t) = \begin{cases} te^{\frac{1}{x^2}-1}, & 0 \leqslant t \leqslant e^{1-\frac{1}{x^2}}, \\ x, & xe^{1-\frac{1}{x^2}} \leqslant t \leqslant x, \\ 0, & t > x \end{cases} \tag{2}$$

The kernel $K(x, t)$ is bounded in the square $\Omega_0\{0 \leqslant x, t \leqslant 1\}$, since $0 \leqslant K(x, t) \leqslant x \leqslant 1$. What is more, it is continuous for $0 \leqslant t \leqslant x$. In this case, equation (1) has an obviously summable solution $\varphi(x) \equiv 0$, and, by virtue of what has been said, this equation does not have any other summable solutions.

On the other hand, direct verification convinces us that equation (1) has an infinity of nonsummable solutions in (0, 1) in the form

$$\varphi(x) = \frac{C}{x}$$

where C is an arbitrary constant and $x \neq 0$.

Indeed, taking into account expression (2) for the kernel $K(x, t)$, we find

$$\int_0^x K(x, t)\varphi(t)\,dt = \int_0^{xe^{1-\frac{1}{x^2}}} te^{\frac{1}{x^2}-1}\frac{C}{t}\,dt +$$

$$+ \int_{xe^{1-\frac{1}{x^2}}}^{x} x\frac{C}{t}\,dt = Cx + Cx\ln e^{\frac{1}{x^2}-1} = \frac{C}{x}$$

Thus we obtain

$$\frac{C}{x} \equiv \frac{C}{x} \quad (x \neq 0)$$

This means that $\varphi(x) = \frac{C}{x}$ is a nonsummable solution of equation (1).

Example 2. Let $0 \leqslant t \leqslant x < a$ ($a > 0$, in particular $a = +\infty$),

$$K(x, t) = \frac{2}{\pi} \frac{xt^2}{x^6 + t^2} \tag{3}$$

The function $K(x, t)$ is even holomorphic everywhere, except at the point (0, 0). However, equation (1) with kernel (3) admits nonsummable solutions. Indeed, the equation

$$\psi(x) = \frac{2}{\pi} \int_0^x \frac{xt^2}{x^6 + t^2} \psi(t)\, dt - \frac{2}{\pi} \frac{\arctan x^2}{x^2} \tag{4}$$

has a summable solution since the function

$$f(x) = -\frac{2}{\pi} \frac{\arctan x^2}{x^2}$$

is bounded and continuous everywhere except at the point $x = 0$.

The function

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ \psi(x) + \frac{1}{x^2}, & x > 0 \end{cases} \tag{5}$$

where $\psi(x)$ is a solution of (4) will now be a nonsummable solution of (1) with kernel (3).

Indeed, for $x > 0$ we have

$$\int_0^x K(x, t)\, \varphi(t)\, dt = \frac{2}{\pi} \int_0^x \frac{xt^2}{x^6 + t^2} \psi(t)\, dt + \frac{2}{\pi} \int_0^x \frac{x\, dt}{x^6 + t^2} \tag{6}$$

By virtue of equation (4), the first term on the right of (6) is

$$\psi(x) + \frac{2}{\pi} \frac{\arctan x^2}{x^2}$$

The second term yields

$$\frac{2}{\pi}\int_0^x \frac{x\,dt}{x^6+t^2}=\frac{2}{\pi}\left(\frac{1}{x^2}\arctan\frac{t}{x^3}\right)\Big|_{t=0}^{t=x}=\frac{2}{\pi x^2}\arctan\frac{1}{x^2}\quad (x>0)$$

Thus

$$\int_0^x K(x,\ t)\,\varphi(t)\,dt=\psi(x)+\frac{2}{\pi}\,\frac{\arctan x^2}{x^2}+\frac{2}{\pi x^2}\arctan\frac{1}{x^2}=$$

$$=\psi(x)+\frac{1}{x^2}=\varphi(x)$$

which means that the function $\varphi(x)$ defined by (5) is a nonsummable solution of equation (1) with kernel (3).

Example 3. The equation

$$\varphi(x)=\int_0^x t^{x-t}\varphi(t)\,dt\quad (0\leqslant x,\ t\leqslant 1)$$

has a unique continuous solution $\varphi(x)\equiv 0$. By direct substitution we see that this equation also has an infinity of discontinuous solutions of the form

$$\varphi(x)=Cx^{x-1}$$

where C is an arbitrary constant.

4. The Method of Successive Approximations

Suppose we have a Volterra-type integral equation of the second kind:

$$\varphi(x)=f(x)+\lambda\int_0^x K(x,\ t)\,\varphi(t)\,dt \tag{1}$$

We assume that $f(x)$ is continuous in $[0,\ a]$ and the kernel $K(x,\ t)$ is continuous for $0\leqslant x\leqslant a,\ 0\leqslant t\leqslant x$.

Take some function $\varphi_0(x)$ continuous in $[0,\ a]$. Putting the function $\varphi_0(x)$ into the right side of (1) in place of $\varphi(x)$, we get

$$\varphi_1(x)=f(x)+\lambda\int_0^x K(x,\ t)\,\varphi_0(t)\,dt$$

The thus defined function $\varphi_1(x)$ is also continuous in the interval $[0, a]$. Continuing the process, we obtain a sequence of functions

$$\varphi_0(x), \ \varphi_1(x), \ \ldots, \ \varphi_n(x), \ \ldots$$

where

$$\varphi_n(x) = f(x) + \lambda \int_0^x K(x, t)\, \varphi_{n-1}(t)\, dt$$

Under the assumptions with respect to $f(x)$ and $K(x, t)$, the sequence $\{\varphi_n(x)\}$ converges, as $n \to \infty$, to the solution $\varphi(x)$ of the integral equation (1) (see [13]).

In particular, if for $\varphi_0(x)$ we take $f(x)$, then $\varphi_n(x)$ will be the partial sums of the series (2), of Sec. 3, which defines the solution of the integral equation (1). A suitable choice of the "zero" approximation $\varphi_0(x)$ can lead to a rapid convergence of the sequence $\{\varphi_n(x)\}$ to the solution of the integral equation.

Example. Using the method of successive approximations, solve the integral equation

$$\varphi(x) = 1 + \int_0^x \varphi(t)\, dt$$

taking $\varphi_0(x) \equiv 0$.

Solution. Since $\varphi_0(x) \equiv 0$, it follows that $\varphi_1(x) = 1$. Then

$$\varphi_2(x) = 1 + \int_0^x 1 \cdot dt = 1 + x,$$

$$\varphi_3(x) = 1 + \int_0^x (1 + t)\, dt = 1 + x + \frac{x^2}{2},$$

$$\varphi_4(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Obviously

$$\varphi_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}$$

Thus, $\varphi_n(x)$ is the nth partial sum of the series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = e^x$. Whence it follows that $\varphi_n(x) \xrightarrow[n \to \infty]{} e^x$. It is easy to verify that the function $\varphi(x)=e^x$ is a solution of the given integral equation.

Using the method of successive approximations, solve the following integral equations:

46. $\varphi(x)=x-\int_0^x (x-t)\,\varphi(t)\,dt, \quad \varphi_0(x)\equiv 0.$

47. $\varphi(x)=1-\int_0^x (x-t)\,\varphi(t)\,dt, \quad \varphi_0(x)\equiv 0.$

48. $\varphi(x)=1+\int_0^x (x-t)\,\varphi(t)\,dt, \quad \varphi_0(x)=1.$

49. $\varphi(x)=x+1-\int_0^x \varphi(t)\,dt;$

(a) $\varphi_0(x)=1$, (b) $\varphi_0(x)=x+1$.

50. $\varphi(x)=\frac{x^2}{2}+x-\int_0^x \varphi(t)\,dt;$

(a) $\varphi_0(x)=1$, (b) $\varphi_0(x)=x$, (c) $\varphi_0(x)=\frac{x^2}{2}+x$.

51. $\varphi(x)=1+x+\int_0^x (x-t)\,\varphi(t)\,dt, \quad \varphi_0(x)=1.$

52. $\varphi(x)=2x+2-\int_0^x \varphi(t)\,dt;$

(a) $\varphi_0(x)=1$, (b) $\varphi_0(x)=2$.

53. $\varphi(x)=2x^2+2-\int_0^x x\varphi(t)\,dt;$

(a) $\varphi_0(x)=2$, (b) $\varphi_0(x)=2x$.

54. $\varphi(x)=\frac{x^3}{3}-2x-\int_0^x \varphi(t)\,dt, \quad \varphi_0(x)=x^2.$

55. Let $K(x, t)$ satisfy the condition

$$\int_0^a \int_0^x K^2(x, t)\,dt\,dx<+\infty.$$

Prove that the equation

$$\varphi(x)-\lambda\int_0^x K(x, t)\varphi(t)\,dt=0$$

has, for any λ, a unique solution $\varphi(x)\equiv 0$ in the class $L_2(0, a)$.

The method of successive approximations can also be applied to the solution of nonlinear Volterra integral equations of the form

$$y(x)=y_0+\int_0^x F[t, y(t)]\,dt \tag{2}$$

or the more general equations

$$\varphi(x)=f(x)+\int_0^x F(x, t, \varphi(t))\,dt \tag{3}$$

under extremely broad assumptions with respect to the functions $F(x, t, z)$ and $f(x)$. The problem of solving the differential equation

$$\frac{dy}{dx}=F(x, y), \quad y|_{x=0}=y_0$$

reduces to an equation of the type (2). As in the case of linear integral equations, we shall seek the solution of equation (3) as the limit of the sequence $\{\varphi_n(x)\}$ where, for example, $\varphi_0(x)=f(x)$, and the following elements $\varphi_k(x)$ are computed successively from the formula

$$\varphi_k(x)=f(x)+\int_0^x F(x, t, \varphi_{k-1}(t))\,dt \quad (k=1, 2, \ldots) \tag{4}$$

If $f(x)$ and $F(x, t, z)$ are quadratically summable and satisfy the conditions

$$|F(x, t, z_2)-F(x, t, z_1)|\leqslant a(x, t)\,|z_2-z_1| \tag{5}$$

$$\left|\int_0^x F(x, t, f(t))\,dt\right|\leqslant n(x) \tag{6}$$

where the functions $a(x, t)$ and $n(x)$ are such that in the main domain $(0 \leqslant t \leqslant x \leqslant a)$

$$\int_0^a n^2(x)\,dx \leqslant N^2, \quad \int_0^a dx \int_0^x a^2(x, t)\,dt \leqslant A^2 \tag{7}$$

it follows that the nonlinear Volterra integral equation of the second kind (3) has a unique solution $\varphi(x) \in L_2(0, a)$ which is defined as the limit of $\varphi_n(x)$ as $n \longrightarrow \infty$:

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x)$$

where the functions $\varphi_n(x)$ are found from the recursion formulas (4). For $\varphi_0(x)$ we can take any function in $L_2(0, a)$ (in particular, a continuous function), for which the condition (6) is fulfilled. Note that an apt choice of the zero approximation can facilitate solution of the integral equation.

Example. Using the method of successive approximations, solve the integral equation

$$\varphi(x) = \int_0^x \frac{1+\varphi^2(t)}{1+t^2}\,dt$$

taking as the zero approximation: (1) $\varphi_0(x) = 0$, (2) $\varphi_0(x) = x$.

Solution. (1) Let $\varphi_0(x) = 0$. Then

$$\varphi_1(x) = \int_0^x \frac{dt}{1+t^2} = \arctan x,$$

$$\varphi_2(x) = \int_0^x \frac{1+\arctan^2 t}{1+t^2}\,dt = \arctan x + \frac{1}{3}\arctan^3 x,$$

$$\varphi_3(x) = \int_0^x \frac{1+\left(\arctan t + \frac{1}{3}\arctan^3 t\right)^2}{1+t^2}\,dt = \arctan x +$$

$$+\frac{1}{3}\arctan^3 x + \frac{2}{3\times 5}\arctan^5 x + \frac{1}{7\times 9}\arctan^7 x,$$

$$\varphi_4(x) = \int_0^x \frac{1+\varphi_3^2(t)}{1+t^2}\,dt = \arctan x + \frac{1}{3}\arctan^3 x +$$

$$+\frac{2}{3\times5}\arctan^5 x+\frac{17}{5\times7\times9}\arctan^7 x+\frac{38}{5\times7\times9^2}\arctan^9 x+$$
$$+\frac{134}{9\times11\times21\times25}\arctan^{11} x+\frac{4}{3\times5\times7\times9\times13}\arctan^{13} x+$$
$$+\frac{1}{7^2\times9^2\times15}\arctan^{15} x, \ \ldots.$$

Denoting $\arctan x = u$ and comparing expressions for $\varphi_n(x)$ with the expansion

$$\tan u=\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{2^{2\nu}(2^{2\nu}-1)}{(2\nu)!}B_{2\nu}u^{2\nu-1}, \quad |u|<\frac{\pi}{2}$$

where B_ν are Bernoulli numbers,* we observe that

$$\varphi_n(x)\underset{n\to\infty}{\longrightarrow}\tan(\arctan x)=x$$

It can easily be verified that the function $\varphi(x)=x$ is a solution of the given integral equation.

(2) Let $\varphi_0(x)=x$. Then

$$\varphi_1(x)=\int_0^x\frac{1+t^2}{1+t^2}dt=x$$

In similar fashion we find $\varphi_n(x)=x$ $(n=2, 3, \ldots)$.

Thus, the sequence $\{\varphi_n(x)\}$ is a stationary sequence $\{x\}$, the limit of which is $\varphi(x)=x$. The solution of this integral equation is obtained directly:

$$\varphi(x)=x$$

56. Use the method of successive approximations to solve the integral equation

$$\varphi(x)=\int_0^x\frac{t\,\varphi(t)}{1+t+\varphi(t)}dt$$

---

* The Bernoulli numbers $B_{2\nu+1}$ with odd index (odd degree) are all equal to zero, except $B_1=-\frac{1}{2}$. The number $B_0=1$, the numbers $B_{2\nu}$ are obtained from the recursion formulas

$$B_{2\nu}=-\frac{1}{2\nu+1}+\frac{1}{2}-\sum_{k=2}^{2\nu-2}\frac{2\nu(2\nu-1)\ldots(2\nu-2k+2)}{k!}B_k$$

57. Use the method of successive approximations to find a second approximation $\varphi_2(x)$ to the solution of the integral equation

$$\varphi(x)=1+\int_0^x [\varphi^2(t)+t\varphi(t)+t^2]\,dt$$

58. Use the method of successive approximations to find a third approximation $\varphi_3(x)$ to the solution of the integral equation

$$\varphi(x)=\int_0^x [t\varphi^2(t)-1]\,dt$$

5. Convolution-Type Equations

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two continuous functions defined for $x \geqslant 0$. The *convolution of these two functions* is the function $\varphi_3(x)$ defined by the equation

$$\varphi_3(x)=\int_0^x \varphi_1(x-t)\,\varphi_2(t)\,dt \tag{1}$$

This function, defined for $x \geqslant 0$, will also be a continuous function. If $\varphi_1(x)$ and $\varphi_2(x)$ are original functions for the Laplace transformation, then

$$\mathscr{L}\varphi_3=\mathscr{L}\varphi_1\cdot\mathscr{L}\varphi_2 \tag{2}$$

i. e., the transform of a convolution is equal to the product of the transforms of the functions (convolution theorem).

Let us consider the Volterra-type integral equation of the second kind

$$\varphi(x)=f(x)+\int_0^x K(x-t)\,\varphi(t)\,dt \tag{3}$$

the kernel of which is dependent solely on the difference $x-t$. We shall call equation (3) an *integral equation of the convolution type*.

Let $f(x)$ and $K(x)$ be sufficiently smooth functions which, as $x \to \infty$, do not grow faster than the exponential function, so that

$$|f(x)| \leqslant M_1 e^{s_1 x}, \quad |K(x)| \leqslant M_2 e^{s_2 x} \tag{4}$$

Applying the method of successive approximations, we can show that in this case function $\varphi(x)$ will also satisfy an upper bound of type (4):

$$|\varphi(x)| \leqslant M_3 e^{s_3 x}$$

Consequently, the Laplace transform of the functions $f(x)$, $K(x)$ and $\varphi(x)$ can be found (it will be defined in the half-plane $\operatorname{Re} p = s > \max(s_1, s_2, s_3)$).

Let

$$f(x) \risingdotseq F(p), \quad \varphi(x) \risingdotseq \Phi(p), \quad K(x) \risingdotseq \tilde{K}(p)$$

Taking the Laplace transform of both sides of (3) and employing the convolution theorem, we find

$$\Phi(p) = F(p) + \tilde{K}(p)\Phi(p) \tag{5}$$

Whence

$$\Phi(p) = \frac{F(p)}{1-\tilde{K}(p)} \quad (\tilde{K}(p) \neq 1)$$

The original function $\varphi(x)$ for $\Phi(p)$ will be a solution of the integral equation (3) (see [25]).

Example. Solve the integral equation

$$\varphi(x) = \sin x + 2\int_0^x \cos(x-t)\varphi(t)\,dt$$

Solution. It is known that

$$\sin x \risingdotseq \frac{1}{p^2+1}, \quad \cos x \risingdotseq \frac{p}{p^2+1}$$

Let $\varphi(x) \risingdotseq \Phi(p)$. Taking the Laplace transform of both sides of the equation and taking account of the convolution theorem (transform of a convolution), we get

$$\Phi(p) = \frac{1}{p^2+1} + \frac{2p}{p^2+1}\Phi(p)$$

Whence

$$\Phi(p)\left[1-\frac{2p}{p^2+1}\right]=\frac{1}{p^2+1}$$

or

$$\Phi(p)=\frac{1}{(p-1)^2}\risingdotseq xe^x$$

Hence, the solution of the given integral equation is

$$\varphi(x)=xe^x$$

Solve the following integral equations:

59. $\varphi(x)=e^x-\int_0^x e^{x-t}\varphi(t)\,dt.$

60. $\varphi(x)=x-\int_0^x e^{x-t}\varphi(t)\,dt.$

61. $\varphi(x)=e^{2x}+\int_0^x e^{t-x}\varphi(t)\,dt.$

62. $\varphi(x)=x-\int_0^x (x-t)\varphi(t)\,dt.$

63. $\varphi(x)=\cos x-\int_0^x (x-t)\cos(x-t)\,\varphi(t)\,dt.$

64. $\varphi(x)=1+x+\int_0^x e^{-2(x-t)}\varphi(t)\,dt.$

65. $\varphi(x)=x+\int_0^x \sin(x-t)\,\varphi(t)\,dt.$

66. $\varphi(x)=\sin x+\int_0^x (x-t)\varphi(t)\,dt.$

67. $\varphi(x)=x-\int_0^x \sinh(x-t)\varphi(t)\,dt.$

68. $\varphi(x) = 1 - 2x - 4x^2 + \int_0^x [3 - 6(x-t) -$

$- 4(x-t)^2] \varphi(t)\, dt.$

69. $\varphi(x) = \sinh x - \int_0^x \cosh(x-t)\varphi(t)\, dt.$

70. $\varphi(x) = 1 + 2\int_0^x \cos(x-t)\varphi(t)\, dt.$

71. $\varphi(x) = e^x + 2\int_0^x \cos(x-t)\varphi(t)\, dt.$

72. $\varphi(x) = \cos x + \int_0^x \varphi(t)\, dt.$

The Laplace transformation may be employed in the solution of systems of Volterra integral equations of the type

$$\varphi_i(x) = f_i(x) + \sum_{j=1}^{s} \int_0^x K_{ij}(x-t)\varphi_j(t)\, dt \quad (i = 1, 2, \ldots, s) \tag{6}$$

where $K_{ij}(x)$, $f_i(x)$ are known continuous functions having Laplace transforms.

Taking the Laplace transform of both sides of (6), we get

$$\Phi_i(p) = F_i(p) + \sum_{j=1}^{s} \tilde{K}_{ij}(p)\Phi_j(p) \quad (i = 1, 2, \ldots, s) \tag{7}$$

This is a system of linear algebraic equations in $\Phi_j(p)$. Solving it, we find $\Phi_j(p)$, the original functions of which will be the solution of the original system of integral equations (6).

Example. Solve the system of integral equations

$$\left.\begin{aligned} \varphi_1(x) &= 1 - 2\int_0^x e^{2(x-t)}\varphi_1(t)\, dt + \int_0^x \varphi_2(t)\, dt, \\ \varphi_2(x) &= 4x - \int_0^x \varphi_1(t)\, dt + 4\int_0^x (x-t)\varphi_2(t)\, dt \end{aligned}\right\} \tag{8}$$

Solution. Taking transforms and using the theorem on the transform of a convolution, we get

$$\begin{cases} \Phi_1(p)=\frac{1}{p}-\frac{2}{p-2}\Phi_1(p)+\frac{1}{p}\Phi_2(p), \\ \Phi_2(p)=\frac{4}{p^2}-\frac{1}{p}\Phi_1(p)+\frac{4}{p^2}\Phi_2(p) \end{cases}$$

Solving the system obtained for $\Phi_1(p)$ and $\Phi_2(p)$, we find

$$\Phi_1(p)=\frac{p}{(p+1)^2}=\frac{1}{p+1}-\frac{1}{(p+1)^2}$$

$$\Phi_2(p)=\frac{3p+2}{(p-2)(p+1)^2}=\frac{8}{9}\cdot\frac{1}{p-2}+\frac{1}{3}\cdot\frac{1}{(p+1)^2}-\frac{8}{9}\cdot\frac{1}{p+1}$$

The original functions for $\Phi_1(p)$ and $\Phi_2(p)$ are equal, respectively, to

$$\varphi_1(x)=e^{-x}-xe^{-x},$$

$$\varphi_2(x)=\frac{8}{9}e^{2x}+\frac{1}{3}xe^{-x}-\frac{8}{9}e^{-x}$$

The functions $\varphi_1(x)$, $\varphi_2(x)$ are solutions of the original system of integral equations (8).

Solve the following systems of integral equations:

73. $$\begin{cases} \varphi_1(x)=\sin x+\int\limits_0^x \varphi_2(t)\,dt, \\ \varphi_2(x)=1-\cos x-\int\limits_0^x \varphi_1(t)\,dt. \end{cases}$$

74. $$\begin{cases} \varphi_1(x)=e^{2x}+\int\limits_0^x \varphi_2(t)\,dt, \\ \varphi_2(x)=1-\int\limits_0^x e^{2(x-t)}\varphi_1(t)\,dt. \end{cases}$$

75. $$\begin{cases} \varphi_1(x)=e^{x}+\int\limits_0^x \varphi_1(t)\,dt-\int\limits_0^x e^{x-t}\varphi_2(t)\,dt, \\ \varphi_2(x)=-x-\int\limits_0^x (x-t)\varphi_1(t)\,dt+\int\limits_0^x \varphi_2(t)\,dt. \end{cases}$$

76. $$\begin{cases} \varphi_1(x) = e^x - \int\limits_0^x \varphi_1(t)\,dt + 4\int\limits_0^x e^{x-t}\varphi_2(t)\,dt, \\ \varphi_2(x) = 1 - \int\limits_0^x e^{t-x}\varphi_1(t)\,dt + \int\limits_0^x \varphi_2(t)\,dt. \end{cases}$$

77. $$\begin{cases} \varphi_1(x) = x + \int\limits_0^x \varphi_2(t)\,dt, \\ \varphi_2(x) = 1 - \int\limits_0^x \varphi_1(t)\,dt, \\ \varphi_3(x) = \sin x + \frac{1}{2}\int\limits_0^x (x-t)\,\varphi_1(t)\,dt. \end{cases}$$

78. $$\begin{cases} \varphi_1(x) = 1 - \int\limits_0^x \varphi_2(t)\,dt, \\ \varphi_2(x) = \cos x - 1 + \int\limits_0^x \varphi_3(t)\,dt, \\ \varphi_3(x) = \cos x + \int\limits_0^x \varphi_1(t)\,dt. \end{cases}$$

79. $$\begin{cases} \varphi_1(x) = x + 1 + \int\limits_0^x \varphi_3(t)\,dt, \\ \varphi_2(x) = -x + \int\limits_0^x (x-t)\,\varphi_1(t)\,dt, \\ \varphi_3(x) = \cos x - 1 - \int\limits_0^x \varphi_1(t)\,dt. \end{cases}$$

6. Solution of Integro-Differential Equations with the Aid of the Laplace Transformation

A linear integro-differential equation is an equation of the form

$$a_0(x)\varphi^n(x)+a_1(x)\varphi^{n-1}(x)+\ldots+a_n(x)\varphi(x)+ \\ +\sum_{m=0}^{s}\int_0^x K_m(x,\ t)\varphi^{(m)}(t)\,dt=f(x) \qquad (1)$$

Here $a_0(x),\ \ldots,\ a_n(x),\ f(x),\ K_m(x,\ t)\ (m=0,\ 1,\ \ldots,\ s)$ are known functions and $\varphi(x)$ is the unknown function.

Unlike the case of integral equations, when solving integro-differential equations (1), initial conditions of the form

$$\varphi(0)=\varphi_0,\ \varphi'(0)=\varphi_0',\ \ldots,\ \varphi^{n-1}(0)=\varphi_0^{(n-1)} \qquad (2)$$

are imposed on the unknown function $\varphi(x)$. In (1), let the coefficients $a_k(x)=\text{const}$ $(k=0,\ 1,\ \ldots,\ n)$ and let $K_m(x,\ t)=$ $=K_m(x-t)$ $(m=0,\ 1,\ \ldots,\ s)$, that is, all the K_m depend solely on the difference $x-t$ of arguments. Without loss of generality, we can take $a_0=1$. Then equation (1) assumes the form

$$\varphi^n(x)+a_1\varphi^{(n-1)}(x)+\ldots+a_n\varphi(x)+ \\ +\sum_{m=0}^{s}\int_0^x K_m(x-t)\varphi^{(m)}(t)\,dt=f(x)\ \ (a_1,\ \ldots,\ a_n\text{—const}) \qquad (3)$$

Also, let the functions $f(x)$ and $K_m(x)$ be original functions and

$$f(x)\risingdotseq F(p),\ K_m(x)\risingdotseq \tilde{K}_m(p)\ \ (m=0,\ 1,\ \ldots,\ s)$$

Then the function $\varphi(x)$ will also have the Laplace transform

$$\varphi(x)\risingdotseq\Phi(p)$$

Take the Laplace transform of both sides of (3). By virtue of the theorem on the transform of a derivative,

$$\varphi^{(k)}(x)\risingdotseq p^k\Phi(p)-p^{k-1}\varphi_0-p^{k-2}\varphi_0'-\ldots-\varphi_0^{(k-1)} \qquad (4) \\ (k=0,\ 1,\ \ldots,\ n).$$

By the convolution theorem

$$\int_0^x K_m(x-t)\varphi^{(m)}(t)\,dt\risingdotseq \\ \risingdotseq\tilde{K}_m(p)\left[p^m\Phi(p)-p^{m-1}\varphi_0-\ldots-\varphi_0^{(m-1)}\right] \qquad (5) \\ (m=0,\ 1,\ \ldots,\ s)$$

Equation (3) will therefore become

$$\Phi(p)\left[p^n + a_1 p^{n-1} + \ldots + a_n + \sum_{m=0}^{s} \tilde{K}(p)\, p^m\right] = A(p) \quad (6)$$

where $A(p)$ is some known function of p.

From (6) we find $\Phi(p)$, which is an operator solution of the problem (3)-(2). Finding the original function for $\Phi(p)$, we get the solution $\varphi(x)$ of the integro-differential equation (3) that satisfies the initial conditions (2).

Example. Solve the integro-differential equation

$$\varphi''(x) + \int_0^x e^{2(x-t)} \varphi'(t)\, dt = e^{2x} \quad (7)$$

$$\varphi(0) = \varphi'(0) = 0 \quad (8)$$

Solution. Let $\varphi(x) \risingdotseq \Phi(p)$. By virtue of (8)

$$\varphi'(x) \risingdotseq p\Phi(p),$$
$$\varphi''(x) \risingdotseq p^2\Phi(p)$$

Therefore, after taking the Laplace transform, equation (7) will assume the form

$$p^2\Phi(p) + \frac{p}{p-2}\Phi(p) = \frac{1}{p-2} \quad (9)$$

or

$$\Phi(p)\frac{p(p-1)^2}{p-2} = \frac{1}{p-2} \quad (10)$$

From (10) we find

$$\Phi(p) = \frac{1}{p(p-1)^2} \risingdotseq xe^x - e^x + 1$$

Hence, the solution $\varphi(x)$ of the integro-differential equation (7) satisfying the initial conditions (8) is defined by the equality

$$\varphi(x) = xe^x - e^x + 1$$

Solve the following integro-differential equations:

80. $\varphi''(x) + \int_0^x e^{2(x-t)} \varphi'(t)\, dt = e^{2x}$; $\varphi(0) = 0$, $\varphi'(0) = 1$.

81. $\varphi'(x)-\varphi(x)+\int\limits_0^x (x-t)\,\varphi'(t)\,dt-\int\limits_0^x \varphi(t)\,dt=x;$

$\varphi(0)=-1.$

82. $\varphi''(x)-2\varphi'(x)+\varphi(x)+2\int\limits_0^x \cos(x-t)\,\varphi''(t)\,dt+$

$+2\int\limits_0^x \sin(x-t)\,\varphi'(t)\,dt=\cos x;\quad \varphi(0)=\varphi'(0)=0.$

83. $\varphi''(x)+2\varphi'(x)-2\int\limits_0^x \sin(x-t)\,\varphi'(t)\,dt=\cos x;$

$\varphi(0)=\varphi'(0)=0.$

84. $\varphi''(x)+\varphi(x)+\int\limits_0^x \sinh(x-t)\,\varphi(t)\,dt+$

$+\int\limits_0^x \cosh(x-t)\,\varphi'(t)\,dt=\cosh x;\ \varphi(0)=\varphi'(0)=0.$

85. $\varphi''(x)+\varphi(x)+\int\limits_0^x \sinh(x-t)\,\varphi(t)\,dt+$

$+\int\limits_0^x \cosh(x-t)\,\varphi'(t)\,dt=\cosh x;\quad \varphi(0)=-1,$

$\varphi'(0)=1.$

7. Volterra Integral Equations with Limits $(x, +\infty)$

Integral equations of the form

$$\varphi(x)=f(x)+\int\limits_x^{\infty} K(x-t)\,\varphi(t)\,dt \tag{1}$$

which arise in a number of problems in physics can also be solved by means of the Laplace transformation. For this purpose, we establish the convolution theorem for the expressions

$$\int_x^\infty K(x-t)\,\varphi(t)\,dt \tag{2}$$

It is known that for the Fourier transformation

$$\mathscr{F}\left\{\int_{-\infty}^{+\infty} g(x-t)\,\psi(t)\,dt\right\}=\sqrt{2\pi}\,G(\lambda)\,\Psi(\lambda) \tag{3}$$

where $G(\lambda)$, $\Psi(\lambda)$ are Fourier transforms of the functions $g(x)$ and $\psi(x)$, respectively.

Put $g(x)=K_-(x)$, i.e.,

$$g(x)=\begin{cases} 0, & x>0,\\ K(x), & x<0\end{cases}$$

$$\psi(x)=\varphi_+(x)=\begin{cases} \varphi(x), & x>0,\\ 0, & x<0\end{cases} \tag{4}$$

Then (3) can be rewritten as

$$\mathscr{F}\left\{\int_x^{+\infty} K(x-t)\,\varphi(t)\,dt\right\}=\sqrt{2\pi}\,\tilde{K}_-(\lambda)_{\mathscr{F}}\,\tilde{\Phi}_+(\lambda)_{\mathscr{L}} \tag{5}$$

(here and henceforward the subscripts $\mathscr{F}$ or $\mathscr{L}$ will mean that the Fourier transform or the Laplace transform of the function is taken).

To pass from the Fourier transform to the Laplace transform, observe that

$$F_{\mathscr{L}}(p)=\sqrt{2\pi}\,[F_+(ip)]_{\mathscr{F}} \tag{6}$$

Hence, from (5) and (6) we get

$$\mathscr{L}\left\{\int_x^\infty K(x-t)\,\varphi(t)\,dt\right\}=\sqrt{2\pi}\,[\tilde{K}_-(ip)]_{\mathscr{F}}\,[\Phi_+(p)]_{\mathscr{L}} \tag{7}$$

We now express $[\sqrt{2\pi}\,\tilde{K}_-(ip)]_{\mathscr{F}}$ in terms of the Laplace transform:

$$[\sqrt{2\pi}\,\tilde{K}_-(ip)]_{\mathscr{F}}=\int_{-\infty}^0 K(x)\,e^{-px}\,dx=\int_0^\infty K(-x)\,e^{px}\,dx$$

Putting $K(-x)=\mathcal{K}(x)$, we get

$$[\sqrt{2\pi}\,\tilde{K}_-(ip)]_{\mathcal{F}}=\tilde{\mathcal{K}}_{\mathcal{L}}(-p)=\int_0^{\infty} K(-x)\,e^{px}\,dx$$

And so

$$\mathcal{L}\left\{\int_x^{\infty} K(x-t)\,\varphi(t)\,dt\right\}=\tilde{\mathcal{K}}_{\mathcal{L}}(-p)\,\Phi_{\mathcal{L}}(p) \tag{8}$$

Let us now return to the integral equation (1). Taking the Laplace transform of both sides of (1), we obtain

$$\Phi(p)=F(p)+\tilde{\mathcal{K}}(-p)\,\Phi(p) \tag{9}$$

(dropping the subscript $\mathcal{L}$) or

$$\Phi(p)=\frac{F(p)}{1-\tilde{\mathcal{K}}(-p)}\quad(\tilde{\mathcal{K}}(-p)\neq 1) \tag{10}$$

where

$$\tilde{\mathcal{K}}(-p)=\int_0^{\infty} K(-x)\,e^{px}\,dx \tag{11}$$

The function

$$\varphi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{F(p)}{1-\tilde{\mathcal{K}}(-p)}\,e^{px}\,dp \tag{12}$$

is a particular solution of the integral equation (1). It must be stressed that the solution (9) or (12) is meaningful only if the domains of analyticity of $\tilde{\mathcal{K}}(-p)$ and $F(p)$ overlap (see [17]).

Example. Solve the integral equation

$$\varphi(x)=x+\int_x^{\infty} e^{2(x-t)}\,\varphi(t)\,dt \tag{13}$$

Solution. In this case, $f(x)=x$, $K(x)=e^{2x}$. Therefore

$$F(p)=\frac{1}{p^2},\quad \tilde{\mathcal{K}}(-p)=\int_0^{\infty} e^{-2x}\,e^{px}\,dx=\frac{1}{2-p},\quad \operatorname{Re} p<2$$

Thus, we obtain the following operator equation:

$$\Phi(p)=\frac{1}{p^2}+\frac{1}{2-p}\Phi(p)$$

so that

$$\Phi(p)=\frac{p-2}{p^2(p-1)} \tag{14}$$

Whence

$$\varphi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{p-2}{p^2(p-1)}e^{px}\,dp \quad (0<\gamma<2) \tag{15}$$

Integral (15) may be evaluated from the Cauchy integral formula. The integrand function has a double pole $p=0$ and a simple pole $p=1$, which appears for $\gamma>1$; this is connected with including or not including in the solution of equation (13) the solution of the corresponding homogeneous equation

$$\varphi(x)=\int_x^{\infty}e^{2(x-t)}\varphi(t)\,dt$$

Let us find the residues of the integrand function at its poles:

$$\operatorname*{res}_{p=0}\left(\frac{p-2}{p^2(p-1)}e^{px}\right)=2x+1,\quad \operatorname*{res}_{p=1}\left(\frac{p-2}{p^2(p-1)}e^{px}\right)=-e^x$$

Consequently, the solution of the integral equation (13) is $\varphi(x)=2x+1+Ce^x$ (C is an arbitrary constant).

Solve the integral equations:

86. $\varphi(x)=e^{-x}+\int_x^{\infty}\varphi(t)\,dt.$

87. $\varphi(x)=e^{-x}+\int_x^{\infty}e^{x-t}\varphi(t)\,dt.$

88. $\varphi(x)=\cos x+\int_x^{\infty}e^{x-t}\varphi(t)\,dt.$

89. $\varphi(x)=1+\int_x^{\infty}e^{\alpha(x-t)}\varphi(t)\,dt \quad (\alpha>0).$

8. Volterra Integral Equations of the First Kind

Suppose we have a Volterra integral equation of the first kind

$$\int_0^x K(x, t)\,\varphi(t)\,dt = f(x), \quad f(0) = 0 \tag{1}$$

where $\varphi(x)$ is the unknown function.

Suppose that $K(x, t)$, $\frac{\partial K(x, t)}{\partial x}$, $f(x)$ and $f'(x)$ are continuous for $0 \leqslant x \leqslant a$, $0 \leqslant t \leqslant x$. Differentiating both sides of (1) with respect to x, we obtain

$$K(x, x)\,\varphi(x) + \int_0^x \frac{\partial K(x, t)}{\partial x}\,\varphi(t)\,dt = f'(x) \tag{2}$$

Any continuous solution $\varphi(x)$ of equation (1), for $0 \leqslant x \leqslant a$, obviously satisfies equation (2) as well. Conversely, any continuous solution of equation (2), for $0 \leqslant x \leqslant a$, satisfies equation (1) too.

If $K(x, x)$ does not vanish at any point of the basic interval $[0, a]$, then equation (2) can be rewritten as

$$\varphi(x) = \frac{f'(x)}{K(x, x)} - \int_0^x \frac{K'_x(x, t)}{K(x, x)}\,\varphi(t)\,dt \tag{3}$$

which means it reduces to a Volterra-type integral equation of the second kind which has already been considered (see [18]).

If $K(x, x) \equiv 0$, then it is sometimes useful to differentiate (2) once again with respect to x and so on.

Note. If $K(x, x)$ vanishes at some point $x \in [0, a]$, say at $x = 0$, then equation (3) takes on peculiar properties that are quite different from the properties of equations of the second kind. (Picard called them equations of the third kind.) Complications arise here similar to those associated with the vanishing of the coefficient of the highest derivative in a linear differential equation.

Example. Solve the integral equation

$$\int_0^x \cos(x - t)\,\varphi(t)\,dt = x \tag{4}$$

Solution. The functions $f(x)=x$, $K(x, t)=\cos(x-t)$ satisfy the above-formulated conditions of continuity and differentiability.

Differentiating both sides of (4) with respect to x, we get

$$\varphi(x)\cos 0-\int_0^x \sin(x-t)\,\varphi(t)\,dt=1$$

or

$$\varphi(x)=1+\int_0^x \sin(x-t)\,\varphi(t)\,dt \tag{5}$$

Equation (5) is an integral equation of the second kind of the convolution type.

We find its solution by applying the Laplace transformation

$$\Phi(p)=\frac{1}{p}+\frac{1}{p^2+1}\Phi(p)$$

whence

$$\Phi(p)=\frac{p^2+1}{p^3}=\frac{1}{p}+\frac{1}{p^3}\doteqdot 1+\frac{x^2}{2}$$

The function $\varphi(x)=1+\frac{x^2}{2}$ will be a solution of equation (5) and hence of the original equation (4) as well. This is readily seen by direct verification.

Solve the following integral equations of the first kind by first reducing them to integral equations of the second kind:

90. $\int_0^x e^{x-t}\varphi(t)\,dt=\sin x.$

91. $\int_0^x 3^{x-t}\varphi(t)\,dt=x.$

92. $\int_0^x a^{x-t}\varphi(t)\,dt=f(x),\ f(0)=0.$

93. $\int_0^x (1-x^2+t^2)\,\varphi(t)\,dt=\frac{x^2}{2}.$

94. $\int\limits_0^x (2+x^2-t^2)\,\varphi(t)\,dt = x^2.$

95. $\int\limits_0^x \sin(x-t)\,\varphi(t)\,dt = e^{x^2/2}-1.$

9. Euler Integrals

The *gamma function*, or Euler's integral of the second kind, is the function $\Gamma(x)$ defined by the equality

$$\Gamma(x) = \int\limits_0^\infty e^{-t}\,t^{x-1}\,dt \tag{1}$$

where x is any complex number, $\operatorname{Re} x > 0$. For $x=1$ we get

$$\Gamma(1) = \int\limits_0^\infty e^{-t}\,dt = 1 \tag{2}$$

Integrating by parts, we obtain from (1)

$$\Gamma(x) = \frac{1}{x}\int\limits_0^\infty e^{-t}\,t^x\,dt = \frac{\Gamma(x+1)}{x} \tag{3}$$

This equation expresses the basic property of a gamma function:

$$\Gamma(x+1) = x\Gamma(x) \tag{4}$$

Using (2), we get

$$\begin{aligned}\Gamma(2) &= \Gamma(1+1) = 1\cdot\Gamma(1) = 1,\\ \Gamma(3) &= \Gamma(2+1) = 2\cdot\Gamma(2) = 2!,\\ \Gamma(4) &= \Gamma(3+1) = 3\cdot\Gamma(3) = 3!\end{aligned}$$

and, generally, for positive integral n

$$\Gamma(n) = (n-1)! \tag{5}$$

We know that

$$\int\limits_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Putting $x=t^{\frac{1}{2}}$ here, we obtain

$$\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt=\sqrt{\pi}$$

Taking into account expression (1) for the gamma function, we can write this equation as

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

Whence, by means of the basic property of a gamma function expressed by (4), we find

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi},$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{1\times 3}{2^2}\sqrt{\pi} \text{ and so on.}$$

Generally, it will readily be seen that the following equality holds:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\times 3\times 5\ldots(2n-1)}{2^n}\sqrt{\pi} \tag{6}$$

(n a positive integer).

Knowing the value of the gamma function for some value of the argument, we can compute, from (3), the value of the function for an argument diminished by unity. For example,

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}$$

For this reason

$$\Gamma\left(\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}}=\sqrt{\pi} \tag{7}$$

Acting in similar fashion, we find

$$\Gamma\left(-\frac{1}{2}\right)=\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}=-2\sqrt{\pi},$$

$$\Gamma\left(-\frac{3}{2}\right)=\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}=\frac{4}{3}\sqrt{\pi},$$

$$\Gamma\left(-\frac{5}{2}\right)=-\frac{8}{15}\sqrt{\pi} \text{ and so on.}$$

It is easy to verify that $\Gamma(0)=\Gamma(-1)=\ldots=\Gamma(-n)=$ $=\ldots=\infty$. Above we defined $\Gamma(x)$ for $\operatorname{Re} x>0$. The indicated method for computing $\Gamma(x)$ extends this function into the left half-plane, where $\Gamma(x)$ is defined everywhere except at the points $x=-n$ (n a positive integer and 0).

Note also the following relations:

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x}, \tag{8}$$

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\pi^{1/2}\Gamma(2x) \tag{9}$$

and generally

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)=$$
$$=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\Gamma(nx)$$

(Gauss-Legendre multiplication theorem).

The gamma function was represented by Weierstrass by means of the equation

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\} \tag{10}$$

where

$$\gamma=\lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}-\ln m\right)=0.57721\ldots$$

is Euler's constant. From (10) it is evident that the function $\Gamma(z)$ is analytic everywhere except at $z=0$, $z=-1$, $z=-2$, ..., where it has simple poles.

The following is Euler's formula which is obtained from (10):

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\} \tag{11}$$

It holds everywhere except at $z=0$, $z=-1$, $z=-2$, ...

96. Show that $\Gamma'(1)=-\gamma$.

97. Show that for $\operatorname{Re} z>0$

$$\Gamma(z)=\int_0^1\left(\ln\frac{1}{x}\right)^{z-1}dx$$

98. Show that

$$\frac{\Gamma'(1)}{\Gamma(1)}-\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}=2\cdot\ln 2$$

99. Prove that

$$\Gamma(z)=\lim_{n\to\infty}\frac{1\times 2\ldots(n-1)}{z(z+1)\ldots(z+n-1)}n^z$$

We introduce Euler's integral of the first kind $B(p, q)$, the so-called *beta function*:

$$B(p,\ q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (\operatorname{Re} p>0,\quad \operatorname{Re} q>0) \tag{12}$$

The following equality holds (it establishes a relationship between the Euler integrals of the first and second kinds):

$$B(p,\ q)=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \tag{13}$$

100. Show that

$$B(p,\ q)=B(q,\ p)$$

101. Show that

$$B(p,\ q)=B(p+1,\ q)+B(p,\ q+1)$$

102. Show that

$$B(p+1,\ q)=\frac{p}{q}B(p,\ q+1)$$

103. Show that

$$\int_{-1}^1(1+x)^{p-1}(1-x)^{q-1}dx=2^{p+q-1}B(p,\ q)$$

104. Evaluate the integral

$$I=\int_0^{\frac{\pi}{2}} \cos^{m-1}x \sin^{n-1}x\,dx \quad (\operatorname{Re} m>0, \quad \operatorname{Re} n>0)$$

10. Abel's Problem. Abel's Integral Equation and Its Generalizations

A particle is constrained to move under the force of gravity in a vertical plane (ξ, η) along a certain path. It is required to determine this path so that the particle, having started from rest at a point on the curve (path) with ordinate x, reaches the ξ axis in time $t=f_1(x)$, where $f_1(x)$ is a given function (Fig. 1).

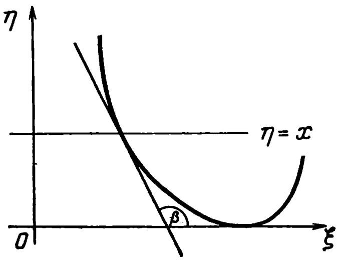

Fig. 1

The absolute velocity of a moving particle is $v=\sqrt{2g(x-\eta)}$. Denote by β the angle of inclination of the tangent to the ξ-axis. Then we will have

$$\frac{d\eta}{dt}=-\sqrt{2g(x-\eta)}\sin\beta$$

whence

$$dt=-\frac{d\eta}{\sqrt{2g(x-\eta)}\sin\beta}$$

Integrating from 0 to x and denoting $\frac{1}{\sin\beta}=\varphi(\eta)$, we get *Abel's equation*

$$\int_0^x \frac{\varphi(\eta)\,d\eta}{\sqrt{x-\eta}}=-\sqrt{2g}\,f_1(x)$$

Denoting $-\sqrt{2g}\, f_1(x)$ by $f(x)$, we finally obtain

$$\int_0^x \frac{\varphi(\eta)}{\sqrt{x-\eta}}\, d\eta = f(x) \tag{1}$$

where $\varphi(x)$ is the required function and $f(x)$ is the given function. After finding $\varphi(\eta)$ we can form the equation of the curve. Indeed,

$$\varphi(\eta) = \frac{1}{\sin\beta}$$

whence

$$\eta = \Phi(\beta)$$

Further

$$d\xi = \frac{d\eta}{\tan\beta} = \frac{\Phi'(\beta)\, d\beta}{\tan\beta}$$

whence

$$\xi = \int \frac{\Phi'(\beta)\, d\beta}{\tan\beta} = \Phi_1(\beta)$$

and consequently, the required curve is defined by the parametric equations

$$\left.\begin{aligned} \xi &= \Phi_1(\beta), \\ \eta &= \Phi(\beta) \end{aligned}\right\} \tag{2}$$

Thus, Abel's problem reduces to a solution of the integral equation

$$f(x) = \int_0^x K(x, t)\, \varphi(t)\, dt$$

with given kernel $K(x, t)$, given function $f(x)$ and unknown function $\varphi(x)$; in other words, it reduces to finding a solution of the Volterra integral equation of the first kind.

The following somewhat more general equation is also called Abel's equation:

$$\int_0^x \frac{\varphi(t)}{(x-t)^{\alpha}}\, dt = f(x) \tag{3}$$

where α is a constant, $0<\alpha<1$ *(Abel's generalized equation)*. We will consider that the function $f(x)$ has a continuous derivative on some interval $[0, a]$. Note that for $\alpha \geqslant \frac{1}{2}$ the kernel of equation (3) is quadratically nonintegrable, i. e., it is not an L_2-function. However, equation (3) has a solution which may be found in the following manner.

Suppose equation (3) has a solution. Replace x by s in the equation and multiply both sides of the resulting equality by $\frac{ds}{(x-s)^{1-\alpha}}$ and integrate with respect to s from 0 to x:

$$\int_0^x \frac{dx}{(x-s)^{1-\alpha}} \int_0^s \frac{\varphi(t)}{(s-t)^{\alpha}}\,dt = \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}}\,ds \tag{4}$$

Changing the order of integration on the left, we obtain

$$\int_0^x \varphi(t)\,dt \int_t^x \frac{ds}{(x-s)^{1-\alpha}(s-t)^{\alpha}} = F(x) \tag{5}$$

where

$$F(x) = \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}}\,ds \tag{6}$$

In the inner integral make the substitution $s=t+y(x-t)$:

$$\int_t^x \frac{ds}{(x-s)^{1-\alpha}(s-t)^{\alpha}} = \int_0^1 \frac{dy}{y^{\alpha}(1-y)^{1-\alpha}} = \frac{\pi}{\sin \alpha\pi}$$

Then from equation (5) we have

$$\int_0^x \varphi(t)\,dt = \frac{\sin \alpha\pi}{\pi} F(x)$$

or

$$\varphi(x) = \frac{\sin \alpha\pi}{\pi} F'(x) = \frac{\sin \alpha\pi}{\pi}\left(\int_0^x \frac{f(s)}{(x-s)^{1-\alpha}}\,ds\right)'_x \tag{7}$$

Thus, the only solution of equation (3) is given by formula (7), which, via integration by parts, can also be rewritten in the form

$$\varphi(x)=\frac{\sin\alpha\pi}{\pi}\left[\frac{f(0)}{x^{1-\alpha}}+\int_0^x\frac{f'(s)}{(x-s)^{1-\alpha}}\,ds\right] \tag{8}$$

This solution has physical meaning only when its absolute value is not less than 1 $\left(\text{since } \varphi(x)=\frac{1}{\sin\beta}\right)$.

We will show that in the case $f(x)\equiv C=\text{const}$, the solution of Abel's problem is a cycloid. (The tautochrone problem: to find the curve along which a particle moving under gravity without friction reaches its lowest position in the same time, irrespective of its initial position.)

In this case $\alpha=\frac{1}{2}$. Hence, by formula (8)

$$\varphi(x)=\frac{1}{\pi}\frac{C}{\sqrt{x}}$$

And therefore

$$\sin\beta=\frac{\pi\sqrt{\eta}}{C}$$

whence

$$\eta=\frac{C^2}{\pi^2}\sin^2\beta$$

Further

$$d\xi=\frac{d\eta}{\tan\beta}=\frac{C^2}{\pi^2}\frac{2\sin\beta\cos\beta}{\tan\beta}\,d\beta=\frac{C^2}{\pi^2}(1+\cos 2\beta)\,d\beta,$$

$$\xi=\frac{C^2}{\pi^2}\left(\beta+\frac{1}{2}\sin 2\beta\right)+C_1$$

Finally,

$$\left.\begin{aligned}\xi&=\frac{C^2}{\pi^2}\left(\beta+\frac{1}{2}\sin 2\beta\right)+C_1,\\ \eta&=\frac{C^2}{2\pi^2}(1-\cos 2\beta)\end{aligned}\right\}$$

(parametric equations of the cycloid).

105. Show that when $f(x)=C\sqrt{x}$ the solution of Abel's problem will be straight lines.

Solve the following integral equations:

106. $\int_0^x \frac{\varphi(t)\,dt}{(x-t)^{\alpha}} = x^n \qquad (0 < \alpha < 1).$

107. $\int_0^x \frac{\varphi(t)\,dt}{\sqrt{x-t}} = \sin x.$

108. $\int_0^x \frac{\varphi(t)\,dt}{\sqrt{x-t}} = e^x.$

109. $\int_0^x \frac{\varphi(t)\,dt}{\sqrt{x-t}} = x^{\frac{1}{2}}.$

110. Solve the two-dimensional Abel equation

$$\iint_D \frac{\varphi(x, y)\,dx\,dy}{\sqrt{(y_0-y)^2-(x_0-x)^2}} = f(x_0, y_0)$$

Here, the domain D is a right isosceles triangle with hypotenuse on the x-axis and vertex at the point (x_0, y_0).

Consider the integral equation (see [21])

$$\int_0^x (x-t)^{\beta}\varphi(t)\,dt = x^{\lambda} \tag{9}$$

($\lambda \geqslant 0$, $\beta > -1$ is real), which in a sense is a further generalization of the Abel equation (3).

Multiply both sides of (9) by $(z-x)^{\mu}$ ($\mu > -1$) and integrate with respect to x from 0 to z:

$$\int_0^z (z-x)^{\mu}\left(\int_0^x (x-t)^{\beta}\varphi(t)\,dt\right)dx = \int_0^z x^{\lambda}(z-x)^{\mu}\,dx \tag{10}$$

Putting $x = \rho z$ in the integral on the right side of (10), we obtain

$$\int_0^z x^{\lambda}(z-x)^{\mu}\,dx = z^{\lambda+\mu+1}\int_0^1 \rho^{\lambda}(1-\rho)^{\mu}\,d\rho =$$

$$= z^{\lambda+\mu+1} B(\lambda+1, \mu+1) = z^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\,\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}$$

$$(\lambda+\mu+1 > \lambda \geqslant 0) \tag{11}$$

Changing the order of integration on the left side of (10), we get

$$\int_0^z \left(\int_0^x (z-x)^\mu (x-t)^\beta \varphi(t)\,dt \right) dx =$$

$$= \int_0^z \left(\int_t^z (z-x)^\mu (x-t)^\beta \,dx \right) \varphi(t)\,dt \tag{12}$$

In the inner integral on the right of (12) put

$$x = t + \rho(z-t)$$

Then

$$\int_t^z (z-x)^\mu (x-t)^\beta \,dx = (z-t)^{\mu+\beta+1} \int_0^1 \rho^\beta (1-\rho)^\mu \,d\rho =$$

$$= (z-t)^{\mu+\beta+1} B(\beta+1,\ \mu+1) = \frac{\Gamma(\beta+1)\,\Gamma(\mu+1)}{\Gamma(\beta+\mu+2)} (z-t)^{\beta+\mu+1} \tag{13}$$

Taking into account (11), (12), (13), we obtain from (10)

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta+\mu+2)} \int_0^z (z-t)^{\mu+\beta+1} \varphi(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+2)} z^{\lambda+\mu+1} \tag{14}$$

Choose μ so that $\mu+\beta+1=n$ (a nonnegative integer). Then from (14) we will have

$$\frac{\Gamma(\beta+1)}{\Gamma(n+1)} \int_0^z (z-t)^n \varphi(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n-\beta+1)} z^{\lambda+n-\beta}$$

or

$$\int_0^z \frac{(z-t)^n}{n!} \varphi(t)\,dt = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\,\Gamma(\lambda+n-\beta+1)} z^{\lambda+n-\beta} \tag{15}$$

Differentiating both sides of (15) $n+1$ times with respect to z, we obtain

$$\varphi(z) = \frac{\Gamma(\lambda+1)\,(\lambda+n-\beta)\,(\lambda+n-\beta-1)\dots(\lambda-\beta)}{\Gamma(\beta+1)\,\Gamma(\lambda+n-\beta+1)} z^{\lambda-\beta-1} \tag{16}$$

or for $\lambda-\beta+k \neq 0$ $(k=0,\ 1,\ \dots,\ n)$

$$\varphi(z)=\frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\,\Gamma(\lambda-\beta)}\,z^{\lambda-\beta-1} \tag{17}$$

This is the solution of integral equation (9).

Observe that if the quantity $\lambda-\beta-1$ is equal to a negative integer, we get $\varphi(z)=0$. In this case, equation (9) does not have a solution in the class of ordinary functions. Its solution is a generalized function (see p. 67).

Example. Solve the integral equation

$$\int_0^x (x-t)\,\varphi(t)\,dt=x^2$$

Solution. In the given case, $\beta=1$, $\lambda=2$. Since $\lambda-\beta+k\neq 0$ $(k=0, 1, 2, \ldots, n)$, it follows, from formula (17) that

$$\varphi(x)=\frac{\Gamma(3)}{\Gamma(2)\,\Gamma(1)}\,x^{2-1-1}=2$$

Solve the integral equations:

111. $\int_0^x (x-t)^{-\frac{1}{3}}\,\varphi(t)\,dt=x^{-\frac{4}{3}}-x^2.$

112. $\int_0^x (x-t)^{\frac{1}{2}}\,\varphi(t)\,dt=\pi x.$

113. $\int_0^x (x-t)^{\frac{1}{4}}\,\varphi(t)\,dt=x+x^2.$

114. $\int_0^x (x-t)^2\,\varphi(t)\,dt=x^3.$

115. $\frac{1}{2}\int_0^x (x-t)^2\,\varphi(t)\,dt=\cos x-1+\frac{x^2}{2}.$

11. Volterra Integral Equations of the First Kind of the Convolution Type

An integral equation of the first kind

$$\int_0^x K(x-t)\,\varphi(t)\,dt=f(x) \tag{1}$$

whose kernel $K(x, t)$ is dependent solely on the difference $x-t$ of arguments will be called an *integral equation of the first kind of the convolution type.*

This class of equations includes, for instance, the generalized Abel equation.

Let us consider a problem that leads to a Volterra integral equation of the convolution type.

A shop buys and sells a variety of commodities. It is assumed that:

(1) buying and selling are continuous processes and purchased goods are put on sale at once;

(2) the shop acquires each new lot of any type of goods in quantities which it can sell in a time interval T, the same for all purchases;

(3) each new lot of goods is sold uniformly over time T.

The shop initiates the sale of a new batch of goods, the total cost of which is unity. It is required to find the law $\varphi(t)$ by which it should make purchases so that the cost of goods on hand should be constant.

Solution. Let the cost of the original goods on hand at time t be equal to $K(t)$ where

$$K(t)=\begin{cases} 1-\frac{t}{T}, & t \leqslant T, \\ 0, & t > T \end{cases}$$

Let us suppose that in the time interval between τ and $\tau+d\tau$ goods are bought amounting to the sum of $\varphi(\tau)\,d\tau$. This reserve diminishes (due to sales) in such a manner that the cost of the remaining goods at time $t>\tau$ is equal to $K(t-\tau)\varphi(\tau)\,d\tau$. Therefore the cost of the unsold part of goods acquired via purchases will, at any time t, be equal to

$$\int_0^t K(t-\tau)\varphi(\tau)\,d\tau$$

Thus, $\varphi(t)$ should satisfy the integral equation

$$1-K(t)=\int_0^t K(t-\tau)\varphi(\tau)\,d\tau$$

We have thus obtained a Volterra integral equation of the first kind of the convolution type.

Let $f(x)$ and $K(x)$ be original functions and let

$$f(x) \risingdotseq F(p), \quad K(x) \risingdotseq \tilde{K}(p), \quad \varphi(x) \risingdotseq \Phi(p)$$

Taking the Laplace transform of both sides of equation (1) and utilizing the convolution theorem, we will have

$$\tilde{K}(p)\,\Phi(p) = F(p) \tag{2}$$

whence

$$\Phi(p) = \frac{F(p)}{\tilde{K}(p)} \quad (\tilde{K}(p) \neq 0) \tag{3}$$

The original function $\varphi(x)$ for the function $\Phi(p)$ defined by (3) will be a solution of the integral equation (1).

Example. Solve the integral equation

$$\int_0^x e^{x-t}\varphi(t)\,dt = x \tag{4}$$

Solution. Taking the Laplace transform of both sides of (4), we obtain

$$\frac{1}{p-1}\Phi(p) = \frac{1}{p^2} \tag{5}$$

whence

$$\Phi(p) = \frac{p-1}{p^2} = \frac{1}{p} - \frac{1}{p^2} \risingdotseq 1 - x$$

The function $\varphi(x) = 1 - x$ is a solution of equation (4).

Solve the integral equations

116. $\int_0^x \cos(x-t)\,\varphi(t)\,dt = \sin x.$

117. $\int_0^x e^{x-t}\varphi(t)\,dt = \sinh x.$

118. $\int_0^x (x-t)^{\frac{1}{2}}\,\varphi(t)\,dt = x^{\frac{5}{2}}.$

119. $\int_0^x e^{2(x-t)}\,\varphi(t)\,dt = \sin x.$

120. $\int_0^x e^{x-t}\varphi(t)\,dt = x^2.$

121. $\int_0^x \cos(x-t)\varphi(t)\,dt = x\sin x.$

122. $\int_0^x \sinh(x-t)\varphi(t)\,dt = x^3e^{-x}.$

123. $\int_0^x J_0(x-t)\varphi(t)\,dt = \sin x.$

124. $\int_0^x \cosh(x-t)\varphi(t)\,dt = x.$

125. $\int_0^x \cos(x-t)\varphi(t)\,dt = x+x^2.$

126. $\int_0^x (x^2-t^2)\varphi(t)\,dt = \frac{x^3}{3}.$

127. $\int_0^x (x^2-4xt+3t^2)\varphi(t)\,dt = \frac{x^4}{12}.$

128. $\frac{1}{2}\int_0^x (x^2-4xt+3t^2)\varphi(t)\,dt = x^2J_4(2\sqrt{x}).$

129. $\int_0^x (x-2t)\varphi(t)\,dt = -\frac{x^3}{6}.$

Note. If $K(x, x)=K(0)\neq 0$, then equation (1) definitely has a solution. In Problem 122 the kernel $K(x, t)$ becomes identically zero for $t=x$, yet the equation has a solution.

As has already been pointed out before, a necessary condition for the existence of a continuous solution of an integral equation of the form

$$\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\varphi(t)\,dt = f(x) \tag{6}$$

consists in the function $f(x)$ having continuous derivatives up to the nth order inclusive and in all its $n-1$ first derivatives vanishing for $x=0$

This "model" equation (6) points to the necessity of matching the orders of vanishing of the kernel for $t=x$ and of the right side $f(x)$ for $x=0$ (the right side must exceed the left by at least unity).

Consider the integral equation

$$\int_0^x (x-t)\varphi(t)\,dt = x \tag{7}$$

Here, $f(x)=x$, $n=2$. Obviously, $f(x)$ has derivatives of all orders, but its first derivative $f'(x)=1\neq 0$; that is, the necessary condition is not fulfilled.

Taking the Laplace transform of both sides of (7) in formal fashion, we get

$$\frac{1}{p^2}\Phi(p) = \frac{1}{p^2}$$

whence

$$\Phi(p) = 1$$

This is the transform of the δ-function $\delta(x)$.

Recall that

$$\delta(x) \doteq 1$$
$$\delta^{(m)}(x) \doteq p^m$$

where m is an integer $\geqslant 0$.

Thus, the solution of the integral equation (7) is the δ-function:

$$\varphi(x) = \delta(x)$$

This is made clear by direct verification if we take into account that the convolution of the δ-function and any other smooth function $g(x)$ is defined as

$$g(x) * \delta(x) = g(x)$$
$$\delta^{(k)}(x) * g(x) = g^{(k)}(x) \quad (k=1, 2, \ldots)$$

Indeed, in our case $g(x)=K(x)=x$ and

$$\int_0^x K(x-t)\,\delta(t)\,dt = K(x) = x$$

Thus, the solution of equation (7) exists, but now in the class of generalized functions (see [4], [22]).

Solve the integral equations:

130. $\int\limits_0^x (x-t)\,\varphi(t)\,dt = x^2 + x - 1.$

131. $\int\limits_0^x (x-t)\,\varphi(t)\,dt = \sin x.$

132. $\int\limits_0^x (x-t)^2\,\varphi(t)\,dt = x^2 + x^3.$

133. $\int\limits_0^x \sin(x-t)\,\varphi(t)\,dt = x + 1.$

134. $\int\limits_0^x \sin(x-t)\,\varphi(t)\,dt = 1 - \cos x.$

Integral equations of the first kind with logarithmic kernel

$$\int\limits_0^x \varphi(t)\ln(x-t)\,dt = f(x), \quad f(0)=0 \tag{8}$$

can also be solved by means of the Laplace transformation.

We know that

$$x^{\nu} \risingdotseq \frac{\Gamma(\nu+1)}{p^{\nu+1}} \quad (\operatorname{Re}\nu > -1) \tag{9}$$

Differentiate the relation (9) with respect to ν:

$$x^{\nu}\ln x \risingdotseq \frac{1}{p^{\nu+1}}\,\frac{d\Gamma(\nu+1)}{d\nu} + \frac{1}{p^{\nu+1}}\ln\frac{1}{p}\,\Gamma(\nu+1)$$

or

$$x^{\nu}\ln x \risingdotseq \frac{\Gamma(\nu+1)}{p^{\nu+1}}\left[\frac{\dfrac{d\Gamma(\nu+1)}{d\nu}}{\Gamma(\nu+1)} + \ln\frac{1}{p}\right] \tag{10}$$

For $\nu=0$ we have (see p. 54)

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$$

where γ is Euler's constant and formula (10) takes the form

$$\ln x \doteqdot \frac{1}{p}(-\gamma-\ln p)=-\frac{\ln p+\gamma}{p} \tag{11}$$

Let $\varphi(x)\doteqdot\Phi(p)$, $f(x)\doteqdot F(p)$. Taking the Laplace transform of both sides of (8) and utilizing formula (11), we get

$$-\Phi(p)\frac{\ln p+\gamma}{p}=F(p)$$

whence

$$\Phi(p)=-\frac{pF(p)}{\ln p+\gamma} \tag{12}$$

Let us write $\Phi(p)$ in the form

$$\Phi(p)=-\frac{p^2F(p)-f'(0)}{p(\ln p+\gamma)}-\frac{f'(0)}{p(\ln p+\gamma)} \tag{13}$$

Since $f(0)=0$, it follows that

$$p^2F(p)-f'(0)\doteqdot f''(x) \tag{14}$$

Let us return to formula (9) and write it in the form

$$\frac{x^\nu}{\Gamma(\nu+1)}\doteqdot\frac{1}{p^{\nu+1}} \tag{9'}$$

Integrate both sides of (9′) with respect to ν from 0 to ∞. This yields

$$\int\limits_0^\infty \frac{x^\nu}{\Gamma(\nu+1)}\,d\nu\doteqdot\int\limits_0^\infty\frac{d\nu}{p^{\nu+1}}=\frac{1}{p\ln p}$$

By the similarity theorem

$$\int\limits_0^\infty \frac{x^\nu a^{-\nu}}{\Gamma(\nu+1)}\,d\nu\doteqdot\frac{1}{p\ln(ap)}=\frac{1}{p(\ln p+\ln a)}$$

If we put $a=e^\gamma$, then

$$\int\limits_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu\doteqdot\frac{1}{p(\ln p+\gamma)} \tag{15}$$

Take advantage of equality (13). By virtue of (15)

$$\frac{f'(0)}{p(\ln p+\gamma)} \doteqdot f'(0)\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu$$

Taking into account (14) and (15), the first term on the right of (13) may be regarded as a product of transforms. To find its original function, take advantage of the convolution theorem:

$$\frac{p^2F(p)-f'(0)}{p(\ln p+\gamma)} \doteqdot \int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu\right)dt$$

Thus, the solution $\varphi(x)$ of the integral equation (8) will have the form

$$\varphi(x)=-\int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu\right)dt-f'(0)\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu$$

where γ is Euler's constant.

In particular, for $f(x)=x$ we get

$$\varphi(x)=-\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}\,d\nu$$

The convolution theorem can also be used for solving nonlinear Volterra integral equations of the type

$$\varphi(x)=f(x)+\lambda\int_0^x \varphi(t)\,\varphi(x-t)\,dt \tag{16}$$

Let

$$\varphi(x)\doteqdot\Phi(p),\quad f(x)\doteqdot F(p)$$

Then, by virtue of equation (16),

$$\Phi(p)=F(p)+\lambda\Phi^2(p)$$

whence

$$\Phi(p)=\frac{1\pm\sqrt{1-4\lambda F(p)}}{2\lambda}$$

The original function of $\Phi(p)$, if it exists, will be a solution of the integral equation (16).

Example. Solve the integral equation

$$\int_0^x \varphi(t)\varphi(x-t)\,dt = \frac{x^3}{6} \tag{17}$$

Solution. Let $\varphi(x) \risingdotseq \Phi(p)$. Taking the Laplace transform of both sides of (17), we get

$$\Phi^2(p) = \frac{1}{p^4}$$

whence

$$\Phi(p) = \pm\frac{1}{p^2}$$

The functions $\varphi_1(x) = x$, $\varphi_2(x) = -x$ will be solutions of the equation (17) [the solution of equation (17) is not unique].

Solve the integral equations:

135. $2\varphi(x) - \int_0^x \varphi(t)\varphi(x-t)\,dt = \sin x.$

136. $\varphi(x) = \frac{1}{2}\int_0^x \varphi(t)\varphi(x-t)\,dt - \frac{1}{2}\sinh x.$

CHAPTER II

FREDHOLM INTEGRAL EQUATIONS

12. Fredholm Equations of the Second Kind. Fundamentals

A linear Fredholm integral equation of the second kind is an equation of the form

$$\varphi(x)-\lambda\int_a^b K(x, t)\varphi(t)\,dt=f(x) \tag{1}$$

where $\varphi(x)$ is the unknown function, $K(x, t)$ and $f(x)$ are known functions, x and t are real variables varying in the interval (a, b), and λ is a numerical factor.

The function $K(x, t)$ is called the *kernel of the integral equation* (1); it is assumed that the kernel $K(x, t)$ is defined in the square $\Omega\{a\leqslant x\leqslant b,\ a\leqslant t\leqslant b\}$ in the (x, t) plane and is continuous in Ω, or its discontinuities are such that the double integral

$$\int_a^b\int_a^b |K(x, t)|^2\,dx\,dt$$

has a finite value.

If $f(x)\not\equiv 0$, equation (1) is *nonhomogeneous*; but if $f(x)\equiv 0$, then (1) takes the form

$$\varphi(x)-\lambda\int_a^b K(x, t)\varphi(t)\,dt=0 \tag{2}$$

and is called *homogeneous*.

The limits of integration, a and b in equations (1) and (2), can be either finite or infinite.

A *solution* of the integral equations (1) and (2) is any function $\varphi(x)$ which, when substituted into the equations, reduces them to identities in $x\in(a, b)$.

Example. Show that the function $\varphi(x)=\sin\frac{\pi x}{2}$ is a solution of the Fredholm-type integral equation

$$\varphi(x)-\frac{\pi^2}{4}\int_0^1 K(x, t)\varphi(t)\,dt=\frac{x}{2}$$

where the kernel is of the form

$$K(x, t)=\begin{cases}\dfrac{x(2-t)}{2}, & 0\leqslant x\leqslant t,\\[2mm] \dfrac{t(2-x)}{2}, & t\leqslant x\leqslant 1\end{cases}$$

Solution. Write the left-hand side of the equation as

$$\varphi(x)-\frac{\pi^2}{4}\int_0^1 K(x, t)\,\varphi(t)\,dt=$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\int_0^x K(x, t)\,\varphi(t)\,dt+\int_x^1 K(x, t)\,\varphi(t)\,dt\right\}=$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\int_0^x \frac{t(2-x)}{2}\varphi(t)\,dt+\int_x^1 \frac{x(2-t)}{2}\varphi(t)\,dt\right\}=$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\frac{2-x}{2}\int_0^x t\varphi(t)\,dt+\frac{x}{2}\int_x^1 (2-t)\,\varphi(t)\,dt\right\}$$

Substituting the function $\sin\frac{\pi x}{2}$ in place of $\varphi(x)$ into this expression, we get

$$\sin\frac{\pi x}{2}-\frac{\pi^2}{4}\left\{(2-x)\int_0^x t\frac{\sin\frac{\pi t}{2}}{2}\,dt+x\int_x^1 (2-t)\frac{\sin\frac{\pi t}{2}}{2}\,dt\right\}=$$

$$=\sin\frac{\pi x}{2}-\frac{\pi^2}{4}\left\{(2-x)\left(-\frac{t}{\pi}\cos\frac{\pi t}{2}+\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right)\Bigg|_{t=0}^{t=x}+\right.$$

$$\left.+x\left[-\frac{2-t}{\pi}\cos\frac{\pi t}{2}-\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right]\Bigg|_{t=x}^{t=1}\right\}=\frac{x}{2}$$

Thus, we have $\frac{x}{2}\equiv\frac{x}{2}$, which, by definition, implies that $\varphi(x)=\sin\frac{\pi x}{2}$ is a solution of the given integral equation.

Check to see which of the given functions are solutions of the indicated integral equations.

137. $\varphi(x)=1,\quad \varphi(x)+\int_0^1 x(e^{xt}-1)\,\varphi(t)\,dt=e^x-x.$

138. $\varphi(x)=e^x\left(2x-\frac{2}{3}\right)$,

$$\varphi(x)+2\int_0^1 e^{x-t}\varphi(t)\,dt=2xe^x.$$

139. $\varphi(x)=1-\dfrac{2\sin x}{1-\frac{\pi}{2}}$, $\quad \varphi(x)-\int_0^\pi \cos(x+t)\,\varphi(t)\,dt=1.$

140. $\varphi(x)=\sqrt{x}$, $\quad \varphi(x)-\int_0^1 K(x, t)\,\varphi(t)\,dt=$

$$=\sqrt{x}+\frac{x}{15}(4x^{3/2}-7).$$

$$K(x, t)=\begin{cases}\dfrac{x(2-t)}{2}, & 0\leqslant x\leqslant t,\\ \dfrac{t(2-x)}{2}, & t\leqslant x\leqslant 1.\end{cases}$$

141. $\varphi(x)=e^x$, $\quad \varphi(x)+\lambda\int_0^1 \sin xt\,\varphi(t)\,dt=1.$

142. $\varphi(x)=\cos x$, $\quad \varphi(x)-\int_0^\pi (x^2+t)\cos t\,\varphi(t)\,dt=\sin x.$

143. $\varphi(x)=xe^{-x}$, $\quad \varphi(x)-4\int_0^\infty e^{-(x+t)}\varphi(t)\,dt=(x-1)e^{-x}.$

144. $\varphi(x)=\cos 2x$, $\quad \varphi(x)-3\int_0^\pi K(x, t)\,\varphi(t)\,dt=\cos x,$

$$K(x, t)=\begin{cases}\sin x\cos t, & 0\leqslant x\leqslant t,\\ \sin t\cos x, & t\leqslant x\leqslant \pi.\end{cases}$$

145. $\varphi(x)=\frac{4C}{\pi}\sin x$, where C is an arbitrary constant,

$$\varphi(x)-\frac{4}{\pi}\int_0^\infty \sin x\,\frac{\sin^2 t}{t}\,\varphi(t)\,dt=0.$$

13. The Method of Fredholm Determinants

The solution of the Fredholm equation of the second kind

$$\varphi(x)-\lambda\int_a^b K(x, t)\,\varphi(t)\,dt=f(x) \qquad (1)$$

is given by the formula

$$\varphi(x)=f(x)+\lambda\int_a^b R(x, t; \lambda) f(t)\,dt \tag{2}$$

where the function $R(x, t; \lambda)$ is called the *Fredholm resolvent kernel* of equation (1) and is defined by the equation

$$R(x, t; \lambda)=\frac{D(x, t; \lambda)}{D(\lambda)} \tag{3}$$

provided that $D(\lambda)\neq 0$. Here, $D(x, t; \lambda)$ and $D(\lambda)$ are power series in λ:

$$D(x, t; \lambda)=K(x, t)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}B_n(x, t)\lambda^n, \tag{4}$$

$$D(\lambda)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}C_n\lambda^n \tag{5}$$

whose coefficients are given by the formulas

$$B_n(x, t)=\underbrace{\int_a^b\cdots\int_a^b}_{n}\begin{vmatrix} K(x, t) & K(x, t_1) & \dots & K(x, t_n) \\ K(t_1, t) & K(t_1, t_1) & \dots & K(t_1, t_n) \\ K(t_2, t) & K(t_2, t_1) & \dots & K(t_2, t_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K(t_n, t) & K(t_n, t_1) & \dots & K(t_n, t_n)\end{vmatrix} dt_1\dots dt_n \tag{6}$$

and

$$B_0(x, t)=K(x, t)$$

$$C_n=\underbrace{\int_a^b\cdots\int_a^b}_{n}\begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) & \dots & K(t_1, t_n) \\ K(t_2, t_1) & K(t_2, t_2) & \dots & K(t_2, t_n) \\ K(t_3, t_1) & K(t_3, t_2) & \dots & K(t_3, t_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K(t_n, t_1) & K(t_n, t_2) & \dots & K(t_n, t_n)\end{vmatrix} dt_1\dots dt_n \tag{7}$$

The function $D(x, t; \lambda)$ is called the *Fredholm minor*, and $D(\lambda)$ the *Fredholm determinant*. When the kernel $K(x, t)$ is bounded or the integral

$$\int_a^b \int_a^b K^2(x, t)\,dx\,dt$$

has a finite value, the series (4) and (5) converge for all values of λ and, hence, are entire analytic functions of λ.

The resolvent kernel

$$R(x, t; \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)}$$

is an analytic function of λ, except for those values of λ which are zeros of the function $D(\lambda)$. The latter are the poles of the resolvent kernel $R(x, t; \lambda)$.

Example. Using Fredholm determinants, find the resolvent kernel of the kernel $K(x, t) = xe^t$; $a = 0$, $b = 1$.

Solution. We have $B_0(x, t) = xe^t$ Further,

$$B_1(x, t) = \int_0^1 \begin{vmatrix} xe^t & xe^{t_1} \\ t_1e^t & t_1e^{t_1} \end{vmatrix} dt_1 = 0,$$

$$B_2(x, t) = \int_0^1 \int_0^1 \begin{vmatrix} xe^t & xe^{t_1} & xe^{t_2} \\ t_1e^t & t_1e^{t_1} & t_1e^{t_2} \\ t_2e^t & t_2e^{t_1} & t_2e^{t_2} \end{vmatrix} dt_1\,dt_2 = 0$$

since the determinants under the integral sign are zero. It is obvious that all subsequent $B_n(x, t) = 0$ Find the coefficients C_n:

$$C_1 = \int_0^1 K(t_1, t_1)\,dt_1 = \int_0^1 t_1e^{t_1}\,dt_1 = 1,$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} t_1e^{t_1} & t_1e^{t_2} \\ t_2e^{t_1} & t_2e^{t_2} \end{vmatrix} dt_1\,dt_2 = 0$$

Obviously, all subsequent C_n are also equal to zero.

In our case, by formulas (4) and (5), we have

$$D(x, t; \lambda) = K(x, t) = xe^t; \quad D(\lambda) = 1 - \lambda$$

Thus,

$$R(x, t, \lambda) = \frac{D(x, t; \lambda)}{D(\lambda)} = \frac{xe^t}{1-\lambda}$$

Let us apply the result obtained to solving the integral equation

$$\varphi(x)-\lambda\int_0^1 xe^t\varphi(t)\,dt=f(x)\quad(\lambda\neq 1)$$

By formula (2)

$$\varphi(x)=f(x)+\lambda\int_0^1\frac{xe^t}{1-\lambda}f(t)\,dt$$

In particular, for $f(x)=e^{-x}$ we get

$$\varphi(x)=e^{-x}+\frac{\lambda}{1-\lambda}x$$

Using Fredholm determinants, find the resolvent kernels of the following kernels:

146. $K(x, t)=2x-t$; $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 1$.

147. $K(x, t)=x^2t-xt^2$; $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 1$.

148. $K(x, t)=\sin x\cos t$; $0\leqslant x\leqslant 2\pi$, $0\leqslant t\leqslant 2\pi$.

149. $K(x, t)=\sin x-\sin t$; $0\leqslant x\leqslant 2\pi$, $0\leqslant t\leqslant 2\pi$.

Practically speaking, only in very rare cases is it possible to compute the coefficients $B_n(x, t)$ and C_n of the series (4) and (5) from formulas (6) and (7), but from these formulas it is possible to obtain the following recursion relations:

$$B_n(x, t)=C_nK(x, t)-n\int_a^b K(x, s)\,B_{n-1}(s, t)\,ds,\tag{8}$$

$$C_n=\int_a^b B_{n-1}(s, s)\,ds\tag{9}$$

Knowing that the coefficient $C_0=1$ and $B_0(x, t)=K(x, t)$, we can use formulas (9) and (8) to find, in succession, C_1, $B_1(x, t)$, C_2, $B_2(x, t)$, C_3 and so on.

Example. Using formulas (8) and (9), find the resolvent kernel of the kernel $K(x, t)=x-2t$, where $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 1$.

Solution. We have $C_0=1$, $B_0(x, t)=x-2t$. Using formula (9), we find

$$C_1=\int_0^1 (-s)\,ds=-\frac{1}{2}$$

By formula (8) we get

$$B_1(x, t)=-\frac{x-2t}{2}-\int_0^1 (x-2s)(s-2t)\,ds=-x-t+2xt+\frac{2}{3}$$

We further obtain

$$C_2=\int_0^1 \left(-2s+2s^2+\frac{2}{3}\right) ds=\frac{1}{3}$$

$$B_2(x, t)=\frac{x-2t}{3}-2\int_0^1 (x-2s)\left(-s-t+2st+\frac{2}{3}\right) ds=0$$

$$C_3=C_4=\ldots=0,\quad B_3(x, t)=B_4(x, t)=\ldots=0$$

Hence,

$$D(\lambda)=1+\frac{\lambda}{2}+\frac{\lambda^2}{6};\ D(x, t; \lambda)=x-2t+\left(x+t-2xt-\frac{2}{3}\right)\lambda$$

The resolvent kernel of the given kernel is

$$R(x, t; \lambda)=\frac{x-2t+\left(x+t-2xt-\frac{2}{3}\right)\lambda}{1+\frac{\lambda}{2}+\frac{\lambda^2}{6}}$$

Using the recursion relations (8) and (9), find the resolvent kernels of the following kernels:

150. $K(x, t)=x+t+1$; $\quad -1\leqslant x\leqslant 1,\ -1\leqslant t\leqslant 1.$

151. $K(x, t)=1+3xt$; $\quad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant 1.$

152. $K(x, t)=4xt-x^2$; $\quad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant 1.$

153. $K(x, t)=e^{x-t}$; $\quad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant 1.$

154. $K(x, t)=\sin(x+t)$; $\quad 0\leqslant x\leqslant 2\pi,\ 0\leqslant t\leqslant 2\pi.$

155. $K(x, t)=x-\sinh t$; $\quad -1\leqslant x\leqslant 1,\ -1\leqslant t\leqslant 1.$

Using the resolvent kernel, solve the following integral equations:

156. $\varphi(x)-\lambda\int\limits_0^{2\pi}\sin(x+t)\,\varphi(t)\,dt=1.$

157. $\varphi(x)-\lambda\int\limits_0^1(2x-t)\,\varphi(t)\,dt=\frac{x}{6}.$

158. $\varphi(x)-\int\limits_0^{2\pi}\sin x\cos t\,\varphi(t)\,dt=\cos 2x.$

159. $\varphi(x)+\int\limits_0^1 e^{x-t}\varphi(t)\,dt=e^x.$

160. $\varphi(x)-\lambda\int\limits_0^1(4xt-x^2)\,\varphi(t)\,dt=x.$

14. Iterated Kernels. Constructing the Resolvent Kernel with the Aid of Iterated Kernels

Suppose we have a Fredholm integral equation

$$\varphi(x)-\lambda\int\limits_a^b K(x,\ t)\,\varphi(t)\,dt=f(x) \tag{1}$$

As in the case of the Volterra equations, the integral equation (1) may be solved by the method of successive approximations To do this, put

$$\varphi(x)=f(x)+\sum_{n=1}^{\infty}\psi_n(x)\,\lambda^n \tag{2}$$

where the $\psi_n(x)$ are determined from the formulas

$$\psi_1(x)=\int\limits_a^b K(x,\ t)\,f(t)\,dt,$$

$$\psi_2(x)=\int\limits_a^b K(x,\ t)\,\psi_1(t)\,dt=\int\limits_a^b K_2(x,\ t)\,f(t)\,dt,$$

$$\psi_3(x) = \int_a^b K(x, t)\,\psi_2(t)\,dt = \int_a^b K_3(x, t)\,f(t)\,dt$$

and so on.

Here

$$K_2(x, t) = \int_a^b K(x, z)\,K_1(z, t)\,dz,$$

$$K_3(x, t) = \int_a^b K(x, z)\,K_2(z, t)\,dz$$

and, generally,

$$K_n(x, t) = \int_a^b K(x, z)\,K_{n-1}(z, t)\,dz \tag{3}$$

$n = 2, 3, \ldots,$ and $K_1(x, t) \equiv K(x, t)$ The functions $K_n(x, t)$ determined from formulas (3) are called *iterated kernels* For them, the following relation holds:

$$K_n(x, t) = \int_a^b K_m(x, s)\,K_{n-m}(s, t)\,ds \tag{4}$$

where m is any natural number less than n.

The resolvent kernel of the integral equation (1) is determined in terms of iterated kernels by the formula

$$R(x, t; \lambda) = \sum_{n=1}^{\infty} K_n(x, t)\,\lambda^{n-1} \tag{5}$$

where the series on the right is called the *Neumann series of the kernel* $K(x, t)$. It converges for

$$|\lambda| < \frac{1}{B} \tag{6}$$

where $B = \sqrt{\int_a^b \int_a^b K^2(x, t)\,dx\,dt}$

The solution of the Fredholm equation of the second kind (1) is expressed by the formula

$$\varphi(x) = f(x) + \lambda \int_a^b R(x, t; \lambda)\,f(t)\,dt \tag{7}$$

The boundary (6) is essential for convergence of the series (5). However, a solution of equation (1) can exist for values of $|\lambda| > \frac{1}{B}$ as well.

Let us consider an example:

$$\varphi(x) - \lambda \int_0^1 \varphi(t)\, dt = 1 \tag{8}$$

Here $K(x, t) \equiv 1$, and hence

$$B^2 = \int_0^1 \int_0^1 K^2(x, t)\, dx\, dt = \int_0^1 \int_0^1 dx\, dt = 1$$

Thus the condition (6) gives that the series (5) converges for $|\lambda| < 1$.

Solving equation (8) as an equation with a degenerate kernel, we get $(1-\lambda)\, C = 1$, where $C = \int_0^1 \varphi(t)\, dt$. For $\lambda = 1$, this equation is unsolvable and hence for $\lambda = 1$ the integral equation (8) does not have any solution. From this it follows that in a circle of radius greater than unity, successive approximations cannot converge for equation (8). However, equation (8) is solvable for $|\lambda| > 1$. Indeed, if $\lambda \neq 1$, then the function $\varphi(x) = \frac{1}{1-\lambda}$ is a solution of the given equation. This may readily be verified by direct substitution.

For some Fredholm equations the Neumann series (5) converges for the resolvent kernel for any values of λ. Let us demonstrate this fact.

Suppose we have two kernels: $K(x, t)$ and $L(x, t)$. We shall call these kernels *orthogonal* if the following two conditions are fulfilled for any admissible values of x and t:

$$\int_a^b K(x, z)\, L(z, t)\, dz = 0, \quad \int_a^b L(x, z)\, K(z, t)\, dz = 0 \tag{9}$$

Example. The kernels $K(x, t) = xt$ and $L(x, t) = x^2t^2$ are orthogonal on $[-1, 1]$.

Indeed,

$$\int_{-1}^{1} (xz)(z^2t^2)\,dz = xt^2 \int_{-1}^{1} z^3\,dz = 0,$$

$$\int_{-1}^{1} (x^2z^2)(zt)\,dz = x^2t \int_{-1}^{1} z^3\,dz = 0$$

There exist kernels which are orthogonal to themselves. For such kernels, $K_2(x, t) \equiv 0$, where $K_2(x, t)$ is the second iterated kernel. It is obvious that in this case all subsequent iterated kernels are also equal to zero and the resolvent kernel coincides with the kernel $K(x, t)$.

Example. $K(x, t) = \sin(x-2t)$; $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant 2\pi$. We have

$$\int_0^{2\pi} \sin(x-2z)\sin(z-2t)\,dz =$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x+2t-3z) - \cos(x-2t-z)]\,dz =$$

$$= \frac{1}{2}\left[-\frac{1}{3}\sin(x+2t-3z) + \sin(x-2t-z)\right]\Bigg|_{z=0}^{z=2\pi} = 0$$

Thus, in this case the resolvent kernel of the kernel is equal to the kernel itself:

$$R(x, t; \lambda) \equiv \sin(x-2t)$$

so that the Neumann series (5) consists of one term and, obviously, converges for any λ.

The iterated kernels $K_n(x, t)$ can be expressed directly in terms of the given kernel $K(x, t)$ by the formula

$$K_n(x, t) = \int_a^b\int_a^b \ldots \int_a^b K(x, s_1)K(s_1, s_2)\ldots K(s_{n-1}, t)\,ds_1,\ ds_2 \ldots ds_{n-1} \tag{10}$$

All iterated kernels $K_n(x, t)$, beginning with $K_2(x, t)$, will be continuous functions in the square $a \leqslant x \leqslant b$, $a \leqslant t \leqslant b$ if the initial kernel $K(x, t)$ is quadratically summable in this square.

If the given kernel $K(x, t)$ is symmetric, then all iterated kernels $K_n(x, t)$ are also symmetric (see [15]).

The following are some examples in finding iterated kernels.

Example 1. Find the iterated kernels for the kernel $K(x, t)=x-t$ if $a=0$, $b=1$.

Solution. Using formulas (2), we find in succession:

$$K_1(x, t)=x-t,$$

$$K_2(x, t)=\int_0^1 (x-s)(s-t)\,ds=\frac{x+t}{2}-xt-\frac{1}{3},$$

$$K_3(x, t)=\int_0^1 (x-s)\left(\frac{s+t}{2}-st-\frac{1}{3}\right)ds=-\frac{x-t}{12},$$

$$K_4(x, t)=-\frac{1}{12}\int_0^1 (x-s)(s-t)\,ds=$$

$$=-\frac{1}{12}K_2(x, t)=-\frac{1}{12}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right),$$

$$K_5(x, t)=-\frac{1}{12}\int_0^1 (x-s)\left(\frac{s+t}{2}-st-\frac{1}{3}\right)ds=$$

$$=-\frac{1}{12}K_3(x, t)=\frac{x-t}{12^2},$$

$$K_6(x, t)=\frac{1}{12^2}\int_0^1 (x-s)(s-t)\,ds=\frac{K_2(x, t)}{12^2}=$$

$$=\frac{1}{12^2}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right).$$

From this it follows that iterated kernels are of the form:

(1) for $n=2k-1$

$$K_{2k-1}(x, t)=\frac{(-1)^k}{12^{k-1}}(x-t)$$

(2) for $n=2k$

$$K_{2k}(x, t)=\frac{(-1)^{k-1}}{12^{k-1}}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right)$$

where $k = 1, 2, 3, \ldots$.

Example 2. Find the iterated kernels $K_1(x, t)$ and $K_2(x, t)$ if $K(x, t) = e^{\min(x, t)}$, $a = 0$, $b = 1$.

Solution. By definition we have

$$\min(x, t) = \begin{cases} x, & \text{if } 0 \leqslant x \leqslant t, \\ t, & \text{if } t \leqslant x \leqslant 1 \end{cases}$$

and for this reason the given kernel may be written as

$$K(x, t) = \begin{cases} e^x, & \text{if } 0 \leqslant x \leqslant t, \\ e^t, & \text{if } t \leqslant x \leqslant 1 \end{cases}$$

This kernel, as may easily be verified, is symmetric, i. e.,

$$K(x, t) = K(t, x)$$

We have $K_1(x, t) = K(x, t)$. We find the second iterated kernel:

$$K_2(x, t) = \int_0^1 K(x, s) K_1(s, t)\, ds = \int_0^1 K(x, s) K(s, t)\, ds$$

Here

$$K(x, s) = \begin{cases} e^x, & \text{if } 0 \leqslant x \leqslant s, \\ e^s, & \text{if } s \leqslant x \leqslant 1, \end{cases}$$

$$K(s, t) = \begin{cases} e^s, & \text{if } 0 \leqslant s \leqslant t, \\ e^t, & \text{if } t \leqslant s \leqslant 1 \end{cases}$$

Since the given kernel $K(x, t)$ is symmetric, it is sufficient to find $K_2(x, t)$ only for $x > t$.

We have (see Fig 2)

$$K_2(x, t) = \int_0^t K(x, s) K(s, t)\, ds + \int_t^x K(x, s) K(s, t)\, ds +$$

$$+ \int_x^1 K(x, s) K(s, t)\, ds$$

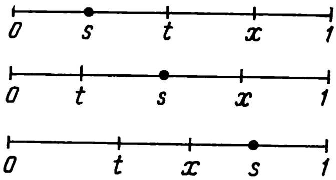

Fig. 2

In the interval $(0, t)$ we have $s < t < x$, and therefore

$$\int_0^t K(x, s) K(s, t)\, ds = \int_0^t e^s e^s\, ds = \int_0^t e^{2s}\, ds = \frac{e^{2t}-1}{2}$$

In the interval (t, x) we have $t < s < x$, and therefore

$$\int_t^x K(x, s) K(s, t)\, ds = \int_t^x e^s e^t\, ds = e^{x+t} - e^{2t}$$

In the interval $(x, 1)$ we have $s > x > t$, and therefore

$$\int_x^1 K(x, s) K(s, t)\, ds = \int_x^1 e^x e^t\, ds = (1-x)\, e^{x+t}$$

Adding the integrals thus found, we obtain

$$K_2(x, t) = (2-x)\, e^{x+t} - \frac{1+e^{2t}}{2} \quad (x > t)$$

We will find the expression for $K_2(x, t)$ for $x < t$ if we interchange the arguments x and t in the expression $K_2(x, t)$ for $x > t$:

$$K_2(x, t) = (2-t)\, e^{x+t} - \frac{1+e^{2x}}{2} \quad (x < t)$$

Thus the second iterated kernel is of the form

$$K_2(x, t) = \begin{cases} (2-t)\, e^{x+t} - \dfrac{1+e^{2x}}{2}, & \text{if } 0 \leqslant x \leqslant t, \\ (2-x)\, e^{x+t} - \dfrac{1+e^{2t}}{2}, & \text{if } t \leqslant x \leqslant 1 \end{cases}$$

Note. If the kernel $K(x, t)$, which is specified in the square $a \leqslant x \leqslant b$, $a \leqslant t \leqslant b$ by various analytic expressions, is not symmetric, then one should consider the case $x < t$ separately. For $x < t$ we have (see Fig. 3)

$$K_2(x, t) = \int_a^b K(x, s) K(s, t)\, ds = \int_a^x + \int_x^t + \int_t^b$$

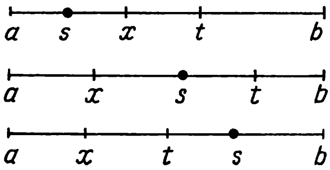

Fig. 3

Example 3. Find the iterated kernels $K_1(x, t)$ and $K_2(x, t)$ if $a = 0$, $b = 1$ and

$$K(x, t) = \begin{cases} x + t, & \text{if } 0 \leqslant x < t, \\ x - t, & \text{if } t < x \leqslant 1 \end{cases}.$$

Solution. We have $K_1(x, t) = K(x, t)$,

$$K_2(x, t) = \int_0^1 K(x, s) K(s, t)\, ds$$

where

$$K(x, s) = \begin{cases} x + s, & 0 \leqslant x < s, \\ x - s, & s < x \leqslant 1, \end{cases} \quad K(s, t) = \begin{cases} s + t, & 0 \leqslant s < t, \\ s - t, & t < s \leqslant 1 \end{cases}$$

Since the given kernel $K(x, t)$ is not symmetric, we consider two cases separately when finding $K_2(x, t)$: (1) $x < t$ and (2) $x > t$.

(1) Let $x < t$. Then (see Fig. 3)

$$K_2(x, t) = I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^x (x - s)(s + t)\, ds = \frac{x^3}{6} + \frac{x^2 t}{2},$$

$$I_2=\int_x^t (x+s)(s+t)\,ds=\frac{5t^3}{6}-\frac{5x^3}{6}+\frac{3}{2}xt^2-\frac{3}{2}x^2t,$$

$$I_3=\int_t^1 (x+s)(s-t)\,ds=\frac{t^3}{6}+\frac{xt^2}{2}-xt+\frac{x}{2}-\frac{t}{2}+\frac{1}{3}$$

Adding these integrals we obtain

$$K_2(x,\ t)=t^3-\frac{2}{3}x^3-x^2t+2xt^2-xt+\frac{x-t}{2}+\frac{1}{3}\quad (x<t)$$

(2) Let $x>t$. Then (see Fig. 2)

$$K_2(x,\ t)=I_1+I_2+I_3$$

where

$$I_1=\int_0^t (x-s)(s+t)\,ds=\frac{3}{2}xt^2-\frac{5t^3}{6},$$

$$I_2=\int_t^x (x-s)(s-t)\,ds=\frac{x^3}{6}-\frac{t^3}{6}-\frac{x^2t}{2}+\frac{xt^2}{2},$$

$$I_3=\int_x^1 (x+s)(s-t)\,ds=-\frac{5}{6}x^3+\frac{3}{2}x^2t+\frac{x-t}{2}-xt+\frac{1}{3}$$

Adding these integrals, we obtain

$$K_2(x,\ t)=-\frac{2}{3}x^3-t^3+x^2t+2xt^2-xt+\frac{x-t}{2}+\frac{1}{3}\quad (x>t)$$

Thus, the second iterated kernel is of the form

$$K_2(x,\ t)=$$

$$=\begin{cases}-\frac{2}{3}x^3+t^3-x^2t+2xt^2-xt+\frac{x-t}{2}+\frac{1}{3}, & 0\leqslant x<t,\\ -\frac{2}{3}x^3-t^3+x^2t+2xt^2-xt+\frac{x-t}{2}+\frac{1}{3}, & t<x\leqslant 1\end{cases}$$

The other iterated kernels $K_n(x,\ t)\ (n=3,\ 4,\ \ldots)$ are found in similar fashion.

Find the iterated kernels of the following kernels for specified a and b.

161. $K(x,\ t)=x-t;\quad a=-1,\ b=1$

162. $K(x,\ t)=\sin(x-t);\quad a=0,\ b=\frac{\pi}{2}\ (n=2,\ 3)$.

163. $K(x, t)=(x-t)^2$; $a=-1$, $b=1$ $(n=2, 3)$.
164. $K(x, t)=x+\sin t$; $a=-\pi$, $b=\pi$.
165. $K(x, t)=xe^t$; $a=0$, $b=1$.
166. $K(x, t)=e^x \cos t$; $a=0$, $b=\pi$.

In the following problems, find $K_2(x, t)$:

167. $K(x, t)=e^{|x-t|}$; $a=0$, $b=1$.
168. $K(x, t)=e^{|x|+t}$; $a=-1$, $b=1$.

We now give an instance of constructing the resolvent kernel of an integral equation with the aid of iterated kernels.

Consider the integral equation

$$\varphi(x)-\lambda \int_0^1 xt\,\varphi(t)\,dt=f(x) \tag{11}$$

Here $K(x, t)=xt$; $a=0$, $b=1$. In consecutive fashion we find

$$K_1(x, t)=xt,$$

$$K_2(x, t)=\int_0^1 (xz)(zt)\,dz=\frac{xt}{3},$$

$$K_3(x, t)=\frac{1}{3}\int_0^1 (xz)(zt)\,dz=\frac{xt}{3^2},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$K_n(x, t)=\frac{xt}{3^{n-1}}$$

According to formula (5)

$$R(x, t; \lambda)=\sum_{n=1}^{\infty} K_n(x, t)\,\lambda^{n-1}=xt\sum_{n=1}^{\infty}\left(\frac{\lambda}{3}\right)^{n-1}=\frac{3xt}{3-\lambda}$$

where $|\lambda|<3$.

By virtue of formula (7) the solution of the integral equation (11) will be written as

$$\varphi(x)=f(x)+\lambda\int_0^1 \frac{3xt}{3-\lambda}\,f(t)\,dt$$

In particular, for $f(x)=x$ we get

$$\varphi(x)=\frac{3x}{3-\lambda}$$

where $\lambda \neq 3$.

Construct resolvent kernels for the following kernels:

169. $K(x, t)=e^{x+t}$; $a=0,\ b=1$.
170. $K(x, t)=\sin x \cos t$; $a=0,\ b=\frac{\pi}{2}$.
171. $K(x, t)=xe^{t}$; $a=-1,\ b=1$.
172. $K(x, t)=(1+x)(1-t)$; $a=-1,\ b=0$.
173. $K(x, t)=x^2t^2$; $a=-1,\ b=1$.
174. $K(x, t)=xt$; $a=-1,\ b=1$.

If $M(x, t)$ and $N(x, t)$ are two orthogonal kernels, then the resolvent kernel $R(x, t; \lambda)$ corresponding to the kernel $K(x, t)=M+N$, is equal to the sum of the resolvent kernels $R_1(x, t; \lambda)$ and $R_2(x, t; \lambda)$ which correspond to each of these kernels.

Example. Find the resolvent kernel for the kernel

$$K(x, t)=xt+x^2t^2, \quad a=-1, \ b=1$$

Solution. As was shown above, the kernels $M(x, t)=xt$ and $N(x, t)=x^2t^2$ are orthogonal on $[-1, 1]$ (see p. 80). For this reason the resolvent kernel of the kernel $K(x, t)$ is equal to the sum of the resolvent kernels of the kernels $M(x, t)$ and $N(x, t)$. Utilizing the results of problems 173 and 174, we obtain

$$R_K(x, t; \lambda)=R_M(x, t; \lambda)+R_N(x, t; \lambda)=\frac{3xt}{3-2\lambda}+\frac{5x^2t^2}{5-2\lambda}$$

where $|\lambda|<\frac{3}{2}$.

Find the resolvent kernels for the kernels:

175. $K(x, t)=\sin x \cos t+\cos 2x \sin 2t$; $a=0,\ b=2\pi$.
176. $K(x, t)=1+(2x-1)(2t-1)$; $a=0,\ b=1$.

This property can be extended to any finite number of kernels.

If the kernels $M^{(1)}(x, t),\ M^{(2)}(x, t),\ \ldots,\ M^{(n)}(x, t)$ are pairwise orthogonal, then the resolvent kernel corresponding to their sum,

$$K(x,\ t)=\sum_{m=1}^{n} M^{(m)}(x,\ t)$$

is equal to the sum of the resolvent kernels corresponding to each of the terms.

Let us use the term "nth trace" of the kernel $K(x,\ t)$ for the quantity

$$A_n=\int_a^b K_n(x,\ x)\,dx,\quad (n=1,\ 2,\ \ldots) \tag{12}$$

where $K_n(x,\ t)$ is the nth iterated kernel for the kernel $K(x,\ t)$.

The following formula holds for the Fredholm determinant $D(\lambda)$:

$$\frac{D'\lambda}{D(\lambda)}=-\sum_{n=1}^{\infty} A_n\lambda^{n-1} \tag{13}$$

The radius of convergence of the power series (13) is equal to the smallest of the moduli of the characteristic numbers.

177. Show that for the Volterra equation

$$\varphi(x)-\lambda\int_0^x K(x,\ t)\,\varphi(t)\,dt=f(x)$$

the Fredholm determinant $D(\lambda)=e^{-A_1\lambda}$ and, consequently, the resolvent kernel for the Volterra equation is an entire analytic function of λ.

178. Let $R(x,\ t;\ \lambda)$ be the resolvent kernel for some kernel $K(x,\ t)$.

Show that the resolvent kernel of the equation

$$\varphi(x)-\mu\int_a^b R(x,\ t;\ \lambda)\,\varphi(t)\,dt=f(x)$$

is equal to $R(x,\ t;\ \lambda+\mu)$.

179. Let

$$\int_a^b \int_a^b K^2(x, t)\,dx\,dt = B^2,$$

$$\int_a^b \int_a^b K_n^2(x, t)\,dx\,dt = B_n^2$$

where $K_n(x, t)$ is the nth iterated kernel for the kernel $K(x, t)$. Prove that if $B_2 = B^2$, then for any n we will have $B_n = B^n$.

15. Integral Equations with Degenerate Kernels. Hammerstein Type Equation

The kernel $K(x, t)$ of a Fredholm integral equation of the second kind is called *degenerate* if it is the sum of a finite number of products of functions of x alone by functions of t alone; i.e., if it is of the form

$$K(x, t) = \sum_{k=1}^{n} a_k(x)\, b_k(t) \tag{1}$$

We shall consider the functions $a_k(x)$ and $b_k(t)$ $(k = 1, 2, \ldots, n)$ continuous in the basic square $a \leqslant x, t \leqslant b$ and linearly independent. The integral equation with degenerate kernel (1)

$$\varphi(x) - \lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x)\, b_k(t)\right] \varphi(t)\,dt = f(x) \tag{2}$$

is solved in the following manner.

Rewrite (2) as

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} a_k(x) \int_a^b b_k(t)\, \varphi(t)\,dt \tag{3}$$

and introduce the notation

$$\int_a^b b_k(t)\, \varphi(t)\,dt = C_k \quad (k = 1, 2, \ldots, n) \tag{4}$$

Then (3) becomes

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x) \tag{5}$$

where C_k are unknown constants, since the function $\varphi(x)$ is unknown.

Thus, the solution of an integral equation with degenerate kernel reduces to finding the constants C_k ($k=1, 2, \ldots, n$). Putting the expression (5) into the integral equation (2), we get (after simple manipulations)

$$\sum_{m=1}^{n} \left\{ C_m - \int_a^b b_m(t) \left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] dt \right\} a_m(x) = 0$$

Whence it follows, by virtue of the linear independence of the functions $a_m(x)$ ($m = 1, 2, \ldots, n$), that

$$C_m - \int_a^b b_m(t) \left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] dt = 0$$

or

$$C_m - \lambda \sum_{k=1}^{n} C_k \int_a^b a_k(t)\, b_m(t)\, dt = \int_a^b b_m(t) f(t) dt \quad (m = 1, 2, \ldots, n)$$

For the sake of brevity, we introduce the notations

$$a_{km} = \int_a^b a_k(t)\, b_m(t)\, dt, \qquad f_m = \int_a^b b_m(t)\, f(t)\, dt$$

and find that

$$C_m - \lambda \sum_{k=1}^{n} a_{km} C_k = f_m \quad (m = 1, 2, \ldots, n)$$

or, in expanded form,

$$\left.\begin{aligned} (1-\lambda a_{11}) C_1 - \lambda a_{12} C_2 - \ldots - \lambda a_{1n} C_n &= f_1, \\ -\lambda a_{21} C_1 + (1-\lambda a_{22}) C_2 - \ldots - \lambda a_{2n} C_n &= f_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -\lambda a_{n1} C_1 - \lambda a_{n2} C_2 - \ldots + (1-\lambda a_{nn}) C_n &= f_n \end{aligned}\right\} \tag{6}$$

For finding the unknowns C_k, we have a linear system of n algebraic equations in n unknowns. The determinant of this system is

$$\Delta(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{vmatrix} \tag{7}$$

If $\Delta(\lambda)\neq 0$, then the system (6) has a unique solution $C_1, C_2, \dots, C_n$, which is obtained from Cramer's formulas

$$C_k=\frac{1}{\Delta(\lambda)}\begin{vmatrix} 1-\lambda a_{11} & \dots & -\lambda a_{1k-1} & f_1 & -\lambda a_{1k+1} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & \dots & -\lambda a_{2k-1} & f_2 & -\lambda a_{2k+1} & \dots & -\lambda a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\lambda a_{n1} & \dots & -\lambda a_{nk-1} & f_n & -\lambda a_{nk+1} & \dots & 1-\lambda a_{nn} \end{vmatrix} \tag{8}$$

$(k=1, 2, \dots, n)$

The solution of the integral equation (2) is the function $\varphi(x)$ defined by the equality

$$\varphi(x)=f(x)+\lambda\sum_{k=1}^{n} C_k a_k(x)$$

where the coefficients C_k $(k=1, 2, \dots, n)$ are determined from formulas (8).

Note. The system (6) may be obtained if both sides of (5) are consecutively multiplied by $a_1(x), a_2(x), \dots, a_n(x)$ and integrated from a to b or if we put (5) into (4) for $\varphi(x)$, replacing x by t.

Example. Solve the integral equation

$$\varphi(x)-\lambda\int_{-\pi}^{\pi}(x\cos t+t^2\sin x+\cos x\sin t)\,\varphi(t)\,dt=x \tag{9}$$

Solution. Write the equation in the following form:

$$\varphi(x)=\lambda x\int_{-\pi}^{\pi}\varphi(t)\cos t\,dt+\lambda\sin x\int_{-\pi}^{\pi}t^2\varphi(t)\,dt+$$
$$+\lambda\cos x\int_{-\pi}^{\pi}\varphi(t)\sin t\,dt+x$$

We introduce the notations

$$C_1=\int_{-\pi}^{\pi}\varphi(t)\cos t\,dt;\quad C_2=\int_{-\pi}^{\pi}t^2\varphi(t)\,dt;\quad C_3=\int_{-\pi}^{\pi}\varphi(t)\sin t\,dt \tag{10}$$

where C_1, C_2, C_3 are unknown constants. Then equation (9) assumes the form

$$\varphi(x)=C_1\lambda x+C_2\lambda\sin x+C_3\lambda\cos x+x \tag{11}$$

Substituting expression (11) into (10), we get

$$C_1=\int_{-\pi}^{\pi}(C_1\lambda t+C_2\lambda\sin t+C_3\lambda\cos t+t)\cos t\,dt,$$

$$C_2=\int_{-\pi}^{\pi}(C_1\lambda t+C_2\lambda\sin t+C_3\lambda\cos t+t)\,t^2\,dt,$$

$$C_3=\int_{-\pi}^{\pi}(C_1\lambda t+C_2\lambda\sin t+C_3\lambda\cos t+t)\sin t\,dt$$

or

$$C_1\left(1-\lambda\int_{-\pi}^{\pi}t\cos t\,dt\right)-C_2\lambda\int_{-\pi}^{\pi}\sin t\cos t\,dt-$$

$$-C_3\lambda\int_{-\pi}^{\pi}\cos^2 t\,dt=\int_{-\pi}^{\pi}t\cos t\,dt$$

$$-C_1\lambda\int_{-\pi}^{\pi}t^3\,dt+C_2\left(1-\lambda\int_{-\pi}^{\pi}t^2\sin t\,dt\right)-$$

$$-C_3\lambda\int_{-\pi}^{\pi}t^2\cos t\,dt=\int_{-\pi}^{\pi}t^3\,dt,$$

$$-C_1\lambda\int_{-\pi}^{\pi}t\sin t\,dt-C_2\lambda\int_{-\pi}^{\pi}\sin^2 t\,dt+$$

$$+C_3\left(1-\lambda\int_{-\pi}^{\pi}\cos t\sin t\,dt\right)=\int_{-\pi}^{\pi}t\sin t\,dt$$

By evaluating the integrals that enter into this system we obtain a system of algebraic equations for finding the unknowns C_1, C_2, C_3:

$$\left.\begin{aligned} C_1 - \lambda\pi C_3 &= 0, \\ C_2 + 4\lambda\pi C_3 &= 0, \\ -2\lambda\pi C_1 - \lambda\pi C_2 + C_3 &= 2\pi \end{aligned}\right\} \qquad (12)$$

The determinant of this system is

$$\Delta(\lambda) = \begin{vmatrix} 1 & 0 & -\pi\lambda \\ 0 & 1 & 4\pi\lambda \\ -2\pi\lambda & -\lambda\pi & 1 \end{vmatrix} = 1 + 2\lambda^2\pi^2 \neq 0$$

The system (12) has a unique solution

$$C_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2}; \quad C_2 = -\frac{8\pi^2\lambda}{1+2\lambda^2\pi^2}; \quad C_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$$

Substituting the values of C_1, C_2, C_3 thus found into (11), we obtain the solution of the given integral equation

$$\varphi(x) = \frac{2\lambda\pi}{1+2\lambda^2\pi^2}(\lambda\pi x - 4\lambda\pi \sin x + \cos x) + x$$

Solve the following integral equations with degenerate kernels:

180. $\varphi(x) - 4\int_0^{\frac{\pi}{2}} \sin^2 x\varphi(t)\,dt = 2x - \pi.$

181. $\varphi(x) - \int_{-1}^{1} e^{\arcsin x}\varphi(t)\,dt = \tan x.$

182. $\varphi(x) - \lambda\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan t\varphi(t)\,dt = \cot x.$

183. $\varphi(x) - \lambda\int_0^1 \cos(q \ln t)\,\varphi(t)\,dt = 1.$

184. $\varphi(x) - \lambda\int_0^1 \arccos t\varphi(t)\,dt = \frac{1}{\sqrt{1-x^2}}.$

185. $\varphi(x)-\lambda\int_0^1\left(\ln\frac{1}{t}\right)^p\varphi(t)\,dt=1 \quad (p>-1).$

186. $\varphi(x)-\lambda\int_0^1(x\ln t-t\ln x)\,\varphi(t)\,dt=\frac{6}{5}(1-4x).$

187. $\varphi(x)-\lambda\int_0^{\frac{\pi}{2}}\sin x\cos t\,\varphi(t)\,dt=\sin x.$

188. $\varphi(x)-\lambda\int_0^{2\pi}|\pi-t|\sin x\,\varphi(t)\,dt=x.$

189. $\varphi(x)-\lambda\int_0^{\pi}\sin(x-t)\,\varphi(t)\,dt=\cos x.$

190. $\varphi(x)-$

$$-\lambda\int_0^{2\pi}(\sin x\cos t-\sin 2x\cos 2t+\sin 3x\cos 3t)\,\varphi(t)\,dt=\cos x.$$

191. $\varphi(x)-$

$$-\frac{1}{2}\int_{-1}^{1}\left[x-\frac{1}{2}(3t^2-1)+\frac{1}{2}t(3x^2-1)\right]\varphi(t)\,dt=1$$

Many problems of physics reduce to nonlinear integral equations of the Hammerstein type (see [24], [28]).

The canonical form of the *Hammerstein-type equation* is

$$\varphi(x)=\int_a^b K(x,\ t)\,f(t,\ \varphi(t))\,dt \tag{1}$$

where $K(x, t)$, $f(t, u)$ are given functions and $\varphi(x)$ is the unknown function.

The following equations readily reduce to equations of type (1):

$$\varphi(x)=\int_a^b K(x,\ t)\,f(t,\ \varphi(t))\,dt+\psi(x) \tag{1$'$}$$

where $\psi(x)$ is the known function, so that the difference between homogeneous and nonhomogeneous equations, which is important in the linear case, is almost of no importance in the nonlinear case. We shall call the function $K(x, t)$ the *kernel of equation* (1).

Let $K(x, t)$ be a degenerate kernel, i. e.,

$$K(x, t) = \sum_{i=1}^{m} a_i(x) b_i(t) \tag{2}$$

Then equation (1) takes the form

$$\varphi(x) = \sum_{i=1}^{m} a_i(x) \int_a^b b_i(t) f(t, \varphi(t))\, dt \tag{3}$$

Put

$$C_i = \int_a^b b_i(t) f(t, \varphi(t))\, dt \quad (i = 1, 2, \ldots, m) \tag{4}$$

where the C_i are as yet unknown constants. Then, by virtue of (3), we will have

$$\varphi(x) = \sum_{i=1}^{m} C_i a_i(x) \tag{5}$$

Substituting into (4) the expression (5) for $\varphi(x)$, we get m equations (generally, transcendental) containing m unknown quantities $C_1, C_2, \ldots, C_m$:

$$C_i = \Psi_i(C_1, C_2, \ldots, C_m) \quad (i = 1, 2, \ldots, m) \tag{6}$$

When $f(t, u)$ is a polynomial in u, i. e.,

$$f(t, u) = p_0(t) + p_1(t)\, u + \ldots + p_n(t)\, u^n \tag{7}$$

where $p_0(t), \ldots, p_n(t)$ are, for instance, continuous functions of t on the interval $[a, b]$, the system (6) is transformed into a system of algebraic equations in $C_1, C_2, \ldots, C_m$. If there exists a solution of the system (6), that is, if there exist m numbers

$$C_1^0, \ C_2^0, \ \ldots, \ C_m^0$$

such that their substitution into (6) reduces the equations to identities, then there exists a solution of the integral equation (3) defined by the equality (5):

$$\varphi(x)=\sum_{i=1}^{m} C_i^0 a_i(x)$$

It is obvious that the number of solutions (generally, complex) of the integral equation (3) is equal to the number of solutions of the system (6).

Example. Solve the integral equation

$$\varphi(x)=\lambda\int_0^1 xt\varphi^2(t)\,dt \tag{8}$$

where λ is a parameter.

Solution. Put

$$C=\int_0^1 t\varphi^2(t)\,dt \tag{9}$$

Then

$$\varphi(x)=C\lambda x \tag{10}$$

Substituting $\varphi(x)$ in the form (10) into the relation (9), we get

$$C=\int_0^1 t\lambda^2C^2t^2\,dt$$

whence

$$C=\frac{\lambda^2}{4}C^2 \tag{11}$$

Equation (11) has two solutions

$$C_1=0, \quad C_2=\frac{4}{\lambda^2}$$

Consequently, integral equation (8) also has two solutions for any $\lambda \neq 0$:

$$\varphi_1(x)\equiv 0, \quad \varphi_2(x)=\frac{4}{\lambda}x$$

There exist simple nonlinear integral equations which do not have real solutions at all.

Consider, for example, the equation

$$\varphi(x)=\frac{1}{2}\int_0^1 e^{\frac{x+t}{2}}(1+\varphi^2(t))\,dt \tag{12}$$

Put

$$C=\frac{1}{2}\int_0^1 e^{\frac{t}{2}}\left(1+\varphi^2(t)\right)dt \tag{13}$$

Then

$$\varphi(x)=Ce^{\frac{x}{2}} \tag{14}$$

For a determination of the constant C we obtain the equation

$$\left(e^{\frac{3}{2}}-1\right)C^2-3C+3\left(e^{\frac{1}{2}}-1\right)=0 \tag{15}$$

It is easy to verify that equation (15) does not have real roots and, hence, the integral equation (12) has no real solutions.

On the other hand, let us consider the equation

$$\varphi(x)=\int_0^1 a(x)\,a(t)\,\varphi(t)\sin\left(\frac{\varphi(t)}{a(t)}\right)dt \tag{16}$$

$$(a(t)>0 \text{ for all } t\in[0,\ 1])$$

In order to determine the constant C, we arrive at the equation

$$1=\int_0^1 a^2(t)\,dt\cdot\sin C \tag{17}$$

If $\int_0^1 a^2(t)\,dt>1$, then equation (17) and, hence, the original integral equation (16) as well have an *infinite* number of real solutions.

Solve the following integral equations:

192. $\varphi(x)=2\int_0^1 xt\,\varphi^3(t)\,dt.$

193. $\varphi(x)=\int_{-1}^1 (xt+x^2t^2)\,\varphi^2(t)\,dt.$

194. $\varphi(x)=\int_0^1 x^2t^2\varphi^3(t)\,dt.$

195. $\varphi(x) = \int_{-1}^{1} \frac{xt}{1+\varphi^2(t)}\, dt.$

196. $\varphi(x) = \int_{0}^{1} (1+\varphi^2(t))\, dt.$

197. Show that the integral equation

$$\varphi(x) = \frac{1}{2}\int_0^1 a(x)\,a(t)\,(1+\varphi^2(t))\,dt$$

$$(a(x) > 0 \text{ for all } x \in [0,\ 1])$$

has no real solutions if $\int_0^1 a^2(x)\,dx > 1$.

16. Characteristic Numbers and Eigenfunctions

The homogeneous Fredholm integral equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x,\ t)\,\varphi(t)\,dt = 0 \tag{1}$$

always has the obvious solution $\varphi(x) \equiv 0$, which is called the *zero (trivial)* solution.

The values of the parameter λ for which this equation has nonzero solutions $\varphi(x) \not\equiv 0$ are called *characteristic numbers** of the equation (1) or of the kernel $K(x,\ t)$, and every nonzero solution of this equation is called an *eigenfunction* corresponding to the characteristic number λ.

The number $\lambda = 0$ is not a characteristic number since for $\lambda = 0$ it follows from (1) that $\varphi(x) \equiv 0$.

If the kernel $K(x,\ t)$ is continuous in the square $\Omega\ \{a \leqslant x,\ t \leqslant b\}$ or is quadratically summable in Ω, and the numbers a and b are finite, then to every characteristic number λ

* Some authors use "eigenvalues" in place of the term "characteristic numbers". We use eigenvalue for the quantity $\sigma = \frac{1}{\lambda}$, where λ is the characteristic number.

there corresponds a finite number of linearly independent eigenfunctions; the number of such functions is called the *index* of the characteristic number. Different characteristic numbers can have different indices.

For an equation with degenerate kernel

$$\varphi(x)-\lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x)\, b_k(t)\right] \varphi(t)\, dt = 0 \tag{2}$$

the characteristic numbers are roots of the algebraic equation

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{vmatrix} = 0 \tag{3}$$

the degree of which is $p \leqslant n$. Here, $\Delta(\lambda)$ is the determinant of the linear homogeneous system

$$\left.\begin{aligned} &(1-\lambda a_{11})C_1-\lambda a_{12}C_2-\dots-\lambda a_{1n}C_n=0, \\ &-\lambda a_{21}C_1+(1-\lambda a_{22})C_2-\dots-\lambda a_{2n}C_n=0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &-\lambda a_{n1}C_1-\lambda a_{n2}C_2-\dots+(1-\lambda a_{nn})C_n=0 \end{aligned}\right\} \tag{4}$$

where the quantities a_{mk} and C_m $(k,\ m=1,\ 2,\ \dots,\ n)$ have the same meaning as in the preceding section.

If equation (3) has p roots $(1 \leqslant p \leqslant n)$, then the integral equation (2) has p characteristic numbers; to each characteristic number λ_m $(m=1,\ 2,\ \dots,\ p)$ there corresponds a nonzero solution

$$\begin{aligned} &C_1^{(1)},\ C_2^{(1)},\ \dots,\ C_n^{(1)} \to \lambda_1, \\ &C_1^{(2)},\ C_2^{(2)},\ \dots,\ C_n^{(2)} \to \lambda_2, \\ &\dots\dots\dots\dots\dots \\ &C_1^{(p)},\ C_2^{(p)},\ \dots,\ C_n^{(p)} \to \lambda_p \end{aligned}$$

of the system (4). The nonzero solutions of the integral equation (2) corresponding to these solutions, i. e., the eigenfunctions, will be of the form

$$\varphi_1(x) = \sum_{k=1}^{n} C_k^{(1)} a_k(x), \qquad \varphi_2(x) = \sum_{k=1}^{n} C_k^{(2)} a_k(x),\ \dots,$$

$$\varphi_p(x) = \sum_{k=1}^{n} C_k^{(p)} a_k(x)$$

An integral equation with degenerate kernel has at most n characteristic numbers and (corresponding to them) eigenfunctions.

In the case of an arbitrary (nondegenerate) kernel, the characteristic numbers are zeros of the Fredholm determinant $D(\lambda)$, i.e., are poles of the resolvent kernel $R(x, t; \lambda)$. It then follows, in particular, that the Volterra integral equation $\varphi(x) - \lambda \int_0^x K(x, t)\varphi(t)\,dt = 0$ where $K(x, t) \in L_2(\Omega_0)$ has no characteristic numbers (for it, $D(\lambda) = e^{-A_1\lambda}$, see Problem 177).

Note. Eigenfunctions are determined to within a multiplicative constant; that is, if $\varphi(x)$ is an eigenfunction corresponding to some characteristic number λ, then $C\varphi(x)$, where C is an arbitrary constant, is also an eigenfunction which corresponds to the same characteristic number λ.

Example. Find the characteristic numbers and eigenfunctions of the integral equation

$$\varphi(x) - \lambda \int_0^{\pi} (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\,\varphi(t)\,dt = 0$$

Solution. We have

$$\varphi(x) = \lambda \cos^2 x \int_0^{\pi} \varphi(t) \cos 2t\,dt + \lambda \cos 3x \int_0^{\pi} \varphi(t) \cos^3 t\,dt.$$

Introducing the notations

$$C_1 = \int_0^{\pi} \varphi(t) \cos 2t\,dt, \qquad C_2 = \int_0^{\pi} \varphi(t) \cos^3 t\,dt \tag{1}$$

we get

$$\varphi(x) = C_1 \lambda \cos^2 x + C_2 \lambda \cos 3x \tag{2}$$

Substituting (2) into (1), we obtain a linear system of homogeneous equations:

$$\left.\begin{aligned} C_1\Big(1-\lambda\int_0^{\pi}\cos^2 t\cos 2t\,dt\Big)-C_2\lambda\int_0^{\pi}\cos 3t\cos 2t\,dt=0,\\ -C_1\lambda\int_0^{\pi}\cos^5 t\,dt+C_2\Big(1-\lambda\int_0^{\pi}\cos^3 t\cos 3t\,dt=0\end{aligned}\right\}\quad(3)$$

But since

$$\int_0^{\pi}\cos^2 t\cos 2t\,dt=\frac{\pi}{4},\quad \int_0^{\pi}\cos 3t\cos 2t\,dt=0$$

$$\int_0^{\pi}\cos^5 t\,dt=0,\quad \int_0^{\pi}\cos^3 t\cos 3t\,dt=\frac{\pi}{8}$$

it follows that system (3) takes the form

$$\left.\begin{aligned}\Big(1-\frac{\lambda\pi}{4}\Big)C_1=0,\\ \Big(1-\frac{\lambda\pi}{8}\Big)C_2=0\end{aligned}\right\}\quad(4)$$

The equation for finding characteristic numbers will be

$$\begin{vmatrix}1-\frac{\lambda\pi}{4} & 0\\ 0 & 1-\frac{\lambda\pi}{8}\end{vmatrix}=0$$

The characteristic numbers are $\lambda_1=\frac{4}{\pi}$, $\lambda_2=\frac{8}{\pi}$.

For $\lambda=\frac{4}{\pi}$, system (4) becomes

$$\begin{cases}0\cdot C_1=0,\\ \frac{1}{2}\cdot C_2=0\end{cases}$$

whence $C_2=0$, C_1 is arbitrary. The eigenfunction will be $\varphi_1(x)=C_1\lambda\cos^2 x$ or, setting $C_1\lambda=1$, we get $\varphi_1(x)=\cos^2 x$.

For $\lambda=\frac{8}{\pi}$, system (4) is of the form

$$\begin{cases}(-1)\cdot C_1=0,\\ 0\cdot C_2=0\end{cases}$$

whence $C_1=0$, C_2 is arbitrary and, hence, the eigenfunction will be $\varphi_2(x)=C_2\lambda\cos 3x$, or, assuming $C_2\lambda=1$, we get $\varphi_2(x)=\cos 3x$.

Thus, the characteristic numbers are

$$\lambda_1=\frac{4}{\pi}, \qquad \lambda_2=\frac{8}{\pi}$$

and the corresponding eigenfunctions are

$$\varphi_1(x)=\cos^2 x, \qquad \varphi_2(x)=\cos 3x$$

A homogeneous Fredholm integral equation may, generally, have no characteristic numbers and eigenfunctions, or it may not have any real characteristic numbers and eigenfunctions.

Example 1. The homogeneous integral equation

$$\varphi(x)-\lambda\int_0^1(3x-2)\,t\,\varphi(t)\,dt=0$$

has no characteristic numbers and eigenfunctions. Indeed, we have

$$\varphi(x)=\lambda(3x-2)\int_0^1 t\,\varphi(t)\,dt$$

Putting

$$C=\int_0^1 t\,\varphi(t)\,dt \tag{1}$$

we get

$$\varphi(x)=C\lambda(3x-2) \tag{2}$$

Substituting (2) into (1), we get

$$\left[1-\lambda\int_0^1(3t^2-2t)dt\right]\cdot C=0 \tag{3}$$

But since $\int_0^1(3t^2-2t)\,dt=0$, equation (3) yields $C=0$ and, hence, $\varphi(x)\equiv 0$.

And so for any λ, this homogeneous equation has only one zero solution $\varphi(x) \equiv 0$ and, hence, it does not have any characteristic numbers or eigenfunctions.

Example 2. The equation

$$\varphi(x) - \lambda \int_0^1 (\sqrt{x}t - \sqrt{t}x)\, \varphi(t)\, dt = 0$$

does not have real characteristic numbers and eigenfunctions.

We have

$$\varphi(x) = C_1 \lambda \sqrt{x} - C_2 \lambda x \tag{1}$$

where

$$C_1 = \int_0^1 t\varphi(t)\, dt, \quad C_2 = \int_0^1 \sqrt{t}\, \varphi(t)\, dt \tag{2}$$

Substituting (1) into (2), we get (after some simple manipulations) the system of algebraic equations

$$\left.\begin{aligned} \left(1 - \frac{2\lambda}{5}\right) C_1 + \frac{\lambda}{3} C_2 = 0, \\ -\frac{\lambda}{2} C_1 + \left(1 + \frac{2\lambda}{5}\right) C_2 = 0 \end{aligned}\right\} \tag{3}$$

The determinant of this system is

$$\Delta(\lambda) = \begin{vmatrix} 1 - \frac{2}{5}\lambda & \frac{\lambda}{3} \\ -\frac{\lambda}{2} & 1 + \frac{2}{5}\lambda \end{vmatrix} = 1 + \frac{\lambda^2}{150}$$

For real λ it does not vanish, so that from (3) we get $C_1 = 0$ and $C_2 = 0$ and, hence, for all real λ the equation has only one solution, namely, the zero solution $\varphi(x) \equiv 0$. Thus, equation (1) does not have real characteristic numbers or eigenfunctions.

Find the characteristic numbers and eigenfunctions for the following homogeneous integral equations with degenerate kernels:

198. $\varphi(x) - \lambda \int_0^{\frac{\pi}{4}} \sin^2 x\varphi(t)\, dt = 0.$

199. $\varphi(x)-\lambda\int\limits_0^{2\pi}\sin x\cos t\varphi(t)\,dt=0.$

200. $\varphi(x)-\lambda\int\limits_0^{2\pi}\sin x\sin t\varphi(t)\,dt=0.$

201. $\varphi(x)-\lambda\int\limits_0^{\pi}\cos(x+t)\,\varphi(t)\,dt=0.$

202. $\varphi(x)-\lambda\int\limits_0^{1}(45x^2\ln t-9t^2\ln x)\,\varphi(t)\,dt=0.$

203. $\varphi(x)-\lambda\int\limits_0^{1}(2xt-4x^2)\,\varphi(t)\,dt=0.$

204. $\varphi(x)-\lambda\int\limits_{-1}^{1}(5xt^3+4x^2t)\,\varphi(t)\,dt=0.$

205. $\varphi(x)-\lambda\int\limits_{-1}^{1}(5xt^3+4x^2t+3xt)\,\varphi(t)\,dt=0.$

206. $\varphi(x)-\lambda\int\limits_{-1}^{1}(x\cosh t-t\sinh x)\,\varphi(t)\,dt=0.$

207. $\varphi(x)-\lambda\int\limits_{-1}^{1}(x\cosh t-t^2\sinh x)\,\varphi(t)\,dt=0.$

208. $\varphi(x)-\lambda\int\limits_{-1}^{1}(x\cosh t-t\cosh x)\,\varphi(t)\,dt=0.$

If the nth iterated kernel $K_n(x, t)$ of the kernel $K(x, t)$ is symmetric, then it may be asserted that $K(x, t)$ has at least one characteristic number (real or complex) and that the nth degrees of all characteristic numbers are real numbers. In particular, for the skew-symmetric kernel $K(x, t)=-K(t, x)$ all characteristic numbers are pure imaginary $\lambda=\beta i$, where β is real (see Problem 220).

The kernel $K(x, t)$ of the integral equation is called *symmetric* if the condition $K(x, t)=K(t, x)$ $(a\leqslant x,\ t\leqslant b)$ is fulfilled.

The following theorems hold for the Fredholm integral equation

$$\varphi(x)-\lambda\int_a^b K(x, t)\,\varphi(t)\,dt=0 \tag{1}$$

with symmetric kernel $K(x, t)$:

Theorem 1. *Equation (1) has at least one real characteristic number.*

Theorem 2. *To every characteristic number λ there corresponds a finite number q of linearly independent eigenfunctions of equation (1), and*

$$\sup q \leqslant \lambda^2 B^2$$

where

$$B^2=\int_a^b\int_a^b K^2(x, t)\,dx\,dt$$

The number q is called the *index* or multiplicity of the characteristic number.

Theorem 3. *Every pair of eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$, corresponding to different characteristic numbers $\lambda_1 \neq \lambda_2$, is orthogonal*; i.e.,

$$\int_a^b \varphi_1(x)\,\varphi_2(x)\,dx=0$$

Theorem 4. *There is a finite number of characteristic numbers in every finite interval of the λ-axis. The upper bound for a number m of characteristic numbers lying in an interval $-l<\lambda<l$ is defined by the inequality*

$$m \leqslant l^2 B^2$$

When the kernel $K(x, t)$ of the integral equation (1) is the Green's function of some homogeneous Sturm-Liouville problem, finding the characteristic numbers and eigenfunctions reduces to the solution of the indicated Sturm-Liouville problem.

Example. Find the characteristic numbers and eigenfunctions of the homogeneous equation

$$\varphi(x)-\lambda\int_0^{\pi} K(x,\ t)\,\varphi(t)\,dt=0$$

where

$$K(x,\ t)=\begin{cases}\cos x\sin t, & 0\leqslant x\leqslant t,\\ \cos t\sin x, & t\leqslant x\leqslant\pi\end{cases}$$

Solution. Represent the equation in the form

$$\varphi(x)=\lambda\int_0^x K(x,\ t)\,\varphi(t)\,dt+\lambda\int_x^{\pi} K(x,\ t)\,\varphi(t)\,dt$$

or

$$\varphi(x)=\lambda\sin x\int_0^x \varphi(t)\cos t\,dt+\lambda\cos x\int_x^{\pi}\varphi(t)\sin t\,dt \qquad (1)$$

Differentiating both sides of (1), we get

$$\varphi'(x)=\lambda\cos x\int_0^x \varphi(t)\cos t\,dt+\lambda\sin x\cos x\,\varphi(x)-$$

$$-\lambda\sin x\int_x^{\pi}\varphi(t)\sin t\,dt-\lambda\sin x\cos x\,\varphi(x)$$

or

$$\varphi'(x)=\lambda\cos x\int_0^x \varphi(t)\cos t\,dt-\lambda\sin x\int_x^{\pi}\varphi(t)\sin t\,dt \qquad (2)$$

Differentiating again, we get

$$\varphi''(x)=-\lambda\sin x\int_0^x \varphi(t)\cos t\,dt+\lambda\cos^2 x\varphi(x)-$$

$$-\lambda\cos x\int_x^{\pi}\varphi(t)\sin t\,dt+\lambda\sin^2 x\,\varphi(x)=$$

$$=\lambda\varphi(x)-\left[\lambda\sin x\int_0^x \varphi(t)\cos t\,dt+\lambda\cos x\int_x^{\pi}\varphi(t)\sin t\,dt\right]$$

The expression in the square brackets is equal to $\varphi(x)$ so that

$$\varphi''(x)=\lambda\varphi(x)-\varphi(x)$$

From (1) and (2) we find that

$$\varphi(\pi)=0, \quad \varphi'(0)=0$$

Thus, the given integral equation reduces to the following boundary-value problem:

$$\varphi''(x)-(\lambda-1)\varphi(x)=0, \tag{3}$$

$$\varphi(\pi)=0, \quad \varphi'(0)=0 \tag{4}$$

The three following cases are possible:

(1) $\lambda-1=0$ or $\lambda=1$. Equation (3) takes the form $\varphi''(x)=0$. Its general solution will be $\varphi(x)=C_1x+C_2$. Utilizing the boundary conditions (4), we obtain (for finding the unknowns C_1 and C_2) the system

$$\begin{cases} C_1\pi+C_2=0, \\ \qquad C_1=0 \end{cases}$$

which has a unique solution: $C_1=0$, $C_2=0$, and hence the integral equation has only the trivial solution

$$\varphi(x)\equiv 0$$

(2) $\lambda-1>0$ or $\lambda>1$. The general solution of equation (3) is of the form

$$\varphi(x)=C_1\cosh\sqrt{\lambda-1}\,x+C_2\sinh\sqrt{\lambda-1}\,x$$

whence

$$\varphi'(x)=\sqrt{\lambda-1}\,(C_1\sinh\sqrt{\lambda-1}\,x+C_2\cosh\sqrt{\lambda-1}\,x)$$

For finding the values of C_1 and C_2, the boundary conditions yield the system

$$\begin{cases} C_1\cosh\pi\sqrt{\lambda-1}+C_2\sinh\pi\sqrt{\lambda-1}=0, \\ \qquad\qquad\qquad\qquad\qquad C_2=0. \end{cases}$$

The system has a unique solution: $C_1=0$, $C_2=0$. The integral equation has the trivial solution $\varphi(x)\equiv 0$. Thus, for $\lambda\geqslant 1$ the integral equation has no characteristic numbers and, hence, no eigenfunctions.

(3) $\lambda - 1 < 0$ or $\lambda < 1$. The general solution of equation (3) is

$$\varphi(x) = C_1 \cos \sqrt{1-\lambda}\, x + C_2 \sin \sqrt{1-\lambda}\, x$$

Whence we find

$$\varphi'(x) = \sqrt{1-\lambda}\,(-C_1 \sin \sqrt{1-\lambda}\, x + C_2 \cos \sqrt{1-\lambda}\, x)$$

In this case, for finding C_1 and C_2 the boundary conditions (4) yield the system

$$\left.\begin{aligned} C_1 \cos \pi \sqrt{1-\lambda} + C_2 \sin \pi \sqrt{1-\lambda} &= 0, \\ \sqrt{1-\lambda}\, C_2 &= 0 \end{aligned}\right\} \tag{5}$$

The determinant of this system is

$$\Delta\lambda = \begin{vmatrix} \cos \pi \sqrt{1-\lambda} & \sin \pi \sqrt{1-\lambda} \\ 0 & \sqrt{1-\lambda} \end{vmatrix}$$

Setting it equal to zero, we get an equation for finding the characteristic numbers:

$$\begin{vmatrix} \cos \pi \sqrt{1-\lambda} & \sin \pi \sqrt{1-\lambda} \\ 0 & \sqrt{1-\lambda} \end{vmatrix} = 0 \tag{6}$$

or $\sqrt{1-\lambda} \cos \pi \sqrt{1-\lambda} = 0$. By assumption $\sqrt{1-\lambda} \neq 0$ and so $\cos \pi \sqrt{1-\lambda} = 0$. Whence we find that $\pi \sqrt{1-\lambda} = \frac{\pi}{2} + \pi n$, where n is any integer. All the roots of equation (6) are given by the formula

$$\lambda_n = 1 - \left(n + \frac{1}{2}\right)^2$$

For values $\lambda = \lambda_n$ the system (5) takes the form

$$\begin{cases} C_1 \cdot 0 = 0, \\ C_2 \quad = 0 \end{cases}$$

It has an infinite number of nonzero solutions

$$\begin{cases} C_1 = C, \\ C_2 = 0 \end{cases}$$

where C is an arbitrary constant. Hence, the original integral equation also has an infinity of solutions of the form

$$\varphi(x)=C\cos\left(n+\frac{1}{2}\right)x$$

which are eigenfunctions of this equation.

Hence, the characteristic numbers and eigenfunctions of the given integral equation will be

$$\lambda_n=1-\left(n+\frac{1}{2}\right)^2, \quad \varphi_n(x)=\cos\left(n+\frac{1}{2}\right)x$$

where n is any integer.

Find the characteristic numbers and eigenfunctions of the homogeneous integral equations if their kernels are of the following form:

209. $K(x, t)=\begin{cases} x(t-1), & 0\leqslant x\leqslant t, \\ t(x-1), & t\leqslant x\leqslant 1. \end{cases}$

210. $K(x, t)=\begin{cases} t(x+1), & 0\leqslant x\leqslant t, \\ x(t+1), & t\leqslant x\leqslant 1. \end{cases}$

211. $K(x, t)=\begin{cases} (x+1)(t-2), & 0\leqslant x\leqslant t, \\ (t+1)(x-2), & t\leqslant x\leqslant 1. \end{cases}$

212. $K(x, t)=\begin{cases} \sin x\cos t, & 0\leqslant x\leqslant t, \\ \sin t\cos x, & t\leqslant x\leqslant \frac{\pi}{2}. \end{cases}$

213. $K(x, t)=\begin{cases} \sin x\cos t, & 0\leqslant x\leqslant t, \\ \sin t\cos x, & t\leqslant x\leqslant \pi. \end{cases}$

214. $K(x, t)=\begin{cases} \sin x\sin(t-1), & -\pi\leqslant x\leqslant t, \\ \sin t\sin(x-1), & t\leqslant x\leqslant \pi. \end{cases}$

215. $K(x, t)=\begin{cases} \sin\left(x+\frac{\pi}{4}\right)\sin\left(t-\frac{\pi}{4}\right), & 0\leqslant x\leqslant t, \\ \sin\left(t+\frac{\pi}{4}\right)\sin\left(x-\frac{\pi}{4}\right), & t\leqslant x\leqslant \pi. \end{cases}$

216. $K(x, t)=e^{-|x-t|}$, $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant 1$.

217. $K(x, t)=\begin{cases} -e^{-t}\sinh x, & 0\leqslant x\leqslant t, \\ -e^{-x}\sinh t, & t\leqslant x\leqslant 1. \end{cases}$

218. Show that if $\lambda_1, \lambda_2, \lambda_1 \neq \lambda_2$ are characteristic numbers of the kernel $K(x, t)$, then the eigenfunctions of the equations

$$\varphi(x) - \lambda_1 \int_a^b K(x, t)\,\varphi(t)\,dt = 0$$

$$\psi(x) - \lambda_2 \int_a^b K(t, x)\,\psi(t)\,dt = 0$$

are orthogonal, i.e.,

$$\int_a^b \varphi(x)\,\psi(x)\,dx = 0$$

219. Show that if $K(x, t)$ is a symmetric kernel, then the second iterated kernel $K_2(x, t)$ has only positive characteristic numbers.

220. Show that if the kernel $K(x, t)$ is skew-symmetric, that is, $K(t, x) = -K(x, t)$, then all its characteristic numbers are pure imaginaries.

221. If the kernel $K(x, t)$ is symmetric, then

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^m} = A_m \quad (m = 2, 3, \ldots)$$

where λ_n are characteristic numbers and A_m are the mth traces of the kernel $K(x, t)$.

Taking advantage of the results of Problems 209, 212, 216, find the sums of the series:

(a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$

(b) $\sum\limits_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}$

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{(1+\mu_n^2)^2}$, where μ_n are the roots of the equation $2\cot\mu = \mu - \frac{1}{\mu}$.

The resolvent kernel of a symmetric kernel is a meromorphic function of λ, for which the characteristic numbers of the integral equation are simple poles. Its residues with respect to the poles λ_i yield eigenfunctions (for any value of t) corresponding to these values of λ_i.

Find the eigenfunctions of the integral equations whose resolvent kernels are defined by the following formulas:

222. $R(x, t; \lambda)=\dfrac{3-\lambda+3(1-\lambda)(2x-1)(2t-1)}{\lambda^2-4\lambda+3}$.

223. $R(x, t; \lambda)=\dfrac{(15-6\lambda)xt+(15-10\lambda)x^2t^2}{4\lambda^2-16\lambda+15}$.

224. $R(x, t; \lambda)=$

$$=\frac{4(5-2\lambda)[3-2\lambda+(3-6\lambda)xt]+5(4\lambda^2-8\lambda+3)(3x^2-1)(3t^2-1)}{4(1-2\lambda)(3-2\lambda)(5-2\lambda)}.$$

Fredholm integral equations with difference kernels. Suppose we have the integral equation

$$\varphi(x)=\lambda\int_{-\pi}^{\pi}K(x-t)\,\varphi(t)\,dt \tag{1}$$

where the kernel $K(x)$ $(-\pi\leqslant x\leqslant\pi)$ is an even function which is periodically extended to the entire x-axis so that

$$K(x-t)=K(t-x) \tag{2}$$

It can be shown that the eigenfunctions of equation (1) are

$$\left.\begin{aligned}\varphi_n^{(1)}(x)&=\cos nx \quad (n=1, 2, \ldots),\\ \varphi_n^{(2)}(x)&=\sin nx \quad (n=1, 2, \ldots)\end{aligned}\right\} \tag{3}$$

and the corresponding characteristic numbers are

$$\lambda_n=\frac{1}{\pi a_n} \quad (n=1, 2, \ldots) \tag{4}$$

where a_n are the Fourier coefficients of the function $K(x)$:

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}K(x)\cos nx\,dx \quad (n=1, 2, \ldots) \tag{5}$$

Thus, to every value of λ_n there correspond two linearly independent eigenfunctions $\cos nx$, $\sin nx$ so that each λ_n

is a double characteristic number. The function $\varphi_0(x) \equiv 1$ is also an eigenfunction of equation (1) corresponding to the characteristic number

$$\lambda_0 = \frac{1}{\pi a_0}, \text{ where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} K(x)\,dx$$

We shall now show that, for example, $\cos nx$ is an eigenfunction of the integral equation

$$\varphi(x) = \frac{\pi^{-1}}{a_n} \int_{-\pi}^{\pi} K(x-t)\,\varphi(t)\,dt \tag{6}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} K(x) \cos nx\,dx$$

Making the substitution $x-t=z$, we find

$$\int_{-\pi}^{\pi} K(x-t)\cos nt\,dt = -\int_{x+\pi}^{x-\pi} K(z)\cos n(x-z)\,dz =$$

$$= \cos nx \int_{x-\pi}^{x+\pi} K(z)\cos nz\,dz + \sin nx \int_{x-\pi}^{x+\pi} K(z)\sin nz\,dz = \pi a_n \cos nx$$

since by virtue of the evenness of $K(x)$ the second integral is zero, and the first integral is a Fourier coefficient a_n multiplied by π in the expansion of the even function $K(x)$.

Thus,

$$\cos nx = \frac{1}{\pi a_n} \int_{-\pi}^{\pi} K(x-t)\cos nt\,dt$$

and this means that $\cos nx$ is an eigenfunction of equation (6).

Similarly, we establish the fact that $\sin nx$ is an eigenfunction of the integral equation (6) corresponding to the same characteristic number $\frac{1}{\pi a_n}$.

225. Find the eigenfunction and the corresponding characteristic numbers of the equation

$$\varphi(x)=\lambda \int_{-\pi}^{\pi} \cos^2(x-t)\,\varphi(t)\,dt.$$

226. Show that the symmetric kernel

$$K(x, t)=\frac{1}{2\pi}\frac{1-h^2}{1-2h\cos(x-t)+h^2} \quad (-\pi \leqslant x,\ t \leqslant \pi)$$

has for $|h|<1$ the eigenfunctions 1, $\sin nx$, $\cos nx$, which correspond to the characteristic numbers 1, $1/h^n$, $1/h^n$.

227. Find the characteristic numbers and eigenfunctions of the integral equation

$$\varphi(x)=\lambda \int_{-\pi}^{\pi} K(x-t)\,\varphi(t)\,dt$$

where $K(x)=x^2\ (-\pi \leqslant x \leqslant \pi)$ is a periodic function with period 2π.

Extremal properties of characteristic numbers and eigenfunctions.

The absolute value of the double integral (Hilbert's integral)

$$|(K\varphi, \varphi)|=\left|\int_a^b\int_a^b K(x, t)\,\varphi(x)\,\varphi(t)\,dx\,dt\right| \tag{1}$$

where $K(x, t)=K(t, x)$ is a symmetric kernel of some integral equation, on the set of normalized functions $\varphi(x)$, i.e., such that

$$(\varphi, \varphi)=\int_a^b \varphi^2(x)\,dx=1$$

has a maximum equal to

$$\max|(K\varphi, \varphi)|=\frac{1}{|\lambda_1|} \tag{2}$$

where λ_1 is the least (in absolute value) characteristic number of the kernel $K(x, t)$. The maximum is attained for $\varphi(x)=$

$=\varphi_1(x)$, which is the eigenfunction of the kernel corresponding to λ_1.

Example. Find the maximum of

$$|(K\varphi, \varphi)|=\left|\int_0^\pi\int_0^\pi K(x, t)\,\varphi(x)\,\varphi(t)\,dx\,dt\right|$$

provided

$$(\varphi, \varphi)=\int_0^\pi \varphi^2(x)\,dx=1$$

if

$$K(x, t)=\cos x\cos 2t+\cos t\cos 2x+1$$

Solution. Solving the homogeneous integral equation

$$\varphi(x)=\lambda\int_0^\pi (\cos x\cos 2t+\cos t\cos 2x+1)\,\varphi(t)\,dt$$

as an equation with a degenerate kernel, we find the characteristic numbers $\lambda_1=\frac{1}{\pi}$ and $\lambda_{2,3}=\pm\frac{2}{\pi}$ and the corresponding eigenfunctions $\varphi_1(x)=C_1$, $\varphi_2(x)=C_2(\cos x+\cos 2x)$, $\varphi_3(x)=C_3(\cos x-\cos 2x)$, where C_1, C_2 and C_3 are arbitrary constants.

The smallest (in absolute value) characteristic number is $\lambda_1=\frac{1}{\pi}$, to which corresponds the eigenfunction $\varphi_1(x)=C_1$. From the normalization condition $(\varphi, \varphi)=1$, we find $C_1=\pm\frac{1}{\sqrt{2\pi}}$. Hence

$$\max\left|\int_0^\pi\int_0^\pi (\cos x\cos 2t+\cos t\cos 2x+1)\,\varphi(t)\,dt\right|=2\pi$$

and it is attained on the functions $\varphi(x)=\pm\frac{1}{\sqrt{2\pi}}$.

228. Find the maximum of

$$\left|\int_a^b\int_a^b K(x, t)\,\varphi(x)\,\varphi(t)\,dx\,dt\right|$$

provided that $\int_a^b \varphi^2(x)\,dx = 1$, if

(a) $K(x, t) = xt, \quad 0 \leqslant x, \ t \leqslant 1;$
(b) $K(x, t) = xt + x^2t^2, \quad -1 \leqslant x, \ t \leqslant 1;$
(c) $K(x, t) = \begin{cases} (x+1)\,t, & 0 \leqslant x \leqslant t, \\ (t+1)\,x, & t \leqslant x \leqslant 1. \end{cases}$

Bifurcation points. Suppose we have a nonlinear integral equation

$$\varphi(x) = \lambda \int_a^b K(x, t, \varphi(t))\,dt. \tag{1}$$

Let $\varphi(x) \equiv 0$ be a solution of the equation, and

$$K(x, t, 0) \equiv 0$$

By analogy with linear integral equations, the nonzero solutions $\varphi(x) \not\equiv 0$ of equation (1) are called *eigenfunctions* and the corresponding values of the parameter λ are called *characteristic numbers* of the equation.

Ordinarily, the integral equations (1) do not have nonzero small solutions for small $|\lambda|$; that is, for small $|\lambda|$ equation (1) has no eigenfunctions with small norm. Small eigenfunctions can appear in the case of increasing $|\lambda|$. Let us introduce the following concept.

The number λ_0 is called a *bifurcation point* of the nonlinear equation (1) if for any $\varepsilon > 0$ there is a characteristic number λ of equation (1) such that $|\lambda - \lambda_0| < \varepsilon$, and to this characteristic number there corresponds at least one eigenfunction $\varphi(x)$ ($\varphi(x) \not\equiv 0$) with norm less than ε: $\|\varphi\| < \varepsilon$. Roughly speaking, a bifurcation point is that value of the parameter λ, in the neighbourhood of which a zero solution of equation (1) branches; i.e., there appear small (in norm) nonzero solutions of equation (1). For linear problems, the bifurcation values coincide with the characteristic numbers.

In problems of technology and physics involving conditions of stability, bifurcation points determine critical forces. Thus, the problem of the bending of a rectilinear rod of unit length and variable rigidity $\rho(x)$ under the action of a force P leads to the solution of the following nonlinear integral equation:

$$\varphi(x) = P\rho(x) \int_0^1 K(x, t)\varphi(t) \sqrt{1 - \left[\int_0^1 K'_x(x, t)\varphi(t)\,dt\right]^2}\,dt \tag{1'}$$

where $\varphi(x)$ is the unknown function.

For small P, equation (1′) has a unique zero solution in the space $C[0, 1]$ (by virtue of the contraction-mapping principle). This means that for small P the rod does not bend. However, a deflection occurs for forces greater than the so-called critical force of Euler. Euler's critical force is the bifurcation value.

By way of an illustration of finding bifurcation points, let us consider the nonlinear equation

$$\varphi(x) = \lambda \int_0^1 [\varphi(t) + \varphi^3(t)]\,dt \tag{2}$$

Put

$$C = \int_0^1 [\varphi(t) + \varphi^3(t)]\,dt$$

Then

$$\varphi(x) = C\lambda \tag{3}$$

and equation (2) reduces to the algebraic equation

$$C = \lambda C + \lambda^3 C^3 \tag{4}$$

From (4) we get

$$C_1 = 0, \quad C_{2,3} = \pm \sqrt{\frac{1-\lambda}{\lambda^3}}$$

whence, by (3),

$$\varphi_1 \equiv 0, \quad \varphi_{2,3} = \pm \sqrt{\frac{1-\lambda}{\lambda}}$$

Thus, for any $0 < \lambda < 1$, equation (2) admits real nonzero solutions. For $\lambda = 1$ it has only the zero solution $\varphi \equiv 0$ (threefold).

Thus, for any $0 < \varepsilon < 1$, the number $\lambda = 1 - \varepsilon$ is a characteristic number of equation (2) to which there correspond two eigenfunctions:

$$\varphi_1 = \frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon}}; \quad \varphi_2 = -\frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon}}$$

where $\varepsilon = 1 - \lambda$. Hence, the point $\lambda_0 = 1$ is a bifurcation point of equation (2). One can also speak of bifurcation points of *nonzero* solutions of nonlinear integral equations.

Find the bifurcation points of the zero solutions of the integral equations:

229. $\varphi(x) = \lambda \int\limits_0^1 xt\,(\varphi(t) + \varphi^3(t))\,dt.$

230. $\varphi(x) = \lambda \int\limits_0^1 (3x-2)\,t\,(\varphi(t) + \varphi^3(t))\,dt.$

(Bifurcation points are discussed in more detail in [10], [9], [28].)

17. Solution of Homogeneous Integral Equations with Degenerate Kernel

The homogeneous integral equation with degenerate kernel

$$\varphi(x) - \lambda \int\limits_a^b \left[\sum_{k=1}^{n} a_k(x)\, b_k(t)\right] \varphi(t)\,dt = 0 \tag{1}$$

where the parameter λ is not its characteristic number (i. e., $\Delta(\lambda) \neq 0$) has a unique zero solution: $\varphi(x) \equiv 0$. But if λ is a characteristic number ($\Delta(\lambda) = 0$), then, besides the zero solution, equation (1) also has nonzero solutions—the eigenfunctions which correspond to that characteristic number. The general solution of the homogeneous equation (1) is obtained as a linear combination of these eigenfunctions.

Example. Solve the equation

$$\varphi(x) - \lambda \int\limits_0^{\pi} (\cos^2 x \cos 2t + \cos^3 t \cos 3x)\,\varphi(t)\,dt = 0$$

Solution. The characteristic numbers of this equation are $\lambda_1=\frac{4}{\pi}$, $\lambda_2=\frac{8}{\pi}$; the corresponding eigenfunctions are of the form

$$\varphi_1(x)=\cos^2 x, \qquad \varphi_2(x)=\cos 3x$$

The general solution of the equation is

$$\begin{aligned} &\varphi(x)=C\cos^2 x && \text{if } \lambda=\frac{4}{\pi},\\ &\varphi(x)=C\cos 3x && \text{if } \lambda=\frac{8}{\pi},\\ &\varphi(x)=0 && \text{if } \lambda\neq\frac{4}{\pi},\quad \lambda\neq\frac{8}{\pi} \end{aligned}$$

where C is an arbitrary constant. The last zero solution is obtained from the general solutions for $C=0$.

Solve the following homogeneous integral equations:

231. $\varphi(x)-\lambda\int\limits_0^{\pi}\cos(x+t)\,\varphi(t)\,dt=0.$

232. $\varphi(x)-\lambda\int\limits_0^{1}\arccos x\varphi(t)\,dt=0.$

233. $\varphi(x)-2\int\limits_0^{\pi/4}\frac{\varphi(t)}{1+\cos 2t}\,dt=0.$

234. $\varphi(x)-\frac{1}{4}\int\limits_{-2}^{2}|x|\,\varphi(t)\,dt=0.$

235. $\varphi(x)+6\int\limits_0^{1}(x^2-2xt)\,\varphi(t)\,dt=0.$

18. Nonhomogeneous Symmetric Equations

The nonhomogeneous Fredholm integral equation of the second kind

$$\varphi(x)-\lambda\int\limits_a^b K(x,t)\,\varphi(t)\,dt=f(x) \tag{1}$$

is called *symmetric* if its kernel $K(x, t)$ is symmetric: $K(x, t) \equiv K(t, x)$.

If $f(x)$ is continuous and the parameter λ does not coincide with the characteristic numbers $\lambda_n (n = 1, 2, \ldots)$ of the corresponding homogeneous integral equation

$$\varphi(x) - \lambda \int_a^b K(x, t)\, \varphi(t)\, dt = 0 \tag{2}$$

then equation (1) has a unique continuous solution, which is given by the formula

$$\varphi(x) = f(x) - \lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda - \lambda_n} \varphi_n(x) \tag{3}$$

where $\varphi_n(x)$ are eigenfunctions of equation (2),

$$a_n = \int_a^b f(x)\, \varphi_n(x)\, dx \tag{4}$$

The series on the right side of formula (3) converges absolutely and uniformly in the square $a \leqslant x,\ t \leqslant b$.

But if the parameter λ coincides with one of the characteristic numbers, say $\lambda = \lambda_k$, of index q (multiplicity of the number λ_k), then equation (1) will not, generally speaking, have any solutions. Solutions exist if and only if the q conditions are fulfilled:

$$(f, \varphi_m) = 0 \quad \text{or} \quad \int_a^b f(x)\, \varphi_m(x)\, dx = 0 \tag{5}$$
$$(m = 1, 2, \ldots, q)$$

that is, if the function $f(x)$ is orthogonal to all eigenfunctions belongirg to the characteristic number λ_k. In this case equation (1) has an infinity of solutions which contain q arbitrary constants and are given by the formula

$$\varphi(x) = f(x) - \lambda \sum_{n=q+1}^{\infty} \frac{a_n}{\lambda - \lambda_n} \varphi_n(x) + $$
$$+ C_1\varphi_1(x) + C_2\varphi_2(x) + \ldots + C_q\varphi_q(x) \tag{6}$$

where $C_1, C_2, \ldots, C_q$ are arbitrary constants.

In the case of the degenerate kernel

$$K(x, t)=\sum_{k=1}^{m} a_k(x) b_k(t)$$

formulas (3) and (6) will contain finite sums in place of series in their right-hand members.

When the right-hand side of equation (1), i. e., the function $f(x)$, is orthogonal to all eigenfunctions $\varphi_n(x)$ of equation (2), the function itself will be a solution of equation (1): $\varphi(x)=f(x)$.

Example 1. Solve the equation

$$\varphi(x)-\lambda \int_0^1 K(x, t) \varphi(t) dt = x, \tag{1}$$

where

$$K(x, t)=\begin{cases} x(t-1), & \text{if } 0 \leqslant x \leqslant t \\ t(x-1), & \text{if } t \leqslant x \leqslant 1 \end{cases}$$

Solution. The characteristic numbers and their associated eigenfunctions are of the form

$$\lambda_n=-\pi^2 n^2, \quad \varphi_n(x)=\sin \pi n x, \quad n=1, 2, \ldots$$

If $\lambda \neq \lambda_n$, then

$$\varphi(x)=x-\lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda+n^2\pi^2} \sin \pi n x \tag{2}$$

will be a solution of equation (1). We find the Fourier coefficients a_n of the right side of the equation:

$$a_n=\int_0^1 x \sin n\pi x dx=\int_0^1 x d\left(-\frac{\cos n\pi x}{n\pi}\right)=\frac{(-1)^{n+1}}{n\pi}$$

Substituting into (2), we get

$$\varphi(x)=x-\frac{\lambda}{\pi} \sum_{n=}^{\infty} \frac{(-1)^{n+1}}{n(\lambda+n^2\pi^2)} \sin n\pi x$$

For $\lambda = -n^2\pi^2$ equation (1) has no solutions since

$$a_n = \frac{(-1)^{n+1}}{n\pi} \neq 0$$

Example 2. Solve the equation

$$\varphi(x) - \lambda \int_0^1 K(x, t)\,\varphi(t)\,dt = \cos \pi x$$

where

$$K(x, t) = \begin{cases} (x+1)\,t, & 0 \leqslant x \leqslant t, \\ (t+1)\,x, & t \leqslant x \leqslant 1 \end{cases}$$

Solution. The characteristic numbers are

$$\lambda_0 = 1, \quad \lambda_n = -n^2\pi^2 \quad (n = 1, 2, \ldots)$$

Their associated eigenfunctions are

$$\varphi_0(x) = e^x, \quad \varphi_n(x) = \sin n\pi x + n\pi \cos n\pi x \ (n = 1, 2, \ldots)$$

If $\lambda \neq 1$ and $\lambda \neq -n^2\pi^2$, then the solution of the given equation will have the form

$$\varphi(x) = \cos \pi x - \lambda \left[\frac{a_0 e^x}{\lambda - 1} + \sum_{n=1}^{\infty} \frac{a_n}{\lambda + n^2\pi^2} (\sin n\pi x + n\pi \cos n\pi x) \right]$$

and since

$$a_0 = \int_0^1 e^x \cos \pi x\, dx = -\frac{1+e}{1+\pi^2}$$

$$a_n = \int_0^1 \cos \pi x (\sin n\pi x + n\pi \cos n\pi x)\, dx = \begin{cases} 0, & n \neq 1, \\ \frac{\pi}{2}, & n = 1 \end{cases}$$

it follows that

$$\varphi(x) = \cos \pi x + \lambda \left[\frac{1+e}{1+\pi^2} \frac{e^x}{\lambda - 1} - \frac{\pi}{2(\lambda + \pi^2)} (\sin \pi x + \pi \cos \pi x) \right]$$

For $\lambda = 1$ and $\lambda = -\pi^2\,(n = 1)$ the equation has no solutions since its right-hand side, that is, the function $\cos \pi x$, is not orthogonal to the corresponding eigenfunctions

$$\varphi_0(x) = e^x,$$
$$\varphi_1(x) = \sin \pi x + \pi \cos \pi x$$

But if $\lambda = -n^2\pi^2$, where $n = 2, 3, \ldots$, then the given equation has an infinity of solutions which are given by formula (6):

$$\varphi(x) = \cos\pi x + \lambda\left[\frac{1+e}{1+\pi^2}\frac{e^x}{\lambda-1} - \frac{\pi}{2(\lambda+\pi^2)}(\sin\pi x + \pi\cos\pi x)\right] + \\ + C(\sin n\pi x + n\pi\cos n\pi x)$$

where C is an arbitrary constant.

In certain cases, a nonhomogeneous symmetric integral equation can be reduced to a nonhomogeneous boundary-value problem. This is possible when the kernel $K(x, t)$ of the integral equation is a Green's function of some linear differential operator. Let us illustrate how this is done.

Example 3. Solve the equation

$$\varphi(x) - \lambda\int_0^1 K(x, t)\varphi(t)\,dt = e^x \tag{1}$$

where

$$K(x, t) = \begin{cases} \dfrac{\sinh x\sinh(t-1)}{\sinh 1}, & 0 \leqslant x \leqslant t, \\ \dfrac{\sinh t\sinh(x-1)}{\sinh 1}, & t \leqslant x \leqslant 1 \end{cases}$$

Solution. Rewrite the equation as

$$\varphi(x) = e^x + \frac{\lambda\sinh(x-1)}{\sinh 1}\int_0^x \sinh t\,\varphi(t)\,dt + \\ + \frac{\lambda\sinh x}{\sinh 1}\int_x^1 \sinh(t-1)\,\varphi(t)\,dt \tag{2}$$

Differentiating twice, we obtain

$$\varphi'(x) = e^x + \frac{\lambda\cosh(x-1)}{\sinh 1}\int_0^x \sinh t\varphi(t)\,dt + \\ + \frac{\lambda\sinh(x-1)}{\sinh 1}\sinh x\varphi(x) + \lambda\frac{\cosh x}{\sinh 1}\int_x^1 \sinh(t-1)\varphi(t)\,dt - \\ - \frac{\lambda\sinh x}{\sinh 1}\sinh(x-1)\varphi(x),$$

$$\varphi''(x)=e^x+\frac{\lambda\sinh(x-1)}{\sinh 1}\int_0^x \sinh t\varphi(t)\,dt+$$

$$+\frac{\lambda\sinh x}{\sinh 1}\int_x^1 \sinh(t-1)\varphi(t)\,dt+\frac{\lambda\cosh(x-1)\sinh x}{\sinh 1}\varphi(x)-$$

$$-\frac{\lambda\cosh x}{\sinh 1}\sinh(x-1)\varphi(x)$$

or

$$\varphi''(x)=e^x+\varphi(x)+\lambda\varphi(x)$$

Putting $x=0$ and $x=1$ in (2), we get $\varphi(0)=1$, $\varphi(1)=e$. The required function $\varphi(x)$ is a solution of the nonhomogeneous boundary-value problem

$$\varphi''(x)-(\lambda+1)\varphi(x)=e^x, \tag{3}$$

$$\varphi(0)=1, \quad \varphi(1)=e \tag{4}$$

Let us consider the following cases:

(1) $\lambda+1=0$, or $\lambda=-1$. Equation (3) is of the form $\varphi''(x)=e^x$. Its general solution is

$$\varphi(x)=C_1x+C_2+e^x$$

Taking into account the boundary conditions (4), we get the following system for finding the constants C_1 and C_2:

$$\begin{cases} C_2+1=1, \\ C_1+C_2+e=e \end{cases}$$

Its solution is of the form $C_1=0$, $C_2=0$, and, hence,

$$\varphi(x)=e^x$$

(2) $\lambda+1>0$, or $\lambda>-1$, $\lambda\neq 0$. The general solution of equation (3) is

$$\varphi(x)=C_1\cosh\sqrt{1+\lambda}\,x+C_2\sinh\sqrt{1+\lambda}\,x-\frac{e^x}{\lambda}$$

The boundary conditions (4) yield the following system for finding C_1 and C_2:

$$\begin{cases} C_1-\frac{1}{\lambda}=1, \\ C_1\cosh\sqrt{1+\lambda}+C_2\sinh\sqrt{1+\lambda}-\frac{e}{\lambda}=e \end{cases}$$

whence

$$C_1 = 1 + \frac{1}{\lambda}, \quad C_2 = \frac{e - \cosh\sqrt{1+\lambda}}{\sinh\sqrt{1+\lambda}} \left(1 + \frac{1}{\lambda}\right)$$

After simple manipulations, the unknown function $\varphi(x)$ is reduced to

$$\varphi(x) = \left(1 + \frac{1}{\lambda}\right) \frac{\sinh\sqrt{1+\lambda}\,(1-x)}{\sinh\sqrt{1+\lambda}} - \frac{e^x}{\lambda}$$

(3) $\lambda + 1 < 0$, or $\lambda < -1$. Denote $\lambda + 1 = -\mu^2$. The general solution of equation (3) is

$$\varphi(x) = C_1 \cos \mu x + C_2 \sin \mu x + \frac{e^x}{1+\mu^2}$$

The boundary conditions (4) yield the system

$$\left.\begin{aligned} &C_1 + \frac{1}{1+\mu^2} = 1, \\ &C_1 \cos \mu + C_2 \sin \mu = e \frac{\mu^2}{1+\mu^2} \end{aligned}\right\} \tag{5}$$

In turn, two cases are possible here:

(a) μ is not a root of the equation $\sin \mu = 0$. Then

$$C_1 = \frac{\mu^2}{1+\mu^2}, \quad C_2 = \frac{(e - \cos \mu)\,\mu^2}{(1+\mu^2) \sin \mu}$$

and, hence,

$$\varphi(x) = \frac{\mu^2}{1+\mu^2} \left[\cos \mu x + \frac{e - \cos \mu}{\sin \mu} \sin \mu x\right] + \frac{e^x}{1+\mu^2}$$

where $\mu = \sqrt{-\lambda - 1}$.

(b) μ is a root of the equation $\sin \mu = 0$, i. e., $\mu = n\pi$ $(n = 1, 2, \ldots)$. System (5) is inconsistent and, consequently, the given equation (1) has no solutions.

In this case, the corresponding homogeneous integral equation

$$\varphi(x) + (1 + n^2\pi^2) \int_0^1 K(x, t)\, \varphi(t)\, dt = 0 \tag{6}$$

will have an infinity of nontrivial solutions, that is, the numbers $\lambda_n = -(1 + n^2\pi^2)$ are characteristic numbers and their associated solutions $\varphi_n(x) = \sin n\pi x$ are eigenfunctions of equation (6).

Solve the following nonhomogeneous symmetric integral equations:

236. $\varphi(x) - \frac{\pi^2}{4}\int_0^1 K(x, t)\varphi(t)\,dt = \frac{x}{2}$,

$$K(x, t) = \begin{cases} \frac{x(2-t)}{2}, & 0 \leqslant x \leqslant t, \\ \frac{t(2-x)}{2}, & t \leqslant x \leqslant 1. \end{cases}$$

237. $\varphi(x) + \int_0^1 K(x, t)\varphi(t)\,dt = xe^x$,

$$K(x, t) = \begin{cases} \frac{\sinh x \sinh(t-1)}{\sinh 1}, & 0 \leqslant x \leqslant t, \\ \frac{\sinh t \sinh(x-1)}{\sinh 1}, & t \leqslant x \leqslant 1. \end{cases}$$

238. $\varphi(x) - \lambda\int_0^1 K(x, t)\varphi(t)\,dt = x - 1$,

$$K(x, t) = \begin{cases} x - t, & 0 \leqslant x \leqslant t, \\ t - x, & t \leqslant x \leqslant 1. \end{cases}$$

239. $\varphi(x) - 2\int_0^{\pi/2} K(x, t)\varphi(t)\,dt = \cos 2x$,

$$K(x, t) = \begin{cases} \sin x \cos t, & 0 \leqslant x \leqslant t, \\ \sin t \cos x, & t \leqslant x \leqslant \frac{\pi}{2}. \end{cases}$$

240. $\varphi(x) - \lambda\int_0^{\pi} K(x, t)\varphi(t)\,dt = 1$,

$$K(x, t) = \begin{cases} \sin x \cos t, & 0 \leqslant x \leqslant t, \\ \sin t \cos x, & t \leqslant x \leqslant \pi. \end{cases}$$

241. $\varphi(x) - \lambda\int_0^1 K(x, t)\varphi(t)\,dt = x$,

$$K(x, t) = \begin{cases} (x+1)(t-3), & 0 \leqslant x \leqslant t, \\ (t+1)(x-3), & t \leqslant x \leqslant 1. \end{cases}$$

242. $\varphi(x)-\int\limits_0^{\pi} K(x, t)\varphi(t)\,dt=\sin x,$

$$K(x, t)=\begin{cases}\sin\left(x+\frac{\pi}{4}\right)\sin\left(t-\frac{\pi}{4}\right), & 0\leqslant x\leqslant t,\\ \sin\left(t+\frac{\pi}{4}\right)\sin\left(x-\frac{\pi}{4}\right), & t\leqslant x\leqslant \pi.\end{cases}$$

243. $\varphi(x)-\int\limits_0^{1} K(x, t)\varphi(t)\,dt=\sinh x,$

$$K(x, t)=\begin{cases}-e^{-t}\sinh x, & 0\leqslant x\leqslant t,\\ -e^{-x}\sinh t, & t\leqslant x\leqslant 1.\end{cases}$$

244. $\varphi(x)-\lambda\int\limits_0^{1} K(x, t)\varphi(t)\,dt=\cosh x,$

$$K(x, t)=\begin{cases}\dfrac{\cosh x\cosh(t-1)}{\sinh 1}, & 0\leqslant x\leqslant t,\\ \dfrac{\cosh t\cosh(x-1)}{\sinh 1}, & t\leqslant x\leqslant 1.\end{cases}$$

245. $\varphi(x)-\lambda\int\limits_0^{\pi}|x-t|\,\varphi(t)\,dt=1.$

19. Fredholm Alternative

For Fredholm integral equations we have the theorems:

Theorem 1 (Fredholm alternative). *Either the nonhomogeneous linear equation of the second kind*

$$\varphi(x)-\lambda\int\limits_a^b K(x, t)\varphi(t)\,dt=f(x) \tag{1}$$

has a unique solution for any function $f(x)$ *(in some sufficiently broad class) or the corresponding homogeneous equation*

$$\varphi(x)-\lambda\int\limits_a^b K(x, t)\varphi(t)\,dt=0 \tag{2}$$

has at least one nontrivial (that is, not identically zero) solution.

Theorem 2. *If the first alternative holds true for equation (1), then it holds true for the associated equation*

$$\psi(x)-\lambda\int_a^b K(t, x)\psi(t)\,dt=g(x) \tag{3}$$

as well. The homogeneous integral equation (2) and its associated equation

$$\psi(x)-\lambda\int_a^b K(t, x)\psi(t)\,dt=0 \tag{4}$$

have one and the same finite number of linearly independent solutions.

Note. If the functions $\varphi_1(x)$, $\varphi_2(x)$, $\ldots$, $\varphi_n(x)$ are solutions of the homogeneous equation (2), then their linear combination

$$\varphi(x)=C_1\varphi_1(x)+C_2\varphi_2(x)+\ldots+C_n\varphi_n(x)=\sum_{k=1}^{n} C_k\varphi_k(x)$$

where the C_k $(k=1, 2, \ldots, n)$ are arbitrary constants, is also a solution of the equation.

Theorem 3. *A necessary and sufficient condition for the existence of a solution $\varphi(x)$ of the nonhomogeneous equation (1) in the latter case of the alternative is the condition of orthogonality of the right side of the equation, i.e., of the function $f(x)$, to any solution $\psi(x)$ of the homogeneous equation (4) associated with (2):*

$$\int_a^b f(x)\psi(x)\,dx=0 \tag{5}$$

Note. When condition (5) is fulfilled, equation (1) will have an infinite number of solutions, since this equation will be satisfied by any function of the form $\varphi(x)+\tilde{\varphi}(x)$, where $\varphi(x)$ is some solution of equation (1) and $\tilde{\varphi}(x)$ is any solution of the corresponding homogeneous equation (2). Besides, if equation (1) is satisfied by the functions $\varphi_1(x)$ and $\varphi_2(x)$, then by virtue of the linearity of the equation their difference, $\varphi_1(x)-\varphi_2(x)$, is a solution of the corresponding homogeneous equation (2).

The Fredholm alternative is particularly important in practical situations. Instead of proving that a given integral equation (1) has a solution, it is often simpler to prove that the appropriate homogeneous equation (2) or its associated equation (4) has only trivial solutions. Whence it follows, by virtue of the alternative, that equation (1) indeed has a solution.

Remarks. (1) If the kernel $K(x, t)$ of the integral equation (1) is symmetric, that is, $K(x, t) \equiv K(t, x)$, then the associated homogeneous equation (4) coincides with the homogeneous equation (2) which corresponds to equation (1).

(2) In the case of the nonhomogeneous integral equation with degenerate kernel

$$\varphi(x)-\lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x)\, b_k(t)\right] \varphi(t)\, dt = f(x)$$

the orthogonality condition (5) of the right side of this equation yields n equalities

$$\int_a^b f(t)\, b_k(t)\, dt = 0 \quad (k=1, 2, \ldots, n)$$

Example 1.

$$\varphi(x)-\lambda \int_0^1 (5x^2-3)\, t^2 \varphi(t)\, dt = e^x$$

Solution. We have

$$\varphi(x) = C\lambda(5x^2-3)+e^x \tag{1}$$

where

$$C=\int_0^1 t^2\varphi(t)\, dt \tag{2}$$

Substituting (2) into (1), we get

$$C = C\lambda \int_0^1 (5t^4-3t^2)\, dt + \int_0^1 t^2 e^t\, dt$$

whence

$$C = e - 2$$

For any λ, the given equation has a unique solution:

$$\varphi(x) = \lambda(e-2)(5x^2-3) + e^x$$

and the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_0^1 (5x^2-3)\, t^2 \varphi(t)\, dt = 0$$

has a unique zero solution $\varphi(x) \equiv 0$.

Example 2.

$$\varphi(x) - \lambda \int_0^1 \sin \ln x \varphi(t)\, dt = 2x$$

Solution. We have

$$\varphi(x) = C\lambda \sin \ln x + 2x$$

where $C = \int_0^1 \varphi(t)\, dt$. Substituting the expression $\varphi(t)$ into the integral, we obtain

$$C = C\lambda \int_0^1 \sin \ln t\, dt + 1$$

whence

$$C\left(1 + \frac{\lambda}{2}\right) = 1$$

If $\lambda \neq -2$, then the given equation has a unique solution $\varphi(x) = \frac{2\lambda}{2+\lambda} \sin \ln x + 2x$; the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_0^1 \sin \ln x \varphi(t)\, dt = 0$$

has only the zero solution $\varphi(x) \equiv 0$.

But if $\lambda = -2$, then the given equation does not have any solutions since the right side $f(x)=2x$ is not orthogonal to the function $\sin\ln x$; the homogeneous equation has an infinity of solutions since it follows from the equation defining C, $0\cdot C=0$, that C is an arbitrary constant; all these solutions are given by the formula

$$\varphi(x)=\tilde{C}\sin\ln x \quad (\tilde{C}=-2C)$$

Example 3.

$$\varphi(x)-\lambda\int_0^{\pi}\cos(x+t)\,\varphi(t)\,dt=\cos 3x$$

Solution. Rewrite the equation in the form

$$\varphi(x)-\lambda\int_0^{\pi}(\cos x\cos t-\sin x\sin t)\,\varphi(t)\,dt=\cos 3x$$

Whence we have

$$\varphi(x)=C_1\lambda\cos x-C_2\lambda\sin x+\cos 3x \tag{1}$$

where

$$\left\{\begin{aligned} C_1&=\int_0^{\pi}\varphi(t)\cos t\,dt,\\ C_2&=\int_0^{\pi}\varphi(t)\sin t\,dt\end{aligned}\right. \tag{2}$$

Substituting (1) into (2), we get

$$\left\{\begin{aligned} C_1&=\int_0^{\pi}(C_1\lambda\cos t-C_2\lambda\sin t+\cos 3t)\cos t\,dt,\\ C_2&=\int_0^{\pi}(C_1\lambda\cos t-C_2\lambda\sin t+\cos 3t)\sin t\,dt\end{aligned}\right.$$

whence

$$\begin{cases} C_1\left(1-\lambda\int\limits_0^\pi \cos^2 t\,dt\right)+C_2\lambda\int\limits_0^\pi \sin t\cos t\,dt = \\ \qquad\qquad = \int\limits_0^\pi \cos 3t\cos t\,dt, \\ -C_1\lambda\int\limits_0^\pi \cos t\sin t\,dt + C_2\left(1+\lambda\int\limits_0^\pi \sin^2 t\,dt\right)= \\ \qquad\qquad = \int\limits_0^\pi \cos 3t\sin t\,dt \end{cases}$$

or

$$\begin{cases} C_1\left(1-\lambda\frac{\pi}{2}\right)=0, \\ C_2\left(1+\lambda\frac{\pi}{2}\right)=0 \end{cases} \tag{3}$$

The determinant of this system is

$$\Delta(\lambda)\begin{vmatrix} 1-\lambda\frac{\pi}{2} & 0 \\ 0 & 1+\lambda\frac{\pi}{2}\end{vmatrix}=1-\frac{\pi^2}{4}\lambda^2$$

(1) If $\lambda\neq\pm\frac{2}{\pi}$ $(\Delta(\lambda)\neq 0)$, then system (3) has a unique solution $C_1=0$, $C_2=0$ and, hence, the given equation has the unique solution $\varphi(x)=\cos 3x$ and the corresponding homogeneous equation

$$\varphi(x)-\lambda\int\limits_0^\pi \cos(x+t)\,\varphi(t)\,dt=0 \tag{4}$$

only has the zero solution $\varphi(x)=0$.

(2) If $\lambda=\frac{2}{\pi}$, then system (3) takes the form

$$\begin{cases} C_1\cdot 0=0, \\ C_2\cdot 2=0 \end{cases}$$

Whence it follows that $C_2=0$ and $C_1=C$, where C is an arbitrary constant. The given equation will have an infinity of solutions which are given by the formula

$$\varphi(x)=\frac{2}{\pi}C\cdot\cos x+\cos 3x$$

or

$$\varphi(x)=\tilde{C}\cdot\cos x+\cos 3x \quad \left(\tilde{C}=\frac{2C}{\pi}\right);$$

the corresponding homogeneous equation (4) has an infinity of solutions:

$$\varphi(x)=\tilde{C}\cdot\cos x$$

(3) If $\lambda=-\frac{2}{\pi}$, then system (3) takes the form

$$\begin{cases} 2\cdot C_1=0, \\ 0\cdot C_2=0 \end{cases}$$

whence $C_1=0$, $C_2=C$, where C is an arbitrary constant.

The general solution of the given equation is of the form

$$\varphi(x)=\frac{2}{\pi}C\cdot\sin x+\cos 3x$$

or

$$\varphi(x)=\tilde{C}\cdot\sin x+\cos 3x \quad \left(\tilde{C}=\frac{2C}{\pi}\right)$$

In this example, the kernel $K(x, t)=\cos(x+t)$ of the given equation is symmetric: $K(x, t)=K(t, x)$; the right side of the equation [that is, the function $f(x)=\cos 3x$] is orthogonal to the functions $\cos x$ and $\sin x$ on the interval $[0, \pi]$.

Investigate for solvability the following integral equations (for different values of the parameter λ):

246. $\varphi(x)-\lambda\int\limits_0^{\pi}\cos^2 x\varphi(t)\,dt=1.$

247. $\varphi(x)-\lambda\int\limits_{-1}^{1}xe^t\varphi(t)\,dt=x.$

248. $\varphi(x)-\lambda\int\limits_0^{2\pi}|x-\pi|\varphi(t)\,dt=x.$

249. $\varphi(x)-\lambda\int\limits_0^1 (2xt-4x^2)\varphi(t)\,dt=1-2x.$

250. $\varphi(x)-\lambda\int\limits_{-1}^1 (x^2-2xt)\varphi(t)\,dt=x^3-x.$

251. $\varphi(x)-\lambda\int\limits_0^{2\pi}\left(\frac{1}{\pi}\cos x\cos t+\frac{1}{\pi}\sin 2x\sin 2t\right)\varphi(t)\,dt=$
$=\sin x.$

252. $\varphi(x)-\lambda\int\limits_0^1 K(x, t)\varphi(t)\,dt=1$

where

$$K(x, t)=\begin{cases}\cosh x\cdot\sinh t, & 0\leqslant x\leqslant t,\\ \cosh t\cdot\sinh x, & t\leqslant x\leqslant 1\end{cases}$$

20. Construction of Green's Function for Ordinary Differential Equations

Suppose we have a differential equation of order n:

$$L[y]\equiv p_0(x)y^n+p_1(x)y^{n-1}+\ldots+p_n(x)y=0 \qquad (1)$$

where the functions $p_0(x)$, $p_1(x)$, $\ldots$, $p_n(x)$ are continuous on $[a, b]$, $p_0(x)\neq 0$ on $[a, b]$, and the boundary conditions are

$$V_k(y)=\alpha_k y(a)+\alpha_k^1 y'(a)+\ldots+\alpha_k^{n-1}y^{n-1}(a)+$$
$$+\beta_k y(b)+\beta_k^1 y'(b)+\ldots+\beta_k^{n-1}y^{n-1}(b)$$
$$(k=1, 2, \ldots, n) \qquad (2)$$

where the linear forms $V_1, \ldots, V_n$ in $y(a)$, $y'(a)$, $\ldots$, $y^{(n-1)}(a)$, $y(b)$, $\ldots$, $y^{(n-1)}(b)$ are linearly independent.

We assume that the homogeneous boundary-value problem (1)-(2) has only a trivial solution $y(x)\equiv 0$.

Definition. Green's function of the boundary-value problem (1)-(2) is the function $G(x, \xi)$ constructed for any point ξ, $a<\xi<b$, and having the following four properties:

(1) $G(x, \xi)$ is continuous and has continuous derivatives with respect to x up to order $(n-2)$ inclusive for $a\leqslant x\leqslant b$.

(2) Its $(n-1)$th derivative with respect to x at the point $x=\xi$ has a discontinuity of the first kind, the jump being equal to $\frac{1}{p_0(x)}$, i.e.,

$$\left.\frac{\partial^{n-1}G(x, \xi)}{\partial x^{n-1}}\right|_{x=\xi+0} - \left.\frac{\partial^{n-1}G(x, \xi)}{\partial x^{n-1}}\right|_{x=\xi-0} = \frac{1}{p_0(\xi)} \tag{3}$$

(3) In each of the intervals $[a, \xi)$ and $(\xi, b]$ the function $G(x, \xi)$, considered as a function of x, is a solution of equation (1):

$$L[G]=0 \tag{4}$$

(4) $G(x, \xi)$ satisfies the boundary conditions (2):

$$V_k(G)=0 \quad (k=1, 2, \ldots, n) \tag{5}$$

Theorem 1. *If the boundary-value problem (1)-(2) has only the trivial solution* $y(x)\equiv 0$, *then the operator* L *has one and only one Green's function* $G(x, \xi)$.

Proof. Let $y_1(x), y_2(x), \ldots, y_n(x)$ be linearly independent solutions of the equation $L[y]=0$. Then, by virtue of Property (3), the unknown function $G(x, \xi)$ must have the following representation on the intervals $[a, \xi)$ and $(\xi, b]$:

$$G(x, \xi)=a_1y_1(x)+a_2y_2(x)+\ldots+a_ny_n(x) \text{ for } a\leqslant x<\xi$$

and

$$G(x, \xi)=b_1y_1(x)+b_2y_2(x)+\ldots+b_ny_n(x) \text{ for } \xi\leqslant x<b$$

Here, $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are some functions of ξ. The continuity of the function $G(x, \xi)$ and of its first $n-2$ derivatives with respect to x at the point $x=\xi$ yields the relations

$$\begin{gathered}
[b_1y_1(\xi)+\ldots+b_ny_n(\xi)]-[a_1y_1(\xi)+\ldots+a_ny_n(\xi)]=0,\\
[b_1y_1'(\xi)+\ldots+b_ny_n'(\xi)]-[a_1y_1'(\xi)+\ldots+a_ny_n'(\xi)]=0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
[b_1y_1^{(n-2)}(\xi)+\ldots+b_ny_n^{(n-2)}(\xi)]-[a_1y_1^{(n-2)}(\xi)+\\
\ldots+a_ny_n^{(n-2)}(\xi)]=0
\end{gathered}$$

and condition (3) takes the form

$$[b_1y_1^{(n-1)}(\xi)+\ldots+b_ny_n^{(n-1)}(\xi)]-[a_1y_1^{(n-1)}(\xi)+ \\ \ldots+a_ny_n^{(n-1)}(\xi)]=\frac{1}{p_0(\xi)}$$

Let us put $c_k(\xi)=b_k(\xi)-a_k(\xi)$ $(k=1, 2, \ldots, n)$; then we get a system of linear equations in $c_k(\xi)$:

$$\left.\begin{array}{l} c_1y_1(\xi)+c_2y_2(\xi)+\ldots+c_ny_n(\xi)=0, \\ c_1y_1'(\xi)+c_2y_2'(\xi)+\ldots+c_ny_n'(\xi)=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1y_1^{(n-2)}(\xi)+c_2y_2^{(n-2)}(\xi)+\ldots+c_ny_n^{(n-2)}(\xi)=0, \\ c_1y_1^{(n-1)}(\xi)+c_2y_2^{(n-1)}(\xi)+\ldots+c_ny_n^{(n-1)}(\xi)=\dfrac{1}{p_0(\xi)} \end{array}\right\} \qquad (6)$$

The determinant of system (6) is equal to the value of the Wronskian $W(y_1, y_2, \ldots, y_n)$ at the point $x=\xi$ and is therefore different from zero. For this reason, system (6) uniquely defines the functions $c_k(\xi)$ $(k=1, 2, \ldots, n)$. To determine the functions $a_k(\xi)$ and $b_k(\xi)$ let us take advantage of the boundary conditions (2). We write $V_k(y)$ in the form

$$V_k(y)=A_k(y)+B_k(y) \qquad (7)$$

where

$$A_k(y)=\alpha_k y(a)+\alpha_k^{(1)}y'(a)+\ldots+\alpha_k^{(n-1)}y^{(n-1)}(a),$$
$$B_k(y)=\beta_k y(b)+\beta_k^{(1)}y'(b)+\ldots+\beta_k^{(n-1)}y^{(n-1)}(b).$$

Then, by conditions (5), we get

$$V_k(G)=a_1A_k(y_1)+a_2A_k(y_2)+\ldots+a_nA_k(y_n)+$$
$$+b_1B_k(y_1)+b_2B_k(y_2)+\ldots+b_nB_k(y_n)=0 \ (k=1, 2, \ldots, n)$$

Taking into consideration that $a_k=b_k-c_k$, we will have

$$(b_1-c_1)A_k(y_1)+(b_2-c_2)A_k(y_2)+\ldots+(b_n-c_n)A_k(y_n)+$$
$$+b_1B_k(y_1)+b_2B_k(y_2)+\ldots+b_nB_k(y_n)=0$$
$$(k=1, 2, \ldots, n)$$

Whence, by virtue of (7),

$$b_1V_k(y_1)+b_2V_k(y_2)+\ldots+b_nV_k(y_n)=$$
$$=c_1A_k(y_1)+c_2A_k(y_2)+\ldots+c_nA_k(y_n) \ (k=1, 2, \ldots, n) \qquad (8)$$

Note that system (8) is linear in the quantities $b_1, \ldots, b_n$. The determinant of the system is different from zero:

$$\begin{vmatrix} V_1(y_1) & V_1(y_2) & \ldots & V_1(y_n) \\ V_2(y_1) & V_2(y_2) & \ldots & V_2(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ V_n(y_1) & V_n(y_2) & \ldots & V_n(y_n) \end{vmatrix} \neq 0 \qquad (9)$$

by virtue of our assumption concerning the linear independence of the forms $V_1, V_2, \ldots, V_n$.

Consequently, the system of equations (8) has a unique solution in $b_1(\xi), b_2(\xi), \ldots, b_n(\xi)$, and since $a_k(\xi) = b_k(\xi) - c_k(\xi)$, it follows that the quantities $a_k(\xi)$ $(k = 1, 2, \ldots, n)$ are defined uniquely. Thus the existence and uniqueness of Green's function $G(x, \xi)$ have been proved and a method has been given for constructing the function.

Note 1. If the boundary-value problem (1)-(2) is self-adjoint, then Green's function is symmetric, i. e.,

$$G(x, \xi) = G(\xi, x)$$

The converse is true as well.

For the conditions of self-adjointness of the boundary-value problem for differential operators of the second and fourth orders see [20], Vol. I.

Note 2. If at one of the extremities of an interval $[a, b]$ the coefficient of the highest derivative vanishes, for example $p_0(a) = 0$, then the natural boundary condition for boundedness of the solution at $x = a$ is imposed, and at the other extremity the ordinary boundary condition is specified (see Example 2 below).

An Important Special Case

Let us consider the construction of Green's function for a second-order differential equation of the form

$$(p(x) y')' + q(x) y = 0,$$
$$p(x) \neq 0 \text{ on } [a, b], \quad p(x) \in C^{(1)}[a, b] \tag{10}$$

with boundary conditions

$$y(a) = y(b) = 0 \tag{11}$$

Suppose that $y_1(x)$ is a solution of equation (10) defined by the initial conditions

$$y_1(a) = 0, \quad y_1'(a) = \alpha \neq 0 \tag{12}$$

Generally speaking, this solution need not necessarily satisfy the second boundary condition; we will therefore assume that $y_1(b) \neq 0$. But functions of the form $C_1 y_1(x)$, where

C_1 is an arbitrary constant, are obviously solutions of equation (10) and satisfy the boundary condition

$$y(a)=0$$

Similarly, we find the nonzero solution $y_2(x)$ of equation (10), such that it should satisfy the second boundary condition, i. e.,

$$y_2(b)=0 \tag{13}$$

This same condition will be satisfied by all solutions of the family $C_2y_2(x)$, where C_2 is an arbitrary constant.

We now seek Green's function for the problem (10)-(11) in the form

$$G(x,\xi)=\begin{cases} C_1y_1(x) & \text{for } a\leqslant x\leqslant \xi, \\ C_2y_2(x) & \text{for } \xi\leqslant x\leqslant b \end{cases} \tag{14}$$

and we shall choose the constants C_1 and C_2 so that the Properties (1) and (2) are fulfilled, i. e., so that the function $G(x,\xi)$ is continuous in x for fixed ξ, in particular, continuous at the point $x=\xi$:

$$C_1y_1(\xi)=C_2y_2(\xi)$$

and so that $G'_x(x,\xi)$ has a jump, at the point $x=\xi$, equal to $\frac{1}{p(\xi)}$:

$$C_2y'_2(\xi)-C_1y'_1(\xi)=\frac{1}{p(\xi)}$$

Rewrite the last two equalities as

$$\left.\begin{aligned} -C_1y_1(\xi)+C_2y_2(\xi)&=0, \\ -C_1y'_1(\xi)+C_2y'_2(\xi)&=\frac{1}{p(\xi)} \end{aligned}\right\} \tag{15}$$

The determinant of system (15) is the Wronskian $W[y_1(x), y_2(x)]=W(x)$ computed at the point $x=\xi$ for linearly independent solutions $y_1(x)$ and $y_2(x)$ of equation (10), and, hence, it is different from zero:

$$W(\xi)\neq 0$$

so that the quantities C_1 and C_2 of system (15) are determined at once:

$$C_1=\frac{y_2(\xi)}{p(\xi)W(\xi)}, \qquad C_2=\frac{y_1(\xi)}{p(\xi)W(\xi)} \tag{16}$$

Substituting the expressions for C_1 and C_2 into (14), we finally get

$$G(x, \xi)=\begin{cases} \dfrac{y_1(x)\, y_2(\xi)}{p(\xi)\, W(\xi)}, & a \leqslant x \leqslant \xi, \\ \dfrac{y_1(\xi)\, y_2(x)}{p(\xi)\, W(\xi)}, & \xi \leqslant x \leqslant b \end{cases} \tag{17}$$

Note 1. The solutions $y_1(x)$ and $y_2(x)$ of equation (10) that we have chosen are linearly independent by virtue of the assumption that $y_1(b) \neq 0$.

Indeed, all solutions linearly dependent on $y_1(x)$ have the form $C_1 y_1(x)$ and, consequently, for $C_1 \neq 0$, do not vanish at the point $x=b$ at which, according to our choice, the solution $y_2(x)$ vanishes.

Note 2. The boundary-value problem for a second-order equation of the form

$$y''(x)+p_1(x)\, y'(x)+p_2(x)\, y(x)=0 \tag{18}$$

and boundary conditions

$$y(a)=A, \quad y(b)=B \tag{19}$$

reduces to the above-considered problem (10)-(11) as follows:

(1) The linear equation (18) is reduced to (10) by multiplying (18) by $p(x)=e^{\int p_1(x)\,dx}$ [we have to take $p(x)\, p_2(x)$ for $q(x)$].

(2) The boundary conditions (19) reduce to zero conditions (11) by a linear change of variables

$$z(x)=y(x)-\frac{B-A}{b-a}(x-a)-A$$

The linearity of equation (18) is preserved in this change, but unlike equation (10), we now obtain the nonhomogeneous equation $L[z]=f(x)$, where

$$f(x)=-\left[A+\frac{B-A}{b-a}(x-a)\right] q(x)-\frac{B-A}{b-a}\, p(x)$$

However, we construct Green's function for the homogeneous boundary-value problem $L[z]=0$, $z(a)=z(b)=0$, which fully coincides with the Problem (10)-(11).

Example 1. Construct Green's function for the homogeneous boundary-value problem

$$y^{\mathrm{IV}}(x)=0, \tag{1}$$

$$\left.\begin{aligned} y(0)=y'(0)=0, \\ y(1)=y'(1)=0 \end{aligned}\right\} \tag{2}$$

Solution 1. We shall first show that the boundary-value problem (1)-(2) has only a trivial solution. Indeed, the fundamental system of solutions for equation (1) is

$$y_1(x)=1, \quad y_2(x)=x, \quad y_3(x)=x^2, \quad y_4(x)=x^3 \tag{3}$$

so that its general solution is of the form

$$y(x)=A+Bx+Cx^2+Dx^3$$

where A, B, C, D are arbitrary constants. The boundary conditions (2) give us four relations for determining A, B, C, D:

$$\begin{aligned} y(0)&=A=0,\\ y'(0)&=B=0,\\ y(1)&=A+B+C+D=0,\\ y'(1)&=B+2C+3D=0 \end{aligned}$$

Whence we have $A=B=C=D=0$.

Thus, the problem (1)-(2) has only a zero solution $y(x)\equiv 0$, and, hence, for it we can construct a (unique) Green's function $G(x, \xi)$.

2. We now construct Green's function. Utilizing the fundamental system of solutions (3), represent the unknown Green's function $G(x, \xi)$ in the form

$$G(x, \xi)=a_1\cdot 1+a_2\cdot x+a_3\cdot x^2+a_4\cdot x^3 \quad \text{for } 0\leqslant x\leqslant \xi, \tag{4}$$

$$G(x, \xi)=b_1\cdot 1+b_2\cdot x+b_3\cdot x^2+b_4\cdot x^3 \quad \text{for } \xi\leqslant x\leqslant 1 \tag{5}$$

where a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 are as yet unknown functions of ξ. Put $c_k(\xi)=b_k(\xi)-a_k(\xi)$ $(k=1, 2, 3, 4)$ and write out the system of linear equations for finding the functions $c_k(\xi)$ [see system (6) on p. 136]:

$$\left.\begin{aligned} c_1+c_2\xi+c_3\xi^2+c_4\xi^3&=0,\\ c_2+c_3\cdot 2\xi+c_4\cdot 3\xi^2&=0,\\ c_3\cdot 2+c_4\cdot 6\xi&=0,\\ c_4\cdot 6&=1 \end{aligned}\right\} \tag{6}$$

Solving the system, we get

$$\left.\begin{aligned} c_1(\xi) &= -\frac{1}{6}\xi^3, \quad c_2(\xi) = \frac{1}{2}\xi^2 \\ c_3(\xi) &= -\frac{1}{2}\xi, \quad c_4(\xi) = \frac{1}{6} \end{aligned}\right\} \tag{7}$$

We further take advantage of Property (4) of Green's function, namely, that it must satisfy the boundary conditions (2), i. e.,

$$\left.\begin{aligned} G(0, \xi) = 0, \quad G'_x(0, \xi) = 0, \\ G(1, \xi) = 0, \quad G'_x(1, \xi) = 0 \end{aligned}\right\}$$

In our case these relations take the form

$$\left.\begin{aligned} a_1 &= 0, \\ a_2 &= 0, \\ b_1 + b_2 + b_3 + b_4 &= 0, \\ b_2 + 2b_3 + 3b_4 &= 0 \end{aligned}\right\} \tag{8}$$

Taking advantage of the fact that $c_k = b_k - a_k$ $(k = 1, 2, 3, 4)$, we find from (7) and (8) that

$$\left.\begin{aligned} & a_1 = 0; \; b_1 = -\frac{1}{6}\xi^3; \; a_2 = 0; \; b_2 = \frac{1}{2}\xi^2; \\ & b_3 = \frac{1}{2}\xi^3 - \xi^2; \; b_4 = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3; \\ & a_3 = \frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3; \; a_4 = -\frac{1}{6} + \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3 \end{aligned}\right\} \tag{9}$$

Putting the values of the coefficients $a_1, a_2, \ldots, b_4$ from (9) into (4) and (5), we obtain the desired Green's function:

$$G(x, \xi) = \begin{cases} \left(\frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3\right)x^2 - \left(\frac{1}{6} - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3\right)x^3, \\ \qquad 0 \leqslant x \leqslant \xi \\ -\frac{1}{6}\xi^3 + \frac{1}{2}\xi^2 x + \left(\frac{1}{2}\xi^3 - \xi^2\right)x^2 + \left(\frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right)x^3 \\ \qquad \xi \leqslant x \leqslant 1 \end{cases}$$

This expression is readily transformed to

$$G(x, \xi) = \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right)\xi^2 - \left(\frac{1}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\xi^3$$
$$\text{for } \xi \leqslant x \leqslant 1$$

so that $G(x, \xi)=G(\xi, x)$, i. e., Green's function is symmetric. This was evident from the start since the boundary-value problem (1)-(2) was self-adjoint.

The reader is advised to establish this by himself. Also we suggest checking to see that the Green's function which we have found satisfies all the requirements (1 to 4) given in the definition.

Example 2. Construct Green's function for the differential equation

$$xy''+y'=0 \tag{1}$$

for the following conditions:

$$\begin{gathered} y(x) \text{ bounded as } x \to 0, \\ y(1)=\alpha y'(1), \quad \alpha \neq 0 \end{gathered} \tag{2}$$

Solution. First find the general solution of equation (1) and convince yourself that the conditions (2) are fulfilled only when

$$y(x) \equiv 0$$

Indeed, denoting $y'(x)=z(x)$ we get $xz'+z=0$, whence $\ln z=\ln c_1-\ln x$, $z=\frac{c_1}{x}$ and, hence,

$$y(x)=c_1 \ln x+c_2 \tag{3}$$

It is clear that $y(x)$ defined by formula (3) satisfies the conditions (2) only for $c_1=c_2=0$, and, hence, Green's function can be constructed for the problem (1)-(2).

Let us write down $G(x, \xi)$ formally as

$$G(x, \xi)=\begin{cases} a_1+a_2 \ln x & \text{for } 0<x \leqslant \xi, \\ b_1+b_2 \ln x & \text{for } \xi \leqslant x \leqslant 1 \end{cases} \tag{4}$$

From the continuity of $G(x, \xi)$ for $x=\xi$ we obtain

$$b_1+b_2 \ln \xi-a_1-a_2 \ln \xi=0$$

and the jump $G'_x(x, \xi)$ at the point $x=\xi$ is equal to $\frac{1}{\xi}$ so that

$$b_2 \cdot \frac{1}{\xi}-a_2 \cdot \frac{1}{\xi}=\frac{1}{\xi}$$

Putting

$$c_1=b_1-a_1, \quad c_2=b_2-a_2 \tag{5}$$

we will have

$$\begin{cases} c_1 + c_2 \ln \xi = 0, \\ \qquad\quad c_2 = 1 \end{cases}$$

whence

$$c_1 = -\ln \xi, \quad c_2 = 1 \tag{6}$$

Now let us use conditions (2). The boundedness of $G(x, \xi)$ as $x \to 0$ gives us $a_2 = 0$, and from the condition $G(x, \xi) = \alpha G'_x(x, \xi)$ we get $b_1 = \alpha b_2$. Taking into account (5) and (6), we get the values of all coefficients in (4):

$$a_1 = \alpha + \ln \xi, \quad a_2 = 0, \quad b_1 = \alpha, \quad b_2 = 1$$

Thus

$$G(x, \xi) = \begin{cases} \alpha + \ln \xi, & 0 < x \leqslant \xi, \\ \alpha + \ln x, & \xi \leqslant x \leqslant 1 \end{cases}$$

Example 3. Find Green's function for the boundary-value problem

$$y''(x) + k^2 y = 0,$$
$$y(0) = y(1) = 0$$

Solution. It is easy to see that the solution $y_1(x) = \sin kx$ satisfies the boundary condition $y_1(0) = 0$, and the solution $y_2(x) = \sin k(x-1)$ satisfies the condition $y_2(1) = 0$; they are linearly independent. Let us find the value of the Wronskian for $\sin kx$ and $\sin k(x-1)$ at the point $x = \xi$:

$$W(\xi) = \begin{vmatrix} \sin k\xi & \sin k(\xi - 1) \\ k \cos k\xi & k \cos k(\xi - 1) \end{vmatrix} =$$
$$= k [\sin k\xi \cos k(\xi - 1) - \sin k(\xi - 1) \cos k\xi] = k \sin k$$

Noting, in addition, that in our example $p(x) = 1$, we get, by (17),

$$G(x, \xi) = \begin{cases} \dfrac{\sin k(\xi - 1) \sin kx}{k \sin k}, & 0 \leqslant x \leqslant \xi, \\ \dfrac{\sin k\xi \sin k(x-1)}{k \sin k}, & \xi \leqslant x \leqslant 1 \end{cases}$$

In the following examples, establish whether a Green's function exists for the given boundary-value problem and if it does, construct it.

253. $y''=0;\quad y(0)=y'(1),\quad y'(0)=y(1).$
254. $y''=0;\quad y(0)=y(1),\quad y'(0)=y'(1).$
255. $y''+y=0;\quad y(0)=y(\pi)=0.$
256. $y^{\mathrm{IV}}=0;\quad y(0)=y'(0)=y''(1)=y'''(1)=0.$
257. $y'''=0;\quad y(0)=y'(1)=0;\quad y'(0)=y(1).$
258. $y'''=0;\quad y(0)=y(1)=0;\quad y'(0)=y'(1).$
259. $y''=0;\quad y(0)=0,\quad y(1)=y'(1).$
260. $y''+y'=0;\quad y(0)=y(1),\quad y'(0)=y'(1).$
261. $y''-k^2y=0\,(k\neq 0);\quad y(0)=y(1)=0.$
262. $y''+y=0;\quad y(0)=y(1),\quad y'(0)=y'(1).$
263. $y'''=0;\quad y(0)=y(1)=0,\quad y'(0)+y'(1)=0.$
264. $y''=0;\quad y'(0)=hy(0),\quad y'(1)=-Hy(1).$
265. $x^2y''+2xy'=0;$ $y(x)$ is bounded for $x\to 0$, $y(1)=\alpha y'(1).$
266. $x^3y^{\mathrm{IV}}+6x^2y'''+6xy''=0;$ $y(x)$ is bounded as $x\to 0$, $y(1)=y'(1)=0.$
267. $x^2y''+xy'-y=0;$ $y(x)$ is bounded as $x\to 0$, $y(1)=0.$
268. $xy''+y'-\frac{1}{x}y=0;$ $y(0)$ is finite, $y(1)=0.$
269. $x^2y''+xy'-n^2y=0;$ $y(0)$ is finite, $y(1)=0.$
270. $x^2(\ln x-1)y''-xy'+y=0;$ $y(0)$ is finite, $y(1)=0.$
271. $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]=0;\quad y(0)=0,$ $y(1)$ is finite.
272. $xy''+y'=0;$ $y(0)$ is bounded, $y(l)=0.$
273. $y''-y=0;\quad y(0)=y'(0),\quad y(l)+\lambda y'(l)=0.$
(Consider the cases: $\lambda=1$, $\lambda=-1$, $|\lambda|\neq 1$.)

21. Using Green's Function in the Solution of Boundary-Value Problems

Let there be given a nonhomogeneous differential equation
$$L[y]\equiv p_0(x)\,y^{(n)}(x)+p_1(x)\,y^{(n-1)}(x)+\ldots+p_n(x)\,y(x)=f(x) \quad (1)$$
and the boundary conditions
$$V_1(y)=0,\quad V_2(y)=0,\ \ldots,\ V_n(y)=0 \qquad (2)$$

As in Sec. 20, we consider that the linear forms $V_1, V_2, \ldots, V_n$ in $y(a), y'(a), \ldots, y^{(n-1)}(a), y(b), y'(b), \ldots, y^{(n-1)}(b)$ are linearly independent.

Theorem. *If $G(x, \xi)$ is Green's function of the homogeneous boundary-value problem*

$$L[y]=0,$$
$$V_k(y)=0, \quad (k=1, 2, \ldots, n).$$

then the solution of the boundary-value problem (1)-(2) is given by the formula

$$y(x)=\int_a^b G(x, \xi) f(\xi) \, d\xi \tag{3}$$

(see [20]).

Example 1. Using Green's function, solve the boundary-value problem

$$y''(x)-y(x)=x, \tag{1}$$
$$y(0)=y(1)=0 \tag{2}$$

(a) Let us first find out whether Green's function exists for the corresponding homogeneous boundary-value problem

$$y''(x)-y(x)=0, \tag{1'}$$
$$y(0)=y(1)=0 \tag{2'}$$

It is obvious that $y_1(x)=e^x$, $y_2(x)=e^{-x}$ is the fundamental system of solutions of the equation (1'). Hence, the general solution of this equation is

$$y(x)=Ae^x+Be^{-x}$$

The boundary conditions (2) are satisfied if and only if $A=B=0$, i. e., $y(x) \equiv 0$. Thus, Green's function exists.

(b) It can readily be verified that

$$G(x, \xi)=\begin{cases} \dfrac{\sinh x \sinh (\xi-1)}{\sinh 1}, & 0 \leqslant x \leqslant \xi, \\ \dfrac{\sinh \xi \sinh (x-1)}{\sinh 1}, & \xi \leqslant x \leqslant 1 \end{cases} \tag{3}$$

is Green's function for the boundary-value problem (1')-(2').

(c) We write the solution of the boundary-value problem (1)-(2) in the form

$$y(x)=\int_0^1 G(x, \xi)\,\xi\, d\xi \tag{4}$$

where $G(x, \xi)$ is defined by formula (3).

Splitting up the interval of integration into two parts and substituting from (3) into (4) the expression for Green's function, we obtain

$$y(x)=\int_0^x \frac{\xi \sinh \xi \sinh (x-1)}{\sinh 1}\, d\xi+\int_x^1 \frac{\xi \sinh x \sinh (\xi-1)}{\sinh 1}\, d\xi=$$

$$=\frac{\sinh (x-1)}{\sinh 1} \int_0^x \xi \sinh \xi\, d\xi+\frac{\sinh x}{\sinh 1} \int_x^1 \xi \sinh (\xi-1)\, d\xi \tag{5}$$

But

$$\int_0^x \xi \sinh \xi\, d\xi=x \cosh x-\sinh x,$$

$$\int_x^1 \xi \sinh (\xi-1)\, d\xi=1-x \cosh (x-1)+\sinh (x-1)$$

and therefore

$$y(x)=\frac{1}{\sinh 1}\{\sinh (x-1)[x \cosh x-\sinh x]+$$

$$+\sinh x[1-x \cosh (x-1)+\sinh (x-1)]\}=\frac{\sinh x}{\sinh 1}-x$$

Here we take advantage of the formula

$$\sinh (\alpha \pm \beta)=\sinh \alpha \cdot \cosh \beta \pm \cosh \alpha \sinh \beta$$

and also the oddness of the function $\sinh x$.

Direct verification convinces us that the function

$$y(x)=\frac{\sinh x}{\sinh 1}-x$$

satisfies equation (1) and the boundary conditions (2).

Example 2. Reduce to an integral equation the following boundary-value problem for the nonlinear differential equation:

$$y'' = f(x, y(x)), \tag{1}$$

$$y(0) = y(1) = 0 \tag{2}$$

Constructing Green's function for the problem

$$y'' = 0, \tag{3}$$

$$y(0) = y(1) = 0 \tag{2}$$

we find

$$G(x, \xi) = \begin{cases} (\xi - 1)x, & 0 \leqslant x \leqslant \xi, \\ (x - 1)\xi, & \xi \leqslant x \leqslant 1 \end{cases}$$

Regarding the right side of equation (1) as the known function, we get

$$y(x) = \int_0^1 G(x, \xi) f(\xi, y(\xi))\, d\xi \tag{4}$$

Thus the solution of the boundary-value problem (1)-(2) reduces to the solution of a nonlinear integral equation of the Hammerstein type (see Sec. 15), the kernel of which is Green's function for the problem (3)-(2). The significance of the Hammerstein-type integral equations lies precisely in the fact that the solution of many boundary-value problems for nonlinear differential equations reduces to the solution of integral equations of this type.

Solve the following boundary-value problems using Green's function:

274. $y'' + y = x;\ y(0) = y\left(\frac{\pi}{2}\right) = 0.$

275. $y^{\mathrm{IV}} = 1;\ y(0) = y'(0) = y''(1) = y'''(1) = 0.$

276. $xy'' + y' = x;\ y(1) = y(e) = 0.$

277. $y'' + \pi^2 y = \cos \pi x;\ y(0) = y(1),\ y'(0) = y'(1).$

278. $y'' - y = 2 \sinh 1;\ y(0) = y(1) = 0.$

279. $y'' - y = -2e^x;\ y(0) = y'(0),\ y(l) + y'(l) = 0.$

280. $y'' + y = x^2;\ y(0) = y\left(\frac{\pi}{2}\right) = 0.$

22. Boundary-Value Problems Containing a Parameter; Reducing Them to Integral Equations

Many situations require the consideration of a boundary-value problem of the following type:

$$L[y]=\lambda y+h(x), \tag{1}$$

$$V_k(y)=0 \quad (k=1, 2, \ldots, n) \tag{2}$$

where

$$L[y] \equiv p_0(x)\, y^{(n)}(x)+p_1(x)\, y^{(n-1)}(x)+\ldots+p_n(x)\, y(x)$$

$$V_k(y) \equiv \alpha_k y(a)+\alpha_k^{(1)} y'(a)+\ldots+\alpha_k^{(n-1)} y^{(n-1)}a+$$
$$+\beta_k y(b)+\beta_k^{(1)} y'(b)+\ldots+\beta_k^{(n-1)} y^{(n-1)}(b) \quad (k=1, 2, \ldots, n)$$

(the linear forms $V_1, V_2, \ldots, V_n$ are linearly independent); $h(x)$ is a given continuous function of x; λ is some numerical parameter.

For $h(x) \equiv 0$ we have the homogeneous boundary-value problem

$$\left.\begin{aligned} &L[y]=\lambda y, \\ &V_k(y)=0 \quad (k=1, 2, \ldots, n) \end{aligned}\right\} \tag{3}$$

Those values of λ for which the boundary-value problem (3) has nontrivial solutions $y(x)$ are called *eigenvalues of the boundary-value problem* (3); the nontrivial solutions are called the associated *eigenfunctions*.

Theorem. *If the boundary-value problem*

$$\left.\begin{aligned} &L[y]=0, \\ &V_k(y)=0 \quad (k=1, 2, \ldots, n) \end{aligned}\right\} \tag{4}$$

has the Green's function $G(x, \xi)$, then the boundary-value problem (1)-(2) is equivalent to the Fredholm integral equation

$$y(x)=\lambda \int_a^b G(x, \xi)\, y(\xi)\, d\xi+f(x) \tag{5}$$

where

$$f(x)=\int_a^b G(x, \xi)\, h(\xi)\, d\xi \tag{6}$$

In particular, the homogeneous boundary-value problem (3) is equivalent to the homogeneous integral equation

$$y(x)=\lambda\int_a^b G(x, \xi)\, y(\xi)\, d\xi \tag{7}$$

Note. Since $G(x, \xi)$ is a continuous kernel, the Fredholm theory is applicable to the integral equation. Therefore the homogeneous integral equation (7) can have at most a countable number of characteristic numbers $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ which do not have a finite limit point. For all values of λ different from the characteristic values, the nonhomogeneous equation (5) has a solution for any continuous right side $f(x)$. This solution is given by the formula

$$y(x)=\lambda\int_a^b R(x, \xi; \lambda)\, f(\xi)\, d\xi+f(x) \tag{8}$$

where $R(x, \xi; \lambda)$ is the resolvent kernel of the kernel $G(x, \xi)$. Here, for any fixed values of x and ξ in $[a, b]$ the function $R(x, \xi; \lambda)$ is a meromorphic function of λ; only characteristic numbers of the homogeneous integral equation (7) may be the poles of this function.

Example. Reduce the boundary-value problem

$$y''+\lambda y=x, \tag{1}$$

$$y(0)=y\left(\frac{\pi}{2}\right)=0 \tag{2}$$

to an integral equation.

Solution. First find the Green's function $G(x, \xi)$ for the corresponding homogeneous problem:

$$\left.\begin{aligned} &y''(x)=0, \\ &y(0)=y\left(\frac{\pi}{2}\right)=0 \end{aligned}\right\} \tag{3}$$

Since the functions $y_1(x)=x$ and $y_2(x)=x-\frac{\pi}{2}$ are, respectively, linearly independent solutions of the equation $y''(x)=0$ that satisfy the conditions $y(0)=0$ and $y\left(\frac{\pi}{2}\right)=0$, we seek

Green's function in the form

$$G(x,\xi)=\begin{cases}\dfrac{y_1(x)\,y_2(\xi)}{W(\xi)}, & 0\leqslant x\leqslant\xi,\\ \dfrac{y_1(\xi)\,y_2(x)}{W(\xi)}, & \xi\leqslant x\leqslant\dfrac{\pi}{2}\end{cases}$$

where

$$W(\xi)=\begin{vmatrix}\xi & \xi-\dfrac{\pi}{2}\\ 1 & 1\end{vmatrix}=\frac{\pi}{2}$$

Thus,

$$G(x,\xi)=\begin{cases}\left(\dfrac{2}{\pi}\xi-1\right)x, & 0\leqslant x\leqslant\xi\\ \left(\dfrac{2}{\pi}x-1\right)\xi, & \xi\leqslant x\leqslant\dfrac{\pi}{2}\end{cases}\tag{4}$$

Further, taking advantage of Green's function (4) as the kernel of an integral equation, we get the following integral equation for $y(x)$:

$$y(x)=f(x)-\lambda\int_0^{\frac{\pi}{2}}G(x,\xi)\,y(\xi)\,d\xi$$

where

$$f(x)=\int_0^{\frac{\pi}{2}}G(x,\xi)\,\xi\,d\xi=$$

$$=\int_0^x\left(\frac{2x}{\pi}-1\right)\xi^2\,d\xi+\int_x^{\frac{\pi}{2}}\left(\frac{2\xi}{\pi}-1\right)x\xi\,d\xi=\frac{1}{6}x^3-\frac{\pi^2}{24}x$$

Thus, the boundary-value problem (1)-(2) has been reduced to the integral equation

$$y(x)+\lambda\int_0^{\frac{\pi}{2}}G(x,\xi)\,y(\xi)\,d\xi=\frac{1}{6}x^3-\frac{\pi^2}{24}x$$

Reduce the following boundary-value problems to integral equations:

281. $y'' = \lambda y + x^2;\quad y(0) = y\left(\frac{\pi}{2}\right) = 0.$

282. $y'' = \lambda y + e^x;\quad y(0) = y(1) = 0.$

283. $y'' + \frac{\pi^2}{4} y = \lambda y + \cos\frac{\pi x}{2};\ y(-1) = y(1),$
$y'(-1) = y'(1).$

284. $y'' + \lambda y = 2x + 1;\ y(0) = y'(1),\ y'(0) = y(1).$

285. $y^{\mathrm{IV}} = \lambda y + 1;\ y(0) = y'(0) = 0,\ y''(1) = y'''(1) = 0.$

286. $y''' + \lambda y = 2x;\ y(0) = y(1) = 0,\ y'(0) = y'(1).$

287. $y'' + \lambda y = e^x;\ y(0) = y'(0),\ y(1) = y'(1).$

23. Singular Integral Equations

We shall call the following integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x, t)\,\varphi(t)\,dt \tag{1}$$

singular if the interval of integration (a, b) is infinite or the kernel $K(x, t)$ is nonintegrable [for example, in the sense of $L_2(\Omega)$].

In the case of singular integral equations, things may occur which do not have any analogy in the case of a finite interval (a, b) and a well-behaved kernel $K(x, t)$ (continuous or lying in $L_2(\Omega)$).

Thus, if the kernel $K(x, t)$ is continuous in $\Omega\ \{a \leqslant x,\ t \leqslant b\}$ and a and b are finite, then the spectrum of the integral equation, that is, the set of characteristic numbers, is discrete and to every characteristic number there corresponds at most a finite number of linearly independent eigenfunctions (the characteristic numbers have a finite multiplicity).

In the case of singular integral equations the spectrum may be continuous, that is, the characteristic numbers may fill whole intervals, and there may be characteristic numbers of infinite multiplicity.

Some examples will illustrate the situation.

We consider the Lalesco-Picard equation

$$\varphi(x)=\lambda\int_{-\infty}^{+\infty} e^{-|x-t|}\varphi(t)\,dt \tag{2}$$

The kernel of this equation, $K(x, t)=e^{-|x-t|}$, possesses an infinite norm. Indeed,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K^2(x, t)\,dx\,dt=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2|x-t|}\,dx\,dt=\int_{-\infty}^{+\infty} dx$$

If the function $\varphi(x)$ is twice differentiable, then the integral equation (2), which can be written in the form

$$\varphi(x)=\lambda\left[e^{-x}\int_{-\infty}^{x} e^{t}\varphi(t)\,dt+e^{x}\int_{x}^{+\infty} e^{-t}\varphi(t)\,dt\right]$$

is equivalent to the differential equation

$$\varphi''(x)+(2\lambda-1)\varphi(x)=0 \tag{3}$$

The general solution of equation (3) is of the form

$$\varphi(x)=C_1e^{rx}+C_2e^{-rx} \tag{4}$$

(C_1, C_2 are arbitrary constants), where

$$r=\sqrt{1-2\lambda} \tag{5}$$

Here, for the integral in the right-hand member of (2) to exist, it is necessary that $|\operatorname{Re} r|<1$, that is, that λ be greater than zero for real λ. Hence, in the domain of real numbers the spectrum of equation (2) fills the infinite interval $0<\lambda<+\infty$. Every point of this interval is a characteristic number of equation (2) of multiplicity 2. However, the associated eigenfunctions do not belong to the class $L_2(-\infty, +\infty)$.

For $\lambda>\frac{1}{2}$, $\sin\sqrt{2\lambda-1}\,x$, $\cos\sqrt{2\lambda-1}\,x$ are, by (4), eigenfunctions; for $\lambda=\frac{1}{2}$ we get $\varphi(x)=C_1+C_2x$. Thus, for $\lambda\geqslant\frac{1}{2}$ there exist eigenfunctions bounded in $(-\infty, +\infty)$. However, if the real part $\sqrt{1-2\lambda}$ is positive and less than unity, then formula (4), for any choice of the constants C_1, C_2 $(C_1^2+C_2^2 \neq 0)$, yields a solution of the integral equation (2) unbounded on $(-\infty, +\infty)$.

This example illustrates the essential role of the class of functions in which the solution of the integral equation is sought.

Thus, if we seek the solution of equation (2) in the class of bounded functions, then, as we shall see, all the values of $\lambda > \frac{1}{2}$ are characteristic.

But if the solution of equation (2) is sought in the class of L_2-functions $(-\infty, +\infty)$, then for any value of λ equation (2) has only the trivial solution $\varphi(x) \equiv 0$, i.e., not one of the values of λ is characteristic for solutions in $L_2(-\infty, +\infty)$.

Let $F(x)$ be a continuous function absolutely integrable on $[0, +\infty]$ and having a finite number of maxima and minima on any finite interval of the x-axis.

Let us construct the Fourier cosine transform of this function:

$$F_1(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F(x) \cos \lambda x \, dx$$

Then

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F_1(\lambda) \cos \lambda x \, d\lambda$$

Adding these two formulas, we get

$$F_1(x) + F(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} [F_1(t) + F(t)] \cos xt \, dt$$

that is, for any choice of the function $F(x)$ satisfying the above-indicated conditions, the function $\varphi(x) = F_1(x) + F(x)$ is an eigenfunction of the integral equation

$$\varphi(x) = \lambda \int_0^{+\infty} \varphi(t) \cos xt \, dt \tag{6}$$

corresponding to the characteristic number $\lambda = \sqrt{\frac{2}{\pi}}$.

Since $F(x)$ is an arbitrary function, it thus follows that for the indicated value of λ equation (6) has an infinite number of linearly independent eigenfunctions.

This peculiarity of equation (6) is due to the fact that (6) is singular [the interval of integration in (6) is infinite].

Example. Consider the integral equation

$$\varphi(x) = \lambda \int_0^\infty \varphi(t) \cos xt \, dt \tag{7}$$

and take

$$F(x) = e^{-ax} \quad (a > 0)$$

Then

$$F_1(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos xt \, dt = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$$

Further,

$$\varphi(x) = F(x) + F_1(x) = e^{-ax} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2} \tag{8}$$

Substituting $\varphi(x)$ into equation (7), we have

$$e^{-ax} + \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2} =$$

$$= \lambda \left[\int_0^\infty e^{-at} \cos xt \, dt + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a \cos xt}{a^2+t^2} dt \right] \tag{9}$$

As has already been pointed out

$$\int_0^\infty e^{-at} \cos xt \, dt = \frac{a}{a^2+x^2}$$

The second integral on the right of (9) may be found by using Cauchy's theorem on residues:

$$\int_0^\infty \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-ax}.$$

From (9) we thus obtain

$$e^{-ax} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2} = \lambda \left[\frac{a}{a^2+x^2} + \sqrt{\frac{\pi}{2}} e^{-ax} \right] \tag{10}$$

It is clear from this that if $\lambda=\sqrt{\frac{2}{\pi}}$, then the function

$$\varphi(x)=e^{-ax}+\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}\not\equiv 0$$

will be a solution of the integral equation (7). Hence, $\lambda=\sqrt{\frac{2}{\pi}}$ is a characteristic number of (7), and the function

$$\varphi(x)=e^{-ax}+\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2} \tag{8}$$

is the corresponding eigenfunction; now, since a is any number greater than 0, the characteristic number $\lambda=\sqrt{\frac{2}{\pi}}$ is associated with an infinity of linearly independent eigenfunctions (8).

In similar fashion, we can show that equation (7) has a characteristic number $\lambda=-\sqrt{\frac{2}{\pi}}$ associated with the eigenfunctions

$$e^{-ax}-\sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}\quad (a>0)$$

288. Show that the integral equation

$$\varphi(x)=\lambda\int_0^\infty \varphi(t)\sin xt\,dt$$

has characteristic numbers $\lambda=\pm\sqrt{\frac{2}{\pi}}$ of infinite multiplicity, and find the associated eigenfunctions.

289. Show that the integral equation with a Hankel kernel

$$\varphi(x)=\lambda\int_0^\infty J_\nu(2\sqrt{xt})\,\varphi(t)\,dt$$

[where $J_\nu(z)$ is a Bessel function of the first kind of order ν] has characteristic numbers $\lambda=\pm 1$ of infinite multiplicity, and find the associated eigenfunctions.

290. Show that for the integral equation

$$\varphi(x)=\lambda\int_x^{\infty}\frac{(t-x)^n}{n!}\varphi(t)\,dt$$

any number λ for which one of the values $\sqrt[n+1]{\lambda}$ has a positive real part is a characteristic number.

291. Show that the Volterra integral equation

$$\varphi(x)=\lambda\int_0^x\left(\frac{1}{t}-\frac{1}{x}\right)\varphi(t)\,dt$$

has an infinity of characteristic numbers $\lambda=\xi+i\eta$, where the point (ξ, η) lies outside the parabola $\xi+\eta^2=0$.

The solution of certain singular integral equations with the aid of Efros' theorem (generalized product theorem). Let

$$\varphi(x)\risingdotseq\Phi(p),$$
$$u(x, \tau)\risingdotseq U(p)\,e^{-\tau q(p)}$$

where $U(p)$ and $q(p)$ are analytic functions. Then

$$\Phi(q, (p))\,U(p)\risingdotseq\int_0^{\infty}\varphi(\tau)\,u(x, \tau)\,d\tau \tag{1}$$

This is the *generalized product theorem (theorem of Efros)*. If $u(x, \tau)=u(x-\tau)$, then $q(p)\equiv p$ and we obtain the ordinary product theorem:

$$\Phi(p)\,U(p)\risingdotseq\int_0^{\infty}\varphi(\tau)\,u(x-\tau)\,d\tau$$

If $U(p)=\frac{1}{\sqrt{p}}$, $q(p)=\sqrt{p}$, then

$$u(x, \tau)=\frac{1}{\sqrt{\pi x}}e^{-\frac{\tau^2}{4x}} \tag{2}$$

Therefore if it is known that $\Phi(p)\risingdotseq\varphi(x)$, then by the Efros theorem we find the original function for $\frac{\Phi(\sqrt{p})}{\sqrt{p}}$:

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} \doteqdot \frac{1}{\sqrt{\pi x}} \int_0^\infty \varphi(\tau) e^{-\frac{\tau^2}{4x}} d\tau \tag{3}$$

Example. Solve the integral equation

$$\frac{1}{\sqrt{\pi x}} \int_0^\infty e^{-\frac{t^2}{4x}} \varphi(t)\, dt = 1 \tag{4}$$

Solution. Let $\varphi(x) \doteqdot \Phi(p)$. Taking the Laplace transform of both sides of (4), we get, by formula (3),

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} = \frac{1}{p}$$

whence

$$\frac{\Phi(p)}{p} = \frac{1}{p^2}, \quad \text{or} \quad \Phi(p) = \frac{1}{p} \doteqdot 1$$

Hence, $\varphi(x) \equiv 1$ is a solution of (4).

Solve the following integral equations:

292 $\frac{1}{\sqrt{\pi x}} \int_0^\infty e^{-\frac{t^2}{4x}} \varphi(t)\, dt = e^{-x}$.

293 $\frac{1}{\sqrt{\pi x}} \int_0^\infty e^{-\frac{t^2}{4x}} \varphi(t)\, dt = 2x - \sinh x$.

294. $\frac{1}{\sqrt{\pi x}} \int_0^\infty e^{-\frac{t^2}{4x}} \varphi(t) dt = x^{\frac{2}{3}} + e^{4x}$.

It is known that

$$t^{\frac{n}{2}} J_n(2\sqrt{t}) \doteqdot \frac{1}{p^{n+1}} e^{-\frac{1}{p}} \quad (n = 0, 1, 2, \ldots)$$

where $J_n(z)$ is a Bessel function of the first kind of order n. In particular,

$$J_0(2\sqrt{t}) \doteqdot \frac{1}{p} e^{-\frac{1}{p}}$$

By virtue of the similarity theorem

$$J_0(2\sqrt{xt}) \doteqdot \frac{1}{p} e^{-\frac{x}{p}}$$

whence it is seen that for the Efros theorem we should then take

$$q(p) \equiv \frac{1}{p}$$

Example. Solve the integral equation

$$\varphi(x) = xe^{-x} + \lambda \int_0^\infty J_0(2\sqrt{xt})\,\varphi(t)\,dt \quad (|\lambda| \neq 1) \tag{5}$$

Solution. Let $\varphi(x) \doteqdot \Phi(p)$. Taking the Laplace transform of both sides of (5) and taking into account the Efros theorem, we find

$$\Phi(p) = \frac{1}{(p+1)^2} + \lambda \frac{1}{p} \Phi\left(\frac{1}{p}\right) \tag{6}$$

Replacing p by $\frac{1}{p}$, we get

$$\Phi\left(\frac{1}{p}\right) = \frac{p^2}{(p+1)^2} + \lambda p\,\Phi(p) \tag{7}$$

From (6) and (7) we find

$$\Phi(p) = \frac{1}{(p+1)^2} + \frac{\lambda}{p}\left[\frac{p^2}{(p+1)^2} + \lambda p \Phi(p)\right]$$

or

$$\Phi(p) = \frac{1}{1-\lambda^2}\left[\frac{1}{(p+1)^2} + \frac{\lambda p}{(p+1)^2}\right]$$

Whence

$$\varphi(x) = e^{-x}\left(\frac{x}{1+\lambda} + \frac{\lambda}{1-\lambda^2}\right)$$

Solve the following integral equations ($\lambda \neq \pm 1$):

295. $\varphi(x) = e^x + \lambda \int_0^\infty \sqrt{\frac{x}{t}}\, J_1(2\sqrt{xt})\,\varphi(t)\,dt.$

296. $\varphi(x) = \cos x + \lambda \int_0^\infty J_0(2\sqrt{xt})\,\varphi(t)\,dt.$

297. $\varphi(x)=\cos x+\lambda\int\limits_0^\infty \frac{x}{t} J_2(2\sqrt{xt})\,\varphi(t)\,dt.$

298. $\varphi(x)=\sin x+\lambda\int\limits_0^\infty \sqrt{\frac{x}{t}}\, J_1(2\sqrt{xt})\,\varphi(t)\,dt.$

Solving certain singular integral equations with the aid of the Mellin transformation.

Let a function $f(t)$ be defined for positive t and let it satisfy the conditions

$$\int\limits_0^1 |f(t)|\, t^{\sigma_1-1}\,dt<+\infty,\qquad \int\limits_1^\infty |f(t)|\, t^{\sigma_2-1}\,dt<+\infty \tag{1}$$

for a proper choice of the numbers σ_1 and σ_2. The function

$$F(s)=\int\limits_0^\infty f(t)\,t^{s-1}\,dt\quad (s=\sigma+i\tau,\ \sigma_1<\sigma<\sigma_2) \tag{2}$$

is the *Mellin transform* of the function $f(t)$. The inversion formula of the Mellin transformation is

$$f(t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} F(s)\,t^{-s}\,ds\quad (t>0,\ \sigma_1<\sigma<\sigma_2) \tag{3}$$

where the integral is taken along the straight line l: Re $s=\sigma$ parallel to the imaginary axis of the s plane and is understood to be the principal value. When the behaviour of the function $f(t)$ as $t\longrightarrow 0$ and $t\longrightarrow\infty$ is known, say from physical reasoning, then the boundaries of the strip $(\sigma_1,\ \sigma_2)$ may be established from the conditions of the absolute convergence of the integral (2). But if the behaviour of $f(t)$ is only known at one end of the interval $(0,\ +\infty)$, say as $t\longrightarrow 0$, then only σ_1 is defined, the straight line of integration l in (3) must be chosen to the right of the straight line $\sigma=\sigma_1$ and to the left of the closest singularity of the function $F(s)$.

The Mellin transformation is closely associated with the transformations of Fourier and Laplace and many theorems which refer to the Mellin transformation can be obtained from the corresponding theorems for the Fourier and Laplace transformations by means of a change of variables.

The convolution theorem for the Mellin transformation is of the form

$$M\left\{\int_0^\infty f(t)\,\varphi\left(\frac{x}{t}\right)\frac{dt}{t}\right\}=F(s)\cdot\Phi(s) \tag{4}$$

From this we can conclude that the Mellin transformation is convenient in the solution of integral equations of the form

$$\varphi(x)=f(x)+\int_0^\infty K\left(\frac{x}{t}\right)\varphi(t)\frac{dt}{t} \tag{5}$$

Indeed, let the functions $\varphi(x)$, $f(x)$ and $K(x)$ admit the Mellin transformation, and let $\varphi(x)\longrightarrow\Phi(s)$, $f(x)\longrightarrow F(s)$, $K(x)\longrightarrow\tilde{K}(s)$; the domains of analyticity $F(s)$ and $\tilde{K}(s)$ have a common strip $\sigma_1<\operatorname{Re}s=\sigma<\sigma_2$. Taking the Mellin transform of both sides of equation (5) and utilizing the convolution theorem (4), we get

$$\Phi(s)=F(s)+\tilde{K}(s)\cdot\Phi(s) \tag{6}$$

whence

$$\Phi(s)=\frac{F(s)}{1-\tilde{K}(s)}\quad(\tilde{K}(s)\neq 1) \tag{7}$$

This is the operator solution of the integral equation (5). Using the inverse formula (3), we find the solution $\varphi(x)$ of this equation:

$$\varphi(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{F(s)}{1-\tilde{K}(s)}x^{-s}\,ds \tag{8}$$

Consider the integral equation

$$\varphi(x)=e^{-\alpha x}+\frac{1}{2}\int_0^\infty e^{-\frac{x}{t}}\varphi(t)\frac{dt}{t}\quad(\alpha>0) \tag{9}$$

Taking the Mellin transform of both sides of (9), we get

$$M\{e^{-\alpha x}\}=\int_0^\infty e^{-\alpha x}x^{s-1}\,dx=\alpha^{-s}\int_0^\infty e^{-z}z^{s-1}\,dz=\frac{\Gamma(s)}{\alpha^s}\equiv F(s),$$

$$M\left\{\frac{1}{2}e^{-x}\right\}=\frac{1}{2}\Gamma(s)\equiv\tilde{K}(s)\quad(\operatorname{Re}s>0)$$

so that the domains of analyticity of $F(s)$ and $\tilde{K}(s)$ coincide. The operator equation corresponding to equation (9) will have the form

$$\Phi(s)=\frac{\Gamma(s)}{\alpha^s}+\frac{1}{2}\Gamma(s)\Phi(s)\tag{10}$$

whence

$$\Phi(s)=\frac{\Gamma(s)}{\alpha^s\left[1-\frac{1}{2}\Gamma(s)\right]}$$

By the inverse formula (8) we obtain

$$\varphi(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s)}{1-\frac{1}{2}\Gamma(s)}\cdot\frac{ds}{(\alpha x)^s}\quad(\sigma>0)\tag{11}$$

We find the integral (11) with the aid of Cauchy's integral formula.

For $\alpha x>1$, we include in the contour of integration the semicircle lying in the right half-plane. In this case, the sole singularity of the integrand lies at the point $s=3$ at which

$$1-\frac{1}{2}\Gamma(s)=0$$

Then

$$\varphi(x)=\frac{2}{(\alpha x)^3\,\psi(3)},\qquad \alpha x>1$$

where $\psi(3)$ is the logarithmic derivative of the Γ-function at the point $s=3$:

$$\psi(3)=\frac{\Gamma'(3)}{\Gamma(3)}=\frac{3}{2}-\gamma$$

(γ is Euler's constant).

For $\alpha x<1$, the singularities of the integrand are the negative roots of the function $1-\frac{1}{2}\Gamma(s)$ so that

$$\varphi(x)=-2\sum_{k=1}^{\infty}\frac{1}{(\alpha x)^{s_k}\psi(s_k)}, \qquad \alpha x<1$$

where $\psi(s_k)$ are values of the logarithmic derivative $\Gamma(s)$ at the points $s=s_k$ $(k=1, 2, \ldots)$. Thus,

$$\varphi(x)=\begin{cases}\dfrac{4}{(3-2\gamma)(\alpha x)^3}, & \alpha x>1,\\ -2\displaystyle\sum_{k=1}^{\infty}\frac{1}{(\alpha x)^{s_k}\psi(s_k)}, & \alpha x<1\end{cases}$$

Let us consider an integral equation of the form

$$\varphi(x)=f(x)+\int_0^{\infty}K(xt)\varphi(t)\,dt \tag{1}$$

(the Fox equation). Multiplying both sides of (1) by x^{s-1} and integrating with respect to x between the limits 0 and ∞, we get

$$\int_0^{\infty}\varphi(x)x^{s-1}\,dx=\int_0^{\infty}f(x)x^{s-1}\,dx+\int_0^{\infty}\varphi(t)\,dt\int_0^{\infty}K(xt)x^{s-1}\,dx$$

Denoting the Mellin transform of the functions $\varphi(x)$, $f(x)$, $K(x)$, by $\Phi(s)$, $F(s)$, $\tilde{K}(s)$, respectively, we get, after simple manipulations,

$$\Phi(s)=F(s)+\tilde{K}(s)\int_0^{\infty}\varphi(t)t^{-s}\,dt \tag{2}$$

It is easy to see that $\int_0^{\infty}\varphi(t)t^{-s}\,dt=\Phi(1-s)$ so that (2) will be written in the form

$$\Phi(s)=F(s)+\Phi(1-s)\tilde{K}(s) \tag{3}$$

Replacing s by $1-s$ in (3), we get

$$\Phi(1-s)=F(1-s)+\Phi(s)\tilde{K}(1-s) \tag{4}$$

From (3) and (4) we find

$$\Phi(s)=F(s)+F(1-s)\tilde{K}(s)+\Phi(s)\tilde{K}(s)\cdot\tilde{K}(1-s)$$

whence

$$\Phi(s)=\frac{F(s)+F(1-s)\tilde{K}(s)}{1-\tilde{K}(s)\cdot\tilde{K}(1-s)} \tag{5}$$

This is the operator solution of equation (1).

Using the inverse Mellin formula, we find

$$\varphi(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)+F(1-s)\tilde{K}(s)}{1-\tilde{K}(s)\tilde{K}(1-s)}x^{-s}\,ds \tag{6}$$

which is a solution of the integral equation (1).

Example. Solve the integral equation

$$\varphi(x)=f(x)+\lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)\cos xt\,dt \tag{7}$$

Solution. We have

$$\tilde{K}(s)=\lambda\sqrt{\frac{2}{\pi}}\int_0^\infty x^{s-1}\cos x\,dx \tag{8}$$

To compute the integral (8) we take advantage of the fact that

$$\int_0^\infty e^{-x}x^{z-1}\,dx=\Gamma(z) \tag{9}$$

If in formula (9) we turn the ray of integration up to the imaginary axis, which by virtue of the Jordan lemma is possible for $0<z<1$, we arrive at the formula

$$\int_0^\infty e^{-ix}x^{z-1}\,dx=e^{-\frac{i\pi z}{2}}\Gamma(z)$$

Separating the real and imaginary parts, we get

$$\int_0^\infty x^{z-1}\cos x\,dx=\cos\frac{\pi z}{2}\cdot\Gamma(z), \tag{10}$$

$$\int_0^\infty x^{z-1}\sin x\,dx=\sin\frac{\pi z}{2}\Gamma(z) \tag{11}$$

Thus, by virtue of (8) and (10),

$$\tilde{K}(s)=\lambda\sqrt{\frac{2}{\pi}}\Gamma(s)\cos\frac{\pi s}{2} \tag{12}$$

Also,

$$\tilde{K}(s)\cdot\tilde{K}(1-s)=\lambda\sqrt{\frac{2}{\pi}}\Gamma(s)\cos\frac{\pi s}{2}\cdot\lambda\sqrt{\frac{2}{\pi}}\Gamma(1-s)\sin\frac{\pi s}{2}=$$
$$=\frac{\lambda^2}{\pi}2\cos\frac{\pi s}{2}\sin\frac{\pi s}{2}\Gamma(s)\Gamma(1-s)=\lambda^2$$

since $\Gamma(s)\cdot\Gamma(1-s)=\frac{\pi}{\sin\pi s}$. Hence, if $M\{f(x)\}=F(s)$, then by formula (5) (for $|\lambda|\neq 1$)

$$\Phi(s)=\frac{F(s)+F(1-s)\tilde{K}(s)}{1-\lambda^2}$$

and therefore

$$\varphi(x)=\frac{1}{2\pi i(1-\lambda^2)}\int_{\sigma-i\infty}^{\sigma+i\infty}\Big[F(s)+$$
$$+F(1-s)\lambda\sqrt{\frac{2}{\pi}}\Gamma(s)\cos\frac{\pi s}{2}\Big]x^{-s}\,ds=$$
$$=\frac{1}{1-\lambda^2}\cdot\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)x^{-s}\,ds+$$
$$+\frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\cos\frac{\pi s}{2}F(1-s)x^{-s}\,ds \tag{13}$$

In the second integral on the right of (13) replace $F(1-s)$ by $\int_0^\infty f(t)t^{-s}\,dt$ and note that $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)x^{-s}\,ds=f(x)$.

Then formula (13) can be rewritten as

$$\varphi(x) = \frac{f(x)}{1-\lambda^2} + \\ + \frac{\lambda}{1-\lambda^2} \sqrt{\frac{2}{\pi}} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) \cos\frac{\pi s}{2} (xt)^{-s}\, ds \int_0^\infty f(t)\, dt \qquad (14)$$

By Mellin's inverse formula,

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s) \cos\frac{\pi s}{2} (xt)^{-s}\, ds = \cos xt$$

so that, finally, we have

$$\varphi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos xt\, dt, \quad (|\lambda| \neq 1)$$

Solve the integral equations:

299. $\varphi(x) = \frac{1}{1+x^2} + \frac{1}{\sqrt{\pi}} \int_0^\infty \varphi(t) \cos xt\, dt.$

300. $\varphi(x) = f(x) + \lambda \sqrt{\frac{2}{\pi}} \int_0^\infty \varphi(t) \sin xt\, dt.$

301. $\varphi(x) = -e^{-x} + \frac{2}{\sqrt{\pi}} \int_0^\infty \varphi(t) \cos xt\, dt.$

CHAPTER III

APPROXIMATE METHODS

24. Approximate Methods of Solving Integral Equations

1. Replacing the kernel by a degenerate kernel. Suppose we have an integral equation

$$\varphi(x)=f(x)+\lambda\int_a^b K(x, t)\varphi(t)\,dt \tag{1}$$

with arbitrary kernel $K(x, t)$. The simplicity of finding a solution to an equation with a degenerate kernel (see Sec. 15) naturally leads one to think of replacing the given arbitrary kernel $K(x, t)$ approximately by a degenerate kernel $L(x, t)$ and taking the solution $\tilde{\varphi}(x)$ of the new equation

$$\tilde{\varphi}(x)=f(x)+\lambda\int_a^b L(x, t)\tilde{\varphi}(t)\,dt \tag{2}$$

as an approximation to the solution of the original equation (1). For the degenerate kernel $L(x, t)$ close to the given kernel $K(x, t)$, we can take a partial sum of Tailor's series for the function $K(x, t)$, a partial sum of the Fourier series for $K(x, t)$ with respect to any complete system of functions $\{u_n(x)\}$ orthonormal in $L_2(a, b)$, etc. We shall indicate some error estimates in the solution (1) that occur when replacing a given kernel by a degenerate kernel.

Let there be given two kernels $L(x, t)$ and $K(x, t)$ and let it be known that

$$\int_a^b |K(x, t)-L(x, t)|\,dt<h$$

and that the resolvent kernel $R_L(x, t; \lambda)$ of the equation with kernel $L(x, t)$ satisfies the inequality

$$\int_a^b |R_L(x, t; \lambda)|\,dt<R$$

and also that $|f(x)-f_1(x)|<\eta$. Then, if the condition

$$1-|\lambda|h(1+|\lambda|R)>0$$

is fulfilled, it follows that the equation

$$\varphi(x)=\lambda\int_a^b K(x,\ t)\,\varphi(t)\,dt+f(x)$$

has a unique solution $\varphi(x)$ and the difference between this solution and the solution $\tilde{\varphi}(x)$ of the equation

$$\tilde{\varphi}(x)=f_1(x)+\lambda\int_a^b L(x,t)\,\tilde{\varphi}(t)\,dt$$

does not exceed

$$|\varphi(x)-\tilde{\varphi}(x)|<\frac{N|\lambda|(1+|\lambda|R)^2 h}{1-|\lambda|h(1+|\lambda|R)}+\eta \tag{3}$$

where N is the upper bound of $|f(x)|$ (see [8]).

For the degenerate kernel $L(x,\ t)$, the resolvent kernel $R_L(x,\ t;\ \lambda)$ is found simply (to within the evaluation of the integrals); namely, if $L(x,\ t)=\sum_{k=1}^{n} X_k(x)\,T_k(t)$, then, putting

$$\int_a^b X_k(x)\,T_s(x)\,dx=a_{sk}$$

we get

$$R_L(x,\ t;\ \lambda)=\frac{D(x,\ t;\ \lambda)}{D(\lambda)} \tag{4}$$

where

$$D(x,\ t;\ \lambda)=\begin{vmatrix} 0 & X_1(t) & \dots & X_n(x) \\ T_1(t) & 1-\lambda a_{11} & \dots & -\lambda a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ T_n(t) & -\lambda a_{n1} & \dots & 1-\lambda a_{nn} \end{vmatrix}, \tag{5}$$

$$D(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{vmatrix} \tag{6}$$

The roots $D(\lambda)$ are the characteristic numbers of the kernel $L(x, t)$.

One more estimate ($\lambda = 1$). Let

$$K(x, t) = L(x, t) + \Lambda(x, t) \tag{7}$$

where $L(x, t)$ is a degenerate kernel and $\Lambda(x, t)$ has a small norm in some metric. Also let $R_K(x, t)$, $R_L(x, t)$ be resolvent kernels of the kernels $K(x, t)$, and $L(x, t)$, respectively, and let $\|\Lambda\|$, $\|R_K\|$, $\|R_L\|$ be the norms of the operators with corresponding kernels. Then

$$\|\varphi - \tilde{\varphi}\| \leqslant \|\Lambda\| \cdot (1 + \|R_K\|) \cdot (1 + \|R_L\|) \cdot \|f\| \tag{8}$$

The norm in formula (8) can be taken in any function space. The following estimate holds true for the norm of the resolvent kernel R of any kernel $K(x, t)$:

$$\|R\| \leqslant \frac{\|K\|}{1 - |\lambda| \cdot \|K\|} \tag{9}$$

And in the space $C(0, 1)$ of continuous functions, on the interval $[0, 1]$,

$$\|K\| = \max_{0 \leqslant x \leqslant 1} \int_0^1 |K(x, t)|\, dt$$

$$\|f\| = \max_{0 \leqslant x \leqslant 1} |f(x)| \tag{10}$$

In the space of quadratically summable functions over $\Omega\{a \leqslant x, t \leqslant b\}$,

$$\|K\| \leqslant \left(\int_a^b \int_a^b K^2(x, t)\, dx\, dt \right)^{\frac{1}{2}},$$

$$\|f\| = \left(\int_a^b f^2(x)\, dx \right)^{\frac{1}{2}} \tag{11}$$

Example. Solve the equation

$$\varphi(x) = \sin x + \int_0^1 (1 - x \cos xt)\, \varphi(t)\, dt \tag{1}$$

by replacing its kernel with a degenerate kernel.

Solution. Expanding the kernel $K(x, t) = 1 - x\cos xt$ in a series, we get

$$K(x, t) = 1 - x + \frac{x^3t^2}{2} - \frac{x^5t^4}{24} + \dots \tag{2}$$

Let us take the first three terms of the expansion (2) for the degenerate kernel $L(x, t)$,

$$L(x, t) = 1 - x + \frac{x^3t^2}{2} \tag{3}$$

and solve the new equation

$$\tilde{\varphi}(x) = \sin x + \int_0^1 \left(1 - x + \frac{x^3t^2}{2}\right) \tilde{\varphi}(t)\, dt \tag{4}$$

From (4) we have

$$\tilde{\varphi}(x) = \sin x + C_1(1 - x) + C_2x^3 \tag{5}$$

where

$$C_1 = \int_0^1 \tilde{\varphi}(t)\, dt, \quad C_2 = \frac{1}{2}\int_0^1 t^2\tilde{\varphi}(t)\, dt \tag{6}$$

Substituting (5) into (6), we get a system for determining C_1 and C_2.

We have

$$C_1 = \int_0^1 [\sin t + C_1(1 - t) + C_2t^3]\, dt = \frac{1}{2}C_1 + \frac{1}{4}C_2 + 1 - \cos 1,$$

$$C_2 = \frac{1}{2}\int_0^1 [t^2 \sin t + C_1(t^2 - t^3) + C_2t^5]\, dt =$$

$$= \frac{1}{24}C_1 + \frac{1}{12}C_2 + \sin 1 - 1 + \frac{1}{2}\cos 1$$

or

$$\left.\begin{aligned} \frac{1}{2}C_1 - \frac{1}{4}C_2 &= 1 - \cos 1, \\ -\frac{1}{24}C_1 + \frac{11}{12}C_2 &= \sin 1 + \frac{1}{2}\cos 1 - 1 \end{aligned}\right\} \tag{7}$$

Solving this system, we get

$$C_1 = 1.0031, \quad C_2 = 0.1674$$

and then

$$\tilde{\varphi}(x) = 1.0031\,(1-x) + 0.1674x^3 + \sin x.$$

The exact solution of the equation $\varphi(x) \equiv 1$.

Now let us estimate $\|\varphi - \tilde{\varphi}\|$ using the formula

$$\|\varphi - \tilde{\varphi}\| \leqslant \|\Lambda\| \cdot (1 + \|R_K\|)(1 + \|R_L\|) \cdot \|f\| \tag{8}$$

In the metric of the L_2-space we get

$$\|\Lambda\| \leqslant \frac{1}{24}\left\{\int_0^1\int_0^1 x^{10}t^8\,dx\,dt\right\}^{\frac{1}{2}} = \frac{1}{72\sqrt{11}} < \frac{1}{238},$$

$$\|K\| \leqslant \left\{\int_0^1\int_0^1 (1 - x\cos xt)^2\,dx\,dt\right\}^{\frac{1}{2}} =$$

$$= \left\{2\cos 1 - \frac{1}{8}\cos 2 + \frac{1}{16}\sin 2 - \frac{5}{6}\right\}^{\frac{1}{2}} < \frac{3}{5},$$

$$\|L\| \leqslant \left\{\int_0^1\int_0^1 \left(1 - x + \frac{x^3t^2}{2}\right)^2 dx\,dt\right\}^{\frac{1}{2}} = \sqrt{\frac{5}{14}} < \frac{3}{5},$$

$$\|f\| = \left\{\int_0^1 \sin^2 x\,dx\right\}^{\frac{1}{2}} = \frac{\sqrt{2 - \sin 2}}{2} < \frac{3}{5}$$

We estimate the norms of the resolvent kernels R_K and R_L using the formulas

$$\|R_K\| \leqslant \frac{\|K\|}{1 - |\lambda| \cdot \|K\|}, \quad \|R_L\| \leqslant \frac{\|L\|}{1 - |\lambda| \cdot \|L\|}$$

where $|\lambda| = 1$. Hence, $\|R_K\| \leqslant \frac{3}{2}$, $\|R_L\| \leqslant \frac{3}{2}$ but then

$$\|\varphi - \tilde{\varphi}\| < \frac{1}{238}\left(1 + \frac{3}{2}\right)\left(1 + \frac{3}{2}\right) \cdot \frac{3}{5} < 0.016$$

Find the solution of the integral equation in the following cases by means of substituting a degenerate kernel for the kernel, and estimate the error:

302. $\varphi(x)=e^{x}-x-\int_{0}^{1} x(e^{xt}-1)\varphi(t)\,dt.$

303. $\varphi(x)=x+\cos x+\int_{0}^{1} x(\sin xt-1)\varphi(t)\,dt.$

304. $\varphi(x)=\frac{1}{2}(e^{-x}+3x-1)+\int_{0}^{1}(e^{-xt^2}-1)x\varphi(t)\,dt.$

305. $\varphi(x)=\frac{x}{2}+\frac{1}{2}\sin x+\int_{0}^{1}(1-\cos xt^2)x\varphi(t)\,dt.$

2. The method of successive approximations. The method of successive approximations (iteration method) consists in the following.

We have an integral equation

$$\varphi(x)=f(x)+\lambda\int_{a}^{b} K(x,\ t)\varphi(t)\,dt \tag{1}$$

We construct a sequence of functions $\{\varphi_n(x)\}$ with the aid of the recursion formula

$$\varphi_n(x)=f(x)+\lambda\int_{a}^{b} K(x,\ t)\varphi_{n-1}(t)\,dt \tag{2}$$

The functions $\varphi_n(x)$ $(n=1,\ 2,\ \ldots)$ are considered as approximations to the desired solution of the equation; the zero approximation $\varphi_0(x)$ may be chosen arbitrarily.

Under certain conditions

$$|\lambda|<\frac{1}{B},\quad B=\sqrt{\int_{a}^{b}\int_{a}^{b} K^2(x,\ t)\,dx\,dt} \tag{3}$$

the sequence (2) converges to the solution of equation (1). The magnitude of the error of the $(m+1)$th approximation is given by the inequality

$$|\varphi(x)-\varphi_{m+1}(x)|\leqslant FC_1B^{-1}\frac{|\lambda B|^{m+1}}{1-|\lambda B|}+\Phi C_1B^{-1}|\lambda B|^{m+1} \tag{4}$$

where

$$F=\sqrt{\int_a^b f^2(x)\,dx},\qquad \Phi=\sqrt{\int_a^b \varphi_0^2(x)\,dx},$$

$$C_1=\sqrt{\max_{a\leqslant x\leqslant b}\int_a^b K^2(x,\ t)\,dt}$$

Solve the following equations using the method of successive approximations:

306. $\varphi(x)=1+\int_0^1 xt^2\varphi(t)\,dt.$

307. $\varphi(x)=\frac{5}{6}x+\frac{1}{2}\int_0^1 xt\varphi(t)\,dt.$

308. Find the third approximation $\varphi_3(x)$ to the solution of the integral equation

$$\varphi(x)=1+\int_0^1 K(x,\ t)\,\varphi(t)\,dt$$

where

$$K(x,\ t)=\begin{cases} t, & x\geqslant t,\\ x, & x\leqslant t\end{cases}$$

and estimate the error.

Observe that the basic difficulty in applying the method of successive approximations consists in computing the integrals in formulas (2). As a rule, it is performed with the aid of the formulas of approximate integration. Therefore, here too it is advisable to replace the given kernel by a degenerate kernel with the aid of a Taylor expansion and only then to introduce the iteration method.

3. The Bubnov-Galerkin method. An approximate solution of the integral equation

$$\varphi(x)=f(x)+\lambda\int_a^b K(x,\ t)\,\varphi(t)\,dt \tag{1}$$

by means of the Bubnov-Galerkin method is sought in the following manner. We choose a system of functions $\{u_n(x)\}$ complete in $L_2(a, b)$ and such that for any n the functions $u_1(x), u_2(x), \ldots, u_n(x)$ are linearly independent and we seek the approximate solution $\varphi_n(x)$ in the form

$$\varphi_n(x) = \sum_{k=1}^{n} a_k u_k(x) \tag{2}$$

The coefficients a_k $(k = 1, 2, \ldots, n)$ are found from the following linear system:

$$(\varphi_n(x),\ u_k(x)) = (f(x)\, u_k(x)) + \lambda \left(\int_a^b K(x,\ t)\, \varphi_n(t)\, dt,\ \ u_k(x) \right)$$

$$(k = 1, 2, \ldots, n) \tag{3}$$

where (f, g) stands for $\int_a^b f(x)\, g(x)\, dx$ and in place of $\varphi_n(x)$ we have to substitute $\sum_{k=1}^{n} a_k u_k(x)$. If the value of λ in (1) is not characteristic, then the system (3) is uniquely solvable for sufficiently large n and as $n \to \infty$ the approximate solution $\varphi_n(x)$ (2) tends, in the metric $L_2(a, b)$, to an exact solution $\varphi(x)$ of equation (1).

Example. Use the Bubnov-Galerkin method to solve the equation

$$\varphi(x) = x + \int_{-1}^{1} xt\varphi(t)\, dt \tag{4}$$

Solution. For the complete system of functions on $[-1, 1]$ we choose the system of Legendre polynomials $P_n(x)$ $(n = 0, 1, 2, \ldots)$. We shall seek the approximate solution $\varphi_n(x)$ of equation (4) in the form

$$\varphi_3(x) = a_1 \cdot 1 + a_2 x + a_3 \frac{3x^2 - 1}{2}$$

Substituting $\varphi_3(x)$ in place of $\varphi(x)$ into equation (4), we get

$$a_1 + a_2 x + a_3 \frac{3x^2 - 1}{2} = x + \int_{-1}^{1} xt \left(a_1 + a_2 t + a_3 \frac{3t^2 - 1}{2} \right) dt$$

or

$$a_1+a_2x+a_3\frac{3x^2-1}{2}=x+x\frac{2}{3}a_2 \tag{5}$$

Multiplying both sides of (5) successively by 1, x, $\frac{3x^2-1}{2}$ and integrating with respect to x between the limits -1 and 1, we obtain

$$2a_1=0,$$
$$\frac{2}{3}a_2=\frac{2}{3}+\frac{4}{9}a_2,$$
$$\frac{2}{5}a_3=0$$

Whence $a_1=0$, $a_2=3$, $a_3=0$, and so $\varphi_3(x)=3x$. It is easy to verify that this is the exact solution of equation (4).

Use the Bubnov-Galerkin method to solve the following integral equations:

309. $\varphi(x)=1+\int\limits_{-1}^{1}(xt+x^2)\varphi(t)\,dt.$

310. $\varphi(x)=1+\frac{4}{3}x+\int\limits_{-1}^{1}(xt^2-x)\varphi(t)\,dt.$

311. $\varphi(x)=1-x(e^x-e^{-x})+\int\limits_{-1}^{1}x^2e^{xt}\varphi(t)\,dt.$

Note. The Bubnov-Galerkin method yields an exact solution for degenerate kernels; for the general case it is equivalent to replacing the kernel $K(x, t)$ by the degenerate kernel $L(x, t)$.

25. Approximate Methods for Finding Characteristic Numbers

1. Ritz method. Suppose we have an integral equation

$$\varphi(x)=\lambda\int\limits_{a}^{b}K(x, t)\varphi(t)\,dt$$

with symmetric kernel $K(x, t)\equiv K(t, x)$.

Choose a sequence of functions $\{\psi_n(x)\}$, $\psi_n(x)\in L_2(a, b)$ such that the system $\{\psi_n(x)\}$ is complete in $L_2(a, b)$ and for any n the functions $\psi_1(x), \psi_2(x), \ldots, \psi_n(x)$ are linearly independent on $[a, b]$. We assume

$$\varphi_n(x)=\sum_{k=1}^{n} a_k\psi_k(x) \tag{1}$$

and we subject the coefficients a_k to the condition $\|\varphi_n\|=1$; under this condition we seek the stationary values of the quadratic form

$$(K\varphi_n, \varphi_n)$$

We arrive at a homogeneous linear system in the coefficients a_k (σ is a Lagrange multiplier):

$$\sum_{k=1}^{n}\{(K\psi_j, \psi_k)-\sigma(\psi_j, \psi_k)\}a_k=0 \tag{2}$$
$$(j=1, 2, \ldots, n)$$

For a nonzero solution (2) to exist the determinant of the system (2) must be equal to zero:

$$\left|\begin{array}{l} (K\psi_1, \psi_1)-\sigma(\psi_1, \psi_1)\,(K\psi_1, \psi_2)-\sigma(\psi_1, \psi_2)\ldots \\ \ldots(K\psi_1, \psi_n)-\sigma(\psi_1, \psi_n) \\ (K\psi_2, \psi_1)-\sigma(\psi_2, \psi_1)\,(K\psi_2, \psi_2)-\sigma(\psi_2, \psi_2)\ldots \\ \ldots(K\psi_2, \psi_n)-\sigma(\psi_2, \psi_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (K\psi_n, \psi_1)-\sigma(\psi_n, \psi_1)\,(K\psi_n, \psi_2)-\sigma(\psi_n, \psi_2)\ldots \\ \ldots(K\psi_n, \psi_n)-\sigma(\psi_n, \psi_n) \end{array}\right|=0 \tag{3}$$

The roots of equation (3) yield approximate values of the eigenvalues of the kernel $K(x, t)$. The largest of the roots of equation (1) yields an underestimate of the largest eigenvalue. Finding σ from (3) and substituting into (2), we seek the nonzero solution a_k $(k=1, 2, \ldots, n)$ of the system (2). Substituting the values of a_k thus found into (1) we get an approximate expression for the eigenfunction corresponding to the eigenvalue that was found.

Example. Using the Ritz method, find the approximate value of the smallest characteristic number of the kernel

$$K(x, t)=xt; \quad a=0, \quad b=1.$$

Solution. For the coordinate system of the functions $\psi_n(x)$, choose the system of Legendre polynomials $\psi_n(x) = P_n(2x-1)$. We confine ourselves to two terms in formula (1) so that

$$\varphi_2(x) = a_1 P_0(2x-1) + a_2 P_1(2x-1).$$

Noting that

$$\psi_1 \equiv P_0(2x-1) = 1, \quad \psi_2 \equiv P_1(2x-1) = 2x-1$$

we find

$$(\psi_1, \psi_1) = \int_0^1 dx = 1; \ (\psi_1, \psi_2) = (\psi_2, \psi_1) = \int_0^1 (2x-1)\,dx = 0;$$

$$(\psi_2, \psi_2) = \int_0^1 (2x-1)^2\,dx = \frac{1}{3}$$

Further

$$(K\psi_1, \psi_1) = \int_0^1 \left(\int_0^1 K(x, t)\,\psi_1(t)\,dt \right) \psi_1(x)\,dx = \int_0^1 \int_0^1 xt\,dx\,dt = \frac{1}{4}$$

$$(K\psi_1, \psi_2) = \int_0^1 \int_0^1 xt\,(2x-1)\,dx\,dt = \frac{1}{12}$$

$$(K\psi_2, \psi_2) = \int_0^1 \int_0^1 xt\,(2t-1)(2x-1)\,dx\,dt = \frac{1}{36}$$

The system (3) then becomes

$$\begin{vmatrix} \frac{1}{4} - \sigma & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{36} - \frac{1}{3}\sigma \end{vmatrix} = 0$$

or

$$\sigma^2 - \sigma\left(\frac{1}{12} + \frac{1}{4}\right) = 0$$

Whence $\sigma_1 = 0$, $\sigma_2 = \frac{1}{3}$. The largest eigenvalue $\sigma_2 = \frac{1}{3}$, hence, the smallest characteristic number $\lambda = \frac{1}{\sigma_2} = 3$.

Use the Ritz method to find the smallest characteristic numbers of the kernels ($a=0$, $b=1$):

312. $K(x, t)=x^2t^2$.

313. $K(x, t)=\begin{cases} t, & x\geqslant t, \\ x, & x\leqslant t. \end{cases}$

314. $K(x, t)=\begin{cases} \frac{1}{2}x(2-t), & x\leqslant t, \\ \frac{1}{2}t(2-x), & x\geqslant t. \end{cases}$

2. The method of traces. Let us use the term mth trace of the kernel $K(x, t)$ for the number

$$A_m=\int_a^b K_m(t, t)\,dt$$

where $K_m(x, t)$ stands for the mth iterated kernel.

The following approximate formula holds true for the smallest characteristic number λ_1, given sufficiently large m:

$$|\lambda_1|\approx\sqrt{\frac{A_{2m}}{A_{2m+2}}} \tag{1}$$

Formula (1) yields the value $|\lambda_1|$ in excess.

Traces of even order for the symmetric kernel are computed from the formula

$$A_{2m}=\int_a^b\int_a^b K_m^2(x, t)\,dx\,dt=2\int_a^b\int_a^x K_m^2(x, t)\,dt\,dx \tag{2}$$

Example. Using the method of traces, find the first characteristic number of the kernel

$$K(x, t)=\begin{cases} t, & x\geqslant t, \\ x, & x\leqslant t \end{cases}\qquad a=0,\quad b=1$$

Solution. Since the kernel $K(x, t)$ is symmetric, it is sufficient to find $K_2(x, t)$ only for $t<x$.

We have

$$K_2(x, t) = \int_0^1 K(x, z) K(z, t)\, dz = \int_0^t z^2\, dz +$$

$$+ \int_t^x zt\, dz + \int_x^1 xt\, dz = xt - \frac{x^2 t}{2} - \frac{t^3}{6}$$

Then, from formula (2), we find for $m = 1$ and $m = 2$, respectively,

$$A_2 = 2 \int_0^1 dx \int_0^x K_1^2(x, t)\, dt = 2 \int_0^1 dx \int_0^x t^2\, dt = 2 \int_0^1 \frac{x^3}{3}\, dx = \frac{1}{6},$$

$$A_4 = 2 \int_0^1 dx \int_0^x K_2^2(x, t)\, dt =$$

$$= 2 \int_0^1 dx \int_0^x \left(x^2 t^2 + \frac{x^4 t^2}{4} + \frac{t^6}{36} + \frac{x^2 t^4}{6} - x^3 t^2 - \frac{x t^4}{3} \right) dt =$$

$$= 2 \int_0^1 \left(\frac{t^3 x^2}{3} + \frac{t^3 x^4}{12} + \frac{t^7}{7 \cdot 36} - \frac{x^3 t^3}{3} - \frac{x t^5}{15} + \frac{x^2 t^5}{30} \right) \Bigg|_{t=0}^{t=x} dx =$$

$$= 2 \int_0^1 \left(\frac{x^5}{3} + \frac{x^7}{12} + \frac{x^7}{7 \cdot 36} - \frac{x^6}{3} - \frac{x^6}{15} + \frac{x^7}{30} \right) dx = \frac{17}{630}$$

Then by formula (1)

$$\lambda_1 \approx \sqrt{\frac{\frac{1}{6}}{\frac{17}{630}}} = 2.48$$

Use the method of traces to find the first characteristic number of the following kernels ($a = 0$, $b = 1$):

315. $K(x, t) = xt$.

316. $K(x, t) = x^2 t^2$.

317. $K(x, t) = \begin{cases} \frac{1}{2} x(2 - t), & x \leqslant t, \\ \frac{1}{2} t(2 - x), & x \geqslant t. \end{cases}$

318. $K(x, t)=\begin{cases}-\sqrt{xt}\ln t, & x\leqslant t,\\ -\sqrt{xt}\ln x, & x\geqslant t.\end{cases}$

3. Kellogg's method. Let $K(x, t)$ be a symmetric kernel which for definiteness we will consider to be positive definite, and let $\omega(x)$ be an arbitrary function in $L_2(a, b)$. We construct a sequence of functions

$$\left.\begin{array}{l} \omega_1(x)=\int\limits_a^b K(x, t)\,\omega(t)\,dt,\\ \omega_2(x)=\int\limits_a^b K(x, t)\,\omega_1(t)\,dt,\\ \dots\dots\dots\dots\dots\dots\\ \omega_n(x)=\int\limits_a^b K(x, t)\,\omega_{n-1}(t)\,dt\\ \dots\dots\dots\dots\dots\dots \end{array}\right\} \qquad (1)$$

and consider the numerical sequence

$$\left\{\frac{\|\omega_{n-1}\|}{\|\omega_n\|}\right\} \qquad (2)$$

Let $\varphi_1(x)$, $\varphi_2(x)$, ... be orthonormal eigenfunctions of the kernel $K(x, t)$ and let $\lambda_1\leqslant\lambda_2\leqslant\dots$ be the associated characteristic numbers. Further, let the function $\omega(x)$ be orthogonal to the functions $\varphi_1(x)$, $\varphi_2(x)$, ..., $\varphi_{k-1}(x)$ but not orthogonal to the eigenfunction $\varphi_k(x)$. Then the sequence (2) has as its limit the kth characteristic number λ_k.

The sequence of functions $\left\{\frac{\omega_n(x)}{\|\omega_n(x)\|}\right\}$ *converges* in this case to some function, which is a linear combination of eigenfunctions that is associated with the characteristic number λ_k. The sequence

$$\left\{\frac{1}{\sqrt[n]{\|\omega_n\|}}\right\} \qquad (3)$$

converges to the same limit as does the sequence (2). If $(\omega, \varphi_1)\neq 0$, we get two approximate formulas for the smallest characteristic number:

$$\lambda_1\approx\frac{\|\omega_{n-1}\|}{\|\omega_n\|}, \qquad (4)$$

$$\lambda_1 \approx \frac{1}{\sqrt[n]{\|\omega_n\|}} \tag{5}$$

Formula (4) yields the value of λ_1 in excess. If the kernel $K(x, t)$ is not positive definite, then formulas (4) and (5) yield an approximate value of the smallest absolute magnitude of the characteristic number of the given kernel. For an apt choice of $\omega(x)$ the Kellogg method is comparatively simple for computation.

The drawback of the method is that we do not know beforehand which of the characteristic numbers has been determined.

Example. Use the Kellogg method to compute the smallest characteristic number of the kernel $K(x, t) = x^2 t^2$, $0 \leqslant x$, $t \leqslant 1$.

Solution. We take $\omega(x) = x$. Then

$$\omega_1(x) = \int_0^1 x^2 t^2 \, t \, dt = \frac{x^2}{4},$$

$$\omega_2(x) = \int_0^1 x^2 \frac{t^4}{4} \, dt = \frac{1}{4} x^2 \cdot \frac{1}{5},$$

$$\omega_3(x) = \int_0^1 \frac{1}{4 \cdot 5} x^2 t^4 \, dt = \frac{1}{4 \cdot 5^2} x^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\omega_n(x) = \frac{1}{4 \cdot 5^{n-1}} x^2$$

Further,

$$\| \omega_n(x) \| \frac{1}{4} \cdot \frac{1}{5^{n-1}} \sqrt{\int_0^1 x^4 \, dx} = \frac{1}{4 \cdot 5^{n-1} \sqrt{5}}$$

Thus, by (4),

$$\lambda_1 \approx \frac{\frac{1}{4} \cdot \frac{1}{5^{n-2}} \cdot \frac{1}{\sqrt{5}}}{\frac{1}{4} \cdot \frac{1}{5^{n-1}} \cdot \frac{1}{\sqrt{5}}} = 5$$

Using the Kellogg method, find the smallest characteristic numbers of the following kernels:

319. $K(x, t) = xt; \quad 0 \leqslant x, \ t \leqslant 1.$

320. $K(x, t) = \sin x \sin t; \quad -\pi \leqslant x, \ t \leqslant \pi.$

321. $K(x, t) = \begin{cases} t, & x \geqslant t, \\ x, & x \leqslant t; \end{cases} \quad 0 \leqslant x, \ t \leqslant 1.$

322. $K(x, t) = \begin{cases} \frac{1}{2} x(2-t), & x \leqslant t, \\ \frac{1}{2} t(2-x), & x \geqslant t; \end{cases} \quad 0 \leqslant x, \ t \leqslant 1.$

9. $\varphi(x)=-x+\int\limits_0^x (t-x)\,\varphi(t)\,dt$

10. $\varphi(x)=1+\int\limits_0^x \varphi(t)\,dt$

11. $\varphi(x)=\cos x-\int\limits_0^x (x-t)\,\varphi(t)\,dt$

12. $\varphi(x)=5-6x+\int\limits_0^x [5-6(x-t)]\,\varphi(t)\,dt$

13. $\varphi(x)=\cos x-x-\int\limits_0^x (x-t)\,\varphi(t)\,dt$

14. $\varphi(x)=x-\sin x+e^x(x-1)+\int\limits_0^x [\sin x-e^x(x-t)]\varphi(t)\,dt$

15. $\varphi(x)=\cos x-2x(1+x^2)-\int\limits_0^x (1+x^2)(x-t)\,\varphi(t)\,dt$

16. $\varphi(x)=xe^x+1-x(x^2-1)-$
$$-\int\limits_0^x \left[x+\frac{1}{2}(x^2-x)(x-t)^2\right]\varphi(t)\,dt$$

17. $\varphi(x)=x(x+1)^2+\int\limits_0^x x(x-t)^2\,\varphi(t)\,dt$

19. $\frac{1}{\sqrt{\lambda}}\sinh\sqrt{\lambda}(x-t)\ (\lambda>0)$ **20.** $e^{(1+\lambda)(x-t)}$

21. $e^{\lambda(x-t)}e^{x^2-t^2}$ **22.** $\frac{1+x^2}{1+t^2}e^{\lambda(x-t)}$

23. $\frac{2+\cos x}{2+\cos t}e^{\lambda(x-t)}$ **24.** $\frac{\cosh x}{\cosh t}e^{\lambda(x-t)}$

25. $a^{x-t}e^{\lambda(x-t)}$

26. $e^{x-t}(x-t+2)$ 27. $\frac{1}{4}e^{x-t}-\frac{9}{4}e^{-3(x-t)}$

28. $2xe^{x^2-t^2}$ 29. $\frac{4t^2+1}{2(2t+1)^2}\left[\frac{8}{4t^2+1}-4e^{-2(x-t)}\right]$;

one of the solutions of the corresponding differential equation $y_1(x)=e^{-2x}$.

31. $\frac{1}{\sqrt{2}}\sinh\sqrt{2}(x-t)$ 32. 1 33. $(x-t)e^{-(x-t)}$

34. $e^{\frac{x-t}{2}}\left[\cosh\frac{\sqrt{5}}{2}(x-t)+\frac{1}{\sqrt{5}}\sinh\frac{\sqrt{5}}{2}(x-t)\right]$

35. $2e^{x-t}(1+x-t)$ 36. $\varphi(x)=e^{2x}$

37. $\varphi(x)=\frac{1}{5}e^{3x}-\frac{1}{5}\cos x+\frac{2}{5}\sin x$

38. $\varphi(x)=3^x(1-e^{-x})$

39. $\varphi(x)=e^x\sin x+(2+\cos x)e^x\ln\frac{3}{2+\cos x}$

40. $\varphi(x)=e^{x^2-x}-2x$ 41. $\varphi(x)=e^{x^2+2x}(1+2x)$

42. $\varphi(x)=e^x(1+x^2)$

43. $\varphi(x)=\frac{1}{1+x^2}+x\arctan x-\frac{1}{2}\ln(1+x^2)$

44. $\varphi(x)=e^{\frac{x^2}{2}}(x+1)-1$ 45. $\varphi(x)=e^{-x}\left(\frac{x^2}{2}+1\right)$

46. $\varphi(x)=\sin x$ 47. $\varphi(x)=\cos x$

48. $\varphi(x)=\cosh x$ 49. $\varphi(x)=1$ 50. $\varphi(x)=x$

51. $\varphi(x)=e^x$ 52. $\varphi(x)=2$ 53. $\varphi(x)=2$

54. $\varphi(x)=x^2-2x$ 56. $\varphi(x)\equiv 0$

57. $\varphi_2(x)=1+x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{13}{24}x^4+\frac{1}{4}x^5+\frac{1}{18}x^6+\frac{1}{63}x^7$

58. $\varphi_3(x)=-x+\frac{x^4}{4}-\frac{x^7}{14}+\frac{x^{10}}{160}$ 59. $\varphi(x)=1$

60. $\varphi(x)=x-\frac{x^2}{2}$ 61. $\varphi(x)=\frac{1}{2}(3e^{2x}-1)$

62. $\varphi(x)=\sin x$ 63. $\varphi(x)=\frac{1}{3}(2\cos\sqrt{3}\,x+1)$

64. $\varphi(x)=2x+1$ 65. $\varphi(x)=x+\frac{x^3}{6}$

66. $\varphi(x)=\frac{1}{2}\sin x+\frac{1}{2}\sinh x$ 67. $\varphi(x)=x-\frac{x^3}{6}$

68. $\varphi(x)=e^x$ **69.** $\varphi(x)=\frac{2}{\sqrt{5}}\sinh\frac{\sqrt{5}}{2}x\cdot e^{-\frac{x}{2}}$

70. $\varphi(x)=1+2xe^x$ **71.** $\varphi(x)=e^x(1+x)^2$

72. $\varphi(x)=\frac{e^x+\cos x+\sin x}{2}$

73. $\varphi_1(x)=\sin x,\ \varphi_2(x)=0$

74. $\varphi_1(x)=3e^x-2,\ \varphi_2(x)=3e^x-2e^{2x}$

75. $\varphi_1(x)=e^{2x},\ \varphi_2(x)=\frac{1-e^{2x}}{2}$

76. $\begin{cases}\varphi_1(x)=(x+2)\sin x+(2x+1)\cos x,\\ \varphi_2(x)=\frac{2+x}{2}\cos x-\frac{2x+1}{2}\sin x\end{cases}$

77. $\varphi_1(x)=2\sin x,\ \varphi_2(x)=2\cos x-1,\ \varphi_3(x)=x$

78. $\varphi_1(x)=\cos x,\ \varphi_2(x)=\sin x,\ \varphi_3(x)=\sin x+\cos x$

79. $\begin{cases}\varphi_1(x)=\left(1+\frac{x}{2}\right)\cos x+\frac{1}{2}\sin x,\\ \varphi_2(x)=1-x+\frac{1}{2}\sin x-\left(1+\frac{x}{2}\right)\cos x,\\ \varphi_3(x)=\cos x-1-\left(1+\frac{x}{2}\right)\sin x\end{cases}$

80. $\varphi(x)=e^x-1$

81. $\varphi(x)=-e^x$

82. $\varphi(x)=\frac{1}{2}x\sin x$

83. $\varphi(x)=1-e^{-x}-xe^{-x}$

84. $\varphi(x)=1-\cos x$

85. $\varphi(x)=1-x+2(\sin x-\cos x)$

86. $\varphi(x)=(1-x)e^{-x}$

87. $\varphi(x)=c+2e^{-x}$

88. $\varphi(x)=\cos x-\sin x$

89. $\varphi(x)=\frac{\alpha}{\alpha-1}-\frac{e^{(\alpha-1)x}}{\alpha-1}$

90. $\varphi(x)=\cos x-\sin x$

91. $\varphi(x)=1-x\ln 3$

92. $\varphi(x)=f'(x)-f(x)\ln a$

93. $\varphi(x)=xe^{x^2}$

94. $\varphi(x)=xe^{-\frac{x^2}{2}}$

95. $\varphi(x)=e^{\frac{x^2}{2}}(x^2+2)-1$

104. $I=\frac{1}{2}\frac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)}$

106. $\varphi(x)=\frac{\Gamma(n+1)\cdot x^{n+\alpha-1}}{\Gamma(1-\alpha)\Gamma(n+\alpha)}$

107. $\varphi(x)=\frac{1}{\pi}\int\limits_0^x \frac{\cos t}{\sqrt{x-t}}\,dt$

108. $\varphi(x)=\frac{1}{\pi}\left(\frac{1}{\sqrt{x}}+e^x\int\limits_0^x e^{-t}t^{-\frac{1}{2}}\,dt\right)$

109. $\varphi(x)=\frac{1}{2}$

110. $\varphi(x, y)=\frac{1}{2\pi^2}\left(\frac{\partial^2 g}{\partial y^2}-\frac{\partial^2 g}{\partial x^2}\right)$, where

$$g(x, y)=\iint\limits_{D_1}\frac{f(u, v)\,du\,dv}{\sqrt{(y-v)^2-(x-u)^2}}$$

and D_1 is a right isosceles triangle with vertex at the point (x, y) and hypotenuse on the Ou-axis of the UOV-plane.

111. $\varphi(x)=\frac{4}{3}-\frac{2x^{2/3}}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$ 112. $\varphi(x)=\frac{2}{\sqrt{x}}$

113. $\varphi(x)=\frac{1}{\Gamma\left(\frac{5}{4}\right)}\left[\frac{1}{\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}}+\frac{2x^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)}\right]$

114. $\varphi(x)=3$

115. $\varphi(x)=\sin x$ 116. $\varphi(x)=1$ 117. $\varphi(x)=e^{-x}$

118. $\varphi(x)=\frac{15}{4}x$ 119. $\varphi(x)=\cos x-2\sin x$

120. $\varphi(x)=2x-x^2$ 121. $\varphi(x)=2\sin x$

122. $\varphi(x)=3!\,(xe^{-x}-x^2e^{-x})$ 123. $\varphi(x)=J_0(x)$

124. $\varphi(x)=1-\frac{x^2}{2}$ 125. $\varphi(x)=1+2x+\frac{x^2}{2}+\frac{x^3}{3}$

126. We have $x^2-t^2=x^2-2xt+t^2+2xt-2t^2=$
$=(x-t)^2+2t(x-t)$. Therefore

$$\frac{x^3}{3}=\int\limits_0^x (x-t)^2\varphi(t)\,dt+2\int\limits_0^x t(x-t)\varphi(t)\,dt$$

Taking transforms and applying the product theorem and the theorem of differentiation of a transform, by virtue of which $t\varphi(t) \doteqdot -\Phi'(p)$, we get

$$\frac{2}{p^4} = \frac{2}{p^3}\Phi(p) - \frac{2}{p^2}\Phi'(p)$$

or

$$\Phi'(p) = \frac{1}{p}\Phi(p) - \frac{1}{p^2}$$

Solving this differential equation, we find

$$\Phi(p) = C \cdot p + \frac{1}{2p}$$

Since $\Phi(p)$ is an image function, $\Phi(p)$ should approach zero as $p \to \infty$, so that $C = 0$ and, hence, $\Phi(p) = \frac{1}{2p}$, whence $\varphi(x) = \frac{1}{2}$.

127. $\varphi(x) = C - x$

128. $\varphi(x) = C + J_0(2\sqrt{x})$

129. $\varphi(x) = C + x$

130. $\varphi(x) = 2 + \delta(x) - \delta'(x)$

131. $\varphi(x) = \delta(x) - \sin x$

132. $\varphi(x) = \delta(x) + 3$

133. $\varphi(x) = 1 + x + \delta(x) + \delta'(x)$

134. $\varphi(x) = 1$

135. $\varphi(x) = J_1(x)$

136. $\varphi(x) = -I_1(x)$, $I_1(x)$ is a modified Bessel function of the first kind. In Problems 141 and 142 the indicated functions are not solutions of the corresponding integral equations, but in Problems 137 to 140 and 143 to 145, the indicated functions are solutions.

146. $R(x, t; \lambda) = \dfrac{2x - t + \left(x + t - 2xt - \frac{2}{3}\right)\lambda}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}$

147. $R(x, t; \lambda) = \dfrac{x^2t - xt^2 + xt\left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)\lambda}{1 + \frac{\lambda^2}{240}}$

148. $R(x, t; \lambda) = \sin x \cos t$

149. $R(x, t; \lambda) = \dfrac{\sin x - \sin t - \pi(1 + 2\sin x \sin t)\lambda}{1 + 2\pi^2\lambda^2}$

150. $R(x, t; \lambda) = \dfrac{x+t+1+2\left(xt+\frac{1}{3}\right)\lambda}{1-2\lambda-\frac{4}{3}\lambda^2}$

151. $R(x, t; \lambda) = \dfrac{1+3xt+\left(3\frac{x+t}{2}-3xt-1\right)\lambda}{1-2\lambda+\frac{1}{4}\lambda^2}$

152. $R(x, t; \lambda) = \dfrac{4xt-x^2-\left(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt\right)\lambda}{1-\lambda+\frac{\lambda^2}{18}}$

153. $R(x, t; \lambda) = \dfrac{e^{x-t}}{1-\lambda}$

154. $R(x, t; \lambda) = \dfrac{\sin(x+t)+\pi\lambda\cos(x-t)}{1-\pi^2\lambda^2}$

155. $R(x, t; \lambda) = \dfrac{x-\sinh t-2(e^{-1}+x\sinh t)\lambda}{1+4e^{-1}\lambda^2}$

156. $\varphi(x) = 1$

157. $\varphi(x) = \frac{1}{6}\left[x+\dfrac{(6x-2)\lambda-\lambda^2 x}{\lambda^2-3\lambda+6}\right]$

158. $\varphi(x) = \cos 2x$

159. $\varphi(x) = \frac{1}{2}e^x$

160. $\varphi(x) = \dfrac{3x(2\lambda-3\lambda x+6)}{\lambda^2-18\lambda+18}$

161. $K_{2n-1}(x, t) = \left(-\frac{4}{3}\right)^{n-1}(x-t)$,

$K_{2n}(x, t) = 2(-1)^n\left(\frac{4}{3}\right)^{n-1}\left(xt+\frac{1}{3}\right) \; (n = 1, 2, 3, \ldots)$

162. $K_2(x, t) = \dfrac{\sin(x+t)}{2}-\frac{\pi}{4}\cos(x-t)$, $K_3(x, t) =$

$= \dfrac{4-\pi^2}{16}\sin(x-t)$

163. $K_2(x, t) = \frac{2}{3}(x+t)^2+2x^2t^2+\frac{4}{3}xt+\frac{2}{5}$

$K_3(x, t) = \frac{56}{45}(x^2+t^2)+\frac{8}{3}x^2t^2-\frac{32}{9}xt+\frac{8}{15}$

164. $K_{2n-1}(x, t) = (2\pi)^{2n-2}(x+\sin t)$

$K_{2n}(x, t) = (2\pi)^{2n-1}(1+x\sin t) \; (n = 1, 2, \ldots)$

165. $K_n(x, t) = xe^t$

166. $K_n(x, t) = (-1)^{n-1}\left(\frac{e^\pi+1}{2}\right)^{n-1} e^x \cos t$

167. $K_2(x, t) = \begin{cases} \frac{e^{x+t}+e^{2-x-t}}{2} + (t-x-1)e^{t-x}, & 0 \leqslant x \leqslant t \\ \frac{e^{x+t}+e^{2-x-t}}{2} + (x-t-1)e^{x-t}, & t \leqslant x \leqslant 1 \end{cases}$

168. $K_2(x, t) = \begin{cases} \frac{e^2+1}{2} e^{t-x}, & -1 \leqslant x \leqslant 0 \\ \frac{e^2+1}{2} e^{t+x}, & 0 \leqslant x \leqslant 1 \end{cases}$

169. $R(x, t; \lambda) = \frac{2e^{x+t}}{2-(e^2-1)\lambda}; \ |\lambda| < \frac{2}{e^2-1}$

170. $R(x; t; \lambda) = \frac{2 \sin x \cos t}{2-\lambda}; \ |\lambda| < 2$

171. $R(x, t; \lambda) = \frac{xe^{t+1}}{e-2\lambda}; \ |\lambda| < \frac{e}{2}$

172. $R(x, t; \lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}; \ |\lambda| < \frac{2}{3}$

173. $R(x, t; \lambda) = \frac{5x^2t^2}{5-2\lambda}; \ |\lambda| < \frac{5}{2}$

174. $R(x, t; \lambda) = \frac{3xt}{3-2\lambda}; \ |\lambda| < \frac{3}{2}$

175. $R(x, t; \lambda) = \sin x \cos t + \cos 2x \sin 2t$

176. $R(x, t; \lambda) = \frac{1}{1-\lambda} + \frac{3(2x-1)(2t-1)}{3-\lambda}; \ |\lambda| < 1$

180. $\varphi(x) = \frac{\pi^2}{\pi-1} \sin^2 x + 2x - \pi$

181. $\varphi(x) = \tan x$

182. $\varphi(x) = \frac{\pi}{2} \cdot \lambda + \cot x$

183. $\varphi(x) = \frac{1+q^2}{1+q^2-\lambda}$

184. $\varphi(x) = -\frac{\pi^2\lambda}{8(\lambda-1)} + \frac{1}{\sqrt{1-x^2}}, \ \lambda \neq 1$

185. $\varphi(x) = \frac{1}{1-\lambda\Gamma(p+1)}$

186. $\varphi(x) = \frac{2\lambda^2 x + \left(\frac{\lambda^2}{4}+\lambda\right) \ln x}{1+\frac{29}{48}\lambda^2} + \frac{6}{5}(1-4x)$

187. $\varphi(x)=\frac{2}{2-\lambda}\sin x;\ \lambda\neq 2$

188. $\varphi(x)=\lambda\pi^3\sin x+x$

189. $\varphi(x)=2\,\frac{2\cos x+\pi\lambda\sin x}{4+\pi^2\lambda^2}$

190. $\varphi(x)=\lambda\pi\sin x+\cos x$

191. $\varphi(x)=\frac{15}{32}(x+1)^2+\frac{5}{16}$

192. $\varphi_1(x)\equiv 0;\ \varphi_{2,3}(x)=\pm\sqrt{\frac{5}{2}}\,x$

193. $\varphi_1(x)\equiv 0;\ \varphi_2(x)=\frac{7}{2}x^2;\ \varphi_{3,4}(x)=\pm\frac{15}{4\sqrt{7}}\,x+\frac{5}{4}x^2$

194. $\varphi_1(x)\equiv 0;\ \varphi_{2,3}(x)=\pm 3x^2$ **195.** $\varphi(x)\equiv 0$

196. Has no solutions.

198. $\lambda_1=\frac{8}{\pi-2};\ \varphi_1(x)=\sin^2 x$

199. No. **200.** $\lambda_1=\frac{1}{\pi};\ \varphi_1(x)=\sin x$

201. $\lambda_1=-\frac{2}{\pi},\ \lambda_2=\frac{2}{\pi};\ \varphi_1(x)=\sin x,$

$\varphi_2(x)=\cos x$

202. There are no real characteristic numbers and eigenfunctions.

203. $\lambda_1=\lambda_2=-3;\ \varphi(x)=x-2x^2$

204 $\lambda_1=\frac{1}{2};\ \varphi_1(x)=\frac{5}{2}x+\frac{10}{3}x^2$

205. $\lambda_1=\frac{1}{4};\ \varphi_1(x)=\frac{3}{2}x+x^2$

206. $\lambda_1=-\frac{e}{2};\ \varphi_1(x)=\sinh x$ **207.** None

208. There are no real characteristic numbers and eigenfunctions.

209. $\lambda_n=-n^2\pi^2;\ \varphi_n(x)=\sin n\pi x\quad (n=1,\ 2,\ \ldots)$

210. $\lambda_0=1;\ \varphi_0(x)=e^x;\ \lambda_n=-n^2\pi^2;$

$\varphi_n(x)=\sin n\pi x+n\pi\cos n\pi x\quad (n=1,\ 2,\ \ldots)$

211. $\lambda_n=-\frac{\mu^2{}_n}{3};\quad \varphi_n(x)=\sin\mu_n x+\mu_n\cos\mu_n x,$ where μ_n is a root of the equation $\mu-\frac{1}{\mu}=2\cot\mu.$

212. $\lambda_n = 4n^2 - 1$; $\varphi_n(x) = \sin 2nx \quad (n = 1, 2, \dots)$

213. $\lambda_n = \left(n + \frac{1}{2}\right)^2 - 1$; $\varphi_n(x) = \sin\left(n + \frac{1}{2}\right)x$

214. $\lambda_n = \frac{1-\mu^2{}_n}{\sin 1}$, $\varphi_n(x) = \sin \mu_n (\pi + x) \quad (n = 1, 2, \dots)$, where μ_n are roots of the equation $\tan 2\pi\mu = -\mu \tan 1$.

215. $\lambda_n = 1 - \mu_n^2$; $\varphi_n(x) = \sin \mu_n x + \mu_n \cos \mu_n x$, where μ_n are roots of the equation $2 \cot \pi\mu = \mu - \frac{1}{\mu}$.

216. $\lambda_n = \frac{1+\mu_n^2}{2}$; $\varphi_n(x) = \sin \mu_n x + \mu_n \cos \mu_n x$, where μ_n are roots of the equation $2 \cot \mu = \mu - \frac{1}{\mu}$.

217. $\lambda_n = -1 - \mu_n^2$; $\varphi_n(x) = \sin \mu_n x$, where μ_n are roots of the equation $\tan \mu = \mu \ (\mu > 0)$.

221. (a) $\frac{\pi^4}{90}$; (b) $\frac{\pi^2}{16} - \frac{1}{2}$; (c) $\frac{1+e^{-2}}{8}$

222. $\varphi_1(x) = 1$; $\varphi_2(x) = 2x - 1$;

223. $\varphi_1(x) = x$; $\varphi_2(x) = x^2$

224. $\varphi_1(x) = 1$; $\varphi_2(x) = x$; $\varphi_3(x) = 3x^2 - 1$

225. $\lambda_0 = \frac{1}{\pi}$, $\varphi_0(x) = 1$; $\lambda_1 = \frac{2}{\pi}$, $\varphi_1^{(1)}(x) = \cos 2x$, $\varphi_1^{(2)}(x) = \sin 2x$

227. $\lambda_0 = \frac{3}{2\pi^3}$, $\varphi_0(x) = 1$; $\lambda_n = (-1)^n \frac{n^2}{4\pi}$
$\varphi_n^{(1)}(x) = \cos nx$, $\varphi_n^{(2)}(x) = \sin nx \quad (n = 1, 2, \dots)$

228. (a) $\frac{1}{3}$, $\varphi(x) = \pm\sqrt{3}\,x$; (b) $\frac{2}{3}$; $\varphi(x) = \pm\sqrt{\frac{3}{2}}\,x$; (c) 1; $\varphi(x) = \pm\sqrt{\frac{2}{e^2-1}}\,e^x$ **229.** $\lambda = 3$

230. There are no bifurcation points.

231. $$\varphi(x) = \begin{cases} C \cdot \sin x, & \lambda = -\frac{2}{\pi} \\ C \cdot \cos x, & \lambda = \frac{2}{\pi} \\ 0, & \lambda \neq \pm\frac{2}{\pi} \end{cases}$$

232. $\varphi(x)=\begin{cases} C\cdot\arccos x, & \lambda=1 \\ 0, & \lambda\neq 1 \end{cases}$ **233.** $\varphi(x)=C$

234. $\varphi(x)=C\,|x|$ **235.** $\varphi(x)=C\,(x-x^2)$

236. $\varphi(x)=\sin\frac{\pi x}{2}$ **237.** $\varphi(x)=x-2+2e^x$

238. $$\varphi(x)=\begin{cases} \dfrac{\sin\mu x+\sin\mu(x-1)-\mu\cos\mu x}{2\mu\cos\frac{\mu}{2}\left(\cos\frac{\mu}{2}+\frac{\mu}{2}\sin\frac{\mu}{2}\right)}, & \lambda>0, \\[2ex] \dfrac{\sinh\mu x+\sinh\mu(x+1)-\mu\cosh\mu x}{2\mu\cosh\frac{\mu}{2}\left(\cosh\frac{\mu}{2}-\frac{\mu}{2}\sinh\frac{\mu}{2}\right)}, & \lambda<0, \\ \text{where } \mu=\sqrt{2\lambda} \end{cases}$$

239. $$\varphi(x)=\cos 2x+4\sum_{n=1}^{\infty}\frac{n\sin 2nx}{(4n^2-1)(4n^2-3)}$$

240. $$\varphi(x)=\begin{cases} \dfrac{\lambda\cos\sqrt{\lambda+1}\,(\pi-x)+\cos\sqrt{\lambda+1}\,\pi}{(\lambda+1)\cos\pi\sqrt{\lambda+1}}, & \lambda>-1 \\[2ex] \dfrac{\lambda\cosh\sqrt{-\lambda-1}\,(\pi-x)+\cosh\sqrt{-\lambda-1}\,\pi}{(\lambda+1)\cdot\cosh\pi\sqrt{-\lambda-1}}, & \lambda<-1 \\[2ex] \dfrac{x^2}{2}-\pi x+1, & \lambda=-1 \end{cases}$$

241. $$\varphi(x)=\begin{cases} \dfrac{3(\sinh\mu+\mu\cosh\mu x)+\sinh\mu(x-1)-2\mu\cosh\mu(x-1)}{(1+2\mu^2)\cdot\sinh\mu+3\mu\cosh\mu} \\ \qquad\qquad \lambda>0\quad(\mu=2\sqrt{\lambda}) \\[2ex] \dfrac{3(\sin\mu+\mu\cos\mu x)+\sin\mu(x-1)-2\mu\cos\mu(x-1)}{(1-2\mu^2)\sin\mu+3\mu\cos\mu} \\ \qquad\qquad \lambda<0\quad(\mu=2\sqrt{-\lambda}) \end{cases}$$

242. $\varphi(x)=-1$ **243.** $\varphi(x)=\dfrac{e\cdot\sinh\sqrt{2}\,x}{\sinh\sqrt{2}+\sqrt{2}\cosh\sqrt{2}}$

244. $$\varphi(x)=\begin{cases} \dfrac{-\sinh 1\cdot\cos\mu x}{\mu\sin\mu}, & \lambda>1\quad(\mu=\sqrt{\lambda-1}) \\[2ex] \dfrac{\sinh 1\cdot\cosh\mu x}{\mu\sinh\mu}, & \lambda<1\quad(\mu=\sqrt{1-\lambda}) \\ \text{no solutions if } \lambda=1. \end{cases}$$

245. $\varphi(x)=\begin{cases} \dfrac{\cosh\mu\left(x-\frac{\pi}{2}\right)}{\cosh\frac{\mu\pi}{2}-\frac{\mu\pi}{2}\sinh\frac{\mu\pi}{2}} & \text{if } \mu=\sqrt{2\lambda},\ \lambda>0, \\ \dfrac{\cos\mu\left(x-\frac{\pi}{2}\right)}{\cos\frac{\mu\pi}{2}+\frac{\mu\pi}{2}\sin\frac{\mu\pi}{2}} & \text{if } \mu=\sqrt{-2\lambda},\ \lambda<0 \end{cases}$

$\varphi(x)\equiv 1$ if $\lambda=0$; μ is not a root of the equations

$$\cosh\frac{\mu\pi}{2}-\frac{\mu\pi}{2}\sinh\frac{\mu\pi}{2}=0,$$
$$\cos\frac{\mu\pi}{2}+\frac{\mu\pi}{2}\sin\frac{\mu\pi}{2}=0.$$

246. $\varphi(x)=1+\frac{2\pi\lambda}{2-\pi\lambda}\cos^2 x,\quad \lambda\neq\frac{2}{\pi}$. No solutions for $\lambda=\frac{2}{\pi}$.

247. $\varphi(x)=\frac{e}{e-2\lambda}x,\ \lambda\neq\frac{e}{2}$. No solutions for $\lambda=-\frac{e}{2}$.

248. $\varphi(x)=x+\frac{2\pi^2\lambda\,|x-\pi|}{1-\pi^2\lambda},\ \lambda\neq\frac{1}{\pi^2}$. No solutions for $\lambda=\frac{1}{\pi^2}$

249. $\varphi(x)=\frac{3x(2\lambda^2x-2\lambda^2-5\lambda-6)+(\lambda+3)^2}{(\lambda+3)^2},\ \lambda\neq-3$. No solutions for $\lambda=-3$.

250. $\varphi(x)=\begin{cases} x^3-\dfrac{3(4\lambda+5)\cdot x}{5(4\lambda+3)} & \text{if } \lambda\neq\frac{3}{2},\ \lambda\neq-\frac{3}{4} \\ x^3-\frac{11}{15}x+Cx^2 & \text{if } \lambda=\frac{3}{2} \end{cases}$

For $\lambda=-\frac{3}{4}$ there are no solutions.

251. $\varphi(x)=\begin{cases} \sin x & \text{if } \lambda\neq 1 \\ C_1\cos x+C_2\sin 2x+\sin x & \text{if } \lambda=1. \end{cases}$

252. $\varphi(x)=-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ if $\lambda=-1$;

$$\varphi(x)=\left\{\frac{(\mu^2-1)\cosh\mu x}{\cosh\mu-\mu\sinh\mu\tanh 1}+1\right\}\frac{1}{\mu^2}$$

if $\lambda=\mu^2-1$, where μ is not a root of the equation $\cosh\mu=\mu\sinh\mu\cdot\tanh 1$; $\varphi(x)=\frac{1}{\mu^2}\left[\frac{(\mu^2+1)\cos\mu x}{\cos\mu+\mu\sin\mu\tanh 1}-1\right]$

if $\lambda=-(\mu^2+1)$ where μ is not a root of the equation $\cos\mu+\mu\sin\mu\cdot\tanh 1=0$. In the remaining cases there are no solutions.

253. $G(x,\xi)=\begin{cases}\xi-1+(\xi-2)x, & 0\leqslant x\leqslant\xi\\ (\xi-1)x-1, & \xi\leqslant x\leqslant 1\end{cases}$

254. It is obvious that the equation $y''(x)=0$ has an infinity of solutions $y(x)=C$ under the conditions $y(0)=y(1)$, $y'(0)=y'(1)$. Therefore, Green's function does not exist for this boundary-value problem.

255. Green's function does not exist.

256. $G(x,\xi)=\begin{cases}\dfrac{x^2}{6}(3\xi-x), & 0\leqslant x\leqslant\xi\\ \dfrac{\xi^2}{6}(3x-\xi), & \xi\leqslant x\leqslant 1\end{cases}$

257. $G(x,\xi)=\begin{cases}\dfrac{x(\xi-1)}{2}(x-x\xi+2\xi), & 0\leqslant x\leqslant\xi\\ \dfrac{\xi}{2}[x(2-x)(\xi-2)+\xi], & \xi\leqslant x\leqslant 1\end{cases}$

258. $G(x,\xi)=\begin{cases}\dfrac{x(x-\xi)(\xi-1)}{2}, & 0\leqslant x\leqslant\xi\\ \dfrac{-\xi(\xi-x)(x-1)}{2}, & \xi\leqslant x\leqslant 1\end{cases}$

259. Green's function does not exist. **260.** Green's function does not exist.

261. $G(x,\xi)=\begin{cases}\dfrac{\sinh k(\xi-1)\sinh kx}{k\sinh k}, & 0\leqslant x\leqslant\xi\\ \dfrac{\sinh k\xi\sinh k(x-1)}{k\sinh k}, & \xi\leqslant x\leqslant 1\end{cases}$

262. $G(x,\xi)=\begin{cases}\dfrac{\cos\left(x-\xi+\frac{1}{2}\right)}{2\sin\frac{1}{2}}, & 0\leqslant x\leqslant\xi\\ \dfrac{\cos\left(\xi-x+\frac{1}{2}\right)}{2\sin\frac{1}{2}}, & \xi\leqslant x\leqslant 1\end{cases}$

263. Green's function does not exist.

264. $$G(x,\ \xi)=\begin{cases}\dfrac{(hx+1)\,[H\,(\xi-1)-1]}{h+H+hH}, & 0\leqslant x\leqslant\xi\\[2mm] \dfrac{(h\xi+1)\,[H\,(x-1)-1]}{h+H+hH}, & \xi\leqslant x\leqslant 1\end{cases}$$

265. $$G(x,\ \xi)=\begin{cases}\alpha+1-\dfrac{1}{\xi}, & 0\leqslant x\leqslant\xi\\[2mm] \alpha+1-\dfrac{1}{x}, & \xi\leqslant x\leqslant 1\end{cases}$$

266. $$G(x,\ \xi)=\begin{cases}\xi-\ln\xi-1-\dfrac{x\,(\xi-1)^2}{2\xi}, & 0\leqslant x\leqslant\xi\\[2mm] x-\ln x-1-\dfrac{\xi\,(x-1)^2}{2x}, & \xi\leqslant x\leqslant 1\end{cases}$$

267. $$G(x,\ \xi)=\begin{cases}\dfrac{x}{2}\left(1-\dfrac{1}{\xi^2}\right), & 0\leqslant x\leqslant\xi\\[2mm] \dfrac{1}{2}\left(x-\dfrac{1}{x}\right), & \xi\leqslant x\leqslant 1\end{cases}$$

268. $$G(x,\ \xi)=\begin{cases}\dfrac{x}{2}\left(\xi-\dfrac{1}{\xi}\right), & 0\leqslant x\leqslant\xi\\[2mm] \dfrac{\xi}{2}\left(x-\dfrac{1}{x}\right), & \xi\leqslant x\leqslant 1\end{cases}$$

269. $$G(x,\ \xi)=\begin{cases}\dfrac{1}{2n\xi}\left[(x\xi)^n-\left(\dfrac{x}{\xi}\right)^n\right], & 0\leqslant x\leqslant\xi\\[2mm] \dfrac{1}{2n\xi}\left[(x\xi)^n-\left(\dfrac{\xi}{x}\right)^n\right], & \xi\leqslant x\leqslant 1\end{cases}$$

270. $$G(x,\ \xi)=\begin{cases}-\dfrac{x\ln\xi}{\xi^2\,(\ln\xi-1)^2}, & 0\leqslant x\leqslant\xi\\[2mm] -\dfrac{\ln x}{\xi\,(\ln\xi-1)^2}, & \xi\leqslant x\leqslant 1\end{cases}$$

271. $$G(x,\ \xi)=\begin{cases}\dfrac{1}{2}\ln\dfrac{1-x}{1+x}, & 0\leqslant x\leqslant\xi\\[2mm] \dfrac{1}{2}\ln\dfrac{1-\xi}{1+\xi}, & \xi\leqslant x\leqslant 1\end{cases}$$

272. $$G(x,\ \xi)=\begin{cases}\ln\dfrac{\xi}{l}, & 0\leqslant x\leqslant\xi\\[2mm] \ln\dfrac{x}{l}, & \xi\leqslant x\leqslant l\end{cases}$$

273. $G(x,\ \xi)=\begin{cases} \left[\frac{1-\lambda}{2(1+\lambda)}e^{\xi-2l}-\frac{1}{2}e^{-\xi}\right]e^{x}, \\ 0\leqslant x\leqslant\xi & (|\lambda|\neq 1) \\ \frac{1-\lambda}{2(1+\lambda)}e^{\xi-2l+x}-\frac{1}{2}e^{\xi-x}, & \xi\leqslant x\leqslant l \end{cases}$

For $\lambda=1$, $G(x,\ \xi)=-\frac{1}{2}e^{-|x-\xi|}$ does not depend on l. For $\lambda=-1$ Green's function does not exist.

274. $y=x-\frac{\pi}{2}\sin x$ **275.** $y=\frac{x^2}{24}(x^2-4x+6)$

276. $y=\frac{1}{4}[(1-e^2)\ln x+x^2-1]$

277. $y=\frac{1}{4\pi}(2x-1)\sin\pi x.$

278. $y=2[\sinh x-\sinh(x-1)-\sinh 1].$

279. $y=\sinh x+(l-x)e^{x}$

280. $y=2\cos x+\left(2-\frac{\pi^2}{4}\right)\sin x+x^2-2$

281. $y(x)=\lambda\int\limits_0^{\frac{\pi}{2}}G(x,\ \xi)\,y(\xi)\,d\xi+\frac{x^4}{12}-\frac{\pi^3}{96}x,$

where $G(x,\ \xi)=\begin{cases}\left(\frac{2\xi}{\pi}-1\right)x, & 0\leqslant x\leqslant\xi \\ \xi\left(\frac{2x}{\pi}-1\right), & \xi\leqslant x\leqslant\frac{\pi}{2}\end{cases}$

282. $y(x)=\lambda\int\limits_0^1 G(x,\ \xi)\,y(\xi)\,d\xi+e^{x}-ex+x-1$

where $G(x,\ \xi)=\begin{cases}(\xi-1)x, & 0\leqslant x\leqslant\xi \\ (x-1)\xi, & \xi\leqslant x\leqslant 1\end{cases}$

283. $y(x)=\lambda\int\limits_{-1}^{1}G(x,\ \xi)\,y(\xi)\,d\xi+\frac{x}{\pi}\sin\frac{\pi x}{2}+\frac{2}{\pi^2}\cos\frac{\pi x}{2}$

where $G(x,\ \xi)=\begin{cases}\frac{1}{\pi}\sin\frac{\pi}{2}(\xi-x), & -1\leqslant x\leqslant\xi \\ \frac{1}{\pi}\sin\frac{\pi}{2}(x-\xi), & \xi\leqslant x\leqslant 1\end{cases}$

284. $y(x) = -\lambda \int_0^1 G(x, \xi)\, y(\xi)\, d\xi + \frac{1}{6}(2x^3 + 3x^2 - 17x - 5)$

where $G(x, \xi) = \begin{cases} (\xi - 2)x + \xi - 1, & 0 \leqslant x \leqslant \xi \\ (\xi - 1)x - 1, & \xi \leqslant x \leqslant 1 \end{cases}$

285. $y(x) = \lambda \int_0^1 G(x, \xi)\, y(\xi)\, d\xi + \frac{x^2}{24}(x^2 - 4x + 6)$

where $G(x, \xi) = \begin{cases} \frac{x^2}{6}(3\xi - x), & 0 \leqslant x \leqslant \xi \\ \frac{\xi^2}{6}(3x - \xi), & \xi \leqslant x \leqslant 1 \end{cases}$

286. $y(x) = -\lambda \int_0^1 G(x, \xi)\, y(\xi)\, d\xi + \frac{1}{12} x(x-1)(x^2 + x - 1)$

where $G(x, \xi) = \begin{cases} \frac{1}{2} x(x - \xi)(\xi - 1), & 0 \leqslant x \leqslant \xi \\ -\frac{1}{2}\xi(\xi - x)(x - 1), & \xi \leqslant x \leqslant 1 \end{cases}$

287. $y(x) = e^x - \lambda \int_0^1 G(x, \xi)\, y(\xi)\, d\xi$

where $G(x, \xi) = \begin{cases} (1 + x)\xi, & 0 \leqslant x \leqslant \xi \\ (1 + \xi)x, & \xi \leqslant x \leqslant 1 \end{cases}$

292. $\varphi(x) = \cos x$ **293.** $\varphi(x) = x^2 - \frac{1}{2}(\cosh x - \cos x)$

294. $\varphi(x) = \frac{\Gamma\left(\frac{5}{2}\right)}{3!} x^3 + \cosh 2x$

295. $\varphi(x) = \frac{1}{1 - \lambda^2}[e^x - \lambda(e^x - 1)]$

296. $\varphi(x) = \frac{1}{1 - \lambda^2}(\cos x + \lambda \sin x)$

297. $\varphi(x) = \frac{1}{1 - \lambda^2}[\cos x + \lambda(x - \sin x)]$

298. $\varphi(x) = \frac{\sin x}{1 - \lambda}$ **299.** $\varphi(x) = \frac{2}{1 + x^2} + \sqrt{\pi}\, e^{-x}$

300. $\varphi(x) = \frac{f(x)}{1 - \lambda^2} + \frac{\lambda}{1 - \lambda^2} \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt\, dt \quad (|\lambda| \neq 1)$

301. $\varphi(x)=e^{-x}+\dfrac{2}{\sqrt{\pi}\,(1+x^2)}$

302. $\tilde{\varphi}(x)=e^{x}-x-0.5010x^2-0.1671x^3-0.0422x^4$; $|\varphi-\tilde{\varphi}|<0.18$; the exact solution $\varphi(x)\equiv 1$

303. $\tilde{\varphi}(x)=\cos x+\dfrac{x}{89}[78-78\cdot\sin 1-24\cdot\cos 1+x(84\sin 1+108\cos 1-84)]$; $|\varphi-\tilde{\varphi}|<0.040$; the exact solution $\varphi(x)\equiv 1$.

304. $\tilde{\varphi}(x)=\dfrac{1}{2}(e^{-x}+3x-1)-0.252x^2+0.084x^3$; $|\varphi-\tilde{\varphi}|<0.016$; the exact solution $\varphi(x)=x$.

305. $\tilde{\varphi}(x)=\dfrac{x}{2}+\dfrac{1}{2}\sin x+\left(\dfrac{58}{9}-\dfrac{16}{3}\sin 1-\dfrac{52}{15}\cos 1\right)x^3$ $|\varphi-\tilde{\varphi}|<0.0057$; the exact solution $\varphi(x)=x$.

306. $\varphi(x)=1+\dfrac{4}{9}x$; $\varphi_0(x)=1$

307. $\varphi_n(x)=\left(1-\dfrac{5}{6^n}\right)x$; $\varphi_0(x)=0$; the exact solution $\varphi(x)=x$.

308. $\varphi_3(x)=1+\dfrac{22}{15}x-\dfrac{1}{2}x^2-\dfrac{2}{9}x^3+\dfrac{1}{24}x^4-\dfrac{1}{120}x^5-\dfrac{1}{720}x^6$; $\varphi_0(x)=1$; the exact solution $\varphi(x)=\cos x+\tan 1\cdot\sin x$.

309. $\varphi_3(x)=6x^2+1$ is the exact solution.

310. $\varphi_3(x)=1$ is the exact solution.

311. $\varphi_3(x)=1$ is the exact solution.

312. $\lambda_1=5\frac{1}{7}$ (exact value $\lambda_1=5$).

313. $\lambda_1=2.486$; $\lambda_2=32.181$. **314.** $\lambda_1=4.59$

315. $\lambda_1=3$ **316.** $\lambda_1=5$ **317.** $\lambda_1=4.19$

318. $\lambda_1=5.78$ **319.** $\lambda_1=3$ **320.** $\lambda_1=4$

321. $\lambda_1=2.475$ **322.** $\lambda_1=4.998$

APPENDIX

SURVEY OF BASIC METHODS FOR SOLVING INTEGRAL EQUATIONS

I. Volterra Integral Equations

Volterra integral equations of the second kind:

$$\varphi(x)=f(x)+\lambda\int_a^x K(x,\ t)\,\varphi(t)\,dt \tag{1}$$

1. Method of resolvent kernels. The solution of equation (1) is given by the formula

$$\varphi(x)=f(x)+\lambda\int_a^x R(x,\ t;\ \lambda)\,f(t)\,dt \tag{2}$$

The function $R(x, t; \lambda)$ is the resolvent kernel of the integral equation (1) and is defined as the sum of the series

$$R(x,\ t;\ \lambda)=\sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,\ t) \tag{3}$$

where the iterated kernels $K_{n+1}(x, t)$ are found from the recursion formula

$$\left.\begin{aligned} &K_{n+1}(x,\ t)=\int_t^x K(x,\ z)\,K_n(z,\ t)\,dz \\ &K_1(x,\ t)=K(x,\ t)\quad(n=1,\ 2,\dots) \end{aligned}\right\} \tag{4}$$

2. Method of successive approximations. The solution of equation (1) is defined as the limit of a sequence $\{\varphi_n(x)\}$, $n=0, 1, 2,\dots$, the general term of which is found from the recursion formula

$$\varphi_n(x)=f(x)+\lambda\int_0^x K(x,\ t)\,\varphi_{n-1}(t)\,dt \tag{5}$$

It is often convenient to take the function $f(x)$ as the zero approximation $\varphi_0(x)$.

3. Volterra integral equations of the second kind of the convolution type

$$\varphi(x)=f(x)+\int_0^x K(x-t)\,\varphi(t)\,dt \tag{6}$$

are solved with the aid of the Laplace transformation.

Let $f(x)$, $K(x)$ and $\varphi(x)$ be original functions and let

$$f(x) \risingdotseq F(p), \quad K(x) \risingdotseq \tilde{K}(p), \quad \varphi(x) \risingdotseq \Phi(p)$$

Taking the Laplace transforms of both sides of equation (6) and utilizing the product theorem, we find

$$\Phi(p)=\frac{F(p)}{1-\tilde{K}(p)}, \quad \tilde{K}(p)\neq 1 \tag{7}$$

The original function $\varphi(x)$ for $\Phi(p)$ will be the solution of equation (6).

4. Volterra integral equations of the first kind

$$\int_0^x K(x, t)\,\varphi(t)\,dt=f(x) \tag{8}$$

where

$$K(x, x)\neq 0$$

are reduced, by differentiation, to Volterra integral equations of the second kind of the form

$$\varphi(x)=\frac{f'(x)}{K(x, x)}-\int_0^x \frac{K'_x(x, t)}{K(x, x)}\,\varphi(t)\,dt \tag{9}$$

5. Volterra integral equations of the first kind of the convolution type

$$\int_0^x K(x-t)\,\varphi(t)\,dt=f(x) \tag{10}$$

are solved with the aid of the Laplace transformation. If $f(x)$, $K(x)$ and $\varphi(x)$ are original functions and

$$f(x) \risingdotseq F(p), \quad K(x) \risingdotseq \tilde{K}(p), \quad \varphi(x) \risingdotseq \Phi(p),$$

then, by taking Laplace transforms of both sides of (10) and applying the convolution theorem, we obtain

$$\Phi(p)=\frac{F(p)}{\tilde{K}(p)} \tag{11}$$

The original function $\varphi(x)$ for the function $\Phi(p)$ will be the solution of equation (10).

II. Fredholm Integral Equations

Fredholm integral equations of the second kind:

$$\varphi(x)-\lambda\int_a^b K(x, t)\varphi(t)\,dt=f(x) \tag{12}$$

1. Method of Fredholm determinants. The solution of equation (12) is given by the formula

$$\varphi(x)=f(x)+\lambda\int_a^b R(x, t; \lambda)f(t)\,dt \tag{13}$$

where the function

$$R(x, t; \lambda)=\frac{D(x, t; \lambda)}{D(\lambda)}; \qquad D(\lambda)\neq 0 \tag{14}$$

is called the *Fredholm resolvent kernel* of equation (12). Here,

$$D(x, t; \lambda)=K(x, t)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}B_n(x, t)\lambda^n \tag{15}$$

$$D(\lambda)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}C_n\lambda^n \tag{16}$$

The coefficients $B_n(x, t)$, C_n are defined by the relations

$$B_n(x, t)=$$

$$=\underbrace{\int_a^b\cdots\int_a^b}_{n}\begin{vmatrix} K(x, t) & K(x, t_1) & \dots & K(x, t_n)\\ K(t_1, t) & K(t_1, t_1) & \dots & K(t_1, t_n)\\ K(t_2, t) & K(t_2, t_1) & \dots & K(t_2, t_n)\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ K(t_n, t) & K(t_n, t_1) & \dots & K(t_n, t_n)\end{vmatrix}dt_1\dots dt_n \tag{17}$$

$$B_0(x, t) = K(x, t),$$

$$C_n = \underbrace{\int_a^b \dots \int_a^b}_{n} \times$$

$$\times \begin{vmatrix} K(t_1, t_1) & K(t_1, t_2) & \dots & K(t_1, t_n) \\ K(t_2, t_1) & K(t_2, t_2) & \dots & K(t_2, t_n) \\ K(t_3, t_1) & K(t_3, t_2) & \dots & K(t_3, t_n) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ K(t_n, t_1) & K(t_n, t_2) & \dots & K(t_n, t_n) \end{vmatrix} dt_1 \dots dt_n \quad (18)$$

The recursion relations are

$$B_n(x, t) = C_n K(x, t) - n \int_a^b K(x, s) B_{n-1}(s, t)\, ds, \quad (19)$$

$$C_n = \int_a^b B_{n-1}(s, s)\, ds \quad (n = 1, 2, \dots),$$

$$C_0 = 1, \ B_0(x, t) = K(x, t) \quad (20)$$

2. The method of successive approximations. The integral equation (12) may be solved by the method of successive approximations. To do this, assume

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \psi_n(x) \lambda^n \quad (21)$$

where $\psi_n(x)$ are determined from the formulas:

$$\psi_1(x) = \int_a^b K(x, t) f(t)\, dt,$$

$$\psi_2(x) = \int_a^b K(x, t) \psi_1(t)\, dt = \int_a^b K_2(x, t) f(t)\, dt,$$

$$\psi_3(x) = \int_a^b K(x, t) \psi_2(t)\, dt = \int_a^b K_3(x, t) f(t)\, dt \text{ and so forth.}$$

Here,

$$K_2(x,\ t)=\int_a^b K(x,\ z)\,K(z,\ t)\,dz,$$

$$K_3(x,\ t)=\int_a^b K(x,\ z)\,K_2(z,\ t)\,dz$$

and, generally,

$$K_n(x,\ t)=\int_a^b K(x,\ z)\,K_{n-1}(z,\ t)\,dz,\ n=2,\ 3,\dots \tag{22}$$

where $K_1(x,\ t)=K(x,\ t)$.

The functions $K_n(x,\ t)$ defined by formulas (12) are called *iterated kernels*.

3. Integral equations with degenerate kernel:

$$\varphi(x)-\lambda\int_a^b\left[\sum_{k=1}^{n}a_k(x)\,b_k(t)\right]\varphi(t)\,dt=f(x) \tag{23}$$

The solution of equation (23) is

$$\varphi(x)=f(x)+\lambda\sum_{k=1}^{n}C_k a_k(x) \tag{24}$$

where the constants C_k are found from the linear system

$$\left.\begin{array}{l}(1-\lambda a_{11})\,C_1-\lambda a_{12}C_2-\dots-\lambda a_{1n}C_n=f_1,\\ -\lambda a_{21}C_1+(1-\lambda a_{22})\,C_2-\dots-\lambda a_{2n}C_n=f_2,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\\ -\lambda a_{n1}C_1-\lambda a_{n2}C_2-\dots+(1-\lambda a_{nn})\,C_n=f_n\end{array}\right\} \tag{25}$$

Here,

$$a_{km}=\int_a^b a_k(t)\,b_m(t)\,dt;\qquad f_m=\int_a^b b_m(t)\,f(t)\,dt \tag{26}$$

$$(k,\ m=1,\ 2,\ \dots,\ n).$$

If the determinant of the system (25) is not zero, then equation (23) has a unique solution (24).

4. Characteristic numbers and eigenfunctions. The value of the parameter $\lambda \neq 0$ for which the homogeneous Fredholm integral equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x, t)\, \varphi(t)\, dt = 0 \tag{27}$$

has a nonzero solution $\varphi(x) \not\equiv 0$ is called the *characteristic number* of equation (21) or of the kernel $K(x, t)$, while the nonzero solution of this equation is called the *eigenfunction* associated with the characteristic number λ.

If the kernel $K(x, t)$ is continuous or quadratically summable in the square $\Omega\{a \leqslant x,\ t \leqslant b\}$, then to every characteristic number λ there correspond a finite number of linearly independent eigenfunctions.

In the case of an equation with a degenerate kernel

$$\varphi(x) - \lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x)\, b_k(t)\right] \varphi(t)\, dt = 0 \tag{28}$$

the characteristic numbers are roots of the algebraic equation

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda a_{n1} & -\lambda a_{n2} & & 1-\lambda a_{nn} \end{vmatrix} = 0 \tag{29}$$

where $\Delta(\lambda)$ is the determinant of the system (25); the degree of this equation is $p \leqslant n$.

If equation (29) has p distinct roots $(1 \leqslant p \leqslant n)$, then the integral equation (27) has p characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_p$ to which correspond the eigenfunctions

$$\varphi_m(x) = \sum_{k=1}^{n} C_k^{(m)} a_k(x) \quad (m = 1, 2, \dots, p) \tag{30}$$

Here, $C_k^{(m)}$ $(k = 1, 2, \dots, n)$ is the solution of the system (25) that corresponds to the characteristic number λ_m $(m = 1, 2, \dots, p)$. For an arbitrary (nondegenerate) kernel, the characteristic numbers are zeros of the Fredholm determinant $D(\lambda)$, i. e., they are poles of the resolvent kernel $R(x, t; \lambda)$.

If the kernel $K(x, t)$ is Green's function of some homogeneous Sturm-Liouville problem, then finding the characteristic numbers and eigenfunctions reduces to solving the indicated Sturm-Liouville problem.

5. Nonhomogeneous symmetric Fredholm integral equations of the second kind:

$$\varphi(x) - \lambda \int_a^b K(x, t)\, \varphi(t)\, dt = f(x), \tag{31}$$
$$K(x, t) = K(t, x).$$

Let λ_n $(n = 1, 2, \ldots)$ be the characteristic numbers and $\varphi_n(x)$ $(n = 1, 2, \ldots)$ the associated eigenfunctions of the kernel $K(x, t)$.

(a) If the parameter $\lambda \neq \lambda_n$ $(n = 1, 2, \ldots)$, then the integral equation (31) has a unique solution continuous on $[a, b]$:

$$\varphi(x) = f(x) - \lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda - \lambda_n} \varphi_n(x) \tag{32}$$

where

$$a_n = \int_a^b f(x)\, \varphi_n(x)\, dx \tag{33}$$

The series on the right of (32) converges absolutely and uniformly on $[a, b]$.

(b) If the parameter λ coincides with one of the characteristic numbers, say $\lambda = \lambda_k$, of index q (the multiplicity of the characteristic number λ_k), then equation (31) has an infinite number of solutions when and only when $f(x)$ is orthogonal to all eigenfunctions of the characteristic number λ_k, i. e., when the q conditions are fulfilled:

$$\int_a^b f(x)\, \varphi_m(x)\, dx = 0 \qquad (m = 1, 2, \ldots, q) \tag{34}$$

All these solutions are given by the formula

$$\varphi(x) = f(x) - \lambda \sum_{n=q+1}^{\infty} \frac{a_n}{\lambda - \lambda_n} \varphi_n(x) + $$
$$+ C_1\varphi_1(x) + C_2\varphi_2(x) + \ldots + C_q\varphi_q(x) \tag{35}$$

where $C_1, \ldots, C_q$ are arbitrary constants, $\varphi_1(x), \ldots, \varphi_q(x)$ are eigenfunctions of the kernel associated with the characteristic number λ_k.

If even one of the q conditions (34) is not fulfilled, then equation (31) has no solutions.

If the function $f(x)$ is orthogonal to all eigenfunctions $\varphi_n(x)$ of the kernel $K(x, t)$, then this function will itself be a solution of equation (31): $\varphi(x) \equiv f(x)$.

6. Fredholm integral equations of the second kind whose eigenfunctions are classical orthogonal functions:

$$\text{(a)} \quad \varphi(x) - \lambda \int_0^l K(x, t)\varphi(t)\,dt = 0 \tag{33}$$

where

$$K(x, t) = \begin{cases} \dfrac{x(l-t)}{l}, & x \leqslant t, \\ \dfrac{t(l-x)}{l}, & t \leqslant x \end{cases}$$

The characteristic numbers are

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2$$

The eigenfunctions are $\varphi_n(x) = \sin \frac{\pi n x}{l}$ $(n = 1, 2, \ldots)$

$$\text{(b)} \quad \varphi(x) - \lambda \int_{-1}^{1} K(x, t)\varphi(t)\,dt + \frac{1}{2}\int_{-1}^{1} \varphi(t)\,dt = 0$$

where

$$K(x, t) = \begin{cases} \dfrac{1}{2}\ln\dfrac{1+x}{1-t}, & x \leqslant t, \\ \dfrac{1}{2}\ln\dfrac{1+t}{1-x}, & t \leqslant x \end{cases}$$

The characteristic numbers are $\lambda_n = n(n+1)$. The eigenfunctions are $\varphi_n(x) = P_n(x)$, where $P_n(x)$ are Legendre polynomials defined by the formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Since $P_0(x)=1$ and $P_1(x)=x$, it follows, using the recursion formula

$$(n+1)P_{n+1}(x)=x(2n+1)P_n(x)+nP_{n-1}(x),$$

that it is possible to find Legendre polynomials of any degree $n=2, 3, \ldots$

$$\text{(c)}\quad \varphi(x)-\lambda\int_0^{\infty} K(x, t)\varphi(t)\,dt=0$$

where

$$K(x, t)=\begin{cases} \frac{1}{2\nu}\left(\frac{x}{t}\right)^{\nu}, & x\leqslant t, \\ \frac{1}{2\nu}\left(\frac{t}{x}\right)^{\nu}, & t\leqslant x \end{cases}$$

The characteristic numbers are $\lambda_n=\alpha_n^2$, where α_n are roots of the transcendental equation $J_\nu(\alpha)=0$. The eigenfunctions are $\varphi_n(x)=J_\nu(\alpha_n x)$, where $J_\nu(x)$ are Bessel functions of the first kind of order ν. Bessel functions of the first kind are defined by the formula

$$J_\nu(x)=\left(\frac{x}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}$$

$$\text{(d)}\quad \varphi(x)-\lambda\int_0^{+\infty} K(x, t)\varphi(t)\,dt=0$$

where

$$K(x, t)=\begin{cases} e^{\frac{x+t}{2}}\int_t^{+\infty}\frac{e^{-\tau}}{\tau}\,d\tau, & x\leqslant t, \\ e^{\frac{x+t}{2}}\int_x^{+\infty}\frac{e^{-\tau}}{\tau}\,d\tau. & t\leqslant x \end{cases}$$

The characteristic numbers are

$$\lambda_n=n+1$$

The eigenfunctions are

$$\varphi_n(x)=e^{-\frac{x}{2}}L_n(x)$$

where $L_n(x)$ are Chebyshev-Laguerre polynomials defined by the formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})$$

Using the recursion formula

$$L_{n+1}(x) = (2n+1-x)L_n(x) - n^2 L_{n-1}(x) \quad (n \geqslant 1),$$

it is possible to obtain Chebyshev-Laguerre polynomials of any degree n, knowing that $L_0(x) = 1$ and $L_1(x) = -x+1$.

(e) $$\varphi(x) - \lambda \int_{-\infty}^{+\infty} K(x, t)\varphi(t)\,dt = 0$$

where

$$K(x, t) = \begin{cases} \frac{1}{\sqrt{\pi}} e^{\frac{x^2+t^2}{2}} \int_{-\infty}^{x} e^{-\tau^2}\,d\tau \int_{t}^{+\infty} e^{-\tau^2}\,d\tau, & x \leqslant t, \\ \frac{1}{\sqrt{\pi}} e^{\frac{x^2+t^2}{2}} \int_{-\infty}^{t} e^{-\tau^2}\,d\tau \int_{x}^{+\infty} e^{-\tau^2}\,d\tau, & t \leqslant x \end{cases}$$

The characteristic numbers are

$$\lambda_n = 2(n+1) \qquad (n = 0, 1, 2, \ldots)$$

The eigenfunctions are

$$\varphi_n(x) = e^{-\frac{x^2}{2}} H_n(x)$$

where $H_n(x)$ are Chebyshev-Hermite polynomials defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} =$$
$$= (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} + \ldots$$

Knowing $H_0(x) = 1$ and $H_1(x) = 2x$, it is possible to obtain Chebyshev-Hermite polynomials of any degree by using the recursion formula

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

BIBLIOGRAPHY

[1] Angot, André, *Compléments de mathématiques*, Paris, 1957.

[2] Berezin, I. S. and N. P. Zhidkov, *Computation Methods*, Vol. 2, "Fizmatgiz", 1960 (Metody vychislenii).

[3] Courant, R. and D. Hilbert, *Methoden der Matematischen Physik*, Berlin, 1937 (*Methods of Mathematical Physics*, rev. ed. New York, 1953/66).

[4] Gelfand, I. M. and G. E. Shilov, *Generalized Functions and Operations on Them*, "Fizmatgiz", 1958 (Obobshchennye funktsii i deistviya nad nimi).

[5] Goursat, É., *Cours d'analyse*, Paris, 1928.

[6] Gyunter, N. M. and P. O. Kuzmin, *Problems in Higher Mathematics*, Vol. 3, "Gostekhizdat", 1947 (Sbornik zadach po vysshei matematike).

[7] Ince, E. L., *Ordinary Differential Equations*, London, 1927.

[8] Kantorovich, L. V. and V. I. Krylov, *Approximate Methods of Higher Analysis*, Moscow-Leningrad, "Gostekhizdat", 1949 (Priblizhennye metody vysshego analiza).

[9] Krasnoselsky, M. A., *Positive Solutions of Operator Equations*, "Fizmatgiz", 1962 (Polozhitelnye resheniya operatornykh uravnenii).

[10] Krasnoselsky, M. A., *Topological Methods in the Theory of Nonlinear Integral Equations*, "Gostekhizdat", 1951 (Topologicheskie metody v teorii nelineinykh integralnykh uravnenii).

[11] Krasnov, M. L. and G. I. Makarenko, *Operational Calculus. Stability of Motion*, "Nauka", 1964 (Operatsionnoe ischislenie).

[12] Lalesco, T., *Introduction à la théorie des équations intégrales*, Paris, 1912.

[13] Lovitt, W. V., *Linear Integral Equations*, New York, 1924.

[14] Mikhlin, S. G., *Applications of Integral Equations to Certain Problems of Mechanics, Mathematical Physics and Engineering*, "Gostekhizdat", 1947 (Prilozheniya integralnykh uravnenii k nekotorym problemam mekhaniki, matematicheskoi fiziki i tekhniki).

[15] Mikhlin, S. G., *Lectures on Linear Integral Equations*, "Fizmatgiz", 1959 (Lektsii po lineinym integralnym uravneniyam).

[16] Mikhlin, S. G. and Kh. L. Smolitsky, *Approximate Methods of Solving Differential and Integral Equations*, "Nauka", 1965 (Priblizhënnye metody resheniya differentsialnykh i integralnykh uravnenii).

[17] Morse, Philip M., Herman Feshbach, *Methods of Theoretical Physics*, New York, 1953.

[18] Petrovsky, I. G., *Lectures on the Theory of Integral Equations*, "Nauka", 1965 (Lektsii po teorii integralnykh uravnenii).

[19] Privalov, I. I., *Integral Equations*, "Gostekhizdat", 1937 (Integralnye uravneniya).

[20] Sansone, Giovanni, *Equazioni differenziali nel campo reale*, P. 1, Bologna, 1948, P. 2, Bologna, 1949.

[21] Schmeidler, W., *Integralgleichungen mit Anwendungen in Physik und Technik*, Leipzig, 1955.

[22] Schwartz, Laurent, *Méthodes mathematiques pour les sciences physiques*, Paris, 1961.

[23] Shelkovnikov, F. A. and K. G. Takoishvili, *Exercises in Operational Calculus*, "Vysshaya shkola", 1961. (Sbornik uprazhnenii po operatsionnomy ischisleniyu).

[24] Smirnov, N. S., *Introduction to the Theory of Nonlinear Integral Equations*, "Gostekhizdat", 1951 (Vvedenie v teoriu nelineinykh integralnykh uravnenii).

[25] Smirnov, V. I., *Course of Higher Mathematics*, Vol. 4, "Fizmatgiz", 1958 (Kurs vysshei matematiki).

[26] Sobolev, S. A., *Equations of Mathematical Physics*, "Nauka", 1966 (Uravneniya matematicheskoi fiziki).

[27] Titchmarsh, E. C., *Introduction to the Theory of Fourier Integrals*, New York, 1937.

[28] Tricomi, F. G., *Integral Equations*, New York, 1957.

[29] Uryson, P. S., *On the Uniqueness of Solution of Linear Volterra Integral Equations. Transactions in Topology and Other Fields of Mathematics*, Vol. 1, pp. 78-84, "Gostekhizdat", 1951 (O edinstvennosti resheniya lineinykh integralnykh uravnenii Volterra).

[30] Wiarda, G. *Integralgleichungen unter besonderer Berüksichtigung der Andwendungen*. Leipzig, 1930.

INDEX

TO THE READER

Mir Publishers would be greateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is: Mir Publishers 2, Pervy Rizhsky Pereulok, Moscow, USSR.

Printed in the Union of Soviet Socialist Republics